NOTES ON THE POINCARÉ–BENDIXSON THEOREM

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Our goal in these notes is to understand the long-time behavior of solutions to ODEs. For this it will be very useful to introduce the notion of *ω-limit sets*. A remarkable result - the Poincaré–Bendixson theorem - is that for **planar** ODEs, one can have a rather good understanding of ω-limit sets. I have been benefited a lot from the textbook *Differential equations, dynamical systems and an introduction to chaos* by Hirsch–Smale–Devaney while preparing for these notes. As alwyas, **this is a preliminary version, if you have any comments or corrections, even very minor ones, please let me know**.

Consider the following ODE, i.e.,

$$u'(t) = F(u(t)), \tag{0.1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. In particular, the Picard–Lindelöf theorem applies. Therefore, we know that given any initial data, there exists a unique maximal solution.

Definition 0.1 (Flow map)**.** Let $u_0 \in \mathbb{R}^n$ and $u(t)$ be the unique maximal solution to the ODE (0.1) with initial data u_0. Suppose t is in the maximal interval of existence. Define

$$\varphi_t(u_0) = u(t).$$

Remark 0.2. The Picard–Lindelöf theorem implies that for every $u_0 \in \mathbb{R}^n$, there exists ϵ such that $\varphi_t(u_0)$ is well-defined for $t \in (-\epsilon, \epsilon)$. Moreover, it is in fact true that $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ is C^1 (whenever it is defined) (**Exercise**: think about why this is true).

Definition 0.3 (ω-limit set)**.** Let u_0 be such that $\varphi_t(u_0)$ is well-defined for all $t \geq 0$. Then its ω-limit set Ω is defined as

$$\Omega := \{x \in \mathbb{R}^n : \varphi_{t_k}(u_0) \to x \text{ for some sequence } t_k \to \infty\}.$$

We can prove some easy properties.

Proposition 0.4. *Let Ω be an ω-limit set associated to the flow $\varphi_t(u_0)$. The following properties hold:*

- *Ω is closed.*
- *If $x \in \Omega$, then $\varphi_t(x) \in \Omega$ (as long as $\varphi_t(x)$ is well-defined).*
- *If Ω is bounded, then it is connected.*

Proof. (1) By definition, we can write

$$\Omega = \cap_{\tau \geq 0} \overline{\{u(t) : t \geq \tau\}}.$$

Since Ω is an intersection of closed set, it is closed.

(2) If $x \in \Omega$, then there exists $s_k \to \infty$ such that $x = \lim_{k\to\infty} \varphi_{s_k}(u_0)$. Then, as long as $\varphi_t(x)$ is well-defined[1],

$$\varphi_t(x) = \varphi_t(\lim_{k\to\infty} \varphi_{s_k}(u_0)) = \lim_{k\to\infty} \varphi_{t+s_k}(u_0).$$

Hence, $\varphi_t(x) \in \Omega$.

(3) This is left as an **Exercise** (cf. HW8).

□

Here are a few examples of ω-limit sets:

(1) Suppose u_0 is an equilibrium point. Then the ω-limit set associated to $\varphi_t(u_0)$ is u_0
(2) Suppose $\underline{u}$ is an equilibrium point and u_0 is a point such that $\varphi_t(u_0) \to \underline{u}$ as $t \to \infty$. Then the corresponding ω-limit set is $\underline{u}$.
(3) Let $\varphi_t(u_0)$ be a periodic solution. Then the corresponding ω-limit set is exactly the image of $\varphi_t(u_0)$.

[1]Note that we have used that φ_t is continuous here, cf. Remark 0.2.

(4) A less obvious example is the following:

$$x'(t) = \sin x(t)\left(-\frac{1}{10}\cos x(t) - \cos y(t)\right),$$

$$y'(t) = \sin y(t)\left(\cos x(t) - \frac{1}{10}\cos y(t)\right).$$

It is easy to check that $(\frac{\pi}{2}, \frac{\pi}{2})$, $(0,0)$, $(0,\pi)$, $(\pi,0)$, (π,π) are all equilibrium points. Moreover, consider any initial data point in $(0,\pi)\times(0,\pi)\setminus\{(\frac{\pi}{2},\frac{\pi}{2})\}$, the solution approaches the square bounded by $x = 0, \pi$, $y = 0, \pi$. (This is not so easy to prove, but you will investigate some aspects of this in the homework.) In particular, the ω-limit set, which is the square as above, consists of four equilibrium points (the vertices of the square) and four segments connecting them.

It is a remarkable fact that on $\mathbb{R}^2$, bound ω-limit sets in general do not get much more complicated than the examples that we have seen!

Theorem 0.5 (Poincaré–Bendixson)**.** *Consider a* ***planar*** *ODE, i.e., assume $n = 2$. Let Ω be a non-empty bounded ω-limit set. Then either Ω contains at least one equilibrium point, or there exists a periodic solution $u(t)$ such that the image of $u(t)$ is exactly Ω.*

Remark 0.6. Notice that this is **false** on $\mathbb{R}^n$ for $n > 2$! One example we have seen is Problem 4 on HW5. (**Exercise**: Show that indeed in the setting of that problem, there exists a solution $u(t)$ such that the ω-limit set does not obey the conclusion of Theorem 0.5.)

The remainder of these notes will be devoted to a proof of Theorem 0.5. **From now on, we will consider** (0.1) **and fix** $n = 2$**.**

1. Transversal line

Given the ODE (0.1) (with $n = 2$), we can define the notion of transversal lines.

Definition 1.1 (Transversal lines)**.** A line segment $S = \{\lambda x_0 + (1-\lambda)x_1 : \lambda \in (0,1)\}$ is said to be *transversal* if for every $x \in \overline{S}$, $F(x)$ is non-vanishing and is not parallel to S.

The following is an easy fact about transversal lines:

Lemma 1.2. *Let S be a transversal line and $x_0 \in S$. Then there exists an open set U with $x \in U$ and a C^1 function $\tau : U \to \mathbb{R}$ such that $\varphi_{\tau(x)}(x) \in S$ for all $x \in U$.*

Proof. Let (y, z) be the coordinates on $\mathbb{R}^2$. Without loss of generality, assume that $x_0 = (0,0)$ and S is a subset of $y = 0$. For ϵ sufficiently small to be determined, define a map $\Psi : B_{\mathbb{R}}(0,\epsilon) \times B_{\mathbb{R}^2}(0,\epsilon) \to \mathbb{R}$ by

$$\Psi(t, x) = \pi\varphi_t(x),$$

where $\pi : \mathbb{R}^2 \to \mathbb{R}$ is the map $(y, z) \mapsto y$.

Fact: Ψ is a C^1 map. (This can be proven using the definitions and carefully studying (0.1), but we will omit the proof.)

Now, by definition $\Psi(0,0) = 0$. Moreover, since S is transversal, $\frac{\partial\Psi}{\partial t}(0,0) = \pi F(x_0) \neq 0$. The conclusion of the proposition hence follows from the implicit function theorem. □

2. Main monotonicity property

The main technical proposition that builds towards the Poincaré–Bendixson theorem is the following monotonicity property.

Proposition 2.1. *Let S be a transversal line and $u(t)$ be a solution to the ODE. Suppose $u(t_0)$, $u(t_1)$ and $u(t_2)$ are three points on S with $t_0 < t_1 < t_2$ such that $u(t_0) \neq u(t_1)$, then they are monotonic along S.*

Proof. Consider the curve $\Gamma \subset \mathbb{R}^2$ given by

$$\Gamma := \{u(t) : t \in [t_0, t_1]\} \cup \{x \in S : x \text{ is between } u(t_0) \text{ and } u(t_1)\}.$$

By the Jordan curve theorem[2], Γ divides $\mathbb{R}^2$ into two components, D_1 and D_2. Since $F(u(t_1))$ is transversal to S, u must either enter D_1 or D_2 after t_1. Suppose that u enters D_1 after time t_1, i.e., suppose there exists $\epsilon > 0$ such that $u(t) \in D_1$ for $t \in (t_1, t_1 + \epsilon)$. We claim that $u(t) \in D_1$ for all $t > t_1$. Assume not, then there exists a time $t_* > t_1$ such that $u(t_*) \in \Gamma$, but this is impossible:

- we cannot have $u(t_*) \in \{u(t) : t \in [t_0, t_1]\}$ since this contradicts the uniqueness of solutions; and
- we cannot have $u(t_*) \in \{x \in S : x \text{ is between } u(t_0) \text{ and } u(t_1)\}$ since F points towards D_1 on that set.

In particular, this implies $u(t_2) \in int(D_1)$, from which it follows that the points $u(t_0)$, $u(t_1)$ and $u(t_2)$ are monotonic along S. □

Proposition 2.2. *Let $u(t)$ be a solution to* (0.1) *and Ω be the corresponding ω-limit set. Let S be a transversal line. Then $S \cap \Omega$ has at most one point.*

Proof. Suppose not, i.e., assume that $x_1, x_2 \in S \cap \Omega$ are distinct. By Lemma 1.2, there exists U_1, U_2 disjoint open neighborhoods of x_1, x_2 respectively such that the conclusion of Lemma 1.2 holds. Since $x_1, x_2 \in \Omega$, there exist sequences of times $\{t_{1,n}\}_{n=1}^{\infty}$ and $\{t_{2,n}\}_{n=1}^{\infty}$ with $t_{1,n}, t_{2,n} \to \infty$ such that $u(t_{1,n}) \to x_1$ and $u(t_{2,n}) \to x_2$. Without loss of generality, we can assume that $u(t_{1,n}) \in U_1$, $u(t_{2,n}) \in U_2$ for all $n \in \mathbb{N}$. Using Lemma 1.2, there exist sequences of times $\{\tilde{t}_{1,n}\}_{n=1}^{\infty}$ and $\{\tilde{t}_{2,n}\}_{n=1}^{\infty}$ with $\tilde{t}_{1,n}, \tilde{t}_{2,n} \to \infty$ such that $u(\tilde{t}_{1,n}) \to x_1$ and $u(\tilde{t}_{2,n}) \to x_2$ <u>and</u> $u(\tilde{t}_{1,n}), u(\tilde{t}_{2,n}) \in S$ for all $n \in \mathbb{N}$. From this one easily concludes that there are three times, say, $\tilde{t}_{1,1} < \tilde{t}_{2,n_*} < \tilde{t}_{1,n_{**}}$ (for some $n_*, n_{**} \in \mathbb{N}$) such that $u(\tilde{t}_{1,1}) \neq u(\tilde{t}_{2,n_*})$, but the three points $u(\tilde{t}_{1,1})$, $u(\tilde{t}_{2,n_*})$ and $u(\tilde{t}_{1,n_{**}})$ are <u>not</u> monotonic along S. This contradicts Proposition 2.1. □

3. Proof of Poincaré–Bendixson theorem

We now prove the Poincaré–Bendixson theorem. We will consider a non-empty, bounded ω-limit set Ω with <u>no</u> equilibrium points. Our goal will be to show that Ω coincides with the image of a periodic solution. This will be done in two steps. In the first step, we show that the flow starting from any point in Ω is periodic. In the second step, we prove that the image of the periodic solution is in fact all of Ω.

Proposition 3.1. *Let Ω be the ω-limit set associated to the solution $\varphi_t(u_0)$. Suppose Ω is non-empty, bounded and has <u>no</u> equilibrium points. If $y \in \Omega$, then $\varphi_t(y)$ is a periodic solution.*

Proof. By Proposition 0.4, $\varphi_t(y) \in \Omega$ as long as it is defined. In particular, by the extension theorem, this implies $\varphi_t(y)$ is defined for all $t \geq 0$. Since $\{\varphi_t(y)\}_{t\geq 0}$ is bounded, by the Bolzano–Weierstrass theorem, there exists a sequence $\{t_k\}_{k=1}^{\infty}$ with $t_k \to \infty$ and a point z such that $\varphi_{t_k}(y) \to z$. Since $\varphi_{t_k}(y) \in \Omega$ for all $k \in \mathbb{N}$ and Ω is closed (by Proposition 0.4), this implies $z \in \Omega$. In particular, z is not an equilibrium point. Therefore, we can find a transversal line S passing through z.

Since $\varphi_{t_k}(y) \to z$, by Lemma 1.2, there exists $\tilde{t}_k \to \infty$ such that $\varphi_{\tilde{t}_k}(y) \to z$ <u>and</u> $\varphi_{\tilde{t}_k}(y) \in S$. Since $\varphi_{\tilde{t}_k}(y) \in S \cap \Omega$, by Proposition 2.2, they must in fact be the same point for all $k \in \mathbb{N}$.

In particular, there exist $\tilde{t}_{k_1} \neq \tilde{t}_{k_2}$ such that $\varphi_{\tilde{t}_{k_1}}(y) = \varphi_{\tilde{t}_{k_2}}(y)$. This implies that $\varphi_t(y)$ is a periodic solution. □

Proposition 3.2. *Let Ω be the ω-limit set associated to the solution $\varphi_t(u_0)$. Suppose Ω is non-empty, bounded and has <u>no</u> equilibrium points and $y \in \Omega$. Then $\Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\} = \emptyset$.*

Proof. It suffices to show that $\Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$ is closed. This is because by Proposition 0.4, Ω is connected. Therefore, if we can write $\Omega = (\cup_{t\geq 0}\{\varphi_t(y)\}) \cup (\Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\})$ as a union of two disjoint closed sets, one of them must be empty. Since $\cup_{t\geq 0}\{\varphi_t(y)\} \neq \emptyset$, it follows that $\Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\} = \emptyset$.

Now take a sequence of points $\{z_k\}_{k=1}^{\infty} \subset \Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$ and suppose $z_k \to z$. Our goal is to show that $z \in \Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$. This would imply that $\Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$ is closed.

First, we note that since Ω is closed, $z \in \Omega$. In particular, z is not an equilibrium point. We can therefore find a traversal line S such that $z \in S$. For every $k \in \mathbb{N}$, since $z_k \in \Omega$, there exists a sequence of times $\{t_{k,\ell}\}_{\ell=1}^{\infty}$ (with $t_{k,\ell} \to \infty$ as $\ell \to \infty$) such that $\varphi_{t_{k,\ell}}(u_0) \to z_k$ as $\ell \to \infty$. By Lemma 1.2, for sufficiently large k, we can find $\{\tilde{t}_{k,\ell}\}_{\ell=1}^{\infty}$ (with $\tilde{t}_{k,\ell} \to \infty$ as $\ell \to \infty$) such that $\varphi_{\tilde{t}_{k,\ell}}(u_0) \to z_k$ as $\ell \to \infty$ <u>and</u> moreover

[2]The Jordan curve theorem says that any continuous simple closed curve in the plane divides the plane into two disjoint components (the "inside" and the "outside").

$\varphi_{\tilde{t}_{k,\ell}}(u_0) \in S$ for all $\ell \in \mathbb{N}$. By Proposition 2.2, all these points must coincide, and therefore $\varphi_{\tilde{t}_{k,\ell}}(u_0) = z$ for all k sufficiently large and for all $\ell \in \mathbb{N}$. In particular, this implies $z_k = z$ for all k sufficiently large.

Since $z_k \in \Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$, it follows that $z \in \Omega \setminus \cup_{t\geq 0}\{\varphi_t(y)\}$. This concludes the proof. □

Combining Propositions 3.1 and 3.2 gives Theorem 0.5.