1 Introduction

If you fake it long enough, there comes a point where you aren’t faking it any more. Here are some tips to help you along the way...

2 Calculus

Derivative The definition of a derivative is as follows. For some function $f(x)$,

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$ 

2.1 Differentiation Rules

It is useful to remember the following rules for differentiation. Let $f(x)$ and $g(x)$ be two functions

2.1.1 Linearity

$$\frac{d}{dx}(af(x) + bg(x)) = af'(x) + bg'(x)$$

for constants $a$ and $b$.

2.1.2 Product rule

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x)$$

2.1.3 Chain rule

$$\frac{d}{dx}g(f(x)) = g'(f(x))f'(x)$$
2.1.4 Quotient Rule

\[ \frac{d}{dx} \frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2} \]

2.1.5 Some Basic Derivatives

\[
\begin{align*}
\frac{d}{dx} x^a &= ax^{a-1} \\
\frac{d}{dx} \frac{1}{x^a} &= -\frac{a}{x^{a+1}} \\
\frac{d}{dx} e^x &= e^x \\
\frac{d}{dx} a^x &= a^x \log a \\
\frac{d}{dx} \log |x| &= \frac{1}{x}
\end{align*}
\]

2.1.6 Convexity and Concavity

It is very easy to get confused about the convexity and concavity of a function. The technical mathematical definition is actually somewhat at odds with the colloquial usage. Let \( f(x) \) be a twice differentiable function in an interval \( I \). Then:

\[
f''(x) \geq 0 \Rightarrow f(x) \text{ convex} \quad (1)
\]

\[
f''(x) \leq 0 \Rightarrow f(x) \text{ concave}
\]

If you think about a profit function as a function of time, a convex function would show increasing marginal returns, while a concave function would show decreasing marginal returns.

This leads into an important theorem (particularly for stochastic demography), known as Jensen’s Inequality. For a convex function \( f(x) \),

\[ \mathbb{E} [f(X)] \geq f(\mathbb{E} [X]). \]

2.2 Taylor Series

\[ T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (x - a)^k \]

where \( f^{(k)}(a) \) denotes the \( k \)th derivative of \( f \) evaluated at \( a \), and \( k! = k(k-1)(k-2)\ldots(1) \).

For example, we can approximate \( e^r \) at \( a = 0 \):

\[ e^r \approx 1 + r + \frac{r^2}{2} + \frac{r^3}{6} \ldots \]
2.3 Jacobian

For a system of equations, $F(x)$ and $G(\lambda)$, the Jacobian matrix is

$$
J = \begin{pmatrix}
\frac{\partial F}{\partial x} & \frac{\partial F}{\partial \lambda} \\
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial \lambda}
\end{pmatrix}.
$$

This is very important for the analysis of stability of interacting models such as those for epidemics and predator-prey systems. The equilibrium of a system is stable if and only if the real parts of all the eigenvalues of $J$ are negative.

2.4 Integration

Linearity

$$
\int \left[ af(x) + bg(x) \right] dx = a \int f(x) dx + b \int g(x) dx
$$

Integration by Parts

$$
\int u \cdot v' \, dx = u \cdot v - \int v \cdot u' \, dx
$$

Some Useful Facts About Integrals

$$
\int \frac{f'(x)}{f(x)} \, dx = \log |f(x)|
$$

$$
\int x^a \, dx = \frac{x^{a+1}}{a+1}, \quad a \neq -1
$$
\[ \int e^x \, dx = e^x \]
\[ \int \frac{dx}{x} = \log |x| \]

2.5 Definite Integrals

\[ \int_a^b f(x) \, dx = [F(x)]_a^b = F(b) - F(a) \]

2.5.1 Expectation

For a continuous random variable \( X \) with probability density function \( f(x) \), the expected value is

\[ \mathbb{E}(X) = \int_{\Omega} x f(x) \, dx \]

where the integral is taken over the set of all possible outcomes \( \Omega \).

For example, the average age of mothers of newborns in a stable population:

\[ A_B = \int_{a}^{\beta} ae^{-\rho a} l(a)m(a) \, da \]

Since (from the Euler-Lotka equation) the probability that a mother will be \( a \) years old in a stable population is \( f(a) = e^{-\rho a} l(a)m(a) \).

Some Properties of Expectation

\[ \mathbb{E}[aX] = a\mathbb{E}[X] \]

For two discrete random variables, \( X \) and \( Y \),

\[ \mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y] \]

2.5.2 Variance

For a continuous random variable \( X \) with probability density function \( f(x) \) and expected value \( \mu \), the variance is

\[ \mathbb{V}(X) = \int_{\Omega} (x - \mu)^2 f(x) \, dx \]

A useful formula for calculating variances:

\[ \mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \]
2.6 Exponents and Logarithms

Properties of Exponentials

\[ x^a x^b = x^{a+b} \]

\[ \frac{x^a}{x^b} = x^{a-b} \]

\[ x^a = e^{a \log x} \]

Complex Case

\[ e^z = e^{a+bi} = e^a e^{bi} = e^a (\cos b + i \sin b) \]

\[ (x^a)^b = x^{ab} \]

\[ x^{-a} = \frac{1}{x^a} \]

The logarithm to the base \( e \), where \( e \) is defined as

\[ e = \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^n \]

Assume that \( \log \equiv \log_e \). Logarithms to other bases will be marked as such. For example: \( \log_{10}, \log_2 \), etc.

This is an important for demography:

\[ \lim_{n \to \infty} \left( 1 + \frac{r}{n} \right)^n = e^r \]

Properties of Logarithms

\[ \log x^a = a \log x \]

\[ \log ab = \log a + \log b \]

\[ \log \frac{a}{b} = \log a - \log b \]
**Complex Numbers**  We encounter complex numbers frequently when we calculate the eigenvalues of projection matrices, so it is useful to know something about them. Imaginary number: $i = \sqrt{-1}$. Complex number: $z = a + bi$, where $a$ is the real part and $b$ is a coefficient on the imaginary part.

It is useful to represent imaginary numbers in their polar form. Define axes where the abscissa represents the real part of a complex number and the ordinate represents the imaginary part (these axes are known as an *Argand diagram*). This vector, $a + bi$ can be represented by the angle $\theta$ and the radius of the vector rooted at the origin to point $(a, b)$. Using trigonometric definitions, $a = r \sin \theta$ and $b = r \cos \theta$, we see that

$$z = a + ib = r(\cos \theta + i \sin \theta).$$

Believe it or not, this comes in handy when we interpret the transient dynamics of a population.

Let $z$ be a complex number with real part $a$ and imaginary part $b$,

$$z = a + bi$$

Then the complex conjugate of $z$ is

$$\bar{z} = a - bi$$

Non-real eigenvalues of demographic projection matrices come in conjugate pairs.

### 3 Linear Algebra

A matrix is a rectangular array of numbers

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

A vector is simply a list of numbers.
A scalar is a single number: $\lambda = 1.05$

We refer to individual matrix elements by indexing them by their row and column positions. A matrix is typically named by a capital (bold) letter (e.g., $\mathbf{A}$). An element of matrix $\mathbf{A}$ is given by a lowercase $a$ subscripted with its indices. These indices are subscripted following the lowercase letter, first by row, then by column. For example, $a_{21}$ is the element of $\mathbf{A}$ which is in the second row and first column.

**Matrix Multiplication**

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  n_1 \\
  n_2
\end{bmatrix}
= \begin{bmatrix}
  a_{11}n_1 + a_{12}n_2 \\
  a_{21}n_1 + a_{22}n_2
\end{bmatrix}
\]

Multiply each row element-wise by the column

For Example,

\[
\begin{bmatrix}
  2 & 3 \\
  4 & 5
\end{bmatrix}
\begin{bmatrix}
  6 \\
  7
\end{bmatrix}
= \begin{bmatrix}
  (2 \cdot 6) + (3 \cdot 7) \\
  (4 \cdot 6) + (5 \cdot 7)
\end{bmatrix}
= \begin{bmatrix}
  33 \\
  59
\end{bmatrix}
\]

**Matrix Addition or Subtraction**

\[
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
+ \begin{bmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{bmatrix}
= \begin{bmatrix}
  a_{11} + b_{11} & a_{12} + b_{12} \\
  a_{21} + b_{21} & a_{22} + b_{22}
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 2 \\
  3 & 4
\end{bmatrix}
+ \begin{bmatrix}
  5 & 6 \\
  7 & 8
\end{bmatrix}
= \begin{bmatrix}
  6 & 8 \\
  10 & 12
\end{bmatrix}
\]

**Multiplying a Matrix by a Scalar**

\[
\lambda \begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
= \begin{bmatrix}
  \lambda a_{11} & \lambda a_{12} \\
  \lambda a_{21} & \lambda a_{22}
\end{bmatrix}
\]

\[
4 \begin{bmatrix}
  2 & 3 \\
  4 & 5
\end{bmatrix}
= \begin{bmatrix}
  8 & 12 \\
  16 & 20
\end{bmatrix}
\]

**Systems of Equations** Matrix notation was invented to make solving simultaneous equations easier.

\[
y_1 = ax_1 + bx_2 \\
y_2 = cx_1 + dx_2
\]

In matrix notation:

\[
\begin{bmatrix}
  y_1 \\
  y_2
\end{bmatrix}
= \begin{bmatrix}
  a & b \\
  c & d
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2
\end{bmatrix}
\]
3.1 Eigenvalues and Eigenvectors

A scalar $\lambda$ is an eigenvalue of a square matrix $A$ and $w \neq 0$ is its associated eigenvector if

$$Aw = \lambda w.$$  

Eigenvalues of $A$ are calculated as the roots of the characteristic equation,

$$\det(A - \lambda I) = 0,$$

where $I$ is the identity matrix, a square matrix with ones along the diagonal and zeros elsewhere.

For example, we can calculate the eigenvalues for the matrix,

$$A = \begin{bmatrix} f_1 & f_2 \\ p_1 & 0 \end{bmatrix}.$$

Solve the characteristic equation $\det(A - \lambda I) = 0$:

$$(A - \lambda I) = \begin{bmatrix} f_1 & f_2 \\ p_1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} f_1 - \lambda & f_2 \\ p_1 & -\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -(f_1 - \lambda)\lambda - f_2 p_1$$

$$\lambda^2 - f_1 \lambda - f_2 p_1 = 0$$

Use the quadratic equation to solve for $\lambda$:

$$\lambda = \frac{-f_1 \pm \sqrt{f_1^2 - 4f_2 p_1}}{2f_1}$$

**Numerical Example**  Define:

$$A = \begin{bmatrix} 1.5 & 2 \\ 0.5 & 0 \end{bmatrix}$$

$$(A - \lambda I) = \begin{bmatrix} 1.5 - \lambda & 2 \\ 0.5 & -\lambda \end{bmatrix}$$

$$\lambda^2 - 1.5 \lambda - 1 = 0$$

$$(\lambda - 2)(\lambda + 0.5) = 0$$

The roots of this are $\lambda = 2$ and $\lambda = -0.5$. A $k \times k$ matrix will have $k$ eigenvalues. If a matrix is non-negative, irreducible, and primitive, one of these eigenvalues is guaranteed to be real, positive, and strictly greater than all the others.
Analytic Formula for Eigenvalues: The $2 \times 2$ Case

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

The eigenvalues are:

\[ \lambda_{\pm} = \frac{T}{2} \pm \sqrt{\left(\frac{T}{2}\right)^2 - D} \]

where $T = a + d$ is the trace and $D = ad - bc$ is the determinant of matrix $A$. 