

EXAMPLE SHEET 2: NONLINEAR WAVE EQUATIONS

Throughout this example sheet, we take $\square = -\partial_t^2 + \Delta$.
 Notice that the examples with * are comparatively advanced.

(1) Consider the equation

$$\eta^{-2} \partial_t^2 \phi - 2\eta^{-2} \sum_{i=1}^3 \partial_i \phi \partial_{t_i}^2 \phi + \eta^{-2} \sum_{i,j=1}^3 \partial_i \phi \partial_j \phi \partial_{i_j}^2 \phi - \sum_{i=1}^3 \partial_i^2 \phi = 0$$

in $I \times \mathbb{R}^3$. Given smooth and compactly initial data such that $\|(\phi_0, \phi_1)\|_{W^{1,\infty}(\mathbb{R}^3) \times L^\infty(\mathbb{R}^3)} \leq \epsilon$, show that if ϵ is sufficiently small, the equation has a unique local-in-time solution.

(2) (Improved breakdown criterion) We have shown in class that for a general class of nonlinear wave equation, if the maximal time of existence $T_* < \infty$, then

$$\lim_{t \rightarrow T_*} \sum_{|\alpha| \leq \lfloor \frac{n}{2} + 1 \rfloor} \|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \rightarrow \infty.$$

Now, assume¹ the following functional inequality: For $s \geq 0$, we have

$$\|fg\|_{H^s(\mathbb{R}^n)} \leq C(\|f\|_{H^s(\mathbb{R}^n)} \|g\|_{L^\infty(\mathbb{R}^n)} + \|g\|_{H^s(\mathbb{R}^n)} \|f\|_{L^\infty(\mathbb{R}^n)})$$

for some $C > 0$.

Show that for the class of equations that we considered in the lectures, in fact if $T_* < \infty$, then

$$\lim_{t \rightarrow T_*} \sum_{|\alpha| \leq 1} \|\partial^\alpha \phi\|_{L^\infty(\mathbb{R}^n)} \rightarrow \infty.$$

(3) (Local existence with a class of Lipschitz data *) Consider the semilinear problem

$$\square \phi = T^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi,$$

where $T^{\alpha\beta}$ are constants. Given functions $\phi_{0,1}, \phi_{0,2}, \phi_{1,1}, \phi_{1,2} \in C_c^\infty$ and consider data of the form

$$(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\phi_{0,1} + (x_1)_+ \phi_{0,2}, \phi_{1,1} + \mathbf{1}_{\{x_1 \geq 0\}} \phi_{1,2}),$$

where $y_+ = y$ if $y \geq 0$ and $y_+ = 0$ otherwise. Notice that for such initial data, the first derivative of ϕ in general has a jump discontinuity across the set $\{x_1 = 0\}$.

(a) Show that in $(1+1)$ -dimensions, there exists

$$T = T(\|\partial \phi_{0,1}\|_{L^\infty}, \|(x_1)_+ \partial \phi_{0,2}\|_{L^\infty}, \|\phi_{0,2}\|_{L^\infty}, \|\phi_{1,1}\|_{L^\infty}, \|\phi_{1,2}\|_{L^\infty}) > 0$$

such that a unique solution $(\phi, \partial_t \phi) \in L^\infty([0, T]; W^{1,\infty}(\mathbb{R})) \times L^\infty([0, T]; L^\infty(\mathbb{R}))$ exists. (Hint: Integrate the equation directly in a manner similar to the second proof of the global regularity for $1+1$ dimensional wave map in the lecture notes.)

(b) We now consider the higher dimensional case. We use the notation the $x = (x_1, x')$, where $x' \in \mathbb{R}^{n-1}$. First, we consider the linear inhomogeneous wave equation

$$\square \phi = F.$$

Prove the following L^2 estimate for solutions:

$$\begin{aligned} & \sup_{t \in [0, T]} \left(\sum_{|\alpha| \leq k} \|\partial_{t, x_1} \partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha| \leq k+1} \|\partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) \right) \\ & \leq C \left(\sum_{|\alpha| \leq k} \|\partial_{t, x_1} \partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) + \sum_{|\alpha| \leq k+1} \|\partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(0) + \int_0^T \sum_{|\alpha| \leq k} \|\partial_{x'}^\alpha F\|(t) dt \right), \end{aligned}$$

¹Notice that this is indeed true and can be proved by Littlewood-Paley theory. See for instance Lemma A.8 in *Nonlinear dispersive equations* by Tao for a proof.

for some constant $C > 0$ depending only on dimensions and k .

(c) Prove the following functional inequality which controls the L^∞ norm :

$$\begin{aligned} & \sup_{t \in [0, T]} \sum_{|\alpha| \leq k} \|\partial_{x'}^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \\ & \leq C \sup_{t \in [0, T]} \left(\sum_{|\alpha| \leq k + \frac{n}{2} + 1} \|\partial_{t, x_1} \partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha| \leq k + \frac{n}{2} + 2} \|\partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) \right) \end{aligned}$$

for some C depending on n and k . (Hint: This should be thought of as a slightly refined version of Sobolev embedding. Use a combination of the one-dimensional Sobolev embedding and the $(n-1)$ -dimensional Sobolev embedding.)

(d) Then prove the following L^∞ estimate for solutions to

$$\square \phi = F :$$

$$\begin{aligned} & \sup_{t \in [0, T]} \sum_{|\alpha| \leq k-1} \|\partial_{t, x_1} \partial_{x'}^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(t) \\ & \leq C \sum_{|\alpha| \leq k-1} \|\partial_{t, x_1} \partial_{x'}^\alpha \phi\|_{L^\infty(\mathbb{R}^n)}(0) + \int_0^T \left(\sum_{|\alpha| \leq k + \frac{n}{2}} \|\partial_{t, x_1} \partial_{x'}^\alpha F\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha| \leq k + \frac{n}{2} + 1} \|\partial_{x'}^\alpha F\|_{L^2(\mathbb{R}^n)}(t) \right) dt \\ & + C \int_0^T \left(\sum_{|\alpha| \leq k + \frac{n}{2} + 2} \|\partial_{x_1} \partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) + \sum_{|\alpha| \leq k + \frac{n}{2} + 3} \|\partial_{x'}^\alpha \phi\|_{L^2(\mathbb{R}^n)}(t) \right) dt. \end{aligned}$$

(Hint: Imitate the proof in the $(1+1)$ -dimensional case and use the previous part.)

(e) Combine the estimates above and perform an iteration in an appropriate function space to conclude that there exists a unique solution in $[0, T] \times \mathbb{R}^n$ for T suitably small.

(4) (Nirenberg's example) Consider the equation

$$\square \phi = (\partial_t \phi)^2 - \sum_{i=1}^3 (\partial_i \phi)^2$$

in $I \times \mathbb{R}^3$. Show that there exists smooth and compactly supported initial data such that the solution blows up in finite time. (Hint: Consider the change of variable $\psi = e^\phi - 1$. When is this change of variables invertible?) On the other hand, given smooth and compactly supported functions (f, g) , show that there exists ϵ_0 (depending on f and g) such that if the initial data satisfy

$$(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\epsilon f, \epsilon g)$$

then the solution is global in time whenever $\epsilon \leq \epsilon_0$.

(5) As a consequence of the general local existence theorem, we know that the equation (in $I \times \mathbb{R}^3$)

$$\square \phi = (\partial_t \phi)^2 - \sum_{i=1}^3 (\partial_i \phi)^2$$

has a local solution in $[0, T]$ when the initial data $(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} \in H^5(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$. Moreover, the time of existence T depends only of the $H^5(\mathbb{R}^3) \times H^4(\mathbb{R}^3)$ norm of $(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}}$.

(a) Show, using the result in the previous problem, that the same result does not hold if we only assume that the initial data satisfy $(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

(b) Suppose that the data are in $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$. Find the smallest s_0 such that if $s > s_0$, then there exists a unique local solution $(\phi, \partial_t \phi) \in L^\infty([0, T]; H^s(\mathbb{R}^3)) \times L^\infty([0, T]; H^{s-1}(\mathbb{R}^3))$ for some $T > 0$ depending only on the $H^s(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$ norm of the data.

(6) Consider the equation

$$\square \phi = |\phi|^2 \phi$$

in $\mathbb{R} \times \mathbb{R}^3$. We have shown in class that finite energy smooth data give rise to global-in-time finite energy smooth solutions. Show that smooth initial data, i.e., $(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1) \in C^\infty \times C^\infty$ (without assumption on the finiteness of the initial energy!) give rise to global smooth solutions.

- (7) (Constraint equations) This problem requires some familiarity with differential geometry. In the lecture, we showed that given initial data set $(\mathbb{R}^n, \hat{g}_{ij}, \hat{k}_{ij})$ for the Einstein vacuum equations, the tensors Ric_{0i} and $(Ric - \frac{1}{2}gR)_{00}$ are determined by the data. Show that in fact the vanishing of these curvature components is equivalent to

$$\hat{\nabla}_i \hat{k}^i_j - \hat{\nabla}_j \hat{k}^i_i = 0,$$

$$\hat{R}(\hat{g}) + (\hat{k}^i_i)^2 - \hat{k}^i_j \hat{k}^j_i = 0,$$

where $\hat{\nabla}$ is the Levi-Civita connection associated to \hat{g} and $\hat{R}(\hat{g})$ is the scalar curvature associated to \hat{g} .

- (8) (Maxwell's equations via electromagnetic potential) When studying the Einstein vacuum equations, we show that the metric only satisfies a system of nonlinear wave equations after choosing an appropriate gauge (coordinates). Moreover, the gauge condition is propagated by the equation. In this problem, we see a similar idea at work in a linear setting - for the Maxwell's equations. Let $A : I \times \mathbb{R}^3 \rightarrow \mathbb{R}^{3+1}$. We write the components of A as (A_0, A_1, A_2, A_3) . We say A is a electromagnetic potential if

$$E_i = \partial_i A_0 - \partial_t A_i, \quad B_i = \sum_{j,k=1}^3 \epsilon_{ijk} \partial_j A_k.$$

- (a) Suppose E and B satisfy the Maxwell's equations, show that A satisfies the following second order equation:

$$\square A_\mu - m^{\nu\sigma} \partial_{\mu\nu}^2 A_\sigma = 0.$$

(In particular, the components of A do not satisfy wave equations.)

- (b) Let $\tilde{A}_\mu = A_\mu + \partial_\mu \phi$. Show that the electromagnetic fields \tilde{E} and \tilde{B} associated to \tilde{A} coincide with E and B .
- (c) Show that if A satisfies the Coulomb gauge condition $\sum_{i=1}^3 \partial_i A_i = 0$, then A_0 satisfies an elliptic equation and A_i satisfy wave equations for $i = 1, 2, 3$ with derivatives of A_0 as a source.

$$0 = \square A_i - m^{\nu\sigma} \partial_{i\nu}^2 A_\sigma = \square A_i + \partial_{it}^2 A_0$$

- (d) Show that if the Coulomb gauge condition is satisfied initially, then it is propagated by the evolution equations $\square A_i = 0$.
- (e) Prove using the above considerations that given sufficiently regular initial data (E^0, B^0) such that E^0 and B^0 are divergence-free, then there exists a unique solution to the Maxwell's equations such that $(E, B) \in L^\infty([0, T]; L^2(\mathbb{R}^3))$ for every $T > 0$.
- (9) (Global regularity for the subcritical/critical defocusing nonlinear wave equation *) In this problem, we discuss the global regularity for the problem

$$\square \phi = |\phi|^{p-1} \phi$$

in $\mathbb{I} \times \mathbb{R}^3$, where $p \leq 5$, i.e., when the problem is subcritical or critical. (Recall that we discussed in class the $p = 3$ case. This problem goes beyond that discussion.) A key estimate that is useful to establish the global regularity is the Strichartz estimates. It is a fundamental estimate for the wave equation which unfortunately we do not have time to cover in class. Nevertheless, it reads as follows: Let ϕ be the solution to the inhomogeneous linear wave equation

$$\square \phi = F$$

with initial data

$$(\phi, \partial_t \phi) \upharpoonright_{\{t=0\}} = (\phi_0, \phi_1).$$

Then, for every q, r satisfying

$$\frac{1}{2} = \frac{1}{q} + \frac{3}{r}, \quad \frac{1}{q} \leq \frac{1}{2} - \frac{1}{r}, \quad 2 \leq q, r \leq \infty, \quad r \neq \infty,$$

there exists $C = C(q, r) > 0$ such that the following estimate holds

$$\|\phi\|_{L^q([0, \infty); L^r(\mathbb{R}^3))} \leq C(\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} + \|F\|_{L^{q'}([0, \infty); L^{r'}(\mathbb{R}^3))}).$$

Here q' and r' are the conjugate exponents of q and r defined by $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Assume this estimate for the rest of this problem.

- (a) Show that for $p < 5$, if $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$, then the solution exists for all time and remains smooth. (Hint: Prove that the time of existence depends only on the $\dot{H}^1 \times L^2$ norm of (ϕ_0, ϕ_1) . To this end, first show using a bootstrap argument that $\|\phi\|_{L^q([0,T];L^r(\mathbb{R}^3))}$ is finite for some well-chosen q and r .)

- (b) Show that for $p = 5$, there exists $\epsilon_0 > 0$ such that if $(\phi_0, \phi_1) \in C_c^\infty \times C_c^\infty$ and

$$\|(\phi_0, \phi_1)\|_{\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)} \leq \epsilon_0,$$

then there exists a global-in-time smooth solution.

- (10) As always, please look for mistakes in this example sheet and let me know about them immediately!