Microlocal Analysis for Hyperbolic Boundary Value Problems

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Contents

Chapter 1. Introduction
1. Notation for manifolds with boundary 7
2. Supported and Extendible distributions 7

Chapter 2. Microlocal Preliminaries
1. The Kohn–Nirenberg calculus on \( \mathbb{R}^d \) 9

Chapter 3. Basic estimates for hyperbolic equations on manifolds with boundary
1. Energy Estimates and Well Posedness 13
2. Strict Hyperbolicity 18
3. Boundaries 23
4. Existence and Uniqueness 24

Chapter 4. Propagation of Singularities
1. Propagation in the bulk 25
2. Propagation of singularities for strictly hyperbolic problems 25
3. The \( \mathcal{b} \)-wavefront set 26
4. Propagation near the boundary 26
5. The generalized bicharacteristic flow 36
6. The Weyl law on a manifold with boundary 36
7. Microlocal defect measures 36

Chapter 5. Equivalence of Glancing Hypersurfaces
1. Symplectic preliminaries 37
2. Folding relations 37
3. The billiard ball maps 37
4. Formal solution 37
5. Completion of the proof 37
6. Some consequences 37

Chapter 6. The Melrose–Taylor Parametrix
1. The Friedlander Model and the Ansatz 39
2. Geometric preliminaries 41
3. Solution of Eikonal Equations 44
4. Solution of the transport equations 49
5. Fourier–Airy Operators 53

Chapter 7. Calderón Projectors – Boundary Integral Operators?? 55
Appendix A. Notation
CHAPTER 1

Introduction

These notes are being prepared for a course in Fall 2017 in the Stanford Department of Mathematics. The ultimate goal of the course is to study the propagation of singularities on manifolds with boundary and parametrices for boundary value problems including the Melrose–Taylor parametrix. Throughout we will focus on the Dirichlet type problems, occasionally digressing to discuss the Dirichlet to Neumann map which can be applied to a wide variety of boundary conditions.

The material considered here comes largely from the work of Melrose [], Melrose–Sjöstrand [], Taylor [], Zworski [] and Farris []

The basic example to which these notes apply is that of the wave operator $P = \partial_t^2 - \Delta_g$ posed on $\mathbb{R} \times M$ where $M$ is a compact manifold with boundary $\partial M$.

1. Notation for manifolds with boundary

2. Supported and Extendible distributions
CHAPTER 2

Microlocal Preliminaries

In this chapter, we briefly review the standard calculus of pseudodifferential operators.  

1. The Kohn–Nirenberg calculus on \( \mathbb{R}^d \)

1.1. Symbols. We introduce two symbol classes. We say that a family of smooth functions with parameter \( h \in [0, 2) \), \( a(x, \xi; h) \in C^\infty(\mathbb{R}^{2d}) \) lies in \( S^m(\mathbb{R}^d \times \mathbb{R}^d) \) if

\[
|\partial_\xi^\alpha \partial_x^\beta a(x, \xi; h)| \leq C_{\alpha\beta} \langle \xi \rangle^{m-|\beta|}, \quad \langle \cdot \rangle := (1 + |\cdot|^2)^{1/2}.
\]

We say that \( a \in S^m \) is positively homogeneous of order \( m \) and write \( a \in S^m_{\text{hom}} \) if there exists \( F : \mathbb{R}^d \to (0, \infty) \) so that for \( s \geq 1 \) and \( |\xi| \geq F(x) \),

\[
a(x, s\xi) = s^m a(x, \xi).
\]

We say that \( a \in S^m \) lies in \( S^m_{\text{phg}} \) if there exist \( a_j(x, \xi) \in S^j_{\text{hom}}, j = m, m-1, \ldots \) and \( C_{\alpha\beta N} > 0 \) so that

\[
\left| \partial_\xi^\alpha \partial_x^\beta \left( a - \sum_{j=0}^{N-1} h^j a_{m-j} \right) \right| \leq C_{\alpha\beta N} h^N \langle \xi \rangle^{m-N}.
\]

For \( a \in S^m \), we then define the operator

\[
\text{Op}_h(a)u := \frac{1}{(2\pi h)^d} \int \int e^{i(x-y, \xi/h)} a(x, \xi; h) u(y) dy d\xi
\]

at first for \( u \in \mathcal{S} \) and then by transposition for \( u \in \mathcal{S}' \).

We define the set of pseudodifferential operators of order \( m \), \( \Psi^m(\mathbb{R}^d) \) and the polyhomogeneous pseudodifferential operators \( \Psi_{h, \text{phg}}^m(\mathbb{R}^d) \) by

\[
\Psi^m(\mathbb{R}^d) := \{ A \in \mathcal{L}(\mathcal{S}', \mathcal{S}) \mid A = \text{Op}_h(a), \text{ for some } a \in S^m(\mathbb{R}^d \times \mathbb{R}^d) \}
\]

\[
\Psi_{h, \text{phg}}^m(\mathbb{R}^d) := \{ A \in \mathcal{L}(\mathcal{S}', \mathcal{S}') \mid A = \text{Op}_h(a), \text{ for some } a \in S^m_{\text{phg}}(\mathbb{R}^d \times \mathbb{R}^d) \}.
\]

Here \( \mathcal{L} \) denotes the set of continuous linear operators.

Remark 1. We note that there are many other ways of quantizing symbols in \( S^m \). However for most purposes, they are the same and in particular, they yield the same classes \( \Psi^m_h(\mathbb{R}^d) \).

For \( A \in \Psi^m_h(\mathbb{R}^d) \), we define the principal symbol map \( \sigma_m : \Psi^m_h \to S^m/hS^{m-1} \) by

\[
\sigma_m(\text{Op}_h(a)) = a + hS^{m-1}.
\]
We sometimes write $a(x, hD)$ for the operator $Op_h(a)$.

We recall a few facts about pseudodifferential operators. Define the norm

$$
\|u\|_{H^s_h} := \|(hD)^s u\|_{L^2}
$$

for all $u \in H^s(\mathbb{R}^d)$.

**Lemma 1.1.** Suppose that $A \in \Psi^m_h(\mathbb{R}^d)$. Then for $s \in \mathbb{R}$, $A : H^s_h(\mathbb{R}^d) \to H^{s-m}_h(\mathbb{R}^d)$ and, moreover, there exists $C > 0$ so that

$$
\|Au\|_{H^{s-m}_h} \leq \sup |\xi|^{-m} \|\sigma(A)\| \|u\|_{H^s_h} + C h \|u\|_{H^{s-1}_h}.
$$

**Lemma 1.2.** Suppose that $a, b \in S^m_{phg}(\mathbb{R}^d)$ and $\text{supp } a \cap \text{supp } b = \emptyset$. Then for all $N > 0$,

$$
Op_h(a) Op_h(b) = O_{H^{-N}_h(\mathbb{R}^d) \to H^N_h(\mathbb{R}^d)}(h^N).
$$

**Lemma 1.3.** Suppose $0 \leq a \in S^m(\mathbb{R}^d)$. Then there exists $c > 0$ so that for all $u \in C^\infty_c(\mathbb{R}^d)$,

$$
\langle Op_h(a) u, u \rangle_{L^2} \geq -c h \|u\|_{H^{m-1}_h}.
$$

Next, we observe that for $a \in S^{m_1}_h$, $b \in S^{m_2}_h$,

$$
\begin{align*}
Op_h(a) Op_h(b) &= Op_h(ab) + hR, & R \in \Psi^{m_1+m_2-1}_h, \\
h^{-1}[Op_h(a), Op_h(b)] &= Op_h(-i\{a, b\}) + hR, & R \in \Psi^{m_1+m_2-2}_h, \\
Op_h(a)^* &= Op_h(\bar{a}) + hR, & R \in \Psi^{m_1-1}_h,
\end{align*}
$$

where

$$
\{a, b\} := \sum_i \partial_{\xi_i} a \partial_{x_i} b - \partial_{x_i} a \partial_{\xi_i} b.
$$

Note that the map $\sigma_m : \Psi^m_{phg} \to S^m$ is well defined if we take $\sigma_m(Op_h(a)) = a_m$ where $a_m$ is from the expansion [1].

**1.2. Pseudodifferential operators on compact manifolds without boundary.** Let $M$ be a compact manifold without boundary. Let

$$
\Psi^{-\infty}(M) := \mathcal{L}(\mathcal{D}'(M), C^\infty(M))
$$

and

$$
h^\infty \Psi^{-\infty} := \{ A \in \Psi^{-\infty} \mid A = O_{H^{-N}_h \to H^N_h(h^N)}, \text{ for all } N > 0 \}.
$$

We say that $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$ lies in $\Psi^m_h(M)$ (respectively $\Psi^m_{phg}(M)$) if

1. For all $\varphi, \psi \in C^\infty_c$ with $\text{supp } \varphi \cap \text{supp } \psi = \emptyset$, $\varphi A\psi \in h^\infty \Psi^{-\infty}$.
2. If $(U, \kappa)$ is a coordinate chart with $\kappa : U \to V \subset \mathbb{R}^d$ a diffeomorphism and $\varphi, \psi \in C^\infty_c(U)$, $\chi \in C^\infty_c(V)$. Then

$$
\chi(\kappa^{-1})^* \varphi A \psi \kappa^* \chi \in \Psi^m_h(\mathbb{R}^d), \text{ (respectively } \Psi^m_{phg}(\mathbb{R}^d)).
$$
1. THE KOHN–NIRENBERG CALCULUS ON $\mathbb{R}^d$

We say that $a \in S^m(T^*M)$ if in any coordinates $(x, \xi) \ a(x, \xi) \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$. Similarly, we say $a \in S^m_{phg}(T^*M)$ if $a(x, \xi) \in S^m_{phg}(\mathbb{R}^d \times \mathbb{R}^d)$. Then there is a symbol map $\sigma_m : \Psi^m_h(M) \rightarrow S^m(T^*M)/hS^{m-1}(T^*M)$ given by the following procedure. Let $(U_\alpha, \kappa_\alpha)$ be an atlas on $M$. Then

$$\sigma(A) = \sum_\alpha \tilde{\kappa}_\alpha^* \sigma((\kappa^{-1}_\alpha)^* \varphi_\alpha A \varphi_\alpha \kappa_\alpha^*)$$

where $\tilde{\kappa}_\alpha : T^*U \rightarrow T^*\mathbb{R}^d$ is the lift of $\kappa_\alpha$ as a symplectomorphism. Note that $\sigma_m$ has the following properties

1. $0 \rightarrow h\Psi^{m-1}_h \rightarrow \Psi^m_h \xrightarrow{\sigma_m} S^m/hS^{m-1} \rightarrow 0$ is exact.
2. $\sigma(AB) = \sigma(A)\sigma(B)$.
3. $\sigma([A, B]) = -i\{\sigma(A), \sigma(B)\}$ where $\{\cdot, \cdot\}$ is the poisson bracket.

Finally, there is a non-canonical quantization map $\text{Op}_h : S^m(T^*M) \rightarrow \Psi^m_h(T^*M)$ with $\text{Op}_h : S^m_{phg}(T^*M) \rightarrow \Psi^m_{h, phg}(T^*M)$ so that

1. For $a \in S^m(T^*M)$, $\sigma_m(\text{Op}_h(A)) = a + hS^{m-1}$.
2. For all $A \in \Psi^m_h(T^*M)$ (respectively $\Psi^m_{h, phg}(M)$), there exists $a \in S^m(T^*M)$ (respectively $S^m_{phg}(M)$) so that $A - \text{Op}_h(a) \in h^\infty \Psi^{-\infty}$.
3. If $a, b \in S^m_{phg}$ and $\text{supp} \ a \cap \text{supp} \ b = \emptyset$.
   $$\text{Op}_h(a) \text{Op}_h(b) \in h^\infty \Psi^{-\infty}(M).$$

We will also write $S^{\text{comp}}(T^*M)$ for symbols compactly supported in $T^*M$ and $\Psi^m_{h, \text{comp}}$ for their quantizations.

We will sometimes have reason to use the standard calculus (i.e. $h = 1$). In this case, we write $\text{Op}$ for the quantization and $\Psi^m$ for the operators resulting from $S^m$. 

CHAPTER 3

Basic estimates for hyperbolic equations on manifolds with boundary

1. Energy Estimates and Well Posedness

1.1. Estimates without a boundary. We will work in the case of second order operators, but the methods developed here apply equally well to higher order equations.

1.1.1. First order operators. We consider the problem

\[ P_t := (D_t - \text{Op}(a_t))u = f, \quad 0 < t < T, \quad u|_{t=0} = u_0 \]

where

(i) \( a_t(x,\xi) = a(t, x, \xi) \) belongs to a bounded set in \( S^1(\mathbb{R}^d \times \mathbb{R}^d) \) for \( 0 \leq t \leq T \)

(ii) \( t \mapsto a_t \) is continuous with values in \( C^\infty(\mathbb{R}^d \times \mathbb{R}^d) \)

(iii) \( \text{Im} \ a(t, x, \xi) \geq -M, \ 0 \leq t \leq T \).

We start with an energy estimate

**Lemma 1.1.** Let \( s \in \mathbb{R} \). Then for \( \lambda \in \mathbb{R} \) large enough and all \( u \in C^1([0, T]; H^s(\mathbb{R}^d)) \setminus C^0([0, T]; H^{s+1}(\mathbb{R}^d)) \) and \( p \in [1, \infty] \)

\[ \left( \frac{1}{2} \int_0^T \| e^{-\lambda t} u(t, \cdot) \|_{H^s}^p \lambda dt \right)^{\frac{1}{p}} \leq \| u(0, \cdot) \|_{H^s} + 2 \int_0^T e^{-\lambda t} \| P_t u \|_{H^s} dt. \]

**Proof.** Then consider \( E(t) = e^{-2\lambda t} \| u(t) \|_{L^2}^2 \).

\[
\begin{align*}
\partial_t E(t) &= 2 \text{Re} \langle \partial_t [e^{-\lambda t} u], e^{-\lambda t} u \rangle \\
&= 2 \text{Re} \ e^{-2\lambda t} \langle \partial_t u, u \rangle - 2\lambda E(t) \\
&= -2 e^{-2\lambda t} \text{Im} \langle D_t u, u \rangle - 2\lambda E(t) \\
&= -2 e^{-2\lambda t} \text{Im} \langle P_t u, u \rangle - 2 e^{-2\lambda t} \text{Im} \langle \text{Op}(a_t) u, u \rangle - 2\lambda E(t) \\
&\leq 2 \| e^{-\lambda t} P_t u \| E^{1/2}(t) + 2(C - \lambda) E(t)
\end{align*}
\]

where in the last line we apply the sharp Gårding inequality \([1.3]\) together with \([i] [iii]\) and the fact that \( u(t) \in H^1 \). Choosing \( \lambda \geq C \), then

\[ \partial_t E(t) \leq 2 \| e^{-\lambda t} P_t u \| E^{1/2}(t). \]

Integrating in time gives

\[
\sup_{0 \leq \tau \leq t} E(\tau) \leq E(0) + 2 \sup_{0 \leq \tau \leq t} E^{1/2}(t) \int_0^t \| e^{-\lambda s} P_s u \| ds.
\]
So,
\[
\left( \sup_{0 \leq \tau \leq t} E^{1/2}(\tau) - \int_0^t \| e^{-\lambda s} P_s u \| ds \right)^2 \leq E(0) + \left( \int_0^t \| e^{-\lambda s} P_s u \| ds \right)^2
\]
and in particular,
\[
e^{-Ct} \| u(t) \| \leq \| u(0) \| + 2 \int_0^t \| e^{-Cs} P_s u \| ds.
\]
Then, taking \( \lambda > 2C \) large enough and using that \( \| e^{-\lambda t/2} \|_{L^p(\mathbb{R})} \leq 2 \) for \( \lambda > 0 \). So,
\[
e^{-\lambda t} \| u(t) \| \leq e^{(C-\lambda)t} \| u(0) \| + 2 \int_0^t e^{-\lambda s} \| P_s u \| e^{(C-\lambda)(t-s)} ds.
\]
and taking \( \lambda > 2C \), and integrating both sides in \( t \),
\[
\left( \int_0^T \left( e^{-\lambda t} \| u(t) \| \right) dt \right)^{1/p} \leq \| e^{-\lambda t/2} \|_{L^p} \| u(0) \| + 2 \left( \int_0^T \left( \int_0^t e^{-\lambda s} \| P_s u \| e^{(C-\lambda)(t-s)} ds \right) dt \right)^{1/p}.
\]
Applying Minkowski’s inequality then gives
\[
\left( \int_0^T \left( e^{-\lambda t} \| u(t) \| \right) dt \right)^{1/p} \leq \| e^{-\lambda t/2} \|_{L^p} \| u(0) \| + \| e^{-\lambda t/2} \|_{L^p} \int_0^T e^{-\lambda s} \| P_s u \| ds
\]
Then, since \( \| e^{-\lambda t/2} \|_{L^p} \leq (2/\lambda)^{1/p} \), the lemma follows for \( s = 0 \).

To finish the proof, apply the \( s = 0 \) case to \( \tilde{A}_t = \text{Op}(\langle \xi \rangle^s) A_t \text{Op}(\langle \xi \rangle^{-s}) \) with \( u = \text{Op}(\langle \xi \rangle^s) u \).

\[ \square \]

**Theorem 1.1** (Well-posedness for first order equations). Let (i)-(iii) hold and \( s \in \mathbb{R} \). Then for all \( f \in L^1((0,T); H^s(\mathbb{R}^d)) \) and \( \phi \in H^s(\mathbb{R}^d) \), there is a unique solution \( u \in C([0,T]; H^s(\mathbb{R}^d)) \) of (3) and (4) holds.

**Proof.** We start with uniqueness. Suppose (3) holds with \( \phi = 0 \) and \( f = 0 \). Then, since \( u \in C([0,1]; H^s(\mathbb{R}^d)) \), \( \text{Op}(a_i) u \in C([0,T]; H^{s-1}(\mathbb{R}^d)) \) and hence \( \partial_t u \in C([0,T]; H^{s-1}(\mathbb{R}^d)) \). That is, \( u \in C^1([0,T]; H^{s-1}) \) and in particular, (4) implies that \( u = 0 \).

To show existence, we apply the energy estimate to the adjoint problem. Suppose \( v \in C_c^\infty((-\infty,T) \times \mathbb{R}^d) \). Then, observe that by (1) applied with \( t \mapsto T - t \) gives for any \( r \in \mathbb{R} \),
\[
\sup_{t \in [0,T]} \| v(t) \|_{H^{-r}(\mathbb{R}^d)} \leq C \int_0^T \| (D_t - \text{Op}(a_i)^*) v \|_{H^{-r}(\mathbb{R}^d)} dt.
\]
Thus, for \( f \in L^1([0,T]; H^r(\mathbb{R}^d)) \), \( \phi \in H^r(\mathbb{R}^d) \),
\[
\int_0^T \langle f(t), v(t) \rangle_{\mathbb{R}^d} dt - i \langle \phi, v(0) \rangle \leq (\| f(t) \|_{L^1([0,T]; H^r(\mathbb{R}^d))} + \| \phi \|_{H^r}) \sup_{[0,T]} \| v(t) \|_{H^{-r}(\mathbb{R}^d)}
\leq C(\| f(t) \|_{L^1([0,T]; H^r(\mathbb{R}^d))} + \| \phi \|_{H^r}) \int_0^T \| (D_t - \text{Op}(a_i)^*) v \|_{H^{-r}(\mathbb{R}^d)} dt.
\]
By the Hahn-Banach theorem there exists \( u \in L^\infty([0, T]; H^r(\mathbb{R}^d)) \) so that
\[
\int_0^T \langle u, (D_t - \text{Op}(a_t))v \rangle dt = \int_0^T \langle f(t), v(t) \rangle dt - i \langle \phi, v(0) \rangle
\]
for all \( v \in C_0^\infty((-\infty, T) \times \mathbb{R}^d) \) in particular, \( u \) solves \( (3) \) as a distribution, so all we need to do is show that \( u \) has the desired regularity.

Let \( f_\epsilon \in \mathcal{S} \) and \( \phi_\epsilon \in \mathcal{S} \) have \( f_\epsilon \to f_0 \) in \( L^1([0, T]; H^s) \) and \( \phi_\epsilon \to \phi_0 \) in \( H^s \). Then, applying the above arguments, we obtain \( u_\epsilon \in L^\infty([0, T]; H^s(\mathbb{R}^d)) \) with \( u_\epsilon \) solving \( (3) \) for \( f_\epsilon \) and \( \phi_\epsilon \). Then, \( u_\epsilon \in L^\infty([0, T]; H^{s+2}) \) which implies \( \text{Op}(a_t)u_\epsilon \in L^\infty([0, T]; H^{s+1}) \). The equation then implies \( D_t u_\epsilon \in L^\infty([0, T]; H^{s+1}) \) and in particular, \( u_\epsilon \in C^0([0, T]; H^s) \). Using the equation again then implies \( \partial_t u_\epsilon \in C^0([0, T]; H^s) \). In particular, \( (4) \) applies to \( u_\epsilon \). Therefore,
\[
\sup_{[0,T]} \|u_\epsilon(t, \cdot) - u_\epsilon'(t, \cdot)\|_{H^s} \leq C (\|\phi_\epsilon - \phi_\epsilon'\|_{H^s} + \int_0^T \|f_\epsilon - f_\epsilon'\|_{H^s} dt),
\]
Hence, since \( \phi_\epsilon \) and \( f_\epsilon \) are Cauchy, \( u_\epsilon \) is Cauchy in \( C^0([0, T]; H^s(\mathbb{R}^d)) \) and in particular converges to \( u_0 \in C^0([0, T]; H^s(\mathbb{R}^d)) \) satisfying \( (4) \) with \( \phi = \phi_0 \) and \( f = f_0 \). By similar arguments after using \( (3) \), \( u_\epsilon \) is also Cauchy in \( C^1([0, T]; H^{s+1}(\mathbb{R}^d)) \) and hence \( u_0 \in C^1([0, T]; H^{s+1}(\mathbb{R}^d)) \). Therefore, \( u_0 \) is the desired solution. \( \square \)

1.2. Second Order Operators. We now consider second order operators. The same techniques apply to \( m^\text{th} \) order operators of the same type, but for simplicity we work only with second order. We want to study,
\[
P = \sum_{j=0}^{2} P_j(t, x, D_x) D_t^j
\]
where
(i) \( P_m = 1 \) and \( P_j \in C^\infty(\mathbb{R}; \mathbb{P}^{2-j}_\text{phg}(\mathbb{R}^d)) \) with \( \sigma_{2-j}(P_j) = p_j(t, x, \xi) \), \( \sigma_2(P) = p \)
(ii) The zeros of the map
\[
f : \tau \mapsto \langle \xi \rangle^{-2} p(t, x, \langle \xi \rangle \tau, \xi)
\]
are uniformly simple on \( \mathbb{R} \times \partial T^* \mathbb{R}^d \). That is,
\[
|\partial_\tau \langle \xi \rangle^{-2} p(t, x, \langle \xi \rangle \tau, \xi)|^2 + |p(t, x, \langle \xi \rangle \tau, \xi)|^2 > 0, \quad \text{for} \quad \tau \in \mathbb{R}, \ (t, x, \xi) \in \mathbb{R}^{d+1} \times \partial T^* \mathbb{R}^d.
\]
Let \( \lambda_1, \lambda_2 \in C^\infty(\mathbb{R} \times \partial T^* \mathbb{R}^d) \) be the zeros of \( f \) on \( \partial T^* \mathbb{R}^d \) and \( \lambda_i \in \mathbb{P}^{1}_\text{phg}(\mathbb{R}^{d+1} \times \mathbb{R}^d) \) with \( \langle \xi \rangle^{-1} |\partial_\tau (|\xi|^{-4} p(t, x, |\xi| \tau, \xi)|^2 + |p(t, x, |\xi| \tau, \xi)|^2 > 0, \quad \text{for} \quad \tau \in \mathbb{R}, \ (t, x, \xi) \in \mathbb{R}^{d+1} \times \partial T^* \mathbb{R}^d.\]
Let \( \Lambda_i = \lambda_i(t, x, D) \).

The goal is then to factor \( P \) in terms of \( D_t - \Lambda_i \) with an error involving no derivatives in time. For this, write
\[
P = \sum_{j=0}^{2} P_j D_t^j = \sum_{j=0}^{2} [(D_t - \Lambda_i) P_j + [P_j, D_t] + \Lambda_i P_j] D_t^{j-1}
\]
where for \( k < 0, D_t^k = 0 \). Then, \( [P_j, D_t] \in C^\infty(\mathbb{R}; \mathbb{P}^{2-j}_\text{phg}) \), so this is a sum of the form
\[
P = (D_t - \Lambda_i) \sum_{j=0}^{2} P_j D_t^{j-1} + \sum_{j=0}^{1} \tilde{P}_j D_t^j
\]
and hence with \( \hat{P}_j \in S^{2-j} \). Iterating this procedure gives

\[
P = (D_t - \Lambda_i)E_i + R
\]

where \( E_i = \sum_{k=0}^{1} E_{ik}D_t^k \) with \( E_{ik} \in C^\infty(\mathbb{R}; \Psi^{1-k}) \) and \( R \in C^\infty(\mathbb{R}; \Psi^2) \). Taking the symbol of both sides we see that

\[
p(t, x, \tau, \xi) = (\tau - \lambda_i)\sigma_1(E_i)(t, x, \tau, \xi) + \sigma_2(R)(t, x, \xi)
\]

In particular, setting \( \tau = \lambda_i \), we see that \( \sigma_2(R)(t, x, \xi) = 0 \) and hence \( R \in C^\infty(\mathbb{R}; \Psi^1) \). Moreover, \( \sigma(E_i) = \tau - \lambda_j j \neq i \).

We summarize this in the next lemma

**Lemma 1.2.** Suppose that \( P \) satisfies (i) and (ii) and let \( \lambda_i \) \( i = 1, 2 \) as above. Then there exist \( \Lambda_j(x, D'), \Lambda_j(x, D') \) with \( \sigma(\Lambda_j) = \lambda_i = \sigma(\tilde{\Lambda}_j) \) so that

\[
P = (D_t - \Lambda_1)(D_t - \Lambda_2) + R(x, D') = (D_t - \tilde{\Lambda}_1)(D_t - \tilde{\Lambda}_2) + \tilde{R}(x, D')
\]

with \( R, \tilde{R} \in S^1 \).

With this factorization in place, we can prove the energy estimate.

**Lemma 1.3.** Let \( s \in \mathbb{R}, T > 0, \) and \( u \in C^1([0, T]; H^s) \cap C^0([0, T]; H^{s+1}) \) with \( Pu \in L^1((0, T); H^s) \). Then

\[
\sup_{0 \leq t \leq T} \sum_{j < 2} \| D_t^j u(t, \cdot) \|_{H^{s+1-j}} \leq C_{s,T} \left( \sum_{j < 2} \| D_t^j u(0, \cdot) \|_{s+1-j} + \int_0^T \| Pu(t, \cdot) \|_s dt \right).
\]

**Proof.** By the estimate (4), there exists \( C > 0 \) uniform in \( p \in [1, \infty], \lambda > 0 \) large so that

\[
\left( \int_0^T \lambda [e^{-\lambda t} \| E_i u(t, \cdot) \|_{H^j}]^p dt \right)^{\frac{1}{p}} \leq C \left( \| E_i u(0, \cdot) \|_{H^s} + \int_0^T e^{-\lambda t} (\| Pu \|_{H^s} + C \| u \|_{H^{s+1}}) dt \right).
\]

We next estimate \( u \) and \( D_t u \) in terms of \( E_1 u \) and \( E_2 u \). For this, observe that

\[
\tau^k = \frac{\lambda_2^k (\tau - \lambda_1) - \lambda_1^k (\tau - \lambda_2)}{\lambda_2 - \lambda_1}
\]

and hence with

\[
Q_1^k = (\lambda_1 - \lambda_2)^{-1}(x, D)\Lambda_1^k \in C^\infty(\mathbb{R}; \Psi^0)
\]

\[
Q_2^k = (\lambda_2 - \lambda_1)^{-1}(x, D)\Lambda_2^k \in C^\infty(\mathbb{R}; \Psi^0)
\]

\[
\sigma_k(Q_1^k E_1 + Q_2^k E_2) = \tau^k
\]

and we have

\[
D_t^k = Q_1^k E_1 + Q_2^k E_2 - \sum_{j=0}^{1} R_j D_t^j, \quad R_j \in C^\infty(\mathbb{R}; \Psi^{k-1-j}).
\]

So, for each \( t \),

\[
\sum_{k=0}^{1} \| D_t^k u \|_{H^{s+1-k}} \leq C \sum_i \| E_i u \|_{H^s} + C \sum_{k=0}^{1} \| D_t^k u \|_{H^{s-k}}.
\]
Next, observe that
\[ \|D_t u\|_{H^{s-1}} \leq \|E_1 u\|_{H^{s-1}} + \|\Lambda_2 u\|_{H^{s-1}} + C \|u\|_{H^{s-1}} \]
and hence
\[ \sum_{k=0}^1 \|D^k_t u\|_{H^{s+1-k}} \leq C \sum_{i=1}^4 \|E_i u\|_{H^s} + C \|u\|_{H^s}. \] (8)

Using (7), together with (4) applied to \( u \) with \( a_t = 0 \),
\[ \left( \int_0^T \lambda \left[ e^{-\lambda t} \sum_{k=0}^1 \|D^k_t u\|_{H^{s+1-k}} \right]^p \right)^{\frac{1}{p}} \leq \sum_{i=1}^4 \|E_i u(0)\|_{H^s} + \|u(0)\|_{H^s} + C \int_0^T e^{-\lambda t} (\|P u\|_{H^s} + \sum_{k=0}^1 \|D^k_t u\|_{H^{s+1-k}}) dt. \] (9)

Letting \( p = 1 \) and \( \lambda \gg 1 \), in (9), the last term can be absorbed in the left hand side and we have
\[ \int_0^T \lambda \left[ e^{-\lambda t} \sum_{k=0}^1 \|D^k_t u\|_{H^{s+1-k}} \right] dt \leq C_{s,T} \left( \sum_{i=0}^1 \|D^i_t u(0)\|_{H^{s+1-i}} + \int_0^T \|P u\|_{H^d} dt \right). \]

Inserting this into the right hand side of (9) gives
\[ \left( \int_0^T \lambda \left[ e^{-\lambda t} \sum_{k=0}^1 \|D^k_t u\|_{H^{s+1-k}} \right]^p \right)^{\frac{1}{p}} \leq C_{s,T} \left( \sum_{i=0}^1 \|D^i_t u(0)\|_{H^{s+1-i}} + \int_0^T \|P u\|_{H^d} dt \right). \]

We now turn to the well posedness for 2nd order equations.

**THEOREM 1.2.** Assume that (i) and (ii) hold. Then for \( f \in L^1((0,T);H^s) \), \( u_j \in H^{s+1-j} \) there exists a unique solution \( u \in C^1([0,T];H^s) \cap C^0([0,T];H^{s+1}) \) to
\[ Pu = f \text{ in } 0 < t < T, \quad D^j_t u|_{t=0} = u_j, \text{ for } j < 2. \]

**PROOF.** As before, the uniqueness follows from the energy estimate. Therefore, we need only study existence. Suppose \( v \in C^\infty_c((\infty,T) \times \mathbb{R}^d) \) and \( u \in C^\infty(\mathbb{R}^{d+1}) \). Then
\[ \int_0^T \langle u, P^* v \rangle dt = \int_0^T (Pu, v) dt - i \sum_{j+k<2} \langle D^k_t u(0), E^k_t P^* j+k+1 v(0) \rangle. \] (10)

Since, \( P^* \) satisfies (i) and (iii), integrating backwards in time and applying (7),
\[ \sum_{k<2} \|D^k_t v(t)\|_{H^{-s-k}} \leq C \int_0^T \|P^* v\|_{H^{-1-s}}. \]

Thus,
\[ \left| \int_0^T (f, v) dt - i \sum_{j+k<2} \langle u_j, E^k_t P^* j+k+1 v(0) \rangle \right| \leq C \left( \int_0^T \|f\|_{H^s} dt + \sum_{j} \|u_j\|_{H^{s+1-j}} \right) \int_0^T \|P^* v\|_{H^{-1-s}}. \]
The Hahn Banach theorem implies that there exists \( u \in L^\infty((0, T]; H^{s+1}) \) such that \( (10) \) holds with \( Pu = f, D_t^j u(0) = u_j \).

As before the remaining task is to improve the regularity of \( u \) and in fact to show that there is a solution with the desired regularity. We proceed as in the proof of Theorem 1.1. That is, it is enough to show that if \( f \in \mathcal{S} \) and \( u_j \in \mathcal{S} \), then the resulting solution has the desired regularity. So, suppose that \( u \) is the function given above solving

\[
P u = f \in \mathcal{S}, \quad D_t^j u|_{t=0} = u_j \in \mathcal{S}.
\]

Then, for any \( s, u \in L^\infty([0, T]; H^{s+1}) \) and using the factorization of \( P \)

\[
(D_t - \Lambda_i)E_i u = f + R_j u
\]

So, \( E_i u \in L^\infty([0, T]; H^s) \) and hence \( D_t E_i u \in L^\infty([0, T]; H^{s-1}) \), so \( E_i u \in C^0([0, T]; H^{s-1}) \). Now, we have by \((5)\) that \( D_t u \in L^\infty([0, T]; H^{s-1}) \) and hence using the equation \( D_t^2 u \in L^\infty([0, T]; H^{s-2}) \). Therefore, \( D_t u \in C^0([0, T]; H^{s-2}) \) and \( u \in C^0([0, T]; H^{s-1}) \). In particular,

\[
u \in \bigcap_{s, j=0}^1 C^j([0, T]; H^{s-j}).
\]

Applying the equation proves that in fact \( u \) is smooth. \( \square \)

### 2. Strict Hyperbolicity

Now, we consider more general second order operators on a manifold \( X \) without boundary.

**Definition 2.1.** Let \( P \in \text{Diff}^m(X) \) with principal symbol \( p \) is strictly hyperbolic with respect to \( \phi \in C^\infty(X; \mathbb{R}) \) if \( p(x, d\phi(x)) \neq 0 \) and the map

\[
\mathbb{R} \ni \tau \mapsto p(x, \xi + \tau d\phi(x))
\]

has \( m \) distinct roots for all \( x \in X \) and \( \pi \in (T^*_x X \setminus 0) \setminus \mathbb{R} d\phi(x) \).

**Example 2.1.** Consider \( X = \mathbb{R}_x \times M_x \) and \( P = -\partial^2_t + \Delta_g \). Then, \( p(t, x, \tau, \xi) = \tau^2 - |\xi|^2_g \). Consider \( \phi(t, x) = t \). Then, \( d\phi = dt \) and so

\[
p(t, x, d\phi, 0) = 1 \neq 0, \quad p(t, x, \tau d\phi, \xi) = \tau^2 - |\xi|^2_g
\]

So, \( P \) is strictly hyperbolic with respect to \( \phi = t \).

Our main aim for the rest of this section is to study well posedness of the Cauchy for \( P \) and in particular,

**Theorem 2.1.** Let \( P \) be a differential operator of order 2 with \( C^\infty \) coefficients in \( X \) and let \( Y \subset X \) be an open, precompact subset. Assume that \( P \) is strictly hyperbolic with respect to \( \phi \in C^\infty(X; \mathbb{R}) \) and define

\[
X_+ = \{ x \in X \mid \phi(x) > 0 \}, \quad X_0 := \{ x \in X \mid \phi(x) = 0 \}.
\]

(i) If \( f \in H^s_{\text{loc}}(X) \) has support in the closure of \( X_+ \) then there exists \( u \in H^{s+1}_{\text{loc}}(X) \) with support in the closure of \( X_+ \) such that \( Pu = f \) in \( Y \).
We then define \( H \) the norm \( F \) use the following notation. For a space of distributions \( u \) is checked easily from the definition. Therefore, we define \( 1 \). We also define \( 2.0.1. \) Lemma We will need the following lemma \( X \) We define spaces analogously on a manifold \( X \). Proof. Let \( U \). Moreover \( u \) is a vector field with \( v \) is a vector field with \( v \phi = 1 \) near \( X_0 \), \( f \in \mathcal{H}^s_{\text{loc}}(X_+) \) and \( u_j \in H^{s+j}_{\text{loc}}(X_0) \), \( j < 2 \), then there exists \( u \in \mathcal{H}_{\text{loc}}^{s+1}(X_+) \) such that \( Pu = f \) in \( X_+ \cap Y \) and \( v^j u = u_j \) in \( X_0 \cap Y \). Moreover \( u \) satisfies for every \( K \subset X \) compact with \( K \). [TODO]estimates

2.0.1. Anisotropic Sobolev spaces. We will work locally assuming that \( X = \mathbb{R}^d \) and \( \phi(x) = x_1 \), but before we do so, it will be useful to have certain anisotropic sobolev spaces. For a distribution \( u \in \mathcal{S}(\mathbb{R}^d) \) and \( m, s \in \mathbb{R} \), we define the norm

\[
\|u\|^2_{m,s} := (2\pi)^{-n} \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^{(1 + |\xi'|)^s} d\xi
\]

We then define \( H^{m,s}(\mathbb{R}^d) \) as the closure of \( \mathcal{S} \) with respect to the \( \| \cdot \|_{m,s} \) norm.

We will actually want to work in the half space \( \mathbb{R}^d_+ := \{(x_1, x') \in \mathbb{R}^d \mid x_1 > 0\} \). For this, we use the following notation. For a space of distributions \( F(\mathbb{R}^d) \), we define

\[
\mathcal{F}(\mathbb{R}^d_+) := \{ U |_{\mathbb{R}^d_+} \mid U \in F(\mathbb{R}^d) \}.
\]

We also define

\[
\dot{\mathcal{F}}(\mathbb{R}^d_+) := \{ U \in F(\mathbb{R}^d) \mid \text{supp } U \subset \mathbb{R}^d_+ \}.
\]

We will mostly be concerned with \( \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) and \( \dot{\mathcal{H}}^{m,s}(\mathbb{R}^d_+) \). For \( u \in \mathcal{H}^{m,s}(\mathbb{R}^d_+) \), we define the norm

\[
\|u\|_{m,s} := \inf \{ \|U\|_{m,s} \mid U \in H^{m,s}(\mathbb{R}^d), U|_{\mathbb{R}^d_+} = u \}.
\]

We define spaces analogously on a manifold \( X \) with boundary.

We will need the following lemma

**Lemma 2.1.** Let \( u \in \mathcal{S}'(\mathbb{R}^d_+) \). Then \( u \in \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) if and only if \( u \in \mathcal{H}^{m-1,s+1}(\mathbb{R}^d_+) \) and \( D_j u \in \mathcal{H}^{m,s-1}(\mathbb{R}^d_+) \);

\[
\frac{1}{2} \|u\|^2_{m,s} \leq \|D_1 u\|^2_{m-1,s} + \|u\|^2_{m-1,s+1} \leq \|u\|^2_{m,s}.
\]

Furthermore, \( u \in \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) if and only if \( u \in \mathcal{H}^{m,s-1}(\mathbb{R}^d_+) \) and \( D_j u \in \mathcal{H}^{m,s-1}(\mathbb{R}^d_+) \) for \( 1 < j \leq d \) and

\[
\|u\|^2_{m,s} = \|u\|^2_{m,s-1} + \sum_{j=2}^d \|D_j u\|^2_{m,s-1}.
\]

**Proof.** Let \( u \in \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) and \( U \in H^{m,s}(\mathbb{R}^d) \) with \( U|_{\mathbb{R}^d_+} = u \). Then,

\[
\|U\|^2_{m,s} = \|D_1 U\|^2_{m-1,s} + \|U\|^2_{m-1,s+1} = \|u\|^2_{m,s-1} + \sum_{j=2}^d \|D_j U\|^2_{m,s-1}
\]

is checked easily from the definition. Therefore, \( u \in \mathcal{H}^{m-1,s+1}, D_1 u \in \mathcal{H}^{m-1,s}, u \in \mathcal{H}^{m,s-1}, \) and \( D_j u \in \mathcal{H}^{m,s-1} \) and

\[
\|D_1 u\|^2_{m-1,s} + \|u\|^2_{m-1,s+1} \leq \|u\|^2_{m,s}
\]

\[
\|u\|^2_{m,s-1} + \sum_{j=2}^d \|D_j u\|^2_{m,s-1} \leq \|u\|^2_{m,s}
\]
We now need to show the other inclusions. Together with the estimates, they are obtained from the following lemma.

**Lemma 2.2.** Let \( \Lambda_{m,s}(\xi) = (\langle \xi' \rangle + i\xi_1)^m \langle \xi' \rangle^s \). Then \( \Lambda_{m,s}(D) \) is an isomorphism of \( \mathcal{S}'(\mathbb{R}^d_+) \) extending by continuity to \( L^2(\mathbb{R}^d_+) \to \dot{H}^{-m,-s}(\mathbb{R}^d_+) \). Moreover, \( \Lambda_{m,s}(D) \) extends by continuity from \( \mathcal{C}^\infty_c(\mathbb{R}^d_+) \) to an isomorphism from \( \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) to \( L^2(\mathbb{R}^d_+) \). In particular,

\[
\|u\|_{m,s} = \|\Lambda_{m,s}(D)u\|_{L^2(\mathbb{R}^d_+)}
\]

and \( \mathcal{H}^{m,s}(\mathbb{R}^d_+) \) consists of those \( u \in \mathcal{S}'(\mathbb{R}^d_+) \) with \( \Lambda_{m,s}u \in L^2 \).

**Proof.** One can easily check that \( \text{supp} \mathcal{F}^{-1}(\Lambda_{m,s}) \subset \mathbb{R}^d_+ \) and \( \text{supp} \mathcal{F}^{-1}(\Lambda_{m,s}) \subset \mathbb{R}^d_- \). (For example by the Paley-Weiner theorem.) Therefore, \( \Lambda_{m,s}(D) \) maps \( \mathcal{S}'(\mathbb{R}^d_+) \) continuously to itself and has inverse \( \Lambda_{-m,-s}(D) \). The extension to \( L^2 \) then follows since \( |\Lambda_{m,s}(\xi)|^2 = \langle \xi \rangle^{2m} \langle \xi' \rangle^{2s} \). The second statement follows by duality.

\( \square \)

We record the following consequence of Lemma 2.2 for later use.

**Corollary 2.1.** Suppose \( u \in \dot{H}^{-m,-s}(\mathbb{R}^d_+) \). Then there exist \( u_0 \in \dot{H}^{1-m,-s-1}(\mathbb{R}^d_+) \) and \( u_1 \in \dot{H}^{1-m,-s} \) so that

\[
u = u_0 + D_1 u_1.\]

**Proof.** Observe that \( \Lambda_{m,s}(D) = (\langle D' \rangle + iD_1) \Lambda_{m-1,s} \). So, since \( \Lambda_{m,s} : L^2 \to \dot{H}^{-m,-s} \) is an isomorphism, there exists \( v \in L^2 \) such that

\[
u = \Lambda_{m,s}v = (\langle D' \rangle + iD_1) \Lambda_{m-1,s}v
\]

and observing that

\[
\langle D' \rangle \Lambda_{m-1,s}v \in \dot{H}^{1-m,-s-1}, \quad i\Lambda_{m-1,s}v \in \dot{H}^{1-m,-s}. \]

\( \square \)

2.0.2. **The local problem.** We start with the case \( X = \mathbb{R}^d \) and \( \phi(x) = x_1 \). Then,

\[
P = \sum_{j=0}^{2} P_j(x_1, x', D_{x'}) D_{x_1}^j, \quad P_j \in \text{Diff}^{2-j}
\]

and by the strict hyperbolicity assumption \( 0 < |P_2| \). Hence, dividing by a nonzero smooth function we may assume that \( P_2 = 1 \). Moreover, since \( P \) is a homogeneous polynomial of degree 2 in \( (\xi_1, \xi') \) for which that map \( \xi_1 \mapsto p(x, \xi_1, \xi') \) has two distinct real zeros, we may assume \( P \) satisfies the assumptions of Theorem 1.2 for \( x \in Y \).

Let \( \lambda_1(x, \xi'), \lambda_2(x, \xi') \) be the roots of \( p(x, \cdot, \xi') \). Fix \( \chi \in C^\infty_c(\mathbb{R}^d) \) with \( \chi \equiv 1 \) in \( Y \) and define

\[
p = \prod_{j=1}^{2} (\xi_1 - \tilde{\lambda}_j), \quad \tilde{\lambda}_i = \chi \lambda_j + (1 - \chi) j |\xi'|.
\]
Now, let \( \tilde{P} \in \text{Diff}^2(\mathbb{R}^d) \) with principal symbol \( \tilde{p} \). Then \( \tilde{P} \) satisfies the hypotheses of Theorem 1.2 for \( x \in \mathbb{R}^d \) and has constant coefficients outside a compact set.

**Lemma 2.3.** Suppose that \( P \) is strictly hyperbolic with respect to \( x_1 \) in \( X \subset \mathbb{R}^d \) and let \( Y \subset X \) be open and precompact. If \( f \in \overline{H}^{s,t}(\mathbb{R}^d_+) \) and \( s \geq 0 \), \( u_j \in H^{s+t+1-j}(\mathbb{R}^{d-1}) \), \( j < 2 \), then there exists \( u \in \overline{H}^{s+1,t}(\mathbb{R}^d_+) \) such that \( D_1^j u \in C^0(\mathbb{R}^d; H^{s+t+1-j}(\mathbb{R}^{d-1})) \) when \( x_1 \geq 0 \) and

\[
P u = f \quad \text{in} \quad Y \cap \mathbb{R}^d_+, \quad D_1^j u = u_j \quad \text{in} \quad Y \cap (\mathbb{R}^{d-1} \times \{0\}), \quad j < 2.
\]

Moreover, for \( \chi \in C_c^\infty(X) \) with \( \chi \equiv 1 \) on \( Y \), \( C \chi > 0 \) so that we can find such a \( u \) with

\[
\|u\|_{H^{s+1,t}(\mathbb{R}^d_+)} \leq C \chi(\|f\|_{s,t} + \sum_j \chi u_j \|H^{s+t+1-j})
\]

**Proof.** Fix \( T > \sup \{x_1 \mid x \in \text{supp} \chi \} \). Then, since \( s \geq 0 \), \( f \in L^1((0,T); H^{s+t+1-j}) \). Hence, by Theorem 1.2 there exists \( v \in C^0([0,T]; H^{s+t+1-j}) \cap C^1([0,T]; H^{s+t}) \) with \( \tilde{P} v = \chi f \) on \( 0 < x_1 < T \) and \( D_1^j v|_{x_1=0} = \chi u_j \). Note that, letting \( X_T := (0,T) \times \mathbb{R}^{d-1} \), by Lemma 2.1 \( v \in \overline{H}^{s+1,t}(X_T) \). Moreover, \( \tilde{P} v \in \overline{H}^{s+1,t}(X_T) \).

Now, observe that by Lemma 2.1

\[
D_1^j v \in \overline{H}^{k+1-j,s+t-k}(X_T), \quad k = 0, j < 2.
\]

Suppose this holds for \( k < s \). Then,

\[
D_1^2 v = \tilde{P} v - \sum_{j < 2} \tilde{P} j D_1^j v \in \overline{H}^{s+t+1-k}(X_T).
\]

In particular, \( D_1^j v \in \overline{H}^{k+1,s+t-k-1} \) and \( v \in \overline{H}^{k+2,s+t-k-1} \) by Lemma 2.1. So, by induction \( v \in \overline{H}^{s+1,t}(X_T) \) as desired. Letting \( u = \chi v \) gives the desired result. \( \square \)

Next we give a local version of the first part of Theorem 2.1.

**Lemma 2.4.** Assume \( P \) is strictly hyperbolic with respect to \( x_1 \) in \( X \subset \mathbb{R}^d \) and let \( Y \subset X \) be open and precompact. If \( f \in \dot{H}^{s,t}(\mathbb{R}^d_+) \), then there exists \( u \in \dot{H}^{s+1,t}(\mathbb{R}^d_+) \) such that \( Pu = f \) in \( Y \).

**Proof.** First, suppose \( s \geq 0 \). Then, by Lemma 2.3 there exists \( v \in \overline{H}^{s+1,t} \{x_1 > -1\} \) with

\[
\tilde{P} v = f, \quad -1 < x_1 < T, \quad D_1^j v|_{x_1=-1} = 0.
\]

Observe then that since \( \text{supp} \ f \subset \mathbb{R}^d_+ \), the energy estimate [6] implies \( \text{supp} \ v \subset \mathbb{R}^d_+ \) and hence, extending \( v \) to \( x_1 < -1 \) by \( 0 \), \( v \in \dot{H}^{s+1,t}(\mathbb{R}^d_+) \).

Thus, we have verified the theory for \( s \geq 0 \). To prove it for \( s < 0 \), we proceed by induction. Assume the theorem holds when \( s \) is replace by \( s + 1 \). Then by Corollary 2.1

\[
f = f_0 + D_1 f_1
\]

where \( f_0 \in \dot{H}^{s+1,t-1} \) and \( f_1 \in \dot{H}^{s+1,t} \). Now, there exist \( u_0 \in \dot{H}^{s+2,t-1} \) and \( u_1 \in \dot{H}^{s+2,t} \) such that

\[
\tilde{P} u_0 = f_0, \quad \tilde{P} u_1 = f_1.
\]
Now, \( U = u_0 + D_1 u_1 \in \dot{H}^{s+1,1} \) and
\[
\tilde{P} U - f = [\tilde{P}, D_1] u_1 \in \dot{H}^{s+1,1-1}
\]
since \([\tilde{P}, D_1] \) of order at most 1 in the \( D_1 \) derivatives and 2 in all derivatives. Then, we have \( v \in \dot{H}^{s+2,1-1} \) with \( \tilde{P} v = [\tilde{P}, D_1] u_1 \) and \( u = U - v \) is the desired solution. \( \square \)

In order to finish the proof of Theorem 2.1 we need some uniqueness for solutions

**Lemma 2.5.** Suppose \( P \) is strictly hyperbolic in \( X \) with respect to \( \phi \). Then every \( x_0 \in X \) has a fundamental system of neighborhoods \( V \) such that \( u \in \mathcal{D}'(V), \phi \geq \phi(x_0) \) in \( \text{supp } u \), and \( Pu = 0 \) in \( V \) implies \( u = 0 \) in \( V \).

**Proof.** Take coordinates so that \( x_0 = 0 \) and \( \phi(x) = \phi(0) + x_1 - |x'|^2 \). This is possible by the Morse lemma. Then let
\[
V_\epsilon = \{ x \mid |x_1| < \epsilon^2, |x'| < \epsilon \}.
\]
Now, \( \xi_1 \mapsto (\xi')^{-2} p(0, (\xi') \xi_1, \xi') \) has distinct real zeros on \( \partial \mathbb{P}^{d-1}_0 \mathbb{R}^d \) so the same is true for \( x \in V_\epsilon \) provided \( 0 < \epsilon < \epsilon_0 \). Moreover, this \( \epsilon_0 \) can be chosen uniformly on compact subset of \( X \).

Now, by Lemma 2.4 for \( g \in C^\infty_0(V_\epsilon) \), we can find \( v \in C^\infty_0(V_\epsilon) \) with \( P^* v = g \) in \( V_\epsilon \) and \( v = 0 \) for \( x_1 > \epsilon^2 - \delta \) where \( \delta > 0 \) is chosen so that \( g = 0 \) on \( x_1 > \epsilon^2 - \delta \). Then,
\[
0 = (Pu, v) = (u, P^* v) = (u, g)
\]
and in particular \( u = 0 \) in \( V_\epsilon \). \( \square \)

We finally prove the Theorem 2.1

**Proof.** Let \( X_\nu \subset Y \) be coordinate patches and \( Y_\nu \subset X_\nu \) be precompact so that \( \bar{Y} \subset \cup_\nu Y_\nu \). For each \( \nu \), by Lemma 2.4 we can find \( u_\nu \in H^{s+1}_\text{loc}(X_\nu) \) vanishing in \( \phi < 0 \) so that \( Pu_\nu = f \) in \( Y_\nu \).

We choose a covering of \( X_0 \cap \bar{Y} \) by open sets \( V_\mu \) with the properties claimed in Lemma 2.5 and choose \( V_\mu \) small enough that that if \( V_\mu \cap V_\nu' \neq \emptyset \), \( V_\mu \cup V_\nu' \subset Y_\nu \) for some \( \nu \). We define \( u = u_\nu \) in \( V_\mu \) whenever \( V_\mu \subset Y_\nu \). Then \( u \) is well defined in \( V = \cup_\mu V_\mu \) and \( Pu = f \) there.

Now, let \( \chi \in C^\infty_0(X) \) with \( \chi \equiv 1 \) in a neighborhood, \( W \) of \( X_0 \cap \bar{Y} \) so that \( \bar{W} \subset V \). Then \( P(\chi u) = f \) in \( W \). In particular, there exists \( g \in H^{s}_\text{loc}(X) \) and \( \epsilon > 0 \) so that \( g = 0 \) when \( \phi < \epsilon \) and \( P(\chi u) = f - g \) in \( Y \). Therefore, we need to prove the statement with \( f \) replaced by \( g \) and \( \phi \) replaced by \( \phi - \epsilon \).

Since \( \bar{Y} \) is compact, and the \( \epsilon_0 \) in the proof of Lemma 2.5 can be chosen uniformly on compact sets, the \( \epsilon \) above can be chosen uniformly with \( \phi \) replaced by \( \phi - t \) for \( t \in [0, \sup_Y \phi(x)] \). Then, since the theorem is trivial if \( Y \subset \{ \phi < 0 \} \), the proof is complete after a finite number of iterations of the argument above.

For the second part of the theorem we appeal to Lemma 2.3 instead of Lemma 2.4.

**Corollary 2.2.** Suppose that for \( a < b \), \( X_{ab} := \{ a < \phi(x) < b \} \) is precompact. Then the solution given by Theorem 2.1 is unique when \( Y = X_{ab} \). \( \square \)
3. Boundaries

Proof. Suppose that \( Pu = 0 \) in \( X_{ab} \), \( \text{supp} \, u \subset \{ \phi(x) \geq 0 \} \). Then, by construction \( \text{supp} \, u \subset \{ \phi > \epsilon \} \) where \( \epsilon \) is uniform on \( X_{ab} \). In particular, iterating finitely many times, \( u = 0 \).

For uniqueness with \( s \geq 0 \) an initial conditions imposed, suppose that \( Pu = 0 \) in \( X_{ab} \), \( u \in H^{1,2}(X) \) with \( D^j u|_{t=0} = 0 \) for \( j < 2 \). Then, extending \( u \) by 0 into \( \phi < 0 \), we have \( u \in H^{1,2}(X) \) we have \( \text{supp} \, u \subset \{ \phi(x) \geq 0 \} \) and hence as before \( u = 0 \). \( \square \)

3. Boundaries

We will focus on the case \( m = 2 \) throughout these notes. In this case, it is possible to associate a Lorentzian metric to \( p \). Multiplying \( P \) by \( p(x, d\phi(x))^{-1} \), we assume that \( p(x, d\phi) = 1 \). Since \( m = 2 \), in coordinates \((x, \xi)\) on \( T^*M \), we can write

\[
p(x, \xi) = \langle G^{-1}(x)\xi, \xi \rangle
\]

where \( G^{-1}(x) \) is a symmetric invertible matrix. In fact, The invertibility of \( G \) follows from the fact that \( p \) is strictly hyperbolic with respect to \( \phi \). In fact, if \( G^{-1}(x)\xi = 0 \), then

\[
p(x, \xi + \tau d\phi) = \langle G^{-1}(x)(\xi + \tau d\phi), (\xi + \tau d\phi) \rangle = \tau^2 \langle G^{-1}(x)d\phi, d\phi \rangle = \tau^2 p(x, d\phi) = \tau^2
\]

which has a double root at 0.

We then define a symmetric bilinear form on \( T^*_x M \) by

\[
\langle \xi, \eta \rangle_g := \langle G^{-1}(x)\xi, \eta \rangle
\]

which we identify in the usual way with a metric on \( T^*_x X \) i.e.

\[
\langle V, W \rangle_g := \langle G(x)V, W \rangle.
\]

We now classify tangent vectors into three types.

Definition 3.1. We say that \( V \in T^*_x X \)

1. is time-like if \( \langle V, V \rangle_g > 0 \)
2. is space-like if \( \langle V, V \rangle_g < 0 \)
3. is null if \( \langle V, V \rangle_g = 0 \).

We then say that a hypersurface \( H \) is spacelike, timelike or null if \( p(\nu) < 0 \), \( p(\nu) > 0 \), or \( p(\nu) = 0 \) respectively where \( \nu \) a conormal to \( H \).

Throughout this section, we will also make the assumptions that

\[
(11) \quad X \ni x \mapsto \phi(x) \text{ is proper,} \quad \partial X \text{ is time-like.}
\]

The proof of Theorems 4.1 and ?? will follow as usual from certain a-priori estimates on the solution.

4. Existence and Uniqueness

References: Hormande:III 24.1

**Theorem 4.1.** Let \( f \in \bar{H}^s_{\text{loc}}(X^o) \), \( u_0 \in H^{s+1}(\partial X) \) where \( s \geq 0 \) and assume \( u_0, f \) vanish on \( \{ \phi < a \} \), that \( p \) is strictly hyperbolic with respect to \( \phi \) and \([11]\) holds. Then there is a unique \( u \in \bar{H}^{s+1}_{\text{loc}}(X^o) \) such that

\[
\begin{align*}
Pu &= f \quad \text{in } X^o \\
u &= u_0 \quad \text{on } \partial X \\
u &= 0 \quad \text{on } \phi < a.
\end{align*}
\]

Moreover \( u \) satisfies for every \( a' < a < b < b' \), [TODO]estimates

\[
\|u\|_{\bar{H}^{s+1}(a'<\phi<b')} \leq C(\|f\|_{\bar{H}^s(a'<\phi<b')} + \|u_0\|_{H^{s+1}(\partial X)}).
\]
CHAPTER 4

Propagation of Singularities

In this chapter we study an operator $P \in \text{Diff}^2(X)$ for $X$ a manifold with boundary $\partial X$ and interior $X^o$. We assume throughout that $\partial X$ is non-characteristic for $P$. That is, $p$ does not vanish on $N^*\partial X$. For propagation in $X^o$, we will allow $P$ to have an imaginary principal symbol. However, when it comes time to study the problem near the boundary, we will insist that the symbol be real valued.

1. Propagation in the bulk

2. Propagation of singularities for strictly hyperbolic problems

We now prove the propagation of singularities result for pseudodifferential operators.

**Theorem 2.1.** Let $X$ be a compact manifold and $P \in \Psi^m_{\text{phg}}(X)$ with $\sigma(P) = p - iq$ with $p,q$ real valued. Suppose that $A, B, B_1 \in \Psi^0_{\text{phg}}(M)$ such that

1. for all $(x_0, \xi_0) \in \text{WF}(A)$, there exist $T > 0$ so that $\exp(-T\langle \xi \rangle^{-m+1}H_p)(x_0, \xi_0) \in \text{ell}(B)$, $\exp(-t\langle \xi \rangle^{-m+1}H_p)(x_0, \xi_0) \in \text{ell}(B_1)$, $0 \leq t \leq T$.
2. $q \geq 0$ on $\text{WF}(B_1)$. Then, for all $u \in \mathcal{D}'(M)$ if $B_1Pu \in H^{s-m-1}(X)$ and $Bu \in H^s$ then $Au \in H^s$ and for all $N > 0$, there exists $C_N > 0$ such that

$$\|Au\|_{H^s} \leq C\|B_1Pu\|_{H^{s-m-1}} + C\|Bu\|_{H^s} + C_N\|u\|_{H^{-N}}.$$

**2.1. Construction of the escape function.** The idea will be to use positivity of the commutator $[P, A]$ to obtain estimates on $u$ in terms of $P$. In order to do this, we will produce a so-called escape function which is increasing along the flow.

**Lemma 2.1.** Let $X$ be a compact manifold without boundary and $A, B, B_1$ be as in Theorem 2.1. Then there exists $0 \leq g \in C^\infty(\partial T^*X)$ such that there exists $\beta \geq 0$ with

$$g > 0 \text{ on } \text{WF}(A), \quad \langle \xi \rangle^{-m+1}H_pg \leq -\beta g \text{ in a neighborhood of } \partial T^*X \setminus \text{ell}(B).$$

**Proof.** Let $\varphi_t(x_0, \xi_0) := \exp(t\langle \xi \rangle^{-m+1}H_p)(x_0, \xi_0)$. We start with the case $\text{WF}(A) = \{(x_0, \xi_0)\}$. We may assume that $\langle \xi \rangle^{-m+1}H_p(x_0, \xi_0) \neq 0$ since otherwise any $g \geq 0$ will due. Now, let $T > 0$ so that $\varphi_{-T}(x_0, \xi_0) \in \text{ell}(B)$. The $\Sigma \subset \partial T^*X$ a hypersurface through $(x_0, \xi_0)$ transverse to $H_p$. Then, there exists a neighborhood $V$ of $(x_0, \xi_0)$ in $\Sigma$ and $\delta > 0$ so that

$$\Phi(t, q) : (-T - \delta, \delta) \times V \ni (t, q) \mapsto \varphi_t(q) \in T^*X.$$
is a diffeomorphism onto its image and
\[ \Phi(-T-\delta,-T+\delta,V) \subset \ell(B), \quad \Phi(-T,\delta,V) \subset \bar{\ell}(B_1) \]
Let \( 0 \leq \psi(t) \in C_c^\infty(-T-\delta,\delta) \) such that \( \psi'(t) \leq -\beta \psi \) off of \((-T-\delta/2,T+\delta/2)\) and \( \chi(0) > 0 \).
Then, fix \( 0 \leq \chi \in C_c^\infty(V) \) and define
\[ g(x,\xi) = (\psi \otimes \chi)(\Phi^{-1}(x,\xi)). \]
and extend \( g \) by 0 outside of the image of \( \Phi \).
Notice that \( g > 0 \) in a neighborhood of \((x_0,\xi_0)\), so by compactness of \( \bar{\text{WF}}(A) \), there exist \((x_n,\xi_n) \in T^*X \) such that
\[ g = \sum_n g_{x_n,\xi_n} > 0 \quad \text{on} \ \bar{\text{WF}}(A), \]
Moreover,
\[ H_\rho g = \sum_n H_\rho g_{x_n,\xi_n} \leq -\beta \sum_n g_{x_n,\xi_n} = -\beta g \ \text{in a neighborhood of} \ T^*X \setminus \bar{\ell}(B). \]
\[ \square \]

We now prove the propagation of singularities estimate

\textbf{Proof.} Fix a volume form on \( M \) and write \( \text{Im} A = \frac{A-A^*}{2i} \), \( \text{Re} A = \frac{A+A^*}{2} \). Let \( G = \text{Op}(\langle \xi \rangle^{s+\frac{1-m}{2}}) \) and \( E = \text{Op}(\langle \xi \rangle^{s+\frac{1-m}{2}}) \) for some metric on \( X \). For \( u \in C^\infty(M) \) consider
\[ \text{Im}(Pu,G^*Gu) = \frac{\text{Im}(\text{Re} Pu,G^*Gu) + \text{Re}(\text{Im} Pu,G^*Gu)}{2} \]
[TODO]finish see Dyatlov–Zworski
\[ \square \]

3. The \( b \)-wavefront set

[TODO]

4. Propagation near the boundary

With Theorem 2.1 in place, we have a complete understanding of how singularities propagate in \( X^o \) (at least provided that \( H_\rho \) is not radial!). Therefore, it remains to understand the behavior near the boundary of \( X \). Throughout this section, we will use [TODO] to change coordinates so that
\[ p = \xi_1^2 - r(x,\xi) \]
where \( p(x,\xi) \) is the symbol of \( p \).
We denote by \( \pi : T^*_\partial X \to T^*\partial X \) the projection through \( N^*\partial X \) and write \( \Sigma := \{ \rho \in T^*X \mid p(\rho) = 0 \} \). We then divide \( T^*\partial X \) into three regions, the \textit{elliptic}, \textit{hyperbolic}, and \textit{glancing} regions
\[ \mathcal{E} := \{ q \in T^*\partial X \mid \pi^{-1}(q) \cap \Sigma = \emptyset \}, \]
(12)
\[ \mathcal{H} := \{ q \in T^*\partial X \mid \#(\pi^{-1}(a) \cap \Sigma) = 2 \}, \]
\[ \mathcal{G} := T^*\partial X \setminus (\mathcal{E} \cup \mathcal{H}). \]
We will study each of these regions separately. There is no propagation in $\mathcal{E}$ and the propagation in $\mathcal{H}$ results in broken bicharacteristics. The propagation through $\mathcal{G}$ is subtle and will require a great deal of analysis.

### 4.1. The elliptic region

Our first task will be to prove the analog of the fact that if $X$ is a compact manifold without boundary, then $WF(u) \subset WF(Pu) \cup \{ p = 0 \}$. Note that the same proof shows that this continues to hold in the interior of a manifold with boundary and in particular, $WF_b(u)|_{X^0} \subset WF_b(Pu)|_{X^0}$. Now, with $\Sigma := \{ p = 0 \} \subset T^*X$, define $\tilde{\Sigma} = \iota(\Sigma) \subset T^*X$ where $\iota : T^*X^0 \to T^*X$ is the natural inclusion extended to $T^*X$. One might hope that analog of the elliptic regularity statement would be

$$WF_b(u) \subset \tilde{\Sigma} \cup WF_b(Pu).$$

However, it is easy to see that such a statement cannot hold without imposing some boundary conditions and so, the correct statement should be

$$WF_b(u) \subset \tilde{\Sigma} \cup WF_b(Pu) \cup WF(u|_{\partial X}).$$

Now, notice that in coordinates where $\partial X = \{ x_1 = 0 \}$

$$\tilde{\Sigma}|_{\partial X} = \{(x', \xi') \in T^*\partial X \mid (0, x_1, \xi_1, \xi') \in \Sigma \} = \mathcal{H} \cup \mathcal{G}$$

and so to obtain (13) it is enough to show

$$WF_b(u) \cap \mathcal{E} \subset (WF_b(Pu) \cup WF(u|_{\partial X})) \cap \mathcal{E}.$$

In order to study the elliptic region, we start with a local problem. In particular, we assume that $X \subset \mathbb{R}^d_+$ with $\partial X = \{ x_1 = 0 \}$. We then consider

$$P = D_{x_1}^2 + b(x, D')D_{x_1} + c(x, D').$$

We will start by proving elliptic regularity for such an operator.

**Lemma 4.1.** Suppose that $u \in \overline{H}_{\text{comp}}^1, Pu \in \overline{H}^{-1}$, and $u|x_1=0 = 0$ and $T^*\{ x_1 = 0 \} \subset \mathcal{E}$. Then,

$$\|u\|_{\overline{H}^{-1}} \leq C(\|Pu\|_{\overline{H}^{-1}} + \|u\|_{L^2}).$$

We first need,

**Lemma 4.2.** For $u \in \dot{H}_{\text{comp}}^1$,

$$(D_{x_1}^2 u, u)_{H^{-1}, H^1} = \|D_{x_1} u\|^2.$$  

**Proof.** Since $u \in \dot{H}_{\text{comp}}^1$, we may extend $u$ by 0 to $U \in H^1(\mathbb{R}^d)$. Then, let $u_\epsilon \to U$ in $H^1$ with $u_\epsilon \in C_0^\infty$ and supp $u_\epsilon \subset \{ x_1 \geq 0 \}$. For example, take $\psi \in C_0^\infty(\mathbb{R}^d)$ with supp $\psi \subset \{ x_1 > 0 \}$, $\int \psi = 1$ and let $u_\epsilon = e^{-d} \psi_\epsilon * u$.

$$D_{x_1}^2 u_\epsilon, u_\epsilon)_{H^{-1}, H^1} = \|D_{x_1} u_\epsilon\|^2.$$  

Now,

$$|D_{x_1}^2 u, u| - (D_{x_1}^2 u_\epsilon, u_\epsilon) = |D_{x_1}^2 (u - u_\epsilon), u| + |D_{x_1}^2 u_\epsilon, u - u_\epsilon|$$

$$\leq \|u - u_\epsilon\|_{H^1}(\|u\|_{H^1} + \|u_\epsilon\|_{H^1}) \to 0.$$
Also,
\[ \|D_x u\|_2^2 \rightarrow \|D_x u\|_2^2, \]
completing the proof. □

We now prove Lemma 4.2.

PROOF. We proceed using standard energy estimates.
\[
(Pu, u)_{\mathcal{H}^{-1}, \mathcal{H}^1} = (D_x^2 u, u) + (b(x, D') D_x u, u) + (c(x, D') u, u)
\]
\[ \|D_x u\|_2^2 + (D_x u, b^* (x, D') u) + (c(x, D') u, u) \]

Now, since \( T^* \{ x_1 = 0 \} \subset \mathcal{E} \), there exists \( \delta > 0 \) so that
\[ c(x, \xi') \geq \frac{b^2(x, \xi')}{4} + \delta |\xi'|^2. \]
Therefore, there exists \( \delta_0 > 0 \) so that
\[ (c(x, D') u, u) - \frac{(1 + \delta_0)}{4} \|b(x, D') u\|_2^2 \geq \frac{\delta}{2} \|\nabla_x u\|_2^2 - C \|u\|_2^2. \]
On the other hand
\[ |(D_x u, b^* (x, D') u)| \leq \|D_x u\|_2 \|b^* (x, D') u\|_2 \leq \frac{1 + \delta_0}{4} \|b(x, D') u\|_2^2 + \frac{1}{1 - \delta_0} \|D_x u\|_2^2. \]
So,
\[ |(Pu, u)| \geq c(\|D_x u\|_2^2 + \|\nabla_x u\|_2^2) - C \|u\|_2^2. \]
In particular,
\[ c \|u\|_{H^1}^2 \leq \|Pu\|_{H^{-1}} \|u\|_{H^1} + (C + c) \|u\|_{H^1}^2 \leq \epsilon^{-1} \|Pu\|_{H^{-1}}^2 + \epsilon \|u\|_{H^1}^2 + C \|u\|_2^2 \]
So, choosing \( 0 < \epsilon < c/2 \),
\[ \|u\|_{H^1}^2 \leq C(\|Pu\|_{H^{-1}}^2 + \|u\|_{H^1}^2). \]
□

Next, let \( E : H^s(\{ x_1 = 0 \}) \rightarrow \mathcal{H}^{s+\frac{1}{2}} \) be the extension operator. Let \( u \in \mathcal{H}^1 \) with \( u|_{\{ x_1 = 0 \}} = u_0 \in H^{\frac{1}{2}} \). Then, \( u - Eu_0 \in \mathcal{H}^1 \) with \( u|_{x_1 = 0} = 0 \) Moreover,
\[ \|Eu_0\|_{\mathcal{H}^1} \leq \|u_0\|_{H^{1/2}}. \]
So,
\[ \|P(u - Eu_0)\|_{\mathcal{H}^{-1}} \leq \|Pu\|_{\mathcal{H}^{-1}} + C \|u_0\|_{H^{1/2}}. \]
In particular,

**Lemma 4.3.** For \( u \in \mathcal{H}^1 \), with \( u|_{x_1 = 0} = u_0 \),
\[ \|u\|_{\mathcal{H}^1} \leq C(\|Pu\|_{\mathcal{H}^{-1}} + \|u_0\|_{H^{1/2}} + \|u\|_{L^2}). \]

Next, we improve the regularity of \( u \).
4. PRODUCTION NEAR THE BOUNDARY

**Lemma 4.4.** Suppose \( u \in \mathcal{L}^2(\mathbb{R}^d_+) \), \( u|_{x_1=0} = u_0 \in H^{1/2} \) and \( Pu \in H^{-1} \). Then,
\[
\| u \|_{H^1} \leq C(\| Pu \|_{H^{-1}} + \| u_0 \|_{H^{1/2}}).
\]

**Proof.** Let \( \chi \in C^\infty_c(\mathbb{R}^{n-1}) \) with \( \chi \equiv 1 \) near 0. Then put \( u_\epsilon = \chi(\epsilon D')u \). Then \( u_\epsilon \to u \in \mathcal{L}^2 \), \( Pu_\epsilon \to Pu \in H^{-1} \), \( u_\epsilon \in H^{0,t} \) for all \( t \). Now,
\[
Pu_\epsilon = \chi(\epsilon D')Pu + [\chi(\epsilon D'), b(x, D')]D_{x_1}u + [\chi(\epsilon D'), c(x, D')]u \in H^{-1,t}
\]
for any \( t \). In particular,
\[
D_{x_1}^2 u_\epsilon \in H^{-1,t-1}, \quad D_{x_1} u_\epsilon \in H^{-1,t}.
\]
Therefore, \( D_{x_1} u_\epsilon \in H^{0,t-1} \) and then, since \( u_\epsilon \in H^{0,t-2} \), \( u \in H^{1,t-2} \). In particular, choosing \( t \geq 2 \), \( u_\epsilon \in H^1 \). Hence,
\[
\| u_\epsilon \|_{H^1} \leq C(\| Pu_\epsilon \|_{H^{-1}} + \| u_\epsilon \|_{L^2} + \| u_\epsilon|_{x_1=0}\|_{H^{1/2}}) \leq 2C(\| Pu \|_{H^{-1}} + \| u \|_{L^2} + \| u_0 \|_{H^{1/2}}).
\]
In particular, there exists a subsequence so that \( u_\epsilon \to \tilde{u} \) in \( \mathcal{P}^l \). But \( u_\epsilon \to u \in \mathcal{L}^2 \), so \( u \in \mathcal{P}^l \) and the estimate continues to hold. \( \square \)

**Lemma 4.5.** Suppose \( s \geq 0 \), \( u \in \mathcal{P}^{s,t}(\mathbb{R}^d_+) \), \( u|_{x_1=0} = u_0 \in H^{s+t+\frac{1}{2}} \). Then
\[
\| u \|_{H^{s+1,t}} \leq C(\| Pu \|_{H^{s-1,t}} + \| u \|_{H^{s,t}} + \| u|_{x_1=0}\|_{H^{s+t+\frac{1}{2}}}).
\]

**Proof.** We start with \( t = 0 \), observe that
\[
P(\langle D' \rangle^s)u = \langle D' \rangle^s Pu + [b(x, D'), \langle D' \rangle^s]D_{x_1}u + [c(x, D'), \langle D' \rangle^s]u
\]
so,
\[
\| \langle D' \rangle^s u \|_{H^1} \leq C(\| \langle D' \rangle^s Pu \|_{H^{-1}} + \| u \|_{H^{0,s}} + \| \langle D' \rangle^s u_0 \|_{H^{1/2}}).
\]
In particular,
\[
\| u \|_{H^{1,s}} \leq C\| Pu \|_{H^{1,s}} + \| u \|_{H^{s,0}} + \| u_0 \|_{H^{s+\frac{1}{2}}}.
\]
Iterating as before, we then obtain Therefore,
\[
\| u \|_{H^{s+1,0}} \leq C\| Pu \|_{H^{s-1,0}} + \| u_0 \|_{H^{s+\frac{1}{2}}}
\]
Now, for \( t \neq 0 \),
\[
\| \langle D' \rangle^t u \|_{H^{s+1,0}} \leq C(\| \langle D' \rangle^t Pu \|_{H^{s-1,0}} + \| u \|_{H^{s,t}} + \| \langle D' \rangle^t u_0 \|_{H^{s+\frac{1}{2}}})
\]
which concludes the proof. \( \square \)

Finally,

**Lemma 4.6.** Suppose \( T^*\{x_1 = 0\} \subset \mathcal{E}, u \in \mathcal{N}(\mathbb{R}^d_+) \),
\[
Pu = f \in \mathcal{N}(\mathbb{R}^d_+), \quad u|_{x_1=0} = u_0.
\]
Then,
\[
WF_b(u)|_{T^*\{x_1=0\}} = WF_b(f)|_{\{x_1=0\}} \cup WF_b(u_0).
\]
PROOF. Here, the inclusion of the right hand side in the left is automatic from the definition of WF. Therefore, we need only show that the right hand side is contained in the left. Suppose that \((0, \xi'_0) \notin WF_b(f)|_{x_1=0} \cup WF_b(u_0)\). Fix \(\delta > 0\) so that
\[
\{0 < x_1 \leq \delta\} \cap \{(x, \xi_1, 0) \in WF(u) \cup WF(f)\} = \emptyset.
\]
In particular, this is possible since WF\(_b\)(u)\(_{(x_1=0)}\) \(\subset T^*\{x_1 = 0\}\) and WF\(_b\)(u) is closed. Therefore, for \(x_1 > 0\) small and \((x, \xi) \in WF(u)\), \(x_1|\xi_1| < |\xi'|\).

Now, let
\[
W' := WF_b(f)|_{x_1=0} \cup WF_b(u_0) \cup \{(x, \xi') | 0 < x_1 \leq \delta, (x, \xi) \in WF(f)\mbox{ for some }\xi_1\}.
\]
This is closed since WF\(_b\)(f) is. Let \(\chi \in S^\infty(\mathbb{R} \times T^*\mathbb{R}^{d-1})\) with \(\chi(x, \xi') = 0\) for \(x_1 \geq \delta\) and order \(-\infty\) in a conic neighborhood of \(W'\). We call such a \(\chi\) a good cutoff. Then, since WF\((f)\) avoids the \(N^*\{x_1 = c\}\) for \(c \leq \delta\), \(\chi(x, D')f \in C^\infty\) [TODO] references.

Now, choose \(s, t\) such that for \(\chi \in S^j\) good,
\[
\chi(x, D')u \in \Pi^s_{\text{loc}}^{t-j}.
\]

Now, for \(\chi \in S^0\), good
\[
P\chi(x, D')u = \chi(x, D')f + [P, \chi(x, D')]u \in \Pi^{s-1,t}.
\]

Indeed,
\[
[P, \chi(x, D')]u = P_0(x, D')\chi_0(x, D') + D_{x_1}P_1(x, D')\chi_1(x, D')
\]
where \(P_i\chi_i\) is a good cutoff in \(S^{1-i}\).

By iteration we obtain then that
\[
\chi(x, D')u \in \Pi^{s+1,t-1}
\]
and in particular,
\[
\chi(x, D')u \in \Pi^{s_0,t_0}
\]
with \(s_0 \geq 0\).

Now, observe that
\[
\chi(x, D')u|_{x_1=0} = \chi(x, D')u_0 \in C^\infty,
\]
So
\[
P\chi(x, D')u \in \Pi^{s_0-1,t_0}, \quad \chi(x, D')u|_{x_1=0} \in C^\infty.
\]
Therefore, by Lemma 4.5, \(\chi(x, D')u \in \Pi^{s_0+1,t_0}\) and in particular, \(\chi(x, D')u \in C^\infty\). Now, since \((0, \xi'_0) \notin W'\), we may take \(\chi(x, \xi')\) good with \(\chi(0, \xi'_0) = 1\) and hence by [TODO] reference, \((0, \xi'_0) \notin WF_b(u)\).

Finally, we return to the general situation and prove our main theorem for singularities there.

**Theorem 4.1.** Suppose that \(P \in \text{Diff}^2(X)\) with \(\partial X\) non-characteristic for \(P\). Suppose \(f \in \mathcal{N}(X)\) and
\[
P u = f \in X^\circ, \quad u|_{\partial X} = u_0
\]
Then,
\[
WF_b(u)|_{\partial X} \cap \mathcal{E} = (WF_b(f)|_{\partial X} \cup WF(u_0)) \cap \mathcal{E}.
\]
[TODO] for this theorem to make sense we need to prove earlier that if $\partial X$ is non-char, $Pu = f \in \mathcal{N}(X)$ then $u \in \mathcal{N}(X)$. We did this in class.

**Proof.** Again, the inclusion of the left-hand side in the right is trivial. So, suppose we work in coordinates where $(0, \xi_0') \notin \WF_b(f)|_{\partial X} \cup \WF(u_0)$. Then, observe that in these coordinates

$$P = P_2(x)D_x^2 + P_1(x, D')D_x + P_0(x, D')$$

with $P_1 \in S^{2-i}$ and $P_2 \neq 0$ near 0. Therefore, let $\chi \in C^\infty_c(\mathbb{R}^{d-1})$ with $\chi \equiv 1$ near 0 and let $\psi \in C^\infty_c(\mathbb{R}^{d-1})$ with $\psi \equiv 1$ in a conic neighborhood of $\{\xi' : |\xi' - \xi_0'| = |\xi_0'|, |\xi| \geq 1\}$. Let

$$\tilde{P} = D_x^2 + \sum_{j=0}^1 (|P_0(x)|^{-1}\chi(x)\psi(D')P_j(x, D') + P_0(0)^{-1}(1 - \chi(x)\psi(D'))P_j(0, \xi_0'/|\xi_0'||D'|))D_x^j$$

In particular, for $g \in \mathcal{N}(X)$,

$$(0, \xi_0') \notin \WF_b([\tilde{P} - P]g).$$

Therefore,

$$(0, \xi_0') \notin \WF_b(\tilde{P}u)$$

Moreover, $T^*\{x_1 = 0\} \subset \mathcal{E}_\rho$ for $\chi$ supported in a small enough neighborhood of 0 and $\psi$ in a small enough neighborhood of $\xi_0'$. Therefore, Lemma 4.6 applies and $(0, \xi_0') \notin \WF_b(u)$ as desired. \qed

[TODO] prove the next thing We next state a quantitative analog of Theorem 4.1.

**Theorem 4.2.** Suppose that $P \in \Diff^2(X)$ with $\partial X$ non-characteristic for $P$. Suppose $f \in \mathcal{N}(X)$ and

$$Pu = f \in X^\circ, \quad u|_{\partial X} = u_0$$

Then, for $A \in \Psi^0_b(X), B \in \Psi^0_b(X)$, with $\WF_b(A) \subset \text{ell}(B)$ and $\WF_b(B) \subset \mathcal{E} \cap \tilde{\Sigma} = \emptyset$, we have

$$\|Au\|_{\mathcal{H}^s} \leq C(\|BPu\|_{\mathcal{H}^{-2}} + \|Bu|_{\partial X}\|_{\mathcal{H}^{-\frac{1}{2}}}.$$}

4.2. The Hyperbolic region. Now that we have microlocal elliptic regularity in place, we want to study the analog of Theorem 2.1. In particular, we want to understand how singularities may propagate inside $\tilde{\Sigma}$. This will happen in two steps. First, we study $\mathcal{H} \subset T^*\partial X$ and only then $\mathcal{G}$. In $\mathcal{H}$, many methods are available to prove propagation of singularities. We will use a factorization method similar to what was used to prove well posedness for hyperbolic equations. For other approaches see [TODO] references.

Recall that we may choose coordinates so that $\partial X = \{x_1 = 0\}$ and

$$p(x, \xi) = \xi_1^2 - r(x, \xi').$$

If $(x_0', \xi_0') \in \mathcal{H}$, then the roots of $\xi_1 \mapsto p(0, x_0', \xi_1, \xi_0')$ are $\pm \sqrt{r_0(x_0', \xi_0')}$ where $r_0(x', \xi') := r(0, x', \xi')$. Note then that the $\pm$ root is a covector pointing in/out of $X$. Moreover, since

$$H_p = 2\xi_1 \partial_{x_1} - H_r,$$

$\gamma(t) : \exp(\pm tH_p)(0, x', \pm \sqrt{r_0(x', \xi')}\xi', \xi')$ is a bicharacteristic for $p$ in $X^\circ$ for $t$ in $(0, \epsilon)$. 
4. PROPAGATION OF SINGULARITIES

**Definition 4.1.** We define a compressed broken bicharacteristic for $p$ as a continuous map

$$
\gamma(t) : I \to \Sigma \setminus \mathcal{G}
$$

where $I \subset \mathbb{R}$ is an interval and

- When $\gamma(t) \in \tilde{\Sigma} \cap T^*X^o$, $\gamma(t) \in C^1$ and $\gamma'(t) = H_p(\gamma(t))$.
- $\{t \in I \mid \gamma(t) \in \mathcal{H}\}$ is a discrete subset of $I$.

Locally near a point $(x', \xi') \in \mathcal{H}$, such a broken bicharacteristic is given by

$$
\gamma(t) = \begin{cases} 
\iota(\exp(tH_p)(0, x', \sqrt{r_0(x', \xi')}, \xi')) & 0 \leq t \leq \epsilon \\
\iota(\exp(tH_p)(0, x', -\sqrt{r_0(x', \xi')}, \xi')) & -\epsilon \leq t \leq 0.
\end{cases}
$$

Our next theorem says that singularities of solutions to $Pu = 0$ with $u|_{\partial X} = 0$ are invariant along such broken bicharacteristics.

**Theorem 4.3.** Suppose that $P \in \text{Diff}^2(X)$ with $\partial X$ non-characteristic for $P$. Suppose $f \in \mathcal{N}(X)$ and

$$
Pu = f \in X^o, \quad u|_{\partial X} = u_0
$$

Suppose that $\gamma : I \to \tilde{\Sigma}$ is a broken bicharacteristic such that $\gamma(I) \cap (\text{WF}_b(f) \cup \text{WF}(u_0)) = \emptyset$, then for any $t_0 \in I$ if $\gamma(t_0) \notin \text{WF}_b(u)$ then

$$
\gamma(I) \notin \text{WF}_b(u).
$$

Notice that for $\gamma(I) \cap T^*\partial X = \emptyset$, Theorem 4.3 is an easy consequence of Theorem 2.1.

Therefore, in the proof we may work locally near a point $t \in I$ such that $\gamma(t) \in \mathcal{H}$. Without loss of generality, we therefore assume $\rho_0 = \gamma(0) \in \mathcal{H}$. Then, since $\rho \in \mathcal{H}$, freezing coefficients as in [14], we may replace $P$ by $\tilde{P}$ such that $T^*\partial X \subset \mathcal{H}$ so that $\tilde{P}u$ has the same wavefront set properties as $Pu$ near $\rho_0$.

We continue to call the operator $P$ for brevity and observe that

$$
p(x, \xi) = \xi_1^2 - r(x, \xi')
$$

with $r(x, \xi') < c|\xi'|^2$. In particular, $P$ is strictly hyperbolic with respect to $x_1$ and we may apply Lemma [1.2]. However, it will be necessary to upgrade this factorization.

**Lemma 4.7.** There exist $\Lambda_{\pm}(x, D')$, $\tilde{\Lambda}_{\pm}(x, D') \in S^1$ with symbols $\sigma(\Lambda_{\pm}) = \sigma(\tilde{\Lambda}_{\pm}) = \pm \sqrt{r}$ so that

$$
P = (D_{x_1} - \Lambda_+)(D_{x_1} - \Lambda_-) + E(x, D') = (D_{x_1} - \tilde{\Lambda}_-)(D_{x_1} - \tilde{\Lambda}_+) + \tilde{E}(x, D')
$$

with $E, \tilde{E} \in S^{-\infty}$.

**Proof.** We have from Lemma [1.2] that there exist $\Lambda_{0,-}(x, D') \in S^1$, $\Lambda_{0,+}(x, D') \in S^1$ and $E_0(x, D') \in S^1$ such that

$$
P = (D_{x_1} - \Lambda_{0,+})(D_{x_1} - \Lambda_{0,-}) + E_0(x, D')
$$

with $\sigma(\Lambda_{0,\pm}) = \pm \sqrt{r(x, \xi')}$. Suppose that there exist $\Lambda_{j,-}(x, D') \in S^1$, $\Lambda_{j,+}(x, D') \in S^1$ and $E_j(x, D') \in S^{1-j}$ such that

$$
P = (D_{x_1} - \Lambda_{j,+})(D_{x_1} - \Lambda_{j,-}) + E_j(x, D')$$
with $\sigma(\Lambda_{j,\pm}) = \pm \sqrt{r(x, \xi')}$. We will show that the error can be improved to $E_{j+1} \in S^{-j}$. For this, let

$$\lambda_{j+1}(x, \xi') = -\frac{\sigma(E_j)(x, \xi')}{2\sqrt{r(x, \xi')}} \in S^{-j}.$$

Then

$$\lambda_{j+1}(x, D')(D_{x_1} - \Lambda_{j,-}) + E_j = \lambda_{j+1}(x, D')(\Lambda_{j,+} - \Lambda_{j,-}) + E_j + \lambda_{j+1}(x, D')(D_{x_1} - \Lambda_{j,+})$$

$$= \lambda_{j+1}(x, D')(\Lambda_{j,+} - \Lambda_{j,-}) + E_j$$

$$+ (D_{x_1} - \Lambda_{j,+})\lambda_{j+1}(x, D') + [\lambda_{j+1}(x, D'), D_{x_1} - \Lambda_{j,+}]$$

$$= E_{j+1,1}(x, D')$$

$$+ (D_{x_1} - \Lambda_{j,+} - \lambda_{j+1}(x, D'))\lambda_{j+1}(x, D') + E_{j+1,2}(x, D')$$

where $E_{j+1,1}, E_{j+1,2} \in S^{-j}$.

In particular,

$$P = (D_{x_1} - \Lambda_{j,+})(D_{x_1} - \Lambda_{j,-}) + E_j(x, D')$$

$$= (D_{x_1} - \Lambda_{j,+})(D_{x_1} - \Lambda_{j,-}) - \lambda_{j+1}(x, D')(D_{x_1} - \Lambda_{j,-})$$

$$+ (D_{x_1} - \Lambda_{j,+} - \lambda_{j+1}(x, D'))\lambda_{j+1}(x, D') + E_{j+1,2}(x, D') - E_{j+1,1}(x, D')$$

$$= (D_{x_1} - \Lambda_{j,+} - \lambda_{j+1}(x, D'))(D_{x_1} - \Lambda_{j,-} + \lambda_{j+1}(x, D')) + E_j(x, D')$$

with $E_j \in S^{-j}$. Let $\Lambda_{j+1,\pm} = \Lambda_{j,\pm} + \lambda_{j+1}(x, D')$. Then defining $\Lambda_{\pm} \sim \Lambda_{0,\pm} \pm \sum_{j \geq 1} \lambda_j$ gives the desired factorization.

Repeating the arguments starting with the $\tilde{\Lambda}$ factorization completes the proof. \qed

Next, we construct an operator $Q(x, D')$ with desirable microlocalization properties such that $[D_{x_1} - \Lambda_+, Q] \in S^{-\infty}$. This will allow us to complete the proof of Theorem 4.3 by effectively microlocalizing $Pu = f$ to a broken bicharacteristic.

**Lemma 4.8.** Fix $\rho_0 \in \mathcal{H}$, $\gamma : I \to \tilde{\Sigma}$ a broken bicharacteristic with $\gamma(0) = \rho_0$ and let $U$ be a conic neighborhood of $\gamma([0, \infty) \cap I)$. Then there exist $\epsilon > 0$ and a conic neighborhood $V \subset T^*\partial X$ so that for any $\tilde{q}_+ \in S^0(T^*\partial X)$ supported in $V$, there exists $Q_+(x, D') \in S^0$ with symbol $q_+$ such that $q_+(0, \rho) = \tilde{q}_+$,

$$WF(Q_+) \cap \{ x_1 \leq \epsilon \} \subset \{(x, \xi') | (x, \sqrt{r(x, \xi')}, \xi') \in U \}$$

and

$$[D_{x_1} - \Lambda_+, Q_+] \in S^{-\infty}.$$

Similarly, let $U$ be a conic neighborhood of $\gamma((-\infty, 0) \cap I)$. Then there exist $\epsilon > 0$, $V \subset T^*\partial X$ a conic neighborhood of $\rho_0$ so that for $\tilde{q}_- \in S^0(T^*\partial X)$ supported in $V$ there exists $Q_-$ with $\sigma(Q_-)(0, \rho) = \tilde{q}_-$,

$$WF(Q_-) \cap \{ x_1 \leq \epsilon \} \subset \{(x, \xi') | (x, -\sqrt{r(x, \xi')}, \xi') \in U \}$$

and

$$[D_{x_1} - \Lambda_-, Q_-] \in S^{-\infty}.$$
4. PROPAGATION OF SINGULARITIES

PROOF. Fix \( \tilde{q}_0 \in S^0(T^*\partial X) \) with \( \text{supp} \, q_0 \subset U \cap T^*\partial X \) and \( q_0(\rho_0) = 1 \). Then, since \( \partial x_1 \) is transverse to \( \{x_1 = 0\} \), there exists \( q_0 \in C^\infty([0, \epsilon) \times T^*\partial X) \) solving

\[
(\partial x_1 - H_{\sqrt{\tau}})q_0 = 0, \quad q_0|_{x_1 = 0} = \tilde{q}_0.
\]

Moreover, by homogeneity, \( q_0 \in C^\infty([0, \epsilon); S^0(T^*\partial X)) \). Let \( Q_0 = q_0(x, D') \). Then,

\[
[D_{x_1} - \Lambda_+, Q_0] = E_0(x, D') \in S^{-1}.
\]

Now, since on \( \gamma((0, \infty) \cap I) \), \( H_p = g(\partial x_1 - H_{\sqrt{\tau}}) \), choosing \( \text{supp} \, \tilde{q}_0 \) small enough, we may assume \( \text{supp} \, q_0 \cap \{x_1 \leq \epsilon\} \) is contained in the RHS of (15) and hence \( \text{WF}(E_0) \cap \{x_1 \leq \epsilon\} \) is contained in the RHS of (15). Assume now that we have \( Q_j \) with \( \sigma(q_j)(\rho_0) = 1 \), \( \text{WF}(Q_j) \subset U \) and

\[
[D_{x_1} - \Lambda_+, Q_j] = E_j(x, D') \in S^{-1-j}.
\]

Let \( q_j \in S^{-j} \) solve

\[
(\partial x_1 - H_{\sqrt{\tau}})q_j = -\sigma(E_j), \quad q_j|_{x_1 = 0} = 0.
\]

Then, \( \text{supp} \, q_j \cap \{x_1 \leq \epsilon\} \) is contained in the RHS of (15). Putting \( Q_{j+1} = Q_j + q_j(x, D') \). Then we have \( \text{supp} \, q_j \cap \{x_1 \leq \epsilon\} \) is contained in the RHS of (15). Putting \( Q_j \sim \sum_j q_j \) completes the proof of (16).

Using \( \partial x_1 + H_{\sqrt{\tau}} \) rather than \( \partial x_1 - H_{\sqrt{\tau}} \) completes the proof of (18). \( \square \)

PROOF OF THEOREM 4.3. Let \( \gamma \) be a broken bicharacteristic with \( \rho_0 = \gamma(0) \in \mathcal{H} \). Since \( f \in \mathcal{N}(X) \), shrinking \( \epsilon > 0 \) if necessary, we may assume that for \( 0 < x_1 \leq \epsilon \), \( (x, \xi) \in \text{WF}(f) \cup \text{WF}(u) \) implies \( \xi^i \neq 0 \). Shrinking \( \epsilon > 0 \) if necessary, let \( Q_+ \in C^\infty([0, \epsilon); \Psi^0(\partial X) \) be as in Lemma 4.8.

Then,

\[
(D_{x_1} - \Lambda_+)Q_+(D_{x_1} - \Lambda_-) = Q_+P + R(x, D')
\]

where \( R \in C^\infty([0, \epsilon]; \Psi^{-\infty}) \). In particular, since for \( 0 < x_1 \leq \epsilon \), \( (x, \xi) \in \text{WF}(f) \cup \text{WF}(u) \) implies \( \xi^i \neq 0 \), we have in particular,

\[
(D_{x_1} - \Lambda_+)Q_+(D_{x_1} - \Lambda_-)u \in C^\infty(x_1 \leq \epsilon).
\]

In particular, for \( x_1 > 0 \), \( \text{WF}(Q_+(D_{x_1} - \Lambda_-)u) \) is arbitrarily close to \( \xi_1 = r(x, \xi^i)^{1/2} \) and contained in \( U \). Therefore, for \( B_+ \) so that \( \gamma((0, \epsilon) \cap I) \subset \ell(B_+) \),

\[
\|Q_+(D_{x_1} - \Lambda_-)u\|_{H^{0,s}(\epsilon/2 \leq x_1 \leq \epsilon)} \leq C\|B_+u\|_{H^{s+1}}.
\]

Now, by Lemma 4.1. [TODO] be more precise here

\[
\|Q_+(D_{x_1} - \Lambda_-)u(x_1, \cdot)\|_{H^s} \leq C(\|Q_+(0, x', D')(D_{x_1} - \Lambda_-)u\|_{H^{s,s}((\xi_1 \leq \epsilon)} + \|Q_+Pu\|_{H^{s,s}} + \|R(x, D')u\|_{H^{0,s}})
\]

\[
\leq C(\|B_+u\|_{H^{s+1}} + \|Q_+Pu\|_{H^s}).
\]

Also, using (21) repeatedly,

\[
\|Q_+(D_{x_1} - \Lambda_-)u\|_{H^{s,0}} \leq C(\|B_+u\|_{H^{s+1}} + \|Q_+Pu\|_{H^s}).
\]

Now, let \( Q_- \) as in (17) with \( Q_-(0, x', D') = Q_+(0, x', D') \). Then,

\[
\|Q_-(D_{x_1} - \Lambda_+) - Q_+(D_{x_1} - \Lambda_-)u\|_{x_1 = 0} = Q_-(0, x', D')(\Lambda_ - \Lambda_+)u_0.
\]
Observe that
\[(D_{x_1} - \tilde{\Lambda}_-)Q_- (D_{x_1} - \tilde{\Lambda}_+) = Q_- P + \tilde{R}\]
with \(\tilde{R} \in C^\infty([0, \varepsilon]; \Psi^{-\infty})\). Therefore, letting \(A \in \Psi^0(\partial X)\) with \(WF(Q_- (0, x', D')) \subseteq \text{ell}(A)\),
\[
\|Q_- (D_{x_1} - \tilde{\Lambda}_+) u(x_1, \cdot)\|_{H^s} \\
\quad \leq C(\|Q_- (0, x', D') (D_{x_1} - \tilde{\Lambda}_+) u(x_1, \cdot)\|_{H^s} + \|Q_- Pu\|_{H^{0,s}} + \|\tilde{R}(x, D') u\|_{H^{0,s}} \\
\quad \leq C(\|Au_0\|_{H^{s+1}} + \|B_+ u\|_{H^{s+1}} + \|Q_+ Pu\|_{H^s} + \|u\|_{H^{-N}}).
\]
Using (22) repeatedly then gives
\[
\|Q_- (D_{x_1} - \tilde{\Lambda}_+) u\|_{H^{s,0}} \leq C(\|Au_0\|_{H^{s+1}} + \|B_+ u\|_{H^{s+1}} + \|Q_+ Pu\|_{H^s} + \|u\|_{H^{-N}}).
\]

Now, let \(\tilde{Q}_0(x, D') \in C^\infty([0, \varepsilon]; \Psi^0)\) have \(WF(Q_0) \subseteq \text{ell}(Q_-) \cap \text{ell}(Q_+)\). Then, by the elliptic parametrix construction
\[
\tilde{Q}_0(x, D') (D_{x_1} - \tilde{\Lambda}_+) = E_- Q_- (D_{x_1} - \tilde{\Lambda}_+) + R_1 \\
\tilde{Q}_0(x, D') (D_{x_1} - \Lambda_-) = E_+ Q_+ (D_{x_1} - \Lambda_-) + R_2
\]
with \(E_\pm \in C^\infty([0, \varepsilon]; \Psi^0)\) and \(R_i \in C^\infty([0, \varepsilon]; \Psi^{-\infty})\). Hence,
\[
\tilde{Q}_0(x, D') (\Lambda_- - \tilde{\Lambda}_+) = E_- Q_- (D_{x_1} - \tilde{\Lambda}_+) - E_+ Q_+ (D_{x_1} - \Lambda_-) + R_3
\]
Finally, since \(\tilde{\Lambda}_+ - \Lambda_- \in C^\infty([0, \varepsilon]; \Psi^1)\) is elliptic,
\[
\|\tilde{Q}_0(x, D') u\|_{H^{s+1}} \leq C(\|Au_0\|_{H^{s+1}} + \|B_+ u\|_{H^{s+1}} + \|Q_+ Pu\|_{H^s} + \|u\|_{H^{-N}})
\]
Now, for \(\tilde{Q}_0\) with \(WF(Q_0) \subseteq \text{ell}(\tilde{Q}_0)\),
\[
PQ_0(x, D') u = Q_0 Pu + [Q_0, P] u
\]
So,
\[
\|PQ_0 u\|_{H^{s-1,0}} \leq C\|Q_+ Pu\|_{H^s} + \|\tilde{Q}_0 u\|_{H^{s,0}}
\]
Therefore,
\[
\|D_{x_1}^2 Q_0 u\|_{H^{s-1,0}} \leq C\|Q_+ Pu\|_{H^s} + \|\tilde{Q}_0 u\|_{H^{s,1}}
\]
and hence
\[
\|D_{x_1} Q_0 u\|_{H^{s,0}} \leq C\|Q_+ Pu\|_{H^s} + \|\tilde{Q}_0 u\|_{H^{s,1}}.
\]
Finally, this implies
\[
\|Q_0 u\|_{H^{s+1,0}} \leq C\|Q_+ Pu\|_{H^s} + \|\tilde{Q}_0 u\|_{H^{s,1}}.
\]
Therefore,
\[
\|Q_0 u\|_{H^{s+1}} \leq C(\|Au_0\|_{H^{s+1}} + \|B_+ u\|_{H^{s+1}} + \|Q_+ Pu\|_{H^s} + \|u\|_{H^{-N}}).
\]
In particular, since \(Q_\pm\) are elliptic at \(\rho_0\), for any \(C(x, D')\) with \(WF(C)\) sufficiently close to \(\rho_0\),
\[
\|C u\|_{H^{s+1}} \leq C(\|Au_0\|_{H^{s+1}} + \|B_+ u\|_{H^{s+1}} + \|Q_+ Pu\|_{H^s} + \|u\|_{H^{-N}}).
\]
Now, suppose that \(\gamma(I) \cap (WF_B(f) \cup WF(u_0)) = \emptyset\) and \(\gamma(\varepsilon) \notin WF_B(u)\). Let \(U\) a conic neighborhood of \(\gamma([0, \varepsilon])\) so that \(U \cap (WF_B(f) \cup WF(u_0)) = \emptyset\). Choose \(\epsilon > 0\) small enough so that \(\gamma((0, \epsilon]) \cap WF(f) = \emptyset\). Then, there exists \(A\) elliptic at \(\rho_0\) so that \(Au_0 \in C^\infty\) and \(B_+(x, D')\) elliptic on \(\gamma(\varepsilon)\) so that \(B_+ u \in C^\infty\). Moreover, for \(Q_+\) supported in \(U\), \(Q_+ Pu \in C^\infty\). In particular, (23)
implies $\rho_0 \notin WF_b(u)$. Switching the roles of $Q_{\pm}$, we also obtain that if $\gamma(-\epsilon) \notin WF_b(u)$, then $\rho_0 \notin WF_b(u)$ which completes the proof.

\[\square\]

[TODO] state estimates

5. The generalized bicharacteristic flow

6. The Weyl law on a manifold with boundary

7. Microlocal defect measures
CHAPTER 5

Equivalence of Glancing Hypersurfaces

1. Symplectic preliminaries

2. Folding relations

3. The billiard ball maps

4. Formal solution

5. Completion of the proof

6. Some consequences
CHAPTER 6

The Melrose–Taylor Parametrix

1. The Friedlander Model and the Ansatz

1.1. The Friedlander Model. As a first step, we consider the Friedlander model. This toy example guides us when we consider the general case. The Friedlander model is given by

\[ P = (hD_{y_1})^2 - y_1 + hD_{y_2} \partial X = \{ y_1 = 0 \}. \]

Suppose that

\[ Pu = 0 \quad u|_{\partial X} = f \]

Then, taking the semiclassical Fourier transform in the \( x' \) variables gives

\[ (-h^2 \partial^2_{y_1} - y_1 + \eta_2) \mathcal{F}_{h,y'} u(y_1, \eta') = 0 \quad \mathcal{F}_{h,y'} u(0, \eta') = \mathcal{F}_h(f)(\eta'). \]

The solution to this problem for \( \mu = 0 \) is

\[ u = (2\pi h)^{-d+1} \int \frac{A(h^{-2/3}(-y_1 + \eta_2))}{A(h^{-2/3}\eta_2)} e^{i\theta(y', \eta')} \mathcal{F}_h(f)(\eta') d\eta' \]

where \( A \) is a solution to the Airy equation. Let \( \zeta_0 := -y_1 + \eta_2 \) and \( \theta_0 = \langle y', \eta' \rangle \).

Our goal will be to model the parametrices on the Friedlander model and therefore to seek solutions \( u \) to

\[ P(x, hD)u = 0 \]

of the form

\[ u = (2\pi h)^{-d+1} \int \left[ g_0 A(h^{-2/3}\zeta) + i h^{1/3} g_1 A'(h^{-2/3}\zeta) \right] e^{i\theta/h} \mathcal{F}_h(f)(\eta') d\eta' \]

where \( f \) is a function on \( \partial X \). We will then correct the boundary values by applying an FIO in the boundary variables which replaces the Fourier transform in the Friedlander model.

Remark 2. Note that it is necessary to add the \( A' \) term since it is not possible to cancel \( A' \) with an amplitude times \( A \). However, since \( A \) solves a second order ODE, it is possible to cancel \( A'' \) with an amplitude times \( A \).

1.2. Eikonal and Transport Equations. First, we consider a general differential operator

\[ P(x, hD) = \sum a_{jk}(x)hD_jhD_k + \sum b_j(x)hD_j + c(x) \]

with \( a_{jk} = a_{kj} \) applied to \( (25) \).
For $A$ an Airy function, we have, letting $f_j$ denote $\partial_j f$, and $\zeta_h = h^{-2/3}\zeta$

$$h D_j \left( g A(\zeta_h)e^{\frac{i}{h}\theta} \right) = \theta_j gA(\zeta_h)e^{\frac{i}{h}\theta} - ihg_jA(\zeta_h)e^{\frac{i}{h}\theta} - ih^{1/3}\zeta_jgA'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$h D_k h D_j \left( g A(\zeta_h)e^{\frac{i}{h}\theta} \right) =$$

$$\left[ (\theta_k \theta_j - \zeta_j \zeta_k) g - ih(\theta_k g_j + \theta_j g_k + \theta_j k g) - h^2 g_{jk} \right] A(\zeta_h)e^{\frac{i}{h}\theta}$$

$$- ih^{1/3} [ (\theta_j \zeta_k + \zeta_j \theta_k) g - ih(\theta_j g_k + \zeta_j g_k + \zeta_j k g) ] A'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$h D_j \left( g A'(\zeta_h)e^{\frac{i}{h}\theta} \right) =$$

$$\theta_j g A'(\zeta_h)e^{\frac{i}{h}\theta} - ihg_jA'(\zeta_h)e^{\frac{i}{h}\theta} - ih^{1/3}\zeta_jgA(\zeta_h)e^{\frac{i}{h}\theta}$$

$$h D_k h D_j \left( g A'(\zeta_h)e^{\frac{i}{h}\theta} \right) = -ih^{1/3} [ (\theta_j \zeta_k + \theta_k \zeta_j) g + ih(g_j \zeta_k \zeta + g_k \zeta \zeta + \zeta_j k g + \zeta_j k g) ] A(\zeta_h)e^{\frac{i}{h}\theta}$$

$$+ [ (\theta_j \theta_k - \zeta_j \zeta_k) g - ih(\theta_j g_k + \theta_k g_k + \theta_k j g) - h^2 j k g ] A'(\zeta_h)e^{\frac{i}{h}\theta}$$

So,

$$P(g_0 A(\zeta_h)e^{\frac{i}{h}\theta})$$

$$= \left[ \left( \langle ad\theta, d\theta \rangle - \zeta \langle ad\zeta, d\zeta \rangle + \langle b, d\theta \rangle + c \rangle g_0 \right) - ih(2 \langle ad\theta, dg_0 \rangle - P_2 g_0 + \langle b, dg_0 \rangle + h^2 P_{2g_0} \right] A(\zeta_h)e^{\frac{i}{h}\theta}$$

$$\left[ -ih^{1/3} \left( \langle 2 \langle ad\theta, d\zeta \rangle + \langle b, d\zeta \rangle \rangle g_0 \right) \right] A'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$P(ih^{1/3} g_1 A'(\zeta_h)e^{\frac{i}{h}\theta}) =$$

$$\left[ \zeta(2 \langle ad\theta, d\zeta \rangle + \langle b, d\zeta \rangle) g_1 \right] A(\zeta_h)e^{\frac{i}{h}\theta}$$

$$\left[ -ih(2 \zeta \langle ad\zeta, dg_1 \rangle + \langle ad\zeta, dg_1 \rangle - \zeta(2 P_2 g_1) \rangle g_1 \right] A'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$+ ih^{1/3} \left( \langle ad\theta, d\theta \rangle - \zeta \langle ad\zeta, d\zeta \rangle + \langle b, d\theta \rangle + c \rangle g_1 \right) A'(\zeta_h)e^{\frac{i}{h}\theta}$$

$$\left[ -ih(2 \langle ad\theta, dg_1 \rangle - (P_2 d\theta) g_1 + \langle b, dg_1 \rangle + h^2 P_{2g_1} \rangle + h^2 P_{2g_1} \right] A'(\zeta_h)e^{\frac{i}{h}\theta}$$

where $a_{jk} = a_{jk}(x), \ P_2 = h^{-2}(P - \langle b, hD \rangle - c(x))$ and $\langle \cdot, \cdot \rangle$ denotes the euclidean inner product.

Now, applying $P$ under the integral in \[\text{(25)}\] gives the eikonal equations

$$\left\{ \begin{array}{l} \langle ad\theta, d\theta \rangle - \zeta \langle ad\zeta, d\zeta \rangle + \langle b, d\theta \rangle + c = 0 \\ 2 \langle ad\theta, d\zeta \rangle + \langle b, d\zeta \rangle = 0 \end{array} \right.$$

Writing

$$\phi^\pm = \theta \pm \frac{2}{3}(-\zeta)^{3/2},$$

the eikonal equations are equivalent to the standard equation

$$p(x, d\phi^\pm) = 0.$$
Then there exists a neighborhood \((31)\) so that
\[
\rho \sim \sum_n g_i^{[n]}(x, \eta') h^n.
\]

Then the transport equations have the form
\[
\begin{align*}
2\langle ad\theta_0, dg_0^{[n]} \rangle + 2\zeta_0 \langle ad\zeta_0, dg_1^{[n]} \rangle + \langle b, dg_0^{[n]} \rangle + \langle ad\zeta_0, d\zeta_0 \rangle g_1^{[n]} - P_2\theta_0 g_0^{[n]} - \zeta_0(P_2\zeta_0)g_1^{[n]} &= F_1^{[n]}(\theta, \zeta, g_i^{[m]<[n]}, \mu) \\
2\langle ad\zeta_0, dg_0^{[n]} \rangle - 2\langle ad\theta_0, dg_1^{[n]} \rangle - \langle b, dg_1^{[n]} \rangle - (P_2\zeta_0)g_0^{[n]} + (P_2\theta_0)g_1^{[n]} &= F_2^{[m]}(\theta, \zeta, g_i^{[m]<[n]}, \mu).
\end{align*}
\]

(28)

More generally, we consider transport equations of the form
\[
\begin{align*}
2\langle ad\theta_0, dg_0 \rangle + 2\zeta_0 \langle ad\zeta_0, dg_1 \rangle + \langle b, dg_0 \rangle + \langle ad\zeta_0, d\zeta_0 \rangle g_1 + B_1g_0 + \zeta_0 B_2g_1 &= F_1 \\
2\langle ad\zeta_0, dg_0 \rangle - 2\langle ad\theta_0, dg_1 \rangle - \langle b, dg_1 \rangle + B_2g_0 - B_1g_1 &= F_2
\end{align*}
\]

Then, these equations are equivalent to
\[
2\langle ad\phi^\pm, dg^\pm \rangle + \langle b, dg^\pm \rangle + G^\pm g^\pm = F^\pm
\]

where
\[
g^\pm = g_0 \pm (-\zeta_0)^{1/2} g_1 \quad G^\pm = B_1 \mp (-\zeta_0)^{1/2} B_2 \quad F^\pm = F_1 \mp (-\zeta_0)^{1/2} F_2.
\]

2. Geometric preliminaries

We will start from the equivalence of glancing hypersurfaces [TODO]prove. In particular, suppose \(S\) is a symplectic manifold of dimension \(2d\) with \(P = \{p = 0\}, Q = \{q = 0\} \subset S\) hypersurfaces (i.e. embedded submanifolds of codimension 1). Suppose that there is \(\rho \in P \cap Q\) so that
\[
dp, dq, \text{ are linearly independent at } \rho,
\]

(31)

\[
\{p, \{p, q\}\}(\rho) = 0, \quad \{p, \{q, p\}\}(\rho) \neq 0.
\]

Then there exists a neighborhood \(U\) of \(\rho\) and a symplectomorphism \(\kappa: U \to T^*\mathbb{R}^d\) with
\[
\kappa(U \cap P) \subset \{\eta^2_1 - y_1 + \eta_2 = 0\}, \quad \kappa(U \cap Q) \subset \{y_1 = 0\}, \quad \kappa(\rho) = (0, 0).
\]

Now, we can quotient \(P\) and \(Q\) by their respective Hamiltonian fibrations. That is, we write for \(\rho_0, \rho_1 \in Q\),
\[
\rho_0 \sim \rho_1 \text{ if there exists } t \in \mathbb{R} \text{ with } \exp(tH_q)(\rho_0) = \rho_1
\]

to define the space \(Q/\mathbb{R}H_q\). We then define a symplectic form on \(Q/\mathbb{R}H_q\) by
\[
\sigma_\theta = \pi^Q_* \sigma|_Q
\]
where $\pi^Q : Q \rightarrow Q/\mathbb{R}H_q$ is the natural projection map. An application of Darboux’s theorem shows that $\sigma_0$ is closed and non-degenerate.

**Example 2.1.** If $S = T^*\mathbb{R}^d$ and $Q = \{x_1 = 0\}$, then $H_q = -\partial_{x_1}$ and therefore, $(0, x'_0, \xi_0) \sim (0, x'_1, \xi_1)$ if $x'_0 = x'_1$ and $\xi_0 = \xi_1$. Hence, in this case $Q/\mathbb{R}H_q$ is canonically isomorphic to $T^*\{x_1 = 0\}$.

In the relevant case, $S = T^*x$ for some manifold $x$, and $Q = \{(x, \xi) \mid x \in \partial X\}$. We write $\pi_X\rho = x_0$ when it is necessary to refer to the base variable.

Now, since $\kappa$ is symplectic and sends $Q$ to $\{x_1 = 0\}$, it sends the flow lines of $H_q$ in $Q$ to those of $H_{x_1}$ in $\{x_1 = 0\}$. In particular, $\kappa$ induces a map $\kappa_0 : Q/\mathbb{R}H_q \rightarrow T^*\{x_1 = 0\}$. Moreover, $\kappa_0$ is a symplectomorphism.

We will actually assume that

\[(32) \quad H_p(\rho) \text{ is not tangent to } T^*_{x_0}X.\]

This is the case for the wave equation.

We do this so that

\[(33) \quad (\kappa_0^{-1})^*(d\eta_i), \quad i = 2, \ldots d \text{ are linearly independent on } T^*_{x_0}\partial X \text{ at } \rho.\]

To see this, observe that by (32) $d\pi^QH_p$ is not tangent to $T^*_{x_0}\partial X$. Now, let $L := \kappa_0(T^*_{x_0}\partial X)$. Then, since

$$\kappa_0H_p(\rho) = g(\rho)(-\partial_{y_1} + \partial_{y_2})$$

for some nonvanishing $g$, $(\kappa_0)_*d\pi^QH_p = g(\rho)\partial_{y_2}$ and we have that $\partial_{y_2}$ is not tangent to $L$ at $0$. In particular, since $L$ is Lagrangian (as the image of a Lagrangian under a symplectomorphism), this implies $dp|_{L}(0) \neq 0$. Therefore, making a symplectic change of variables on $T^*\{x_1 = 0\}$ fixing $(y_2, \eta_2)$, it is possible to arrange that (33) holds. We then extend this change of variables (independently of $y_1, \eta_1$) to $T^*\mathbb{R}^d$ leaving the normal form completely unchanged.

Now, define the map

$$Y_0 : P \ni \rho \mapsto (\eta_2(\kappa(\rho)), \ldots \eta_d(\kappa(\rho))) \in \mathbb{R}^{d-1}$$

and

$$Y : P \ni \rho \mapsto (\pi_X(\rho), Y_0(\rho)) \in X \times \mathbb{R}^{d-1}.$$

**2.1. Folds.** Let $M$ and $N$ be smooth manifolds. We say that $f : M \rightarrow N \in C^\infty$ is a *fold at* $m$ if $\dim \ker df(m) = \dim \text{Coker } df(m) = 1$ and the Hessian of $f$ at $m_0$ is not equal to 0. The fold set of $f$ is defined as the set $\mathcal{F} := \{m \in M \mid f \text{ is a fold at } m\}$.

As an aside we prove the following

**Lemma 2.1.** Let $f : M \rightarrow N$ be a fold at $m$. Then there exists coordinates $t$ near $m$ on $M$ and $s$ near $f(m)$ on $N$ so that

$$f(t_1, \ldots t_n) = (t_1, \ldots t^2_n).$$
Proof. Choose coordinates $y$ on $N$ so that with $f = (f_1, \ldots, f_n)$ $df_n(m) = 0$. This is possible since $\dim \operatorname{Coker} df > 0$. Moreover, since $\dim \operatorname{Coker} df = 1$, $df_i = 1, \ldots, n - 1$ are linearly independent and we may use $x_i = f_i$ $i = 1, \ldots, n - 1$ as coordinates on $M$ so that

$$f(x) = (x_1, \ldots, x_{n-1}, f_n(x)).$$

and $x(m) = 0$. Then, $df_n(0) = 0$ and since the Hessian of $f$ is non-zero, $\partial^2_{x_n} f_n(0) \neq 0$. So, by the implicit function theorem, $\partial x_n f_n(y) = 0$ has a unique solution $x_n = g(x')$ where $x' = (x_1, \ldots, x_{n-1})$ and $g(0) = 0$. Setting $x_n = x_n - g(x')$, we have $\partial x_n f_n = 0$ when $x_n = 0$ so

$$f_n(x) = f_n(x', 0) + x_n^2 F(x)$$

where $F \in C^\infty$ with $F(0) \neq 0$. Now, replacing $y_n$ by $y_n - f_n(y', 0)$, we have

$$f(x) = (x_1, \ldots, x_{n-1}, x_n^2 F(x)).$$

Finally, switching the sign of $y_n$ if necessary, we may assume $F(0) > 0$ and replace $x_n$ by $x_n F^{1/2}(x)$ to obtain

$$f(x) = (x_1, \ldots, x_{n-1}, x_n).$$

As a consequence, we obtain

Lemma 2.2. Suppose $f : M \to N$ has a fold at $m$. Then there is a neighborhood $V$ of $m$ and a $C^\infty$ map $\iota : V \to V$ so that $\iota^2 = \operatorname{Id}$, $\iota \neq \operatorname{Id}$ and $f \circ \iota = f$. That is, $\iota$ is an involution preserving $f$.

Proof. Take $\iota(y_1, \ldots, y_n) = (y_1, \ldots, -y_n)$ with the coordinates from Lemma 2.1.

2.2. The structure of $Y$.

Lemma 2.3. The map $Y$ is a fold at $\rho$ with fold set meeting $Q$ transversally at $\eta_2 = 0$.

Proof. Consider $\rho \in \{p = 0\} \cap \{H_0 = 0\}$. Now, since $H_0 H_q p \neq 0$, $\{H_0 = 0\}$ is a smooth hypersurface transverse to $H_q$ and we may take coordinates $(x_1, x')$ so that $H_q p = x_1$ and $H_q = \partial_{x_1}$ and $0 \to \rho$. Therefore, $p = \frac{1}{2} x_1^2 + g(x')$. Now, $dp(0) \neq 0$. Therefore, $dg(0) \neq 0$ and we may change coordinates so that $x_2 = 2g$ so that

$$p = \frac{1}{2} (x_1^2 + x_2), \quad H_q = \partial_{x_1}.$$  

So, on $P \cap Q$,

$$\ker d\pi^Q(\rho) = \partial_{x_1}, \quad \operatorname{Coker} d\pi^Q(\rho) = \partial_{x_2}.$$  

Moreover, putting $\phi(s) = (s, -s^2, 0)$, $\phi' = H_q \in \ker d\pi^Q$ and

$$\pi_{\operatorname{Coker} d\pi^Q}(\pi^Q \circ \phi(s))'' = -2 \neq 0.$$  

Therefore, $\pi^Q|_{P \cap Q}$ has a fold at $\rho$.

Now, $Y|_{P \cap Q}$ is $\pi^Q$ followed by replacement of the fiber variables by $\xi_i$. This is well defined since $\kappa$ sends the flow lines of $H_q$ in $Q$ to those of $H_{x_1}$ in $\{x_1 = 0\}$. That is, $(\xi_2, \ldots, \xi_d)$ depend only on $\pi^Q(\rho)$. By the transversality of $\pi^Q$, this replacement is a diffeomorphism and hence preserves the fold. Now, to see that $Y$ itself has a fold, observe that $dq$ and $dp$ are independent at $\rho$ and
hence \( dq|_P(\rho) \neq 0 \). Therefore, \( q = x_1 \) can be used as a coordinate on \( P \). Hence, since \( dY\partial_{x_1} \) is independent of \( dY|_{P \cap Q} \), \( Y \) has a fold at \( \rho \). This also implies that the fold set intersects \( Q \) transversally. To see that this happens at \( \eta_2 = 0 \), simply observe that \( Q \cap P \cap \{H_q p = 0\} \subset Y^{-1}(\{\eta_2 = 0\}) \).

3. Solution of Eikonal Equations

3.1. The transverse case. We first suppose that \( H_p \) is transverse to \( Q = \{x_1 = 0\} \) at \( \rho_0 \in Q \cap P \) and solve the standard eikonal equation

\[
p(x, d\phi(x, \eta)) = 0, \quad \phi(0, x', \eta) = \langle x', \eta \rangle.
\]

We choose coordinates so that \( \rho_0 = (0, 0) \).

In this case, (since \( H_q p(\rho_0) \neq 0 \)) the leaves \( \Lambda^\eta_0 = \{\eta' = \eta_0'\} \) are transverse to \( P \) in \( P \setminus Q \) and hence there is a local diffeomorphism \( T^*\{x_1 = 0\} \to P \cap Q \) near \( \rho_0 \). In particular, \( \pi^Q|_{P \cap Q} \) is a diffeomorphism near \( \rho_0 \). Moreover, since \( H_p q(\rho_0) \neq 0 \), the \( H_p \) flow-out of \( \Lambda^\eta_0 \cap P \), denoted \( \Lambda^p_\eta \) gives a foliation of \( P \) by Lagrangian leaves.

To see that these leaves are indeed Lagrangian, observe that

\[
\sigma(H_p, \cdot) = dp(\cdot).
\]

So, if \( V \) is tangent to \( P \cap Q \), it is in particular tangent to \( \{p = 0\} \) and hence

\[
\sigma(H_p, V) = 0
\]

which together with \( \sigma|_{\Lambda^\eta_0} = 0 \) implies \( \Lambda^p_\eta \) is Lagrangian.

Next, note that \( \Lambda^p_\eta \) project diffeomorphically to \( X \) and hence we may use the base as coordinates on \( \Lambda^p_\eta \). Now, let \( \alpha = \xi dx \) denote the canonical one form (so that \( d\alpha = \sigma \)). Then, \( \alpha|_{\Lambda^\eta_0} \) is closed and hence there exists \( \phi(x, \eta) \in C^\infty \) fixed by

\[
\begin{cases}
  d\phi(\cdot, \eta) = \alpha & \text{on } \Lambda^p_\eta \\
  \phi(0, \eta) = 0 & \text{on } T^*_\pi(\rho_0)\{x_1 = 0\}.
\end{cases}
\]

The second condition fixes a normalization for \( \phi \) on each leaf \( \Lambda^p_\eta \). Now, \( \phi \) is smooth and by construction

\[
\{\Lambda^p_\eta = (x, dx\phi)\}.
\]

Therefore,

\[
p(x, dx\phi) = 0, \quad dx\phi(0, x', \eta) = \eta = 0
\]

and in particular, since \( \phi(0, 0, \eta) = 0 \), \( \phi(0, x', \eta) = \langle x', \eta \rangle \).

Another way to see this is by using Darboux’s theorem to find a symplectomorphism so that

\[
\kappa(U \cap P) \subset \{\eta_1 = 0\}, \quad \kappa(U \cap Q) \subset \{y_1 = 0\}.
\]

We then define \( Y \) as above and set

\[
L^p_{\eta_0} := \{\rho \in P \mid Y(\rho) = (\cdot, \eta_0)\}.
\]
Then, \( L_{\eta_0}^p \) is Lagrangian since
\[
\kappa(U \cap L_{\eta_0}^p) = V \cap \{\eta_1 = 0\} \cap \{(x_2, \ldots, x_d) = \eta_0\}
\]
with \( V \) an open neighborhood of \((0, 0)\). Note also that \( L_{\eta_0}^p \) foliate \( P \) since \( \eta' = \eta_0 \) foliate \( \{\eta_1 = 0\} \).

Now, \( \alpha|_{L_{\eta_0}^p} \) is closed and in particular, since \( Q \) is transverse to \( L_{\eta_0}^p \), there exists \( \Phi_{\eta_0} \in C^\infty(L_{\eta_0}^p) \) with
\[
d\Phi_{\eta_0} = \alpha|_{L_{\eta_0}^p}, \quad \Phi_{\eta_0}|_{Q \cap L_{\eta_0}^p} = f(\eta_0).
\]
Since \( L_{\eta_0}^p \) foliate \( P \), we therefore have for any smooth \( f \), a smooth function \( \Phi \in C^\infty(P) \) with
\[
d(\Phi|_{L_{\eta_0}^p}) = \alpha|_{L_{\eta_0}^p}, \quad \Phi|_{A \cap L_{\eta_0}^p} = f(\eta_0).
\]
That is, since \( Y(P) \subset \{\eta_1 = 0\} \), \( \theta = \theta(x, \xi') \) for any coordinates \( x \) on \( X \) and \( \eta' \) as above. Choosing an appropriate normalization on \( Q \) then recovers the above result.

### 3.2. The folding case.
In the folding case, when we try to solve (34), we will need to allow \( \phi \) to be singular. We will work to construct a solution \( \phi \) with a simple singularity of the form
\[
\phi = \theta \pm \frac{2}{3} (-\zeta)^{3/2}.
\]
This will translate directly to a solution of our original Eikonal equations.

The analog of the Lagrangians \( \Lambda_{\eta_0}^p \) will be played by
\[
L_{\eta_0}^p := \{ \rho \in P \mid Y(\rho) = (\cdot, \eta_0) \}, \quad \eta_0 \in \mathbb{R}^{d-1}.
\]
Then,
\[
\kappa(U \cap L_{\eta_0}^p) = V \cap \{x_1^2 - x_1 + x_2 = 0\} \cap \{(x_2, \ldots, x_d) = \eta_0\}
\]
with \( V \) an open neighborhood of \((0, 0)\).

Let \( T \subset P \) be a \( d-1 \) dimensional submanifold transverse to \( L_{\eta_0}^p \) for each \( \eta_0 \). Then for any \( f \in C^\infty(T; \mathbb{R}) \), we can find \( \Phi \in C^\infty(P) \) so that for any \( \eta_0 \),
\[
d(\Phi|_{L_{\eta_0}^p}) = \alpha|_{L_{\eta_0}^p}, \quad \Phi|_T = f.
\]
Therefore,
\[
p(x, d\Phi|_{L_{\eta_0}^p}) = 0
\]
and if we could use \((x, \eta_0)\) as coordinates on \( P \), we would be done. However, the vanishing of \( H_pq \) means that we cannot do this. It will be convenient to choose \( T \) contained in the fold surface and \( f \equiv 0 \).
There exist $\theta, \zeta \in C^\infty(Y(P); \mathbb{R})$ so that
$$\Phi = Y^*(\theta \pm \frac{2}{3}(-\zeta)^{3/2}).$$

In addition, $Y^*\zeta$ is a defining function for the fold and
$$\zeta = \eta_2, \quad \text{on } Y(P) \cap (T^* \{x_1 = 0\}).$$

Finally,
$$\partial_{x_1} \zeta \neq 0, \quad \text{on } Y(P) \cap (T^* \{x_1 = 0\}).$$

Proof. Let $\iota$ be the fold involution for $Y$ and consider
$$\Phi_e := \frac{1}{2}(\Phi + \iota^*\Phi), \quad \Phi_o := \frac{1}{2}(\Phi - \iota^*\Phi)$$
so that $\Phi = \Phi_e + \Phi_o$, $\Phi_e$ is $\iota$ even, and $\Phi_o$ is $\iota$ odd. Observe that $\rho$ is in the fold surface, $\mathcal{F}$ if and only if $\rho = \iota(\rho)$. Therefore, $\Phi_o|\mathcal{F} = 0$. Moreover, since
$$d\Phi|_{L^p_{\eta_0}} = d\alpha|_{L^p_{\eta_0}},$$
we have
$$d\Phi_o|_{L^p_{\eta_0}} = \frac{1}{2}(\alpha|_{L^p_{\eta_0}} - \iota^*\alpha|_{L^p_{\eta_0}})$$
and in particular, $d\Phi_o|_{L^p_{\eta_0}}$ vanishes at $\mathcal{F}$. Therefore, $\Phi_o$ vanishes to second (and hence third) order at $\mathcal{F}$.

Now, choose coordinates as in Lemma 2.1. Then
$$Y(x_1, \ldots, x_{2n-1}) = (x_1, \ldots, x_{2n-1})$$
and $\iota(x_1, \ldots, x_{2n-1}) = (x_1, \ldots, -x_{2n-1})$. Since $\Phi_e$ is even, in $x_{2n-1}$,
$$\Phi_e = \theta(x_1, \ldots, x_{2n-2}, x_{2n-1}^2) = \theta(Y(x))$$
for some $\theta \in C^\infty$.

On the other hand,
$$\Phi_o = \tilde{\zeta}(x_1, \ldots, x_{2n-2}, x_{2n-1}^2) = \pm \tilde{\zeta}(Y(x))y_{2n-1}(Y(x))^{3/2}.$$ Setting
$$\zeta(y_1, \ldots, y_{2n-1}) = \left(\frac{3}{2}\tilde{\zeta}\right)^{2/3}$$
we would have the first claim if $\tilde{\zeta}$ is non-vanishing at $y_{2n-1} = 0$.

To see this, we will prove (35). Note that $\xi_2$ is independent of the choice of $T$ and $\kappa$ reducing to normal form. Therefore, we aim to show that $\Phi_o$ is independent of these choices and hence that it agrees with the solution in the model case. First, fix $\kappa$ and let $\Phi_1, \Phi_2$ be two solutions associated with different $T_1$ and $T_2$. Then,
$$w = \Phi_1 - \Phi_2$$
is constant on each $L^p_{\eta_0}$ and in particular, is a function only of $\eta_0$. Since the fold involution $\iota$ preserves $L^p_{\eta_0}$, the $\iota$ odd part of $w$ is a function of only $\eta_0$. Since the $\iota$ odd part vanishes on $\mathcal{F}$ and $\mathcal{F}$ intersects every $L^p_{\eta_0}$, it vanishes identically. This implies that $\Phi_o$ is independent of the choice of $T$. 


Now, \[ p_n(x, d((\kappa^{-1})^*\Phi)|_{P_n \cap \{\eta = \eta_0\}}) = 0 \]
with \[ p_n = \eta_1^2 - y_1 + \eta_2. \]
Parametrizing \( P_n \cap \{\eta = \eta_0\} \) by \((y', \eta_1)\), so that \( dx_1 = d(\eta_0 + \eta_1^2) = 2\eta_1 d\eta_1 \),
\[ \partial_{\eta_1} (\kappa^{-1})^*\Phi(y', \eta_1; \eta_0) = 2\eta_1^2. \]
So, the \( \eta_1 \) odd part of \((\kappa^{-1})^*\Phi\) is given by \( \tfrac{2}{3}\eta_1^3 \) which, at \( \eta = \eta_0, y_1 = 0 \) is \( \pm \tfrac{2}{3}(-\eta_2)^{3/2} \).

Moreover, on \( P \cap Q \), the fold of \( Y \) is that of \( \pi^Q \). In particular, \( \iota(0, x', \xi_1, \xi') = (0, x', -\xi_1, \xi') \).
In addition, if \( \rho \in P \cap Q \), then \[ \eta_1(\kappa(\rho)) = -\eta_1(\iota(\rho)), \quad (y', \eta')(\kappa(\rho)) = (y', \eta')(\iota(\rho)). \]
Therefore, on \( Q \), the \( \iota \) odd part of \( \Phi \) is independent of the choice of \( T \) and of the reduction to normal form, \( \kappa \) and is given by \( Y^*(\pm \tfrac{2}{3}(-\eta_2)^{3/2}) \). This also implies that \( \tilde{\zeta} \) from (37) does not vanish near the fold and hence shows that \( \zeta \) defines the fold set.

Next, observe that
\[ Y(P) = \{ \eta_2 \leq x_1 f(x, \eta) \} \]
with \( f(\pi(\rho), 0) \neq 0 \) Therefore, since \( \zeta \) defines the fold set,
\[ \zeta = e(x, \eta)(\eta_2 - x_1 f(x, \eta)) \]
for some \( 0 \neq e \in C^\infty \). In particular, \( \partial_{x_1} \zeta|_{x_1=\eta_2=0} \neq 0 \) completing the proof of (36).

We next show that \( \theta \) is a non-degenerate phase function and can be chosen so that \( \theta|_{x_1=0} \) generates \( \kappa_\theta \).

**Lemma 3.2.** With \( \theta \) as above,
\[ d_{x'} \left( \frac{\partial \theta}{\partial \xi_j'} \right), j = 1, \ldots, d - 1 \text{ are linearly independent on } \zeta \leq 0 \text{ near } \rho_0 \]
and \( \Phi \) can be chosen so that \( \theta|_{x_1=0} \) generates \( \kappa_\theta^{-1} \) on \( \eta_2 \leq 0 \).

**Proof.** Let \((x, \xi)\) be coordinates near \( \rho_0 \) on \( T^*X \) and use \((y, \eta)\) as coordinates near \((0, 0)\) on \( T^*\mathbb{R}^d \). Then,
\[ \xi' dx' = d_{x'} Y^*(\theta \pm \tfrac{2}{3}(-\zeta)^{3/2}). \]
Now, \( d_{x'} Y^* = Y^* d_{x'} \) since \( Y \) does not change the base coordinates. Moreover, at \( x_1 = 0, \zeta = \xi_2, \) so \( d_{x'} \xi|_{x_1=0} = 0 \). In particular, at \( x_1 = 0, \)
\[ \xi' dx' = Y^* d_{x'} \theta. \]
That is,
\[ \partial_{x'} \theta(x', \eta'(\rho)) = \xi'. \]
But the map \( \xi' \to \eta' \) is a diffeomorphism, so (38) follows.
To show that $\theta$ can be chosen so that $\theta|_{x_1=0}$ generates $\kappa^{-1}_\theta$, we need to show that
\[
\kappa_\theta(x', \partial_x \theta(x', \eta')) = (\partial_\eta \theta(x', \eta'), \eta'), \quad x_1 = 0.
\]
Suppose $\tilde{\theta}$ generates $\kappa_\theta$. Then, on $L^p_\eta \cap Q$,
\[
d_x Y^* \tilde{\theta} = d_x Y^* \theta.
\]
In particular, $\tilde{\theta}$ and $\theta$ differ by a normalization $T$ and hence, choosing $T$ appropriately in the solution $\Phi$, we can arrange that $\theta = \tilde{\theta}$. The restriction to $\eta_2 \leq 0$ comes from the fact that $L^p_\eta \cap Q = \emptyset$ if $\eta_2 > 0$.

Our next task is to extend $\theta$ and $\zeta$ from $\zeta \leq 0$. Notice that $Y(P) = \{\zeta \leq 0\}$ is of the form 
\[
\{\eta_2 \leq x_1 f(x, \eta')\}
\]
where $f(Y(\rho)) \neq 0$. It will not be possible to solve the eikonal equations exactly in $\zeta > 0$. However, we will be able to solve them in formal power series both at $\zeta = 0$ and at $x_1 = 0$. It turns out that due to the behavior of the Airy function at the turning point $x = 0$, this will be enough to construct parametrices.

**Lemma 3.3.** There exist $\theta$ and $\zeta \in C^\infty$ in a neighborhood of $\pi(\rho) \times 0 \in X \times \mathbb{R}^{d-1}$ so that 
\[
\zeta|_{Y(Q)} = \zeta_2, \quad d_x \partial_\eta \theta \text{ are linearly independent, } \partial_{x_1} \zeta|_{Y(Q)} \neq 0
\]
and (26) holds in $\zeta \leq 0$ and in Taylor series at $x_1 = 0$.

**Proof.** At this point, we have solved (26) with smooth functions $\zeta_0, \theta_0$ having the above properties and defined in $\zeta_0 \leq 0$. We next extend $\theta_0$ and $\zeta_0$ as real so that $\zeta_0(x, \eta')|_{x_1=0} = \eta_2$ continues to hold and $\partial_{x_1} \zeta_0|_{x_1=0} \neq 0$. Then $\theta_0, \zeta_0$ solve the eikonal equations (26) modulo infinite order errors. That is,
\[
\begin{align*}
\langle ad \theta_0, d \theta_0 \rangle - \zeta_0 \langle ad \zeta_0, d \zeta_0 \rangle + \langle b, d \theta_0 \rangle + c &= e_1 \\
2 \langle ad \theta_0, d \zeta_0 \rangle + \langle b, d \zeta_0 \rangle &= e_2
\end{align*}
\]
with $e_1$ and $e_2$ vanishing identically in $\zeta_0 \leq 0$.

Our aim is to solve the eikonal equations also in formal power series at $x_1 = 0$. That is, we define
\[
\theta = \theta_0 + \theta', \quad \zeta = \zeta_0 + \zeta'
\]
where $\theta', \zeta'$ vanish in $\zeta_0 \leq 0$ and have
\[
\theta' \sim \sum_{k=1}^{\infty} x_k^1 \theta_k(x', \eta'), \quad \zeta' \sim \sum_{k=1}^{\infty} x_k^1 \zeta_k(x', \eta').
\]
We will solve for $\theta_k$ and $\zeta_k$ iteratively. At each step it is crucial that the errors and the previous functions vanish on $\zeta_0 \leq 0$.

The equations we want to solve in formal power series are
\[
\begin{align*}
\langle ad \theta', d \theta' \rangle + 2 \langle ad \theta', d \theta_0 \rangle - \zeta'(\langle ad \zeta_0, d \zeta_0 \rangle - (\zeta_0 + \zeta'))((ad \zeta', d \zeta') + 2 \langle ad \zeta', d \zeta_0 \rangle) + \langle b, d \theta' \rangle &= -e_1 \\
2 \langle ad \theta', d \zeta' \rangle + 2 \langle ad \theta', d \zeta_0 \rangle + 2 \langle ad \theta_0, d \zeta' \rangle + \langle b, d \zeta' \rangle &= -e_2
\end{align*}
\]
Using that $\zeta_0 = \eta_0 + O(x_1)$, we obtain, modulo terms of size $x$, the equations
\[
\begin{align*}
\left\{ \begin{array}{l}
(a_{11} \theta_1 + 2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \theta_1 - a_{11} \zeta_0 (\zeta_1 + 2 \partial_x \zeta_0) \zeta_1 = F_{1,1} \\
[2a_{11} (\zeta_1 + \partial_x \zeta_0) \theta_1 + (2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \zeta_1] = F_{2,1}
\end{array} \right.
\]
and for $k > 1$, modulo $x^k$, the equations
\[
\begin{align*}
\left\{ \begin{array}{l}
k x_1^{k-1} \left[ (2a_{11} \theta_1 + 2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \theta_k - 2a_{11} \zeta_0 (\zeta_1 + \partial_x \zeta_0) \zeta_k \right] = F_{1,k} \\
k x_1^{k-1} \left[ 2a_{11} (\zeta_1 + \partial_x \zeta_0) \theta_k + (2a_{11} \theta_1 + 2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \zeta_k \right] = F_{2,k}
\end{array} \right.
\]
where $F_{i,k}$ vanish to order $x_1^{k-1}$ and to infinite order at $\zeta_0 = 0$ and depend on $\theta_j, \zeta_j$ $0 \leq j \leq k - 1$.

We start by solving for $(\zeta_1, \theta_1)$. For this, we write
\[
\theta_1 = \zeta_1 (\zeta_1, x', \eta') = \frac{F_{2,1} - (2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \zeta_1}{2a_{11} (\zeta_1 + \partial_x \zeta_0)}
\]
Provided $\zeta_1 = O(\zeta_0^\infty)$, near $\zeta_0 = 0$, we have $\theta_1 = O(\zeta_0^\infty)$.

Now, since $H_{x_1,p} = 2 \sum_j 2a_{1j} \zeta_j + b_1 = 0$ at $x_1 = \zeta_0 = 0$, we have $2 \sum_j a_{1j} \partial_x \theta_0 + b_1 = 0$ there. In particular,
\[
\partial_{\zeta_1} \left[ (a_{11} \theta_1 + 2 \sum_j a_{1j} \partial_x \theta_0 + b_1) \theta_1 - a_{11} \zeta_0 (\zeta_1 + 2 \partial_x \zeta_0) \zeta_1 \right] = -2a_{11} \zeta_0 \partial_x \zeta_0 + O(\zeta_1) + O(\zeta_0^2)
\]
Dividing through by $\zeta_0$ and applying the inverse function theorem then gives a solution $\zeta_1$.

Now, we need to solve for $(\theta_k, \zeta_k)$. The equations for $(\theta_k, \zeta_k)$ are linear and the matrix has inverse with norm $\zeta_0^{-1}$ and hence since the errors vanish to order $\zeta_0^\infty$, we may solve for $(\theta_k, \zeta_k)$.

**4. Solution of the transport equations**

Recall that in order to construct a parametrix to our original problem, we arrived at transport equations of the form
\[
\left\{ \begin{array}{l}
2 \langle ad \theta, dg_0 \rangle + 2 \zeta \langle ad \zeta, dg_1 \rangle + \langle b, dg_0 \rangle + \langle ad \zeta, d \zeta \rangle g_1 + B_1 g_0 + \zeta B_2 g_1 = F_1 \\
2 \langle ad \zeta, dg_0 \rangle - 2 \langle ad \theta, dg_1 \rangle - \langle b, dg_1 \rangle + B_2 g_0 - B_1 g_1 = F_2
\end{array} \right.
\]
Writing
\[
g^\pm = g_0 \pm (-\zeta)^{1/2} g_1 \quad G^\pm = B_1 \mp (-\zeta)^{1/2} B_2 \quad F^\pm = F_1 \mp (-\zeta)^{1/2} F_2
\]
then in $\zeta \leq 0$, the transport equations are equivalent to
\[
2 \langle ad x, \phi^\pm, d_x g^\pm \rangle + \langle b, d_x g^\pm \rangle + G^\pm g_x^\pm = F^\pm
\]
As before, $g^\pm, G^\pm$, and $F^\pm$ pull back to smooth functions on $P$ under $Y$. this is because they are smooth functions of $(x, \eta', (-\zeta)^{1/2})$. Writing $g, G, F$ for these lifts to $P$, we then have
\[
2 \langle ad x \phi, g \rangle + \langle b, d_x g \rangle + G g = F
\]
Note now that $2\langle ad_x \Phi, \partial \rangle + \langle b, \partial \rangle = H_p$. To see this, observe that $\pi_X : P \to X$ is a diffeomorphism near $\rho$. Therefore, we may use $x$ as coordinates on $P$. In particular, in these coordinates,

$$H_p = \sum_{j=1}^d \partial_{\xi_j} p \partial_{x_j}.$$ 

Now, in coordinates, $\partial_{x_j} \Phi = \xi_j$. Therefore,

$$\partial_{\xi_j} p = \sum_i a_{ij} \xi_j + b_j = \sum_i a_{ij} \partial_{x_j} \Phi + b_j$$

which gives

$$H_p = 2\langle ad_x \Phi, \partial \rangle + \langle b, \partial \rangle.$$ 

Hence, in $\zeta \leq 0$, we need to solve

$$H_p g + Gg = F$$

with specified initial data for $g_0$ and $g_1$. In particular, $g_1|_{x_1=0} = 0$ and $g_0(\rho) = 1$. For $g$, this amounts to $g|_{P \cap Q} \in C^\infty(Q/\mathbb{R}H_q)$ and $g(\rho) = 1$. In fact, we will want to solve these equations with slightly more general boundary conditions that we describe below.

We now work to simplify (41) before proceeding to solve the equations. Clearly, we can solve

$$H_p \tilde{g} + G\tilde{g} = F, \quad \tilde{g}(\rho) = 0$$

by integrating a smooth, nonvanishing vector field. Subtracting $\tilde{g}$ from $g$ we need only solve the homogeneous problem. Next we remove the order 0 term by solving

$$H_p r = G, \quad r(\rho) = 0.$$ 

Then to solve the original problem it is enough to solve the equation

$$H_p u = 0.$$ 

In particular,

$$g = \tilde{g} + \exp(-r)u$$

has $(H_p + G)g = F$. The boundary conditions on $u$ are, however, more complicated:

$$[\exp(-r)u]_{P \cap Q} = [\tilde{g}]_{P \cap Q}, \quad u(\rho) = 1.$$ 

Here,

$$[f]_O = \frac{1}{2\tau}[f - \iota_Q^* f], \quad [f]_E = \frac{1}{2}[f + \iota_Q^* f]$$

where $\iota_Q$ is the involution induced by $\pi^Q$ and $\tau$ is a $Q$-odd function with $d\tau \neq 0$ on $Y^{-1}(\zeta = 0)$.

More generally, we will want to solve the equations with

$$g_1 = cg_0 + d \quad \text{on } T^* B, \quad g_0(\rho) = c_0.$$ 

Note, $g_1 = [g]_O$ and $g_0 = [g]_E$, so $[g]_O = c[g]_E + d$ which implies

$$[\exp(-r)u]_{P \cap Q} = c[\exp(-r)u]_E + f, \quad u(\rho) = c_0$$

where $c$ and $f$ are given $\iota_Q$ even functions.

Next, observe that the existence of a solution $u$ to (42) with given data $u_0$ on $Q \cap P$ amounts to the $\iota_P$ evenness of $u_0$ where $\iota_P$ is the involution induced by $\pi^P$.  


Therefore, it suffices to find an \( \iota_P \) even function \( u_0 \) on \( Q \cap P \) so that
\[
[\exp(-r)u_0]_O = c[\exp(-r)u_0]_E + f, \quad u_0(\rho) = c_0.
\]
Moreover, since
\[
[\alpha \beta]_E = [\alpha]_E[\beta]_E + \tau^2[\alpha]_O[\beta]_O \quad \text{and} \quad [\exp(-r)]_E(\rho) \neq 0,
\]
and \( [\exp(-r)]_E(\rho) \neq 0 \), it is enough to find an \( \iota_P \) even function, \( v \) with
\[
[v]_O = c[v]_E + f, \quad v(\rho) = c_0.
\]
for some given \( \iota_P \) even \( c \) and \( f \).

**Lemma 4.1.** For any \( \iota_Q \) even \( c, f \in C^\infty(Q \cap P) \), \( c_0 \in \mathbb{C} \), there exists a smooth \( \iota_P \) even function \( v \) on \( Q \cap P \) so that
\[
[v]_O = c[v]_E + f, \quad v(\rho) = c_0
\]
in a neighborhood of \( \rho \).

**Proof.** For this, we work in the normal form so that
\[
\iota_Q(y_2, y''_2, \eta_1, \eta') = (y_2, y''_2, -\eta_1, \eta'), \quad \iota_P(y_2, y''_2, \eta_1, \eta') = (y_2 - 2\eta_1, y''_2, -\eta_1, \eta').
\]
In this coordinates, then we seek an \( \iota_P \) even function solving \( \iota_Q \). To simplify notation we write \( t_1 = \eta_1 \) and \( t_2 = y_2, t' = (y''_2, \eta') \) so that
\[
\iota_Q(t_1, t_2, t') = (-t_1, t_2, t'), \quad \iota_P(t_1, t_2, t') = (-t_1, t_2 - 2t_1, t').
\]
Since \( v \) is \( \iota_P \) even, \( v = v(t_1^2, t_2 - t_1, t') \). We start by solving for \( v \) in formal power series at \( t_1 = 0 \). Assume that \( f = t_1^{2p} f_p(t_2, t') \) and \( v = t_1^{2p} v_p(t_2 - t_1, t') \). Then,
\[
\frac{v_p(t_2 - t_1, t') - v_p(t_2 + t_1, t')}{2t_1} = c(t_1^{2p} f_p(t_2, t') v_p(t_2 - t_1, t') + v_p(t_2 + t_1, t') + f_p(t_2, t').
\]
Since we are interested in Taylor series at \( t_1 = 0 \), we send \( t_1 \to 0 \) and obtain
\[
\partial_t v_p(t_2, t') = -c(0, t_2, t') v_p(t_2, t') + f_p(t_2, t').
\]
In particular, if this is satisfied, then
\[
[v]_O = c[v]_E + f + O(t^{2p+1}).
\]
Moreover, since \( c, f_p \) are \( \iota_Q \) even, the error is actually \( O(t^{2p+2}) \).

We may clearly solve \( \iota_Q \) with \( v_p(0, t') \) given. Therefore, there is a formal power series,
\[
v_f = \sum_{p=0}^{\infty} v_p(t_2 - t_1, t') t_1^{2p}
\]
satisfying \( \iota_Q \) in formal power series.

Now, we apply Borel’s lemma to sum the series \( v_f \) asymptotically to obtain a smooth function \( v_f \) that is \( \iota_P \) even and satisfies
\[
[v_f]_O = c[v_f]_E + g + e, \quad e \in C^\infty, \quad \iota_P e = e, \quad \partial_t^\infty e(t) = O_\alpha(t^\infty).
\]
Hence, by linearity, we need only solve
\begin{equation}
[v]_O = c[v]_E + e, \quad e \in C^\infty, \quad \iota_Q^e = e, \quad \partial^\gamma_t e(t) = O_{\alpha(t)}^e, \quad v|_{t_1=0} = 0
\end{equation}

Since we are interested in solving near 0, we may also assume that supp\(e \subset \{t_2 < 1\}\), \(c\) is supported near \(t_2 = 0\), and supp\(v \subset \{t_2 < 1\}\).

Now, observe that \(\iota_Q^e = e, \ \iota_Q^c = c \ i_p v = v, \ i_p t_1 = -t_1, \ \iota_p v = \frac{1}{1 - t_1 c(t)}[(1 + t_1 c(t))\iota_Q^e + 2t_1 c(t) e(t)]\). Since \(\iota_Q^e = e, \ \iota_Q^c = c \ i_p v = v, \ i_p t_1 = -t_1, \ \iota_p v = \frac{1}{1 - t_1 c(t)} [(1 + t_1 c(t))\iota_Q^v + 2t_1 c(t) e(t)]\)

\begin{align*}
v &= \frac{1}{1 - t_1 c(t)} \left[ (1 + t_1 c(t))\iota_Q^v + 2t_1 c(t) e(t) \right] \\
&= \frac{1 - t_1 \beta^c(t)}{1 - t_1 \beta^c(t)} \beta^v + \frac{-2t_1 \beta^c(t)}{1 - t_1 \beta^c(t)} \beta^e(t)
\end{align*}

where \(\beta = i_p \circ \iota_Q\) is the induced billiard ball map. Letting
\begin{align*}
a(t) := \frac{1 - t_1 \beta^c(t)}{1 + t_1 \beta^c(t)}, \quad b(t) := \frac{-2t_1 \beta^c(t)}{1 - t_1 \beta^c(t)}
\end{align*}

we then have
\begin{equation}
v = \sum_{m=0}^M B_m (\beta^{m+1})^e + A_M (\beta^{M+1})^v
\end{equation}

where 
\begin{align*}
B_m = (\beta^m)^* b \cdot \prod_{k=0}^{m-1} (\beta^k)^* a, \quad A_M = \prod_{k=0}^M (\beta^m)^* a.
\end{align*}

Now, note that
\(\beta(t_1, t_2, t') = (t_1, t_2 + 2t_1, t')\)
and hence, since supp\(v, \ supp\ e \subset \{t_2 < 1\}\), for \(t_1 > 0, t_2 > -1\), and \(M > 2t_1^{-1}, (\beta^M)^* v = (\beta^M)^* e = 0\). In particular, the the right hand side of \(49\) is independent of \(M > 2t_1^{-1}\) when \(t_1 > 0\).

In particular, in \(t_1 > 0, t_2 > -1\),
\begin{equation}
v = \sum_{m=0}^{2t_1^{-1}} B_m (\beta^{m+1})^e
\end{equation}
solves
\begin{equation}
[v]_O = c[v]_E + e, \quad on \ t_1 > 0.
\end{equation}

We need to show that the sum \(50\) converges uniformly with all of its derivatives.
We clearly have
\[ |1 - b(t)| \leq c|t_1| \]
and hence, for \( m \leq M \leq 2t_1^{-1} \),
\[ |B_m(t)| \leq (1 + c|t_1|)^M < C < \infty. \]
Therefore, the coefficients on \((\beta^{m+1})e\) in (50) are uniformly bounded. Now, \( e \) vanishes to all orders at \( t_1 = 0 \) and is supported in \( t_2 < 1 \). Therefore, for all \( N > 0 \), there exists \( c_N > 0 \) so that
\[ |(\beta^m)^*g(t)| \leq C_N|t_1|^{2N}(1 + |t_2 + 1 + mt_1|)^{-N} \leq c_N|t_1|^N(1 + m)^{-N}. \]
In particular, the sum (50) converges rapidly and hence defines a continuous function vanishing to all orders at \( t_1 = 0 \).

Now, \( B_m \) is a product of at most \( M \) terms. Thus, when it is differentiated, in \((t_2, t')\), it produces at most \( M \) choose \( N \) terms each of which is bounded by \( c_N|t_1| \). In particular, differentiation \( N \) times produces at most \( M^N \) sums of the form (50) except that sum number of the factors in \( B_m \) and those involving \( e \) are replaced by derivatives. Since the derivatives of \( g \) satisfy the same type of bounds as \( g \) and the coefficients in each sum are uniformly bounded, we have that any finite number of derivatives the sum (50) converges uniformly to a function vanishing rapidly at \( t_1 = 0 \).

This defines a smooth function \( v \) in \( t_1 \geq 0 \), \( t_2 > -1 \) satisfying (46). The invariance \( \iota_P^*v = v \) then defines a smooth function \( v \) in \(-2 < t_1 < 2 \) vanishing in \(|t_1| > 1 \). On \( t_1 \geq 0 \), \( v \) solves (48). Moreover, \( v = \iota_P^*v \). Therefore, \( v \) solves (47) and in particular (46).

The final step in the construction of the amplitude functions is to extend them into \( \zeta \geq 0 \). As with the eikonal equations, we will only do this in Taylor series at \( \zeta = 0 \) and \( x_1 = 0 \). We do this by extending \( g_0, g_1 \) arbitrarily as smooth functions and then adding a formal power series at \( x_1 = 0 \) [TODO]Flesh out.

5. Fourier–Airy Operators
CHAPTER 7

Calderón Projectors – Boundary Integral Operators??
APPENDIX A

Notation

- $\text{Diff}^m(M)$ differential operators of order $m$ on a manifold $M$. 