Area Exam Syllabus with Notes

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Contents

1 To Do

- Needs to be updated with final remarks from Rafe
- For every theorem, go in and add notes for how to prove

2 Topic 1: Riemannian Geometry

- Fermi Coordinates - definition for internal coordinates, fermi coordinates off a higher codimension submanifold, metric in these coordinates, vanishing of christoffel symbols, gauss lemma, Laplacian in Fermi coordinate codim 1
- Geodesics - Existence via energy, Hopf-Rinow theorem, cut-locus, PDE for geodesics in coordinates, Jacobi equation for family of geodesics

- Curvature - Myers theorem, Bishop-Gromov inequality, Bochner formula, comparison theorem for distance functions ($\Delta\rho$)
- Submanifolds - Gauss-Codazzi equations, first and second variation of area, gauss-bonnet theorem
- Minimal Submanifolds - Definition, statement of regularity for $n \leq 8$, Examples in $\mathbb{R}^3$, non-existence of stable minimal submanifolds when ambient $\mathrm{Ric}_g \geq 0$, Barta theorem on stable minimal

2.1 Fermi Coordinates

(Lee Smooth manifolds)

2.1.1 Riemann Normal Coordinates

- Internally, we have
$$F(\vec{v}) = \exp_{p,g}(\vec{v}) = \gamma_v(1)$$
where $\gamma_v(t)$ is the geodesic with $\gamma_v(0) = 0$ and $\dot{\gamma}_v(0) = v$.
- The exponential map is smooth and
$$d(\exp_p)_0 : T_0(T_pM) \cong T_pM \to T_pM$$
is the identity map of T_pM with the normal identification of $T_0(T_pM)$ with T_pM. This follows because
$$d(\exp_p)_0(v) = \frac{d}{dt}(\exp_p \circ \tau)(t)\Big|_{t=0} = \frac{d}{dt}\exp_p(tv)\Big|_{t=0} = v$$
where $\tau(t) = tv$ is a curve in T_pM starting at 0 with initial velocity v
- Note that this immediately implies that the coordinate basis is orthonormal since if we have $\{e_i\}$ an ONB for $\mathbb{R}^n$, we can identify this as a basis for T_pM via some linear map B, and then
$$d(\exp_p)_0 : T_0T_pM \to T_pM$$
and with the identification of $\{e_i\} \sim \{B(e_i)\}$, we know that
$$d(\exp_p)_0(e_i) = e_i$$
since the exponential map is the identity. Moreover this makes the metric at the point δ_{ij} since we're writing the metric at T_pM in terms of the pull back via the exponential map (and a linear identification of the basis, B) to $\mathbb{R}^n$, i.e.
$$g_{ij} = g_p((d\exp_p)_0(e_i), (d\exp_p)_0(e_j)) = g_p(e_i, e_j)$$
but we've chosen $\{e_i\}$ to be an orthonormal basis (with respect to g_p! Considered as a metric on $\mathbb{R}^n$), so we get $g_{ij} = \delta_{ij}$
- To clarify about B, every ONB $\{b_i\}$ for T_pM (w.r.t g_p) determines a linear map
$$B : \mathbb{R}^n \to T_pM, \qquad B(x_1, \ldots, x_n) = x^ib_i$$
and then
$$d\exp_p(B_*(\partial_i)) = b_i$$
so for the composite map $\varphi = \exp \circ B$, which is the actual chart, we have that the pushed-forward coordinate basis is orthonormal at p.
- The christoffels also vanishes, since we've pulled back the metric. Consider a geodesic in $\mathbb{R}^n$, which in these normal coordinates, we write as
$$\ddot{x}^k + \Gamma^k_{ij}\dot{x}^i\dot{x}^j = 0$$
In normal coordinates, we have
$$x(t) = t\vec{v}$$

so the above is

$$0 + \Gamma^k_{ij}(x(t))v^i v^j = 0$$

and by choosing $\vec{v} = e_i$ and $\vec{v} = e_i \pm e_j$ we can conclude $\Gamma^k_{ij} = 0$. Note also that christoffels can be computed in terms of the metric and derivatives of the metric, and this can be inverted to give derivatives of the metric in terms of the christoffels. So $\partial_k g_{ij} = 0$ at p as well **Note: we can make $\vec{v}$ arbitrary and still glean info about Γ^k_{ij} only when $t = 0$.** For t non-zero, the same tricks don't work since changing $\vec{v}$ will make us evaluate Γ^k_{ij} at different points.

2.1.2 Fermi Coordinates

- We can also do this to get a tubular neighborhood: Let $P \hookrightarrow M$ being a submanifold. We can look at $\exp_p : N_p(P) \to M$. Tubular neighborhoods are of the form

$$U = \exp_p \left(\{(x, v) \in NP \ : \ |v|_g < \delta(x)\right)$$

for some continuous δ. For P compact, we can get a tubular neighborhood of radius ϵ

- We have similar results where we can formulate a map

$$\varphi : U_0 \to M$$

where $U_0 \subseteq \mathbb{R}^n$ and we should imagine $\varphi(U_0)$ as a tubular neighborhood of $P^p \subseteq M^n$. In particular if $\{E_i\}$ is an ONF for $N(P)_p$, $V \subseteq NP$ an open subset and $U = E(V)$ a normal neighborhood of P, then

$$\varphi : E(q, v^i E_i) \mapsto (x^1(q), \ldots, x^p(q), v^1, \ldots, v^{n-p})$$

- In these coordinates we have
  - $P \cap U_0 = \{x^{p+1} = \cdots = x^n = 0\}$
  - For all $q \in P \cap U_0$, the metric splits and

$$g_{ij} = \begin{cases} 0 & 1 \leq i \leq p, \qquad p+1 \leq j \leq n \\ \delta_{ij} & p+1 \leq i, j \leq n \end{cases}$$

  - For all $q \in P \cap U_0$, For $v = v^i E_i \in N_q P$, the geodesic starting at q with velocity v has the coordinate expression

$$\gamma_v(t) = (x^1(q), \ldots, x^p(q), tv^1, \ldots, tv^{n-p})$$

  - For all $q \in P \cap U_0$, the christoffels vanish, $\Gamma^k_{ij} = 0$ for $p + 1 \leq i, j \leq n$
  - For all $q \in P \cap U_0$, We also have $\partial_i g_{jk} = 0$ for $p + 1 \leq i, j, k \leq n$

Note that we can choose to parameterize q via riemann normal coordiantes on P, which will make the metric nicer but the christoffels will be non-trivial, since they record the second fundamental form, which can't be flattened at a point via coordinate transformation (**Note**: This might be useful for Renormalized Volume formula for $\ddot{S}^i$ - handle those pesky second fundamental form terms)

2.1.3 Gauss Lemma

(Lee, section 6.9)

- Kind of a cool proof
- Work in GNC so that

$$\partial_r \Big|_q = \frac{q^i}{b} \partial_{x^i}$$

where $b = r(q) = \sqrt{\sum_i (q^i)^2}$

- We want to prove

Theorem 2.1. *Let (M,g) an RM, U a geodesic ball centered at $p \in M$ and ∂_r radial vector field on $U\backslash\{p\}$ then ∂_r is unit vector field and orthogonal to geodesic spheres*

Proof: From the above, we have that

$$g(\partial_r, \partial_r) = \frac{1}{b^2}\sum (q^i)^2 = 1$$

(note that q is some point away from our base point, p) since $g(\partial_{x_i}, \partial_{x_i}) = 1$. Now consider a geodesic sphere, which corresponds to

$$\Sigma_t = \{s \mid \sum_i x^i(s)^2 = t^2\}$$

for any $w \in T\Sigma_t$ at q. we want to show that $g(w, \partial_r\Big|_q) = 0$. Choose $\sigma(s)$ a curve in Σ_t with $\sigma(0) = q$ and $\sigma'(0) = w$, then form

$$\Gamma(s,t) = \frac{t}{b}\vec{\sigma}(s)$$

for fixed s, this is a geodesic corresponding to initial velocity $\Gamma_s'(0) = \vec{\sigma}(s)/b$ which is unit speed since $\sigma \in \Sigma_t$. Now let $S = \Gamma_*(\partial_s)$ and $T = \Gamma_*(\partial_t)$, then

$$S(0,0) = 0, \qquad T(0,0) = v = \vec{q}/b, \qquad S(0,b) = w, \qquad T(0,b) = \gamma_v'(b) = \partial_r\Big|_q$$

here, we've noted that $q = \gamma_{bv}(1)$ for some $v \in T_pM$ but since $|q| = b$ we have the $v = q/b$ is unit length

We have that $\langle S, T\rangle = 0$ at $(s,t) = (0,0)$ and equal to $\langle w, \partial_r\Big|_q\rangle$ when $(s,t) = (0,b)$ so it suffices to prove that

$$\frac{d}{dt}\langle S, T\rangle = 0$$

for all t and $s = 0$. We see this as

$$\begin{aligned}\frac{d}{dt}\langle S, T\rangle &= \langle \nabla_T S, T\rangle + \langle S, \nabla_T T\rangle \\ &= \langle \nabla_S T, T\rangle + 0 \\ &= \frac{1}{2}\partial_s |T|^2 = 0\end{aligned}$$

since T has unit speed and using that S and T are push forwards of coordinate vectors and therefore commute

2.1.4 Laplacian in Fermi Coordinates, codim 1

- The set up is that we choose fermi coordinates over $P^n \subseteq M^{n+1}$. Moreover, our coordinates parameterizing P are riemann normal coordinates so that

  $$g^{ij} = \delta_{ij}\Big|_p$$

  and more generally

  $$g^{a\nu} \equiv 0, g^{\nu\nu} \equiv 1$$

  where a denotes index for coordinates on P and ν the coordinate on P. We'll let $\partial_{i+1} = \nu$ be the normal for brevity

- We compute this as

  $$\Delta f = g^{ab}(D^2 f)(a,b)$$

  using that

  $$(D^2 f)(a,b) = Hess\ f(\partial_a, \partial_b) = \nabla_{\partial_a}\nabla_{\partial_b} f - \nabla_{\nabla_{\partial_a}\partial_b} f$$

  we know that

  $$1 \le a, b \le n, \qquad \nabla_{\partial_a}\partial_b = (\nabla_{\partial_a}\partial_b)^{\|} + (\nabla_{\partial_a}\partial_b)^{\perp} = (\nabla_{\partial_a}\partial_b)^{\|} + A(\partial_a, \partial_b)$$
  $$\nabla_\nu \nu = 0$$

because the metric is diagonal in these coordinates, this is all we need to compute, i.e.

$$\begin{aligned}\Delta f &= g^{ij}(D^2 f)(i,j)\\ &= g^{ab}(D^2 f)(a,b) + g^{\nu\nu}(D^2 f)(\nu,\nu)\\ &= g^{ab}(f_{ab} - (\nabla_{\partial_a}\partial_b)^{\|}(f) - A_{ab}f_\nu) + g^{\nu\nu}[f_{\nu\nu} - (\nabla_\nu \nu)(f)]\\ &= \Delta_{P,z}(f) - H(z)f_\nu + f_{\nu\nu}\end{aligned}$$

where z is the distance from P

2.2 Geodesics

2.2.1 Existence

- **Existence of Geodesics via Energy**

  Try to minimize

  $$L(\gamma) = \int_0^1 g^{ij}\gamma_i'\gamma_j' = \int_0^1 F(\gamma')$$

  (note that a minimizer of this is the same as a minimizer of the length, $\int \sqrt{g^{ij}(\gamma',\gamma')}$)

  Notice that becuase g is a metric, we have that

  $$F(\gamma') \geq \kappa||\gamma'||^2$$

  (where $||v||^2$ is now the standard euclidean norm), and also

  $$L(\gamma') \geq \kappa'||\gamma'||^2_{L^2}$$

  (Note this is like 8.2 in Evans, "Existence of Minimizers" except here our $w(x)$ is a system $\vec{\gamma}(t)$).

  Because $L(\gamma)$ is essentially $||\gamma'||_{L^2}$, we see that if we have

  $$\gamma \xrightarrow{W^{1,2}} \gamma^* \implies L(\gamma) \xrightarrow{W^{1,2}} L(\gamma^*)$$

  i.e. convergence in H^1 gives convergence of the functional. All we need is weakly lower semicontinuity though for the record.

  Finally, note that

  $$F(p,z,x) = g^{ij}p_i p_j$$

  is convex in p in the sense that

  $$\sum_{i,j} F_{p_i p_j}(w,v,x)\xi_i\xi_j = \sum_{ij} g^{ij}\xi_i\xi_j \geq \kappa||\xi||^2$$

  so we do have convexity. This in turn gives us lower-semi continuity of the functional, i.e.

  $$u_k \rightharpoonup u \implies L(u) \leq \liminf_k L(u_k)$$

  I guess this is evident again since the functional is essentially the L^2 norm up to coefficients. Now conclude with existence and uniqueness of minimizers, which is

  Theorem 2.2. *Assume that F is coercive and convex in the variable p, suppose also that the admissible set of functions, A is non-empty. Then there exists a minimizer of L in A*

- **Existence via PDE**

  In coordinates, we have that

  $$\vec{x}(t) = (x_1(t), \ldots, x_n(t))$$

  and the geodesic equation becomes

  $$\nabla_{\dot{x}}\dot{x} = \ddot{x}_k + \Gamma_{i,j}\dot{x}_i\dot{x}_j = 0$$

  This is a second order PDE so existence and uniqueness gives this the existence of a geodesic locally through a point (with some prescribed velocity vector of course). You can also see here. (Petersen also has stuff on this)

2.2.2 Hopf-Rinow

Statement (See Petersen)

Theorem 2.3. *TFAE*

1. *M is geodescially complete, i.e. all geodesics defined for all time*
2. *M is geodesically complete at some given p*
3. *M satisfies the Heine-Borel property, i.e. all closed bounded sets are compact*
4. *M is metrically complete*

Proof: $1 \implies 2$ is trivial, $3 \implies 4$ is easy since any given convergent sequence is bounded. Take its closure and you'll get a convergent sequence to a limit point (maybe we assume that M is topologically closed from the start, being either a closed manifold or manifold with boundary). $4 \implies 1$, suppose that 1 was not true. The claim is that $\gamma : [0, b) \to M$ must leave every compact set (for if it stayed in some compact subset of M, it could be extended by local existence of a geodesic, given the point and velocity). But by taking $\{\gamma(b - \epsilon_n)\}$, we generate a cauchy sequence of points (cauchy since γ is unit speed and hence distance isn't blowing up!) with no limit point, a contradiction to metric completeness.

For $2 \implies 3$, the idea that a set, K being bounded means that its a finite distance, D from our point p (due to triangle inequality and geodesic completeness), so it suffices to show that K is contained in a larger compact subset (since closed subset of compact is compact). But the exponential map is a diffeomorphism so

$$\exp_p(\overline{B}(0, D)) = S$$

is a closed compact set and $S \supset K$ finishing the proof (Petersen shows that $\exp(\overline{B}(0, r)) = \overline{B}_M(p, r)$ for all r, which takes a bit). This finishes the chain of implications

2.2.3 Cut-Locus

(See Petersen)

- A curve σ is a segment from $p \to q$ is $\ell(\sigma) = d(p, q)$
- We define the segment domain:

$$\operatorname{seg}(p) = \{v \in T_pM \ : \ \exp_p(tv) : [0, 1] \to M \ \text{ is a segment}\}$$

  Hopf-Rinow implies that when M is complete $M = \exp_p(\operatorname{seg}(p))$. Moreover $\operatorname{seg}(p)$ is a closed star-shaped domain

- We define the "interior" of $\operatorname{seg}(p)$ as

$$\operatorname{seg}^0(p) = \{sv \ : \ s \in [0, 1), \ v \in \operatorname{seg}(p)\}$$

- In Petersen, the cut locus is defined as $seg(p) - seg^o(p)$, which is all points (really, all starting vectors v, with associated points $\gamma(1)$ of $\gamma(t) = \exp_p(tv)$) that can be reached via a distance minimizing path, but cannot be extended onward. I.e. $seg^o(p)$ is the interior of all points attainable from p via a geodesic, but because this is the interior, they can be extended a bit with still a distance minimizing curve. The boundary of $seg(p)$ is then all points such that a geodesic from $p \to z$ cannot be extended further. The classification of this is:

  Theorem 2.4. *If v is in the cut locus, $seg(p) - seg^o(p)$, then either*

  - *$\exists w \neq v$ such that $\exp_p(v) = \exp_p(w)$ or*
  - *$D \exp_p \Big|_v$ is singular*

  The idea is that being inside the cut locus is where $r(x) = d(x, p)$ is smooth

Proof: Let $\gamma(t) = \exp_p(tv)$. For $t > 1$, find length minimizing curves such that $\sigma_t(1) = \gamma(t)$ and $\sigma_t(0) = p$. Since $\gamma(t)$ is not a segment (i.e. length minimizer) the initial velocity can't match up, e.g. $\dot{\sigma}_t(0)$ is not proportional to $\dot{\gamma}(0)$. Take a sequence $t_n \downarrow 1$ and set

$$w = \lim_{n\to\infty} \dot{\sigma}_{t_n}(0)$$

Then $|w| = |\dot{\gamma}(0)|$ because this is simply the length from $p \to \gamma(1)$. If $w \neq \dot{\gamma}(0)$, we have the first case. If $w = \dot{\gamma}(0)$, then $D\exp_p$ is singular: if not $D\exp_p$ is non-singular, and hence an embedding near v (in particular an embedding for tv for t close to 1). But note that by definition of segment:

$$\exp_p(\dot{\sigma}_{t_n}(0)) = \exp_p(t_n\dot{\gamma}(0))$$

and injectivity of $\exp_p$ near $\dot{\gamma}(0) = v$ means that $\dot{\sigma}_{t_n} = t_n v$, a contradiction because then $\gamma(t)$ could've been extended $[0,1] \to [0,t_n]$ for some $t_n > 1$. $\square$

I guess the general idea is: go a bit past $\exp_p(v) = x$ and find a distance minimizer to points past $\gamma(1) = x$. Bring these points back close to x and get a w as the starting velocity from the limit of paths coming back to $x = \gamma(1)$. If $w \neq v$, then we're done. If $w = v$, and $D\exp_p$ is non-singular then taking that limit of paths past x and then back to it wasn't so bad and we could've extended $\gamma(t)$ a bit so $v \notin cut(p)$.

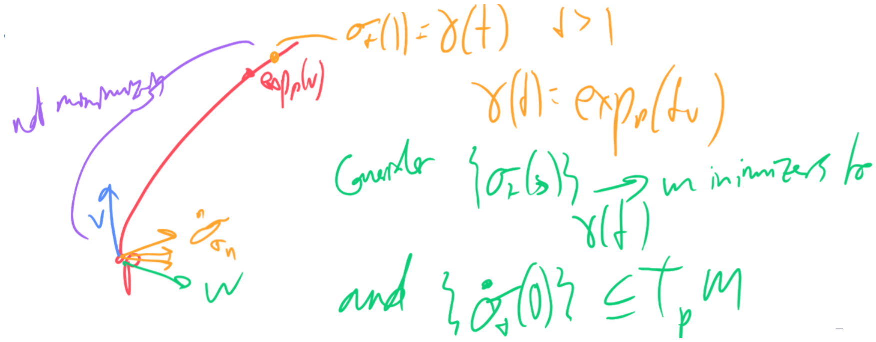

Figure 1

- We also have the injectivity radius which is the largest ϵ such that (See Petersen)

$$\exp_p : B(0,\epsilon) \to B(p,\epsilon)$$

is injective. The main lemma is

Lemma 2.5. *Suppose v in the cut locus and $|v| = inj(p)$, then either*

  - *There is precisely one other vector w with*

$$\exp_p(w) = \exp_p(v)$$

  and

$$\frac{d}{dt}\Big|_{t=1} \exp_p(tw) = -\frac{d}{dt}\Big|_{t=1} \exp_p(tv)$$

  or

  - *$x = \exp_p(v)$ is a critical value for $\exp_p : seg(p) \to M$ (i.e. some v' in the preimage of x under the exponential map has a singlar map $D\exp_p\Big|_{v'}$)*

The proof is that if we're at a regular value, then we find two paths to x. If they are not negatives of each other, then we find some w with $g(w, \dot{\gamma}_1(1)) < 0$, $g(w, \dot{\gamma}_2(1)) < 0$ and we can use w to find a point a little away from x that has two geodesics going to it. This is a contradiction to the injectivity of the exponential map on $seg^0(p)$.

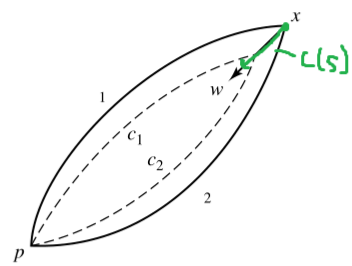

Figure 5.8

Figure 2

2.2.4 Jacobi Fields

(See Petersen)

- There are a few variations of this: the Jacobi field for the distance function r is given by
$$\nabla_{\partial_r}\nabla_{\partial_r}J = -R(J,\partial_r)\partial_r$$
This is a consequence of the first order jacobi equation
$$L_{\partial_r}J = 0$$
where $L.$ is the lie derivative. This essentially tells us that $[\partial_r, J] = 0$. These are important since they give vector fields which are in some sense independent along integral curves of the distance function.

- Another important Jacobi equation is that if we have
$$H : [0,1]^2 \to M, \qquad \boxed{H(s,t) =: \gamma(s,t)}$$
a one parameter family of geodesics. Then
$$0 = \frac{\partial^3\gamma}{\partial s\partial^2 t} \implies 0 = R(\partial_s\gamma, \partial_t\gamma)\partial_t\gamma + \partial_t^2\partial_s\gamma$$
(This comes from Synge's formula for the second variation of the energy of a geodesic) (remember that derivatives do not commute if the connection is non-trivial!). Setting $J = \frac{\partial\gamma}{\partial s}$ we see that
$$\ddot{J} + R(J,\dot{\gamma})\dot{\gamma} = 0$$
J is determined by $J(0)$ and $\dot{J}(0)$ given the above

- When $J(0) = 0$, we can construct
$$H(s,t) = \exp_p(t(\dot{\gamma}(0) + s\dot{J}(0)))$$
which by differentiating in s, we can check gives $J(0) = \frac{\partial\gamma}{\partial s}(0,0) = 0$

- The main ideas are that
$$J(t) = \frac{\partial\gamma}{\partial s}(0,t) = D\exp_p\Big|_{\dot{\gamma}(0)}(t\dot{J}(0))$$
so jacobi fields reflect the differential of the exponential map - in particular if $D\exp_p$ is nonsingular then we can construct a jacobi field with prescribed end point $J(1)$. In particular, note that we're evaluating at $t \neq 0$, i.e. we get the differential along the path away from the starting point!

- The other idea is that for L the length functional, and f the distance from an initial point p, one can show that
$$f(\gamma(s,1)) = L(\gamma_s) := \frac{1}{2}\left(\int_0^1 |\partial_t\overline{\gamma}(s,t)|\right)^2$$
$$df(J(s,1)) = g(J(s,1), \dot{\gamma}_s(1))$$
$$Hess\ f(J(1),J(1)) = g(\dot{J}(1), J(1))$$

2.3 Curvature

2.3.1 Myers Theorem

(See Otis' Notes/Peterson)

Theorem 2.6. *(M, g) a complete RM with*

$$Ric \geq (n-1)k > 0$$

then $diam(M, g) \leq \pi/\sqrt{k}$, furthermore M has finite fundmantal group

Proof: The idea is that we can create variations along any geodesic $\gamma : [0, \ell] \to M$. Write it as

$$V_i(t) = \sin(\pi t/\ell)E_i(t), \qquad i = 2, \ldots, n$$

where $\{\dot{\gamma}, E_2, \ldots, E_n\}$ from an ONB along $T_{\gamma(t)}M$. Recall the energy functional

$$\int g(\dot{\gamma}, \dot{\gamma})$$

Geodesics are stationary points of energy since they minimize. Thus we can use this in contrast to the following computation:

$$\begin{aligned}
\sum_{i=2}^{n} \frac{d^2E}{ds^2}(V_i) &= \sum_{2}^{n} \int_0^l |\dot{V}_i|^2 dt - \int_0^l R(V_i, \dot{\gamma}, \dot{\gamma}, V_i) \\
&= (n-1)\left(\frac{\pi}{l}\right)^2 \int_0^l \cos^2\left(\frac{\pi}{l}t\right) - \sum_{i=2}^{n} \int_0^l \sin^2\left(\frac{\pi}{l}t\right) \sec(E_i, \dot{\gamma}) \\
&= (n-1)\left(\frac{\pi}{l}\right)^2 \int_0^l \cos^2\left(\frac{\pi}{l}t\right) dt - \int_0^l \sin^2(\pi t/l)\mathrm{Ric}(\dot{\gamma}, \dot{\gamma}) \\
&< 0
\end{aligned}$$

So one of the variations must have had $\frac{d^2E}{ds^2}(V_i) < 0$, which is a contradiction to stability of a geodesic. Here, sec denotes sectional curvature. Recall that the second variation formula is

$$\int_0^l [g(\partial_{st}\gamma, \partial_{st}\gamma) - R(\partial_s\gamma, \partial_t\gamma, \partial_t\gamma, \partial_s\gamma)]dt + g(\partial_s^2\gamma, \partial_t\gamma)\Big|_0^l$$

(note that the curvature term comes from differentiating γ enough times and knowing that

$$\partial_s\partial_t\gamma = \nabla_{\gamma_*(\partial_s)}\nabla_{\gamma_*(\partial_t)}\gamma$$

and then commuting terms at the cost of a curvature term)

Main Idea: Construct a specific variation

$$V_i(t) = \sin(\pi t/\ell)E_i(t)$$

where $E_i(t)$ is a basis along $\gamma(t)$ complementing $\gamma'(t)$. Note that by summing over a basis, we can go from Riemann curvature to Ricci. We also use the formula for the second variation of energy

2.3.2 Bishop-Gromov inequality

- Mostly Peterson p. 265, Schoen-Yau is also okay
- Decompose the volume form into polar coordinates (after pulling back via the exponential map)

  $$dVol = \lambda(r, \vec{\theta})dr \wedge dvol_{n-1}$$

  Then we have that

  $$\partial_r\lambda = \lambda(\Delta r)$$

  For constant curvature k, we have that $\lambda_k = sn_k^{n-1}(r)$, where sn_k is a solution to

  $$\ddot{\varphi} + k\varphi = 0$$

  for k positive or negative (e.g. sin or sinh).

- We have the following lemma

Lemma 2.7. *Suppose that (M, g) has $Ric \geq (n-1) \cdot k$ for $k \in \mathbb{R}$, then*

$$\Delta r \leq (n-1)\frac{sn_k'(r)}{sn_k(r)}$$
$$dvol \leq dvol_k$$

Proof: I take the following proof from Lee: we know that

$$\partial_r(\Delta r) + \frac{(\Delta r)^2}{n-1} \leq -(n-1)k$$

This is the traced Riccati equation (see Lee section) (The original riccati equation is

$$D_t \mathcal{H}_r + \mathcal{H}_r + R_{\gamma'} = 0$$

where γ is a unit speed geodesic along a path and $\mathcal{H}_r(w) = \nabla_w \partial_r$ so that

$$g(\mathcal{H}_r(w), v) = (\nabla^2 r)(w, v)$$

the proof is to evaluate this on both ∂_r and vectors in the tangent space of $r = r_0$ along some radial geodesic.) Dividing by $n-1$ this gives

$$\partial_r\left(\frac{\Delta r}{n-1}\right) + \left(\frac{\Delta r}{n-1}\right)^2 \leq -k = \partial_r(\lambda_k) + \lambda_k^2$$

immediately, we have

$$\frac{\partial_r\left(\frac{\Delta r}{n-1}\right)}{k + \left(\frac{\Delta r}{n-1}\right)^2} \leq \frac{\partial_r \lambda_k}{k + \lambda_k^2}$$

to see this, note that the RHS is -1, and then we can clear denominators and get the Riccati bound. Thus

$$F(\lambda(r)) \leq F(\lambda_k(r)) + C$$

where $\lambda(r) = \Delta r/(n-1)$ and $F(x) = \frac{1}{k}\tanh(x/k)$. Since tanh is increasing, we want to conclude that $\lambda(r) \leq \lambda_k(r)$, but we have to ground this inequality at a point. Note that

$$\Delta r = \partial_r \log(r^{n-1}\sqrt{\det g})$$

This is Lee (Lemma 11.13), and follows from computing the laplacian in geodesic normal coordinates. In particular, if we write $\sqrt{\det g} = c_0 + O(r)$, then

$$\Delta r = \frac{(n-1)}{r} + \frac{O(1)}{1+r}$$

as $r \to 0$. As a result, we see that Δr diverges like $(n-1)/r$ for r small. This holds for any Riemannian manifold, so we see that as $r \to 0$, $(\Delta r)/(n-1) \to \infty$, so that $F(\lambda(r)) \to \frac{1}{k}$. Thus, both $F(\lambda(r))$ and $F(\lambda_k(r))$ will agree as $r \to 0$, i.e. $\lambda(r), \lambda_k(r) \to +\infty$. As a result, we can choose $C = 0$, and get

$$F(\lambda(r)) \leq F(\lambda_k(r))$$

and using monotonicity of F, we have

$$\lambda(r) \leq \lambda_k(r)$$

For the second inequality, we want to show that if

$$dVol = \lambda(r, \theta) dr \wedge d\theta$$

then

$$\partial_r \lambda = \lambda \Delta r$$

Here, we've decomposed the metric into fermi geodesic normal coordinates with r the radial coordinate and $\nabla_{\partial r}\theta_i = 0$, i.e. these are coordinates given by the exponential map pulled back to polar coordinates. Then $\lambda = \sqrt{\det g}$ in the usual representation of $dVol = \sqrt{\det g} dx^1 \wedge \cdots dx^n$. We have

$$\begin{aligned}
\Delta r &= \frac{1}{\lambda}\partial_\alpha(g^{\alpha\beta}\lambda\partial_\beta(r)) \\
&= \frac{1}{\lambda}\partial_\alpha(g^{\alpha r}\lambda \cdot 1) \\
&= \frac{1}{\lambda}\partial_r(g^{rr}\lambda) \\
&= \frac{1}{\lambda}\lambda_r
\end{aligned}$$

having used that $g^{r\theta_i} = 0$ and $g^{rr} \equiv 1$. With this, we know that

$$\partial_r \lambda = (\Delta r)\lambda \leq (n-1)\frac{sn_k'(r)}{sn_k(r)}\lambda$$

while

$$\partial_r \lambda_k = (n-1)\frac{sn_k'(r)}{sn_k(r)}\lambda_k$$

just by computation of the metric on the appropriate sized sphere/euclidean space/hyperbolic space. Moreover by choice of normal coordinates, we know that $g = g_k$ at p because everything is orthonormal there. Therefore, the volume form agrees at p, being the determinant of the metric. Thus

$$\lim_{r\to 0}(\lambda - \lambda_k) = 0$$
$$\partial_r(\lambda - \lambda_k) \leq (n-1)\frac{sn_k'(r)}{sn_k(r)}(\lambda - \lambda_k)$$

and so we see that $\lambda \leq \lambda_k$ for all r, meaning that $dVol_g \leq dVol_{g_k}$

- Comparison theorem:

Theorem 2.8 (Bishop-Gromov Volume Inequality). *Suppose (M, g) complete RM with $Ric \geq (n-1)\cdot k$. Then*

$$r \mapsto \frac{vol(B(p, r))}{v(n, k, r)}$$

is non-increasing in r with limit 1 as $r \to 0$

Proof: Move to exponential coordinates where

$$dVol_M = \lambda dr \wedge d\theta$$

and λ_k is the prefactor in the model case of $\text{Ric} = (n-1)k$. Now compute

$$\frac{vol(B(p, R))}{v(n, k, R)} = \frac{\int_0^R \int_{S^{n-1}} \lambda dr \wedge d\theta}{\int_0^R \int_{S^{n-1}} \lambda_k dr \wedge d\theta}$$

differentiate wrt R and get

$$\frac{d}{dR}\left(\frac{vol(B(p, R))}{v(n, k, R)}\right) =$$

$$\frac{1}{v(n,k,R)^2}\left[\int_{S^{n-1}} \lambda(R,\theta) \cdot \int_0^R \int_{S^{n-1}} \lambda_k(r,\theta) dr \wedge d\theta - \int_{S^{n-1}} \lambda_k(R,\theta) \cdot \int_0^R \int_{S^{n-1}} \lambda(r,\theta) dr \wedge d\theta\right]$$
$$= \frac{1}{v(n,k,R)^2}\left[\int_0^R \left(\int_{S^{n-1}} \lambda(R,\theta) \cdot \int_{S^{n-1}} \lambda_k(r,\theta)\right) dr - \int_0^R \left(\int_{S^{n-1}} \lambda_k(R,\theta) \cdot \int_{S^{n-1}} \lambda(r,\theta)\right) dr\right]$$

so to show non-increasing, it **suffices** to check that the integrand is non-increasing, i.e. it suffices to check

$$\frac{\int_{S^{n-1}} \lambda(r,\theta)d\theta}{\int_{S^{n-1}} \lambda_k(r,\theta)d\theta} = \frac{1}{\omega_{n-1}} \int_{S^{n-1}} \frac{\lambda(r,\theta)}{\lambda_k(r,\theta)} d\theta$$

is non-increasing in r (note equality holds since $\lambda_k(r,\theta)$ is independent of θ). Differentiate λ/λ_k and we get this result via

$$\begin{aligned}\partial_r \left(\frac{\lambda}{\lambda_k}\right) &= \frac{\lambda_k \partial_r \lambda - \lambda \partial_r \lambda_k}{\lambda_k^2} \\ &\leq \frac{\lambda_k (n-1) \frac{sn_k'}{sn_k} \lambda - \lambda (n-1) \frac{sn_k'}{sn_k} \lambda_k}{\lambda_k^2} \\ &= 0\end{aligned}$$

which follows from previous lemmas about λ_k. $\square$

2.3.3 Bishop-Gromov V.2

- Many theorems, e.g. the Hessian of the distance function, Jacobi Field, Metric comparison - all depend on sectional curvature bounds, but can't be upgraded to holding for Ricci bounds
- Surprisingly, can get bounds on laplacian of distance function, conjugate point theorem, and volume comparison
- Riccati equation (To be done)
- Riccati Comparison theorem
- Laplacian Comparison

Theorem 2.9 (11.15)**.** *Let (M,g) a Riemannian manifold and suppose there is c such that $Ric_M(v,v) \geq (n-1)c$ for any unit vector v. Given $p \in M$, let U be a normal neighborhood of p and let r the radial distance function on U. Then the following holds:*

$$\Delta r \leq (n-1)\frac{s_c'(r)}{s_c(r)}$$

where

$$s_c(t) = \begin{cases} t & c = 0 \\ R\sin(t/R) & c = \frac{1}{R^2} > 0 \\ R\sinh(t/R) & c = -\frac{1}{R^2} < 0 \end{cases}$$

and $U_0 = U$ if $c \leq 0$, while $U_0 = \{q \in U \ : \ r(q) < \pi R\}$ if $c = \frac{1}{R^2} > 0$

Proof: Let $q \in U_0 \backslash \{p\}$ arbitrary. Consider $\gamma : [0,b] \to U_0$ unit speed raidal geodesic from $p \to q$. We show that bound at $\gamma(t)$ for all $0 \leq t \leq b$. Recall the Riccati for H_r (hessian of r)

$$D_t H_r + H_r^2 + R_{\gamma'} = 0$$

where H_r^2 and $R_{\gamma'}$ are endormorphisms thought of as $H_r^2(w) = H_r(H_r(w))$ and $R_{\gamma'}(w) = R(w,\gamma')\gamma'$ using duality. Note also (11.2) that

$$H_r(w) := \nabla_w \partial_r \implies g(H_r(v),w) = (\nabla^2 r)(v,w)$$

Then by taking trace (which commutes with covariant derivatives)

$$\frac{d}{dt}\Delta r + \mathrm{tr}(H_r^2) + \mathrm{tr}(R_{\gamma'}) = 0$$

In an ONF we have

$$\mathrm{tr} R_{\gamma'} = g^{ii} g(R_{\gamma'}(E_i), E_i) = g^{ii} R(E_i, \gamma', \gamma', E_i) = \mathrm{Ric}(\gamma', \gamma')$$

For the H_r^2 term, write

$$\mathring{H}_r = H_r - \frac{\Delta_r}{n-1}\pi_r$$

where π_r is the projection onto the level set of r and the above is traceless. Then

$$\text{tr}(\mathring{H}_r^2) = \text{tr}(H_r^2) - \frac{\Delta r}{n-1}[\text{tr}(H_r \circ \pi_r) + \text{tr}(\pi_r \circ H_r)] + \frac{(\Delta r)^2}{(n-1)^2}\text{tr}(\pi_r^2)$$

Now note that

$$\text{tr}(\pi_r^2) = \text{tr}(\pi_r) = n-1$$

and also $H_r(\partial_r) = \nabla_{\partial_r}\partial_r = 0$ and so $H_r \circ \pi_r = H_r$ and by definition of H_r/Gauss Lemma, we also have $g(H_r(w), \partial_r) = 0$ so $\pi_r \circ H_r = H_r$. Thus, we have

$$\text{tr}(\mathring{H}_r^2) = \text{tr}(H_r^2) - \frac{(\Delta r)^2}{n-1}$$

Which at the original equation for Δr gives

$$\frac{d}{dt}\left(\frac{\Delta r}{n-1}\right) + \left(\frac{\Delta r}{n-1}\right)^2 + \frac{\text{tr}(\mathring{H}_r^2) + \text{Ric}(\gamma', \gamma')}{n-1} = 0$$

Now let $H(t) = s_c'(t)/s_c(t)$ so that

$$H'(t) + H(t)^2 + c = 0$$

for any c. Now by a **Riccati Comparison Theorem**, we have that for

$$\tilde{S}(t) = \frac{\text{tr}(\mathring{H}_r^2) + \text{Ric}(\gamma', \gamma')}{n-1} \geq c$$

that we can conclude

$$\tilde{H}(t) = \frac{\Delta r}{n-1} \leq H(t) = \frac{s_c'(t)}{s_c(t)}$$

all along $\gamma(t)$.

- Conjugate Point Comparison

Theorem 2.10 (11.16). *Let (M, g) a Riemannian n-manifold and suppose there is $c = \frac{1}{R^2}$ such that $Ric(v, v) \geq (n-1)c$ for all unit v. Then evey geodesic segment of length at least πR has a conjugate point*

Proof: Let U normal neighborhood. Laplacian comparison shows

$$\partial_r \log(r^{n-1}\sqrt{\det g}) = \Delta r \leq (n-1)\frac{s_c'(r)}{s_c(r)} = \partial_r \log(s_c(r)^{n-1})$$

where $r < \pi R$. Since $r^{n-1}\sqrt{\det g}/s_c(r)^{n-1} \to 0$ as $r \downarrow 0$, we get that

$$\frac{r^{n-1}\sqrt{\det g}}{s_c(r)^{n-1}} \leq 1, \qquad \sqrt{\det g} \leq \frac{s_c(r)^{n-1}}{r^{n-1}}$$

Suppose U has q where $r \geq \pi R$ and let $\gamma : [0, b] \to U$ be the unit speed radial geodesic there. Because $s_c(\pi R) = 0$, the above shows that

$$\det g(\gamma(t)) \to 0$$

as $t \uparrow \pi R$, so by continuity $\det g = 0$ at $\gamma(b)$ a contradiction. I.e. no normal neighborhood can have point where $r \geq \pi R$.

Now suppose $\gamma : [0, b] \to M$ a unit speed geodesic with $b \geq \pi R$ and assumse for contradiction that γ has no conjugate points. Let $p = \gamma(0)$, $v = \gamma'(0)$ and $\gamma(t) = \exp_p(tv)$. Because γ has no conjugate points, we can find $W \subset T_P M$ containing

$$L = \{tv \ : \ 0 \leq t < b_0\} \subseteq T_p M$$

on which $\exp_p$ is a local diffeo. Let $\tilde{g}$ be the puled-back metric on $\exp_p^* g$ on W, which as the same curvature estimates as g. Then $\tilde{g}(t) = tv$ is a radial $\tilde{g}$-geodesic in W of length greater than or equal to πR, a contradiction to the preceeding paragraph. $\square$

- Injectivity radius comparison:

  Corollary 2.10.1 (11.17)**.** *Let (M,g) an n-manifold and suppose $c = 1/R^2$ such that $Ric \geq (n-1)c$. Then for every point $p \in M$, we have inj$(p) \leq \pi R$.*

  Proof: Follows from the previous theorem, since every radial geodesic segment in a geodesic ball is minimizing, but the previous t heorem shows that no geodesic segment of length πR or greater is minimizing. Thus no geodesic ball has radius greater than πR

- Bishop-Gromov

  Theorem 2.11. *Let (M,g) an n-manifold and suppose $Ric \geq (n-1)c$. Let $p \in M$ given and for every $\delta > 0$, let $V_g(\delta)$ denote the volume of $B_\delta(p)$ in (M,g). Let $V_c(\delta) = Vol(B_\delta(0), g_c)$, where g_c is the metric on euclidean space, hyperbolic space, or sphere with constant sectional curvature c (euclidean for $c = 0$, hyperbolic for $c < 0$ and sphere for $c > 0$). Then for all $0 \leq \delta \leq$ inj(p), we have*

$$V_g(\delta) \leq V_c(\delta)$$

  furthermore

$$F(\delta) = \frac{V_g(\delta)}{V_c(\delta)}$$

  is a non-increasing function of δ such that $\lim_{\delta \to 0^+} F(\delta) = 1$. If (M,g) complete, the above is true for all $\delta > 0$, not just $\delta \leq$ inj(p). If $V_g(\delta) = V_c(\delta)$, then g has constant sectional curvature on the entire metric ball $B_\delta(p)$

  Proof: Just do the petersen proof

2.3.4 Bochner Formula

- Really not that bad - the formula is

$$\frac{1}{2}\Delta|\nabla u|^2 = |D^2u|^2 + \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u)$$

- To prove this, we compute

$$\Delta \nabla u$$

  in an ONB,

$$\begin{aligned}
\Delta \nabla u &= g(\nabla_{E_i}\nabla_{E_i}\nabla u, E_j)E_j \\
g(\nabla_{E_i}\nabla_{E_i}\nabla u, E_j) &= g(\nabla_{E_i}\nabla_{E_j}\nabla u, E_i) \\
&= R(E_i, E_j, \nabla u, E_i) + g(\nabla_{E_j}\nabla_{E_i}\nabla u, E_i) \\
&= R(E_i, E_j, \nabla u, E_i) + E_j g(\nabla_i \nabla u, E_i)
\end{aligned}$$

  since $\nabla_{E_j} E_i = 0$ at a given point. In the second line, we used that

$$\begin{aligned}
g(\nabla_{E_i}\nabla_{E_i}\nabla u, E_j) &= E_i g(\nabla_{E_i}\nabla u, E_j) \\
&= E_i g(\nabla_{E_j}\nabla u, E_i) \\
&= g(\nabla_{E_i}\nabla_{E_j}\nabla u, E_i)
\end{aligned}$$

  since the hessian of a function is symmetry, i.e.

$$g(\nabla_V \nabla u, W) = Hess\ u(V, W) = Hess\ u(W, V)$$

  (This is the main trick!) Tracing over i, we get

$$\mathrm{Ric}(E_j, \nabla u) + E_j \Delta u$$

  and so

$$\Delta \nabla u = \nabla \Delta u + \mathrm{Ric}(\nabla u, \cdot)$$

  And so

$$\begin{aligned}
\frac{1}{2}\Delta|\nabla u|^2 &= \frac{1}{2}\Delta\langle \nabla u, \nabla u\rangle \\
&= \frac{1}{2}\langle \Delta \nabla u, \nabla u\rangle + \langle D^2u, D^2u\rangle \\
&= \langle \nabla \Delta u, \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u) + |D^2u|^2
\end{aligned}$$

2.3.5 Comparison theorem for distance functions

- Source: Schoen-Yau
- Tbh probably should've been up earlier in this
- I want to prove two theorems
- Laplacian Comparison

  Theorem 2.12. *Let (M, g) with $Ric \geq -(n-1)k^2$ and n dimensional manifold ($k \geq 0$). Let N be the n-dimensional simply connected space of constant sectional curvature $-k^2$. Let ρ_M and ρ_N be distance functions on M and N (with respect to some fixed points on M and N respectively). If $x \in M$ and ρ_M differentiable at x, then for any $y \in N$ with $\rho_N(y) = \rho_M(x)$, we have*

  $$\Delta\rho_M(x) \leq \Delta\rho_N(y)$$

  The proof of this is the Hessian comparison theorem

- Hessian comparison

  Theorem 2.13. *Let M_1, M_2 two n-dim complete RMs. Assume $\gamma_i : [0, a] \to M_i$ are two geodesics parameterized by arc length and γ_i does not intersect cut locus of $\gamma_i(0)$. Let ρ_i the distance function from $\gamma_i(0)$ on M_i and K_i the secontional curvature of M_i. **Assume** that at $\gamma_1(t)$ and $\gamma_2(t)$ for $0 \leq t \leq a$, we have*

  $$K_1\left(X_1, \frac{\partial}{\partial\gamma_1}\right) \geq K_2\left(X_2, \frac{\partial}{\partial\gamma_2}\right)$$

  for X_i any unit vector in $T_{\gamma_i}M_i$ perpendicauly to $\frac{\partial}{\partial\gamma_i}$. Then

  $$H(\rho_1)(X_1, X_1) \leq H(\rho_2)(X_2, X_2)$$

  where $X_i \in T_{\gamma_i(a)}M_i$ is unit norm and perpendicalur at a, e.g.

  $$\langle X_i, \frac{\partial}{\partial\gamma_i}\rangle(\gamma_i(a)) = 0$$

  Proof: Let $\{E_k^i\}$ an ONB of vector fields parallel along γ_i with $E_n^i = \partial/\partial\gamma_i$. Then

  $$H(\rho_i)(X_i, X_i) = \int_0^a [\left|\frac{\partial}{\partial\gamma_i}\tilde{X}_i\right|^2 - R_i(\tilde{X}_i, \frac{\partial}{\partial\gamma_i}, \frac{\partial}{\partial\gamma_i}, \tilde{X}_i)]dt$$

  where $\tilde{X}_i$ is a jacobi field along γ_i with $\tilde{X}_i(\gamma_i(0)) = 0$ and $\tilde{X}_i(\gamma_i(a)) = X_i$. Since

  $$g(X_i, \frac{\partial}{\partial\gamma_i}) = 0 \implies g(\tilde{X}_i, E_n^i) = 0 \qquad \forall p \in \gamma_i$$

  Set

  $$\tilde{X}_2 = \sum_{j=1}^{n-1}\lambda_j(t)E_j^2$$

  Now choose $\{E_j^1\}$ so that

  $$X_1 = \tilde{X}_1(a) = \sum_{j=1}^{n-1}\lambda_j(a)E_j^1(\gamma_1(a))$$

  (note this is possible since all of X_1, $\tilde{X}_1$, X_2, and $\tilde{X}_2$ are all unit norm). Define

  $$Z(t) = \sum_{j=1}^{n-1}\lambda_j(t)E_j^1$$

then Z has the same value as $\tilde{X}_1$ at $t = 0$ and $t = a$ and

$$|\nabla_{\partial/\partial\gamma_2}\tilde{X}_2| = |\sum_j \lambda_j'(t)E_j^2| = |\sum_j \lambda_j'(t)E_j^1| = |\nabla_{\partial/\partial\gamma_1}X|, \qquad |Z| = |\tilde{X}_2|$$

Since **Jacobi fields minimize the Index form among all vector fields with the same boundary values**, (by integration by parts, this is essentially just plugging in the Jacobi equation, see here for more), we get

$$\begin{aligned} H(\rho_1)(X_1, X_1) &= I_0^a(\tilde{X}_1) \le I_0^a(Z) \\ &= \int_0^a \left|\nabla_{\partial/\partial\gamma_1} Z\right|^2 - R(Z, \frac{\partial}{\partial\gamma_1}, \frac{\partial}{\partial\gamma_1}, Z) \\ &= \int_0^a \left|\nabla_{\partial/\partial\gamma_2}\tilde{X}_2\right|^2 - K_1(Z, \frac{\partial}{\partial\gamma_1}) \\ &\le \int_0^a \left|\nabla_{\partial/\partial\gamma_2}\tilde{X}_2\right|^2 - K_2(\tilde{X}_2, \frac{\partial}{\partial\gamma_1}) \\ &= I_0^a(\tilde{X}_2) = H(\rho_2)(X_2, X_2) \end{aligned}$$

□

- I guess the idea of this proof is: Extend X_1 and X_2 to Jacobi fields, and simultaneously, find a parallel ONF on $\gamma_1(t), \gamma_2(t)$. Compare the extensions $\tilde{X}_1$, $\tilde{X}_2$, by using the coefficients from $\tilde{X}_2$ w.r.t. E_j^2 and attach to E_j^1. Now use the condition of being a Jacobi field to say

$$H(\rho_1)(X_1, X_1) = I_0^a(\tilde{X}_1) \le I_0^a(Z) \le I_0^a(\tilde{X}_2) = H(\rho_2)(X_2, X_2)$$

- Proof of Index form: for $p \in M$ let σ a minimal geodsic joining p and x. Let $X \in T_xM$ such that $g(\partial_r, X) = 0$. Extend $X \to \tilde{X}$ a Jacobi field along σ so that $\tilde{X}(\sigma(0)) = 0$, $\tilde{X}(\sigma(r)) = X$ and $[\tilde{X}, \partial_r] = 0$. Then

$$\begin{aligned} H(\rho)(X, X) &= \tilde{X}\tilde{X}\rho - (\nabla_{\tilde{X}}\tilde{X})\rho \\ &= \tilde{X}g(\tilde{X}, \partial_r) - g(\nabla_{\tilde{X}}\tilde{X}, \partial_r) \\ &= g(\tilde{X}, \nabla_{\tilde{X}}\partial_r) \\ &= (\tilde{X}, \nabla_{\partial_r}\tilde{X}) \end{aligned}$$

Having used the commutator relation for the last. But then

$$\begin{aligned} H(\rho)(X, X) &= \int_0^r \frac{d}{dt} g(\tilde{X}, \nabla_{\partial_r}\tilde{X})dt \\ &= \int_0^r |\nabla_{\partial_r}\tilde{X}|^2 + g(\tilde{X}, \nabla_{\partial_r}\nabla_{\partial_r}\tilde{X})dt \\ &= \int_0^r |\nabla_{\partial_r}\tilde{X}|^2 + g(R(\tilde{X}, \partial_r)\partial_r, \tilde{X})dt \end{aligned}$$

having used the Jacobi equation in the last line.

- Now the laplacian comparison theorem follows from tracing the hessian comparison
- We can use the above idea: e.g. Jacobi fields tell us about the Hessian of the distance function, to compute $\Delta\rho$: For any $X \in \gamma(\rho)$, we can take the parallel extension along $\gamma : [0, \rho] \to M$. Then the Jacobi field with the boundary conditions $Y(0) = 0$ and $Y(\rho) = X$ will be

$$Y(t) = sn_k(t)X(t)$$

Using this, we compute

$$\Delta\rho = \sum_{i=1}^{n-1} H(\rho)(X_i, X_i) = (n-1)k \coth k\rho$$

- As a corollary

Theorem 2.14. *Let M an n-dimensional complete RM with $Ric(M) \geq -(n-1)k^2$ and ρ any distance funtion on M. Then*

$$\Delta\rho \leq \frac{n-1}{\rho}(1+k\rho)$$

wherever ρ is smooth

- This follows because

  $$\Delta\rho \leq (n-1)k\coth k\rho$$

  and

  $$k\rho\coth k\rho \leq 1+k\rho$$

2.4 Submanifolds

2.4.1 Gauss-Codazzi Equations

(Lee Intro to Riemannian Manifolds, chapter 8)

- So the Gauss formula is just the splitting of the connection, e.g. if we have $(M,g) \hookrightarrow (\tilde{M},\tilde{g})$, then

  $$\tilde{\nabla}_X Y = \nabla_X Y + A(X,Y)$$

  This isn't exactly true, because we would need to show that

  $$\nabla_X Y = (\tilde{\nabla}_X Y)^{\|}$$

  where ∇ is the Levi-Civita connection on M as a submanifold with the induced metric. To show this agreement, we can actually show that $\tilde{\nabla}^{\|}$ is **symmetric**, i.e.

  $$\tilde{\nabla}^{\|}_X Y - \tilde{\nabla}^{\|}_Y X = [X,Y]$$

  and compatible with the metric, i.e

  $$Xg(Y,Z) = g((\tilde{\nabla}_X Y)^{\|}, Z) + g((\tilde{\nabla}_X Z)^{\|}, Y)$$

  We'll prove this latter fact. We know that for $X,Y,Z \in TM$, we have

  $$\begin{aligned} Xg(Y,Z) = \tilde{X}\tilde{g}(\tilde{Y},\tilde{Z}) &= \tilde{g}(\tilde{\nabla}_{\tilde{X}}\tilde{Y},\tilde{Z}) + \tilde{g}(\tilde{X},\nabla_{\tilde{X}}\tilde{Z}) \\ &= \tilde{g}(\tilde{\nabla}_X\tilde{Y},\tilde{Z}) + \tilde{g}(\tilde{\nabla}_X\tilde{Z},\tilde{Y}) \end{aligned}$$

  since we're evaluating on M where $X \in TM$. But

  $$\tilde{g}(\tilde{\nabla}_X\tilde{Y},\tilde{Z}) = \tilde{g}((\tilde{\nabla}_X\tilde{Y})^{\perp},\tilde{Z}) + \tilde{g}((\tilde{\nabla}_X\tilde{Y})^{\|},\tilde{Z})$$

  but $\tilde{Z} = Z \in TM$ on M, so

  $$\tilde{g}(\tilde{\nabla}_X\tilde{Y},\tilde{Z}) = \tilde{g}((\tilde{\nabla}_X\tilde{Y})^{\|},Z) = g((\tilde{\nabla}_X\tilde{Y})^{\|},Z)$$

  but now

  $$(\tilde{\nabla}_X\tilde{Y})^{\|} = (\tilde{\nabla}_X Y)^{\|}$$

  because since $X \in TM$, we only need to know an extension of Y that lies in M, i.e. the extension to $N(M)$ doesn't matter so we can replace $\tilde{Y} \to Y$ when taking the connection. Thus

  $$\tilde{g}(\tilde{\nabla}_X\tilde{Y},\tilde{Z}) + \tilde{g}(\tilde{\nabla}_X\tilde{Z},\tilde{Y}) = g((\tilde{\nabla}_X Y)^{\|},Z) + g((\tilde{\nabla}_X Z)^{\|},Y)$$

- Also have the Weingarten map: for $N \in N(M)$ and $M \hookrightarrow \tilde{M}$, we define

  $$X,Y \in TM, \qquad g(W_N(X),Y) = A_N(X,Y) = g(N, II(X,Y))$$

  (remember N is arbitrary element of the normal bundle!

- We have the Weingarten equation

  $$(\tilde{\nabla}_X N)^{\|} = -W_N(X)$$

- And from this, we have the Gauss equation

  Theorem 2.15. *For $M \hookrightarrow \tilde{M}$ and $W, X, Y, Z \in TM$, we have*
  $$\tilde{R}(W, X, Y, Z) = R(W, X, Y, Z) - g(A(W, Z), A(X, Y)) + g(A(W, Y), A(X, Z))$$

  Proof: The proof for this is lengthy but straight forward: start with
  $$\tilde{R}(W, X, Y, Z) = g(\tilde{\nabla}_W \tilde{\nabla}_X Y, Z) - g(\tilde{\nabla}_X \tilde{\nabla}_W Y, Z) - g(\tilde{\nabla}_{[W,X]} Y, Z)$$
  Now decompose $\tilde{\nabla}_X Y = \nabla_X Y + A(X, Y)$ and note that
  $$g(\tilde{\nabla}_W II(X, Y), Z) = Wg(II(X, Y), Z) - g(II(X, Y), \tilde{\nabla}_W Z) = -g(II(X, Y), II(W, Z))$$
  and proceed

- We can also define a connection on the normal bundle for tangential vector fields, given by
  $$X \in TM, \qquad N \in NM$$
  $$\nabla_X^\perp N = (\tilde{\nabla}_X N)^\perp$$

- Now define $F \to M$ the bundle with fiber being bilinear maps $T_pM \times T_pM \to N_pM$, like the second fundamental form. Then we can define ∇^F on smooth sections of F (call it B), given by
  $$(\nabla_X^F B)(Y, Z) = \nabla_X^\perp (B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z)$$
  which is just what would happen from the product rule

- The Codazzi equations with the above, we have

  Theorem 2.16. *For $M \hookrightarrow \tilde{M}$, $W, X, Y \in TM$, we have*
  $$(\tilde{R}(W, X)Y)^\perp = (\nabla_W^F A)(X, Y) - (\nabla_X^F A)(W, Y)$$

  Proof: you just compute this using the definition of ∇^F. □

2.4.2 First and Second variation of area

(Colding Minicozzi)

- The first variation is relatively easy to compute. We use that
  $$\frac{d}{dt}\sqrt{\{(g_t)_{ij}\}}\Big|_{t=0} = \text{tr}_g(\dot{g})$$
  where
  $$(g_t)_{ij} = g((F_t)_*(\partial_i), (F_t)_*(\partial_j))$$
  then we have that
  $$g^{ij}\partial_t g((F_t)_*(\partial_i), (F_t)_*(\partial_j)) = \text{div}_\Sigma F_t$$
  we can decompose this as
  $$\text{div}_\Sigma F_t = \text{div}_\Sigma (F_t)^\perp + \text{div}_\Sigma (F_t)^\parallel$$
  When integrating the latter part, we hope that the manifold is closed so that stokes theorem gets rid of it. For the first term, we have
  $$\text{div}_\Sigma (F_t)^\perp = -\langle F_t, H\rangle$$
  by computing, so
  $$\begin{aligned} \nu(t) &= \sqrt{\det(g(E_i(t), E_j(t)))} \\ \implies \nu'(0) &= \text{tr}_g(\{\partial_t g(E_i(t), E_j(t))\}) \\ &= -\langle F_t, H\rangle + \text{div}_\Sigma F_t^\parallel \end{aligned}$$
  integrating (and assuming compact variation or no boundary of Σ) gives
  $$\boxed{\frac{d}{dt} Vol(F(\Sigma, t)) = -\int_\Sigma \langle F_t, H\rangle}$$

- For the second variation we need to use that

$$\nu(t) = \det(g_{ij}(t))$$
$$\implies \frac{d^2}{dt^2}\nu(t) = \frac{1}{2}\left[\mathrm{tr}(\ddot{g}) - \mathrm{tr}(\dot{g}^2) + \frac{1}{2}\mathrm{tr}(\dot{g}))^2\right]$$

for Σ minimal, the last term vanishes, so it suffices to compute the first two. We have

$$\begin{aligned}\ddot{g}_{ij} &= g(\nabla_{F_t}\nabla_{F_t}E_i, E_j) = g(\nabla_{F_t}\nabla_{E_i}F_t, E_j) + 2g(\nabla_{F_t}E_i, \nabla_{F_t}E_j) \\ &= R(F_t, E_i, F_t, E_j) + g(\nabla_{E_i}\nabla_{F_t}F_t, E_j) + g(\nabla_{F_t}E_i, \nabla_{F_t}E_j)\end{aligned}$$

tracing over, we get

$$\mathrm{tr}(\ddot{g}) = \mathrm{Ric}(F_t, F_t) + \mathrm{div}_\Sigma(F_{tt}) + 2|\langle A(\cdot,\cdot), F_t\rangle|^2 + 2|\nabla_\Sigma^N F_t|^2$$

We already know that

$$\dot{g}_{ij} = -2\langle A(E_i, E_j), F_t\rangle$$

and so

$$\mathrm{tr}(\dot{g}^2) = 4|\langle A(\cdot,\cdot), F_t\rangle|^2$$

and in sum, we have

$$\frac{d^2}{dt^2}\nu(t) = -|\langle A(\cdot,\cdot), F_t\rangle|^2 + |\nabla_\Sigma^N F_t|^2 - \mathrm{Ric}^\perp(F_t, F_t) + \mathrm{div}_\Sigma(F_{tt})$$

which is the desired result (Pretty sure I can rederive this on the spot

2.4.3 Gauss-Bonnet Theorem

(See Lee, chapter 9)

- Statement

Theorem 2.17. *Suppose (M, g) oriented, 2-dim RM. Suppose γ positively oriented curved polygon in M and Ω is the interior. Then*

$$\int_\Omega K dA + \int_\gamma \kappa_N ds + \sum_{i=1}^k \epsilon_i = 2\pi$$

where K is Gaussian curvature of g, dA volume form, ϵ_i are the exterior angles of γ, and κ_N is the geodesic curvature of γ

Proof: We know from **The rotation index theorem** (really work with closed curves, or like winding numbers but in the metric setting) that for a closed curve $\gamma : [a, b] \to M$, we have $\theta : [a, b] \to \mathbb{R}$ a tangent angle function and

$$2\pi = \theta(b) - \theta(a) = \sum_{i=1}^k \epsilon_i + \sum_{i=1}^k \int_{a_{i-1}}^{a_i} \theta'(t)dt$$

here, $\{a_i\}$ is some admissible partition of $[a, b]$ such that θ is C^1 on (a_i, a_{i+1}). Again, this is essentially the winding number theorem from complex analysis but with geometry.

From here, we want to show that

$$\sum_{i=1}^k \int_{a_{i-1}}^{a_i} \theta'(t)dt = \int_\Omega K dA + \int_\gamma \kappa_N d\gamma$$

Let $\{E_1, E_2\}$ positively oriented g-orthonormal frame in the interior and along γ (we get this by assuming that $\overline{\Omega}$ can be mapped into some euclidean chart, and then doing Graham-Schmidt there). Then

$$\gamma'(t) = \cos(\theta(t))E_1\Big|_{\gamma(t)} + \sin(\theta(t))E_2\Big|_{\gamma(t)}$$
$$N(t) = -\sin(\theta(t))E_1\Big|_{\gamma(t)} + \cos(\theta(t))E_2\Big|_{\gamma(t)}$$

Differentiating, we get

$$D_t\gamma' = \nabla_{\gamma'}\gamma' = \theta' N + \cos(\theta)\nabla_{\gamma'}E_1 + \sin(\theta)\nabla_{\gamma'}E_2$$

for $v \in T\Omega$, define

$$\omega(v) = g(E_1, \nabla_v E_2)$$

then one can show that

$$\nabla_v E_1 = -\omega(v)E_2, \qquad \nabla_v E_2 = \omega(v)E_1$$

Moreover by definition of geodesic curvature:

$$\kappa_N = (D_t\gamma', N) = \theta' - \omega(\gamma')$$

as one can check. And so

$$\sum_{i=1}^{k}\int_{a_{i-1}}^{a_i}\theta'(t)dt = \sum_{i=1}^{k}\int_{a_{i-1}}^{a_i}[\kappa_N(t) + \omega(\gamma')]dt$$

note that

$$\int_{a_{i-1}}^{a_i}\omega(\gamma')dt = \int_\gamma \omega$$

and so it suffices to show that

$$\int_\gamma \omega = \int_\Omega K dA$$

By stokes, it suffices to show that

$$d\omega = KdA$$

evaluate the right hand side on an ONB, we get

$$\begin{aligned} KdA(E_1, E_2) &= K = R(E_1, E_2, E_2, E_1) \\ &= (\text{work from computation and using definition of riemann curvature and } \omega) \\ &= E_1(\omega(E_2)) - E_2(\omega(E_1)) - \omega([E_1, E_2]) \\ &= d\omega(E_1, E_2) \end{aligned}$$

The last equality is just a theorem in differential forms/definition of the exterior derivative of a 1 form

2.5 Minimal Submanifolds

2.5.1 Definition

In aribtrary codimension, a Riemannian submanifold $(Y, h) \hookrightarrow (M, g)$ (for $h = g\Big|_Y$) is minimal if the mean curvature vector vanishes, e.g.

$$h^{ab}(\nabla^g_{v_a} v_b)^\perp = 0 \in N(Y)$$

2.5.2 Statement of regularity for $n < 8$

- Minimal hypersurfaces inside of $\mathbb{R}^n$ for $n < 8$ are smooth
- When $n = 8$, we know that the Simon's cone is minimal but has a singular point at the origin

  $$S = \{x \in \mathbb{R}^8 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}$$

  see here for a proof - there is no short proof for minimality it's a length computation so i'll omit
- A more beefed up version from Otis' notes

  Proposition. *If $E \subseteq (M^n, g)$ is a local minimizer of perimeter for $3 \le n \le 7$, then after changing E by a set of measure 0, the topological boundary of E, ∂E, is a smooth hypersurface*

2.5.3 Examples of Minimal Surfaces in $\mathbb{R}^3$

(see Colding-Minicozzi, know how to compute a few at least)

- Helicoid
$$(x, y, z) = (t\cos s, t\sin s, s)$$
it's complete, embedded, simply connected, singly-periodic. Also the only (nonflat) ruled sminimal surface, i.e. it can be written as
$$X(s,t) = \beta(t) + s\delta(t)$$

- Catenoid, given by
$$\{(x,y,z) \mid x_1^2 + x_2^2 = \cosh^2(z)\}$$
tbh, better to write it in polar coordinates as
$$r = \cosh(z)$$
because then we have
$$(\zeta, \tau) \mapsto (\cosh(\zeta), \tau, \zeta) = (r, \theta, z)$$
and so
$$F_\zeta = \sinh(\zeta)\partial_r + \partial_z$$
$$F_\tau = \partial_\tau$$
so that using the cylindrical coordinate metric of
$$g = dr^2 + r^2 d\theta + dz^2$$
we get that the induced metric is
$$h_{\zeta\zeta} = \sinh^2(\zeta) + 1 = \cosh^2(\zeta), \qquad h_{\zeta\tau} = 0, \qquad h_{\tau\tau} = r^2 = \cosh^2(\zeta)$$
one can further show by noting that the only non-trivial christoffels are
$$\nabla_{\partial_r}\partial_\theta = r^{-1}\partial_\theta, \qquad \nabla_{\partial_\theta}\partial_\theta = -\frac{1}{r}\partial_r$$
As always, this follows from the Koszul formula
$$g(\nabla_X Y, Z) = \frac{1}{2}\left[Xg(Y,Z) + Yg(X,Z) - Zg(X,Y) + (\text{commutators})\right]$$
that we get
$$F_{\zeta\zeta} = \cosh(\zeta)\partial_r$$
$$F_{\zeta\tau} = \nabla_{F_\zeta} F_\tau = \frac{\sinh(\zeta)}{r}\partial_\theta = \tanh(\zeta)\partial_\theta$$
$$F_{\tau\tau} = -r\partial_r = -\cosh(\zeta)\partial_r$$
Furthermore, we have
$$\nu = \frac{\partial_r - \sinh(\zeta)\partial_z}{\sqrt{1+\sinh^2}} = \operatorname{sech}(\zeta)\partial_r - \tanh(\zeta)\partial_z$$
and so the second fundamental form is given by
$$A_{\zeta\zeta} = 1, \quad A_{\zeta\tau} = 0, \quad A_{\tau\tau} = -1$$
which tells us that
$$h^{ab}A_{ab} = h^{\zeta\zeta}A_{\zeta\zeta} + h^{\tau\tau}A_{\tau\tau} = \operatorname{sech}^2 \cdot 1 - \operatorname{sech}^2 \cdot 1 = 0$$
showing minimality!

- Plane - proof: the second fundamental form is trivial since we can just take a coordinate parameterization for which the connection is trivial

2.5.4 Non-existence of STABLE minimal submanifolds with $\mathrm{Ric}_g \geq 0$

(this should be in Otis' allen-Cahn notes)

- This follows from the Jacobi operator, where

$$\frac{d^2}{dt^2}V(\Sigma_t) = \int_\Sigma \varphi J(\varphi)$$

where

$$J(\varphi) = -[\Delta_\Sigma + (|A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu))]\varphi$$

- Integration by parts gives

$$\frac{d^2}{dt^2}V(\Sigma_t) = \int_\Sigma |\nabla_\Sigma \varphi|^2 - [|A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu)]\varphi^2$$

now let $\varphi = 1$, then the above quantity is ≤ 0 when $\mathrm{Ric} \geq 0$. In fact, we must have $|A_\Sigma|^2 \equiv 0$, so either Σ was flat from the start, i.e. a plane. If $|A_\Sigma|^2$ doesn't vanish somewhere or $\mathrm{Ric}(\nu,\nu) > 0$, then we have that the above integral is < 0, a contradiction to stabiliy

2.5.5 Barta theorem on stable minimal hypersurfaces

- Statement

Proposition. *A 2 sided minimal hypersurface $\Sigma^n \to (M^{n+1}, g)$ is stable $\iff$ there is $u \in C^\infty(\Sigma\backslash\partial\Sigma)$ with $u > 0$ on $\Sigma\backslash\partial\Sigma$ so that $L_\Sigma u \leq 0$*

Here L_Σ is the negation of the Jacobi operator, i.e.

$$L_\Sigma = \Delta_\Sigma + |A_\Sigma|^2 + \mathrm{Ric}(\nu,\nu)$$

- **Proof:** ($\to$) Suppose Σ stable. If compact, then the first eigenfunction of L_Σ satisfies

$$L_\Sigma \varphi = (-\lambda)\varphi, \qquad \lambda \geq 0$$

The sign of λ comes from the stability condition and integration by parts. Moreover $\varphi > 0$, else $|\varphi|$ would also be an eigenfunction everywhere, contradicting uniqueness of the soltuion.

If Σ non-compact, choose $p \in \Omega_1 \subset \Omega \subset \cdots \subset \Sigma$, an exhaustian by comapct regions with smooth boundary. Let φ_i the first eigenfunction of $L_\Sigma\Big|_{\Omega_i}$ normalized so that $\varphi_i(p) = 1$. Because the domains are getting larger, we use the variational characterization of λ, i.e.

$$\lambda = \inf_{f \in C^\infty(\Sigma)\backslash\{0\} f\big|_{\partial\Sigma} = 0} \frac{\int_\Sigma |\nabla f|^2 - Vf^2}{\int_\Sigma f^2}$$

to see that

$$0 \leq \lambda(\Omega_{i+1}) \leq \lambda(\Omega_i)$$

This tells us that

$$\lambda_i \xrightarrow{i\to\infty} \lambda_* \geq 0$$

Moreover, locally we have

$$L_\Sigma \varphi_i + \lambda(\Omega_i)\varphi_i = 0$$

We now note that the Harnack inequality tells us that the value of φ_i at points away from p is connected to the distance to p, e.g.

$$\sup_K \varphi_i \leq C\varphi_i(p) = C$$

so for K compact, we get that $\varphi_i\Big|_K$ is bounded. Because the $\lambda(\Omega_i)$ is converging, we use schauder theory to get

$$||\varphi_i||_{C^{k,\alpha}(K)} \leq C$$

$$\implies \varphi_i \xrightarrow{C^\infty_{loc}} \varphi_*$$

where the second line is via arzela ascoli. This gives us that

$$L_\Sigma \varphi^* + \lambda^* \varphi^* = 0$$

everywhere and $\varphi^*(p) = 1$. Thus, the maximum principle tells us that $\varphi^* > 0$ on $\Sigma \backslash \partial\Sigma$. This finishes one direction of the proof
(Main idea: Use first eigenfunctions which have the right sign. Then use stability condition to figure out $L\varphi \leq 0$. If non-compact, use exhaustion by compact sets, along with harnack inequality and schauder theory and Arzela-Ascoli to get convergence in the limit)

- **Proof:** ($\leftarrow$) Now assume a positive function $u > 0$ with $L_\Sigma u \leq 0$. Then we want to show that $\lambda(\Omega) \geq 0$ for any $\Omega \subset\subset \Sigma\backslash\partial\Sigma$, as this will give us the stability condition by exhausting by compact sets (corresponding to only using function with comapct support.

  Let $w = \log(u)$, then

$$\nabla w = \frac{\nabla u}{u}$$
$$\Delta w = \frac{\Delta u}{u} - |\nabla w|^2 \leq -V - |\nabla w|^2$$

  where we've written

$$L_\Sigma = \Delta_\Sigma + V$$

  Take the above inequality and multiply by $f \in C_c^\infty(\Omega)$ and get

$$\begin{aligned}\int_\Omega V f^2 + |\nabla w|^2 f^2 &\leq \int_\Sigma -(\Delta w) f^2 \\ &= \int_\Sigma \langle \nabla w, \nabla f^2 \rangle \\ &\leq \int 2|f||\nabla f||\nabla w| \\ &\leq |\nabla w|^2 f^2 + |\nabla f|^2\end{aligned}$$

  Thus taking the top and bottom line, we get

$$\int |\nabla f|^2 - V f^2 \geq 0$$

  which is stability. (This feels like a trick of looking at $\log(u)$ the given function and then computing $\int V f^2 + |\nabla w|^2 f^2$ and then peter-paul) $\square$

3 Topic 2: PDE

3.1 Microlocal Analysis

Comes from: Chapter 2 in Andras/Melrose's notes + Rafe Elliptic theory of edge operators

3.1.1 Symbol Calculus

- See here, section II on Pseudo-Differential calculus (in particular, Chapter 6 on symbolic calculus) is quite good
- Can also see Melroses' notes chapter 2, which is where I'll be taking most of this from
- Symbols:

  Definition 3.1. *The space $S^m_\infty(\mathbb{R}^p, \mathbb{R}^n)$ of symbols of order m consist of funtions $a \in C^\infty(\mathbb{R}^p, \mathbb{R}^n)$ such that*

$$|D_z^\alpha D_\xi^\beta a(z, \xi)| \leq C_{\alpha,\beta}(1 + |\xi|)^{m - |\beta|}$$

  where $C_{\alpha,\beta}$ works for all points in $\mathbb{R}^p \times \mathbb{R}^n$, and such constants exist for any $\alpha \in \mathbb{N}^p$, $\beta \in \mathbb{N}^n$

This is also extended for $S^m_\infty(\Omega;\mathbb{R}^n)$ where we have the estimate for the interior.

- Define the following norm

$$||a||_{N,m} = \sup_{\substack{z\in\text{int}(\Omega)\\ \xi\in\mathbb{R}^n}} \max_{|\alpha|+|\beta|\le N} (1+|\xi|)^{-m+|\beta|}|D^\alpha_z D^\beta_\xi a(z,\xi)| < \infty$$

  this makes $S^m_\infty(\Omega;\mathbb{R}^n)$ into a frechet space

- Note the following symbol calculus properties:

$$\begin{gathered} S^m_\infty(\Omega;\mathbb{R}^n) \hookrightarrow S^{m'}_\infty(\Omega;\mathbb{R}^n) \qquad \forall m' \ge m \\ S^m_\infty(\Omega;\mathbb{R}^n)\cdot S^{m'}_\infty(\Omega;\mathbb{R}^n) \subset S^{m+m'}_\infty(\Omega;\mathbb{R}^n) \\ D^\alpha_z : S^m_\infty(\Omega;\mathbb{R}^n) \to S^m_\infty(\Omega;\mathbb{R}^n) \\ D^\beta_\xi : S^m_\infty(\Omega;\mathbb{R}^n) \to S^{m-|\beta|}_\infty(\Omega;\mathbb{R}^n) \end{gathered}$$

- We can also invert up to trivial error

  Lemma 3.2. *If $a\in S^m_\infty(\Omega;\mathbb{R}^n)$ elliptic, then there exists $b\in S^{-m}_\infty(\Omega;\mathbb{R}^n)$ such that*

$$ab - 1 \in S^{-\infty}_\infty(\Omega;\mathbb{R}^n)$$

  Recall that $a(z,\xi)$ is elliptic if

$$|a(z,\xi)| \ge \epsilon(1+|\xi|)^m, \qquad \forall |\xi| \ge C_\epsilon, \quad \epsilon > 0$$

  and also

$$S^\infty_\infty(\Omega;\mathbb{R}^n) = \cup_m S^m_\infty, \qquad S^\infty_\infty = \cap_m S^m_\infty$$

  Proof: Let ϕ be a radial bump function on $\mathbb{R}^n$, and define

$$b(z,\xi) = \begin{cases} \frac{1-\phi(\xi/2C)}{a(z,\xi)} & |\xi| \ge C \\ 0 & |\xi| \le C \end{cases}$$

  Then $b\in C^\infty$, and the symbol estimates follow by noting

$$|\xi| \ge C \implies D^\alpha_z D^\beta_\xi = a^{-1-|\alpha|-|\beta|} G_{\alpha\beta}$$

  for $G_{\alpha\beta}$ a symbol of order $(|\alpha|+|\beta|)m - |\beta|$. This is proved by induction. Finally, we have that

$$b\cdot a = \begin{cases} 1-\phi(\xi/2C) & |\xi| \ge C \\ 0 & |\xi| < C \end{cases}$$

  so since ϕ is a bump function (i.e. compactly supported), this clearly satisfies the symbol decay estimates

3.1.2 Pseudodifferential operators

- We consider

$$a(x,y,\xi) = (1+|x-y|^2)^{w/2}\tilde{a}(x,y,\xi), \qquad \tilde{a}\in S^m_\infty(\mathbb{R}^{2n}_{(x,y)},\mathbb{R}^n_\xi)$$

  for some w.

- If $a\in C^\infty$, then $a\in(1+|x-y|^2)^{w/2}S^m(\mathbb{R}^{2n};\mathbb{R}^n)$ if and only if

$$|D^\alpha_x D^\beta_y D^\gamma_\xi a(x,y,\xi)| \le C_{\alpha,\beta,\gamma}(1+|x-y|)^w(1+|\xi|)^{m-|\gamma|}, \qquad \forall \alpha,\beta,\gamma$$

  in particular, if $m < -n$, then $a(x,y,\xi)u(y)$ is absolutely integrable for any u Schwartz. Thus, we can define

$$\begin{gathered} A : \mathscr{S}(\mathbb{R}^n) \to (1+|x|^2)^{w/2} C^0_\infty(\mathbb{R}^n) \\ A(u) = (2\pi)^{-n}\int e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)dyd\xi \end{gathered}$$

Proposition. *The map defined for $m < -n$ given by*

$$a \mapsto (2\pi)^{-n} \int e^{i(x-y)\cdot\xi} a(x,y,\xi) d\xi \in (1+|x|^2+|y|^2)^{w/2} C^0_\infty(\mathbb{R}^{2n})$$

extends by continuity to

$$I : (1+|x-y|^2)^{w/2} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^{2n})$$

for each $w, m \in \mathbb{R}$ using the topology of $S^{m'}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$ for $m' > m$

$I(a)$ is **the schwartz kernel**

Proof: It suffices to show that

$$|I(a)(\phi)| \leq C||\tilde{a}||_{N,m}||\phi||_k \qquad \forall \phi \in \mathscr{S}(\mathbb{R}^{2n})$$

where $a = (1+|x-y|^2)^{w/2}\tilde{a}$ and $\tilde{a} \in S^{-\infty}_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$. This follows by integration by parts, noting that

$$(1+\xi D_x)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$
$$(1-\xi D_y)e^{i(x-y)\cdot\xi} = (1+|\xi|^2)e^{i(x-y)\cdot\xi}$$

- Finally, we have the following lemma

Lemma 3.3. *If $a \in (1+|x-y|^2)^{w/2} S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$, then the operator A, with Schwartz kernel $I(a)$, is continuous as a map $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$*

Proof: We compute

$$\begin{aligned} Au(\psi) &= \int\int\int e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\psi(x)d\xi dy dx \\ &= \int\int\int (1+|\xi|^2)^{-2q}(1-\xi D_y)^{2q} e^{i(x-y)\cdot\xi} a(x,y,\xi)u(y)\psi(x)d\xi dy dx \\ &\sum_{|\gamma|\leq 2q} \int\int\int e^{i(x-y)} a_\gamma D^\gamma_y u(y)\psi(x) d\xi dy dx \end{aligned}$$

(Note: ψ doesn't matter even though initially we have to pair it with ψ) where a_γ consists of derivatives of a coming from integration by parts of $(1-\xi D_y)^{2q}$, as well as the prefactor of $(1+|\xi|^2)^{-2q}$. Taking $-q+m < -n-|w|$, we get that Au is given by the convergent integral

$$Au(x) = \sum_{|\gamma|\leq 2q} \int\int e^{i(x-y)\cdot\xi} a_\gamma(x,y,\xi) D^\gamma_y u(y) d\xi dy A : \mathscr{S}(\mathbb{R}^n) \to (1+|x|^2)^{w/2} C^0_\infty(\mathbb{R}^n)$$

(note: C^0_∞ is banach space of bounded continuous function on $\mathbb{R}^n$). Moreover

$$D_{x_j} Au(x) = (2\pi)^{-n} \sum_{|\gamma|\leq 2q} \int\int e^{i(x-y)\cdot\xi}(\xi_j + D_{x_j})a_\gamma \cdot D^\gamma_y u(y) dy d$$
$$x_j Au(x) = (2\pi)^n \sum_{|\gamma|\leq 2q} \int\int e^{i(x-y)\cdot\xi}(-D_{\xi_j} + y_j)a_\gamma D^\gamma_y u(y) dy d\xi$$

and so by the same arguments, we get

$$x^\alpha D^\beta_x Au \in (1+|x|^2)^{w/2} C^0_\infty(\mathbb{R}^n), \qquad \forall \alpha, \beta \in \mathbb{N}^n$$

which gives $Au \in \mathscr{S}(\mathbb{R}^n)$. □

- Let $\Psi^m_\infty(\mathbb{R}^n)$ be the space of linear operators $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ corresponding to

$$(1+|x-y|^2)^{-w/2} a \in S^m_\infty(\mathbb{R}^{2n};\mathbb{R}^n)$$

for some w. These are **pseudodifferential operators of the traditional type, of type '1,0'**

3.1.3 Edge operators

- **Chapter 1**
- Set up is as follows: X compact manifold with boundary. ∂X is total space of fibration

$$\begin{tikzcd} F^q \arrow[r] & \partial X \arrow[d, "\pi"] \\ & Y^k \end{tikzcd}$$

  where F^q is compact fibre.

- $\mathcal{V}_e$ smooth vector fields unrestricted in the interior and lie tangent at the boundary **to the leaves of the fibration**
- Edge operators: generated by $\mathcal{V}_e$ and $C^\infty(X)$. Locally with coords (x, y, z), x defining function for ∂X, y coordinates on ∂X lifted from Y, and z coordinates on ∂X from the fiber F,

$$L = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x, y, z)(x\partial_x)^j (x\partial_y)^\alpha \partial_z^\beta$$

  i.e. either the operator vanishes at the boundary, or is of the form $a_\beta(0, y, z)\partial_z^\beta$ on the boundary

- For hyperbolic space, the fiber is empty, and $Y^k = \partial X = \mathbb{R}^n$ in $\mathbb{H}^{n+1}$
- L is elliptic, if it is elliptic as a combination of $x\partial_x,\ x\partial_{y^\alpha},\ \partial_{z^\beta}$. If y terms absent - **totally characteristic**. If z terms absent - **uniformily degenerate**
- Example: scalar laplacian on $\mathbb{H}^{n+1}$ is

$$\Delta_{\mathbb{H}^{n+1}} = x^2\partial_x^2 + (2 - n)x\partial_x + x^2\Delta_{\mathbb{R}^n}$$

  which is elliptic under the edge operator characterization

- **Chapter 2**
  - Not much new in this section, except we denote $\mathrm{Diff}_e^*(X)$ as "differential operators of edge type" which consists of

$$L \in \mathrm{Diff}_e^*(X) \iff L = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(x, y, z)(x\partial_x)^j (x\partial_y)^\alpha \partial_z^\beta$$

    and also the symbol map

$${}^e\sigma_m(L)(x, y, z; \xi, \eta, \zeta) = \sum_{j+|\alpha|+|\beta|=m} a_{j,\alpha,\beta}(x, y, z)\xi^j \eta^\alpha \zeta^\beta$$

    and L elliptic if the above is non-zero whenever $(\xi, \eta, \zeta) \neq 0$.

- **Polyhomogeneity (Appendix 2A)** :
  - $\mathcal{V}_b$ is space of b-operators spanned by $\{x^1\partial_{x_1}, \ldots, x^k\partial_{x_k}, \partial_{y^\alpha}\}$ - here $\{x^k\}$ gives a set of boundary defining functions for the (multiple) boundary hypersurfaces $\{M_{i^1}, \ldots, M_{i^k}\}$ and also y is a set of coordinates for the intersecting corner. Then $\mathcal{V}_b$ is spanned by these guys and the **conormal space** is:

$$A^0(X) = \{u :\ V_1 \cdots V_\ell u \in L^\infty(X), \quad \forall V_i \in \mathcal{V}_b, \qquad \forall \ell\}$$

  - In practice with hyperbolic space, there is no corner, so there's only one x^i and one boundary, and y is still the coordinate on $\mathbb{R}^n = \partial\mathbb{H}^{n+1}$, so the b operators are essentially $\{x\partial_x, \partial_y\}$, which differs from the edge operators which use $x\partial_y$ (since edge operators are only tangent with respect to the fiber on the boundary).
  - For $\{s_i\} \subseteq \mathbb{C}$, $\{p_i\} \subseteq \mathbb{N}_0$, we have

$$A^{s,p}(X) = x^s(\log(x))^p A^0(X)$$

    For one boundary and one codimension, then $x^s(\log x)^p a(y) \in A^{s,p}(X)$ if s is complex, $p \in \mathbb{N}_0$ and $a(y)$ smooth

- $A^*_{phg}(X)$ consists of all **conormal distributions admitting asymptotic expansions** of the form

$$u \in A^*_{phg}(X) \iff u \sim \sum_{\Re(s_j)\to\infty} \sum_{p=0}^{p_j} x^{s_j}(\log x)^p a_{j,p}(x,y), \qquad a_{j,p} \in C^\infty$$

  (by taylor expansion, we can just make $a_{j,p} = a_{j,p}(y)$)

- Edge calculus
  - Won't give the full description but the edge calculus is a ring of pseudodifferential operators with $Diff^*_e(X)$ as a subring.
  - Small calculus: PsiDos with kernels vanishing to all orders at the side boundary faces $B_{10}(X^2_e)$ and $B_{01}(X^2_e)$
  - Large calculus: PsiDos with kernels having polyhomogeneous conormal singularities at these faces
  - Consider lifts of elements of $Diff^*_e(X)$ to X^2_e, e.g. the identity map (multiply by 1). Schwartz kernel is

$$K_I = \delta(x-\tilde{x})\delta(y-\tilde{y})\delta(z-\tilde{z})\mu, \qquad \mu = \sqrt{dxdydzd\tilde{x}d\tilde{y}d\tilde{z}}$$

    where μ is a half density (i.e. something that when inner producted with something else gives a measure)
  - Recall $\beta_2 : X^2_e \to X^2$ which is the projection in spherical coordinates and the identity on $X^2 \backslash S$. Then

$$\beta_2^* K_I = \delta(\theta_0 - \theta_{k+1})\delta(\theta')\delta(z-\tilde{z})r^{(-k+1)/2}\nu, \qquad \nu = \sqrt{drd\theta d\tilde{y}dzd\tilde{z}}$$

  - We define the small calculus

$$\Psi^*_e(X;\Omega^{1/2}) := A^{\mathcal{E}}_{phg}I^*(X^2_e;\Delta_e;r^{-(k+1)/2}\Omega^{1/2})$$

    where $\mathcal{E} = (\emptyset, \emptyset, (0,0))$
  - Recall that $\mathcal{E}$ is an index set indicating vanishing behavior at the boundary. Also Δ_e is the embedding of the diagonal in X^2_e. The point is: **Every $A \in \Psi^*_e$ corresponds, after factoring out the singular half-density, to a distribution κ_A conormal along the lifted diagonal of X^2_e, vanishing to infinite order at the side boundaries and smooth at the front face**
  - By standard parametrix construction and symbol theory

    Theorem 3.4 (3.8)**.** *If $A \in \Psi^m_e$ elliptic then there exists $B \in \Psi^{-m}_e$ such that $AB - I \in \Psi^{-\infty}_e$ and $BA - I \in \Psi^{-\infty}_e$. This parametrix B is well defined up to an element of $\Psi^{-\infty}_e$.*

    I won't prove this but its important to note that the remainder terms in $\Psi^{-\infty}_e$ are **not compact operators** because their schwartz kernels are smooth only on X^2_e and not on X^2. They are smooth on the interior though

3.1.4 Theory of parameterices (in the large calculus)

(Chapter 5 in Rafe's paper)

- Context: $L \in Diff^*_e(X)$, so an edge operator
- $N(L)$ - This is the *restriction* of the *lift* through the *left* of L to X^2_e
- Background on X^2_e (Chapter 2)
  - Constructed from $X^2 = X \times X$, except we blow up around a submanifold S and

$$X^2_e = (X^2 \backslash S) \sqcup (N^+(S)/\mathbb{R}^+)$$

    this has 3 boundary hypersurfaces, given by the **left** and **right** boundaries corresponding to boundaries of each copy of X, as well as the **front face**, which is the spherical normal bundle (quotiented)
  - Picture should be two planes in $\mathbb{R}^3$ intersecting in a line, except you replace the line with a quarter of a cylinder (line blows up to a cylinder). Front face then looks like boundary of cylinder

- In particular, if we have

$$\begin{array}{ccc} F & \longrightarrow & \partial X \\ & & \downarrow \\ & & Y \end{array}$$

with coordinates (x, y, z) (x bdf for X, y coordinates on Y lifted to ∂X, z fibre coordinates for F), then we define coordinates (x, y, z) and $(\tilde{x}, \tilde{y}, \tilde{z})$ on each of the copies of X. Moreover

$$S := \{x = \tilde{x} = 0, \ y = \tilde{y}\}$$

- We have non-singular coordinates

$$r = \sqrt{x^2 + \tilde{x}^2 + |y - \tilde{y}|^2}, \qquad \theta = (x, y - \tilde{y}, \tilde{x})/r = (\theta_0, \theta', \theta_{k+1})$$

Smooth and independent when lifted to X_e^2

- Near front face, X_e^2 locally diffeomorphic to $\mathbb{R}^+ \times S_{++}^{k-1} \times F_z \times \partial X_{\tilde{y},\tilde{z}}$ (S_{++}^{k-1} is the quarter sphere, $(\theta_0, \theta_{k+1} \geq 0)$, so when F is trivial and the projection is the identity map, then it really looks the hyperbolic case, i.e. radial coordinate and angular coordinate restricted to quarter sphere)
- Blow down map

$$\beta_{(2)} : X_e^2 \to X^2$$

given by

$$\beta_{(2)}((r, \theta, \tilde{y}, z, \tilde{z})) = (r\theta_0, \tilde{y} + r\theta', z, r\theta_{k+1}, \tilde{y}, \tilde{z})$$

- If we compose the above with the map

$$\beta_L : (r\theta_0, \tilde{y} + r\theta', z, r\theta_{k+1}, \tilde{y}, \tilde{z}) \to (x, y, z)$$
$$\beta_R : (r\theta_0, \tilde{y} + r\theta', z, r\theta_{k+1}, \tilde{y}, \tilde{z}) \to (\tilde{x}, \tilde{y}, \tilde{z})$$

then these are the left and right blow downs.

- Also have projective coordinates $(s, u, \tilde{x}, \tilde{y}, z, \tilde{z})$ and $(x, y, t, v, z, \tilde{z})$

$$s = x/\tilde{x}, \qquad u = \frac{y - \tilde{y}}{\tilde{x}}$$
$$t = \tilde{x}/x, \qquad v = \frac{\tilde{y} - y}{x}$$

top row is smooth away from B_{01} and bottom smooth away from B_{10}. In both systems, $\tilde{x} = 0$ and $x = 0$ define the front face, because $x = s \cdot \tilde{x}$ in the first and $\tilde{x} = x \cdot t$ in the second. Also

$$x\partial_x = s\partial_s = x\partial_x - t\partial_t - v\partial_v$$
$$x\partial_y = s\partial_u = x\partial_y - \partial_v$$

- Front face, $B_{11}(X_e^2)$ fibres over ∂X (E.g. think of this as a sphere over a point) (Still a bit confused about this), $N(L)$ restricts to an elliptic operator *on the interior of each leaf of this fibration.* The **leaves are diffeomorphic to** $S_{++}^{k+1} \times F$ (checks out in the two half plane case)

- Chapter 2: Define $N(L)$ as the restriction of L to the front face of B_{11} of the lift of L to X_e^2

$$L = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(s\tilde{x}, \tilde{y} + \tilde{x}u, z)(s\partial_s)^j (s\partial_u)^\alpha \partial_z^\beta$$

then

$$N(L) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(0, \tilde{y}, z)(s\partial_s)^j (s\partial_u)^\alpha \partial_z^\beta$$

- We define the dual normal operator

$$N(\hat{L}) = \sum_{j+|\alpha|+|\beta| \leq m} a_{j,\alpha,\beta}(s\partial_s)^j (is\eta)^\alpha \partial_z^\beta$$

now for $t = s|\eta|$ and $\hat{\eta} = \eta/|\eta| \in S^*_{\tilde{y}}Y$, we rewrite the above as

$$L_0 = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(t\partial_t)^j (it\hat{\eta})^\alpha \partial_z^\beta$$

This is a family of **b-operators** on $\mathbb{R}^+ \times F$ depending smoothly on $(\tilde{y}, \tilde{\eta}) \in S^*Y$

- We characterize these operators as

Definition 3.5. *(5.3) $L_0 \in Diff_b^m(\mathbb{R}^+ \times F)$ is said to be of Bessel type if it has the form*

$$L_0 = \sum_{j+|\alpha|+|\beta|\leq m} a_{j,\alpha,\beta}(z)(t\partial_t)^j (it)^\alpha \partial_z^\beta$$

and is called elliptic if the associated symbol

$$\sum_{j+\ell+|\beta|=m} a_{j,\alpha,\beta}(z)\tau^j \sigma^\ell \zeta^\beta$$

is elliptic

- The prototypical example is on $\mathbb{R}^+$ is $L_0 = (t\partial_t)^2 - t^2$, with generic behavior of exponential growth/decay as $t \to \infty$

- Define
$$H^{r,\delta,\ell} = \{u \ : \ \phi(s)u \in t^\delta H^r, \quad (1-\phi(t))u \in t^{-\ell}H^r\}$$
where H^r is normal L^2 sobolev space in the t, z variables and $\phi(t) \in C_0^\infty(\mathbb{R}^+)$ equals 1 near $t = 0$.

- Recall the set Λ
  - Formally,
$$\Lambda = \{\Re(\zeta) + \frac{1}{2} \ : \ \zeta \in \text{spec}_b(L)\}$$
  - Note that $\text{spec}_b(L)$ is the set of $\zeta \in \mathbb{C}$ for which $I_\zeta(L)$ fails to be invertible on $L^2(F)$ for some $y \in Y$
  - Recall the definition of $I_\zeta(L)$

    Definition 3.6. *The indicial family $I_\zeta(L)$ of $L \in Diff_e^*(X)$ is the family of operators given by*

$$L(x^\zeta(\log x)^p f(x,y,z)) = x^\zeta(\log x)^p I_\zeta(L) f(0,y,z) + O(x^\zeta(\log x)^{p-1}), \qquad \forall f \in C^\infty(X), \ \ \zeta \in \mathbb{C}, \ \ p \in \mathbb{N}_0$$

  - For example, if we take L to be the hyperbolic laplacian, then

$$L = \Delta = x^2\partial_x^2 + (2-n)x\partial_x + x^2\Delta_{\mathbb{R}^{n-1}} = (x\partial_x)^2 + (1-n)(x\partial_x) + x^2\Delta_{\mathbb{R}^{n-1}}$$

    and so for $p = 0$

$$L(x^\zeta f(x,y,z)) = L(x^\zeta f(0,y,z) + O(x^{\zeta+1}) = [\zeta^2 + (1-n)\zeta]f(0,y,z) + O(x^{\zeta+1})$$

    i.e. the indicial operator is $\zeta^2 + (1-n)\zeta$, which has roots of $\zeta = 0, n-1$.
  - Lemma on Fredholmness

    Lemma 3.7 (5.5)**.** *The map*
$$L_0 : H^{r+2,\delta,\ell} \to H^{r,\delta,\ell-m}$$
    for L_0 of Bessel type and elliptic is Fredholm provided $\delta \notin \Lambda$

    Proof: For $\delta \notin \Lambda$, we prove Fredholm properties by constructing right and left parametrices for L_0, which are bounded on the appropriate spaces, and which are inverses up to compact error. It suffices to construct a right parametrix since the left parameterix will differ by compact error as in the normal pseudodifferential case (i.e. same procedure works).

    Paramtrix constructed by patching together local parametrices near $t = 0$ and $t = \infty$. Near $t = 0$, only the b-structure is relevant, and we have a parametrix via existence in the small calculus (won't go into details, too long. Though the gist is that its the standard parametrix construction being careful of the faces and degeneracy there) so that $G : t^\delta H_b^r \to t^\delta H_b^{r+m}$ is bounded (recall that H_b is the sobolev space with basis of derivatives $\{x\partial_x, \partial_{z^1}, \dots, \partial_{z^q}\}$).

Near $t = \infty$, we use ellipticity so that the principal symbol

$$\tilde{\sigma}(L_0) = i^m(a_{m,0,0}t^m\tau^m + a_{0,m,0}t^m) + \sum_{|\beta|=m} a_{0,0,\beta}\partial_z^\beta$$

where τ is dual to t and the above satisfies

$$\langle\tilde{\sigma}(L_0)u, u\rangle \geq Ct^m(1+\tau^m)||u||^2$$

for all $t \geq s_0$ for some s_0. Hence the operator norm of $\tilde{\sigma}(L_0)^{-1}$ is bounded by $Ct^{-m}(1+\tau)^{-m}$. Define the parametrix near ∞ by

$$H_\infty(u) = \int e^{it\tau}\tilde{\sigma}(L_0)^{-1}\hat{u}(\tau, z)dz$$

(I guess the idea is that t behaves as a distinguished coordinate on $\mathbb{R}^+ \times F$, so we just transform over the fiber F) where $\hat{u}$ is the fourier transform. By choice of ϕ, we have that

$$(1-\phi(t))H_\infty : H^{r,\delta,\ell-m} \to H^{r+m,\delta,\ell}$$

is bounded (since we have sufficient decay on the symbol). Patching H_∞ to the parametrix near 0 via a bump function, we obtain H such that

$$LH = I - K$$

for some error K. K is a finite sum of terms, booth smooth of order at least one, decaying like t^{-1} as $t \to \infty$ and like t^ϵ for some $\epsilon > 0$ as $t \to 0$ (this must be in the proof of the small calculus parametrix on H_b^r, for the parametrix away, note the prefactor of $(1-\phi(t))$ and the fact that $\phi(0) = 1$). Hence both maps in

$$K : H^{r,\delta,\ell} \to H^{r+1,\delta+\epsilon,\ell+1} \hookrightarrow H^{r,\delta,\ell}$$

are bounded. Since the second inclusion is **compact**, K itself is compact, completing the proof. $\square$

(Chapter 4 - parametrices of b operators)

- Extremal cases of edge theory: when either Y, the base, or F, the fiber, is a point. $\mathcal{V}_e$ then consists of either vector fields tangent to ∂X or vanishing on ∂X, denoted by $\mathcal{V}_b$ or $\mathcal{V}_o$. We consider $\mathcal{V}_b$ and a trivial base (this fits the paradigm of $\mathbb{H}^{n+1} = \mathbb{R}^n \times \mathbb{R}^+$)
- Recall
$$\mathcal{V}_b = \{x\partial_x, \partial_{z_1}, \dots, \partial_{z_q}\}$$
- Note that $\text{spec}_b(L)$ is automatically discrete and so for $L \in \text{Diff}_b^m(X)$ elliptic, we consider
$$L : x^\delta H_b^{\ell+m}(X, \Omega^{1/2}X) \to x^\delta H_b^\ell(X, \Omega^{1/2}X)$$
- Also define
$$\Lambda = \{\text{Re}(\zeta) + 1/2 \ : \ \zeta \in \text{spec}_b(L)\}$$
- We have that L is fredholm away from the spectrum:

  Theorem 3.8 (4.4)**.** *For $\delta \notin \Lambda$,*
$$L : x^\delta H_b^{\ell+m}(X, \Omega^{1/2}X) \to x^\delta H_b^\ell(X, \Omega^{1/2}X)$$
  is Fredholm

  We'll prove this by constructing parametrices which are inverses up to comapct error.

- Proof of (4.4)

* We have
$$L = \sum_{j+|\beta|\leq m} a_{j,\beta}(x,z)(x\partial_x)^j \partial_z^\beta$$
* Using theory of the small calculus, we find a parametrix: **The lift of L through the left to X_b^2 is tranversally elliptic to the lifted diagonal Δ_b**. Thus, factoring out the singualr half-density, we get
$$A_0 \in \Psi_b^{-m}(X) \quad \text{s.t.} \quad LA_0 \equiv I \mod \Psi_b^{-\infty}(X)$$
i.e.
$$LA_0 - I = R_0 \text{ s.t. } R_0 \in \Psi_b^{-\infty}(X)$$
* R_0 is not compact since **it does not vanish on the front face**, so we have to modify it. Note that
$$A_0 : x^\delta H^{\ell - m} \to x^\delta H_b^\ell$$
is bounded for every ℓ and δ (this is some corollary), while
$$R_0 : x^\delta H_b^\ell \to x^\delta H_b^s$$
for every ℓ, s, δ.
* We now modify our parametrix and solve for $A_1 \in \Psi_b^{\infty,*}$ such that
$$L(A_0 + A_1) = I - R_1$$
such that $R_1 \in \Psi_b^{-\infty,\mathcal{F}}$ where $\mathcal{F}$ is a collection of index sets with $F \in \mathcal{F} \implies F_{11} = 1$, i.e. R_1 **vanishes to first order at the front face of X_b^2**. This is accomplished by solving
$$(LA_1)\Big|_{B_{11}} = I(L)(A_1\Big|_{B_{11}}) = R_0\Big|_{B_{11}}$$
* This is solved via the Mellin transform (won't go in detail, too specific)
* We now have A_1 which is bounded and is a right inverse with error R_1 which vanishes simply on B_{11}. We can find a left parametrix similarly, A_2 so that
$$L(A_0 + A_1 + A_2) = I - R_2$$
where R_2 again vanishes to first order on the front face. This vanishing implies that R_2 is compact, so L is Fredholm! (Again, note that our error always vanishes on the front face, not the side faces, the left vs. right issue is just left vs. right parametrix) $\square$

– One can also do a modified neumann series to make it so that R_2 vanishes to infinite order on either the left or right face - this is what let's us deduce a polyhomogeneous expansion (Again, details too much)

3.1.5 Construction of parametrics for elliptic pseudodifferentialoperators

- Elliptic

Definition 3.9. *We say $A \in \Psi_\infty^m(\mathbb{R}^n)$ elliptic if it is invertible modulo an error in $\Psi_\infty^{-\infty}$ (i.e. symbol is $\in S_\infty^{-\infty}$) with the approximate inverse of order $-m$, i.e.*
$$\exists B \in \Psi_\infty^{-m}(\mathbb{R}^n) \quad s.t. \quad A \circ B - Id \in \Psi_\infty^{-\infty}(\mathbb{R}^n)$$

- Recall the quotient space of symbols
$$S_\infty^{m-[1]}(\mathbb{R}^n;\mathbb{R}^n) = S_\infty^m(\mathbb{R}^n;\mathbb{R}^n)/S_\infty^{m-1}(\mathbb{R}^n;\mathbb{R}^n)$$
along with the principle symbol map
$$\sigma_m : \Psi_\infty^m \to S_\infty^{m-[1]}$$
$$\sigma_m(A) = [\sigma_L(A)] = [\sigma_R(A)]$$
where σ_L, σ_R are the left and right symbol maps which give an x (respectively, y) independent symbol to the pseudodifferential operator A

- Recall the notion of asymptotic summation for symbols $a \in S^m$, i.e.

$$a \in S^m, \quad a \sim \sum_{j=0}^{\infty} a_j \iff \forall N \geq 0, \qquad a - \sum_{j=0}^{N-1} a_j \in S_{\infty}^{m-N}$$

the actual sum $\sum_{j=0}^{\infty} a_j$ may not converge, but we're saying that the partial expansion holds

- We have the following theorem

Theorem 3.10. *TFAE for $A \in \Psi_{\infty}^m(\mathbb{R}^n)$*

1. *A elliptic*
2. *$\exists [b] \in S^{-m-[1]}$ such that*
$$\sigma_m(A) \cdot [b] \equiv 1 \in S_{\infty}^{0-[1]}$$
3. *$\exists b \in S_{\infty}^{-m}(\mathbb{R}^n; \mathbb{R}^n)$ such that*
$$\sigma_L(A) \cdot b - 1 \in S_{\infty}^{-\infty}(\mathbb{R}^n; \mathbb{R}^n)$$
4. $$\exists \epsilon > 0 \;\; s.t. \;\; |\sigma_L(A)(x, \xi)| \geq \epsilon(1 + |\xi|)^m, \qquad \forall \quad |\xi| > \frac{1}{\epsilon}$$

Proof: We now the equivalence of the last two by constructing inverse of elliptic symbols. Now since the symbol map is multiplicative

$$A \circ B - Id \in \Psi_{\infty}^{-\infty} \implies \sigma_m(A) \cdot \sigma_{-m}(B) \equiv 1 \in S_{\infty}^{0-[1]}$$

which is 1 $\implies$ 2. Assume 2, and recall $\sigma_m(A) = [\sigma_L(A)]$. Find a rep of the equivalence class b_1 such that

$$\sigma_L(A) \cdot b_1 = 1 + e_1, \quad e_1 \in S_{\infty}^{-1}$$

Now define

$$b = b_1 \cdot \text{“}(1 + e_1)^{-1}\text{”}$$

where

$$(1 + e_1)^{-1} \sim \sum_{j \geq 0} (-1)^j e_1^j$$

i.e. we can find an f that has the asymptotic expansion of

$$f \sim \sum_{j \geq 1} (-1)^j e_1^j$$

so that

$$\sigma_L(A) \cdot b_1(1 + f) = 1 + e_{\infty}$$

for $e_{\infty} \in S^{-\infty}$ and $b = b_1(1 + f) \in S_{\infty}^{-m}$. Then

$$\sigma_L(A) \cdot b_1(1 + f) = 1 + e_{\infty}$$

which proves (3). Moreover, since $e_{\infty} \in S_{\infty}^{-\infty}$, we have that

$$\forall N, \qquad \sup(1 + |\xi|^N)|e_{\infty}| < \infty$$

and in particular $|e_{\infty}| < \frac{1}{2}$ for all $|\xi| > C$ for some C. And so

$$|\sigma_L(A)| \geq |\sigma_L(A) b_1(1 + f)||b_1(1 + f)|^{-1} \geq \frac{1}{2}(1 + |\xi|)^m$$

since $b = b_1(1 + f)$ is a symbol of order $-m$. But this gives the first condition on ellipticity!

Now suppose that (3) holds, i.e. $\exists b$ s.t. $\sigma_L(A) \cdot b - 1 \in S_{\infty}^{-\infty}$. Then for $B_1 = q_L(b)$, we have

$$\sigma_0(A \circ B_1) = [q_m(A)] \cdot [b] \equiv 1 \implies A \circ B_1 - Id = E_1 \in \Psi_{\infty}^{-1}$$

doing the same trick of
$$F \sim \sum_{j\geq 1} E_1^j(-1)^j$$
we see that $(Id + F) \in \Psi^0_\infty$ and
$$A \circ B_1 \circ (Id + F) \in \Psi^{-\infty}_\infty$$
so that $B = B_1 \circ (Id + F) \in \Psi^{-m}_\infty$, giving (1). □

- This is kind of a lengthy proof, but the idea is to use the definition of a symbol, definition of ellipticity, and then the neumann series trick for inverting $(1 + \epsilon)$.
- The above theorem also shows how to construct a parametric when your symbol is elliptic
- Left vs. Right parametrix doesn't matter:

  Lemma 3.11. *$A \in \Psi^m_\infty(\mathbb{R}^n)$ elliptic if and only if there exists $B' \in \Psi^{-m}_\infty(\mathbb{R}^n)$ such that*
  $$B' \circ A = Id + E', \qquad E' \in \Psi^{-\infty}_\infty$$
  and if B satisfies the above as well, then $B - B' \in \Psi^{-\infty}_\infty$

  Proof: Certainly, if we have an inverse then $\sigma_{-m}(B') \cdot \sigma_m(A) \equiv 1$, and by the previous lemma, we know that A is elliptic. If A elliptic, then taking the b such that
  $$b \cdot \sigma_L(A) = 1 \in S^{0-[1]}_\infty$$
  we can take $B = q_L(b)$ and get
  $$B \circ A - Id = E_1 \in \Psi^{-1}_\infty$$
  and do the same trick, i.e.
  $$(1 + F) \sim \sum_{j\geq 0} (-1)^j E_1^j$$
  and so for $B' = (Id + F') \circ B$, we have
  $$B' \circ A = (Id + F')B \circ A \in \Psi^{-\infty}_\infty$$
  Now if we take B the original inverse, we get
  $$B' \circ A \circ B = B'(Id + E), \qquad E \in \Psi^{-\infty}_\infty$$
  but also
  $$B' \circ A \circ B = (Id + E')B, \qquad E \in \Psi^{-\infty}_\infty$$
  taking the difference of the two, we see that $B - B' \in \Psi^{-\infty}_\infty$

3.2 Calculus of Variations

Existence of minimizers, Euler-Lagrange equation, Noether's theorem (Chapter 8, Evans)

3.2.1 Existence of Minimizers

- Need a few conditions
- Coercivity: there exist constants $\alpha > 0, \beta \geq 0$ such that
  $$L(p, z, x) \geq \alpha |p|^q - \beta$$
  i.e.
  $$I[w] \geq \delta |Dw|^q_{L^q} - \gamma$$
  where $\gamma = \beta |U|$, which hints at defining $I[w]$ for functions in $W^{1,q}(U)$

- Define
$$\mathcal{A} = \{w \in W^{1,q}(U) \mid w\Big|_{\partial U} = g \quad \text{(in trace sense)}\}$$
as **the admissible set of functions** for I. Remember that to make sense of $w\Big|_{\partial U}$, we take a sequence of functions $\{w_i\} \subseteq C^\infty(\overline{U})$ which converge to w in $W^{1,q}$ norm. Then the limit of their boundary values gives the trace

- Lower semicontinuity: though we work in $W^{1,q}$, we note that $u, Du \in L^q(U)$, so using that L^q is reflexive (Banach-Alaoglu), we get a sequence of $\{u_k\}$ such that $Du_k \rightharpoonup Du$ and $u_k \rightharpoonup u$ in L^q, i.e. $u_k \rightharpoonup u$ in $W^{1,q}$.

  But now we need that I is continuous w.r.t. the weak topology, else a minimizing sequence won't converge to the minimizing value. Instead, we ask
$$I[u] \leq \liminf_{j\to\infty} I[u_{k_j}]$$
  for some weakly convergent subsequence. This is sufficient to get that u is a minimizer.

  Definition 3.12. *We say that $I[\cdot]$ is (sequentially) weakly lower semicontinuous on $W^{1,q}(U)$ provided that*
$$I[u] \leq \liminf_{k\to\infty} I[u_k]$$
  whenever $u_k \rightharpoonup u$ in $W^{1,q}(U)$

- Convexity: A consequence of having a minimizer is
$$L_{p_i p_j}(Du, u, x)\xi^i \xi^j \geq 0$$
  for all $\xi \in \mathbb{R}^n$ and all $x \in U$. I.e. this is necessary. This suggests that convexity in p is useful, i.e.

  Theorem 3.13. *Assume L smooth, bounded below, and in addition*
$$p \mapsto L(p, z, x)$$
  is convex for all z, x. Then $I[\cdot]$ is weakly lower semicontinuous on $W^{1,q}(U)$

  Proof: suppose $u_k \rightharpoonup u \in W^{1,q}$ and $l = \liminf_k I[u_k]$. Then we want $I[u] \leq l$. By passing to a subsequence, we can assume
$$l = \lim_k I[u_k]$$
  We have
$$I[u] = \int L(Du, u, x) = \int [L(Du, u, x) - L(Du_k, u, x)] + [L(Du_k, u, x) - L(Du_k, u_k, x)] + L(Du_k, u_k, x)$$
  So
$$I[u] - I[u_k] = \int [L(Du, u, x) - L(Du_k, u, x)] + [L(Du_k, u, x) - L(Du_k, u_k, x)]$$
  Intuitively, we use convergence in L^q of both Du_k and u_k to get that this difference tends to 0, **and** we've assumed that L is smooth so this seems okay. However, since U is open (potentially the L estimates drop off as we approach the boundary), we need some uniform continuity. Let $G_\epsilon \subset U$ a subset so that $u_k \to u$ uniformly on G_ϵ, $|U - G_\epsilon| < \epsilon$ and
$$x \in G_\epsilon \implies |u(x)| + |Du(x)| < \frac{1}{\epsilon}$$
  by Egorov's theorem and that $u \in W^{1,q}$, such a set exists. Then
$$\begin{aligned} I[u_k] &= \int_U L(Du_k, u_k, x)dx \\ &\geq \int_{G_\epsilon} L(Du_k, u_k, x)dx \\ &\geq \int_{G_\epsilon} L(Du, u_k, x)dx + \int_{G_\epsilon} D_p L(Du, u_k, x) \cdot (Du_k - Du) \end{aligned}$$

here, we've assumed that $L \geq 0$ (else we could shift L up by a constant. And the last inequality is true for general convex functions, i.e. f convex means that

$$f(y) \geq f(x) + Df(x) \cdot (y - x)$$

see theorem 1.2.3 here. Now we know that

$$\lim_{k \to \infty} \int_{G_\epsilon} L(Du, u_k, x) dx = \int_{G_\epsilon} L(Du, u, x) dx$$

since we have uniform convergence since L is smooth, and $u \to u_k$ uniformly and u_k, Du_k are bounded on G_ϵ (this is what allows us to pass the limit through the integral). Moreover

$$\lim_{k \to \infty} \int_{G_\epsilon} D_p L(Du, u_k, x) \cdot (Du_k - Du) = 0$$

since again $D_p L(Du, u_k, x)$ is bounded and $Du_k - Du$ in L^q. This tells us that

$$l = \lim_k I[u_k] \geq \int_{G_\epsilon} L(Du, u, x)$$

and sending $\epsilon \to 0$ we get $l \geq I[u]$. □

Note that since $u_k \to u \in L^q$ strongly (this is Rellich) we didn't need any convexity assumption about L in the middle variable

- Existence of minimizer:

  Theorem 3.14. *Assume L satisfies coercivity, and is convex in p, and $\mathcal{A}$, the admissible set is non-empty. Then there is at least one $u \in A$ with*

  $$I[u] = \min_{w \in \mathcal{A}} I[w]$$

  Proof: Set $m = \inf_w I[w]$. If m is finite, select a minimizing sequence. WLOG assume $L \geq 0$, and so

  $$I[w] \geq \alpha \int_U |Dw|^q$$

  By this, we get that $\sup_k ||Du_k||_{L^q} \leq C$. We want to show that $\{u_k\}$ is bounded in $W^{1,q}$, so it suffices to bound $||u||_{L^q}$. Now fix $w \in \mathcal{A}$, then

  $$\begin{aligned} ||u_k||_{L^q} &\leq ||u_k - w||_{L^q} + ||w||_{L^q} \\ &\leq C||Du_k - Dw||_{L^q} + ||w||_{L^q} \\ &\leq C||Du_k|| + C||Dw|| + ||w|| \\ &\leq C' \end{aligned}$$

  having used Poincare inequality since $u_k\Big|_{\partial U} = w\Big|_{\partial U}$ by deifnition of admissible set (i.e. same boundary conditions). Thus we have $\{u_k\}$ is bounded in $W^{1,q}$.

  By our work before, we can find a subsequence, also called $\{u_k\}$, which converges weakly in $W^{1,q}$ to some u. We show that $u \in \mathcal{A}$. Note the boundary condition is preserved, and we can consider $u_k - w \in W_0^{1,q}$. $W_0^{1,q} \subseteq W^{1,q}$ is a closed linear subspace, so it is weakly closed (Mazur's theorem), and hence $u - w \in W_0^{1,q}(U)$, i.e. the trace is correct. Also via our previous theorem

  $$I[u] \leq \liminf_k I[u_k] = m \implies I[u] = m$$

- Uniqueness

  Theorem 3.15. *Suppose that*

  1. *$L = L(p, x)$ does not depend on z*

2. There exists $\theta > 0$ such that

$$L_{p_i p_j}\xi^i\xi^j \geq \theta|\xi|^2$$

i.e. uniform convexity for $p \mapsto L(p,x)$

Then a minimizer $u \in \mathcal{A}$ of I is unique

Proof: Suppose $u, \tilde{u}$ both minimizers. Then consider $\frac{u+\tilde{u}}{2} = v \in \mathcal{A}$. By uniform convexity, we have

$$L(p,x) \geq L(q,x) + D_pL(q,x)\cdot(p-q) + \frac{\theta}{2}|p-q|^2$$

setting $q = Dv$ and $p = Du$, we get

$$I[v] + \int_U D_pL(Dv,x)\cdot\left(\frac{Du - D\tilde{u}}{2}\right) + \frac{\theta}{8}\int_U |Du - D\tilde{u}|^2 \leq I[u]$$

repeating this with $q = Dv$ and $p = \tilde{u}$, we get

$$I[v] + \int_U D_pL(Dv,x)\cdot\left(\frac{D\tilde{u} - Du}{2}\right) + \frac{\theta}{8}\int_U |Du - D\tilde{u}|^2 \leq I[\tilde{u}]$$

so that

$$I[v] + \frac{\theta}{8}\int_U |Du - D\tilde{u}|^2 \leq \frac{I[u] + I[\tilde{u}]}{2}$$

i.e. $I[v]$ does strictly better unless $Du \equiv D\tilde{u}$. Since the boundary conditions are the same, we have that $u \equiv \tilde{u}$ a.e.

(Idea: the average does better than either individual function. Now use convexity to get

$$L(p,x) \geq L(q,x) + D_pL(q,x)\cdot(p-q) + \frac{\theta}{2}|p-q|^2$$

and set $p = Du/p = D\tilde{u}$ and $q = Dv$)

3.2.2 Euler-Lagrange Equation

- Suppose we have a functional

$$L : R^n \times R \times \overline{U} \to \mathbb{R}$$
$$(p,z,x) \mapsto L(p,z,x) \in \mathbb{R}$$
$$I[w] := \int_U L(Dw(x), w(x), x)$$

  for smooth functions $w : \overline{U} \to \mathbb{R}$ satisfying a boundary condition like

$$w\Big|_{\partial U} = g$$

- Suppose we have u, a minimizer of I subject to the boundary condition. Then by the usual calculus of variations equation, we get

$$L_z(Du,u,x) = \sum_i \partial_{x_i}(L_{p_i}(Du,u,x))$$

  which I guess we can write as

$$\partial_z L(Du,u,x) = (\mathrm{div}_x \nabla_p L)(Du,u,x)$$

- Great examples

$$I(u) = \int |\nabla u|^2 \leftrightarrow \Delta u = 0$$

$$I(u) = \int \frac{1}{2}a^{ij}u_{x_i}u_{x_j} - uf \leftrightarrow \partial_{x_i}(a^{ij}u_{x_j}) + f = 0 = \mathrm{div}(a^{ij}u_{x_j}) + f$$

$$I(u) = \int \sqrt{1+|Du|^2} \leftrightarrow \mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|Du|^2}}\right) = 0$$

  Respectively, these are Laplace's equation, diverence form inhomogenous elliptic PDE, and minimal surface equation

3.2.3 Noether's theorem

- We define a domain variation as follows: let $\varphi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be $\varphi(x, \tau)$ a smooth family of vector fields satisfying $\varphi(x, 0) = x$. Then for small $|\tau|$, $\varphi(x, t)$ is smooth diffeo. We also define

$$v(x) = \varphi_t(x, 0), \qquad U(t) = \varphi(U, t)$$

- Given $u : \mathbb{R}^n \to \mathbb{R}$ smooth, consider smooth function variations $w : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $w(x, t)$ such that

$$w(x, 0) = u(x)$$

we write

$$m(x) = w_t(x, 0)$$

and call it a *multiplier*

- We define

Definition 3.16. *We say $I[\cdot]$ is invariant under the domain variations φ, and the function variations w provided that*

$$\int_U L(Dw(x,t), w(x,t), x)dx = \int_{U(t)} L(Du, u, x)dx$$

for all small $|t|$ and all open sets $U \subseteq \mathbb{R}^n$

The idea is to make w some function of $\varphi^*(u)$.

- Noether's theorem

Theorem 3.17. *Suppose $I[\cdot]$ invariant under φ and w corresponding to some u smooth, then*

$$\sum_i \partial_{x_i}(mL_{p_i}(Du, u, x) - L(Du, u, x)v^i) = m\left(\sum_i \frac{d}{dx^i}(L_{p_i}(Du, u, x)) - L_z(Du, u, x)\right)$$

for $v = \dot{\varphi}(x)$ and $m = \dot{w}(x)$.

If u is a crit point of $I[\cdot]$, and hence solve $-div_x(DL_p) + L_z = 0$, then we have the other divergence identity

$$[\sum_i \partial_{x_i}(mL_{p_i}(Du, u, x) - L(Du, u, x)v^i) = 0$$

i.e. multiply by m converts the Euler-lagrange PDE into a true divergence form PDE

Proof: Differentiate

$$\int_U L(Dw(x,t), w(x,t), x)dx = \int_{U(t)} L(Du, u, x)dx$$

w.r.t. time, and get

$$\int_U D_pL \cdot Dm + L_z m dx = \int_{\partial U} Lv \cdot \nu dS$$

now integrate by parts the LHS of the desired equation for u, using the above identity. ☐

- Examples

  1. (Translation invariance) If $L = L(p, z)$ then we can set

$$\varphi(x, t) := x + te_k, \qquad w(x, t) := u(x + te_k)$$

  where $v = e_k$ and $m = u_{x_k}$. Then for u a critical point, we get

$$\sum_i (L_{p_i}(Du, u, x)u_{x_k} - L(Du, u, x)\delta_{ik})_{x_i} = 0$$

  i.e.

$$\text{div}(L_{p_i} \cdot u_{x_k}) = L(Du, u, x)_{x_k}$$

  for each k

2. (Scaling invariance), consider
$$I[w] = \int_u |Dw|^p dx$$
whose minimizers solve
$$\mathrm{div}(|Du|^{p-2} Du) = 0$$
which is the p-laplacian equation. This is invariant for
$$\varphi(x,t) = e^t x, \qquad w(x,t) = e^{t(n-p)/p} u(e^t x)$$
then
$$v = x, \qquad m = Du \cdot x + \frac{n-p}{p} u$$
and we get the corresponding divergence identity for u
$$\mathrm{div}\left(\left[Du \cdot x + \frac{n-p}{p} u \right] p |Du|^{p-2} \nabla u - |Du|^p \vec{x} \right) = 0$$

3.3 Functional Analysis

Lax-Milgram, Fredholm Alternative, existence of weak solutions to 2nd order elliptic equations (Section 6.2 of Evans)

3.3.1 Lax-Milgram

Theorem 3.18. *Suppose H real hilbert space and*
$$B : H \times H \to \mathbb{R}$$
$$(i) \quad |B(u,v)| \leq \alpha ||u|| \, ||v||$$
$$(ii) \quad \beta ||u||^2 \leq B(u,u)$$
Let $f : H \to \mathbb{R}$ bounded linear functional on H, then there exists a unique element $u \in H$ such that
$$B(u,v) = f(v)$$
for all $v \in H$

Proof: As always, our main tool is Riesz-Representation (aka Hahn-Banach). For each $u \in H$, Riesz gives us a $w \in H$ such that
$$B(u,v) = (w,v)$$
Write $Au = w$, so
$$B(u,v) = (Au, v)$$
we claim that $A : H \to H$ is a bounded linear operator. Linearity is easy, boundedness follows by (i) used as
$$||Au||^2 = (Au, Au) = B(u, Au) \leq \alpha ||u|| \, ||Au|| \implies ||Au|| \leq \alpha ||u||$$
We now prove: A is one to one and $R(A)$, the range is closed. These follow from (ii):
$$\beta ||u||^2 \leq B(u,u) = (Au, u) \leq ||Au|| \cdot ||u||$$
$$\implies ||Au|| \geq \beta ||u||$$
We now show $R(A) = H$, if not, then there exists an orthogonal complement, i.e. let $S = R(A)^\perp$, then for all $u \in H$ and $z \in S$, we have
$$(Au, z) = B(u, z) = 0 \implies$$
but now set $u = z$ and use (ii). This completes the proof that $A : H \to H$ is bounded linear bijection.

Now take $f \in H^*$. Riesz tells us that $f = w$ for some $w \in H$, i.e.
$$f(u) = (w, u) \qquad \forall u \in H$$
Let $z = A^{-1}(w)$, then
$$f(u) = (Az, u) = B(z, u)$$
and so z is the element we want. Uniqueness of z follows from (ii)

3.3.2 Fredholm-Alternative (+ weak existence to elliptic solutions)

- Let

$$L = -a^{ij}(x)\partial_{x_i}\partial_{x_j} + b^i(x)\partial_{x_i} + c(x)$$

be a space dependent linear operator and

$$B(u,v) = \int_U \sum_{i,j} a^{ij} u_{x_i} v_{x_j} + \sum_i v^i u_i v + cuv$$

be the corresponding bilinear form

- Definitions

Definition 3.19. *(i) The operator L^*, the formal adjoint of L is*

$$L^* v = -(a^{ij} v_{x_j})_{x_i} - b^i v_{x_i} + [c - b^i_{x_i}]v$$

(ii) The adjoint bilinear form

$$B^* : H_0^1 \times H_0^1 \to \mathbb{R}$$

is defined by

$$B^*(v,u) = B(u,v)$$

for all $u, v \in H_0^1$.

(iii) $v \in H_0^1$ is a weak solution of the adjoint problem

$$L^* v = f \quad \in U$$
$$v = 0 \quad \in \partial U$$

provided

$$B^*(v,u) = (f,u), \qquad \forall u \in H_0^1$$

- Energy Estimates

Theorem 3.20. *For*

$$B(u,v) = \int_U a^{ij} u_i v_j + b^i u_i v + cuv$$

the operator corresponding to L, there exist constants such that

$$|B(u,v)| \leq \alpha ||u||_{H_0^1} ||v||_{H_0^1}$$
$$\beta ||u||_{H_0^1} \leq B(u,u) + \gamma ||u||^2_{L^2}$$

$\forall u, v \in H_0^1$

Proof: The first estimate we can prove manually using that the coefficients are in L^∞. For the second condition, bound

$$\int_U |Du|^2 \leq B(u,u) + C||u||^2_{L^2}$$

via simple bounds: peter-paul inequality to put more weight on $||u||^2_{L^2}$, and then use poincare inequality, i.e. $||Du||_{L^2} \sim ||u||^2_{H_0^1}$. □

Note: this does not satisfy the conditions of Lax-Milgram, so often we consider modified operator $L_\gamma = L + \gamma$ and $B_\gamma(u,v) = B(u,v) + \gamma(u,v)$

- First existence for weak solution of elliptic PDE

Theorem 3.21. *There is $\gamma \geq 0$ such that $\forall \mu \geq \gamma$ and each $f \in L^2$, we can find $u \in H_0^1$ solving*

$$Lu + \mu u = f \quad \in U$$
$$u = 0 \quad \in \partial U$$

Proof: Define B_μ as above. Then B_μ satisfies the hypotheses of Lax-Milgram. Fix $f \in L^2$, and define $v \mapsto (f, v)$ is a bounded linear functional on L^2 and H_0^1. Lax-Milgram gives us $u \in H_0^1$ such that

$$B_\mu(u, v) = (f, v) \qquad \forall v \in H_0^1$$

and so u is our weak solution. $\square$

- Existence for weak solutions (Fredholm alternative)

Theorem 3.22. *Precisely one of the following holds:*

1. *For each $f \in L^2$, there exists a unique weak solution of the boundary problem (*)*

$$\begin{aligned} Lu &= f \quad \in U \\ u &= 0 \quad \in \partial U \end{aligned}$$

or else, a weak solution $u \not\equiv 0$ of the homogenous problem (**)

$$\begin{aligned} Lu &= 0 \quad \in U \\ u &= 0 \quad \in \partial U \end{aligned}$$

If the homogenous problem has the solution, the dimension of solutions (call this space N) is finite and equals the dimension of N^*, the weak solutions of

$$\begin{aligned} L^*u &= 0 \quad \in U \\ u &= 0 \quad \in \partial U \end{aligned}$$

By contrast, $Lu = f$ with 0 dirichlet data has a weak solution

$$(f, v) = 0, \qquad \forall v \in N^*$$

The dicotomy between the nontrivial dirichlet problem and the homogenous problem is called the ***Fredholm Alternative***

Proof: Choose γ so that Lax-Milgram applies to B_γ. Then for each $g \in L^2$, we have a $u \in H_0^1$ such that

$$B_\gamma(u, v) = (g, v) \qquad \forall v \in H_0^1(U)$$

now write $u = L_\gamma^{-1}(g)$ when the above holds.

Note that if we have a solution of (*), the boundary value problem $Lu = f$, if and only if

$$B_\gamma(u, v) = (\gamma u + f, v) \qquad \forall v \in H_0^1 \iff u = L_\gamma^{-1}(\gamma u + f)$$

This is because $B_\gamma(u, v) = (L_\gamma u, v)$ after integration by parts. Write this as

$$u - Ku = h, \qquad Ku = \gamma L_\gamma^{-1} u, \quad h = L_\gamma^{-1}(f)$$

We now want to show that $K : L^2 \to L^2$ is bounded, compact, linear operator and then apply the functional analytic Fredholm alternative. We compute

$$\beta ||u||_{H_0^1}^2 \le B_\gamma(u, u) = (g, u) \le ||g||_{L^2} ||u||_{H_0^1}$$

so

$$||Kg||_{H_0^1} \le C ||g||_{L^2}$$

but $H_0^1 \subset\subset L^2$ is a compact embedding by Rellich so we deduce that K is compact.

Now apply the functional analytic Fredholm Alternative, framing our original problem as

$$\begin{aligned} u - Ku &= h \\ \text{or} \quad u - Ku &= 0 \end{aligned}$$

In the former case, u is unique and defined for any $h \in L^2$, in the latter, the space of solutions is finite dimensional and the Fredholm alternative (for compact operators) says it is equal to the dimension of the dual space of solutions to

$$v - K^* v = 0$$

Unravelling the definitions, we get our first two claims in the theorem.

For the last claim, we want to solve

$$u - Ku = h$$

which by the functional fredholm alternative, tells us

$$(h, v) = 0 \qquad \forall v \text{ s.t. } v - K^* v = 0$$

but unravelling the definitions again, we have

$$0 = (h, v) = \frac{1}{\gamma}(Kf, v) = \frac{1}{\gamma}(f, K^* v) = \frac{1}{\gamma}(f, v)$$

which is what we wanted to prove

- **Functional Analytic Fredholm Alternative** (see Brezis)

  Theorem 3.23. *Let $T : E \to E$ be a compact operator, then*

  1. $\ker(I - T)$ *is finite dimensional*
  2. *$Ran(I - T)$ is closed, and more precisely $Ran(I - T) = \ker(I - T^*)^{\perp}$*
  3. $\ker(I - T) = \{0\} \iff Ran(I - T) = E$
  4. $\dim \ker(I - T) = \dim \ker(I - T^*)$

  Remark The adjoint can be defined on a dense subset $D(T^*) \subseteq E$ and exists by Hahn-Banah. It satisfies the fundmantel relationship

  $$\langle v, Tu \rangle_{E^*, E} = \langle T^* v, u \rangle_{E^* E}, \qquad \forall u \in D(T), \quad \forall v \in D(T^*)$$

3.4 Eigenvalues of the laplacian

Cheeger theorem for lower bounds, orthogonality of eigenfunctions on closed manifold, Faber-Kahn theorem on lower bound for first eigenvalue for $\Omega \subseteq \mathbb{R}^n$, Statement of variation of first eigenvalue of laplacian with respect to the domain , co-area formula and usage in proof

3.4.1 Cheeger theorem for lower bounds

(Schoen Yau, Chapter 3)

- Dirichlet Eigenvalue problem: Let the domain be $H^1_0(M)$ and eigenvalues

  $$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \ldots$$

  along with functions $\{\phi_i\}$ such that

  $$\Delta \phi_i = -\lambda \phi_i, \qquad \phi_i \Big|_{\partial M}$$

  then $\{\phi_i\}$ forms an ONB for H^1_0

- Neumann Eigenvalue problem: Let the domain by $H^1(M)$, with eigenvalues

  $$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$$

  and $\{\phi_i\}$ the corresponding eigenfunctions which forms an ONB for H^1 with

  $$\Delta \phi_i = -\lambda_i \phi_i, \qquad \frac{\partial \phi_i}{\partial \nu}\Big|_{\partial M} = 0, \qquad \phi_i \in C^\infty(M)$$

- Rayleigh quotient characterization

$$\lambda_1 = \inf\Big\{\frac{\int |\nabla f|^2}{\int |f|^2}\ \Big|\ f \in H\Big\}$$

in particular

$$\forall C \in \mathbb{R}, \qquad C \leq \lambda_1 \iff \int |\nabla f|^2 \geq C \int |f|^2, \qquad \forall f \in H$$

where H is the relevant set of functions we consider for the Neumann/Dirichlet problem

- Let M be compact RM

Definition 3.24. *If $\partial M \neq \emptyset$, define*

$$h_D(M) = \inf\{\frac{Vol(\partial\Omega)}{Vol(\Omega)} \mid \Omega \subset\subset M\}$$

If $\partial M = \emptyset$, define

$$h_N(M) = \inf\Big\{\frac{Vol(H)}{min(Vol(M_1), Vol(M_2))} \mid H, \textit{ hypersurface dividing } M \textit{ into } M_1, M_2 \textit{ with } \partial M_1 = \partial M_2 = H\Big\}$$

- Cheeger theorem

Theorem 3.25. *Let M compact RM. In the Dirichlet case, $\lambda_1 \geq \frac{1}{4}h_d^2(M)$, in the Neumann case $\lambda_1 \geq \frac{1}{4}h_N^2(M)$*

Proof: Let f eigenfunction for λ_1, then WLOG $f > 0$. Integrating

$$-f\Delta f = \lambda f^2$$

by parts and using $f\Big|_{\partial M} = 0$, we have

$$\int_M |\nabla f|^2 = \lambda_1 \int f^2$$

Suppose that for some $\mu > 0$

$$\int_M |\nabla \varphi| \geq \mu \int |\varphi|, \qquad \forall \varphi \in C^\infty(M) \text{ s.t. } \varphi\Big|_{\partial M} = 0$$

then $\lambda_1 \geq \frac{1}{4}\mu^2$. This follows by holder inequality with $\varphi = f^2$. Now by Co-Area

$$\begin{aligned}
\int_M |\nabla \varphi| &= \int_{\mathbb{R}} (\int_{\varphi=\sigma} 1) d\sigma \\
&= \int_{\mathbb{R}} \text{Area}(\varphi = \sigma) d\sigma \\
&= \int_{\mathbb{R}} \frac{\text{Area}(\varphi = \sigma)}{\text{Vol}(\varphi \geq \sigma)} \text{Vol}(\varphi \geq \sigma) d\sigma \\
&\geq \inf_\sigma \frac{\text{Area}(\varphi = \sigma)}{\text{Vol}(\varphi \geq \sigma)} \int_{\mathbb{R}} \text{Vol}(\varphi \geq \sigma) d\sigma \\
&= \inf_\sigma \frac{\text{Area}(\varphi = \sigma)}{\text{Vol}(\varphi \geq \sigma)} \int_M |\varphi| \\
&\geq h_D(M) \int_M |\varphi|
\end{aligned}$$

where the last inequality follows from the definition of $h_D(M)$.

Neumann Case: In this case, the second eigenfunction has 2 nodal domains (**this is courant's nodal domain theorem, which is sharp for the second eigenvalue since if there was only one domain, f restricted to the boundary would be 0 and the neumann derivative would be 0, meaning that f is 0. But**

2 **is an upper bound by Courant, so it must be exactly** 2) M_+ and M_- on which f is positive and negative. Assume M_+ has smaller volume. Then

$$h_D(M_+) \geq h_N(M)$$

Since f does not change sign and $f\Big|_{\partial M_+} = 0$, f is the eigenfuction w.r.t. dirichlet boundary conditions and first eigenvalue λ_1. But then

$$\lambda_1 \geq \frac{1}{4}h_D^2(M_+) \geq \frac{1}{4}h_N^2(M)$$

which ends the proof. □

Seems like the trick is using $\int |\nabla\varphi| \geq \mu \int |\varphi$ then $\lambda_1 \geq \frac{1}{4}\mu^2$ and then doing co-area to get a bound on μ. For Neumann, using courant nodal domain and reducing it to Dirichlet

3.4.2 Orthogonality of eigenfunctions on closed manifolds

- I guess the idea is that for dirichlet eigenfunctions

$$\int \varphi_1\varphi_2 = \int \frac{1}{\lambda_1}(-\Delta\varphi_1)\varphi_2 = \frac{1}{\lambda_1}\int \varphi_1(-\Delta\varphi_2) = \frac{\lambda_2}{\lambda_1}\int \varphi_1\varphi_2$$

which means that $\lambda_1 = \lambda_2$ or $\int \varphi_1\varphi_2 = 0$

3.4.3 Faber-Kahn theorem for lower bound for first eigenvalues

- The idea is to use the co-area formula and isoperimetric inequality, which we restate
- Isoperimetric inequality

Theorem 3.26. *Let M be a RM Ω a domain with compact closure in M, then there exists a constant C independent of Ω such that*

$$C(Vol(\Omega))^{(n-1)/n} \leq Vol(\partial\Omega)$$

- Co-Area Formula

Theorem 3.27. *Let M compact RM with boundary, $f \in W^{1,1}(M)$, then for any nonnegative measurable $g : M \to \mathbb{R}$, we have*

$$\int_M g = \int_{\mathbb{R}} \left(\int_{f=\sigma} \frac{g}{|\nabla f|}\right) d\sigma$$

- Note that for M compact, the sobolev inequality follows from the isoperimetric inequality and the co-area formula
- Now we use these to prove Faber-Kahn

Theorem 3.28. *Let $\Omega \subset \mathbb{R}^n$ a domain, $B(R)$ a ball in $\mathbb{R}^n$ of radius R such that $Vol(\Omega) = vol(B(R))$. Then we have $\lambda_1(\Omega) \geq \lambda_1(B(R))$*

Proof: Let f the first eigenfunction with dirichlet conditions on Ω, i.e. $\Delta f = -\lambda_1(\Omega)f$ and $f > 0$. Construct $g : B(R) \to \mathbb{R}^+$ such that

$$Vol(f \geq C) = Vol(g \geq C) \qquad \forall C > 0$$

g can be chosen to be radial, such that $g(R) = 0$. From this, we get

$$\int_\Omega f^2 = \int_{\mathbb{R}^+} Vol(f^2 \geq C)dc = \int_{\mathbb{R}^+} Vol(g^2 \geq C) = \int_{B(R)} g^2$$

we now show $\int_\Omega |\nabla f|^2 \geq \int_{B(R)} |\nabla g|^2$ which by dividing and using the rayleigh quotient characterization of the first eigenvalue will prove it. To show this, we comput

$$Vol(g = c) = \int_{g=c} 1 = \left(\int_{g=c} |\nabla g| \int_{g=c} \frac{1}{|\nabla g|}\right)^{1/2}$$

by the isoperimetric inequlity, and Holder's inequality

$$\int_{f=c} |\nabla f| \int_{f=c} \frac{1}{|\nabla f|} \geq \left(\int_{f=c} 1 \right)^2 = (Vol(f=c))^2 \geq (Vol(g=c))^2$$
$$= \int_{g=c} |\nabla g| \int_{g=c} \frac{1}{|\nabla g|}$$

And now by Co-area, we have

$$-\frac{dVol(f>c)}{dc} = \int_{f=c} \frac{1}{|\nabla f|}$$
$$-\frac{dVol(g>c)}{dc} = \int_{g=c} \frac{1}{|\nabla g|}$$

Obviously the two expressions are equal since $Vol(g>c) = Vol(f>c)$ for all c, which when combined gives

$$\int_{f=c} |\nabla f| \geq \int_{g=c} |\nabla g|$$

and applying Co-area again, we get

$$\int_\Omega |\nabla f|^2 = \int_{\mathbb{R}^+} \left(\int_{f=c} |\nabla f| \right) dc \geq \int_{\mathbb{R}^+} \left(\int_{g=c} |\nabla g| \right) dc = \int_{B(R)} |\nabla g|^2$$

which by rayleigh quotient characterization finishes it. □

The idea is to construct g radial such that $Vol(g>c) = Vol(f>c)$. Then compare their rayleigh quotients by comparing L^2 norms and L^2 norms of ∇f and ∇g. Note that $\int_\Omega f^2 = \int_{\mathbb{R}^+} Vol(f^2 \geq c)$ should follow by co-area? Unsure but it makes sense though, could be a fubini theorem trick

- Note that equality will only hold if $\Omega = B(R)$, since this requires equality in the isoperimetric inequality.

3.4.4 Variation of first eigenvalue w.r.t. domain

- Idea is that we have $\Phi : [0,1] \times \Omega \to \mathbb{R}^n$ such that $\Phi(0,\cdot) = Id\Big|_\Omega$, and for small t, $\Phi(t)$ is a diffeo, e.g.

$$\Phi(t) = I + tV$$

Let $\Omega_t = \Phi(t, \Omega)$ and $\lambda_k(t) = \lambda_k(\Omega_t)$ the kth dirichlet eigenvalue of the laplacian on Ω_t. Let u_t be the associated eigenfunction so that $||u_t||_{H_0^1(\Omega_t)} = 1$

Theorem 3.29. *Let Ω bounded open set, assume $\lambda_k(\Omega)$ simple. Then $t \mapsto \lambda_k(t)$, $t \mapsto u_t \in L^2$ are differentiable with*

$$\lambda_k'(0) = -\int_\Omega div(|\nabla u|^2 V)dx$$

If, moreover, Ω is C^2, then

$$\lambda_k'(0) = -\int_{\partial\Omega} \left(\frac{\partial u}{\partial n}\right)^2 \langle V, n\rangle d\sigma$$

Proof: Consider a C^1 extension of $u : \Omega \to \mathbb{R}$ to $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$. Define

$$G(t) := \frac{\int_{\Omega_t} |\nabla \tilde{u}|^2 dA}{\int_{\Omega_t} \tilde{u}^2 dA}$$

Then we have that

$$G(t) \geq \lambda_1(t), \qquad G(0) = \lambda_1(0) \implies G'(0) = \lambda_1'(0)$$

because $G - \lambda_1$ is a smooth function, positive, which achieves 0 (global min) at $t = 0$. We compute

$$\Phi_t(x) = x + tV + O(t^2), \qquad \Phi_t^{-1}(x) = x - tV + O(t^2)$$

we compute

$$G'(0) = \frac{\left(\partial_t \int_{\Omega_t} |\nabla \tilde{u}|^2 dA)\right|_{t=0} \cdot 1 - \int_\Omega |\nabla u|^2 \cdot \partial_t \left(\int_{\Omega_t} \tilde{u}^2 dA\right)}{(\int_{\Omega_t} \tilde{u}^2 dA)^2}$$

$$G'(0) = (\partial_t \int_{\Omega_t} |\nabla \tilde{u}|^2 dA)\Big|_{t=0}$$

here we've used that

$$\partial_t \int_{\Omega_t} \tilde{u}^2 dA = \int_{\partial\Omega} \tilde{u}^2 \mathrm{div}(V) = 0$$

since u vanishes on $\partial\Omega$. By the same reasoning, we have

$$\int_{\Omega_t} |\nabla \tilde{u}|^2 dA = \int_\Omega \sum_i u_i(\Phi_t)^2 \Phi_t^*(dA)$$

$$\implies \partial_t \int_{\Omega_t} |\nabla \tilde{u}|^2 dA\Big|_{t=0} = \int_\Omega 2u_i V(u_i) dA + \int_\Omega u_i^2 \mathrm{div}(V) dA$$

$$= \int_\Omega u_n^2 \langle V, n\rangle$$

after integrating by parts. This gives the desired answer!

3.4.5 Co-Area formula

- Statement (Otis' Math 258 notes)

  Theorem 3.30. *For (M, g) an RM and $u : M \to \mathbb{R}$ locally lipschitz and h a measurable function, then*

  $$\int_M h|\nabla u| = \int_{\mathbb{R}} \left(\int_{u^{-1}(s)} h\right) ds$$

  Proof: Suppose $\nabla u \neq 0$, choose coordinates so that $u(x) = x^n$ (i.e. run an implicit function theorem argument by choosing a direction to have non-zero gradient, e.g. $u_{x_i} \neq 0$, and then we can replace $x_i \to u(x)$). Let this transformation be F, then we have

  $$g(F_*(\partial_{x_n}), F_*(\partial_{x_n})) = g(du, du) = |\nabla u|^{-2}$$
  $$g(F_*(\partial_{x_n}), F_*(\partial_{x_j})) = 0, \qquad j < n$$
  $$g(\partial_{x_j}, \partial_{x_i}) = \delta_{ij}, \qquad i, j < n \implies d\mu_g = |\nabla u|^{-1} d\mu_{u^{-1}(s)} ds$$
  $$\implies \int_M w = \int_{\mathbb{R}} \left(w |\nabla u|^{-1} d\mu_{u^{-1}(s)}\right) ds$$

- There feels like some there are some holes in this proof, Not totally complete, but good enough (See here) (See Leon's updated lectures on geometric measure theory, or Evans Gariepy, "Measure theory and Fine properties of functions)

- So we're not really pushing forward ∂_{x_i} by the map $(x_1, \ldots, x_{n-1}) \to (x_1, \ldots, x_{n_1}, u)$ or anything. Really we're just saying the coordinates are

  $$(x_1, \ldots, x_{n-1}, u)$$

  where $\{x_1, \ldots, x_{n-1}\}$ are on level sets $u = c$. Not sure how $g(\partial_{x_n}, \partial_{x_n}) = |\nabla u|^{-2}$ comes about

3.5 Schauder Estimates

Statement of estimates, Simon's proof of estimates by scaling, applications (e.g. regularity of harmonic functions), Improved estimates for functions orthogonal to kernel

3.5.1 Statement of Estimates

- The main ones I'll be considering are first and second order estimates

  Theorem 3.31. *Suppose $\Omega \subseteq \mathbb{R}^n$ open subset and $u \in C^{2,\alpha}(\Omega)$ a bounded solution of*

  $$Lu = a^{ij}u_{ij} + b^k u_k + cu = f$$

  for $f \in C^\alpha$, with coefficients in C^α and a^{ij} positive definite. Then

  $$||u||^*_{C^{2,\alpha}} \leq K(||u||_{C^0(\Omega)} + ||f||^{(2)}_{C^\alpha,\Omega}$$

  here,

  $$||u||^*_{C^{2,\alpha}} := \max(1, d^{2+\alpha})||u||_{C^{2,\alpha}(\Omega)}$$

  and

  $$||f||^{(2)}_{C^\alpha} = \sup_{x\in\Omega} d_x^2|f(x)| + \sup_{x,y\in\Omega} d^{2+\alpha}_{x,y}\frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

  where d_x is the distance from x to the boundary.

- We also want first order estimates

  Theorem 3.32. *Suppose $\Omega \subseteq \mathbb{R}^n$ open subset and $u \in C^{2,\alpha}(\Omega)$ a bounded solution of*

  $$Lu = a^{ij}u_{ij} + b^k u_k + cu = div(V)$$

  for $f \in C^\alpha$, with coefficients in C^α and a^{ij} positive definite. Then

  $$||u||^*_{C^{1,\alpha}} \leq K(||u||_{C^0(\Omega)} + \sum_i ||V_i||^{(1)}_{C^\alpha,\Omega}$$

3.5.2 Schauder Estimates by scaling

- Instead of proving the first two estimates, we will prove Simon's estimate by scaling (skipping some details)

- Simon's estimate is a bit strong because he only requires the semi-norm on f, e.g. $[f]_\alpha$ as opposed to $||f||_{C^\alpha}$. Simon's estimate certainly implies the classical estimates but its a bit curious because

  $$\Delta u = 1$$

  would give

  $$||u||_{C^{2,\alpha}} \leq K||u||_{C^0}$$

  when we use $[f]_\alpha$ and

  $$||u||_{C^{2,\alpha}} \leq K(||u||_{C^0} + 1)$$

  when we use $||f||_\alpha$ (Otis is also uncertain about the strength of the schauder estimate with the semi-norm vs. the full C^α norm. Should probably just leave it for now)

- First note the interpolation inequality (not proven)

  $$\rho^{|\gamma|}\sup_{x\in B_\rho(x_0)}|D^\gamma u| \leq \epsilon\rho^{2n+\alpha}[D^2u]_{\alpha,B_\rho(x_0)} + C\sup_{x\in B_\rho(x_0)}(|u|)$$

  where $0 \leq \gamma_i \leq 2$. The proof will not be stated, but it is similar to one dimensional interpolation inequalities

- By the above, it suffices to prove

  $$[D^2u]_\alpha \leq [Lu]_\alpha$$

  for then, adding the above bounds for all values of γ, we get

  $$||u||_{C^{2,\alpha}} \leq K[D^2u]_\alpha + C||u||_{C^0} \leq K[Lu]_\alpha + C||u||_{C^0}$$

- We will prove this for our particular case of an elliptic second order differential operator, first with constant coefficients

Theorem 3.33. *Suppose $L = a^{ij}\partial_i\partial_j$ is the homogenous second order differential operator with elliptic coefficients. Then*

$$[D^2u]_{\alpha,\mathbb{R}^n} \leq C[Lu]_{\alpha,\mathbb{R}^n}$$

Proof: This relies on the following lemma

- **Lemma 3.34.** *If L is a κ-homogeneous operator of order $|k| = m$, elliptic, and $Lu = 0$ in $\mathbb{R}^n$, and if there are constants such that*

$$\sup_{B_R(0)} |u| \leq CR^q \qquad \forall R \geq 1$$

then u is a polynomial

Proof: For $\tilde{\Omega} \subset\subset \Omega$ and $Lu = 0$ inside, then we have

$$\sup_{\tilde{\Omega}} |D^k u| \leq C(k, L, \Omega, \tilde{\Omega}) \int_{\Omega} |u|^2, \qquad k \geq 1$$

This follows by the closed graph theorem for the graph $\{(f, Bf)\}$ where $B : \ker(L) \to C^0(\tilde{\Omega})$ given by $Bw = D^\delta w\Big|_{\tilde{\Omega}}$. The bound follows.

Now by homogeneity, we have

$$\sup_{B_{R/2}} |D^\gamma u|^2 \leq C(L, n, \gamma) R^{-2|\gamma|} |B_R(y)|^{-1} \int_{B_R(y)} |u|^2$$

Sending $R \to \infty$, we see that $D^\gamma u = 0$ for all $k = |\gamma|$ sufficiently large, which implies that u is a polynomial. $\square$

- **Proof of theorem**:

Suppose the bound is false, then after normalizing, we have a sequence of function $\{u_k\}$ such that $[D^2 u_k]_\alpha = 1$ and

$$[Lu]_\alpha < k^{-1}$$

Find some β such that $|\beta| = 2$, along with points x_k and precisions, h_k, such that

$$h_k^{-\alpha} |D^\beta u_k(x_k + h_k e_i) - D^\beta u_k(x_k)| \geq \frac{1}{4n}$$

i.e. we point pick so that the finite holder difference is non-trivial. We can shift $x_k \to 0$ and also rescale u_k's so that to get a new sequence $\{u_k^*\}$ such that

$$[D^2 u_k^*]_\alpha = 1, \qquad |D^\beta u_k^*(e_i) - D^\beta u_k^*(0)| \geq C, \qquad [Lu_k^*]_\alpha \leq k^{-1}$$

Now the idea is to define

$$\tilde{u}_k = u_k^* - (\text{its second order taylor polynomial})$$

Since L is homogeneous of order 2, we know that

$$L\tilde{u}_k = Lu_k^* + C$$

for some constant C, and so

$$[L\tilde{u}_k]_\alpha = [Lu_k^*]_\alpha$$

By Arzela ascoli, get a sequence of $\tilde{u}_k \to v$ converging in C^2 such that $v \in C^{2,\alpha}$ (Never thought about this, but it is true by triangle inequality)

$$D^\beta v(e_i) \neq 0, \quad D^\beta v(0) = 0, \quad [Lv]_\alpha = 0, \quad [D^2 v]_\alpha \leq 1$$

But $[Lv]_\alpha = 0$ implies that v is a constant. By construction of the $\tilde{u}_k$'s, we see that

$$Lv(0) = \lim_{k\to\infty} L\tilde{u}_k(0) = \lim_{k\to\infty} 0 = 0$$

(since we've subtracted off the second order taylar polynomial and are applying a second order operator). Thus $Lv \equiv 0$, and by the lemma v is a polynomial. But now

$$\sup_{t \in \mathbb{R}\setminus\{0\}} \frac{|D^\beta v(te_i) - D^\beta v(0)|}{t^\alpha} = \sup_{t \in \mathbb{R}\setminus\{0\}} \frac{|D^\beta v(te_i)|}{t^\alpha} = \infty$$

since $\alpha < 1$ and $D^\beta v(te_i)$ is a non-constant polynomial in t. This contradicts $[D^2 v]_\alpha < \infty$, so the initial bound must hold! □

(Ellipticity only comes in here via the lemma! From melrose)

- We now prove the slight generalization to lower order terms and variable coefficients
- First note that local estimates follow very similarly, and we have

Theorem 3.35. *Suppose L is 2-homogeneous, L is elliptic, and the 2nd order taylor series of u exists (with remainder term). Then*

$$[D^2 u]_{C^\alpha(B_{\theta\rho}(0))} \leq C\left([Lu]_{C^\alpha(B_\rho(0))} + \rho^{-2n-\alpha}||u||_{L^\infty(B_\rho(0))}\right)$$

where C depends only on n, θ, and L

I will not prove this in full since its essentially a modification of the original lemma on $\mathbb{R}^n$ with the following lemma. The idea is to prove

$$[D^2 u]_{C^\alpha(B_{\delta\sigma}(y))} \leq \epsilon[D^2 u]_{C^\alpha(B_\sigma(y))} + C[Lu]_{C^\alpha(B_\sigma(y))}$$

i.e. for any $\sigma, \epsilon > 0$, $y \in \mathbb{R}^n$, there exists **some** $\delta = \delta(\epsilon) \in (0,1)$ and $C = C(\epsilon)$. This is proved using the same techniques as the $\mathbb{R}^n$ estimates. Now note that

$$[D^2 u]_{C^\alpha(B_{\theta\sigma}(y))} \leq \epsilon[D^2 u]_{C^\alpha(B_\sigma(y))} + C([Lu]_{C^\alpha(B_\sigma(y))} + \sigma^{-2n-\alpha}||u||_{L^\infty(B_\sigma(y))})$$

for **every** $\theta \in (0,1)$ and every $B_\sigma(y) \subset B_\rho(0)$ where $C = C(\epsilon, \theta)$. This follows via interpolation by looking at the quotient $|h|^{-\alpha}|D^\gamma u(x + he_i) - D^\gamma u(x)|$ and breaking it into the cases where $0 < |h| < \delta(1-\theta)\sigma$ and $|h| \geq \delta(1-\theta)\sigma$. Note that the latter case is what gives the $||u||_{L^\infty}$ term.

Now the proof follows from the following lemma:

Lemma 3.36. *If S is a monotone subadditive function on convex subsets of a ball $B = B_{\rho_0}(y_0)$ (i.e. so that $S(A) \leq \sum_j S(A_j)$ whenever $A_1, \dots, A_N$ are given convex subsets of B with $A \subset \cup_{j=1}^N A_j$) and if $\theta_0 \in (0, \frac{1}{2}]$, $k > 0$ are given constants, then there is $\epsilon = \epsilon(\theta_0, k) \in (0,1)$ such that if $E \geq 0$ a constant and*

$$\sigma^k S(B_{\theta_0\sigma}(y)) \leq \epsilon\sigma^k S(B_\sigma(y)) + E$$

for all balls $B_\sigma(y) \subset B$ then for any $B_\rho(y)B$, we have

$$\rho^k S(B_{\theta\rho}(y)) \leq CE$$

for each $\theta \in (0,1)$ where $C = C(n, \theta_0, \theta, k)$

Indeed, we let S be the holder semi-norm on some ball and $E = C([Lu] + \rho^{-2n-\alpha}||u||_{L^\infty})$ and we conclude

- Variable coefficients and lower order terms:

Theorem 3.37. *If $L = a^{ij}(x)\partial_i\partial_j + b^k(x)\partial_k(x) + c(x)u$, then*

$$[D^2 u]_{C^\alpha(B_{\theta\rho}(0))} \leq C\left([Lu]_{C^\alpha(B_\rho(0))} + \rho^{-2n-\alpha}||u||_{L^\infty(B_\rho(0))}\right)$$

Proof: The idea is to write

$$L = P_y + R_y = \left(a^{ij}(y)\partial_i\partial_j\right) + \left([a^{ij}(x) - a^{ij}(y)]\partial_i\partial_j + b^k(x)\partial_k + c(x)\right)$$

for any $y \in B_\rho(0)$. Immediately, we have the estimate for P_y since locally its a constant coefficient homogeneous operator. Note that for each $\eta \in (0,1)$, (Honestly, no idea why this holds for all η, seems like you can just send $\eta \to 0$ and kill the $[D^2u]_\alpha$ term we have

$$[R_y u]_{C^\alpha(B_\sigma(y))} \le C\Lambda\eta^\alpha[D^2u]_{C^\alpha(B_\sigma(y))} + C\Lambda\sum_{i=0}^{1}\sigma^{|\delta|-2n-\alpha}\sup_{B_\sigma(y)}|D^\delta u|$$

where $\Lambda \ge |a^{ij}(x)|, |b^k(x)|, |c(x)|$ for all x. Now for η sufficiently small and using $P_y u = Lu - R_y u$, we have

$$[D^2u]_{C^\alpha(B_{\sigma/2}(y))} \le \epsilon[D^2u]_{C^\alpha(B_{\eta\sigma}(y))} + C([Lu]_{C^\alpha(B_\rho(y))} + \sigma^{-|\gamma|}\sup_{B_\rho(y)}|D^\gamma u|)$$

Now interpolation gives

$$[D^2u]_{C^\alpha(B_{\sigma/2}(y))} \le \epsilon[D^2u]_{C^\alpha(B_\sigma(y))} + C([Lu]_{C^\alpha(B_\rho(y))} + \sigma^{-2n-\alpha}\sup_{B_\rho(y)}|u|)$$

but this is the same type of inequality as in the local estimates, so we can use the monotone subadditive functional lemma to conclude the bound we want.

3.5.3 Applications

- Harmonic equations

  $$\Delta u = 0$$

  We can get regularity quite easily from this: Assuming that $u \in C^0$ (or really, just L^∞) then we have on the interior

  $$||u||_{C^{2,\alpha}} \le K(||\Delta u||_{C^\alpha} + ||u||_{C^0}) \implies ||u||_{C^{2,\alpha}} \le K||u||_{C^0}$$

  So really, $u \in L^\infty$ lets us upgrade to $u \in C^{2,\alpha}$. Differentiate this equation (say we're in the euclidean setting), let's us do

  $$\Delta u_{x_i} = 0$$

  and so repeating the schauder argument with $||u_{x_i}||_{C^{2,\alpha}} \le K||u_{x_i}||_{C^0}$ lets us conclude regularity for higher derivatives. Repeating this ad infinitum tells us that u is smooth.

- Another application would be: for a family of solutions to some elliptic linear PDE, L, then we can use Schauder estimates + Arzela-Ascoli + uniform boundedness of C^0 norms lets us construct a subsequence which converges to some limiting function

- Cool source for more applications

- Regularity of eigenfunctions

  $$(\Delta + \lambda)u = 0, \qquad L = \Delta + \lambda$$

  so eigenfuntions will be regular by the same argument as above

- Regularity of Allen-Cahn solutions: we know that solutions to Allen-Cahn satisfy $|u| < 1$ everywhere by the maximum principle. We use the $C^{1,\alpha}$ interior schauder estimate to get

  $$||u||_{C^{1,\alpha}} \le K(||Lu||_{C^0} + ||u||_{C^0})$$

  here we reframe the semi-elliptic PDE as a linear operator:

  $$\Delta u - u(u^2 - 1) = 0, \qquad L = \Delta + (1 - u^2)$$

  and so $Lu = 0$ (since our coefficients are in L^∞) and we get that

  $$||u||_{C^{1,\alpha}} \le K(||u||_{C^0}) \le K$$

  we can differentiate Allen-Cahn and repeat this process, e.g.

  $$\Delta u_{x_i} - W''(u)u_{x_i} = 0, \qquad L_1 = \Delta + (1 - 3u^2)$$

  and now we can use $C^{2,\alpha}$ estimates on u_{x_i} since $(1 - 3u^2) \in C^\alpha$. This gives

  $$||u_{x_i}||_{C^{2,\alpha}} \le K||u_{x_i}||_{C^0} \le K||u||_{C^{1,\alpha}} \le K^2$$

  repeating this with more derivatives gives smoothness of solutions to Allen-Cahn

3.5.4 Improved Estimates

The idea is that suppose we have the standard Schauder interior estimate for functions $u : \mathbb{R}^n \to \mathbb{R}$ and an elliptic operator L

$$||u||_{C^{2,\alpha}} \leq K(||Lu||_{C^\alpha} + ||u||_{C^0})$$

Suppose that L has no kernel on whatever space we're working on (could be $C^{2,\alpha}$ or some subset thereof). Then we can prove

$$\boxed{||u||_{C^{2,\alpha}} \leq K||Lu||_{C^\alpha}}$$

by contradiction.

Proof: It suffices to prove that

$$||u||_{C^0} \leq K||Lu||_{C^0}$$

Suppose not, then we have a sequence of functions $\{u_j\}$ such that

$$\frac{1}{j}||u_j||_{C^0} \geq ||Lu_j||_{C^0}$$

By linearity, normalize each u_j so that $||u_j||_{C^0} = 1$ and there exist $\{p_j\}$ so that $|u_j(p_j)| \geq \frac{1}{2}$. This tells us that $||Lu_j||_{C^0} \to 0$ and schauder estimates also give

$$||u_j||_{C^{2,\alpha}} \leq K\left(1 + \frac{1}{j}\right)||u_j||_{C^0} \leq 2K$$

so we have uniform $C^{2,\alpha}$ bounds. The same holds for $\tilde{u}_j(p) = u_j(p + p_j)$. Then we have uniform $C^{2,\alpha}$ bounds which means we get convergence in C^2 (by Arzela-Ascoli) along a subsequence to some

$$u_\infty(p) = \lim_{j\to\infty} \tilde{u}_j(p)$$

Moreover, we see that $|u_\infty(0)| \geq \frac{1}{2}$ by definition of $\tilde{u}_j(p)$. Then we have that

$$\{\tilde{u}_j\} \xrightarrow{C^2} u_\infty \implies L\tilde{u}_j \xrightarrow{C^2} Lu_\infty = 0$$

and so we've found $u_\infty \in \ker(L)$ but $u_\infty \neq 0$. This is a contradiction to the fact that $u_\infty(p) \neq 0$! $\square$

Sometimes we work on domains with boundary (i.e. the domain of $\{u_j\}$ lacks translation invariance by arbitrary p_j), in which case we need to be worried that the point picking argument doesn't give $|p_j| \to \infty$. Then we need something like the maximum principle to tell us that if $|u_j(p_j)| > 1/2$ then p_j is bounded.

3.5.5 Schauder Estimates Via Green's Function

Here, I'll talk a little bit about how Schauder estimates for harmonic functions can be derived from the green's formula.

- Define the fundamental solution

$$\Gamma(x-y) = \Gamma(|x-y|) = \begin{cases} \frac{1}{n(2-n)\omega_n}|x-y|^{2-n}, & n > 2 \\ \frac{1}{2\pi}\log|x-y|, & n = 2 \end{cases}$$

- We compute

$$D_i\Gamma = \frac{1}{n\omega_n}(x_i - y_i)|x-y|^{-n}$$

$$D_{ij}\Gamma = \frac{1}{n\omega_n}[|x-y|^2\delta_{ij} - n(x_i - y_i)(x_j - y_j)]|x-y|^{-n-2}$$

  And we can bound

$$|D_i\Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{1-n}$$

$$|D_{ij}\Gamma(x-y)| \leq \frac{1}{n\omega_n}|x-y|^{-n}$$

$$|D^\beta\Gamma(x-y)| \leq C|x-y|^{2-n-|\beta|}, \qquad C = C(n, |\beta|)$$

- We have
$$u(y) = \int_{\partial\Omega} \left(u\frac{\partial \Gamma}{\nu}(x-y) - \Gamma(x-y)\frac{\partial u}{\partial \nu}\right) ds + \int_{\Omega} \Gamma(x-y)\Delta u dx, \qquad y \in \Omega$$
This follows from the general formula of
$$\int_{\Omega} v\Delta u - u\Delta v = \int_{\partial\Omega} v\frac{\partial u}{\partial \nu} - u\frac{\partial u}{\partial \nu}$$
and then noting that $\Delta\Gamma = \delta_y$

- We get the following on u harmonic just by differentiating
$$|Du(y) \leq \frac{n}{d_y} \sup_{\Omega} |u|$$
where $d_y = \text{dist}(y, \partial\Omega)$. This also follows from the mean value property for u_i, which are harmonic
$$u_i(y) = \frac{1}{\omega_n R^n}\int_B u_i(x)dx = \frac{1}{\omega_n R^n}\int_B \langle \nabla u, \nabla x_i\rangle dx = \frac{1}{\omega_n R^n}\int_{\partial B} u\langle e_i, \nu\rangle \leq \frac{C}{R}\sup_{\partial B}|u|$$

- In general, we have (either by differentiating the green's function or doing the mean value trick)

  Theorem 3.38. *Let u be harmonic in Ω and Ω' any compact subset, then for any multi-index α, we have*
$$\sup_{\Omega'} |D^\alpha u| \leq \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u|$$

4 Topic 3: Miscellaneous

- The Allen-Cahn equation - 1-D solution, BV functions, Modica-Mortola result for BV compactness, Stability operator, Stable solutions on $\mathbb{R}^2$, Modica Inequality, Monotonicity formula for $E(B_R(0), u)$, kernel classification of $L_* = \Delta_g - W''(g)$, Exponential decay of solutions away from nodal set, Solutions on S^n (nodal set is equator, two parallels), Γ-convergence of nodal sets to minimal hypersurfaces
- Wave equation - solution to linear wave equation, fundamental solution, Finite speed of propogation, monotonicity of energy functional, existence of solutions to linear wave equation, uniqueness of solutions to linear wave equation
- Ginzburg Landau - Comparison to Allen-Cahn, canonical 2D solutions, convergence of $\{u_\epsilon\} \to u^*$ an S^1 valued function, dependence of u^* on dirichlet data, harmonic functions in $C(\Omega\backslash\{p_i\}_{i=1}^n, S^1)$
- Poincare-Einstein metrics - graham normal form, boundary defining functions, evenness of metric, correspondence between conformal infinity and einstein metric to the interior, Examples of PE spaces

4.1 Allen-Cahn Equation

4.1.1 1-D solution

For $\epsilon = 1$, it is the heteroclinic given by
$$f''(t) = f(f^2 - 1)$$
and for non-zero ϵ, just replace $f(t) \to f(t/\epsilon)$, where
$$f(t) = \tanh(t/\sqrt{2})$$
how to prove this? Note that if we frame
$$\begin{aligned} f''(t) &= W'(f(t)) \\ \implies f''f' &= W'f' \\ \implies \frac{1}{2}(f')^2 &= W(f) + C \end{aligned}$$

when $C = 0$, the above is equivalent to

$$f' = \pm\frac{1}{\sqrt{2}}(1 - f^2)$$

which gives the solution. We can fix the sign by assuming that $f'(t) > 0$ for all t (note: $f \in [-1, 1]$ by energy arguments, and we want f to be an energy minimizer, also if $f = \pm 1$, then maximum principle tells us $f \equiv 1$, so f' has a definitive sign). Then $f(t) = \tanh(t/\sqrt{2})$ can be checked to be a solution (or integration). When $C \neq 0$, one can show that $f(t)$ has infinite energy by noting that the Allen-Cahn integration is given by

$$\int \frac{1}{2}(f')^2 + W(f)dt =$$

if energy is finite, then there exist points such that $\frac{(f')^2}{2} + W(f) \to 0$, but then from $(f')^2 = 2W(f) + C$, we see that $C \to 0$, so $C = 0$.

4.1.2 BV functions

- For $\Omega \subseteq (M, g)$, a BV function is $u \in L^1(\Omega)$ such that Du is a TM-valued Radon measure, i.e. for any $X \in C^1_c(\Omega; TM)$

$$\int_\Omega u\mathrm{div}_g(X) = -\int_\Omega g(X, Du)$$

  Then $|Du|$ is a usual Radon measure defined by

$$\int_{\Omega'} |Du| = \sup\{\int_{\Omega'} u\mathrm{div}_g(X)d\mu_g \ : \ X \in C^1_c(\Omega'; TM), \ \ ||X||_{L^\infty} \leq 1\}$$

  for any $\Omega' \subseteq \Omega$, and we have the norm

$$||u||_{BV(\Omega)} = ||u||_{L^1(\Omega)} + \int_\Omega |Du|$$

4.1.3 Modica-Mortola result for BV compactness/Γ-convergence of nodal sets to minimal hypersurfaces

The main result is this proposition from Otis' notes:

Proposition. *For $\Omega \subseteq (M, g)$ a precompact open set, suppose $\{u_\epsilon\}$ satisfy $E_\epsilon(u_\epsilon; \Omega) \leq C$. Then there is a subsequence $\epsilon_k \to 0$ and $u_0 \in BV_{loc}(\Omega)$ with $u_0 \in \{\pm 1\}$ a.e. and*

$$u_{\epsilon_k} \to u_0$$

in $L^1_{loc}(\Omega)$. Moreover

$$\sigma P(\{u_0 = 1\}; \Omega') \leq \liminf_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k}; \Omega')$$

where $\sigma = \Phi(1) - \Phi(-1) = \int_{-1}^1 \sqrt{2W(s)}ds$ for any Ω' compactly contained in Ω

Proof: We sketch the proof as follows:

1. Define

$$\Phi(t) = \int_0^t \sqrt{2W(s)}ds$$

   One can compute that

$$|\Phi(t)| \leq \alpha + \beta W(t)$$

   and so uniform energy bounds imply that

$$||\Phi(u_\epsilon)||_{L^1} \leq C$$

   We also have

$$E_\epsilon(u; \Omega) \geq \int_\Omega |\nabla_g \Phi(u_\epsilon)| d\mu_g$$

and so $\Phi(u_\epsilon)$ is in $W^{1,1}$. By BV compactness, we find $v_0 \in BV_{loc}(\Omega)$ so that a subsequence of $\Phi(u_\epsilon)$ convergens in L^1_{loc} to v_0 and

$$\int_{\Omega'} |Dv_0| \leq \liminf_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k};\Omega')$$

The function Φ is invertible (needs to be checked) and Φ^{-1} is uniformily continuous. Moreover $W(t) \geq ct^4$ for t sufficiently large, so we get that

$$||u_\epsilon||_{L^4}^4 = \int_\Omega u_\epsilon^4 \leq \int_{\Omega\cap\{u_\epsilon<t_0\}} u_\epsilon^4 + \int_{\Omega\cap\{u_\epsilon\geq t_0\}} u_\epsilon^4$$

The first integral is bounded and for the second one

$$\int_{\Omega\cap\{u_\epsilon\geq t_0\}} u_\epsilon^4 \leq c^{-1}\int_{\Omega\cap\{u_\epsilon\geq t_0\}} W(u_\epsilon) \leq C$$

and so $u_\epsilon \in L^4$ with uniform bounds coming from $|\Omega|$ and C, the energy bound. It suffices (because we only care about our results up to a subsequence) to find a further subsequence from our original so that $u_{\epsilon_k} \to u_0 := \Phi^{-1}(v_0)$ with convergence in $L^1_{loc}(\Omega)$ and that $u_0 \in BV_{loc}(\Omega)$ with $u_0 = \pm 1$ a.e. in Ω. The fact that $u_0 = \pm 1$ a.e. follows from Chebyshev:

$$\frac{\delta^2}{4}\mu\{x\in\Omega' \ : \ |u_{\epsilon_k}(x)^2 - 1| > \delta\} \leq \int_{\Omega'} W(u_\epsilon) \leq C\epsilon$$

(remember the energy is $\epsilon|\nabla u|^2/2 + \frac{1}{\epsilon}W(u)$) and the above holds for any $\delta > 0$. Now compute

$$\begin{aligned}
\Phi(u_0) &= \Phi(1)\chi_{u_0=1} + \Phi(-1)\chi_{u_0=-1} \\
&= [\Phi(1) - \Phi(-1)]\chi_{u_0=1} + \Phi(-1)[\chi_{u_0=-1} + \chi_{u_0=-1}] \\
&= [\Phi(1) - \Phi(-1)]\chi_{u_0=1} + \Phi(-1)\chi_\Omega
\end{aligned}$$

a.e. in Ω, hence

$$\int_{\Omega'} |D\Phi(u_0)| = (\Phi(1) - \Phi(-1))P(\chi_{u_0=1};\Omega') = (\Phi(1) - \Phi(-1))\int_{\Omega'} |Du_0|$$

which completes the proof $\square$

2. **Details**:
Below is the content of exercise 3.1 in the notes

- We show $|\Phi(t)| \leq \alpha + \beta W(t)$. Note that it suffices to show this for $t > 1$

$$\Phi(t) = \int_0^t \sqrt{2W(s)}ds = \frac{1}{\sqrt{2}}\int_0^t |1-s^2|ds = C + \frac{1}{\sqrt{2}}\int_1^t (s^2-1) = C + \frac{1}{\sqrt{2}}\left(\frac{t^3}{3} - t\right)$$

  but clearly a cubic is less than a quadratic (e.g. $W(t)$) for t large, so this is fine
- We note that Φ is continuous and increasing away from its critical points. Thus it is invertible. For refernce, the derivative is

$$\Phi'(t) = \sqrt{2W(t)} = \frac{1}{\sqrt{2}}|1-t^2|$$

  Hence

$$(\Phi^{-1})'\Big|_{\Phi(t)} = \frac{1}{|1-t^2|}$$

  away from $t = \pm 1$, this is bounded above, so we have a uniform lipschitz bound. Near ± 1, note that Φ^{-1} is still continuous, so we can just take some δ which works as a modulus of continuity about a small neighborhood of ± 1 and then we compare this with the uniform bound we get elsewhere.

- We want to show that that $u_{\epsilon_k} \to u_0 =: \Phi^{-1}(v_0)$ in measure on Ω. Recall that convergence in measure is given by

$$\lim_{k\to\infty} \mu(|u_{\epsilon_k} - u_0| \geq \delta) = 0$$

for any $\delta \geq 0$. Set

$$v_k = \Phi^{-1}(u_{\epsilon_k}), \qquad v_0 = \Phi^{-1}(u_0)$$

then by uniform continuity of Φ^{-1}, we have

$$|u_{\epsilon_k} - u_0| \geq \delta \implies |v_k - v_0| \geq \delta'(\delta)$$

i.e.

$$\{|u_{\epsilon_k} - u_0| \geq \delta\} \subseteq \{|v_k - v_0| \geq \delta'(\delta)\}$$

but we know that (up to a subsequence

$$\lim_{k\to\infty} \mu\{|v_k - v_0| \geq \delta'(\delta)\} = 0$$

by convergence in BV. Thus we get convergence in measure

- We show that if (X, μ) a finite measure space, if $\{f_i\}$ measureable functions converging to f in measure and $||f_i||_{L^p(X)} \leq C$ for some $C > 0$, $p > 1$, then $f_i \to f \in L^1$

  Note that uniform L^p estimates for $p > 1$ immediately give uniform L^1 estimates (call the bound C) for f_i, just by dividing X into the set where $|f_i| > 1$ and $|f_i| \leq 1$. We now compute

$$\int_X |f_i - f| = \left(\int_{X\cap|f_i|\geq k,\text{ or }|f|\geq k} + \int_{X\cap|f_i|\leq k,\text{ and }|f|\leq k}\right)|f - f_i| = I_1 + I_2$$

  Note that by uniform L^1 estimates, we have

$$\mu\{|f_i| > k\} \leq \frac{C}{k}$$

  which tends to 0 with k large. Call the first set A and the second set above B. We know that

$$|A| \leq \frac{2C\mu(X)}{k} \implies I_1 \leq \int_A |f| + |f_i| = \int |f|\chi_A + \int |f_i|\chi_A \leq ||f||_p||\chi_A||_q + ||f_i||_p||\chi_A||_q \leq \tilde{C}\cdot k^{-1/q}$$

  so I_1 is negligible. For I_2 we decompose B into $|f - f_i| \geq \delta$ and $|f - f_i| \leq \delta$. On the first set, we know that its measure is small, say less than ϵ for i large, and also

$$|f - f_i| \leq 2K$$

  so the integrand of I_2 over the first set is less than $\epsilon 2K \to 0$. For the second set, it's less than $\delta\mu(X)$ which can also be sent to 0 with $\delta \to 0$. Thus we get L^1 convergence

- We now conclude that $u_{\epsilon_k} \to u_0$ in $L^1(\Omega')$ and thus (after passing to a further subsequence via diagonal argument) a.e. in Ω. The latter fact follows because convergence in L^p for $p \geq 1$ gives a pointwise convergent subsequence a.e. The former fact follows because we have uniform L^4 bounds on $\{u_{\epsilon_k}\}$, as well as convergence in measure of $u_{\epsilon_k} \to u_0$, so we get L^1 convergence

- Note that $|u_0| = \pm 1$ a.e. because we have that $u_{\epsilon_k} \to u_0$ and the convergence in measure of $u_{\epsilon_k}^2 \to 1$. This is because L^1 convergence implies convergence in measure from Chebyshev's inequality and the triangle inequality. To see that $u_0 \in BV_{loc}$, we know that $u_0 \in L^1$. Moreover, to show boundedness of Du as a TM-valued Radon measure, we write

$$\begin{aligned} u_0 &= \Phi^{-1}(v_0) \\ &= 1\chi_{v_0=\Phi(1)} + (-1)\chi_{v_0=\Phi(-1)} \\ &= 2\chi_{v_0=\Phi(1)} - 1 \\ &= 1 - 2\chi_{v_0=\Phi(-1)} \end{aligned}$$

we use the last two expressions as follows: using the first, we have

$$\int u_0 \mathrm{div}_g(X) = \int_{v_0=\Phi(1)} 2\mathrm{div}_g(X) = \frac{2}{\Phi(1)} \int_{v_0=\Phi(1)} v_0 \mathrm{div}_g(X)$$
$$\int u_0 \mathrm{div}_g(X) = \int_{v_0=\Phi(-1)} -2\mathrm{div}_g(X) = \frac{-2}{\Phi(-1)} \int_{v_0=\Phi(-1)} v_0 \mathrm{div}_g(X)$$

okay but now note that $\Phi(1) = -\Phi(-1)$ from the definition, so we add the two lines and get

$$\begin{aligned} 2\int u_0 \mathrm{div}_g(X) &= \frac{2}{\Phi(1)} \left(\int_{v_0=\Phi(1)} + \int_{v_0=\Phi(-1)} \right) v_0 \mathrm{div}_g(X) \\ &= \frac{2}{\Phi(1)} \int_{\Omega} v_0 \mathrm{div}_g(X) \end{aligned}$$

the last expression is bounded by $\frac{2}{\Phi(1)}||Dv||_{L^1}$, so we see that u_0 has a bounded BV norm! This was tricky and not obvious (Another way to note this is that $u_0 = \frac{1}{\Phi(1)}v_0$ a.e. since we know $u_0 = \pm 1$ everywhere corresponding to when $v_0 = \Phi(\pm 1)$)

We also have the following proposition, which I will sketch

Proposition. *If $E \subseteq \Omega$ set of finite perimeter, then there is a sequence $u_\epsilon \in H^1(\Omega) \cap L^4(\Omega)$ with*

$$\sigma P(E;\ \Omega) = \lim_{\epsilon \to 0} E_\epsilon(u_\epsilon;\ \Omega)$$

and $u_\epsilon \to \chi_\epsilon - \chi_{\Omega \setminus E}$ in $L^1(\Omega)$

Proof: When ∂E is smooth, basically we just paste in the heteroclinic, $g_\epsilon(t)$ cutoff to be $\pm$ outside of some finite distance range K. Then the standard fermi computation does it, i.e. let $u_\epsilon = \varphi(\epsilon^{-1}t)$ where φ is the cut off heteroclinic at the normal scale and cut off outside of $(k\ln(\epsilon), -k\ln(\epsilon))$. Then

$$\begin{aligned} E_\epsilon(u_\epsilon;\Omega) &= \int_\Omega \frac{1}{2\epsilon}(\varphi')^2 + \epsilon^{-1}W(\varphi)d\mu_g \\ &= \int_{k\epsilon\ln(\epsilon)}^{-k\epsilon\ln(\epsilon)} \int_{\Sigma_t} (\cdots) d\mu_{\Sigma_t} dt \\ &= \int_{k\epsilon\ln(\epsilon)}^{-k\epsilon\ln(\epsilon)} \int_{\Sigma_t} (\cdots)(\mu(\Sigma) + O(t)) dt \\ &= \int_{k\ln(\epsilon)}^{-k\ln(\epsilon)} \int_{\Sigma_t} (\cdots)\mu(\Sigma) dt + O(\epsilon) \\ &= (\sigma + O(\epsilon^k))\mu(\Sigma) + O(\epsilon) \end{aligned}$$

so this works. Here Σ is ∂E and σ is the energy of the heteroclinic on $\mathbb{R}$, and we note that as $\epsilon \to 0$, the energy of our cut-off heteroclinic converges to the energy of the heteroclinic itself (log cutoff in 1-D is okay). The first $O(\epsilon)$ comes from changing coordinates $t \to t/\epsilon$

4.1.4 Stability operator

For u_ϵ a solution to the ϵ dependent AC-equation, we have

$$Q_{u_\epsilon}(\psi,\psi) = \int_M \left(\epsilon|\nabla\psi|^2 + \frac{1}{\epsilon}W''(u_\epsilon)\psi^2 \right) d\mu_g$$

Main proposition:

Proposition. *If u_ϵ stable solution to AC, then*

$$\int_{|\nabla u_\epsilon| \neq 0} (|\nabla\psi|^2|\nabla u_\epsilon|^2 - ((|D^2u_\epsilon|^2 - |\nabla|\nabla u_\epsilon||^2) + Ric_g(\nabla u_\epsilon, \nabla u_\epsilon))\psi^2) d\mu_g \geq 0$$

Proof: The way to prove this is to use a bochner formula for u_ϵ, plug in $\psi\sqrt{|\nabla u_\epsilon|^2+\delta^2}$ into stability, and send $\delta \to 0$. $\square$

Corollary 4.0.1. *Suppose (M^n, g) with positive ricci curvature. Then there are no non-trivial stable solution to allen-cahn*

Proof: Note that

$$|\nabla|\nabla u_\epsilon||^2 \leq |D^2 u|^2$$

when $|\nabla u| \neq 0$. Now plug in $\psi = 1$ (I would prove this by computing $\nabla|\nabla u_\epsilon|^2 = |\nabla u_\epsilon|(\nabla|\nabla u_\epsilon|)$)

4.1.5 Stable solutions on $\mathbb{R}^2$

We have the following theorem (we work with $\epsilon = 1$ since we can rescale on $\mathbb{R}^2$):

Theorem 4.1 (Ghoussoub-Gui, 98)**.** *Suppose $u \in C^2(\mathbb{R}^2)$ a stable solution to AC with $|u| \leq 1$. Then*

$$u(x) = \mathbb{H}(\langle a, x\rangle - b)$$

for some $a \in S^1$ and $b \in \mathbb{R}^2$, where $\mathbb{H}$ is the heteroclinic

Proof: First note that stability implies (by the previous proposition and $\mathrm{Ric}_{\mathbb{R}^2} \equiv 0$)

$$\int_{\mathbb{R}^n} |\nabla\varphi|^2|\nabla u|^2 \geq \int |B|^2\varphi^2|\nabla u|^2$$

where

$$|B|^2 = |\nabla u|^{-2}(|D^2u|^2 - |\nabla|\nabla u||^2)$$

when $|\nabla u| \neq 0$ and $|B|^2 = 0$ when $|\nabla u| = 0$. This is the enhanced second fundamental form, since we can think of $(-u_x, -u_y, 1)/\sqrt{1+|\nabla u|^2}$ as the normal.

Now there are a few steps to show: **the heteroclinic is the unique stable solution on $\mathbb{R}^n$ with vanishing enhanced second fundamental form**

1. Note a solution to AC cannot have $\nabla u = 0$, else it is the constant ± 1.
2. When $\nabla u \neq 0$, $|D^2u|^2 - |\nabla|\nabla u||^2 \geq 0$ (to see this, try computing $\nabla|\nabla u|^2 = |\nabla u|\nabla|\nabla u|$ in two ways!) (Here's the computation: It suffices to show that

$$|\nabla|\nabla u|^2|^2 \leq 4|\nabla u|^2\ |D^2u|^2$$

Simply because the LHS is

$$|\nabla|\nabla u|^2|^2 = |2|\nabla u|\ \nabla|\nabla u||^2 = 4|\nabla u|^2\ |\nabla|\nabla u||^2$$

and so if we're allowed to divide by $|\nabla u|$ then this is clearly equivalent to the original statement. We compute

$$\nabla|\nabla u|^2 = \sum_{i,j} 2u_i u_{ij} e_j$$

$$\begin{aligned}\implies |\nabla|\nabla u|^2|^2 &= 4\sum_{i,j,k} u_j u_k u_{ij} u_{ik} = 4\langle \nabla u \circ (\nabla u)^T, (D^2u \circ D^2u)\rangle \\ &\leq 4|\nabla u \circ (\nabla u)^T||D^2u \circ D^2u|\end{aligned}$$

but now note that

$$|\nabla u \circ (\nabla u)^T| \leq |\nabla u|^2, \qquad |D^2u \circ D^2u| \leq |D^2u|^2$$

and so

$$|\nabla|\nabla u|^2|^2 \leq 4|\nabla u|^2\ |D^2u|^2$$

)

3. If $|B|^2 = 0$ on all of $\mathbb{R}^n$, then $\frac{\nabla u}{|\nabla u|}$ is a parallel vector field on $\{\nabla u \neq 0\}$, which can be seen by computing $\left|D\left(\frac{\nabla u}{|\nabla u|}\right)\right|$.

4. By unique continuation, a solution to AC with $|B|^2 = 0$ on all of $\mathbb{R}^n$ must be
$$u(x) = \tilde{u}(\langle a, x - x_0\rangle)$$
for some one dimensional function $\tilde{u}$.

5. If in addition to $|B|^2 = 0$, we also assume that u is stable then $\tilde{u}$ must be the heteroclinic (So for 1-D, solutions to Allen-Cahn are either heteroclinic or periodic (Poincare-Bendixson). Look at the first order ODE to get this. Periodic solutions cannot be stable. Take AC and differentiate it
$$u'' = W'(u) \to u''' = W''(u)u'$$
This is a homogeneous ODE i.e. even though u depends on time t, we see that both side just depend on u and its derivatives, hence poincare-bendixson applies. Suppose u' on some interval has 3 zeros (u' always has zeros because u periodic, so just choose interval large enough to have 3 zeros), then u' is in $\ker(L)$ for $L = \partial_t^2 - W''(u)$, but stability would then say that u' is the lowest eigenfunction (on an interval which we're periodic on), but this can't be the case, if u' has 3 zeros, because first eigenfunctions are always positive on the interior. Thus u' can't be periodic and we have to have that its the heteroclinic

this finishes our classification of stable solutions with vanishing second fundamental form.

Also note that by schauder estimates, because $|u| \leq 1$, we have $|\nabla u| \leq C$ on $\mathbb{R}^n$ for $C = C(n)$ for any AC solution.

Now when $n = 2$, we consider cutoff functions $\{\varphi_i\}$ such that $\varphi_i \to 1$ pointwise on $\mathbb{R}^2$ and $\int_{\mathbb{R}^2} |\nabla \varphi_i|^2 \to 0$. We do this with a log-cutoff, φ_R. But now we use stability
$$\liminf_{R\to\infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 |\nabla u|^2 \leq C^2 \liminf_{R\to\infty} \int_{\mathbb{R}^2} |\nabla \varphi_R|^2 = 0$$
Moreover, because $\varphi_R \to 1$ pointwise and by fatou, we have (by stability)
$$\int_{\mathbb{R}^2} |\nabla \varphi|^2 |\nabla u|^2 d\mu \geq \int_{\mathbb{R}^2} |B|^2 \varphi^2 |\nabla u|^2 d\mu$$
$$\implies \int_{\mathbb{R}^2} |B|^2 |\nabla u|^2 \leq 0$$
so $|B|^2 \equiv 0$, which tells us that the solution is one dimensional.

4.1.6 Modica Inequality

(Problem 1 in appendix B of Otis notes)

We want to show that
$$P = |\nabla u|^2 - 2W(u) \leq 0$$
for $u \in C^2_{loc}(\mathbb{R}^n)$. We proceed as follows:

1. Show that $\inf_{\mathbb{R}^n} |\nabla u| = 0$ - if not then we could find a path via gradient flow, call it $\gamma(t)$, such that $\gamma' = \nabla u$ at any given point, and then
$$u(\gamma(t_2)) - u(\gamma(t_1)) = \int_{t_1}^{t_2} \frac{d}{dt} u(\gamma(t)) = \int_{t_1}^{t_2} \nabla u \cdot \gamma' = \int_{t_1}^{t_2} |\nabla u|^2 \geq (t_2 - t_1)C^2$$
which would tend to infinity as $t_2 - t_1 \to \infty$. But then we'd get that $|u| > 1$ in finite time, a contradiction

2. We show that
$$|\nabla u|^2 \Delta P \geq \frac{1}{2} |\nabla P|^2 + W'(u) \nabla u \cdot \nabla P$$
this just comes from computing the laplacian of P and using a bochner formula for u along with $\Delta u = W'(u)$

3. We prove that $P \leq 0$ if $\sup_{\mathbb{R}^n} P$ is attained somewhere in $\mathbb{R}^n$. Assume $\sup_{\mathbb{R}^n} P = P(0) > 0$. Then the laplacian must be negative so
$$0 \geq |\nabla u|^2 \Delta P > \frac{1}{2}|\nabla P|^2 + W'(u)\nabla u \cdot \nabla P$$
but $\nabla P = 0$ at the critical point, we have
$$0 > 0$$
a contradiction. (It's actually a bit more refined than this. Instead of direct comparison, frame this as a second order operator on P, then use maximum principle to conclude that P is locally constant. Take care to handle cases when $|\nabla u| \neq 0$ and $|\nabla u| = 0$. Good solution in my notes

4. If P does not attain its supremum, we still have $P \leq 0$. THe idea is take $\{x_i\}$ such that $P(x_i) \to \sup_{\mathbb{R}^n} P > 0$. Consider $u_i(x) = u(x - x_i)$, which all have uniform $C^{2,\alpha}$ bounds by schauder estimates, the allen-cahn equation, and $|u_i| \leq 1$. Then by arzela-ascoli, we get convergence to some u_∞ on $\mathbb{R}^n$, such that $P_\infty(0)$ is the maximum of P_∞

This finishes the proof. Note that if $P = 0$ somewhere, then we have $|\nabla u|^2 = 2W(u)$ at that point. Note that $W(u)$ is always non-zero, since $|u| \neq 1$ by maximum principle. Thus $|\nabla u| \neq 0$ locally and we have
$$0 \geq -|\nabla u|^2 \Delta P + \frac{1}{2}|\nabla P|^2 + W'(u)\nabla u \cdot \nabla P$$
Thus the maximum principle applies (note $-|\nabla u|^2$ is uniformily elliptic coefficient in a small nbd of p) and we conclude that $P = 0$ locally. Clearly $P = 0$ is an open condition from this argument, but it's also a closed condition so $P \equiv 0$ everywhere. Now define
$$\varphi = H^{-1}(u)$$
Then we compute
$$\nabla u = H'(\varphi)\nabla\varphi = \frac{1}{\sqrt{2}}(1 - \mathbb{H}^2)\nabla\varphi$$
and so
$$|\nabla u|^2 = 2W(u) \iff \frac{1}{2}(1 - H(\varphi)^2)^2|\nabla\varphi|^2 = \frac{1}{2}(1 - \mathbb{H}(\varphi)^2)^2 \implies |\nabla\varphi| \equiv 1$$
everywhere. Moreover, note that
$$\begin{aligned} \Delta u = H''(\varphi)|\nabla\varphi|^2 + H'(\varphi)\Delta\varphi &= W'(H(\varphi)) \cdot 1 + H'(\varphi)\Delta\varphi \\ \implies \Delta u &= W'(u) + H'(\varphi)\Delta\varphi \\ \implies \Delta\varphi &\equiv 0 \end{aligned}$$
so φ is a harmonic function with bounded derivatives. Moreover the derivatives are harmonic themselves. By liouville's theorem for bounded harmonic functions on$\mathbb{R}^n$ (applied to the derivatives), we have that φ_i must be constant functions, i.e. φ is linear! This gives us that u is the heteroclinic potentially shifted.

Note that for a manifold, and ϵ dependent allen-cahn, we define
$$P = \epsilon|\nabla u|^2 - 2\epsilon^{-1}W(u)$$
and one can show that if $\mathrm{Ric}_g \geq 0$ then $P \leq 0$. In general, even without the Ric constraint, we can prove $P \leq C$.

4.1.7 Monotonicity formula for $E(B_R(0), u)$

(Still part of problem 1 in the appendix)

We show that the energy functional
$$E_R(u) := R^{1-n}\int_{B_R(0)} \left(\frac{1}{2}|\nabla u|^2 + W(u)\right)$$
satisfies a monotonicity formula. We compute
$$\frac{dE_R(u)}{dR} = -(n-1)R^{-1}E_R + R^n\int_{\partial B_R(0)} e(u)\langle x, \nu\rangle d\mu$$

where $e(u)$ is the energy density and of course $\langle x, \nu \rangle = 1$, but we put it this way to integrate by parts and get

$$\frac{dE_R}{dR} = R^{-n} \int_{B_R(0)} (-P) d\mu + R^{1-n} \int_{\partial B_R(0)} (\partial_\nu u)^2$$

but since $-P \geq 0$, we see that

$$\frac{dE_R}{dR} \geq 0$$

which is the monotonicity. (Again, there's more nuance to this but I do this in the minimal surface reading notes)

4.1.8 Kernel classification of $L_* = \Delta_g - W''(g)$

(Problem 2 in Summer school notes)

The short is that $\ker(L_*) = \dot{g}(t)$ where

$$L_* = \Delta_{\mathbb{R}^{n+1}} - W''(g(t))$$

on $\mathbb{R}^{n+1}$ coordinatized as $(s_1, \ldots, s_n, t)$. I'll sketch the proof

1. First note that $L_*(\dot{g}) = 0$ since

$$\ddot{g} = W'(g) \implies \dot{\ddot{g}} = W''(g)\dot{g} = \Delta_{\mathbb{R}^{n+1}}(\dot{g})$$

2. The claim is true for $n = 1$ - Don't really understand Otis' hint. The proof from Del-Pino, Kowalczyk, and Wei is to write

$$w(t) = \rho(t)\dot{g}$$

Then we compute

$$\int L_*(w)w = \int |w'|^2 + W''(g)w^2 = \int \dot{\rho}^2 \dot{g}^2$$

where the equality comes from integration by parts. Thus the left most is 0 if and only if $\dot{\rho} = 0$ everywhere since $\dot{g}^2$ is always non-zero. Thus $\rho = c$. (Actually need to apply a cutoff because the second integration by parts involves terms with $\dot{\rho}$, and a priori, not sure we can integrate things like $\rho\dot{\rho}\dot{g}\ddot{g}$ - see Otis' email)

Otis answer: Consider $f = (\log(\mathcal{H}(t))''$ and multiply by $u(t)^2$ (an arbitrary function with compact support). Conclude that

$$\int_{-\infty}^{\infty} u'(t)^2 + W''(\mathcal{H}(t))u(t)^2 \, dt = \int_{-\infty}^{\infty} (\mathcal{H}'(t)^{-1}\mathcal{H}''(t)u(t) - u'(t))^2 \, dt$$

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} W''(\mathcal{H}(t))u(t)^2 - \mathcal{H}'(t)^{-2}\mathcal{H}''(t)^2u(t)^2 + 2\mathcal{H}'(t)^{-1}\mathcal{H}''(t)u(t)u'(t) \\ &= \int_{-\infty}^{\infty} u'(t)^2 + W''(\mathcal{H}(t))u(t)^2 - \mathcal{H}'(t)^{-2}\mathcal{H}''(t)^2u(t)^2 + 2\mathcal{H}'(t)^{-1}\mathcal{H}''(t)u(t)u'(t) - u'(t)^2 \\ &= \int_{-\infty}^{\infty} u'(t)^2 + W''(\mathcal{H}(t))u(t)^2 - (\mathcal{H}'(t)^{-1}\mathcal{H}''(t)u(t) - u'(t))^2 \end{aligned}$$

Choose $u(t) = w(t)\varphi_R$ where φ_R cuts off from R to $2R$. Letting $R \to \infty$, conclude that

$$w'(t) = w(t)\mathcal{H}'(t)^{-1}\mathcal{H}''(t)$$

$$\int_{-\infty}^{\infty} \varphi'(t)^2 w(t) dt = \int_{-\infty}^{\infty} (\mathcal{H}'(t)\mathcal{H}''(t)\varphi(t)w(t) - \varphi w'(t) - w(t)\varphi'(t))^2 \, dt$$

(For what it's worth, I think my solution replacing $w = \rho\dot{g} \to w_R = \rho\dot{g}\varphi_R$ with φ_R a cut off at $\pm(R+1)$ decaying linearly with derivatives ≤ 2 works fine)

3. When $n = 1$, there exists $\mu > 0$ so that if $\int u(t)\dot{g} = 0$, then

$$\int_{-\infty}^{\infty} (u')^2 + W''(g)u(t)^2 \geq \mu \int_{-\infty}^{\mathbb{R}} u^2$$

To see this, note that the left hand side is

$$\int_{\mathbb{R}} L_*(u)u$$

by integration by parts. We actually want to prove

$$\langle L_*(u), u\rangle \geq K||u||_{H^1}^2$$

Suppose not. Then we get a sequence of u_j each orthogonal to $\dot{g}$ such that

$$\langle L_*(u_j), u_j\rangle \leq \frac{1}{j}||u_j||_{H^1}^2$$

normalize by $||u||_{H^1}^2$ and get

$$\langle L_*(u_j), u_j\rangle \leq \frac{1}{j}$$

Since u_j is bounded in H^1, it is in particular bounded in L^2 and by duality and banach alaoglu, we get weakly convergent subsequences, $u_j' \to g$ and $u_j \to f$ each converging in L^2. By choosing another subsequence, we get $g = f'$. Moreover, because this convergence happens in L^2, we get that $||f||_{H^1} = 1$. However,

$$\langle L_*(f), f\rangle = 0$$

But also if we write $f(t) = \rho(t)\dot{g}(t)$, then we get that $\rho \equiv c \neq 0$, and so

$$\langle f, \dot{g}\rangle = \int f\dot{g} = c\sigma_0$$

but then

$$0 = \lim_{j\to\infty} \langle u_j, \dot{g}\rangle = c\sigma_0$$

so $c = 0$ and $f \equiv 0$, but $||f||_{H^1} \neq 0$ a contradiction. THus the bound holds

4. For $n \geq 2$, we can write

$$w(s,t) = c(s)\dot{g}(t) + \overline{w}(s,t)$$

where

$$c(s) = -\frac{1}{\sigma_0}\langle w(s,\cdot), \dot{g}\rangle_{L^2(t)}$$

so that $\langle \overline{w}, w\rangle_{L^2} = 0$. We show that c is **bounded and harmonic**. This follows by writing our $L_*(w) = 0$

$$L_* = \partial_t^2 + \Delta_{\mathbb{R}^{n-1}} - W''(g)$$
$$L_*(w) = \dot{g}\Delta_{\mathbb{R}^{n-1}}c + \Delta_{\mathbb{R}^{n-1}}\overline{w} + (\partial_t^2 - W''(g))\overline{w} = 0$$

multiply the above by $\dot{g}$ and integrate, and use that

$$\int (\partial_t^2 - W''(g))(\overline{w})\dot{g} = 0$$
$$\int \Delta_{\mathbb{R}^{n-1}}(\overline{w})\dot{g} = 0$$

we see that

$$\Delta_{\mathbb{R}^n} c = 0$$

Since w is a bounded function, we have that $c(s)$ is a bounded function as well. Thus c must be a constant.

5. Now we prove that $\overline{w} = 0$. Consider $\sigma \in (0, \sqrt{2})$, $\delta \in (0,1)$, and $\eta > 0$. Define

$$\gamma(s,t) = e^{-\sigma t} + \eta \cosh(\delta t) \sum_{i=1}^{n-1} \cosh(\delta s^i)$$

Then we compute

$$L_*(\gamma) = (\sigma_0^2 - W''(g))e^{-\sigma t} + \eta[2\delta^2 - W''(g)] \cosh(\delta t) \sum_{i=1}^{n-1} \cosh(\delta s^i)$$

we see that the above is < 0 since $W''(g) \to 2$ for $|t|$ large, e.g. $|t| > \Lambda > 0$ for some $\Lambda > 0$.

Note that $L_*(\overline{w}) = 0$ as well. This tells us that on $t > \Lambda$, we get

$$|\overline{w}| \leq ||\overline{w}||_{L^\infty}(e^{\sigma(\Lambda - t)} + \eta \cosh(\delta t) \sum_i \cosh(\delta s^i))$$

which follows by the maximum principle applied to

$$\overline{w} - ||\overline{w}||_{L^\infty}(e^{\sigma(\Lambda - |t|)} + \eta \cosh(\delta t) \sum_i \cosh(\delta s^i))$$

which we see is ≤ 0 on $|t| = \Lambda$. Sending $\eta \to 0$, we get

$$|\overline{w}(s,t)| \leq Ce^{-\sigma t}$$

By Schauder estimates, we get the same decay in t for $|\nabla w|$.

6. Consider

$$V(s) = \int_{\mathbb{R}} \overline{w}(s,t)^2 dt$$

This integral converges uniformily in s, so we can differentiate under the integral sign, and get

$$\Delta_{\mathbb{R}^{n-1}} V = \int_{\mathbb{R}} 2[(\Delta_s \overline{w})\overline{w} + |\nabla_s \overline{w}|^2]$$

Note that we have

$$[\partial_t^2 + \Delta_{\mathbb{R}^{n-1}} - W''(g)]\overline{w} = 0$$

so that the above becomes

$$\begin{aligned} \Delta_{\mathbb{R}^{n-1}} V &= \int_{\mathbb{R}} [-\overline{w}''\overline{w} + W''(g)\overline{w}^2 + |\nabla_s \overline{w}|^2] \\ &= \int_{\mathbb{R}} (\overline{w}')^2 + W''(g)\overline{w}^2 + |\nabla_s \overline{w}|^2 \end{aligned}$$

okay but now we use the second part of this problem to get

$$\int_{\mathbb{R}} (\overline{w}')^2 + W''(g)\overline{w}^2 \geq \mu \int_{\mathbb{R}} \overline{w}^2$$

and so

$$\Delta_{\mathbb{R}^{n-1}} V - \mu V \geq \int |\nabla_s \overline{w}|^2 \geq 0$$

7. Now applying the maximum principle to V as follows: we know V is bounded since $\overline{w}$ has uniform decay. If V achieves its maximum on $\mathbb{R}^n$, call it α, then this is a contradiction to the maximum principle, since maxima can only be attained on the boundary of a set. If V doesn't achieve this, take translates $\{x_i\}$ such that $V(x_i) \to \alpha$, and consider $V_i(x) := V(x + x_i)$. By schauder estimates applied to V and Arzela-ascoli, we get C^2 convergence to some V_∞ which achieves its max at 0 and satisfies

$$\Delta_{\mathbb{R}^{n-1}} V_\infty - \mu V_\infty \geq 0$$

so its maximum on any ball must be achieved on the boundary. But the maximum is achieved at 0, a contradiction. Thus $V_\infty \equiv 0$, which tells us that $V \equiv 0$, and so $\overline{w} \equiv 0$

4.1.9 Exponential decay of solutions away from nodal set

See Guaraco, "Min-Max for the Allen-Cahn equation and other topics (Princeton 2019)"

The idea is to use the same function listed above but in an ϵ-dependent way

$$\gamma_\epsilon(t) = e^{-\sigma t/\epsilon} + b\cosh(t/\epsilon)$$

and then send $b \to 0$. Note that t is the signed distance from $\{u = 0\}$ which we assume is a hypersurface with some regularity. Moreover, we can do this in some small neighborhood around the level set, say $|t| < -k\epsilon \ln(\epsilon)$.

A bit more formally following the exercise: Suppose we work on the positive side of u, then consider $v = u - 1$, which satisfies

$$0 = \epsilon^2 \Delta_g(v) - cv$$
$$L = \epsilon^2\Delta - c, \qquad c = u(u+1)$$

Then we have that

$$\begin{aligned}
L(\gamma_\epsilon) &= \sigma^2 a e^{-\sigma t/\epsilon} + b\cosh(t/\epsilon) - H_t(-\sigma a\epsilon e^{-\sigma t/\epsilon} + \epsilon b\cosh(t/\epsilon)) - c(ae^{-\sigma t/\epsilon} + b\cosh(t/\epsilon)) \\
&= [\sigma^2 + H_t\epsilon - c]ae^{-\sigma t/\epsilon} + [1 - \epsilon - c]b\cosh(t/\epsilon)
\end{aligned}$$

It is clear that because $0 < c < 2$ and $\lim_{t\to 0} \epsilon H_t = 0$, then we have that $L(\gamma_\epsilon) > 0$ as long as $\sigma > \sqrt{2}$. Thus if we take $L(\gamma_\epsilon - (u-1))$ we see that this is positive, and so the maximum must be achieved on the boundary, i.e.

$$\gamma_\epsilon - (u-1) \leq$$

(Very confused about this... Seems like the signs don't work out). (This paper has a good proof - lemma 4.1 and 4.2 though the way its written is a bit confusing here)

Here's an alternate proof: I'll first prove the following

Lemma 4.2. *Given $\delta \in (0,1)$ there exists $\rho_\delta > 0$ such that for any solution of Allen-Cahn with $|u| < 1$ we have*

$$B(x, 2\rho_\delta) \subseteq \mathbb{R}^n - u^{-1}(0) \implies |u^2 - 1| \leq \delta \text{ in } B(x, \rho_\delta)$$

Proof: Let ϕ_R the first eigenfunction to $-\Delta$ pn $B_R(0)$ with dirichlet conditions. Normalize so that $\phi_R(0) = \sup_{B(0,R)} \phi_R = 1$. Recall that the associated eigenvalue is $\lambda_R = \lambda_1/R^n$. For $\delta \in (0,1)$, chose $R_0 > 0$ such that

$$-W'(t)R_0^n \geq \lambda_1 t$$

for all $t \in [0, 1-\delta]$. Assume that $R > R_0$ and $B(x, 2R) \subseteq \mathbb{R}^n - u^{-1}(0)$. Also assume WLOG that $u > 0$ in $B(x, 2R)$.

We claim that $u \geq 1 - \delta$ in $B(x, R)$. If not, there we'd find $\overline{x} \in B(x, R)$ such that $u(\overline{x}) < 1 - \delta$. In this case, we deifne $\epsilon > 0$ to be the largest positive real such that

$$u \geq \epsilon\phi_R(\cdot - \overline{x})$$

in $B(\overline{x}, R)$. We have $\epsilon \leq 1 - \delta$ by definition and we can find a point $z \in B(\overline{x}, R)$ such that

$$u(z) = \epsilon\phi_R(z - \overline{x}) \leq 1 - \delta$$

By construction of R, we have

$$-\epsilon\Delta\phi_R = \frac{\lambda_1}{R^n}\epsilon\phi_R < W'(\epsilon\phi_R)$$

and since $\Delta u = W'(u)$, we conclude

$$-\Delta(\epsilon\phi_R - u) < 0$$

(Actually there's a sign error here that I don't know how to correct).

With this, we prove exponential decay

Lemma 4.3. *There exist constants $C > 0$ and $\alpha > 0$ such that for any AC solution with $|u| < 1$ we have*

$$|u^2 - 1| + |\nabla u| + |D^2 u| \leq Ce^{-\alpha \, dist(x, u^{-1}(0))}$$

Proof: Fix $\alpha > 0$ such that $\alpha^2 < W''(1)$ and choose $\delta \in (0, 1)$ close to 1 so that $W''(t) \geq \alpha^2$ for all $t \in [1 - \delta, 1]$. Accordinag to the above, we know that

$$\forall R > R_0, \qquad u \geq 1 - \delta \qquad \forall z \in B(\overline{x}, R) \quad \text{s.t.} \quad B(\overline{x}, 2R) \subseteq \mathbb{R}^n - u^{-1}(0)$$

(if $u < 0$ then replace the above with $u \leq \delta - 1$). Therefore, we have

$$-\Delta(1 - u) = -\frac{W'(u) - W'(1)}{u - 1}(1 - u) \leq -\alpha^2(1 - u)$$

However, we also have

$$(\Delta + \alpha^2)e^{-\alpha\sqrt{1+r^2}} \geq 0$$

for $r = |x - \overline{x}|$. This with the maximum principle gives the exponential decay of $(1 - u^2)$ away from the zero set. $\square$

4.1.10 Solutions on S^n (nodal set is equator, two parallels)

Exercise 19 in Marco's notes:

Consider $A_\tau = S^n \cap \{|x_{n+1}| < \tau\}$, and $S^n \backslash A_\tau = D_\tau^+ \cup D_\tau^-$. We can take energy minimizers with dirichlet conditions on $x_{n+1} = \tau$ - choose the ones which are positive on $D_\tau^\pm$ and negative on A_τ. Moreover, by symmetry of the equation and domain, these solutions will only depend on x^{n+1} (if not, could reflect the solutions about hypersurfaces and get another energy minimizer, contradicting uniqueness). In particular, this tells us that if u_τ is the pasted solution, then

$$\nabla u_\tau \Big|_{x_{n+1} = \tau} = C_\tau \nu$$

i.e. no variation along the latitude. Furthermore, we have

$$\int_{A_\tau} W'(u_\tau) = \epsilon^2 \int_{A_\tau} \Delta u_\tau = \epsilon^2 \int_{x_{n+1} = \pm\tau} \langle \nabla u_\tau, \nu \rangle = \epsilon^2 C_\tau |\partial A_\tau|$$

Note that $|\partial A_\tau|$ is continuous in $n - 1$ dimensional measure, so we use this to leverage that the neumann data is continuous in τ. In particular, we have

$$\begin{aligned}
C_\tau - C_\sigma &= \frac{1}{|\partial A_\tau|} \int_{A_\tau} W'(u_\tau) - \frac{1}{|\partial A_\sigma|} \int_{A_\sigma} W'(u_\sigma) \\
&= \frac{1}{|\partial A_\tau|} \left[\int_{A_\tau} W'(u_\tau) - W'(u_\sigma) \right] + \frac{1}{|\partial A_\tau|} \left[\left(\int_{A_\tau} - \int_{A_\sigma} \right) W'(u_\sigma) \right] + \left(\frac{1}{|\partial A_\tau|} - \frac{1}{|\partial A_\sigma|} \right) \int_{A_\sigma} W'(u_\sigma)
\end{aligned}$$

again, $W'(u_\tau)$ is always bounded in magnitude by 1 so we can bound the second and third term by $L|\tau - \sigma|$ in absolute value. For the first term, suppose that $||u_\tau - u_{\tau+h}||_{L^1} \not\to 0$ as $h \to 0$ (say on the domain A_τ). Then we'd get a subsequence such that $||u_\tau - u_{\tau+h_i}||_{L^1} \geq c > 0$. All of these function satisfy uniform $C^{2,\alpha}$ estimates since ϵ is fixed, so by Arzela-ascoli, we can find a uniformily convergent subseuqnece of $u_{\tau+h_i} \to u_\tau$ in C^2. But this contradicts the lack of L^1 convergence, since our functions are bounded and on a bounded domain. Thus

$$||u_\tau - u_{\tau+h}||_{L^1} \xrightarrow{h \to 0} 0$$

and in particular

$$\left[\int_{A_\tau} W'(u_\tau) - W'(u_\sigma) \right] \to 0$$

as $\sigma \to \tau$. Thus **the neumann data is continuous in τ.**

Fix ϵ small so that $u_{1/2}$ is non-zero on both $A_{1/2}$ and $D_{1/2}^\pm$. This is an eigenvalue issue which gives existence. We know that for $\tau \to 1$, the minimizers on $D_\tau^\pm = 0$, so u_τ goes from being negative on A_τ to 0 on $D_\tau^\pm$, meaning that the neumann derivative is positive if you move towards the disks. Similarly for τ close to 0, the minimizer on A_τ will be 0, in which case the neumann derivative is negative if you start from A_τ and move towards to the disks. By continuity of the neumann derivative w.r.t. τ, which gives continuity of the "jump" along ∂A_τ, we conclude that there exists some τ such that the neumann data match up.

More details here: from Surim

4.2 Wave Equation

4.2.1 Solution to linear wave equation

- Recall that the linear wave equation is

  $$\Box_m = -\partial_t^2 + \sum_i \partial_{x_i}^2$$

  applied to functions $\phi : I \times \mathbb{R}^n \to \mathbb{R}$

- We are interested in studying the system

  $$\begin{cases} \Box\phi = 0 \\ (\phi, \partial_t \phi)\Big|_{t=0} = (\phi_0, \phi_1) \end{cases}$$

- Assume that $(\phi_0, \phi_1) \in C_c^\infty(\mathbb{R}^n) \times C_c^\infty(\mathbb{R}^n)$, then we fourier transform the wave equation and get

  $$-\hat{\phi}_{tt} - 4\pi^2 |\xi|^2 \hat{\phi} = 0$$

  which has a general solution of

  $$\hat{\phi}(t,\xi) = A(\xi)\sin(2\pi t|\xi|) + B(\xi)\cos(2\pi t|\xi|)$$
  $$A(\xi) = \frac{\hat{\phi}_1(\xi)}{2\pi|\xi|}, \qquad B(\xi) = \hat{\phi}_0(\xi)$$

  which comes from evaluating $\hat{\phi}(0,\xi)$ and also $\partial_t \hat{\phi}(0,\xi)$ and using the initial conditions or rather their fourier transforms

- More generall,y we just require

  $$(\phi_0, \phi_1) = (\phi, \partial_t \phi)\Big|_{t=0} \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$$

4.2.2 Fundamental solution

Still Jonathan's notes

- The fundamental solution is a distribution $E \in \mathscr{D}'(\mathbb{R}^n)$ such that

  $$\Box_m E = \delta_0$$

  We also have the *forward* fundamental solution, E_+, which in addition to the above, also satisfies

  $$\text{supp}(E_+) \subseteq \{(t,x) \in \mathbb{R} \times \mathbb{R}^n \ : \ 0 \le |x| \le t\}$$

- Uniqueness of forward fundamental solution

  Proposition. *A forward fundamental solution on $\mathbb{R}^{n+1}$, if it exists, is unique.*

  Proof: Let E_+ and E both be fundamental solutions. Then

  $$E = E * \delta_0 = E * (\Box E_+) = (\Box E) * E_+ = \delta_0 * E_+ = E_+$$

  note that this doesn't use the fact of the supports being forward, but actually we need this to make sense of $\Box E * E_+$ as a meaningful distribution, i.e. their supports have to be compatible

- Lemma

  Lemma 4.4 (3.4, (3))**.** *Let $u_1, u_2 \in D'(\mathbb{R}^n)$. If $(-supp(u_1)) \cap (supp(u_2) + K)$ is compact for any compact set K, then there exists a unique distribution u such that*

  $$u * \varphi = u_1 * (u_2 * \varphi)$$

  for every $\varphi \in C_c^\infty(\mathbb{R}^n)$

Note that if E and E_+ have the same support restriction in the previous proposition, then $-supp(E) \cap [supp(E_+) + K]$ will be compact since it is at most K shifted (if K is 0 the intersection is just $(x, t) = (0, 0)$)

- We can now use the forward fundamental solution to construct the solution to the linear wave equation

  Proposition. *Let E_+ be a forward fundamental solution to the linear wave equation on $\mathbb{R}^{n+1}$. Then if $(\phi_0, \phi_1) \in C^\infty(\mathbb{R}^n) \times C^\infty(\mathbb{R}^n)$, the unique solution to $\Box\phi = 0$ is given by*

$$\phi(t, x) = -E_+ * (\phi_1 \delta_{\{t=0\}}) - (\partial_t E_+) * (\phi_0 \delta_{\{t=0\}})$$

  More generally, for $\Box\phi = F$ for $F \in C^\infty(\mathbb{R}^n)$, for $F \in C^\infty(\mathbb{R}^n)$, ϕ is given for $t \geq 0$ by the following formula

$$\phi(t, x) = -E_+ * (\phi_1 \delta_{\{t=0\}}) - (\partial_t E_+) * (\phi_0 \delta_{\{t=0\}}) + (F \mathbb{1}_{\{t \geq 0\}}) * E_+$$

  Proof: Just compute manually

$$\begin{aligned}
\phi \mathbb{1}_{\{t \geq 0\}} &= (\phi \mathbb{1}_{\{t \geq 0\}}) * \delta_0 = (\phi \mathbb{1}_{\{t \geq 0\}}) * (\Box E_+) = (\phi \mathbb{1}_{\{t \geq 0\}}) * (\partial_t^2 E) + (\Delta \phi \mathbb{1}_{\{t \geq 0\}}) * E_+ \\
&= (\phi \mathbb{1}_{\{t \geq 0\}}) * (-\partial_t^2 E) + (\partial_t^2 \phi \mathbb{1}_{\{t \geq 0\}}) * E_+ \\
&= -(\partial_t \phi \mathbb{1}_{\{t \geq 0\}}) * (\partial_t E_+) - (\phi \delta_{\{t=0\}}) * (\partial_t E) + (\partial_t^2 \phi \mathbb{1}_{\{t \geq 0\}}) * E_+ \\
&= -(\partial_t \phi \delta_{t=0}) * E_+ - (\phi \delta_{\{t=0\}}) * (\partial_t E_+)
\end{aligned}$$

  here, we've noted that

$$\partial_t (\phi \mathbb{1}_{\{t \geq 0\}}) = \phi_t \mathbb{1}_{\{t \geq 0\}} + \phi \delta_{\{t=0\}}$$

  and similarly with other terms. $\square$

- Here's what the forward fundamental solution actually is

  Proposition (Forward Fundamental Expression)**.** *The unique forward fundamental solution is given by*

$$E_+(t, x) = -\frac{\pi^{(1-n)/2}}{2} \mathbb{1}_{\{t \geq 0\}} \chi_+^{-(n-1)/2}(t^2 - |x|^2)$$

  Before we prove this, we need to recall the following:

- Recall the function

$$\chi_+^a(x) = \frac{x_+^a}{\Gamma(a+1)}$$

  which is an extension of

$$x_+^a = \mathbb{1}_{\{x \geq 0\}} x^a$$

  for $a \in \mathbb{C}$. Note furthermore that

$$\frac{d}{dx} \chi_+^a(x) = \chi_+^{a-1}(x)$$

  and also the following holds

  Lemma 4.5. *1. For any $k \in \mathbb{N}$, we have*

$$\chi_+^{-k}(x) = \delta_0^{(k-1)}(x)$$

  2. For any $k \in \mathbb{N}$, we have

$$\chi_+^{-1/2-k}(x) = \frac{1}{\sqrt{\pi}} \left(\frac{d}{dx}\right)^k \left(\frac{1}{x_+^{1/2}}\right)$$

  where we understand the derivatives as distributional derivatives

- Now we partial prove lemma "Forward Fundamental Expression":
  Proof: Support properties are fine. We now show that $\Box E_+ = c_n \delta_0$ for some constant $c_n \neq 0$, which essentially captures the behavior we want.

Away from 0, we have as a distribution in $\mathscr{D}'(\mathbb{R}^n\backslash\{0\})$ that

$$\Box\left(\mathbb{1}_{\{t\geq 0\}}\chi_+^{-(n-1)/2}(t^2-|x|^2)\right)=\mathbb{1}_{\{t\geq 0\}}\Box\left(\chi_+^{-(n-1)/2}(t^2-|x|^2)\right)$$

because we're away from 0. Using the chain rule, we get

$$\begin{aligned}\mathbb{1}_{\{t\geq 0\}}\Box\left(\chi_+^{-(n-1)/2}(t^2-|x|^2)\right)&=\mathbb{1}_{\{t\geq 0\}}\left(-\partial_t[2t\chi_+^{-(n-1)/2}(t^2-|x|^2)]-2\sum_{i=1}^n\partial_{x_i}\left[x_i\chi_+^{-(n-1)/2}(t^2-|x|^2)\right]\right)\\&=\mathbb{1}_{\{t\geq 0\}}\left(-2(n+1)\chi_+^{-(n+1)/2}(t^2-|x|^2)-4(t^2-|x|^2)\chi_+^{-(n+3)/2}(t^2-|x|^2)\right)\\&=0\end{aligned}$$

having used the chain rule and also

$$\frac{d}{dx}\chi_+^a(x)=\chi_+^{a-1}(x)$$

The last equality holds since

$$x\chi_+^a=\frac{xx_+^a}{\Gamma(a+1)}=(a+1)\frac{x_+^{a+1}}{\Gamma(a+2)}=(a+1)\chi_+^{a+1}$$

This tells us that $\Box E_+$ is supported at $\{0\}$. Since it is a distribution and bounded in a distributional sense it is a linear combination of δ and finitely many derivatives. By scaling, we claim that it is only a multiple of δ_0, e.g. no derivatives.

To see this, for any $\lambda\in\mathbb{R}$ and φ define $\varphi_\lambda(t,x)=\varphi(\lambda t,\lambda x)$, then we have that

$$\langle\Box E_+,\varphi_\lambda\rangle=\langle\Box E_+,\varphi\rangle$$

This just comes from the scaling properties of E_+ with its own definition (i.e. $\chi_{\{t\geq 0\}}$ isn't affected by scaling and $\chi_+^\alpha(t^2-|x^2|)$ scales like $\lambda^{2\alpha}$ but this is exactly the extra scaling needed to cancel out the volume form scaling).However

$$\langle\partial^\alpha\delta_0,\varphi_\lambda\rangle=\lambda^{|\alpha|}\langle\partial^\alpha\delta_0,\varphi_\lambda\rangle$$

and so $\Box E_+=c_n\delta_0$ and not a combination of any of its derivatives. To see that $c_n\neq 0$ (Not sure about this, maybe I'll do it later)

4.2.3 Finite speed of propogation

- Two proofs, one using fundamental solution, one using energy

  Proposition. *Suppose $\phi\in C^\infty(\mathbb{R}^{n+1})$ is a solution to the linear wave equation with data $(\phi_0,\phi_1)=(0,0)$ in $\{y\in\mathbb{R}^n\ :\ |x-y|\leq t\}$ then $\phi(t,x)=0$*

  Note that what this is saying is if we fix a (t,x), then get that information about $\phi_0(y)$ and $\phi_1(y)$ for those such y, then we have that $\phi(t,x)=0$.

  Proof: Using the fundamental solution, we note that because $\text{supp}(E_+)\subseteq\{(t,x)\ :\ 0\leq|x|\leq t\}$, then

  $$\phi(t,x)=-E_+*(\phi_1\delta_{\{t=0\}})-(\partial_tE_+)*(\phi_0\delta_{\{t=0\}})$$

  the first integral is

  $$\int_{|y|\leq t}E_+(y)\phi_1(x-y)\delta_{\{t=0\}}$$

  so if $\phi_1(z)=0$ when $|z-x|\leq t$, i.e. $|y|\leq t$, then clearly we have the statement. Same holds for the second integral. $\square$

  Proof: (Energy) By energy methods, we have that

  $$E(t;x,R):=\frac{1}{2}\int_{B(x,R-t)}[(\partial_t\phi)^2+\sum_i(\partial_i\phi)^2](t,y)dy$$

Suppose we're given a (t,x). Let $R=t$, and note that

$$\int_{B(x,t)}[(\partial_t\phi)^2+\sum_i(\partial_i\phi)^2](0,y)dy=0$$

Then for any $s<t$, we have

$$\frac{1}{2}\int_{B(x,t-s)}[(\partial_t\phi)^2+\sum_i(\partial_i\phi)^2](s,y)dy=0$$

which means that $\nabla\phi\equiv 0$ on this cone about x, but with initial condition of $\phi_0=0$ and $\phi_1=0$ on this cone as well. This means that $\phi(s,y)=0$ for any $y\in B(x,t-s)$ for any $s<t$, and by continuity, $\phi(t,x)=0$ as well. $\square$

4.2.4 Monotonicity of energy functional

- Energy functional is given by

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^n}\left[(\partial_t\phi)^2+\sum_i(\partial_{x_i}\phi)^2\right](t,x)dx$$

- If we integrate $\phi\Box\phi=0$, then we'll see that energy is conserved. Alternatively, take the above, compute the fourier transform, and then compute explicitly using the formula for $\hat{\phi}$ and we'll see that the energy is time independent

- We also have the following improved estimate

  Proposition. *Define, for $0\le t\le R$, the functional*

$$E(t;x,R):=\frac{1}{2}\int_{B(x,R-t)}[(\partial_t\phi)^2+\sum_i(\partial_i\phi)^2](t,y)dy$$

  then $E(t;x,R)$ is non-increasing as a function of t

  Proof: The idea is to start with $\partial_t\phi\Box\phi=0$, and then integrate by parts to the future of $\{t_1\}\times B(x,R)$ and to the past of $\{t_2\}\times B(x,R-t_2)\bigcup(\{s\times\partial B(x,R-s)\}$. You can imagine this as a trapezoid in some sense, where the flat sides give $E(t_1,x,R)$ and $E(t_2,xR)$. The slanted sides, e.g. integration over $\bigcup_{s\in[t_1,t_2]}(\{s\}\times\partial B(x,R-s))$ has a definite sign, e.g.

$$\begin{aligned}&\int_{t_1}^{t_2}\int_{B(x,R-t)}\left(\partial_t\phi\Box\phi\right)(t,y)dydt\\ &=-E_(t_2,x,R)+E(t_1,x,R)-\frac{1}{2}\int_{t_1}^{t_2}\int_{\partial B(x,R-t)}\left[(\partial_t\phi)^2+\sum_i(\partial_i\phi)^2-2\partial_t\phi\sum_i\frac{(y-x)_i}{|y-x|}\partial_i\phi\right]\end{aligned}$$

  By Cauchy schwarz, the last term is positive and so we get

$$E(t_2,x,R)\le E(t_1,x,R)$$

4.2.5 Existence of solutions to linear wave equation

- So we've shown existence by the fourier method (i.e. transform the linear equation and then plug in the solution)
- Have also shown existence by convolving with fundamental solution

4.2.6 Uniqueness of solutions to linear wave equation

- Have also shown this via fourier method, and fundamental solution
- Note also that we can use energy methods to show that given 2 solutions Φ and Ψ to the linear wave equation with the same initial conditions, (ϕ_0,ϕ_1), then $\Phi=\Psi$ for all (t,x). This follows by noting that $\Phi-\Psi$ has 0 initial conditions, and energy is conserved so $\nabla(\Phi-\Psi)\equiv 0$ everywhere

4.3 Ginzburg-Landau

Main source will be Betheul-Brezis-Helein. Tbh I think I can talk about whatever with this. Maybe mention some results by Pigati Stern on approximating minimal surfaces

4.3.1 Comparison to Allen-Cahn

- Ginzburg-Landau Functional is given by

$$E_\epsilon(u) = \int_G \frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(|u|^2 - 1)$$

for $u \in H^1(G; \mathbb{C})$, where G is some domain

- We often want a boundary condition like

$$g(x) = x$$

on ∂G. And set H^1_g the subset of complex valued functions on D with this boundary condition

- Let u be such that

$$\min_{u \in H^1_g} E_\epsilon(u)$$

is achieved. Then

$$\begin{cases} \epsilon^2 \Delta = u_\epsilon = u_\epsilon(|u_\epsilon|^2 - 1) & \text{in } G \\ u_\epsilon = g & \text{on } \partial G \end{cases}$$

- Main difference is that u_ϵ is complex valued, so normal PDE techniques do not work since we now have a system of PDEs, from the above equation and decomposing

$$u(p) = v(p) + iw(p), \qquad v, w \in \mathbb{R}$$

- From Pigati-Stern, if we define

$$F_\epsilon(v) = \frac{1}{|\log \epsilon|} \int |dv|^2 + \frac{1}{4\epsilon^2}(1 - |v|^2)^2$$

on complex valued maps $v : M \to \mathbb{C}$, then one can show that $v^{-1}(0)$ converges to nontrivial, stationary, *rectifiable* $(n-2)$-varifolds, but it is not known as of 2019 whether we can find *integral* varifolds this way

- Integrality is nice since its a closer analogue to finding codimension 2 hypersurfaces

4.3.2 convergence of $\{u_\epsilon\} \to u^*$ an S^1 valued function

- The inspiration for Ginzburg-Landau was to regularize the following problem:

  Given some boundary data $g : \partial G \to S^1$, can we find a minimizer of the dirichlet energy over S^1 valued functions with the right boundary data, i.e.

$$\min_{u \in H^1_g(G; S^1)} \int_G |\nabla u|^2$$

- These are called harmonic maps, and their existence/regularity is heavily dependent on

$$d = \deg(g, \partial G)$$

i.e. how much the boundary data winds

- Minimizer solves

$$-\Delta u_0 = u_0 |\nabla u_0|^2$$

And when $d = 0$, the solution is given by

$$u_0 = e^{i\varphi_0}$$

for φ_0 a real harmonic function (unique mod $2\pi\mathbb{Z}$) and $e^{i\varphi_0} = g$ on ∂G

- In this case, our solutions to the Ginzburg Landau functional converge in some sense, e.g. $||u_\epsilon - u_0|| \to 0$ in various norms
- When $d \neq 0$, we have
$$H^1_g(G;S^1) = \{u : G \to S^1 \mid u\Big|_{\partial G} = g\} = \emptyset$$
for if not, then a minimizer of the energy exists, and is smooth up to ∂G. Thus there would be $u \in C(\overline{G};S^1)$. But then degree theory tells us that if we have this extension to the interior, then g can be homotopied in S^1 to a constant, a contradiction to the fact that $d = \deg(g, \partial G) \neq 0$ (i.e. if $\overline{G}$ is start shaped, then WLOG $x = 0$ is the central point, we could define something like $H(t,x) = u(tx)$. This is the a homotopy of $g = H(1,x)\Big|_{\partial G}$ to $u(0)$)
- The idea is to extend our domain of functions to
$$H^1_g(G;\mathbb{C})$$
and still search for functions which are almost harmonic but we regularize (i.e. add a penalty) to force $|u|$ to be close to 1, e.g.
$$\frac{1}{\epsilon^2}\int_G (|u|^2 - 1)^2$$
which make u want to be S^1 valued in the limit
- We are lead to look at solutions u_ϵ which minimizer E_ϵ as before, the issue is that
$$\int_G |\nabla u_\epsilon|^2 \to +\infty$$
as $\epsilon \to 0$. Since otherwise if this energy was bounded, we could find a weakly convergent subsetqence $u_\epsilon \to \tilde{u} \in H^1_g$, impossible since $H^1_g = \emptyset$.
- However, for a subsequence ϵ_n, we can define
$$u_*(x) = \lim_n u_{\epsilon_n}(x), \qquad \text{a.e. } x \in G$$
which can be viewed as a generalized solution of finding an S^1 valued harmonic map with prescribed boundary data
- The main theorem is:

Theorem 4.6 (0.1)**.** *Assume G starshaped and $d = \deg(g, \partial G)$. Then this is a subsequence $\epsilon_n \to 0$ and exactly d points $\{a_1, \ldots, a_d\}$ in G and a smooth harmonic map u_* from $G\backslash\{a_1, \ldots, a_d\}$ into S^1 with $u_*\Big|_{\partial G} = g$ on ∂G such that*
$$u_{\epsilon_n} \to u_* \quad \text{in } C^k_{loc}(G\backslash \cup_i \{a_i\}) \ \forall k \quad \text{and in } C^{1,\alpha}(\overline{G}\backslash \cup_i \{a_i\}), \ \forall \alpha < 1$$
In addition each singularity has degree $+1$ and there exist complex constants $\{\alpha_i\}$ with $|\alpha_i| = 1$ such that
$$\left| u_*(z) - \alpha_i \frac{(z - a_i)}{|z - a_i|} \right| \leq C|z - a_i|^2$$

4.3.3 Dependence of u^* on dirichlet data

- In the previous section, we've seen how u^* depends on g - u^* converges to a function smooth away from the boundary and d singularities.
- We can also prescribe the location of the singularities and minimizer of that class
- Let $\{b_1, \ldots, b_d\}$ be distinct points in G. Define
$$G_\rho = G\backslash \left(\sqcup_i B(b_i, \rho)\right)$$
and consider
$$\mathcal{E}_p = \{v \in H^1(G_\rho; S^1) \ \text{ s.t. } \ v\Big|_{\partial G} = g, \qquad \deg(v, \partial B(b_i, \rho)) = 1\}$$

then there exists a unique minimizer of

$$\min_{u \in \mathcal{E}_p} \int_{G_\rho} |\nabla u|^2$$

and

$$\int_{G_\rho} |\nabla u|^2 = \pi d |\log \rho| + W(b) + O(\rho)$$

for $W(b) = W(b_1, \ldots, b_d)$ a renormalized energy for the configuration $\{b_1, \ldots, b_d\}$ (That I won't define)

- As $\rho \to 0$, u_ρ converges to some u_0 that has the following properties:
  - u_0 is a smooth harmonic map in $G \backslash \cup_i \{b_i\}$
  - $u_0 = g$ on ∂G
  - We have

    $$\left| u_0(z) - \beta_i \frac{(z - b_i)}{|z - b_i|} \right| \leq C |z - b_i|$$

    as $z \to b_i$ for all b_i
  - There is an explicit formula for u_0 given by

    $$u_0(z) = e^{i\varphi(z)} \prod_{i=1}^{d} \frac{(z - b_i)}{|z - b_i|}$$

    where φ solves

    $$\begin{cases} \Delta \varphi = 0 & \text{in } G \\ \varphi = \varphi_0 & \text{on } \partial G \end{cases}$$

    and φ_0 is define by

    $$e^{i\varphi(z_0)} = g(z) \prod_{i=1}^{d} \frac{|z - b_i|}{(z - b_i)}$$

4.3.4 canonical 2D solutions

- As indicated in the previous section, the canonical 2-D solution looks like (in the small ϵ limit)

  $$u_\epsilon \approx e^{i\varphi(z)} \prod_{i=1}^{d} \frac{(z - b_i)}{|z - b_i|}$$

  where $\{b_i\}$ are either prescribed points and we take minimizers on $G \backslash \cup_i B(b_i, \rho)$
- Or,

  $$u_\epsilon \approx \alpha_i \frac{(z - a_i)}{|z - a_i|}$$

  near each a_i, of which there are only finitely many. Here, α_i is some set of constants in S^1

4.3.5 Harmonic functions in $C(\Omega \backslash \{p_i\}_{i=1}^n, S^1)$

- Essentially covered in the previous section, but the point is that harmonic functions mapping into $\mathbb{C}$ satisfies

  $$-\Delta u_0 = u_0 |\nabla u_0|^2$$

- If we prescribe the singularities $\{p_1, \ldots, p_n\}$, then

  $$u_0(z) = e^{i\varphi(z)} \prod_{i=1}^{n} \frac{z - p_i}{|z - p_i|}$$

  where $\varphi(z)$ solves

  $$\begin{cases} \Delta \varphi = 0 & \text{in } G \\ \varphi = \varphi_0 & \text{on } \partial G \end{cases}$$

and if we prescribe boundary data for u_0 on ∂G, called $g(z)$, then

$$g(z) = e^{i\varphi_0(z)} \prod_{i=1}^{n} \frac{z - p_i}{|z - p_i|}, \qquad z \in \partial G$$

which defines φ_0

4.4 Poincare-Einstein Metrics

- See Rafe's first paper on RV or Graham-Witten
- A metric on a manifold (or rather the Riemannian manifold itself) is said to be Poincare-Einstein if M is a manifold with boundary, $g = \rho^{-2}\overline{g}$ where ρ is a boundary defining function for ∂M, $\overline{g}$ is a smooth metric on $\overline{M}$ and non-degenerate up to the boundary, and g is einstein. Recall that g being einstein means that

  $$\mathrm{Ric}_g = \kappa g$$

  for κ a scalar value.
- The space of all PE metrics (with some fixed regularity) on the interior of a given manifold with boundary is a banach manifold
- The conformal infinity map from the space of PE metrics on the interior to the space of conformal structures on ∂M (which is also a banach manifold) is Fredholm degree 0.

4.4.1 Graham normal form/Evenness of PE metrics

- Few sources for this: Witten-Graham is good, so is Graham-Lee which says a bit technically but was probably the first to do this. Fefferman-Graham is the most thorough but also the most technical, just read the section on Poincare-Einstein metrics
- Most of what I wrote is from "The ambient metric", Fefferman-Graham
- We get the definition fo asymptotically hyperbolic metrics (which Poincare-Einstein metrics are)

  Definition 4.7. *An asymptotically hyperbolic metric g_+ is said to be in **normal form** relative to a metric g in the conformal class if*

  $$g_+ = r^{-2}(dr^2 + g_r)$$

  where g_r is a 1-parameter family of metrics on ∂M such that $g_0 = g$.

- We also have the following theorem on evenness

  Theorem 4.8. *Let $(\partial M, [g])$ a smooth manifold of dimension ≥ 2 equipped with a conformal class. Then there exists an even Poincare metric for $(\partial M, [g])$. If g_+^1 and g_+^2 are two even Poincare metrics for $(\partial M, [g])$ defined on M^o, then there are subsets $U_1, U_2 \subseteq M^o$ containing ∂M, and an even diffeomorphism $\phi : U_1 \to H_2$ such that $\phi\Big|_{\partial M}$ is the identity map and*

  - *If $n = \dim \partial M$ is odd, then $g_+^1 - \phi^* g_+^2$ vanishes to infinite order at the boundary*
  - *If $n = \dim \partial M$ is even then $g_+^1 - \phi^* g_+^2 = O(r^{n-2})$*

  (This is saying that given a conformal class on the boundary we can find a PE metric on the interior that agree up to high order up to diffeomorphism). Similarly, we also have

  Theorem 4.9. *Let M a smooth manifold of dimension $n \geq 2$ and g a smooth metric on M*

  - *There exists an even PE metric, g_+, for $(\partial M, [g])$ which is in normal form relative to g*
  - *Suppose that g_+^1, g_+^2 are even PE metrics for $(M, [g])$ both of which are in normal form relative to g. If n odd, then $g_+^1 - g_+^2$ vanishes to infinite order at every point of ∂M. If n even, then $g_+^1 - g_+^2 = O(r^{n-2})$*

Eveness of metrics

- The idea is that the Einstein condition is second order, and very roughly, we have

$$\Delta_g g = \kappa g + \text{l.o.t}$$

The actual proof of this is difficult, but because Δ_g is second order (though for asymptotically hyperbolic, we expect it to look like $\Delta_g = (r\partial r)^2 + w(r\partial_r) + \text{l.o.t}$) and so we can solve for g in an even manner

- Actually I won't prove this but just leave it at a statement, the proof is quite long

4.4.2 Boundary defining functions

- Given M^n with boundary $N = \partial N$, and a conformal class of metrics $[k]$, the conformal infinity on N
- M is conformally compact, if $\overline{M}$ is a manifold with compact boundary and

$$\exists \rho : \overline{M} \to \mathbb{R}^{\geq 0} \quad \text{s.t.} \quad \{\rho = 0\} = \partial M$$

Moreover, we define $\overline{g} := \rho^2 g$, the **compactified metric** so that $\overline{g}$ is a metric on all of $\overline{M}$ and

$$\nabla^{\overline{g}} \rho \Big|_{\partial M} \neq 0$$

- ρ is called a boundary definition function (bdf)
- Note that if $\varphi : \overline{M} \to \mathbb{R}^+$, then $\rho^* = \varphi\rho$ is also a bdf, so these aren't special
- A bdf is **special** if

$$||d \log(\rho)||_g^2 = \frac{||d\rho||_g^2}{\rho^2} = 1$$

- Proposition:

Proposition. *For M conformally copmact and a choice of representative $k_0 \in [k]$, there exists a unique special bdf for M such that*

$$\overline{g}\Big|_N = k_0$$

Proof: The proof of this is in Witten-Graham (Lemma 2.1) - Consider ρ any defining function. Let $\overline{g}_0 = r_0^2 g$ and $r = r_0 e^{\omega}$ so that $\overline{g} = e^{2\omega}\overline{g}_0$ and

$$dr = e^{\omega}(dr_0 + r_0 d\omega)$$

Then

$$||dr||_{\overline{g}}^2 = ||dr_0||^2 + 2r_0 g_0(dr_0, d\omega) + r_0^2 ||d\omega||_{\overline{g}_0}^2$$

Setting this equal to 1, we get

$$2g_0(dr_0, d\omega) + r_0 ||d\omega||_{\overline{g}_0}^2 = \frac{1 - ||dr_0||_{\overline{g}_0}^2}{r_0}$$

This is a first order PDE so there is a solution near the boundary with any prescribed data of $\omega\Big|_N$

4.4.3 (Statement of) Correspondence between conformal infinity and einstein metric to the interior

Taking most of this from "Conformal invariants" by Fefferman-Graham. But I think a lot of this is in "The ambient metric" as well

- Let M^n a manifold with conformal structure $[g]$ and $M^+ = M \times [0, 1]$. Identify M with $M \times \{0\}$. We want to find a Poincare-metric g^+ such that
  - g^+ has $[g]$ as conformal infinity
  - $\text{Ric}(g^+) = -ng^+$
  - If we decompose

    $$g^+ = r^{-2}\left[dr^2 + g_{ij}^+(x, r)dx^i dx^j\right]$$

    for r a defining function, then we also require g_{ij}^+ an even function of r

- We have the following existence theorem

Theorem 4.10. *Let n be the dimension of M*

1. *If n is odd, then up to diffeomorphism fixing M, there is a unique power series solution to g^+ satisfying the above constraints. If $[g]$ is real analytic, then the power series converges so that g^+ exists and r^2g^+ is analytic up to the boundary*
2. *If n even, then there are conformal structures for which there is no formal power series solution of our constraints*
3. *If $Ric(g^+) = -ng^+$ is replaced with $Ric(g^+) + ng^+$ vanish to order $n-2$, then there is a formal power series solution for g^+ uniquely determined up to order $n-2$ and up to diffeomorphism fixing the boundary (M is the boundary in $M \times [0,1] = M^+$)*

4.4.4 Examples of PE spaces

See this paper by Anderson "Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics"

- Hyperbolic space in the ball model. The metric is

$$g = \frac{g_{euc}}{(1-r^2)^2}$$

and one can check that a bdf is given by

$$\rho = 2\frac{1-r}{1+r}$$

by imposing the condition

$$\rho^{-2}||d\rho||_g^2 = \partial_r(\log(\rho))g^{rr} = 1$$

- AdS-Schwarzschild metric: Let $M = \mathbb{R}^2 \times S^2$, and consider the metric

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^2(1)}$$

for

$$V = 1 + r^2 - \frac{2m}{r}$$

The mass is $m > 0$ and $r \in [r_+, \infty)$ where r_+ is the largest root of $V(r) = 0$. Moreover, the θ parameter is restricted to $[0, \beta]$ such that

$$\lim_{r \to r^+} V^{1/2}\frac{d(V^{1/2})}{dr}\beta = 2\pi$$

otherwise, the metric has a cone singularity along and normal to $\Sigma = \{r = r_+\}$. It follows that g_m smooth everywhere when

$$\beta = \frac{4\pi r_+}{1 + 3r_+^2}$$

As always, schwarzschild metrics are einstein, and the conformal infinity is given by $S^1(\beta) \times S^2(1)$, coming from the restriction of our domain in the $\mathbb{R}^2$ component

- The above can also be done is $S^2 \to \Sigma_g$ a surface of genus $g \geq 2$. Then again, we have $M = \mathbb{R}^2 \times \Sigma$ is the starting space with the metric

$$g_m = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{\Sigma_g}$$

and $\theta \in [0, \beta)$ but this time

$$\beta = \frac{4\pi r_+}{-1 + 3r_+^2}$$

to give smoothness. The conformal infinity is given by $S^1(\beta) \times \Sigma$ (Would be nice to find a bdf for this, probably $r - r_+$ works but then that would make the other parts of the metric degenerate at the boundary? Ask Otis) ($r = r_+$ is a coordinate singularity, so not the actual boundary - the actual boundary corresponds to $r \to \infty$, so need to think about the compactification there) (First change coordinates to get rid of coordinate singularity in the potential) (Actually bdf corresponds to $r \to +\infty$)