Area Exam Presentation: 3/17/22

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1 Motivating Example

- $\mathbb{H}^3$ ball model
$$g_{\mathbb{H}^3} = \frac{dr^2 + r^2 dg_{S^2}}{(1-r^2)^2}$$
want to compute the volume of some $Y^2 \hookrightarrow \mathbb{H}^3$ (Draw picture!)
- If Y^2 closed, then its fine because
$$g_{\mathbb{H}^3} \leq K(dr^2 + r^2 d_{S^2})$$
- If Y^2 is noncompact, then consider $\overline{Y}^2 \subseteq \mathbb{H}^3$, then $g_{\mathbb{H}^3}$ has unbounded coefficients
- Example: $\mathbb{H}^2 \subseteq \mathbb{H}^3$ represented as the geodesic disk. The restricted metric on $\mathbb{H}^2$ is
$$h := g\Big|_{\mathbb{H}^2} = \frac{4}{(1-r^2)^2}[dr^2 + r^2 d\theta^2] \implies \int_{\mathbb{H}^2} dVol_h = \int_0^1 \int_0^{2\pi} \frac{4}{(1-r^2)^2} r dr d\theta = \infty$$
(you can check that this integral diverges, because near the boundary it tends like $(1-r)^{-2}$
- Despite our foolish idea, suppose we wanted to still extract some information related to the volume computation
- Let $\rho = \frac{2(1-r)}{1+r}$. Notice that
  - $\rho^{-1}(0) = \{r = 1\}$
  - Define everywhere on $\overline{\mathbb{H}}^3$
  $$\overline{g} := \rho^2 g_{\mathbb{H}^2} = \frac{16}{(1+r)^4}[dr^2 + r^2 d\theta]$$
  - We have
  $$||d\log(\rho)||_g^2 = ||d\rho||_{\overline{g}}^2 = 1$$
  $$\nabla^{\overline{g}} \rho\Big|_{r=1} = g^{rr}(\partial_r \rho)\partial_r = -\partial_r$$
  $$\overline{g}\Big|_{\{\rho=0\}} = d\theta$$
  so ρ is *like* a distance function from the boundary (e.g. norm of gradient is 1 everywhere, like eikonal equation, and it vanishes to first order on the boundary), and it recreates the standard S^1 metric on the boundary.

(These conditions actually determine ρ, though its unclear at the moment why we'd want this)

- Consider the expansion of

$$\int_{\rho>\epsilon} dA = \int_{\rho>\epsilon} \frac{4r}{(1-r^2)^2} dr d\theta$$
$$= 4\pi \int_{r=0}^{(2-\epsilon)/(2+\epsilon)} \frac{d}{dr}\frac{1}{1-r^2} dr$$

since $\rho > \epsilon \leftrightarrow \frac{2-\epsilon}{2+\epsilon} > r$. Integrating, we get

$$\int_{\rho>\epsilon} dA = 4\pi \left[(1-r^2)^{-1}\right]_{r=0}^{(2-\epsilon)/(2+\epsilon)} = 4\pi\left[\frac{4+4\epsilon+\epsilon^2}{8\epsilon} - 1\right] = 4\pi\left[\frac{1}{2\epsilon} - \frac{1}{2} + \frac{\epsilon}{8}\right]$$

Taking the constant term in ϵ then yields

$$\underset{\epsilon\to 0}{FP} \int_{\rho>\epsilon} dA = 4\pi \cdot \frac{-1}{2} = \boxed{-2\pi}$$

($\underset{\epsilon\to 0}{FP}$ means "finite part" as $\epsilon \to 0$)

- More generally,
  - (Replace) $\mathbb{H}^3 \leftrightarrow M^{n+1}$, **asymptotically hyperbolic, conformally compact** (TBD but i.e. has a compact boundary with some given conformal class of metrics, $[k]$), $Y^m \subseteq M^{n+1}$ minimal
  - $\rho = \frac{2(1-r)}{1+r} \leftrightarrow \rho_Y$ such that
    - $\rho_Y^{-1}(0) = \partial Y$
    - $||d\log(\rho)||_g^2 = 1$

- Do the same process and define the **Hadamard Regularization of Volume**

$$\boxed{\mathrm{RV}(Y) := \underset{\epsilon\to 0}{FP} \int_{\rho_Y>\epsilon} dA_Y}$$

- We call $\mathrm{RV}(Y)$ the **Renormalized Volume**

- (Potential questions: Can this always be defined? When does ρ_Y exist? Does it depend on the choice of $\epsilon \to 0$? Good questions!)

- An equivalent way to compute Renormalized Volume is to define

$$f(z) = \int x^z dA_Y$$

- Happens to be meromorphic in z with poles at $\{\ldots, -1, 0, 1, \ldots, m\}$.

- Define **Riesz Regularization** of volume as

$$\mathrm{RV}(Y) = \underset{z=0}{FP} \int x^z dA_Y$$

- (The idea is that for $\Re(z) > m$ this integral converges and $\underset{z=0}{FP}$ means remove the pole at $z=0$ and evaluate.)

- (At first glance, doesn't seem equivalent, but it ends up being a computation)

- **Advantage**: Differentiating RV(Y) in Riesz form gives more geometric information because (informally)

$$\mathrm{RV}(Y_t) = \int x_t^z dA_{Y_t} \xrightarrow{\partial_t\big|_{t=0}} \int z x^{z-1} \dot{x} dA_Y + \int x^z \partial_t (dA_{Y_t})$$

(Cross out latter due to minimality, for the former, we have)

Lemma 1.1. For $f(x,s)$ nice in x, s, we have that

$$\underset{z=0}{FP} \int_Y z x^{z-1} f(x,s) dA_Y = \int_\gamma [f(x,s)\sqrt{\det \overline{g}\Big|_Y}(x,s)]^{(m)}$$

  - $[\cdot]^{(m)}$ denotes the coefficient of x^m in the asymptotic expansion of the function
  - This means bulk integrals become boundary integrals with the presence of a factor of zx^{z-1}.
  - Integrand for boundary integral often geometric things like volume form, $\dot{x}$, etc.

2 Formal Background

- Given an ambient space, M^n, with boundary $N = \partial M$, and a conformal class of metrics $[k]$, the **conformal infinity**, for N
- (M, g) is Einstein if

$$\mathrm{Ric}_g = kg$$

for some $k \in \mathbb{R}$ (Being conformally compact and Einstein means that g is even to high order in some expansion)

- **Definition:** M is conformally compact (CC), if
  - $\overline{M}$ is a manifold with compact boundary
  - $$\exists\, \rho : \overline{M} \to \mathbb{R}^{\geq 0}, \quad \text{s.t.} \quad \{\rho = 0\} = \partial M$$
  - $\overline{g} := \rho^2 g$ is a metric on $\overline{M}$
  - And
  $$\nabla^{\overline{g}} \rho\Big|_{\partial M} \neq 0$$
- **Definition:** ρ as above is called a boundary definition, and each ρ is associated to a boundary metric in the conformal class $[k]$ given by

$$h := \overline{g}\Big|_{\partial M}$$

- **Remark** If $\varphi : \overline{M} \to \mathbb{R}^+$ smooth, then $\rho^* = \varphi\rho$ is a bdf
- **Definition:** A bdf is **special** if

$$||d\log(\rho)||_g^2 = ||d\rho||_{\overline{g}}^2 = 1$$
$$\nabla^{\overline{g}} \rho\Big|_{\{\rho=0\}} \neq 0$$

(So special bdfs are most like distance functions to the boundary)

- **Proposition.** For M conformally compact and a choice of representative $k_0 \in [k]$, there exists a unique special bdf for M such that

$$\overline{g}\Big|_N = k_0$$

Proof: Requires more machinery than I have time for

- **Proposition:** For n even, Renormalized volume is independent of the choice of special bdf (i.e. independent of the choice of $k_0 \in k$)

2.1 Example

- Poincare Ball model of hyperbolic space $\mathbb{H}^3$

$$g = \frac{4}{(1-r^2)^2}\left[dr^2 + r^2 d\phi^2 + r^2 \sin^2\phi d\theta^2\right]$$

  is Einstein.

- Want special bdf, ρ, for $\mathbb{H}^3$. Assume rotational symmetry and enforce

$$1 = ||d\log(\rho)||_g^2 = \frac{\rho_r^2}{\rho^2} g^{rr} = \partial_r(\log(\rho))^2 \frac{(1-r^2)^2}{4}$$
$$\implies \rho = A\frac{1-r}{1+r}$$

- Suppose we want to prescribe the standard S^2 metric on the boundary, then

$$\overline{g} := \rho^2 g$$
$$\overline{g}\Big|_{r=1} = g_{S^2} = \frac{4A^2}{16}[d\phi^2 + \sin^2\phi d\theta^2] \implies A = 2$$
$$\overline{\nabla}\rho\Big|_{r=1} = -\partial_r$$

3 Theoretical Motivation

Physical Motivation

- String theory: happens (in one old model) on $M \times S^5$, M Einstein and conformally compact. (Draw M at least, ball model is good choice)
- "Wilson Loop Operator", $W(\gamma)$ for $\gamma \subseteq \partial M$, $\to$ find "string" whose "world sheet", $Y \subseteq \overline{M}$, $\partial Y = \gamma$
- In good approximation

$$\langle W(\gamma)\rangle \approx \sum_{\substack{\text{Y minimal} \\ \partial Y = \gamma}} \exp(-T\mathrm{RV}(Y))$$

- (Won't say more about this)

Mathematical Motivation:

- Renormalized Volume is a conformal invariant of the *boundary metric* for even dimensional manifolds
  - i.e. if we change the metric on $\gamma = \partial Y$ by a conformal factor, then $\mathrm{RV}(Y)$ stays the same
- $\mathrm{RV}(Y)$ reflects topological/geometric information

  Proposition (Alexakis, Mazzeo 2008). Suppose (M, g) Einstein with conformally compact boundary. Suppose $\gamma \subseteq \partial M$ and $Y^2 \hookrightarrow M$ with $\partial Y = \gamma$ and Y intersecting the boundary orthogonally, then

$$\mathrm{RV}(Y) = -2\pi\chi(Y) + \frac{1}{2}\int_Y 2|H|^2 - |\mathring{k}|^2 dA + \int_Y W_{1212} dA$$

  Some remarks:

  - This formula is very specific for Y two dimensional
  - $\int |H|^2$ is the Willmore energy and is conformally invariant in two dimensions
  - $\int_Y |\mathring{k}|^2$, where $\hat{k}$ is trace-free second fundamental form is also conformally invariant
  - W_{1212} is the ambient Weyl curvature, and also conformally invariant, and is the asymmetric part of the Riemann curvature tensor (note: this vanishes when $Y \subseteq \mathbb{H}^{n+1}$)
  - "Intersecting the boundary orthogonally" is guaranteed when Y is minimal!

4 Results Overview

- For convenience, we work in $M = \mathbb{H}^{n+1}$ the half space model with
$$g_{\mathbb{H}^{n+1}} = \frac{dx^2 + dy_1^2 + \cdots + dy_n^2}{x^2}$$
where x is *almost a special bdf* for $\mathbb{H}^{n+1}$, and
$$\overline{g} = x^2 g_{\mathbb{H}^{n+1}} = g_{Euc}$$
I'll explain technicalities of why x is not a special bdf but we can still use it to compute renormalized volume. I.e. we can treat x like a special bdf for Y, despite it not satisfying the "special" condition of $||dx||_{\overline{g}\big|_Y} = 1$

4.1 Graphicality

- $Y^m \subseteq \mathbb{H}^{n+1}$ minimal, conformally compact with boundary $\gamma = \partial Y = Y \cap \partial\mathbb{H}^{n+1}$. We require that Y be embedded in some neighborhood of the boundary γ.
- Consider cylinder over the boundary: (Draw this in Half-space model)
$$\Gamma = \gamma \times \mathbb{R}^+ = \{(x, s) \mid s \in \gamma\}$$
- Describe Y near the boundary as a graph over Γ via the exponential map (Draw Y!)
$$Y \cap \{x \leq \epsilon\} = \{\overline{\exp}_\Gamma(u(s, x))\}$$
where $\overline{\exp}$ denotes the exponential map taken with respect to the Euclidean metric, restricted to elements of $N(\Gamma)$.
- u satisfies a degenerate elliptic equation coming from Y being minimal, and just like the ambient metric is even to high order

Theorem 4.1. For $u(s, x) = u^i(s, x)\overline{N}_i(s)$ with $\{\overline{N}_i\}$ ONB for Γ,

$$\boxed{u^i(s,x) = \begin{cases} u_2^i(s)x^2 + u_4^i(s)x^4 + \cdots + u_m^i(s)x^m + u_{m+1}^i(s)x^{m+1} + \ldots & \text{m even} \\ u_2^i(s)x^2 + u_4^i(s)x^4 + \cdots + u_{m+1}^i(s)x^{m+1} + U^i(s)x^{m+1}\log(x) + u_{m+2}(s)x^{m+2} + \ldots & \text{m odd} \end{cases}}$$

for smoothly varying coefficients $u_k(s)$ and $U(s)$.

Remarks:

  - Done before by [Lin], [Guan, Spruck, Szapiel], [Tonegawa], [Han, Sehn, Wang], [Jiang]
  - PDE can be thought of as an ODE in x
$$(x\partial_x)(x\partial_x - (m+1))u(s, x) + (\text{Terms even in } x) = 0$$
  - Getting regularity is difficult: requires geometric arguments with maximum principle, and microlocal analysis (edge operators)
  - **Underlying Idea**: g is even to high order, MSS (minimal surface system) elliptic, so evenness preserved for Y to high order
  - Above expansion is **asymptotic**, not convergent (i.e. can give partial series with remainder vanishing to next order)

- **Corollary:** Renormalized Volume is well defined mathematically
(The reason behind this is that RV requires an asymptotic expansion of the volume form, which is true if we have an even graphical expression
$$\text{RV}(Y) = \underset{z=0}{FP} \int_Y x^z dA_Y = \underset{z=0}{FP} \int_Y x^z \sqrt{\det g\big|_Y}\, dx \wedge ds$$

4.2 Variations Codimension 1

- Consider variations of Y. Describe smooth family of minimal submanifolds as

$$Y_t = \exp_{p,Y}(\phi_t(p)\nu(p))$$

- $\dot\phi := \partial_t \phi_t\Big|_{t=0}$ satisfies the Jacobi equation

$$\Delta_Y \dot\phi + \mathrm{Ric}_{\mathbb{H}^{n+1}}(\nu,\nu)\dot\phi + |A_Y|^2\dot\phi = 0$$

so $\dot\phi$ satisfies a regularity theorem

$$\boxed{\dot\phi(s,x) = \begin{cases} \dot\phi_0(s) + \dot\phi_2(s)x^2 + \cdots + \dot\phi_m(s)x^m + O(x^{m+1}) & \text{m even} \\ \dot\phi_0(s) + \dot\phi_2(s)x^2 + \cdots + \dot\phi_m(s)x^{m+1} + \Phi(s)x^{m+1}\log(x) + O(x^{m+2}) & \text{m odd} \end{cases}}$$

Idea: g is even to high order, Y is also even via graphical expansion, so $\Delta_Y^\perp$, $\tilde{A}_Y$, and Ric record this evenness. Now a similar ODE argument gives the actual expansion and regularity.

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Theorem 4.2. First variation of Renormalized volume in codimension 1:

$$\boxed{\begin{aligned} n \text{ even} &\implies \frac{d}{dt}\mathrm{RV}(Y_t)\Big|_{t=0} = -(n+1)\int_\gamma \dot\phi_0(s)u_{n+1}(s)\, dA_\gamma(s) \\ n \text{ odd} &\implies \frac{d}{dt}\mathrm{RV}(Y_t)\Big|_{t=0} = -(n+1)\int_\gamma \Big[\dot\phi_0(s)u_{n+1}(s) + F(\dot\phi_0,u_2)(s)\Big]\, dA_\gamma(s) \end{aligned}}$$

and the second variation:

$$\begin{aligned} n \text{ even} &\implies \frac{d^2}{dt^2}\mathrm{RV}(Y_t)\Big|_{t=0} = \int_\gamma \Big((1-n)\dot\phi_0(s)\dot\phi_{n+1}(s) + \dot\phi_0(s)^2\left[(n-1)(n-2) - 8nu_2u_{n+1}(s)\right]\Big)\, dA_\gamma(s) \\ n \text{ odd} &\implies \frac{d^2}{dt^2}\mathrm{RV}(Y_t)\Big|_{t=0} = \int_\gamma \Big[(1-n)\dot\phi_0(s)\dot\phi_{n+1}(s) + \dot\phi_0(s)^2\left[(n-1)(n-2) - 8nu_2u_{n+1}(s)\right] \\ &\qquad - \dot\phi_0(s)\Big[4(n+2)\dot\phi_0(s)u_2(s)U(s) + \dot\Phi(s)\Big] + F_2(\dot\phi_0,u_2)\Big]dA_\gamma(s) \end{aligned}$$

5 Sketch of Proof of First variation (Skip if time is short)

We have

$$\mathrm{RV}(Y_t) = \underset{z=0}{FP}\int_{Y_t} x^z dA_{Y_t}$$

Let

$$F_t : Y \to Y_t$$
$$F_t(p) = \exp_{\overline{g}_Y,p}(\phi_t(p)\nu(p))$$

Then

$$\begin{aligned} \mathrm{RV}(Y_t) &= \underset{z=0}{FP}\int_Y F_t^*(x)^z F_t^*(dA_{Y_t}) \\ \frac{d}{dt}\mathrm{RV}(Y_t)\Big|_{t=0} &= \underset{z=0}{FP}\int_Y zx^{z-1}\dot x dA_Y + \underset{z=0}{FP}\int_Y x^z(\dot\phi H_Y)dA_Y \\ &= \int_\gamma [\dot x\sqrt{\det \overline{g}|_Y}]^{(m)}dA_\gamma \end{aligned}$$

We also compute

$$\dot{x} = dx(\dot{F}) = dx(\dot{\phi}\nu) = -\dot{\phi}dx(\nu)$$

Via some work

$$\begin{aligned}
dx(\nu) &= xu_x(1 + \text{ even up to } x^m) \\
&= x(2u_2x + 4u_4x^3 + \text{odd} + mu_mx^{m-1} + (m+1)u_{m+1}x^m + \dots)(1 + \text{even up to } x^m) \\
\dot{\phi} &= \dot{\phi}_0x^{-1} + \dot{\phi}_2x + \dots + \dot{\phi}_mx^{m-1} + \dot{\phi}_{m+1}x^m + \dots \\
\sqrt{\det \overline{g}|_Y} &= 1 + (\text{ even }) + O(x^m)
\end{aligned}$$

and so (Notice that by parity, the only order m term comes from $(m+1)u_{m+1}x^m$ combined with 0th order terms for everything else):

$$\begin{aligned}
[\dot{x}\sqrt{\det \overline{g}|_Y}]^{(m)} &= -(m+1)u_{m+1}\dot{\phi}_0 \\
\implies \mathrm{RV}' &= \int_\gamma -(m+1)\dot{\phi}_0u_{m+1}
\end{aligned}$$

In particular, we've gleaned this formula by investigating the following geometric terms, $\dot{x}$, ν, $dx(\nu)$, etc.

6 Future Work