REVIEW ON FOURIER ANALYSIS AND SOBOLEV THEORY

(1) Given a function $f \in L^1(\mathbb{R}^n)$, define the *Fourier transform* by the formula

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx.$$

Show that if $f(x) = e^{-\pi a|x|^2}$ where $a > 0$, then $\hat{f}(\xi) = a^{-\frac{n}{2}} e^{-\frac{\pi|\xi|^2}{a}}$. (Hint: Start with $n = 1$. First prove that if $x^\alpha h \in L^1$, then

$$\partial^\alpha \hat{h} = [(2\pi i x)^\alpha h]\hat{}.$$

Using this identity, prove that for f defined above, $e^{\frac{\pi\xi^2}{a}} \hat{f}(\xi)$ is independent of ξ. Then conclude the $n = 1$ case using the fact that $\int e^{-\pi a x^2} dx = a^{-\frac{1}{2}}$.)

(2) (Fourier inversion formula) Define the *inverse Fourier transform* by

$$\check{g}(x) := \int_{\mathbb{R}^n} g(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

In this exercise, follow the steps below to show that this name is justified, i.e., for f such that $f, \hat{f} \in L^1(\mathbb{R}^n)$, we have

$$(\hat{f})\check{} = f \tag{1}$$

almost everywhere.

(a) First show that

$$\int_{\mathbb{R}^n} \hat{f}(\xi) g(\xi) d\xi = \int_{\mathbb{R}^n} f(x)\hat{g}(x) dx.$$

(b) Show that

$$(\hat{f})\check{}(x) = \lim_{\epsilon\to 0} \int_{\mathbb{R}^n} e^{-\epsilon^2\pi|\xi|^2 + 2\pi i x\cdot\xi} \hat{f}(\xi) d\xi.$$

(c) Let $g(\xi) = e^{-\epsilon^2\pi|\xi|^2 + 2\pi i x\cdot\xi}$. Show using problem 1 that

$$\hat{g}(y) = \epsilon^{-n} e^{-\frac{\pi|x-y|^2}{\epsilon^2}}.$$

(d) Let ϕ be a smooth function such that $\int_{\mathbb{R}^n} \phi = 1$. Show that if β is an L^p function, then $\int_{\mathbb{R}^n} \beta(y)\epsilon^{-n}\phi(\frac{x-y}{\epsilon})\, dy$ converges to $\beta(x)$ in L^p as $\epsilon \to 0$.

(e) Combine the previous parts and conclude (1) using

$$\int_{\mathbb{R}^n} e^{-\pi|x|^2} dx = 1.$$

(3) (Plancherel's theorem) Let $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Show that

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

(Hint: Use 2a.)

(4) Show that the Fourier transform maps the Schwarz class into itself. Here we recall that the Schwarz class is defined as

$$\mathcal{S} := \{f \in C^\infty : \sup_x |x^\alpha \partial_x^\beta f| \le C_{\alpha,\beta} \text{for all multi-indices } \alpha, \beta\}.$$

Here, we have used the multi-index notation, i.e., for $\alpha = (\alpha_1, \alpha_2, ...\alpha_n)$, we denote by x^α the function $x_1^{\alpha_1} \cdot ... \cdot x_n^{\alpha_n}$ and ∂_x^α the differential operator $\partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n}$. (Hint: Use the formal identity

$$(x^\alpha \partial_x^\beta f)\check{} = (-1)^{|\alpha|}(2\pi i)^{|\beta|-|\alpha|}\partial_\xi^\alpha(\xi^\beta f).)$$

(5) Define the function space $H^s(\mathbb{R}^n)$ as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{\dot{H}^s(\mathbb{R}^n)} := \||\xi|^s \hat{f}\|_{L^2(\mathbb{R}^n)}.$$

First, show that this is indeed a norm. Then, show that when s is a non-negative integer, there exists a constant C (depending only on s) such that

$$C^{-1} \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{\dot{H}^s(\mathbb{R}^n)} \leq C \sum_{|\alpha|=s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

(Here, we have used the multi-index notation as before). Also, define H^s as the completion of $C_c^\infty(\mathbb{R}^n)$ under the norm

$$\|f\|_{H^s(\mathbb{R}^n)} := \|(1+|\xi|)^s \hat{f}\|_{L^2(\mathbb{R}^n)}$$

and show that if s is a non-negative integer, we have

$$C^{-1} \sum_{|\alpha|\leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)} \leq \|f\|_{H^s(\mathbb{R}^n)} \leq C \sum_{|\alpha|\leq s} \|\partial^\alpha f\|_{L^2(\mathbb{R}^n)}.$$

(6) (Hausdorff-Young) Prove that for $p \geq 2$

$$\|\hat{f}\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^{p'}(\mathbb{R}^n)},$$

where $1 \leq p' \leq 2$ is defined via the relation $\frac{1}{p} + \frac{1}{p'} = 1$. (Hint: Consider the case $p = \infty$. Then use Plancheral's theorem together with Reisz-Thorin interpolation theorem.)

(7) (Sobolev embedding theorem I) Prove that there exists a constant $C = C(n,s) > 0$ such that for every $f \in H^s(\mathbb{R}^n)$ with $s > \frac{n}{2}$, we have

$$\|f\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

(Hint: Use the Hausdorff-Young inequality and the Plancheral theorem.)

(8) (Sobolev embedding theorem II) Show that when $\frac{1}{p} = \frac{1}{2} - \frac{s}{n}$ and $p < \infty$, there exists a constant $C = C(n,p) > 0$ such that

$$\|f\|_{L^p(\mathbb{R}^n)} \leq C\|f\|_{H^s(\mathbb{R}^n)}.$$

(Hint: Use the Hardy-Littlewood-Sobolev inequality:

$$\|f * \frac{1}{|x|^\alpha}\|_{L^q(\mathbb{R}^n)} \leq C\|f\|_{L^p(\mathbb{R}^n)}$$

for $\frac{1}{p} = \frac{1}{q} + \frac{n-\alpha}{n}$, $1 < p < q < \infty$, $0 < \alpha < n$.)