Population Growth in Continuous Time

Formal Demography
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Outline

Introduction: The Renewal Equation

We start with Lotka's derivation of the characteristic equation which is simply a model for the number of births at time t, $B(t)$ as a function of the number of births prior to t

The total number of births at time t, $B(t)$, is composed of two components (1) births to women already alive at time $t = 0$ and (2) births to women born since $t = 0$

$$B(t) = \int_0^t N(a,t)m(a)da + G(t) \tag{1}$$

where, $N(a,t)$ Number of women age a alive at time t, $m(a)$ is age-specific fertility rate of women age a, $G(t)$ are births from women alive at $t = 0$

From now on, we will focus on $t > \beta$, where β is the age of last reproduction in the population

Note also that we can integrate from either zero or α the age of first reproduction (or to ∞ for that matter) since $m(a) = 0 \; \forall \; a \notin [\alpha, \beta]$

Equation 1 is the renewal equation

It shows how the present births were generated by previous births – that is, how the population renews itself

Renewal 2

$N(a,t)$ results from the survivors of the births $B(t-a)$:

$$N(a,t) = B(t-a)l(a),$$

where $l(a)$ is the probability of surviving to exact age a.

Substitute back into equation 1 and assume $t > 50$. Note that this is equivalent to integrating from the age of first reproduction, α, to the age of last reproduction (β):

$$B(t) = \int_{\alpha}^{\beta} B(t-a)l(a)m(a)da \tag{2}$$

where $m(a)$ is the birth rate (m for "materinity") of women age a

This is what is known as a homogenous integral equation

The fact that equation 2 is homegenous is good news because solutions to homogenous integral equations have the property of linearity

This means that if $B(t)$ is a solution, then so is $cB(t)$, for some constant c

It also means that if $B_1(t)$ and $B_2(t)$ are both solutions, then $B_1(t) + B_2(t)$ will also be a solution

Renewal 3

Solve the homogeneous integral equation (2). Use $B(t) = Be^{rt}$ as a trial solution

$$Be^{rt} = \int_{\alpha}^{\beta} Be^{r(t-a)} l(a)m(a)da = \int_{\alpha}^{\beta} Be^{rt}e^{-ra} l(a)m(a)da \quad (3)$$

Now divide both sides by Be^{rt} to get:

$$1 = \int_{\alpha}^{\beta} e^{-ra} l(a)m(a)da \quad (4)$$

This is *the Characteristic Equation* of Euler and Lotka

The unique value of r that equates the two sides of 4 is known as the *intrinsic rate of increase*

This model relates the following fundamental quantities in demography:

- The schedule of age-specific mortality
- The schedule of age-specific fertility
- The age-structure of the population
- The rate of increase of the population

Three Equations Characterize a Stable Population

A stable population has an exponential birth series $B(t) = e^{rt}$

The number of age a people alive at time t, $N(a,t)$, is simply the number of births a years ago times the probability of surviving to age a:

$$N(a,t) = B(t-a)l(a)$$

Substitute this birth series into the equation for $N(a)$

$$N(a,t) = Be^{r(t-a)}l(a) = Be^{rt}e^{-ra}l(a) = B(t)e^{-ra}l(a) \tag{5}$$

Integrate this across all ages and rearrange

$$\int_0^\infty N(a,t)da = B(t)\int_0^\infty e^{-ra}l(a)da$$

$$\frac{B(t)}{\int_0^\infty N(a,t)da} = \frac{B(t)}{N(t)} = \frac{1}{\int_0^\infty e^{-ra} l(a)da} = b \tag{6}$$

This is the crude birth rate b of the population

Return to (5), divide both sides by the total population size $N(t)$ to get an expression for the proportionate age structure $c(a,t)$

$$c(a,t) = \frac{N(a,t)}{N(t)} = \frac{B(t)}{N(t)} e^{-ra} l(a)$$

But we know that in a stable population, $B(t)/N(t) = 1/\int_0^\infty e^{-ra} l(a)da$

Therefore, the age structure in a stable population is given by

$$c(a,t) = be^{-ra} l(a) = c(a) \tag{7}$$

The Trinity of the Stable Model

So what are the three equations that characterize the stable population?

They are the equations for (1) age structure, (2) the crude birth rate, and (3) the characteristic equation:

$$c(a) = be^{-ra}l(a)$$

$$b = \frac{1}{\int_{\alpha}^{\beta} e^{-ra}l(a)da}$$

$$1 = \int_{\alpha}^{\beta} e^{-ra}l(a)m(a)da$$

Analyzing the Characteristic Equation

The characteristic equation (eq. 4) has exactly one real solution, which we will call $r = r_0$

While there is one real solution, there are infinitely many solutions in general

All solutions other than r_0 will come in complex conjugate pairs and r_0 will be greater than the real part of all of the complex solutions, r_j

Define the net maternity schedule as $\phi(x) = l(x)m(x)$

Assume that the $\phi(x)$ schedule is fixed, we can thus re-write the characterisitc equation as a function of r

$$f(r) = \int_{\alpha}^{\beta} e^{-rx}\phi(x)dx. \tag{8}$$

As r approaches negative infinity, $f(r)$ approaches infinity

As $r \to 0$, $f(r) \to 0$

Differentiate 8, yielding

$$\frac{d}{dr}f(r) = -\int_{\alpha}^{\beta} xe^{-rx}\phi(x)dx \tag{9}$$

Note that x, e^{-rx}, and $\phi(x)$ are all non-negative functions between our limits of integration α and β

This means that the derivative is always less than zero, meaning $f(r)$ decreases monotonically

Differentiate again

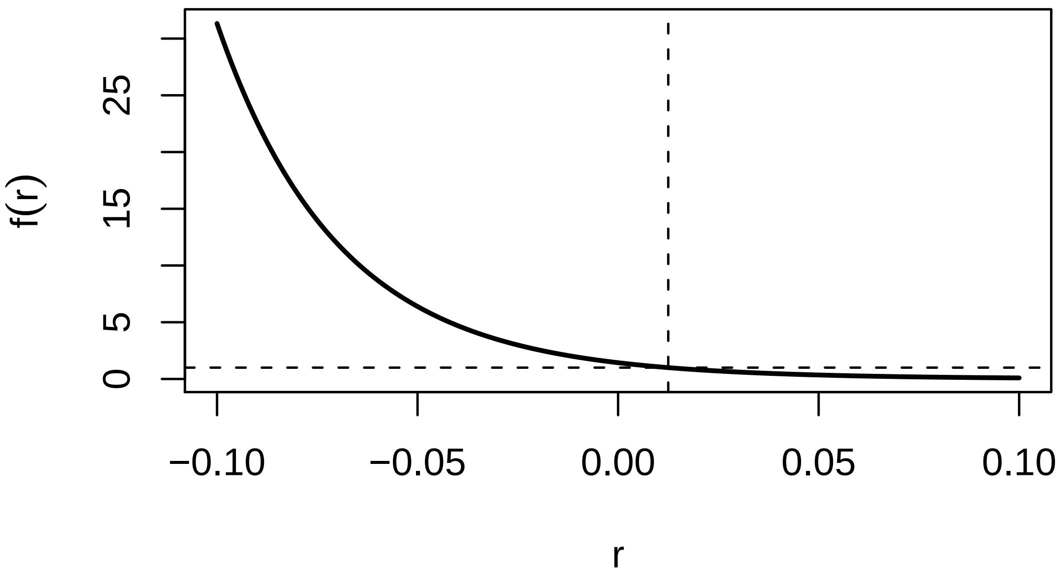
$$\frac{d}{dr}f(r) = \int_{\alpha}^{\beta} x^2e^{-rx}\phi(x)dx$$

We see that $f(r)$ is a concave function of r

In order for a function to take values from ∞ to 0 in a monotonically decreasing way, it can only cross $f(r) = 1$ once

We have therefore proved that there is only one real root to the characteristic equation

Plot of $f(r)$



Plot of $f(r)$ against r, showing where the curve crosses $f(r) = 1$ (horizontal dashed line and the corresponding value of r (vertical dahsed line, $r = 0.0125$. Net maternity data for the Netherlands in 1965 (Keyfitz 1977)

General Solution to the Integral Equation

The linearity of the characteristic equation – stemming from its being a homogenous integral equation – means that if e^{rt} is a solution, then (subject to convergence), so is exponential series of the form

$$B(t) = Q_0 e^{r_0 t} + Q_1 e^{r_1 t} + Q_2 e^{r_2 t} + \dots$$

The Q_i are arbitrary constants

Non-Real Roots

The characteristic equation has only one real root, but it can have an infinite number of complex roots

Complex roots always come as pairs of complex conjugates

That is, for some real constants u and v, if $r_j = u + iv$ is a root, so is $r_j^* = u - iv$

This is important because we can't have imaginary births in our birth series and that fact that complex roots come in conjugate pairs means that there are not imaginary parts in the solution ($r_j + r_j^* = 2u$)

Insert complex $r_j = u + iv$ into the characteristic equation:

$$1 = \int_{\alpha}^{\beta} e^{-ux-ivx}\phi(x)dx \tag{10}$$

We know from Euler's formula that we can write an exponentiated complex number as

$$e^{-ivx} = \cos(vx) - i\sin(vx)$$

We can therefore write equation 10 as two equations

$$1 = \int_{\alpha}^{\beta} e^{-ux}\cos(vx)\phi(x)dx \tag{11}$$

$$0 = \int_{\alpha}^{\beta} e^{-ux} i\sin(vx)\phi(x)dx \tag{12}$$

Equation 12 equals zero because the solution to the characteristic equation must only yield real numbers (since it is equal to unity) – thus the imaginary part of the solution with complex roots must equal zero

On Cosines

We know that $\cos(vx) = \cos(-vx)$, so this shows that if v satisfies the characteristic equation, so will $-v$

This shows that the complex roots come in conjugate pairs

To show that all the r_j must be smaller than r_0, we compare equations 4 and 12

Both these equations must be equal (since they both equal 1!)

Both integrals are essentially the same except that equation 11 contains an extra term, $\cos(vx)$

The maximum value of $\cos(vx)$ will be unity at $vx = 0$ and multiples of 2π, but less than unity everywhere else

The presence of the $\cos(vx)$ in the integral must reduce the value of the integral

If v were equal to r_0, then the integral of equation 4 would be less than one

To bring the integral up so that the equality was satisfied, the exponential term in the integral of equation 11 must clearly be greater the exponential term in equation 4: $e^{-vx} > e^{-r_0 x}$ for all x

Thus, $v < r_0$

Behavior as t Gets Large

We have seen that the exponential birth series is a solution to the characteristic equation

$$B(t) = Q_0 e^{r_0 t} + Q_1 e^{r_1 t} + Q_2 e^{r_2 t} + \dots$$

We have also seen that only r_0 is real and that the real parts of all the other $r_i,\ i > 0$ are less than r_0

This means that as t gets large, the term containing $r_i,\ i > 0$ will get smaller and smaller in comparison to the first term

Eventually, the population will follow a trajectory governed by r_0

So the terms $Q_i e^{r_i t}$ decay eventually to zero as t gets large

Since the $r_i,\ i > 0$ are complex, and the complex roots can be represented as linear combinations of sine and cosine functions, this means that the population trajectory will show $damped\ oscillations$

The Period of Oscillations

The real part of the complex roots need to simultaneously satisfy both equations 11 and 12

Coale (1972) investigated the period of the damped oscillations as a population converged to stability

He assumed that the distribtion of net maternity $\phi(x)$ was symmetric

As Coale (1972) points out, the lowest frequency value that satisfies both equations simultaneously is κ_1, the mean of the net maternity distribution

This means that the period of the damped oscillations that characterize the approach to stability will have a period equal to generation time (i.e., the mean age of the net maternity distribution)

When $\phi(x)$ is not symmetrical, the relationship is only approximate, but it is a very good approximation

Approximating r

Divide both sides of the characteristics equation by R_0, the net reproduction ratio

$$\int_{\alpha}^{\beta} \frac{e^{-rx}\phi(x)}{R_0} dx = \frac{1}{R_0} \tag{13}$$

The term $\phi(x)/R_0$ is a probability density for mother's age at birth

Note that

$$R_0 = \int_{\alpha}^{\beta} \phi(x) dx$$

This means that the left-hand side of equation 13 is a moment generating function $M(-r)$

This is true because we define a moment generating function for a probability density $f(x)$ as follows: if there exists some $h > 0$ such that $M(r) = \mathbb{E}(e^{rx})$, for $|r| < h$, then $M(r)$ is a moment generating function

By the definition of expection, this means

$$M(r) = \int_{-\infty}^{\infty} e^{rx} f(x) dx$$

Having established that equation 13 is a moment generating function, we can write the exponential term in it as a Taylor series expansion

$$e^{rx} = 1 + rx + \frac{1}{2} r^2 x^2 + \frac{1}{3!} r^3 x^3 + \dots$$

Note that the nth raw moment μ_n is defined as $\mu_n = \int x^n f(x) dx$

This means that we can write the moment-generating function as:

$$M(t) = 1 + r\mu_1 + r^2\mu_2 + r^3\mu_3 + \dots$$

It is generally more convenient to work with cumulants rather than moments, so we take the logarithm of equation 13 to get the cumulant generating function $K(-r)$:

$$K(-r) = -\log(R_0)$$

Note that the first two cumulants are equal to the mean and the variance of $f(x)$

$$r\kappa_1 - \frac{1}{2!}r^2\kappa_2 + \frac{1}{3!}\kappa_3 - \ldots = \log R_0. \tag{14}$$

If r_0 is small, then we can ignore the higher-order terms in r, yielding

$$\frac{1}{2}r^2\kappa_2 - r\kappa_1 + \log R_0 = 0. \tag{15}$$

This is a quadratic polynomial in r and we can solve it using the quadratic equation

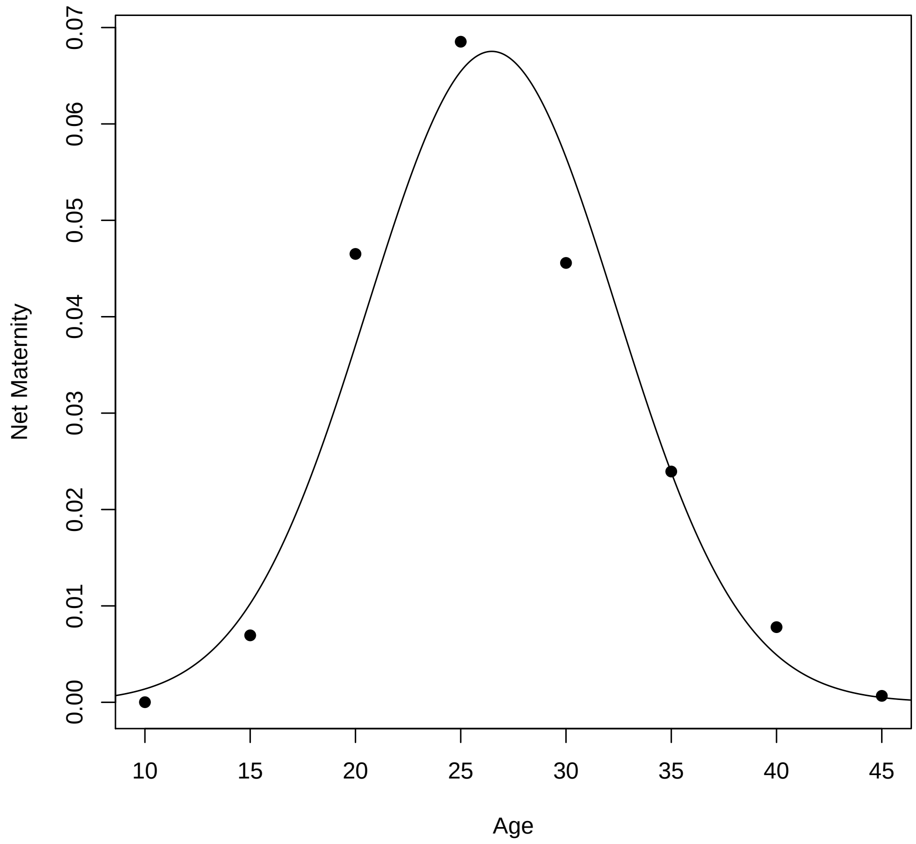
The probability distribution that we are discussing here, $\phi(x)/R_0$, is the distribution of net maternity

Thus, the first two cumulants κ_1 and κ_2 are the mean age of childbearing and the variance around this mean respectively

Note that by assuming that only the first two cumulants are needed, we assume that the distribution of net maternity is normal

Chances are, that's not true, but it's of little consequence

Normal Approximation to Net Maternity Schedule




```
## Net maternity schedule for the Netherlands (1965) -- used in Keyfitz (1977)
## R_0 = sum(phix)
phix <- c(0.0001, 0.0498, 0.3334, 0.4912, 0.3267, 0.1716, 0.0559, 0.0048)
```

```
age <- seq(10,45,by=5)

## calculate mean age of childbearing
kappa1 <-  sum(age*phix/sum(phix))  # by definition of expectation

## calculate variance in age at childbearing by definition of
## variance: V(X) = E(X^2) - E(X)^2
EA2 <- sum(age^2*phix/sum(phix))
kappa2 <- EA2-kappa1^2
plot(age,phix/(5*sum(phix)), pch=19, xlab="Age", ylab="Net Maternity")
## need to divide by 5 because net maternity is given in 5 year intervals

## overlay normal density: N(kappa1, sqrt(kappa2))
x <- seq(5,50,length=1000)
lines(x,dnorm(x,kappa1,sqrt(kappa2)))
```

Net Reproduction Ratio

Let R_i be defined as:

$$R_i = \int_{\alpha}^{\beta} x^i \phi(x) dx, \qquad i = 0, 1, 2, \ldots$$

When $i = 0$, we have the definition for R_0, the *Net Reproduction Ratio* (NRR)

Using this equation, we can also calculate the mean age of childbearing $\kappa_1 = R_1/R_0$ and the variance in age of childbearing $\kappa_2 = R_2/R_0 - (R_1/R_0)^2$.

The NRR is the ratio of population size from one *generation* to the next

There is a direct link between NRR and r, the intrinsic rate of increase, namely:

$$NRR = e^{rT} \tag{16}$$

where T is the length of a generation.

Equation 16 defines T, but it is clear that it will depend on the mortality and fertility schedules somehow

There are two distinct ways to calculate the generation time from age schedules of mortality and reproduction

The first is the the average age of childbearing in the stable population (A_B); the second is the cohort average age of childbearing (μ)

$$A_B = \frac{\int_\alpha^\beta x e^{-rx} \phi(x) dx}{\int_\alpha^\beta e^{-rx} \phi(x) dx} = \int_\alpha^\beta x e^{-rx} \phi(x) dx$$

$$\mu = \frac{R_1}{R_0} = \frac{\int_\alpha^\beta x \phi(x) dx}{\int_\alpha^\beta \phi(x) dx}$$

In practice, these are very similar and a reasonable approximation of generation time is $T \approx \frac{A_B + \mu}{2}$

This says that the length of a generation is a mixture of the average age of mothers in a cohort and the average age of mothers in a period

Re-arrange equation 16

$$r = \frac{\log(NRR)}{T}$$

A new definition:

$$GRR = \int_0^\infty m(a)da$$

The Net Reproduction Ratio can be expressed as

$$NRR = GRR \cdot l(A_M)$$

where A_M is the mean age of childbearing:

$$A_M = \frac{\int_\alpha^\beta am(a)da}{\int_\alpha^\beta m(a)da}$$

Now define G as the proportion of all births that are girls

$$NRR = TFR \cdot G \cdot l(A_M)$$

Substitute the expression for NRR back in to equation 16 and solve for r

$$r = \frac{\log TFR + \log G + \log l(A_M)}{T}$$

This is a pretty good approximation of the intrinsic rate of increase of the population