Quantum Sabine Law for Resonances in Transmission Problems

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The Plan

- Physical problems and mathematical interpretation
- Brief review of scattering resonances
- Heuristic analysis
- Theorems on the distribution of resonances
- Selections from the proof
Problem from Physics: Quantum Corrals

Figure: STM Image of a Quantum Corral (Barr, Zaletel, and Heller)
Problems from Physics: Transparent Obstacles
Problems from Physics: Concert Halls
Mathematical Models

- Quantum Corrals/Concert Halls

\[
(\partial_t^2 - \Delta + \delta_{\partial\Omega} \otimes V)u = F,
\]
in \( \mathbb{R}^d \)

\[
(\partial_t^2 - \Delta + i\delta_{\partial\Omega} \otimes V \partial_t)u = F,
\]
in \( \mathbb{R}^d \)

- Transparent Obstacles

\[
\begin{cases}
(\partial_t^2 - c^2 \Delta)u_1 = F_1 \\
(\partial_t^2 - \Delta)u_2 = F_2
\end{cases}
\]
in \( \mathbb{R}^d \setminus \overline{\Omega} \)

\[
uu_1 = u_2
\]
on \( \partial\Omega \)

\[
\partial_{\nu} u_1 = \alpha \partial_{\nu} u_2
\]
on \( \partial\Omega \)
These are all transmission problems

In (G–Smith ’14) we show that

$$(\partial^2_t - \Delta + \delta_{\partial\Omega} \otimes V)u = F, \quad \text{in } \mathbb{R}^d$$

is equivalent to

$$\begin{cases}
(\partial^2_t - \Delta)u_1 = F_1 & \text{in } \Omega \\
(\partial^2_t - \Delta)u_2 = F_2 & \text{in } \mathbb{R}^d \setminus \overline{\Omega} \\
u_1 = u_2 & \text{on } \partial\Omega \\
\partial_\nu u_1 - \partial_\nu u_2 + Vu_1 = 0 & \text{on } \partial\Omega
\end{cases}$$

For

$$(\partial^2_t - \Delta + i\delta_{\partial\Omega} \otimes V\partial_t)u = 0, \quad \text{in } \mathbb{R}^d$$

replace the second boundary condition with

$$\partial_\nu u_1 - \partial_\nu u_2 + iVu_1 = 0 \quad \text{on } \partial\Omega$$
History of Transmission Problems

- Wave equation transmission problems - Melrose–Taylor (1975, 1981), Safarov (1987), ...
- Physical analysis - Barr, Zaletel, Heller (2010), Zaletel (2010)...
Videos of solutions to transmission problems

Link to videos
Conversion to a stationary problem

Taking the Fourier transform

\[ \hat{u} = \mathcal{F}_{t \rightarrow \lambda} u = \int_0^\infty e^{i\lambda t} u(t, x) dt. \]

under the assumption that \( u \equiv 0 \) on \( t \leq 0 \). Gives for \( \text{Im} \, \lambda \gg 1 \),

\[ (P - \lambda^2) \hat{u} = \hat{F}. \]

So, we study the meromorphic continuation of

\[ (P - \lambda^2)^{-1} : L^2_{\text{comp}} \rightarrow L^2_{\text{loc}} \]

from \( \text{Im} \, \lambda > 0 \). Poles of this continuation are called \textit{scattering resonances}. 
Scattering resonances

Scattering resonances can often be characterized by solutions to

\[(P - \lambda^2)u = 0, \quad u \text{ is } \lambda \text{ outgoing.}\]

Moreover, the solution to the wave equation

\[(\partial_t^2 + P)u = 0\]

can often be expanded (at least in odd dimensions) roughly in the form

\[u \sim \sum_{\lambda \in \Lambda} e^{-it\lambda} u_\lambda\]

where \(\Lambda\) denotes the set of scattering resonances. So, for \(\lambda \in \Lambda\), \(\text{Re } \lambda\) and \(-\text{Im } \lambda\) denote respectively the frequency and decay rate of the associated resonance state, \(e^{-it\lambda} u_\lambda\).
How do waves behave in the presence of an interface

$$c^2|\langle \xi_1, \xi' \rangle|^2 = 1$$  Interface  $$|(\tilde{\xi}_1, \xi')|^2 = 1$$

$$u_1 = e^{i\langle x, \xi \rangle} + Re^{i(-x_1 \xi_1 + \langle x', \xi' \rangle)}$$

$$u_2 = T e^{i(x_1 \tilde{\xi}_1 + \langle x', \xi' \rangle)}$$

$x_1 < 0$  

$x_1 > 0$  

$1$  

$|R|$  

$|T|$
Geometry in transmission problems

- **Incoming ray**
- **Reflected ray**
- **Refracted ray**
- \( |\xi| = 1 \)
- \( |\xi| = c^{-1} \)
- \( \partial \Omega \)

**Total Internal Reflection**
- \( |\xi| = 1 \)
- \( |\xi| = c^{-1} \)
- \( \partial \Omega \)

- **C = 1**
- **C > 1**
- **C < 1**
The Sabine Law

Decay like

\[ \exp \left[ t \left( \log |R|^2 / 2c^{-1}l \right) \right], \quad \text{where} \]

- \( R(x', \xi') \) is the reflection coefficient associated to each point in \( B^* \partial \Omega \).
- \( l(x', \xi') \) is the length of the billiards trajectory originating at \((x', \xi')\).
- \( (\cdot) \) denotes averaging over iterations of the billiard ball map.

\[ \Im \lambda = \frac{\log |R|^2}{2c^{-1}l} \]

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The billiard ball map

Let $B^* \partial \Omega$ be the coball bundle of the boundary. Let $\beta : B^* \partial \Omega \to \overline{B^* \partial \Omega}$ be the billiard ball map.

Figure: Let $q = (x, \xi) \in B^* \partial \Omega$. The solid black arrow on the left denotes the covector $\xi \in B^*_x \partial \Omega$ and that on the right $\xi(\beta(q)) \in B^*_{\pi(x)(\beta(q))} \partial \Omega$. 
Theorem (Cardoso–Popov–Vodev ’99,’01)

Let $\Omega \subseteq \mathbb{R}^d$ be strictly convex with smooth boundary.

- Suppose that $c > 1$. Then there exists $C > 0$ and $\lambda_0 > 0$ such that for $\lambda \in \Lambda$ with $\text{Re} \lambda > \lambda_0$,
  \[ \text{Im} \lambda \leq -C. \]

- Suppose that $c > 1$. Then there exist $C, C_1 > 0$, $\alpha_0 > 0$, and $\lambda_0 > 0$ such that if $\alpha > \alpha_0$ then for $\lambda \in \Lambda$ with $\text{Re} \lambda > \lambda_0$
  \[ \text{Im} \lambda \geq -C \quad \text{or} \quad \text{Im} \lambda \leq -C_1(\text{Re} \lambda)^{1/3}. \]

- Suppose that $c < 1$. Then there exists $C > 0$, $\alpha_0 > 0$, and $\lambda_0 > 0$ such that if $\alpha < \alpha_0$ then for $\lambda \in \Lambda$ with $\text{Re} \lambda > \lambda_0$,
  \[ \text{Im} \lambda \geq -C \quad \text{or} \quad \text{Im} \lambda \leq -C_1(\text{Re} \lambda)^{1/3}. \]
A few definitions for the transparent obstacle problem

\[ u_1 = u_2, \quad \partial_\nu u_1 = \alpha \partial_\nu u_2. \]

Reflectivity
\[
 r(x', \xi') := \frac{\sqrt{1 - |\xi'|_g^2} - \alpha \sqrt{c^2 - |\xi'|_g^2}}{\alpha \sqrt{c^2 - |\xi'|_g^2} + \sqrt{1 - |\xi'|_g^2}}
\]

\[
 r_N(q) := \frac{\sum_{j=1}^{N} \log |r(\beta_j(q))|^2}{N}
\]

Length
\[
 l(q_1, q_2) := |\pi_x(q_1) - \pi_x(q_2)|
\]

\[
 l_N(q) := \frac{\sum_{j=1}^{N} l(\beta_j^{-1}(q), \beta_j(q))}{N}
\]
Distribution of resonances: transparent obstacle problem

**Theorem (G ’15)**

Let $\Omega \subset \mathbb{R}^d$ be strictly convex with smooth boundary and suppose that $c \neq 1$. Then for all $M, \epsilon > 0$ there exists $\lambda_0 > 0$ such that for $\lambda \in \Lambda$ with $\text{Re} \lambda \geq \lambda_0$ and $\text{Im} \lambda \geq -M \log \text{Re} \lambda$, 

\[
\sup_{N>0} \inf_{|\xi'| \leq 1} \frac{r_N}{2c^{-1}l_N} - \epsilon \leq \text{Im} \lambda \leq \inf_{N>0} \sup_{|\xi'| \leq 1} \frac{r_N}{2c^{-1}l_N} + \epsilon.
\]

Moreover, for every $\alpha$ and $c$ this bound is sharp when $\Omega = B(0, 1) \subset \mathbb{R}^2$.

Recall the Sabine law is given by

\[
\text{Im} \lambda = \frac{\log |R|^2}{2c^{-1}l} = \frac{r_N}{2c^{-1}l_N},
\]
Resonances for the transmission problem

Figure: Numerically computed resonances for various $c$ and $\alpha$ when $\Omega = B(0, 1)$. 
Setup for the $\delta$ potential

We rescale, letting $\lambda = \frac{z}{h}$ with $0 < h \ll 1$ so that

$$(-h^2 \Delta - z^2 + h(h\delta_\partial \Omega \otimes V))u = 0.$$ 

Then, let $|V| \leq h^\alpha$ for some $\alpha \geq -1$ and denote

$$\Lambda_{\log} = \left\{ z/h \in \Lambda \mid z \in [1 - Ch, 1 + Ch] + i[-Mh \log h^{-1}, Mh \log h^{-1}] \right\}.$$ 

Redefine

$$r(x', \xi') := \frac{hV}{2i\sqrt{1 - |\xi'|^2} - hV}.$$
Distribution of resonances for the $\delta$ potential

**Theorem (G ’14,’15)**

Let $\Omega \subset \mathbb{R}^d$ be strictly convex with smooth boundary and suppose that $V \in h^\alpha \psi^1(\partial \Omega)$ is self adjoint and elliptic near $|\xi'|_g = 1$ with $\sigma(V) \geq 0$. Suppose that $\alpha > -5/6$. Then for all $\epsilon$, $N_1 > 0$ there exists $\epsilon_1 > 0$, $h_0 > 0$ such that for $0 < h < h_0$

$$\Lambda_{\log}(h) \subset \left\{ \frac{\text{Im } z}{h} \leq \inf_{N \leq N_1} \sup_{|\xi'| < 1 - \epsilon_1} \frac{r_N}{2l_N} + \epsilon \right\}.$$

Moreover, these estimates are sharp in this case of $\Omega = B(0, 1) \subset \mathbb{R}^2$ with $V \equiv 1$.

Recall the Sabine law

$$\frac{\text{Im } z}{h} = \frac{\log |R|^2}{2l} = \frac{r_N}{2l_N}.$$
Theorem (G ’15)

Under the same assumptions on $V$ and $\Omega$. Suppose that $-\frac{5}{6} \geq \alpha \geq -1$. Then for all $\epsilon > 0$, $M > 0$, there exists $h_0 > 0$ such that for $0 < h < h_0$

$$\Lambda_{\log}(h) \subset \bigcup_{j=1}^{M} \left\{ B_- - \epsilon \leq \frac{h^{2/3} \text{Im } z}{\text{Im } \Phi_-(\zeta_j)} \leq B_+ + \epsilon \right\} \bigcup \left\{ \frac{h^{2/3} \text{Im } z}{\text{Im } \Phi_-(\zeta_{M+1})} \geq B_- - \epsilon \right\}$$

where $\zeta_j$ are the zeroes of $\text{Ai}(s)$, $\Phi_-(s) = \frac{A_-'(s)}{A_-(s)}$, and

$$B_+ := \sup_{|\xi'|_g = 1} \frac{2^{1/3} Q(x, \xi')^{4/3}}{|\sigma(V)(x, \xi')|^2}, \quad B_- := \inf_{|\xi'|_g = 1} \frac{2^{1/3} Q(x, \xi')^{4/3}}{|\sigma(V)(x, \xi')|^2}.$$ 

Moreover, these estimates are sharp in this case of $\Omega = B(0, 1) \subset \mathbb{R}^2$ with $V \equiv 1$. 
\[ \sup_{|\xi'| \leq 1 - \delta} \frac{r_N}{l_N} \]

\[ B_5 \quad B_4 \quad B_3 \quad B_2 \quad B_1 \]

\[ \Re z \]

\[ \Im z \quad \alpha > -5/6 \]

\[ \sup_{|\xi'| \leq 1 - \delta} \frac{r_N}{l_N} \]

\[ B_5 \quad B_4 \quad B_3 \quad B_2 \quad B_1 \]

\[ \Re z \]

\[ \Im z \quad \alpha \leq -5/6 \]

**Figure:** Pictorial representation of the resonance free regions for delta potentials. Note that \( B_i \) occur at \( \Im z \sim h^{-2\alpha - 2/3} \).
Bands under a pinching condition

For \(-1 \leq \alpha \leq -5/6\), and

\[
\sup_{|\xi'|_g=1} \frac{Q(x, \xi')^{4/3}}{|\sigma(V)(x, \xi')|^2} \sup_{|\xi'|_g=1} \frac{|\sigma(V)(x, \xi')|^2}{Q(x, \xi')^{4/3}} > \frac{\Im \Phi_-(\zeta_j)}{\Im \Phi_-(\zeta_{j+1})}, \quad j \leq N
\]

There are \(N\) bands of resonance with \(\Im z \sim \hbar^{-2\alpha-2/3}\).

Figure: This figure shows \(\Im \Phi_-(s)\). The black dots are placed at \((\zeta_j, \Im \Phi_-(\zeta_j))\).
Resonances for mildly frequency dependent $\delta$ potentials

Figure: Plot of resonances for a $\delta$ potential not depending on frequency.
Resonances for highly frequency dependent $\delta$ potentials

**Figure:** Plot of $\log(\text{Re} \lambda)$ vs. $\log(-\text{Im} \lambda)$ for highly frequency dependent delta potential with $\alpha = -1$. 
Is this still a Sabine Law?

Note that

\[ \frac{r_1}{2l_1} = \frac{1}{2l(q, \beta(q))} \log |R(\beta(q))|^2 \]

\[ = \frac{-Q(x, \xi') \sqrt{1 - |\xi'|^2}}{|\sigma(hV)|^2} + O(h^{\alpha-1} \sqrt{1 - |\xi'|^2}), \]

\[ \text{Im } \Phi_-(s) \sim -i\sqrt{-s}, \quad s \to -\infty \]

This matches the bands found in our theorem modulo the fact that modes cannot concentrate closer than \( h^{2/3} \) to \( |\xi'|_g = 1 \) and that a quantization involving the zeros of the Airy function happens at that scale.
A generalized boundary stabilized equation

We again rescale $\lambda = z/h$ and analyze

\[
\begin{aligned}
(-h^2 \Delta - z^2)u &= 0 \quad \text{in } \Omega \\
h \partial_\nu u + Bu &= hv \quad \text{on } \partial\Omega
\end{aligned}
\]

where $B = hV + hN_2$ and $N_2$ is the outgoing Dirichlet to Neumann map for the exterior problem. It is enough to give estimates on $\|u|_{\partial\Omega}\|_{H^1_h}$ in terms of $\|v\|_{L^2(\partial\Omega)}$ since this implies estimates on $\|h \partial_\nu u\|_{L^2(\partial\Omega)}$ and hence $\|u\|_{L^2(\Omega)}$. 
Steps in analysis: Resonance free regions

1. Reduction to the boundary
2. Decomposition into several microlocal regions.

First, recall the boundary layer potentials and operators

\[ G(\lambda)f(x) := \int_{\partial \Omega} R_0(\lambda)(x, y)f(y)dS(y) , \quad x \in \partial \Omega \]

\[ N(\lambda)f(x) := \int_{\partial \Omega} \partial_{\nu_y} R_0(\lambda)(x, y)f(y)dS(y) , \quad x \in \partial \Omega \]

\[ S\ell(\lambda)f(x) := \int_{\partial \Omega} R_0(\lambda)(x, y)f(y)dS(y) , \quad x \in \Omega \]

\[ D\ell(\lambda)f(x) := \int_{\partial \Omega} \partial_{\nu_y} R_0(\lambda)(x, y)f(y)dS(y) , \quad x \in \Omega . \]
Reduction to a boundary integral equation

In particular,

\[ u = h^{-1} S \ell(z/h) h \partial_\nu u - D \ell(z/h) u \big|_{\partial \Omega} \]
\[ = -h^{-1} S \ell Bu \big|_{\partial \Omega} + S \ell v - D \ell u \big|_{\partial \Omega} \]
\[ = -S \ell V u \big|_{\partial \Omega} + S \ell v. \]

Letting \( x \to \partial \Omega \) and using the boundary condition, we have

\[ u \big|_{\partial \Omega} = -GVu \big|_{\partial \Omega} + Gv \]
\[ (I + GV) u \big|_{\partial \Omega} = Gv \]

Moreover, with \( \psi = u \big|_{\partial \Omega} \)

\[ \frac{2 \Re z \Im z}{h} \|u\|_{L^2(\Omega)}^2 - \Im \langle B \psi, \psi \rangle_{L^2(\partial \Omega)} = -\Im \langle hv, \psi \rangle_{L^2(\partial \Omega)}. \]
Important Microlocal Regions

| Hyperbolic          | $\mathcal{H}_\epsilon := \{(x', \xi') \in T^*\partial\Omega : |\xi'| \leq 1 - \text{ch}^\epsilon\} $ |
|---------------------|--------------------------------------------------------------------------------------------------|
| Glancing            | $\mathcal{G}_\epsilon := \{(x', \xi') \in T^*\partial\Omega : ||\xi'| - 1| \leq \text{ch}^\epsilon\} $ |
| Elliptic            | $\mathcal{E}_\epsilon := \{(x', \xi') \in T^*\partial\Omega : |\xi'| \geq 1 + \text{ch}^\epsilon\} $ |

Analysis of each component separately

- $\mathcal{H}_\epsilon$ - dynamics appear via Egorov’s theorem
- $\mathcal{G}_\epsilon$ - use of Melrose–Taylor Parametrix to produce microlocal models for $S\ell$ and $G$
- $\mathcal{E}_\epsilon$ - elliptic pseudodifferential operator
Decomposition of boundary layer operators

\[ G := \gamma R_0 \gamma^* = G_\Delta + G_B + G_g \]

- \( G_\Delta \in h^{1-\frac{\epsilon}{2}} \Psi_\epsilon \),
  \[ \sigma(G_\Delta) = \frac{ih}{2\sqrt{1 - |\xi'|^2}} \]

- \( G_B \) a semiclassical FIO associated to \( \beta \)
  \[ \sigma(G_B) = \frac{he^{i\pi/4}e^{i(z-1)l(q,\beta(q))/h}}{2(1 - |\xi'|^2)^{1/4}(1 - |eta(\xi')|^2)^{1/4}} |dx' \wedge d\xi'|^{1/2} \]

- \( G_g \) microlocalized near \( G_\epsilon \) - can be understood using the Melrose–Taylor parametrix.
Analysis away from glancing

Away from glancing it is enough to consider

\[(I + GV)\psi = Gv.\]

Then,

\[(I - RT)G^{1/2}_\Delta V\psi = RG^{1/2}_\Delta Gv.\]

\[R := -(I + G^{1/2}_\Delta VG^{1/2}_\Delta)^{-1}G^{1/2}_\Delta VG^{1/2}_\Delta\]

\[T := G^{-1/2}_\Delta G_B G^{-1/2}_\Delta\]

Iterate until

\[\|(RT)^N\|_{L^2 \to L^2} < 1.\]
Microlocal Models Near glancing

**Theorem (G ’14,’15)**

Microlocally near glancing,

\[
G = h^{2/3} 2\pi e^{\pi i/6} JA_i(h^{1/3} D_1)A_-(h^{1/3} D_1)B_0 J^{-1}
\]

\[
S_\ell^* S_\ell = h^2 JB_1 \Psi S_\ell(h^{1/3} D_1) J^{-1}
\]

where

\[
\sigma(JB_0 J^{-1}) = \frac{1}{(2Q)^{1/3}}, \quad \sigma(JB_1 J^{-1}) = \frac{1}{2Q},
\]

and J is a semiclassical FIO conjugating the billiard ball map to normal form. Here Q is the symbol of the second fundamental form.
Analysis Near Glancing

\[(I + GV)\psi = Gv\]
\[u = -S\ell V\psi + S\ell v\]

\[
\frac{2 \text{Re} z \text{Im} z}{h} \|u\|^2_{L^2(\Omega)} - \text{Im}\langle B\psi, \psi\rangle_{L^2(\partial\Omega)} = -\text{Im}\langle hv, \psi\rangle_{L^2(\partial\Omega)}.
\]