Math 258 Fall 2022: Ricci Flow with Yi Lai

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Contents

Introduction

These are notes from Professor Yi Lai's Math 258 taught in Fall 2022 at Stanford. Thank you to her for a great class! In addition, thank you to Yujie Wu and Shuli Chen for their comments and pictures incorporated into these notes. These notes are not perfect, but may serve as an instructive reference for advanced ideas in Ricci Flow. Professor Lai also has a hand written version of these notes on her website.

1 Lecture 1: 9-27-22

Schedule

- RF short time existence
- Basic RF identities
- Maximum principles
- RIcci solitons
- Perelman's $\mathcal{F}$, $\mathcal{W}$, functionals
- Perelman's no-local collapsing theorem
- Bamler's compactness theory of Ricci Flows

Today: Review Riemannian Geometry - ricci curvature and its linearization

1.1 Riemannian Geometry

- Riemannian curvature tensor

$$\begin{aligned} X &= X^i \partial_{x_i} \\ R(X,Y)Z &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= R^{\ell}_{ijk} X^I Y^j Z^k \frac{\partial}{\partial x^\ell} \\ &= R_{ijk\ell} g^{\ell s} X^i Y^j Z^k \frac{\partial}{\partial x^s} \end{aligned}$$

We note that in the third line, Rm is a $(1,3)$ tensor, while in the fourth line it is a $(0,4)$ tensor.

- Identities

$$R_{ijkl} = -R_{jikl} = -R_{ijkl} = R_{klij}$$
$$R_{ijkl} + R_{kijl} + R_{jkil} = 0 \qquad \text{(first bianchi identity)}$$
$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0 \qquad \text{second bianchi identity}$$

- Ricci curvature

$$\mathrm{Ric}(X,Y) = \mathrm{Tr} Rm(\cdot, X)Y = \mathrm{Ric}_{ij} X^i Y^j$$
$$\mathrm{Ric}_{ij} = R^s_{sij} = g^{st} R_{sijt}$$
$$\mathrm{Ric}_{ij} = \mathrm{Ric}_{ji}, \qquad \mathrm{Ric} \in S^2(T_* M)$$

- Scalar Curvature

$$R = \mathrm{tr}_g \mathrm{Ric} = g^{ij} \mathrm{Ric}_{ij} = g^{ij} g^{st} R_{sijt}$$
$$g^{st} \nabla_s \mathrm{Ric}_{ti} = \frac{1}{2} \nabla_i R \qquad \text{second contracted Bianchi identity}$$

1.2 Space of algebraic curvature tensors

Let (V, g) Euclidean vector space $n = \dim V < \infty$ (e.g. T_pM, g_p). Let $\{e_i\}$ an onb, $S^2(\wedge_2 V) = \{$symmetric 2-forms on$\wedge_2$ $V\}$. Define

$$S^2_B(\wedge V) = \{\mathrm{Rm} = R_{ijkl}(e_i \wedge e_j) \otimes (e_k \wedge e_l) \mid R_{ijkl} \text{ satisfies } *$$

(here $*$ denotes the curvature symmetries and the first bianchi identity. In fact the B subscript stands for Bianchi).

We have an algebraic curvature operator

$$Rm : \wedge_2(V) \to \wedge_2(V)$$
$$e_i \wedge e_j \to -\frac{1}{2} R_{ijkl} e_k \wedge e_l$$

Example: Standard sphere, $(S^n, K = 1)$, then $\forall p \in S^n$, $Rm \in S^2_B(\wedge_2 T_p S^n)$, and

$$Rm = \mathrm{Id}$$

because sphere has constant curvature

$$\begin{aligned} Rm(e_1 \wedge e_2) &= -\frac{1}{2} R_{12kl} e_k \wedge e_l \\ &= -\frac{1}{2} R_{1212} e_1 \wedge e_2 = e_1 \wedge e_2 \\ &= -\frac{1}{2} R_{1221} e_2 \wedge e_1 \end{aligned}$$

1.3 Decomposition of Curvature Tensors

We have

$$S^2_B(\wedge_2 V) = \langle Id \rangle \oplus \langle \mathring{\mathrm{Ric}} \rangle \oplus \langle \mathrm{Weyl} \rangle$$

Note that if K is constant then $Rm = KId$, and hence $Rm \in < Id >$

$$R_{ijkl} = k(\delta_{jk}\delta_{il} - \delta_{ik}\delta_{jl})$$

e.g. spacetime curvature (i.e. riemannian manifolds with $K = k \in \mathbb{R}$).

Now suppose that $Rm \in \langle \mathring{\mathrm{Ric}} \rangle$, suppose $\mathrm{Ric} = (n-2)A$, then $\mathrm{tr}(A) = 0$ and

$$R_{ijkl} = A_{il}\delta_{jk} + A_{jk}\delta_{il} - A_{ik}\delta_{jl} - A_{jl}\delta_{ik}$$

If $Rm \in \langle \mathrm{Weyl} \rangle$, we have $\mathrm{Ric}(Rm) = 0$. (Would be good to get projection maps, presumably something like

$$Rm = Rg \otimes g + g \otimes (\mathrm{Ric} - Rg) + \text{else}$$

or something).

When $n = 2$, $S^2_B(\wedge_2 V) = \langle Id \rangle$, e.g. 2-dimensional riemann manifold.
When $n = 3$

$$S^2_B(\wedge_2 V) = \langle Id \rangle \oplus \langle \mathring{\mathrm{Ric}} \rangle$$

choose an onb $\{e_i\}$ such that

$$\mathrm{Ric} = \begin{bmatrix} \rho_1 & & \\ & \rho_2 & \\ & & \rho_3 \end{bmatrix}$$

So the curavture operator is also diagonal and

$$Rm = \begin{bmatrix} k_1 & & \\ & k_2 & \\ & & k_3 \end{bmatrix}$$

where the rows and columns are $e_2 \wedge e_3$, $e_3 \wedge e_1$, $e_2 \wedge e_1$, and $\rho_1 = k_2 + k_3$, $\rho_2 = k_1 + k_3$, $\rho_3 = k_1 + k_2$.

Also

$$k_1 = K(e_2 \wedge e_3), \quad k_2 = K(e_1 \wedge e_3), \quad k_3 = K(e_1 \wedge e_2)$$

Corollary 1.0.1. (M^3, g) Riemannian manifold. $K \geq 0 \iff Rm \geq 0$

Note that $Rm \geq 0$ always gives $K \geq 0$ by definition of sectional curvature. The other direcion is only true when $n \leq 3$

1.4 Einstein Equation + Ricci Flow

Let (M^n, g) a riemannian manifold

$$\text{Ric} = \lambda g$$

$\lambda \in \mathbb{R}$ - can prove via Schur's lemma that if λ is a scalar function (not necessarily constant), then it **has** to be constant everywhere.

We define Ricci Flow (Hamilton, 1982), $(M^n, (g_t)_{t \in I})$ such that

$$\partial_t g_t = -2\text{Ric}_{g_t}$$

Ex: If $\text{Ric} = \lambda g$, then

$$g_t = \begin{cases} -2\lambda t g, & t > 0, \quad \text{if } \lambda < 0 \\ g & t \in \mathbb{R}, \quad \text{if } \lambda = 0 \\ -2\lambda t g & t < 0 \quad \text{if } \lambda > 0 \end{cases}$$

As an example, Yi draws pictures corresponding to a 2-hold torus with $K = -1$ (expanding surface), a torus with $K = 0$ (Constant), and a sphere with $K = 1$ (round, shrinking sphere) 1 Example: If $(M_i, (g_{i,t})_{t \in I})_{i=1,2}$

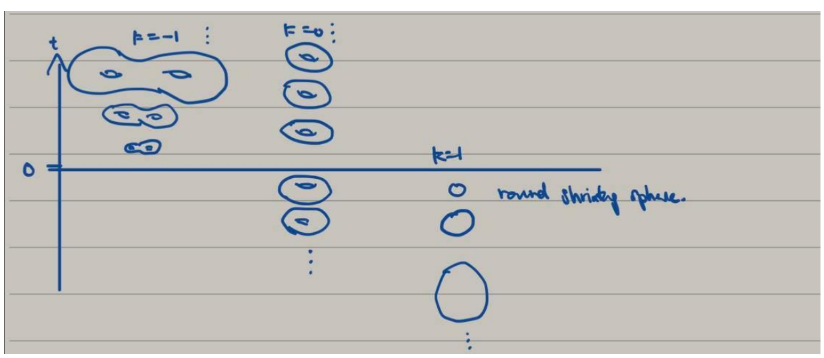

Figure 1

are ricci flows (RFs) then $(M_1 \times M_2, (g_{1,t} + g_{2,t})_{t \in I})$ is a ricci flow.

Example:

$$S^n \times \mathbb{R}^m, \qquad g_t = -2(n-1)t g_{S^n} + g_{\mathbb{R}^m}, \qquad t < 0$$

See here 2

Figure 2

1.5 Symmetries of Ricci FLow

- Time-shift
$$g'_t = g_{t-t_0}, \qquad t \in I + t_0$$
is also a ricci flow
- Parabolic rescaling
$$g'_t = \wedge^2 g_{\lambda^{-2}t} \quad t \in \lambda^2 I,$$
Check that Ricci flow equation is
$$\partial_t g'_t = -2\text{Ric}_{g_{\lambda^{-2}t}} = -2\text{Ric}_{\lambda^2 g_{\lambda^{-2}t}} = -2\text{Ric}_{g'_t}$$
- Diffeomorphism invariance: If $\phi : M \to N$ diffeo, then
$$(N, g_t) \text{ RF} \iff (M, \phi^* g_t) \text{ RF}$$
- Under rescaling $g'_t = r^2 g_{r^{-2}t}$, we say the scale of quantity is k if it changes by r^k under the rescaling.

  Then for any function f, vector V, we have

$k=2$	$k=1$	$k=0$	$k=-1$	$k=-2$
g_{ij}	$\lvert V\rvert$	Ric		g^{ij}
$\lvert V\rvert^2$	$\sqrt{t}$	Γ^k_{ij}		R_{ijkl}
t		R^l_{ijk}		$\lvert Rm\rvert, \lvert\text{Ric}\rvert, R$
		$\nabla^2 f,\ df$		$\lvert\nabla^2 f\rvert$
		$\nabla^k h$		$\nabla f = (df)^\sharp$

1.6 Short time existence and uniqueness

Initial Value problem: Given (M, g), find $T > 0$, and $(g_t)_{t\in[0,T)}$ such that
$$\partial_t = -2\text{Ric}_{g_t} \tag{1}$$
$$g_0 = g$$

Theorem 1.1 (Hamilton)**.** Suppose M compact

- Existence: The above system (1) has solution for some $T > 0$
- Uniqueness: If $(g_{i,t})_{t\in[0,T_i)}$ with $g_{i,0} = g$ for $i = 1, 2$ both RFs, then
$$g_{1,t} = g_{2,t} \qquad \forall t \in [0, \min(T_1, T_2))$$

This tells us that there exists a *maximal* Ricci Flow on each (M, g) compact, which will be unique by the above

2 Lecture 2: 9-29-22

Goal:

- Analytic Properties of Ricci Flow
- Ricci-DeTurk Flow, Harmonic map heat flow

2.1 Diffeomorphism Invariance

$$\mathrm{Ric}_{\phi^* g} = \phi^* \mathrm{Ric}_g$$

If we assume we have a flow ϕ_s associated to a vector field X, then

$$(D(\mathrm{Ric}_g))(\mathcal{L}_X g) = \frac{d}{ds}(\mathrm{Ric}_{\phi_s^* g}) = \frac{d}{ds}(\phi_s^* \mathrm{Ric}_g) = \mathcal{L}_X(\mathrm{Ric})$$

Note that on the left hand side, we have a priori 3 derivatives of X, since we have to differentiate the metric twice to get the ricci curvature. On the right most side, we have 0 derivatives of X. In sum,

$$D(\mathrm{Ric}_g)(h) = \frac{d}{ds}\mathrm{Ric}_{g+sh}$$

One can use these two equations to derive the second contracted bianchi identity!

2.2 Some Operators

Recall: A linear differential operator, L, is elliptic if the principal symbol, $\sigma(L)(\xi)$ is an isomorphism for all $\xi \in T^*M$.

Ex: $L = \Delta_g : C^\infty(M) \to C^\infty(M)$, (M^n, g). Then in local coordinates we have

$$\Delta_g = g^{ij}\partial_i \partial_j$$
$$\sigma[\Delta](\xi) = g^{ij}\xi_i\xi_j = |\xi|_g^2 \neq 0$$

We also have the Lie Derivative:

$$\delta^* : C^\infty(T^*M) \to C^\infty(S_2 T * M)$$

given by

$$\delta^* W = \frac{1}{2} L_{W^\sharp} g$$

where $W^\sharp$ dual to W. In local coordinates, we have

$$(\delta^* W)_{jk} = \frac{1}{2}(\nabla_j W_k + \nabla_k W_j) = \frac{1}{2}(\partial_j W_k + \partial_k W_j)$$

(I guess in geodesic normal coordinates at least) so that

$$\sigma[\delta^*](\xi) = \frac{1}{2}(\xi_j W_k + \xi_k W_j)$$

The dual to δ^* is called the divergence,

$$\delta : C^\infty(S_2 T^* M) \to C^\infty(T^*M)$$
$$(\delta h)_k = -g^{ij}\nabla_i h_{jk}$$

Note that

$$\begin{aligned}
(D\mathrm{Ric})(\mathcal{L}_X g) &= \mathcal{L}_X \mathrm{Ric} \\
X = W^\sharp \implies (D\mathrm{Ric})(\mathcal{L}_{W^\sharp} g) &= \mathcal{L}_{W^\sharp} \mathrm{Ric} \\
((D\mathrm{Ric}) \circ \delta^*)(W) &= \frac{1}{2}\mathcal{L}_{W^\sharp} \mathrm{Ric} \\
0 &= \sigma[(D\mathrm{Ric}) \circ \delta^*](\xi) = \sigma[D\mathrm{Ric}](\xi) \circ \sigma[\delta^*](\xi)
\end{aligned}$$

the last line follows since again it seems that $D\mathrm{Ric}\circ\delta^*$ is a 3rd order operator, but we showed that because of the third line, this is actually first order, so $\sigma[D\mathrm{Ric}\circ\delta^*] = \sigma_3[D\mathrm{Ric}\circ\delta^*] = 0$ since its actually first order. In particular, this tells us that

$$\mathrm{Im}(\sigma[\delta^*](\xi)) \subseteq \ker\sigma[D\mathrm{Ric}](\xi)$$

the left hand side is dimension n, so this says that D**Ric is not elliptic**. We now show

Lemma 2.1.

$$D(-2\mathrm{Ric})(h)_{jk} = \Delta_{jk} + g^{pq}(\nabla_j\nabla_k h_{qp} - \nabla_q\nabla_j h_{kp} - \nabla_q\nabla_k h_{jp})$$

This follows by computing the formula for the first variation of the christoffel symbol. Rewrite this as

$$\begin{aligned}
D(-2\mathrm{Ric})(h_{jk}) &= \Delta h_{jk} + g^{pq}(\nabla_j\nabla_k h_{qp} - \nabla_j\nabla_q h_{kp} - \nabla_k\nabla_q h_{jp}) \\
&\quad + g^{pq}(2R^r_{qjk}h_{rp} - R_{jp}h_{kq} - R_{kq}h_{jq} \\
&= \Delta_L h_{jk} + g^{pq}(\nabla_j\nabla_k h_{qp} - \nabla_j\nabla_q h_{kp} - \nabla_k\nabla_q h_j p) \\
&= \Delta_L h_{jk} + \nabla_j\nabla_k g^{pq}h_{qp} - \nabla_j g^{pq}\nabla_q h_{kp} - \nabla_k g^{pq}\nabla_g h_{jp} \\
&= \Delta_L h_{jk} - \nabla_j\nabla_k \mathrm{tr}(h) - \nabla_j(\delta h)_k - \nabla_k(\delta h)_j \\
&= \Delta_L h_{jk} - \nabla_j\nabla_k \mathrm{tr}(h) - 2\delta^*(\delta h)_{jk}
\end{aligned}$$

where

$$\Delta_L = \Delta_g + g^{pq}(2R^r_{qjk}h_{rp} - R_{jp}h_{kq} - R_{kq}h_{jq}$$

is the Lichnerowicz laplacian, and in the fourth line we've used that the metric is compatible with the connection (torsion free or something) to commute connection with metric coefficients. We also have

Proposition 1. Choose a background metric $\overline{g}$, and let

$$W_j = g_{jk}g^{pq}(\Gamma^k_{pq} - \overline{\Gamma}^k_{pq})$$

then

$$D(-2\mathrm{Ric} + \nabla_i W_j + \nabla_j W_i)(h) = \Delta_L h + \text{first order terms in } h$$

i.e. the operator on the left is **strongly elliptic**

Proof: We compute

$$\begin{aligned}
D(W_j)(h) &= g_{jk}g^{pq}D(\Gamma^k_{pq})(h) + \text{zero order terms in } h \\
&= g_{jk}g^{pq}\cdot\frac{1}{2}g^{kl}\left(\nabla_q h_{lp} + \nabla_p h_{lq} - \nabla_l h_{pq}\right) \\
&= \frac{1}{2}g^{pq}\left(\nabla_q h_{jp} + \nabla_p h_{jq} - \nabla_j h_{pq}\right) + \text{z.o.t} \\
&= (\delta h)_j - \frac{1}{2}\nabla_j \mathrm{tr}(h) + \text{z.o.t}
\end{aligned}$$

where z.o.t. denotes "zeroth order terms." This tells us that

$$D(\nabla_i W_j + \nabla_j W_i)(h) = \nabla_i(\delta h)_j + \nabla_j(\delta h)_i - \nabla_i\nabla_j \mathrm{tr}(h) + (\text{first order terms})$$

2.3 Ricci DeTurk Flow

Choose a background metric $\overline{g}$. Then a metric, $\tilde{g}$, satisfies Ricci-DeTurk Flow if

$$(\partial_t \tilde{g}_t)_{ij} = -2(\mathrm{Ric}_{\tilde{g}_t})_{ij} + \nabla_i W_j + \nabla_j W_i$$

where the connections are taken w.r.t. $\tilde{g}_t$ and where

$$(W_t)_l = \tilde{g}_{lk}\tilde{g}^{ij}(\tilde{\Gamma}^k_{ij} - \overline{\Gamma}^k_{ij})$$

where the t subindex denotes time. Note that the Ricci-DeTurk Flow equation is a **strongly elliptic** PDE, so it should satisfy short time existence and uniqueness.

We now want to compare Ricci flow and Ricci-DeTurk flow. Recall for $\chi : (M_1, g_1) \to (M_2, g_2)$ a map between two riemannian manifolds, we have that

$$\Delta_{g_1,g_2}\chi = \sum_{i=1}^{n} \left(\nabla^{g_2}_{d\chi(e_i)} d\chi(e_i) - d\chi(\nabla^{g_1}_{e_i} e_i) \right)$$

where $\{e_i\}$ is an onb on M_1. Now consider (M, g) and $(M, \tilde{g}_t)$, let

$$W^\sharp := \Delta_{\tilde{g}_t, g} Id$$

then

$$\partial_t \tilde{g}_t = -2\text{Ric}_{\tilde{g}_t} + \mathcal{L}_{W^\sharp} \tilde{g}_t$$

2.4 Harmonic Map Heat Flow

Let $\{\chi_t\}$ a family of diffeos such that $\chi_0 = Id$. Then we say that χ_t satisfies the harmonic map heat flow if

$$\partial_t \chi_t = \Delta_{g_t, \overline{g}} \chi_t$$

when $\{g_t\}$ is a Ricci Flow. We now connect the Harmonic Map Heat Flow to Ricci De Turk Flow 3

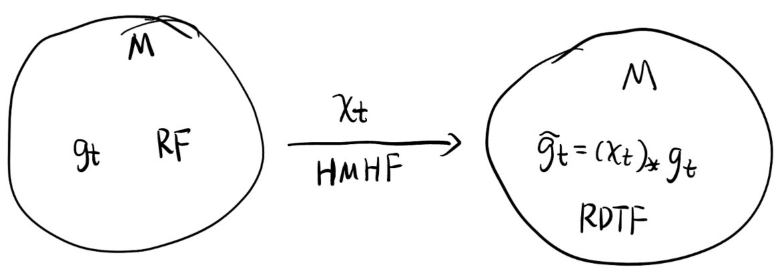

Figure 3

Proposition 2. If (M, g_t) is a Ricci Flow and $\{\chi_t\}$ a harmonic map heat flow w.r.t. $\{g_t\}$, then $(M, \tilde{g}_t = (\chi_t)_* g_t)$ is a Ricci De Turk Flow, and vice versa, i.e. if $\tilde{g}_t$ a Ricci-De-Turk Flow and $\{\chi_t\}$ a harmonic map heat flow (still w.r.t. $\{g_t\}$) then $g_t = (\chi_t)^* \tilde{g}_t$ is a Ricci Flow

A natural question, if $\{\chi_t\}$ is always defined w.r.t $\{g_t\}$, then how can we go from $\tilde{g}_t$, a RDTF to $\{g_t\}$ a RF without having $\{g_t\}$ in the first place? To resolve this, we compute

$$\begin{aligned} \partial_t g_t &= \partial_t(\chi_t^* \tilde{g}_t) = \chi_t^*(\partial_t \tilde{g}_t) + \chi_t^*(\mathcal{L}_{\partial_t \chi_t} \tilde{g}_t) \\ &= \chi_t^*(\partial_t \tilde{g}_t) + \chi_t^*(\mathcal{L}_{\Delta_{\tilde{g}_t, \overline{g}} Id} \tilde{g}_t) \partial_t \chi_t \\ &= \Delta_{g_t, \overline{g}} \chi_t \\ \partial_t \chi_t \circ \chi_t^{-1} &= (\Delta_{g_t, \overline{g}} \chi_t) \circ \chi_t^{-1} = \Delta_{\tilde{g}_t, \overline{g}} Id \end{aligned}$$

where the last line follows from the diffeomorphism invariance of the laplacian. But now plugging this identity into the second line (and using the definition of RDTF flow), we get

$$\partial_t g_t = \chi_t^*(-2\text{Ric}(\tilde{g}_t)) = -2\text{Ric}(\chi_t^* \tilde{g}_t) = -2\text{Ric}_{g_t}$$

This tells us that given the correspondence between $\{g_t\} \leftrightarrow \{\tilde{g}_t\}$, the harmonic map heat flow, $\{\chi_t\}$ actually satisfies both of

$$\partial_t \chi_t = \Delta_{g_t, \overline{g}} \chi_t, \qquad g_t \text{ a RF} \tag{2}$$
$$\partial_t \chi_t = (\Delta_{\tilde{g}_t, \overline{g}} Id) \circ \chi_t, \qquad \tilde{g}_t \text{ a RDTF} \tag{3}$$

i.e. $\{\chi_t\}$ satisfying either of the above is equivalent.

We now show existence of Ricci Flow: If we solve for $\tilde{g}_t$ a Ricci flow, then use the above to solve for $\{\chi_t\}$ a harmonic map heat flow, we have via our proposition

$$g_t := \chi_t^* \tilde{g}_t$$

is a Ricci Flow. This gives short time existence.

Uniqueness: Essentially the same idea, but we formulate it in full: given $\{g_t^i\}$ ricci flows for $i = 1, 2$ and g_0^1, g_0^2, use (2) to solve for χ_t^i. Then via our proposition,

$$\tilde{g}_t^i := (\chi_t^i)_* g_t$$

are RDTF flows with $\tilde{g}_0^1 = \tilde{g}_0^2$. But now uniqueness of RDTF flow tells us that

$$\tilde{g}_t^1 = \tilde{g}_t^2$$

for all t in our maximal interval. But then by (3), we have that

$$\chi_t^1 = \chi_t^2$$

for all t because the harmonic map heat flow is strongly parabolic. Finally, this gives

$$g_t^1 = g_t^2$$

for all t.

2.5 Solving non-linear strongly parabolic PDEs

We have a few non-linear strongly parabolic PDES: RDTF (Ricci De Turk Flow), HMHF (Harmonic Map Heat Flow).

Let (M, g) compact Riemannian Manifold, (E, h) euclidean (real?) Vector Bundle over M with metric connection (e.g. $S^2 T_* M$). Moreover, let $(U_t)_{t \in [0, \tau)}$ smooth family of sections of E, (RDTF: $\tilde{g}_t$). Want to solve

$$\begin{aligned} \partial_t u_t &= a^{ij}(u_t, x, t) \nabla_{ij}^2 u_t + f(u_t, \nabla u_t, x, t) \\ u_0 &= \tilde{u} \end{aligned}$$

Assume

$$a^{ij} \geq C g^{ij}$$

for some uniform $C > 0$. We have short time existence and uniqueness

Theorem 2.2. The above system has a unique solution for some $\tau > 0$

3 Lecture 3: 10-4-22

Today's goals:

- Non-linear parabolic PDE
- Evolution of length, distance
- Evolution of volume form

3.1 Solving non-linear strongly parabolic PDEs

We have

$$\partial_t u_t = a^{ij}(u_t, x, t)\nabla^2_{ij} u_t + f(u_t, \nabla u_t, x, t) \tag{4}$$
$$u_0 = \tilde{u}$$

for $\tilde{u}$ given. We have ellipticity

$$a^{ij} \geq cg^{ij}$$

for some $c > 0$. Then

Theorem 3.1. System (4) has a unique solution for some small $\tau > 0$.

Proof: Let

$$U_\tau := \{u \in C^{2m+2,2\alpha;m+1,\alpha}(M \times [0,\tau]; E) \mid u(\cdot, 0) = 0\}$$

where $C^{k,\beta;k',\beta'}$ denotes regularity separately in spatial and time directions. Here, E is some bundle, e.g. bundle of symmetric 2-forms. Similarly

$$V_\tau = C^{2m,2\alpha;m,\alpha}(M \times [0,\tau], E)$$

for some τ small and determined later. We now consider the differential map

$$F_\tau : U_\tau \to V_\tau$$
$$u \mapsto \partial_t - a^{ij}(u_t, x, t)\nabla^2_{ij} u_t - f(u_t, \nabla u_t, x, t)$$

Our goal is to find a u such that $F(u) = 0$ and $u \in U_\tau$. Of course, we do this by some implicit function

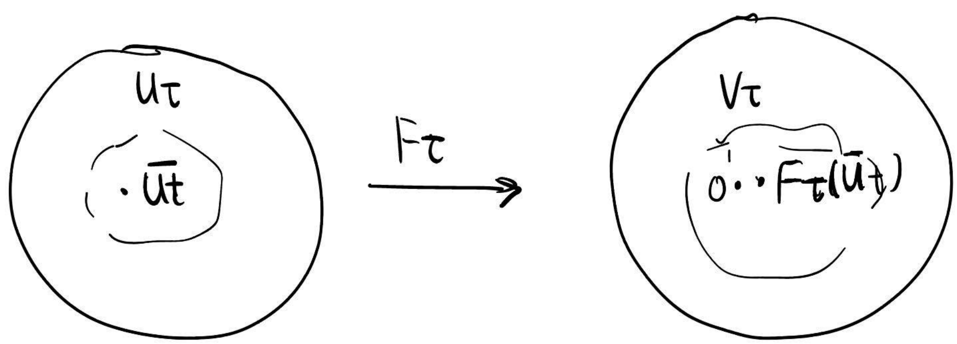

Figure 4

theorem or contraction map. Let

$$\overline{u}_t = tf(0,0,x,t)$$

Then we see that

$$F_t(\overline{u}_t)(\cdot, t = 0) = 0$$

So

$$\lim_{\tau \to 0} ||F_\tau(\overline{u}_t)|| = 0$$

we now want to show that F_τ is non-degenerate as τ goes to 0 so we can truly find a zero. Consider the linearization of F_τ at $u = \overline{u} = \overline{u}_\tau$

$$L_\tau = (DF_\tau)_{\overline{u}} = \partial_t - a^{ij}(\overline{u}, x, t)\nabla^2_{ij} - b^i \nabla_i - C$$
$$L_\tau : U_\tau \to V_\tau$$

$$\text{parabolic schauder estimate} \implies ||\hat{u}||_{U_\tau} \leq C(||L_\tau \hat{u}||_{V^\tau} + ||\hat{u}||_{C^0})$$

here C is **independent of** τ. But now we claim that

$$||\hat{u}||_{C^0} \leq C||L_\tau \hat{u}||_{V^\tau}$$

Proof: Denote $A = ||L_\tau \hat{u}||_{V_\tau}$. Then

$$\begin{aligned} \partial_t \hat{u} - a^{ij}\nabla_{ij}\hat{u} - b^i \nabla_i \hat{u} - c\hat{u} &\leq ||L_\tau \hat{u}||_{C^0} \leq A \\ (\partial_t - a^{ij}\nabla_{ij} - b^i \nabla_i)\hat{u} &\leq A + c\hat{u} \leq A + C||\hat{u}||_{C^0} \\ \text{maximum principle} \quad \implies \hat{u}(\cdot, t) &\leq (A + C||\hat{u}||_{C^0}) \cdot t \end{aligned}$$

This last line follows by comparing $\hat{u}$ with the following function

$$u \text{ s.t. } \partial_t u = A + C||\hat{u}||_{C^0} = \tilde{C}$$

i.e. $\partial_t u$ is a constant. This is a bit opaque, but I guess the idea is to

$$\begin{aligned} (\partial_t - a^{ij}\nabla_{ij} - b^i \nabla_i)\hat{u} &\leq A + C||\hat{u}||_{C^0} \\ \partial_t u &= A + C||\hat{u}||_{C^0} \end{aligned}$$

and to subtract the two or something. Now choose τ very small such that $c \cdot \tau \leq \frac{1}{2} \implies \hat{u}_t \leq C \cdot A$.

We now do the same argument but with the reverse sign, i.e.

$$\hat{u}_t \geq -CA$$

to get that $||\hat{u}_t|| \leq C \cdot A$, which means that

$$\begin{aligned} ||\hat{u}||_{V^\tau} &\leq C||L_\tau \hat{u}||_{V_\tau} \\ \implies ||L_\tau^{-1}|| &\leq C \end{aligned}$$

Seems like the <u>crux of this proof is the parabolic maximum principle</u>. $\square$

We also need to check

$$||D^2 F_\tau|| \leq C$$

But Yi asks that we do it on our own. Once we have this, the **Inverse function theorem** implies that F_τ is invertible on

$$S = B_{U_\tau}(\overline{u}, r_0) \subseteq U_\tau$$

where r_0 is independent of τ. But now the invertibility of L_τ says that balls of a given radius in U_τ will yield balls of comparable radius under L_τ, i.e.

$$F_\tau(B_{U_\tau}(\overline{u}, r_0)) \supseteq B_{V_\tau}(F_\tau(\overline{u}), cr_0)$$

for some $c > 0$ independent of τ. But now, note that for τ sufficiently small, we have $F_\tau(\overline{u}_t) \to 0$ as $\tau \to 0$, so if we choose τ small so that $||F_\tau(\overline{u})|| \leq \frac{1}{2}cr_0$, then we're done.

3.2 Evolution of Lengths

Let $\gamma : [a, b] \to M$, a C^1 curve then

$$\frac{d}{dt}\ell_t(\gamma) = \frac{d}{dt}\int_a^b |\dot{\gamma}(s)|_{g_t} ds = -\int_a^b \frac{\text{Ric}(\dot{\gamma}(s), \dot{\gamma}(s))}{|\dot{\gamma}(s)|} ds$$

assuming that $\{g_t\}$ is a Ricci Flow

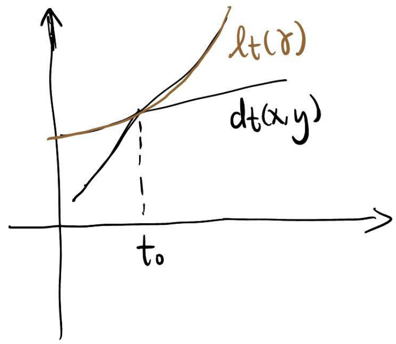

Figure 5

3.3 Distance Distortions

Let $x, y \in M$ compact, $t_0 \in I$. Let γ be a minimizing geodesic (w.r.t. g_{t_0}) parameterized w.r.t. arclength from $x \to y$. Then

$$d_t(x,y) \leq \ell_t(\gamma)$$

The right hand side is an upper barrier of $d_t(x,y)$ at t_0, with equality holding at $t = t_0$.

Now by the same argument as in viscosity solutions, we have that

$$\frac{d}{dt^-}\Big|_{t=t_0} d_t(x,y) \geq \frac{d}{dt}\Big|_{t=t_0} \ell_t(\gamma) = \int_0^{d=d_{t_0}(x,y)} -\mathrm{Ric}(\dot\gamma,\dot\gamma)ds$$
$$\frac{d}{dt^+}\Big|_{t=t_0} d_t(x,y) \leq \frac{d}{dt}\Big|_{t=t_0} \ell_t(\gamma) = \int_0^{d=d_{t_0}(x,y)} -\mathrm{Ric}(\dot\gamma,\dot\gamma)ds$$

having used that the geodesics are unit speed parameterized w.r.t. g_{t_0}. Then

Theorem 3.2. If $k_1 g_t \leq \mathrm{Ric} \leq k_2 g_t$ for all $t \in I$, then for $t_1 < t_2$ we have

$$e^{-k_2(t_2-t_1)} d_{t_1}(x,y) \leq d_{t_2}(x,y) \leq e^{-k_1(t_2-t_1)} d_{t_1}(x,y)$$

Remark $k_1, k_2 \in \mathbb{R}$ arbitrary, i.e. not necessarily positive nor negative. But if $k_1 > 0$, then we essentially have a shrinker and is $k_2 < 0$ we have an expander.
Proof: Integrate our left and right hand derivative bounds. $\square$

We also have

Theorem 3.3 (Hamilton, distance shrinking estimate)**.** If $\mathrm{Ric}_{g_t} \leq r^{-2} g_t$ on $B_t(x,r) \cup B_t(y,r)$, then

$$\frac{d}{dt^-} d_t(x,y) \geq -c_n r^{-1}$$

where c_n is a dimensional constant but not dependent on the ambient manifold.

Corollary 3.3.1. If $\mathrm{Ric} \leq kg$ in $B_t(x,r) \cup B_t(y,r)$ then for $t_1 < t_2$, we have

$$d_{t_2}(x,y) \geq d_{t_1}(x,y) - c_n\sqrt{k}(t_2 - t_1)$$

We now prove the Hamilton theorem

Proof: Choose $\gamma : [0,d] \to M$, a minimizing geodesic paramterized w.r.t. arclength between x, y, w.r.t. g_t.
Case 1: $d_t(x,y) \leq 2r$. Then

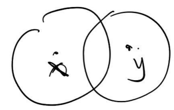

Figure 6

$$\frac{d}{dt^-}d_t(x,y) \geq -\int_0^d \mathrm{Ric}(\dot\gamma,\dot\gamma)ds \geq -r^{-2}d_t(x,y) \geq -2r^{-1}$$

by using our assumption of $-\mathrm{Ric}_{g_t} \geq -r^{-2}g_t$.

Case 2: $d_t(x,y) > 2r$, then define $\{\gamma_u(s)\}$, $u \in (-\epsilon,\epsilon)$, a variation of $\gamma = \gamma_0(s)$, such that $\gamma_u(0) = x$ and $\gamma_u(1) = y$. We now look at the variational vector field

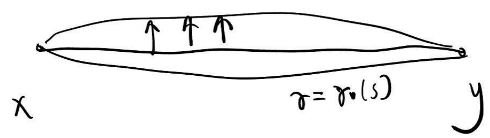

Figure 7

$$V(s) = \frac{d}{du}\Big|_{u=0}\gamma_u(s), \qquad \text{s.t.} \qquad V(0) = V(d) = 0$$

because γ_0 is a minimizing geodesic, then we compute

$$E(u) = \frac{1}{2}\int_0^d |\gamma_u'(s)|^2 ds$$

In particular

$$0 \leq \frac{d^2}{du^2}\Big|_{u=0}E(u) = \int_0^d [|V'(s)|^2 - R(V(s),\dot\gamma,\dot\gamma,V)]ds$$

from the second variation formula. Now we use this to derive our result, in particular, choose a parallel orthornormal frame $\{e_1(s),\ldots,e_n(s)\}$, with $e_1(s) = \dot\gamma(s)$. Let $\varphi : [0,d] \to [0,1]$, a bump function with $\varphi \equiv 1$ on $[r, d-r]$ and $|\varphi'| \leq \frac{10}{r}$

Now let $V(s) = \varphi(s)e_i(s)$ for $i = 2,\ldots,n$. Then

$$0 \leq \int_0^d [|\dot\varphi|^2 - \varphi^2 R(e_i,e_1,e_1,e_i)]ds$$

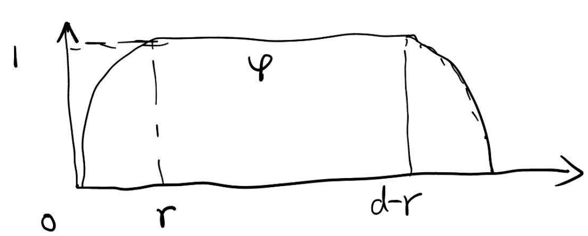

Figure 8

sum over $i = 2, \dots, n$. Then we have that

$$
\begin{aligned}
0 &\leq \int_0^d (n-1)|\varphi'(s)|^2 - \varphi(s)^2 \mathrm{Ric}(e_1, e_1) ds \\
\implies \int_0^d \varphi^2(s)\mathrm{Ric}(\dot\gamma, \dot\gamma) ds &\leq (n-1) \int_0^d |\varphi'(s)|^2 ds
\end{aligned}
$$

But also by construction of the bump function we have

$$
\begin{aligned}
\int_0^d (1-\varphi^2(s))\mathrm{Ric}(\dot\gamma, \dot\gamma) ds &= \int_{[0,r]\cup[d-r,d]} (1-\varphi^2(s))\mathrm{Ric}(\dot\gamma, \dot\gamma) ds \\
&\leq 2rr^{-2} = \frac{2}{r} \\
\implies \int_0^d \mathrm{Ric}(\dot\gamma, \dot\gamma) ds &\leq \frac{100(n-1)}{r} + \frac{2}{r} = \frac{c_n}{r} \\
\implies \frac{d}{dt^-} d_t(x, y) &\geq \frac{-c_n}{r}
\end{aligned}
$$

Ending the proof. □

We now show how volume changes under ricci flow:

Theorem 3.4. For a ricci flow, $\frac{d}{dt} dVol_{g_t} = -R dVol_{g_t}$

Proof: Use the ricci flow equation

$$
\partial_t g_t = -2\mathrm{Ric}_{g_t}
$$

4 Lecture 4: 10-6-22

Today:

- Uhlenbeck's trick
- Gradient of heat flows
- Evolution of curvature tensor

4.1 Uhlenbeck's trick

Let $\{e_i(t_0)\}_{i=0}^n$ onb of $(T_pM, (g_{t_0})_p)$ such that

$$
\frac{d}{dt} e_i(t) = \mathrm{Ric}_t(e_i(t)), \qquad e_i(t_0) = e_i(t_0)
$$

where we make sense of $\mathrm{Ric}_t :$ Vectors $\to$ Vectors by sharping it we note that

$$
\frac{d}{dt} g_t(e_i(t), e_j(t)) = -2\mathrm{Ric}_t(e_i(t), e_j(t)) + \mathrm{Ric}_t(e_i(t), e_j(t)) + \mathrm{Ric}(e_i(t), e_j(t)) = 0
$$

so $\{e_i(t)\}$ is an onb of $(T_pM, (g_t)_p)$ for some small interval in time about t_0.

This inspires us to look at the geometry of M in this time dependent but orthornormal frame. We define

$$
\begin{aligned}
\mathrm{proj}_M &: M \times I \to M \\
\mathrm{proj}_I &: M \times I \to I \\
T^{spat}(M \times I) &= \mathrm{proj}^*(TM) = \ker(dt) \subseteq T(M \times I)
\end{aligned}
$$

$$
\{\text{time-dependent vector field on } M,\ \{X_t\}_{t\in I}\} \overset{1-1}{\leftrightarrow} \{X \in \Gamma(T^{spat}(M \times I)) :\ \text{section of } T^{spat}(M \times I)\}
$$

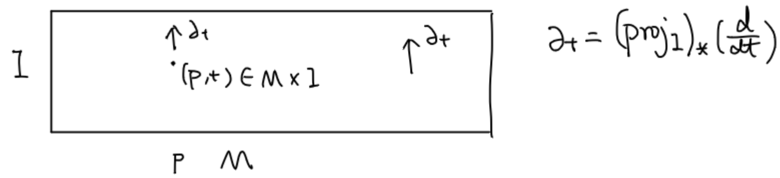

Figure 9

the idea is that X_t just a vector field on M, so it cannot have ∂_t components but it still has time dependence so we can lift it to $T(M \times I)$ with 0 component on TI (see fig 9) For $X \in \Gamma(T^{spat}(M \times I))$, define a connection $\tilde{\nabla}$ by

$$\begin{aligned}\tilde{\nabla}_v X &= \nabla_v^{g_t} X = \nabla_v X \in T^{spat}_{(p,t)}(M \times I)\\ \tilde{\nabla}_{\partial_t} X &= \partial_t X - \mathrm{Ric}_t(X)\\ \tilde{\nabla}_{\partial_t} e_i(t) &= 0\end{aligned}$$

Here, I believe $\partial_t X$ means differentiate the coefficients of X at a fixed point $p \in M$.

Theorem 4.1. $\tilde{\nabla}$ is a metric connection on $(T^{spat}(M \times I), \mathrm{proj}_M^* g_t)$

Proof: Let $\{X_t\}_{t\in I}, \{Y_t\}_{t\in I} \in \Gamma(T^{spat}(M \times I))$. Then

$$\begin{aligned}\frac{d}{dt} g_t(X_t, Y_t) &= -2\mathrm{Ric}_t(X_t, Y_t) + g_t(\partial_t X_t, Y_t) + g_t(X_t, \partial_t Y_t)\\ &= g_t(\tilde{\nabla}_{\partial_t} X_t, Y_t) + g_t(X_t, \tilde{\nabla}_{\partial_t} Y_t)\end{aligned}$$

□

Corollary 4.1.1. We have

$$\begin{aligned}\tilde{\nabla}_v(X \otimes Y) &= (\tilde{\nabla}_v X) \otimes Y + X \otimes (\tilde{\nabla}_v Y)\\ (\tilde{\nabla}_v \alpha)^\sharp &= tilde\nabla_v(\alpha^\sharp)\end{aligned}$$

where α is a 1-form

4.2 Applications of $\tilde{\nabla}$: Gradients of heat flow

Let $u \in C^2(M \times I)$ and

$$\partial_t u_t = \Delta_{g_t} u_t$$

implicitly coupled with Ricci flow. Then

$$\partial_t du_t = d\partial_t u_t = d\Delta_{g_t} u_t = \Delta_{g_t} du_t + \mathrm{Ric}(du_t)$$

where the last equality is by Bochner's formula. Here d is the exterior derivative on just the spatial component.

Note:

$$\begin{aligned}(\tilde{\nabla}_{\partial_t} \alpha_t)(v) &= \partial_t(\alpha_t(v)) - \alpha_t(\tilde{\nabla}_{\partial_t} v)\\ &= \partial_t(\alpha_t(v)) - \alpha_t(\partial_t(v) - \mathrm{Ric}_t(v))\\ &= (\partial_t \alpha_t)(v) + \mathrm{Ric}(\alpha_t)\\ \implies \tilde{\nabla}_{\partial_t} \alpha_t &= \partial_t \alpha_t + \mathrm{Ric}(\alpha_t)\end{aligned}$$

Applying this for $\alpha = du_t$, we get

$$\begin{aligned} \tilde{\nabla}_{\partial_t} du_t &= \partial_t(du_t) + \mathrm{Ric}_t(du_t) \\ \implies \tilde{\nabla}_{\partial_t} du_t &= \Delta_{g_t} du_t \end{aligned}$$

Next we reduce

$$\begin{aligned} \tilde{\nabla}_{\partial_t}(\nabla u_t) &= \tilde{\nabla}_{\partial_t}(du_t)^\sharp \\ &= (\tilde{\nabla}_{\partial_t} du_t)^\sharp \\ &= (\Delta_{g_t} du_t)^\sharp \\ &= \Delta_{g_t} \nabla_{g_t} u \\ &= \Delta \nabla u \end{aligned}$$

where we've applied commutativity of the connection and sharping multiple times. Now we note

$$\begin{aligned} \partial_t |\nabla u|^2 &= 2\langle \tilde{\nabla}_{\partial_t} \nabla u, \nabla u\rangle_{g_t} \\ &= 2\langle \Delta \nabla u, \nabla u\rangle_{g_t} \\ &= \Delta |\nabla u|^2 - 2|\nabla^2 u|^2 \end{aligned}$$

where the last line probably follows by a bochner formula. We also compute

$$\begin{aligned} \partial_t |\nabla u|^2 &= 2|\nabla u| \partial_t |\nabla u| \\ \Delta |\nabla u|^2 &= 2(\Delta |\nabla u|)|\nabla u| + 2|\nabla |\nabla u||^2 \\ \partial_t |\nabla u| &= \Delta |\nabla u| + 2(|\nabla |\nabla u||^2 - |\nabla^2 u|^2) \end{aligned}$$

by Kato's inequality, we have

$$|\nabla |\nabla u||^2 - |\nabla^2 u|^2 \leq 0$$

which follows from $|\nabla |u|| \leq |\nabla u|$. This implies that

$$\partial_t |\nabla u| \leq \Delta |\nabla u|$$

as a consequence of the above, if $|\nabla u|(\cdot, 0) \leq C$, then by the maximum principle, we have

$$|\nabla u(\cdot, t)| \leq C$$

4.3 Application 2: Evolution of Riemann Curvature Tensor

First, choose X, Y, Z time **independent** vector fields on M that commute with each other and ∂_t, i.e.

$$0 = \partial_t X = \partial_t Y = \partial_t Z = [X, Y] = [Y, Z] = [X, Z] = [\partial_t, X] = [\partial_t, Y] = [\partial_t, Z]$$

(I think last three are superfluous requirements?) Moreover, at (p_0, t_0), we want

$$\nabla^{g_{t_0}} X = \nabla^{g_{t_0}} Y = \nabla^{g_{t_0}} Z = 0$$

We compute the curvature of $\tilde{\nabla}$

$$\begin{aligned} \langle R(\partial_t, X)Y, Z\rangle &= \langle \tilde{\nabla}_{\partial_t} \tilde{\nabla}_X Y - \tilde{\nabla}_X \tilde{\nabla}_{\partial_t} Y, Z\rangle \\ &= \langle \partial_t(\nabla_X Y) - \mathrm{Ric}(\nabla_X Y) - \tilde{\nabla}_X(\partial_t Y - \mathrm{Ric}_t(Y)), Z\rangle \end{aligned}$$

at $(p_0, t_0$, we know that $\nabla_X Y = 0$, and also use time independence to get

$$\begin{aligned} \langle R(\partial_t, X)Y, Z\rangle &= \langle \partial_t \nabla_X Y + \nabla_X(\mathrm{Ric}(Y)), Z\rangle \\ &= \frac{1}{2}\partial_t \left(X\langle Y, Z\rangle + Y\langle X, Z\rangle - Z\langle X, Y\rangle\right) + \nabla_X \mathrm{Ric}(Y, Z) \end{aligned}$$

Note that even though $\nabla_X Y = 0$ and $\partial_t X = \partial_t Y = 0$, $\partial_t \nabla_X Y$ may be non zero since $\nabla = \nabla^{g_t}$ is a time dependent connection. In the last line, we used the Koszul formula and also the fact that $\nabla_X Y = \nabla_X Z = 0$ and comptability of the connection to get $(\nabla_X \mathrm{Ric})(Y,Z) = \nabla_X(\mathrm{Ric}(Y,Z))$.

Now we use the Ricci flow equation and get

$$\begin{aligned}
\langle R(\partial_t, X)Y, Z\rangle &= \frac{1}{2}\partial_t\left(X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle\right) + \nabla_X \mathrm{Ric}(Y,Z)\\
&= -X\mathrm{Ric}(Y,Z) - Y\mathrm{Ric}(X,Z) + Z\mathrm{Ric}(X,Y) + \nabla_X\mathrm{Ric}(Y,Z)\\
&= -\nabla_X\mathrm{Ric}(Y,Z) - \nabla_Y\mathrm{Ric}(X,Z) + \nabla_Z\mathrm{Ric}(X,Y) + \nabla_X\mathrm{Ric}(Y,Z)\\
&= -\nabla_Y\mathrm{Ric}(X,Z) + \nabla_Z\mathrm{Ric}(X,Y)
\end{aligned}$$

now we take $\{e_i\}$ an orthonormal basis with $\nabla e_i = 0$ at (p_0, t_0). Then

$$\begin{aligned}
\langle R(\partial_t, X)Y, Z\rangle &= -\nabla_Y\mathrm{Ric}(X,Z) + \nabla_Z\mathrm{Ric}(X,Y)\\
&= \sum_{i=1}^n -\nabla_Y R(X,e_i,e_i,Z) + \nabla_Z R(X,e_i,e_i,Y)\\
&= \sum_{i=1}^n -\nabla_{e_i} R(X,e_i,Y,Z) \qquad \text{2nd Bianchi Identity}
\end{aligned}$$

2nd Bianchi identity is

$$\nabla_i R_{jklm} + \nabla_j R_{kilm} + \nabla_k R_{ijlm} = 0$$

In sum, this tells us that

$$\tilde{R}(\partial_t, X)Y = \sum_{i=1}^n -(\nabla_{e_i} R)(X, e_i)Y$$

Now we recall the definition of the covariant derivative of a tensor, still for X, Y, Z, nice time-independent vectors with our initial assumption:

$$\begin{aligned}
(\tilde{\nabla}_{\partial_t} R)(X,Y)Z &= \tilde{\nabla}_{\partial_t}(R(X,Y)Z) - R(X, \tilde{\nabla}_{\partial_t} Y)Z - R(X,Y)\tilde{\nabla}_{\partial_t} Z\\
&= \tilde{\nabla}_{\partial_t}(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z) + R(\mathrm{Ric}(X),Y)Z + R(X,\mathrm{Ric}(Y))Z + R(X,Y)\mathrm{Ric}(Z)\\
&= \nabla_X\tilde{\nabla}_{\partial_t}\nabla_Y Z + \tilde{R}(\partial_t, X)\nabla_Y Z - \nabla_Y\tilde{\nabla}_{\partial_t}\nabla_X Z - \tilde{R}(\partial_t, Y)\nabla_X Z\\
&+ [R(\mathrm{Ric}(X),Y)Z + R(X,\mathrm{Ric}(Y))Z + R(X,Y)\mathrm{Ric}(Z)]\\
&= \nabla_X(\nabla_Y\tilde{\nabla}_{\partial_t} Z + \tilde{R}(\partial_t, Y)Z) - \nabla_Y(\nabla_X\tilde{\nabla}_{\partial_t} Z + \tilde{R}(\partial_t, X)Z)\\
&+ [R(\mathrm{Ric}(X),Y)Z + R(X,\mathrm{Ric}(Y))Z + R(X,Y)\mathrm{Ric}(Z)]
\end{aligned}$$

now we use $\tilde{\nabla}_{\partial_t} Z = \mathrm{Ric}(Z)$ for our time independent vector field, and we get

$$\begin{aligned}
(\tilde{\nabla}_{\partial_t} R)(X,Y)Z &= \nabla_X(\tilde{R}(\partial_t, Y)Z) - \nabla_Y\tilde{R}(\partial_t, X)Z + R(\mathrm{Ric}(X),Y)Z + R(X,\mathrm{Ric}(Y))Z\\
&= -\nabla_X(\nabla_{e_i} R)(Y,e_i)Z + \nabla_Y(\nabla_{e_i} R)(X,e_i)Z + R(\mathrm{Ric}(X),Y) + R(X,\mathrm{Ric}(Y))Z
\end{aligned}$$

But now

$$\begin{aligned}
-\nabla_X(\nabla_{e_i} R)(Y,e_i)Z &= \nabla_{e_i}(\nabla_X R)(Y,e_i)Z - (R(X,e_i)R)(Y,e_i)Z\\
&= \nabla_{e_i}(\nabla_X R)(Y,e_i)Z - R(X,e_i)(R(Y,e_i)Z)\\
&+ R(R(X,e_i)Y,e_i)Z + R(Y,R(X,e_i)e_i)Z + R(Y,e_i)R(Y,e_i)Z
\end{aligned}$$

where we now interpret $R(X, e_i)$ as a curvature tensor acting on tensors, e.g. R itself in $(R(X,e_i)R)$. Finally, we sum over i, and do the same expansion for

$$\begin{aligned}
\nabla_Y(\nabla_{e_i} R)(X,e_i)Z &= \nabla_{e_i}(\nabla_Y R)(X,e_i)Z - R(Y,e_i)(R(X,e_i)Z) + R(R(Y,e_i)X,e_i)Z\\
&+ R(X,R(Y,e_i)e_i)Z + R(X,e_i)R(Y,e_i)
\end{aligned}$$

subtracting these two, we get

$$\begin{aligned}-\nabla_X(\nabla_{e_i}R)(Y,e_i)Z+\nabla_Y(\nabla_{e_i}R)(X,e_i)Z &= \nabla_{e_i}((\nabla_{e_i}R)(X,Y)Z)+2[R(X,e_i),R(Y,e_i)]Z\\ &\quad - R(R(X,e_i)Y,e_i)Z-R(R(Y,e_i)X,e_i)Z\end{aligned}$$

where we've noted that

$$-R(X,e_i)(R(Y,e_i)Z)+R(Y,e_i)R(X,e_i)Z=[R(X,e_i),R(Y,e_i)]Z$$

This tells us that

$$(\tilde{\nabla}_{\partial_t}R)(X,Y)(z)=\Delta R(X,Y)Z+Q(R)$$

where $Q(R)$ denotes quadratic terms in R. In general

$$\boxed{\tilde{\nabla}_{\partial_t}R=\Delta R+Q(R)}$$

5 Lecture 5: 10-11-22

Goal for today

- Evolution of Ric and R
- Scalar weak/strong maximum principle

Recall that

$$\begin{aligned}\nabla_{\partial_t}Rm &= \Delta Rm+Q(Rm)\\ Q(Rm)_{ijkl} &= -R_{ijst}R_{stkl}+2R_{istl}R_{jstk}-2R_{jstl}R_{istk}\end{aligned}$$

Here, ∇ is the special connection we constructed from last time using Uhlenbeck's trick.

For general evolution of metrics $\{g_t\}$ (i.e. not necessarily Ricci flow), we have

$$\begin{aligned}\partial_t R_{ijkl} &= \nabla_i(\partial_t\Gamma^l_{jk})-\nabla_j(\partial_t\Gamma^l_{ik})+(\text{lower order terms})\\ \frac{d}{dt}g_{ij}(t) &:= h_{ij}\\ \partial_t R_{ijkl} &= \frac{1}{2}(\nabla_i\nabla_k h_{jl}+\nabla_i\nabla_j h_{kl}-\nabla_i\nabla_l h_{kl})+\cdots+(\text{lower order terms})\\ &\overset{(h=-2\mathrm{Ric})}{=}\frac{1}{2}\nabla_i\nabla_k\mathrm{Ric}_{jl}+\cdots\\ &= \Delta Rm\end{aligned}$$

in a loose sense. I guess the point is that we can see the evolution equation from the normal formula for variation of curvature tensor under a family of metrics.

5.1 Evolution of Ric

We have that

$$\begin{aligned} Q(Rm)_{ijki} &= 2\mathrm{Ric}_{st}R_{jstk}=2Rm(\mathrm{Ric})_{jk}\\ \implies \nabla_{\partial_t}\mathrm{Ric} &= \Delta\mathrm{Ric}+2Rm(\mathrm{Ric})\end{aligned}$$

Another way to obtain this is recall that

$$\begin{aligned}
D(-2\mathrm{Ric})(h)_{jk} &= \Delta_L h_{jk} + \nabla_j \nabla_k \mathrm{Tr}(h) + \nabla_j(\delta h)_k + \nabla_k(\delta h)_j \\
&= \Delta_L h_{jk} + \nabla_j(\delta E h)_k + \nabla_k(\delta E h)_j \\
E(h)_{ij} &= h_{ij} - \frac{1}{2}\mathrm{Tr}(h) g_{ij} \\
\delta^* : \text{ Lie derivative } &= \Delta_L h_{jk} + (\delta^*(\delta E h))_{jk} \\
\delta E(\mathrm{Ric}) &= \delta(\mathrm{Ric}_{ij} - \frac{1}{2} R g_{ij}) = (\delta \mathrm{Ric} - \frac{1}{2} dR)_{ij} \\
&= 0 \qquad \text{(second contracted Bianchi identity)}
\end{aligned}$$

here, Δ_L is the Lichnerowicz laplacian, $E(h)$ is the einstein operator. This tells us tbhat

$$\begin{aligned}
D(-2\mathrm{Ric})(-2\mathrm{Ric}) &= \Delta_L(-2\mathrm{Ric}) + 0 = -2\Delta_L \mathrm{Ric} \\
\implies \partial_t(\mathrm{Ric})\Delta_L \mathrm{Ric}
\end{aligned}$$

because $D(-2\mathrm{Ric})(-2\mathrm{Ric}) = \partial_t(-2\mathrm{Ric})$. This follows from work we did on previous days for computing $D(\mathrm{Ric})$

5.2 Evolution of R

We have

$$\begin{aligned}
Rm(\mathrm{Ric})_{ii} &= R_{ikli}\mathrm{Ric}_{kl} = \mathrm{Ric}_{kl}\mathrm{Ric}_{kl} = \mathrm{Ric}^2_{kl} \\
\implies \nabla_t R &= \partial_t R = \Delta R + 2Rm(\mathrm{Ric})_{ii} \\
\implies \partial_t R &= \Delta R + 2|\mathrm{Ric}|^2
\end{aligned}$$

In two dimensions, we know that

$$\mathrm{Ric} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$$

for K the scalar curvature, so $|\mathrm{Ric}|^2 = 2k^2$ and $R = 2k$, which tells us that

$$\begin{aligned}
\partial_t R &= \Delta R + R^2 \\
&= 2\Delta K + 4k^2
\end{aligned}$$

For M closed

$$\begin{aligned}
\partial_t \int R dg_t = \int (\partial_t R) dg_t + \int R(d_t g_t) \\
&= \int \Delta R + R^2 - R \cdot R \\
&= \int \Delta R \\
&= 0
\end{aligned}$$

by closedness. This tells us that $\int R_t dg_t$ is an invariant on 2D ricci flow - i.e. genus can't change

5.3 Scalar Weak maximum principle

Theorem 5.1. Let M a compact, $\{g_t\}_{t\in[0,T)}$ any smooth family of Riemannian metrics. Moreover, suppose we have

$$\begin{aligned}
f; \mathbb{R} \times [0,T) &\to \mathbb{R} \\
(X_t)_{t\in[0,T)} &: \text{ vector fields} \\
u &\in C^\infty(M \times [0,T)) \\
\overline{u} &\in C^\infty([0,T))
\end{aligned}$$

Figure 10

If

$$\partial_t u_t \leq \Delta u_t + X_t \cdot \nabla u_t + f(u_t, t) \tag{5}$$
$$\text{and } u_t \leq \overline{u} \qquad \text{on} \quad \partial(M \times [0,T)) = \partial M \times [0,T] \cup M \times \{0\} \tag{6}$$
$$\text{and } \partial_t \overline{u}(t) \geq f(\overline{u}(t), t), \quad \text{then} \tag{7}$$
$$u_t \leq \overline{u} \qquad \text{everywhere}$$

Proof:
Case 1, Assume (7), (6) have strict inequality. Let

$$t^* = \max\{t \in [0,T) \mid u_t \leq \overline{u} \ \text{ on } [0,t]\}$$

then by (**??**) and M compact, we have $t^* > 0$, then there exists an $x^* \in \text{Int}(M)$ such that $u(x^*, t^*) = \overline{u}(t^*)$. THis implies

$$\implies \partial_t u(x^*, t^*) \geq \partial_t \overline{u}(t^*), \quad \nabla u(x^*, t^*) = 0, \quad \Delta u(x^*, t^*) \leq 0$$

so at (x^*, t^*), we have

$$\begin{aligned}\partial_t \overline{u}(t^*) - f(\overline{u}(t^*), t^*) &\leq \partial_t u(x^*, t^*) = f(u(x^*, t^*), t^*) + X_t \cdot \nabla u \\ &\leq \Delta u_t \\ &\leq 0\end{aligned}$$

since $\nabla u(x^*, t^*) = 0$ and $\Delta u(x^*, t^*) = \Delta u_{t^*}(x^*) \leq 0$. Here, $u_t = u(\cdot, t)$.

Case 2, here we handle non-strict inequality by creating a perturbation. Let

$$\overline{u}_\epsilon(t) = \overline{u}(t) + \epsilon t + \epsilon^2$$

then (**??**) and (**??**) will have strict inequality. For (**??**) its evident, for (**??**), we have

$$\begin{aligned}\partial_t \overline{u}_\epsilon &= \partial_t \overline{u} + \epsilon \\ f(\overline{u}_\epsilon(t), t) - f(\overline{u}(t), t) &\leq C|\overline{u}_\epsilon(t) - \overline{u}(t)| \\ &= C(\epsilon t + \epsilon^2) \leq \frac{1}{2}\epsilon + C\epsilon^2 \\ &\leq \frac{3}{4}\epsilon^2\end{aligned}$$

here C is a bound on the gradient of f, and we choose τ such that for $t \leq \tau$, we have

$$Ct \leq C\tau \leq \frac{1}{2}$$

which allows the last line to hold, assuming ϵ sufficiently small. So by case 1 ,we have $u_t \leq \overline{u}_\epsilon(t)$, now let $\epsilon \to 0$, then we have

$$u_t \leq \overline{u}(t) \qquad \text{on} \quad t \in [0, \tau]$$

now extend to $[0,T]$ for T maximal by an open-closed argument and potentially repeating this construction.

5.4 Scalar Strong maximum principle

Lemma 5.2. Let M be a compact manifold with boundary, $\{g_t\}$, $\{X_t\}$ as before, $u \in C^\infty(M \times [0,T])$. If $u \geq 0$ and

$$\partial_t u \geq \Delta u_t + X_t \cdot \nabla u_t \qquad \forall t \in [0,T]$$

(Note this is a homogeneous inequality with no 0 order terms) and if $\exists x_0 \in \mathrm{Int}(M)$ such that $u(x_0,T) = 0$, then there exists a neighborhood of x_0, U, and $\epsilon > 0$ such that $u \equiv 0$ on $U \times [T-\epsilon, T]$.

Corollary 5.2.1. Same assumption and set up as the above but $u \equiv 0$ on $M \times [0,T]$, by an open closed argument

"**Proof:** " - WLOG, assume M is covered by a coordinate chart. Consider

$$V := \{(x,t) \in M \times [0,T] \ : \ u(x,t)\}$$

then use the above lemma to show that

$$V \cap (M \times \{t\}) = M$$

which makes sense if M is connected. For the time component, repeat the lemma but considering everything on $[0, T-\epsilon]$, i.e. replace $T \to T-\epsilon$, should be a similar openness argument but in the time direction. $\square$

Proof of Lemma: Suppose that no such neighborhood existed, then

$$\exists (x^*, t^*) \quad \text{near} \quad (x_0, T), \ \ u(x^*, t^*) > 0$$

Claim: $\exists \varphi^\infty(M \times [t^*, t])$ such that

$$\varphi \geq 0 \tag{8}$$
$$u_{t^*} \geq \varphi_{t^*} \tag{9}$$
$$\varphi \equiv 0 \qquad \text{on} \qquad \partial M \times [t^*, T] \tag{10}$$
$$\varphi(x_0, T) > 0 \tag{11}$$
$$\partial_t \varphi_t \leq \Delta \varphi_t + X_t \cdot \nabla \varphi_t \tag{12}$$

This φ is a barrier function. Assume the claim is true, then

$$\partial_t (u_t - \varphi_t) \geq \Delta(u_t - \varphi_t) + X_t \cdot \nabla(u_t - \varphi_t)$$

then the weak maximum principle tells us that

$$u(x_0, T) - \varphi(x_0, T) \geq 0 \implies u(x_0, T) > 0$$

a contradiction, since we've assumed that $u(x_0, T) = 0$.

Now the point is to construct such a barrier function, φ, which satisfies the claim. Note that (8), (9), (10), and (11) can be satisfied easily by constructing a bump function about x^*, t^* and scaling it by $\frac{1}{2} u(x^*, t^*)$. Thus, the work is in showing (12).

Proof of claim: Let

$$\varphi(x,t) = e^{-A(t+1)} \phi(|x - x^*| - s(t - t^*))$$

where $A,\ S \in \mathbb{R}$, and $\phi : \mathbb{R} \to \mathbb{R}$ is decreasing and ϕ is a smoothed heaviside function with $\phi(t) \equiv 1, \qquad t \leq 0$ and $\phi(t) \equiv 0, \qquad t \geq \epsilon |x_0 - x^*|$. Moreover, on $[0, \epsilon |x_0 - x^*|]$, we require that

$$-\phi'' \leq C\phi, \qquad (\phi')^2 \leq C\phi' \leq C\phi$$

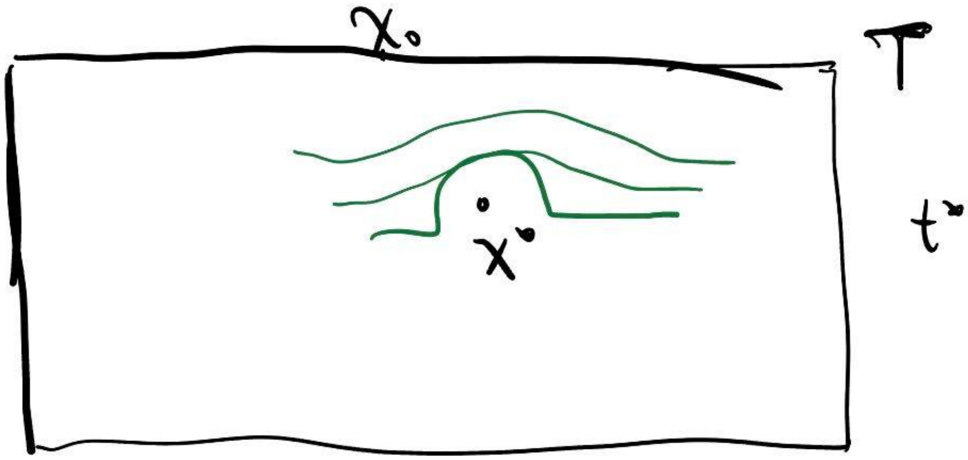

Figure 11

Yi says that we can do this by inserting quadratic transitions at $t = 0, t = 1$ and then smooth appropriately.

Assume WLOG that $B(x_0, 2|x_0 - x^*|) \subseteq \mathrm{Int}(M)$. Choose

$$s \in \left(\left[1 - \frac{1}{2}\epsilon\right] \frac{|x_0 - x^*|}{T - t^*}, (2 - \epsilon)\frac{|x_0 - x^*|}{T - t^*} \right)$$

Then

$$\begin{aligned} \phi(|x_0 - x^*| - s(T - t^*)) &> 0 \\ \phi(|x - x^*| - s(t - t^*)) &= 0, \qquad \forall x \in \partial M,\ \forall t \in [t^*, T] \\ &\implies \varphi = 0 \quad \text{on} \quad \partial M \times [t^*, T] \end{aligned}$$

Now to enforce $u_{t^*} \geq \varphi_{t^*}$, we take A to be very large and $\epsilon \ll 1$. Now to verify (12), we have

$$\begin{aligned} \partial_t \varphi &= e^{-A(t+1)}(-s\phi' - A\phi) \\ |\nabla \varphi| &= e^{-A(t+1)} C|\phi'| \\ \Delta \varphi &\geq e^{-A(t+1)} \phi'' \\ (12) &\iff \partial_t \varphi_t \leq \Delta \varphi_t + X_t \cdot \nabla \varphi_t \\ &\iff C\phi' - A\phi \leq \phi'' - C|\phi'| \\ &\iff C|\phi'| - \phi'' \leq A\phi \end{aligned}$$

so taking $A \gg 1$, this is true, and we get (12) finishing the proof of the lemma. $\square$

We now state the **Scalar Strong Maximum Principle**

Theorem 5.3. Suppose M connected (not necessarily compact), $\{g_t\}$, X_t, u, $\overline{u}$, f, all as before. Suppose that

$$u_t(x) \leq \overline{u}(t) \qquad \forall t \in [0, T]$$

and

$$u(x_0, T) = \overline{u}(T), \qquad \text{for some } x_0 \in \mathrm{Int}(m)$$

then $u_t \equiv \overline{u}(t)$ everywhere on M.

Proof: , let

$$Z = \{(x,t) \in M \times [0,T] \mid u(x,t) = \overline{u}(t)\}$$

Let

$$\begin{aligned} v_t &= \overline{u}_t - u_t \geq 0 \\ \partial_t v_t &\geq \Delta v_t + X_t \cdot \nabla v_t + f(\overline{u}(t),t) - f(u_t,t) \\ &\geq \Delta v_t + X_t \cdot \nabla v_t - C v_t \end{aligned}$$

Now let

$$\begin{aligned} \tilde{v}_t &= e^{Ct} v_t \\ \implies \partial_t \tilde{v}_t &\geq \Delta \tilde{v}_t + X_t \cdot \nabla \tilde{v}_t \\ \overset{\text{Cor}}{\implies} \tilde{V}_t &= 0 \qquad \text{on } M \times [0,T] \\ v_t &= 0 \\ u_t &= \overline{u}_t \qquad \text{everywhere} \end{aligned}$$

6 Lecture 6: 10-13-22

Goal:

- Application of weak and strong maximum principles
- Curvature derivative estimates
- Maximal existence time

6.1 Application of WMP and SMP

Let (M, g_t) a ricci flow. Then we have

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2 \geq \Delta R \tag{13}$$

where R is the scalar curvature. If we assume that we can diagonalize the Ricci curvature (always true I think?)

$$\mathrm{Ric} = \begin{pmatrix} \rho_1 & \dots & \dots \\ & \dots & \\ \dots & \dots & \rho_n \end{pmatrix} \implies R = \rho_1 + \dots + \rho_n$$

which implies that

$$2|\mathrm{Ric}|^2 \geq \frac{2}{n} R^2$$

by AM-GM or something. Then we have

Theorem 6.1. For $t_1 \leq t_2 \in I$ and any $T \in \mathbb{R}$, we have

1. If $R(\cdot, t_1) \geq A \quad \implies \quad R(\cdot, t_2) \geq A$
   Proof: (apply WMP to $\overline{R}(t) = A$, $\partial_t \overline{R}(t) = 0$
2. If $R(\cdot, t_1) \geq \frac{n}{2(T-t_1)}$, then $R(\cdot, t_2) \geq \frac{n}{2(T-t_2)}$
   Proof: (apply WMP to $\overline{R}(t) = \frac{n}{2(T-t)}$, so that $\partial_t \overline{R}(t) = \frac{2}{n} \overline{R}^2(t)$, and use our statement about bounding $2|\mathrm{Ric}|^2$)

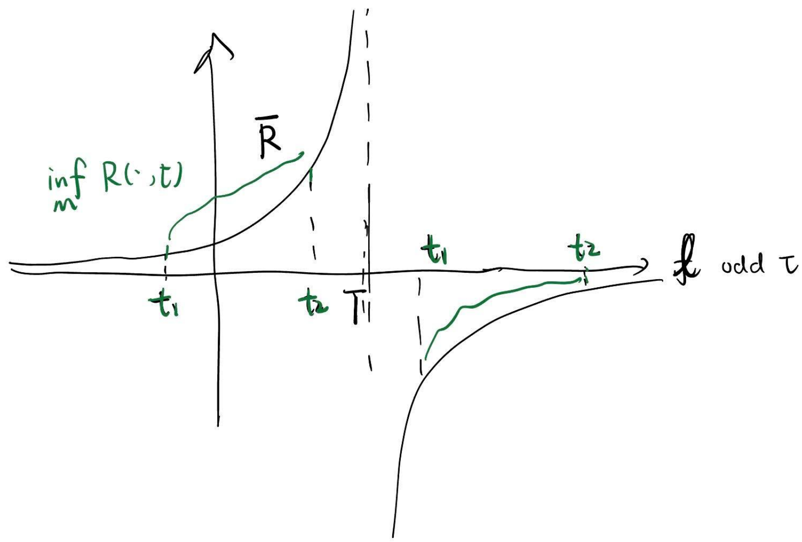

Figure 12

3. If $T \in I$, then $R(\cdot, t) \geq \frac{n}{2(T-t)}$ for all $t > T$
Proof: Let $\{t_i\} \downarrow T$, then we know that $R(\cdot, t_i) \geq -C$ just because I is compact, so we get uniform lower bounds on scalar curvature. This tells us that
$$R(\cdot, t_i) \geq -C \geq \frac{n}{2(T-t_i)}$$
where i is very large, i.e. when $T - t_i \to 0^-$. Now apply the previous statement

4. If g_t is defined on $(-\infty, t_0]$ (ancient flow), then $R \geq 0$.
Proof: Let $T \downarrow -\infty$ in the previous statement

5. If $R(\cdot, t_0) \geq \frac{n}{2T} > 0$, then $t_0 + T \notin I$ (the solution cannot exist up to $t_0 + T$)
Proof: If the above holds, then our second statement tells us that
$$R(\cdot, t_0) \geq \frac{n}{2(T + t_0 - t)}$$
ad the above tends to infinity as $t \to t_0 + T$ from below

Now we do applications of the strong maximum principle

Theorem 6.2. Assume M is connected but possibly non-compact

1. Assume $I = [0, T]$, $R \geq 0$ everywhere. If $R(x_0, T) = 0$ for some $x_0 \in M$, then $\text{Ric} \equiv 0$ for all $t \in [0, T]$
Proof: The strong maximum principle plus $\partial_t R \geq \Delta R \implies R \equiv 0$. But now come back to evolution equation
$$\partial_t R = \Delta R + 2|\text{Ric}|^2 \implies \text{Ric} \equiv 0$$
since both $\partial_t R = \Delta R = 0$

2. If M is compact, $I = (-\infty, \infty)$ (eternal flow), then $\text{Ric} \equiv 0$ (and $\frac{d}{dt} g_t = -2\text{Ric} = 0$ so $g_t = g_0$)
Proof: First, eternal flow $\implies$ ancient $\overset{\text{compact}}{\implies} R \geq 0$.
Claim EIther $R \equiv 0$ or $R > 0$ everywhere. This should follow from the strong maximum principle. Assuming the claim is true, then if $R \equiv 0$, then by the previous computation, we have $\text{Ric} \equiv 0$. If $R > 0$ everywhere (M compact means a positive lower bound on R) then by a previous statement, it can only exist for a finite time, contradicting that this is an eternal flow.

Proof of claim: Suppose not, then $\exists R(x_1,t_1) > 0$ and $R(x_2,t_2) = 0$. By our first statement in this theorem, if we have $R(x_2,t_2) = 0$, then

$$\begin{aligned}
&\implies \text{Ric} = 0 \qquad \forall t \le t_2 \\
&\implies g_t = g_{t_2} \qquad \forall t \ge t_2 \implies \text{Ric} = 0
\end{aligned}$$

The second line follows since the Ricci flow equation will be constant on $(-\infty, t_2)$

6.2 In 2-dimension

6.2.1 Lower Bound

In two dimensions, we have

$$\partial_t R = \Delta R + R^2$$

in this case. We note that $R \le 0$ is preserved by Ricci flow in this dimension. To see this, apply weak maximum principle to the comparison function

$$\overline{R}(t) = 0, \qquad \partial_t \overline{R}(t) = \overline{R}^2(t)$$

so that 0 is an upper barrier. Note that this is not true in dimension $n \ge 3$, since $R^2 \ne |\text{Ric}|^2$ in general

6.2.2 Normalized Volume

For (M, g_t), M compact, $I = [0, \infty)$ (immortal flow), we define the normalized volume $\overline{V}(t) = t^{-n/2} V(t)$ (The scaling is supposed to be intuitive since $t \sim r^2$ since we have a parabolic flow, i.e. $t^{-n/2} \sim r^{-n}$ and $V(t) \sim r^n$). Thus, normalized volume is a scaling invariant and

$$\begin{aligned}
\frac{d}{dt}\overline{V}(t) &= t^{-n/2}\left(-\frac{n}{2t}V(t) + V'(t)\right) \\
&= t^{-n/2}\int\left(-\frac{n}{2t} - R\right) dVol_t
\end{aligned}$$

The second line follows since

$$V'(t) = \frac{1}{2}\text{tr}(\dot{g})dVol$$

and

$$\dot{g} = -2\text{Ric} \implies \text{tr}(\dot{g}) = -2R$$

Recall that $R \ge -\frac{n}{2t}$ holds for $t \in (0, \infty)$, so that

$$\frac{d}{dt}\overline{V}(t) \le 0$$

thus $\overline{V}(t)$ is non-increasing and positive, so it has a limit as $t \to \infty$

$$\overline{V}_\infty = \lim_{t\to\infty} \overline{V}(t)$$

6.2.3 Solitons

Let (M, g), and $\text{Ric}_g = \mathcal{L}_X g + \lambda g$ for $\lambda \in \mathbb{R}$ and X some smooth vector field. We have

$$g(t) = \begin{cases} (-2\lambda t)\phi_{-t}^*(g) & \lambda > 0, \quad t \in (-\infty, 0) \quad \text{(shrinking)} \\ \phi_{-t}^*(g) & \lambda = 0, \quad t \in (-\infty, \infty) \quad \text{(steady)} \\ (-2\lambda t)\phi_{-t}^*(g) & \lambda < 0, \quad t \in (0, \infty) \quad \text{(expanding)} \end{cases}$$

Where ϕ_t is the flow corresponding to X and $\phi_0 = Id$. Then g_t satisfies a ricci flow!! This is a soliton

Theorem 6.3. A compact steady soliton must be Einstein (i.e. with $\lambda = 0 \implies \mathrm{Ric} \equiv 0$)

Proof: g_t is an eternal compact Ricci flow, so $\mathrm{Ric} \equiv 0$

Theorem 6.4. A compact expanding solution must be Einstein.

Proof: $\overline{V}(g_t)$ is a constant in the expanding case because

$$\overline{V}(g_t) = \overline{V}((-2\lambda t)\phi_{-t}^* g) = \overline{V}(\phi_{-t}^* g) = \overline{V}(g)$$

Thus

$$0 = \frac{d}{dt}\overline{V}(g_t) = t^{-n/2} \int \left(-\frac{n}{2t} - R\right) dVol_t$$

we know that

$$-\frac{n}{2t} - R \leq 0$$

so for the derivative to be exactly 0, we have

$$R = \frac{-n}{2t}$$

everywhere. Now plugging this into the evolution of scalar curvature, we have

$$\partial_t R = \Delta R + 2|\mathrm{Ric}|^2 = 0 + 2|\mathring{\mathrm{Ric}}|^2 + \frac{2}{n}R^2$$

but

$$\partial_t R = \frac{n}{2t^2} = \frac{2}{n}R^2$$

so we see that

$$\mathring{\mathrm{Ric}} \equiv 0$$

which implies that

$$\mathrm{Ric} = \frac{R}{n} g_t$$

which is einstein. Now plugging in R, we have

$$\mathrm{Ric} = -\frac{1}{2t} g_t \qquad \forall t$$

6.3 Evolution of Curvature tensor

We have

$$\begin{aligned}
\partial_t |Rm|^2 &= 2\langle \nabla_{\partial t} Rm, Rm\rangle = 2\langle \Delta Rm + Q(Rm), Rm\rangle \\
&= \Delta |Rm|^2 - 2|\nabla Rm|^2 + 2\langle Q(Rm), Rm\rangle \\
&\leq \Delta |Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \\
&\leq \Delta |Rm|^2 + C|Rm|^3
\end{aligned}$$

Now we want to apply the weak maximum principle. Consider the comparison equation

$$\begin{aligned}
\partial_t \overline{u}(t) &= C\overline{u}(t)^{3/2} \\
\implies \overline{u}(t) &= \frac{1}{\left(\frac{C}{2}(T-t)\right)^2}
\end{aligned}$$

so if $|Rm|^2(\cdot, 0) \leq A$, then the weak maximum principle gives

$$|Rm|^2(\cdot, t) \leq \frac{1}{\left(\frac{C}{2}\left(\frac{2}{C\sqrt{A}} - t\right)\right)^2}$$

Exercise Study the equation of $|Rm|$, then if

$$|Rm|(\cdot, 0) \leq A$$

we have that

$$|Rm|(\cdot, t) \leq \frac{1}{A^{-1} - \frac{C}{2}t}$$

(this is a little different than just taking the square root of the previous bound)

Theorem 6.5. Let (M, g_t) a Ricci flow, M compact, then either

- $\sup_{M\times[0,T)} |Rm| < \infty$ OR
- $\lim_{t\uparrow T} \max_M |Rm|(\cdot, t) = \infty$ and

$$\max_M |Rm|(\cdot, t) \geq \frac{C_n}{T-t}$$

(exercise, which should be an application of previous exercises)

6.4 Curvature derivative estimates

Let $(M, g_t)_{t\in[0,T)}$ a ricci flow, M compact. Then

$$\begin{aligned}
\nabla_{\partial_t}\nabla Rm &= \nabla\nabla_{\partial_t} Rm + \tilde{R}(\partial_t, \cdot)Rm = \nabla\nabla_{\partial_t} Rm + \nabla Rm * Rm \quad = \nabla(\Delta Rm + Q(Rm)) + \nabla Rm * Rm \\
&= \Delta\nabla Rm + Rm * \nabla Rm + \nabla Rm * Rm \\
&= \Delta\nabla Rm + Rm * \nabla Rm
\end{aligned}$$

here

$$(A * B)_{jl} = g^{ik} A_{ij} B_{kl}$$

so for example

$$Rm(\text{Ric}) = Rm * Rm$$

Moreover, we'll use that

$$|A * B| \leq C|A||B|$$

where the norm is some tensor bound.

We also have

$$\begin{aligned}
\partial_t |\nabla Rm|^2 &= 2\langle \nabla_{\partial_t}\nabla Rm, \nabla Rm\rangle = 2\langle \Delta\nabla Rm + \nabla Rm * Rm, \nabla Rm\rangle \\
&\leq \Delta|\nabla Rm|^2 - 2|\nabla^2 Rm| + \nabla Rm * \nabla Rm * Rm \\
&\leq \Delta|\nabla Rm|^2 + C|\nabla Rm|^2 \cdot |Rm|
\end{aligned}$$

Now our goal is to derive bounds on $|\nabla Rm|$ in terms of bounds on $|Rm|$.

Suppose $|Rm| \leq A$ on $M \times [0, T)$. Define

$$F = |Rm|^2 + t|\nabla Rm|^2$$

then we have

$$\begin{aligned}
(\partial_t - \Delta)F &\leq |\nabla Rm|^2 + t(C|\nabla Rm|^2|Rm|) + (-2|\nabla Rm|^2 + C|Rm|^3) \\
&\leq |\nabla Rm|^2 + CtA|\nabla Rm|^2 - 2|\nabla Rm|^2 + C|Rm|^3 \\
&\leq C|Rm|^3 \leq CA^3 \\
F(\cdot, 0) &= |Rm|^2 \leq A^2
\end{aligned}$$

having used our bounds on both $|Rm|^2$ and $|\nabla Rm|^2$. In the second to third line, we chose t small so that $CtA < 1$ to cancel the first 3 terms (or rather, bound above by 0). Now the weak maximum princple on the function $F(\cdot, t)$ gives

$$F(\cdot, t) \leq CA^2 \qquad \forall t \in \left[0, \frac{1}{A}\right]$$

Now taking a square root, we get

$$|\nabla Rm| \leq \frac{\sqrt{C}A}{\sqrt{t}}, \qquad \forall t \in \left[0, \frac{1}{A}\right]$$

so if the curvature norm has a bound, then $|\nabla Rm|$ also has some bound in a small interval. Here, we've chosen

$$\overline{F}(t) = CA^3 t + A^2$$

7 Lecture 7: 10-18-22

Today's goals

- Curvature derivative estimates
- Maximal existence time
- Vector-valued maximum principle

7.1 Curvature derivative estimate

Theorem 7.1. Let $(M, \{g_t\}_{t \in [0,T)})$ compact ricci flow. Suppose $|Rm| \leq A$ on $t \in [0, T)$, then

$$|\nabla^\ell_{\partial_t} \nabla^k Rm| \leq \frac{C_{k,\ell} A}{t^{\ell + k/2}}$$

on $t \in [0, 1/A]$. Here ∇ denotes the Uhlenbeck connection

Proof: Step 1: Assume $\ell = 0$. Last time we did $k = 1$. Prove by induction, so assume this is true for k. Then

$$\nabla_{\partial_t} \nabla^k Rm = \Delta \nabla^k Rm + \sum_{i+j=k} \nabla^i Rm * \nabla^j Rm$$

(this formula can also be proved by induction).

$$\begin{aligned}
\nabla_{\partial_t} \nabla^{k+1} Rm &= \nabla \nabla_{\partial_t} \nabla^k Rm + (\tilde{R}(\cdot, \cdot) \cdot \nabla^k Rm) \\
&= \nabla(\Delta \nabla^k Rm + \sum_{i+j=k} \nabla^i Rm * \nabla^j Rm) + \nabla Rm * \nabla^k Rm \\
&= \Delta \nabla^{k+1} Rm + Rm * \nabla^{k+1} Rm + \sum_{i+j=k} \nabla^i Rm * \nabla^j Rm \\
&= \Delta \nabla^{k+1} Rm + \sum_{i+j=k+1} \nabla^i Rm * \nabla^j Rm
\end{aligned}$$

where in the second to third line we apply Bochner's formula. Using this, we have

$$\partial_t |\nabla^k Rm|^2 \leq \Delta |\nabla^k Rm|^2 - 2|\nabla^{k+1} Rm|^2 + C \sum_{i+j=k} |\nabla^i Rm| \cdot |\nabla^j Rm| \cdot |\nabla^k Rm|$$

Now using a similar trick to last time, we note the negative sign in front of $2|\nabla^{k+1} Rm|^2$, and construct a barrier of

$$F = t|\nabla^{k+1} Rm| + |\nabla^k Rm|$$

This implies that the theorem is true for $k+1$. Jere, we've used

$$\nabla_{\partial_t} \nabla \nabla^k Rm = \nabla \nabla_{\partial_t} \nabla^k Rm + \tilde{R} * \nabla^k Rm$$

where $\tilde{R}$ is the curvature for $\tilde{\nabla}_{\partial_t}$, but also equals ∇Rm.

Now we induct on ℓ. After rescaling, assume $A = 1$. Under this, we define

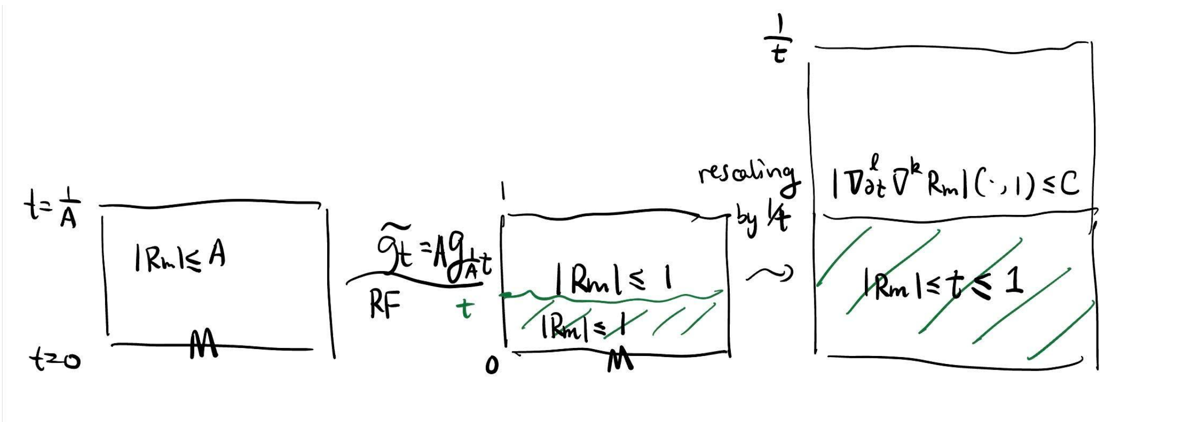

Figure 13

$$\tilde{g}_t = A g_{t/A}$$

Under this flow, we have $|Rm| \leq 1$ and $t \in (0,1]$, so we want to show

$$|\nabla^{\ell}_{\partial_t} \nabla^k Rm| \leq \frac{C_{k,\ell}}{t^{\ell + k/2}} \qquad t \in (0,1]$$

By another rescaling, it suffices to prove it at $t = 1$ - assume we want to show this at time $t \in (0,1]$. We choose another rescaling of the form above but by $1/t$, which sends $t \to 1$ and $1 \to 1/t$. So that

$$|Rm| \leq t \leq 1, \qquad |\nabla^{\ell}_{\partial_t} \nabla^k Rm|(\cdot, 1) \leq C$$

Now note that

$$\nabla_{\partial_t} \nabla^k Rm = \Delta \nabla^k Rm + \sum_{i+j=k} \nabla^i Rm * \nabla^j Rm$$

so $\nabla_{\partial_t} \nabla^k Rm$ is the $*$-composition of $\nabla^i Rm$. So

$$|(\nabla_{\partial_t} \nabla^k Rm)(\cdot, 1)| \leq C_{1,k}$$

which shows it for $(\ell, k) = (1, k)$ with $K \in \mathbb{N}$. By induction, we have

$$\nabla^{\ell}_{\partial_t} \nabla^k Rm = \nabla_{\partial_t} \nabla^{\ell-1}_{\partial_t} \nabla^k Rm$$

now write $\nabla^{\ell-1}_{\partial_t} \nabla^k Rm$ as a $*$-composition of $\nabla^i Rm$. Thus

$$|(\nabla^{\ell}_{\partial_t} \nabla^k Rm)(\cdot, 1)| \leq C_{\ell,k}$$

which finishes the theorem. $\square$

Corollary 7.1.1. Suppose $|Rm| \leq r^{-2}$, on $[0, r^2]$¡ then

$$|\nabla_{\partial_t}^{\ell} \nabla^k Rm|(\cdot, r^2) \leq \frac{C_{k,\ell}}{r^{2\ell+k+2}}$$

Proof: Take $A = r^{-2}$ in theorem. □

Note that this corollary gives a scale invariant bound since $|\nabla_{\partial_t}^{\ell} \nabla^k Rm|(\cdot, r^2)$ is order $r^{-(2\ell+k+2)}$. This means that we'll get such an inequality up to any order on the interval $[0, r^2]$

7.2 Shi's derivative estimates (local bounds on $|\nabla_{\partial_t}^{\ell} \nabla^k Rm|$)

THe previous section worked for $(M, \{g_t\})$ a ricci flow with M compact. In this setting, we assume $(M, \{g_t\}_{t\in[0,T)})$ a ricci flow, but not necessarily compact.

Theorem 7.2. For (M, g_t) a Ricci flow (not necessarily compact). Choose $x_0 \in M$ and $r^2 \leq t_0 < T$ so that $B_{t_0}(x, r) \subset\subset M$ (i.e. relatively compact). Assume

$$|Rm| \leq r^{-2} \qquad \text{on} \quad B_{t_0}(x_0, r) \times [t_0 - r^2, t_0]$$

Then

$$|\nabla_{\partial_t}^{\ell} \nabla^k Rm| \leq \frac{C_{k,\ell}}{t^{2\ell+k+2}}$$

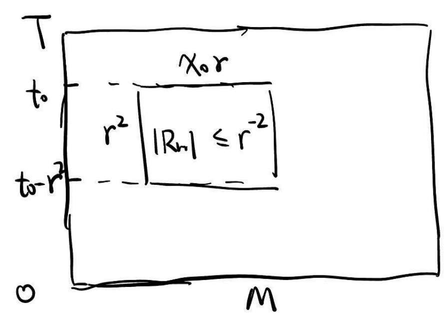

Figure 14

Note that

$$B_{t_0}(x, r) \times [t_0 - r^2, t_] =: P(x_0, t_0; r, -r^2)$$

is called the "backward parabolic neighborhood centered at x_0 of scale r"

7.3 Maximal Existence Time

Lemma 7.3 (Equivalence of Metrics)**.** Suppose $(M, \{g_t\})$, RF, not necessarily compact, $|\text{Ric}| \leq k$ everywhere, then $\forall t_1 \leq t_2 \in I$, we have

$$e^{-K(t_2-t_1)} g_{t_1} \leq g_{t_2} \leq e^{K(t_2-t_1)} g_{t_1}$$

i.e. this says that $C^{-1} g_1 \leq g_2 \leq C g_1$.

Proof: Exercise, should probably just use ricci flow equation and integrate and use bound.

RemarkThis gives us that

$$d_{t_1}(x, y) e^{-K(t_2-t_1)} \leq d_{t_2}(x, y) \leq e^{K(t_2-t_1)} d_{t_1}(x, y)$$

We now show a theorem

Theorem 7.4. If $(M, \{g_t\}_{t\in[0,T)})$ a Rf, not necessarily compact, $T < \infty$. Assume $\sup_{M\times[0,T)} |Rm| < \infty$. Then g_t can be extended smoothly onto $M \times [0, T]$

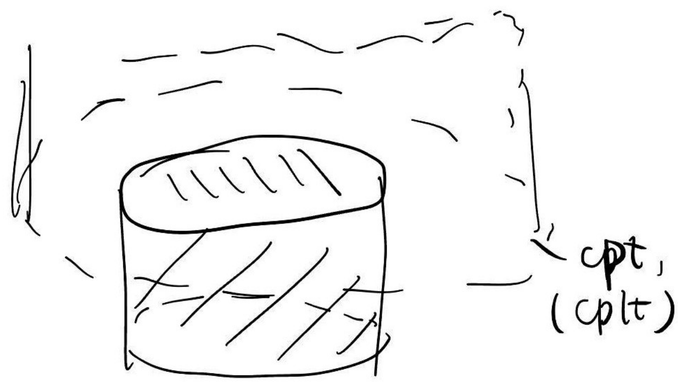

Figure 15

i.e. we can find g_T such that $g_t \to g_T$ smoothly.

Proof: Shi's estimate gives that for any compact subset $U \subseteq M$¡ there exists a $C_{\ell,k}(U)$ such that

$$|\nabla^{\ell}_{\partial_t} \nabla^k Rm| \le C_{\ell,k}(U)$$

on U. Let $p \in M$, $(U, \{x^i\})$ a set of local coordinates about p, so

$$g_t = g_{ij}(x,t)dx^i dx^j$$
$$\text{Lemma} \implies C^{-1} g_s \le g_t \le C g_s \qquad \forall t, s \in I$$

We also have

$$|\partial_t g_{ij}| = 2|\text{Ric}_{ij}| \le C$$

so there exists $g_{ij}(\cdot, T)$ such that

$$g_{ij}(\cdot, t) \xrightarrow{C^0} g_{ij}(\cdot, T)$$

Note that the $C^{-1}g_s \le g_t \le Cg_s$ comparison guarantees that $g_{ij}(\cdot, T)$ is a metric. Now we look at

$$\begin{aligned} |\partial_t \Gamma^k_{ij}(\cdot, t)| &\le C|\nabla \text{Ric}| \\ &\le C \end{aligned}$$

by our formula for the christoffel symbols, and then using the ricci flow equation. In the second line, we use Shi's estimates. This tells us that

$$|\Gamma^k_{ij}| \le C$$

for all t uniformily in t. We also compute

$$(\nabla_k \text{Ric})_{ij} = \partial_k(\text{Ric}_{ij}) - \Gamma^{\ell}_{ki}\text{Ric}_{i\ell} - \Gamma^{\ell}_{kj}\text{Ric}_{j\ell}$$

But again ∇Ric is bounded, so the above gives

$$|\partial(\text{Ric}_{ij})| \le C$$

Now we note that

$$|\partial_t \partial_k g_{ij}| = 2|\partial_k \text{Ric}_{ij}| \le C$$

so we can integrate in time and get
$$|\partial_k g_{ij}| \le C$$
uniformily in time. Now by induction, we can show that
$$|\partial_t^q \partial_{k_1} \cdots \partial_{k_p} g_{ij}| \le C_{p,q,i,j}$$
which allows us to upgrade our convergence of $g(\cdot,t) \to g(\cdot,T)$ from C^0 convergence to **smooth** convergence. Thus we have smooth convergence locally about any point, so we have global smooth convergence (though not uniformily). □

Corollary 7.4.1. We have $(M, \{g_t\}_{t\in[0,T)})$ with M compact and $T < \infty$ maximal, then
$$\max_M |Rm|(\cdot,t) \xrightarrow{t\uparrow T} \infty$$

As an interesting application, we have the following example:

Example:
For $(M^2, \{g_t\}_{t\in[0,T)})$, compact, if $K_{g_0} \le 0$, then $T = \infty$

Proof: Recall that our assumption gives
$$K_{g_t} \le 0$$
by curvature bounds. Moreover,
$$|Rm| \le C|K|$$
but we know that
$$\inf K \le 0$$
because scalar/gaussian curvature is non-decreasing in RF.

7.4 Vector valued maximum principle

For $(M, \{g_t\}_{t\in[0,T)})$ a family of smooth metrics. Let E be a vector bundle on M, rank $k < \infty$. Then $E \times [0,T]$ is a vector bundle on $M \times [0,T]$. Let ∇ a connection on $E \times [0,T]$ compatible with the induced horizontal metric on $E \times [0,T]$.

Remark In the above, we note that the Uhlenbeck connection is a such a connection on $E \times [0,T]$

Now let $C \subseteq E \times [0,T)$ closed such that

- $$C_{x,t} := C \cap \pi^{-1}(x,t) \text{ is convex } \forall (x,t) \in M \times [0,T]$$
- For all t, $C_{x,t}$ are parallel (fixed t) (i.e. $\forall\gamma(s)$ a curve in $M \times \{t\}$, if $e(0) \in C_{\gamma(0,t}$ and $\nabla_{\dot\gamma(s)} e(s) = 0$ then $e(s) \in C_{\gamma(s),t}$

8 Lecture 8: 10-20-22

8.1 Vector valued maximum principle

Our set up is as follows: we have $(M, \{g_t\}_{t\in[0,\tau)})$ smooth family of metrics, and E a vector bundle on $\pi : E \to M$. ∇ is a connection on $E \times [0,T]$ compatible with the space time metric (i.e. just metric on M plus dt^2) induced by π.
$C \subseteq E \times [0,T]$ closed, such that

1. $C_{x,t} := C \cap \pi^{-1}(x,t)$ is convex for all $(x,t) \in M \times [0,T]$

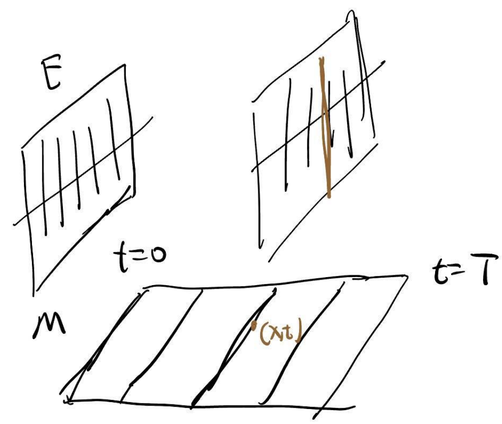

Figure 16

2. $\forall t,\ C_{x,t}$ are parallel

Now let ϕ: a smooth vector field on $E \times [0,T]$, parallel to the fiber of E. Suppose C is preserved by the flow of $\nabla_{\partial_t} u = \phi(u)$. This means that if $u(t) \in \pi^{-1}(x,t)$ for x fixed, if $u(t_0) \in C_{x,t_0}$, $\nabla_{\partial_t} u(t) = \phi(u(t))$, then $u(t) \in C_{x,t}$ for all $t \geq t_0$.

Then for $u \in C^\infty(M \times [0,T]; E \times [0,T])$, suppose we have

$$\nabla_{\partial_t} u = \Delta u + \phi(u)$$

The weak vector-valued maximum principle is

Theorem 8.1 (WMP)**.** Suppose $u(x,t) \in C_{x,t}$, for all $(x,t) \in \partial_{par}(M \times [0,T])$, then $u(x,t) \in C_{x,t}$ for all $(x,t) \in M \times [0,T]$

where ∂_{par} denotes the parabolic boundary, i.e.

$$\partial_{par}(M \times [0,T]) = (\partial M) \times [0,T] \cup M \times \{0\}T$$

We also have the strong vector-valued maximum principle

Theorem 8.2 (SMP)**.** Suppose $u(x,t) \in C_{x,t}$, for all $(x,t) \in M \times [0,T]$, and $u(x_0,t_0) \in \partial C_{x_0,t_0}$ for some $x_0 \in M$, $t_0 > 0$, then $u(x,t) \in \partial C_{x,t}$ for all $(x,t) \in M \times [0,t_0]$

8.1.1 Application of Weak Maximum Principle

Suppose $(M, \{g_t\}_{[0,T]}$ a Ricci flow. Let M compact and $E : S_B(\wedge_2 \mathbb{R}^n) \to M$ be the bundle of algebraic curvature tensors over M. Let ∇ be the uhlenbeck connection on $E \times [0,T]$. Let

$$u = Rm$$

Then the relevant ODE is

$$\nabla_{\partial_t} F = Q(F)$$

and the relevant PDE which is satisfied by Rm itself, is

$$\partial_t Rm = \Delta Rm + Q(Rm)$$

Now let C be given by $C_{x,t} \cong C_t \subseteq S_B(\wedge_2 \mathbb{R}^n)$ some closed convex subset preserved by the ODE.

With this, we have the following:

Theorem 8.3. If $(M^3, \{g_t\}_{t\in[0,T]}$ a Ricci Flow, M compact. Then

1. $sec_{g_0} \geq 0 \implies sec_{g_t} \geq 0, \forall t \geq 0$
2. $\text{Ric}_{g_0} \geq 0 \implies \text{Ric}_{g_t} \geq 0, \forall t \geq 0$

Similarly

Theorem 8.4. If $(M^3, \{g_t\}_{t\in[0,T]})$ a Ricci flow with $\partial M = \emptyset$, then

1. $sec_{g_t} \geq 0$, $sec_{g_T} \not> 0 \implies sec_{g_t} \not> 0, \ \forall t \in [0,T]$
2. $\text{Ric}_{g_t} \geq 0$, $\text{Ric}_{g_T} \not> 0 \implies \text{Ric}_{g_t} \not> 0, \ \forall t \in [0,T]$

Here, we write $\sec_{g_t} > 0$ if $\sec_{g_t} > \lambda(t) g_t$ for some $\lambda(t) > 0$. So $\sec_{g_T}$ can be ≥ 0 but not strictly greater than 0 (i.e. $\not>$) if its 0 along one direction, but not the others (i.e. non-zero but lacks positive definiteness). In this case, the tensor splits.

Proof: For $(x_0, t_0) \in M \times [0,T]$ we can choose an o.n.b. $\{e_i\}$ in $(T_{x_0}M, g_{t_0})$ such that

$$Rm = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}, \quad \text{Ric} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}$$

using the fact that the dimension is 3. Here,

$$\rho_1 = k_2 + k_3$$
$$\rho_2 = k_1 + k_3$$
$$\rho_3 = k_1 + k_2$$

We also compute

$$Q(Rm) = 2 \begin{bmatrix} k_1^2 + k_2k_3 & 0 & 0 \\ 0 & k_2^2 + k_1k_3 & 0 \\ 0 & 0 & k_3^2 + k_1k_2 \end{bmatrix}$$

Now extend $\{e_i\}$ to be an o.n.b. in a neighborhood of x_0 such that

$$\nabla e_i = 0 = \Delta e_i$$

at x_0 and evolve e_i by $\nabla_{\partial_t} e_i = 0$. Thus, the ODE of $\nabla_t Rm = \phi(Rm)$ becomes

$$\partial_t (Rm(e_i, e_j)) = (\nabla_{\partial_t} Rm)(e_i, e_j) = Q(Rm)(e_i, e_j)$$

so we compute

$$\partial_t k_1(t) = 2(k_1^2 + k_2k_3)$$
$$\partial_t k_2(t) = 2(k_2^2 + k_1k_3)$$
$$\partial_t k_3(t) = 2(k_3^2 + k_1k_2)$$

If $k_1(x_0, 0), k_2(x_0, 0), k_3(x_0, 0) \geq 0$, then evolving by the above gives

$$k_1(x_0, t), k_2(x_0, t), k_3(x_0, t) \geq 0$$

for all t. Choose

$$C_{x,t} = C_t = \{Rm \in S_b(\wedge_2 \mathbb{R}^n), \ Rm \geq 0\}$$

(here Rm is just denoting some arbitrary tensor, not the actual Riemann curvature tensor). Then $C_{x,t}$ is preserved by the ODE, so $Rm \geq 0$ is preserved by Ricci Flow. This proves the sectional curvature statement.

For $\text{Ric} \geq 0$, choose $C_{x,t} = C_t = \{Rm \ : \ \text{Ric} \geq 0\}$. Recall that

$$\partial_t \text{Ric} = \Delta \text{Ric} + 2Rm(\text{Ric})$$

for our specific manifold and Ricci flow. Moreover, for every element $F \in C_t$, we have that the following ODE is satisfied

$$\partial_t F = 2Rm(F)$$

where Rm is again the curvature tensor associated to our Ricci flow $\{g_t\}$.

8.2 Linear Support Functions

Definition 8.5. For $C \subseteq \mathbb{R}^k$, closed, convex, a linear support function for C is an affine linear function

$$\begin{aligned} \alpha : \mathbb{R}^k &\to \mathbb{R} \\ v &\mapsto \vec{a} \cdot v + b \end{aligned}$$

such that $|\vec{a}| = |\nabla \alpha| = 1$ and $C \subseteq \{\alpha \geq 0\}$ and $C \cap \ker \alpha \neq \emptyset$.

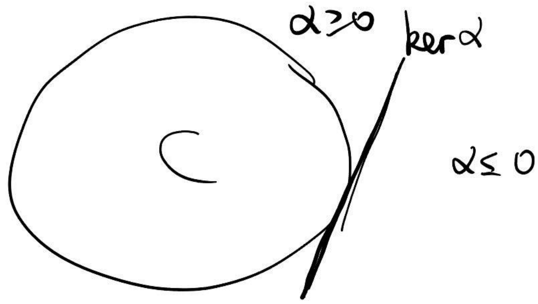

Figure 17

Now we have

Lemma 8.6. The signed distance is given by

$$\text{dsigned}(p, C) = \inf_{\alpha : LSF} \alpha(p)$$

and the infinum can be achieved by a linear support function α such that if $q \in \partial C$ is the closest point to p then $\alpha(p)\nabla\alpha = p - q$

here, LSF is"Linear Support Function" and

$$\text{dsigned}(p, C) := \begin{cases} d(p, \mathbb{R}^k \backslash C) & p \in C \\ -d(p, C) & p \notin C \end{cases}$$

Note that when ∂C smooth, dsigned is the signed distance to ∂C in the usual sense. With this we prove the

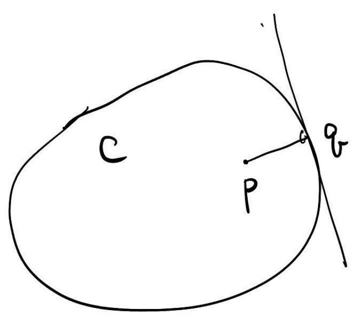

Figure 18

vector valued WMP and SMP's:

Proof: Let

$$s(x,t) = \text{dsigned}(u(x,t), C_{x,t})$$

Here the WMP holds if and only if: $s(\cdot,0) \geq 0$, then $s(\cdot,t) \geq 0$ for all $t \geq 0$

The SMP holds if and only if: $s(x,t) \geq 0$ for all $(x,t) \in M \times [0,T]$ and $s(x_0,T) = 0$, then $s(x,t) \equiv 0,\ \forall (x,t) \in M \times [0,T]$

To show this, we want to prove the following lemma

Lemma 8.7. $\exists C > 0$ such that $(\partial_t - \Delta)s \geq -C \cdot s$

Note that if this is true, then the WMP and SMP hold by comparing $s(x,t)$ to 0.

Proof of Lemma: Assume for simplicity that $s(x,t)$ is smooth. Let α be an LSF such that $q \in \ker(\alpha)$, q is the closest point to $p = u(x_0,t_0)$. Let $\Omega_{t_0} = \{\alpha \geq 0\} \subseteq E_{x_0,t_0}$. Then

$$s(x_0,t_0) = \text{dsigned}(u(x_0,t_0), C_{x_0,t_0}) = \alpha(u(x_0,t_0))$$

Let $\{\Omega_t\}$ be the flow of Ω_{t_0} by the ODE $\nabla_{\partial_t} u = \phi(u)$. Then $C_{x_0,t} \subseteq \Omega_t$ for all $t \leq t_0$. Then

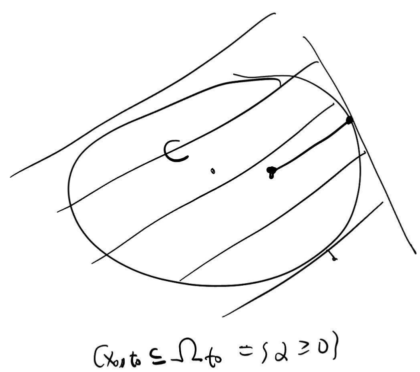

Figure 19

$$s(x_0,t) \leq \text{dsigned}(u(x_0,t), C_{x_0,t}) \leq \text{dsigned}(u(x_0,t), \Omega_t)$$

and equality holds at $t = t_0$. This tells us that

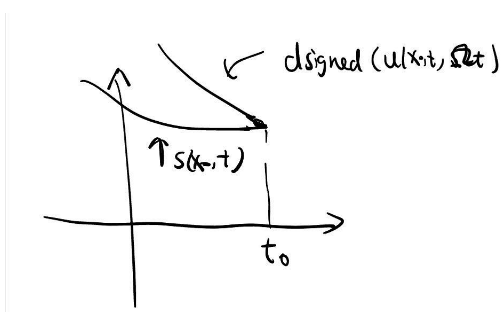

Figure 20

$$\frac{d}{dt}\Big|_{t=t_0} s(x_0,t) \geq \frac{d}{dt}\Big|_{t=t_0} \text{dsigned}(u(x_0,t), \Omega_t) = \alpha(\nabla_{\partial_t} u - \phi(q))$$

were the last equality is an exercise. Now we fix t_0, and extend α to be a LSF on C_{x,t_0} by radially parallel transport. Then

$$\alpha(u(x,t_0)) \geq \text{dsigned}(u(x,t_0), C_{x,t_0}) = s(x,t_0)$$

with equality holding at $x = x_0$. Then we have

$$\begin{aligned}
\Delta\alpha(u(x,t_0) &\geq \Delta s(x,t_0) \qquad \text{at} \quad x = x_0 \\
\alpha(\Delta u(x,t_0)) &\geq \Delta s(x,t_0) \qquad \text{at} \quad x = x_0 \\
(\partial_t - \Delta)s(x_0,t_0) &\geq \alpha(\nabla_{\partial_t} u - \phi(q) - \Delta u)(x_0,t_0) \\
&= \alpha(\phi(u) - \phi(q))(x_0,t_0) \\
&\geq -C|u(x_0,t_0) - q| = -Cs(x_0,t_0)
\end{aligned}$$

which proves the lemma

9 Lecture 9: 10-25-22

Today

- Rigidity of the SMP (strong maximum principle)

Theorem 9.1. We have $(M, \{g_t\}_{t\in[0,T]})$ and

$$\nabla_{\partial_t} u = \Delta u + \phi(u), \qquad u(x,t) \in C_{x,t} \quad \forall t \in [0,T]$$

where $C_{x,t}$ is convex, parallel, and preserved by the ODE. Suppose $u(x_0,t_0) \in \partial C_{x_0,t_0}$ for $t_0 > 0$. Let α be a linear support function for C_{x_0,t_0} and $\alpha(u(x_0,t_0)) = 0$. Then

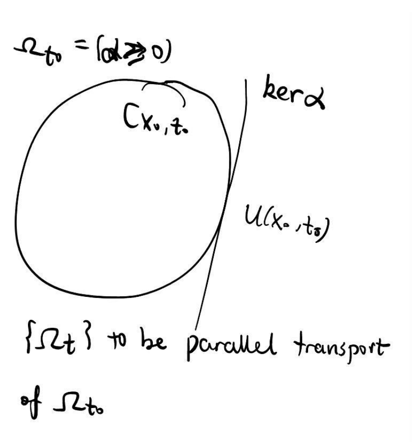

Figure 21

$$\nabla_v u,\ \nabla^2_{v,v} u,\ \nabla_{\partial_t} u - \phi(u) \in \ker(\alpha)$$

for any $v \in T_{x_0}M$, when the above is evaluated at (x_0,t_0).

Let $\Omega_{t_0} = \{\alpha \geq 0\}$, and define $\{\Omega_t\}$ to be the paralell transport of Ω_{t_0}.

Proof: We have $u(x_0,t_0) \in C_{x_0,t} \subseteq \Omega_t$. Moreover

$$d_{signed}(u(x_0,t), \Omega_t) \geq 0$$

and equality holds at $t = t_0$.Thus

$$\begin{aligned} 0 \geq \frac{d}{dt}\Big|_{t=t_0} d_{signed}(u(x_0,t),\Omega_t) &= \alpha(\nabla_{\partial_t} u - \phi(q)) \\ &= \alpha(\nabla_{\partial_t} u - \phi(u)) \\ &= \alpha(\Delta u) \qquad \text{at } (x_0,t_0) \end{aligned}$$

here, $q \in \partial C_{x_0,t_0}$ is the closest point of $u(x_0,t_0)$ and $q = u(x_0,t_0)$.

Now fix t_0, extend α by parallel transport, then

$$u(x_0,t_0) \in C_{x,t_0} \subseteq \{\alpha \geq 0\}, \qquad \alpha(u(x,t_0)) \geq 0$$

and equality in the right hand equation holds at $x = x_0$. Now

$$\begin{aligned} &0 = \partial_v(\alpha(u(\cdot,t_0))) = \alpha(\nabla_v u(x_0,t_0)) \implies \nabla_v u \in \ker\alpha \\ &0 \leq \partial^2_{v,v}(\alpha(u(\cdot,t_0))) = \alpha(\nabla^2_{v,v} u(x_0,t_0)) \implies 0 \leq \alpha(\Delta u(x_0,t_0)) \leq 0 \\ &\quad \implies \alpha(\Delta u) = 0 \end{aligned}$$

where $\alpha(\Delta u) \leq 0$ comes from the viscosity argument and differentiating with respect to t from before. $\square$

Theorem 9.2. With the same set up as in the previous theorem: Moreover if $C_{x_0,t}$ is parallel in t and if one of the following conditions is satisfied

1. $\partial C_{x_0,t_0}$ is smooth at $u(x_0,t_0)$
2. $t_0 < T$

Then $\nabla_v u,\ \nabla^2_{v,v} u,\ \nabla_{\partial_t} u, \phi(u) \in \ker(\alpha)$

Proof: If 1 is true, then by SMP $u(x_0,t) \in \partial C_{x_0,t}$ for all $t \leq t_0$, so

$$\nabla_{\partial_t}\Big|_{t=t_0} u(x_0,t) \in T_{u(x_0,t_0)}\partial C_{x_0,t} = \ker\alpha$$

use

$$\nabla_{\partial_t} u - \phi(u) \in \ker(\alpha) \implies \phi(u) \in \ker(\alpha)$$

If 2 is true, then $\alpha(u(x_0,t)) \geq 0$ in $(t_0-\delta, t_0+\delta)$ for some $\delta > 0$ and equality holds at $t = t_0$, this implies that

$$\partial_t\Big|_{t=t_0} \alpha(u(x_0,t)) = 0 = \alpha(\nabla_{\partial_t} u(x_0,t_0))$$

finishing the proof $\square$.

9.1 Application of theorem to RF

Let $n = 3$, $(M^3, \{g_t\}_{t\in[0,T]})$ RF, not necessarily compact. Let $u = Rm$ and

$$\nabla_{\partial_t} Rm = \Delta Rm + Q(Rm)$$

and

$$C_{x,t} = C_t = C = \{Rm \in S_B(\wedge_2\mathbb{R}^3) \ : \ \mathrm{Ric}(Rm) \geq 0\}$$

Suppose $\mathrm{Ric}_{g_t} \geq 0$ everywhere, but $\mathrm{Ric} > 0$ fails at (x_0,t_0), t_0 (i.e. has null direction). Then $(M,\{g_t\})$ is either flat or locally splits off a line $\forall t \in [0,t_0]$ (we'll prove this!). First note that by the SMP for $\mathrm{Ric} \geq 0$, if $\mathrm{Ric} > 0$ fails at (x_0, t_0) then it fails at all (x, t), $t \leq t_0$.

Proof: Let α be defined by

$$\alpha : Rm \in S_B(\wedge_2\mathbb{R}^3) \mapsto \mathrm{Ric}(Rm)(e,e) \in \mathbb{R}$$

where $0 \neq e \in T_{x_0}M$ is a vector such that $\mathrm{Ric}_{x_0,t_0}(e,e) = 0$ (i.e. $\mathrm{Ric}(e,\cdot) = 0$). Then α is a linear support function on C and $\alpha(Rm(x_0,t_0)) = 0$. We have that

$$0 = \nabla_{\partial_t}\mathrm{Ric}(e,e) = 2Rm * \mathrm{Ric}(e,e)$$

This is because our theorem gives

$$\nabla_v u, \nabla^2_{v,v}u, \phi(u), \nabla_{\partial_t}u \in \ker(\alpha)$$

and we've set $u = Rm$. We also know that (just from Ricci flow properties)

$$\nabla_{\partial_t}Rm = \Delta Rm + Q(Rm)$$
$$\xRightarrow{\mathrm{tr}} \nabla_{\partial_t}\mathrm{Ric} = \Delta\mathrm{Ric} + 2Rm * \mathrm{Ric}$$

Now at (x_0, t_0) choose an o.n.b. $e = e_1, e_2, e_3$ such that

$$\mathrm{Ric} = \begin{bmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{bmatrix}$$

Moreover

$$0 = \frac{d}{dt}\rho_1 = \rho_1(\rho_2 + \rho_3) + (\rho_2 - \rho_3)^2 \implies \rho_2 = \rho_3$$
$$\implies \mathrm{Ric} = \begin{bmatrix} 0 & & \\ & \rho_2 & \\ & & \rho_2 \end{bmatrix}$$

The first equation comes from

$$\nabla_{\partial_t}\mathrm{Ric}(e,e) = \Delta\mathrm{Ric}(e,e) + 2Rm * \mathrm{Ric}(e,e)$$

and then using $\mathrm{Ric}(e,e) = 0$. This tells us that the nullity of Ric is either 1 or 3.

Case 1: If $null(\mathrm{Ric}) = 3$, then $R(x_0,t_0) = 0$. The strong maximum principle applied to R gives that $R(x,t) = 0$ for all x, $t \leq t_0$. Moreover $\mathrm{Ric} \equiv 0$, and in three dimensions this means that $Rm \equiv 0$. Thus $(M, g_t) = (M, g_0)$ and g_0 is flat.
Case 2: If $null(\mathrm{Ric}) = 1$, we can assume that this is the case everywhere (i.e. $\forall(x,t)$)

Proof: So there exists a smooth, unit vector field, e, such that $\mathrm{Ric}(e,e) = 0$, $\mathrm{Ric}(e,\cdot) = 0$. Recall that

$$\nabla^2_{v,v}\mathrm{Ric}(e,e) = 0 = \nabla_v\mathrm{Ric}(e,e)$$

<u>Goal</u>: $\nabla e = 0$ (e is a parallel vector field). Then

$$0 = \partial_v(\nabla_v\mathrm{Ric}(e,e)) = \nabla^2_{v,v}\mathrm{Ric}(e,e) + 2\nabla_v\mathrm{Ric}(\nabla_v e, e)$$

but we know that

$$\nabla^2_{v,v}\mathrm{Ric}(e,e) = 0$$

which implies that

$$\nabla_v\mathrm{Ric}(\nabla_v e, e) = 0$$

but now we can compute

$$0 = \partial_v(\mathrm{Ric}(e, \nabla_v e)) = \nabla_v\mathrm{Ric}(e, \nabla_v e) + \mathrm{Ric}(\nabla_v e, \nabla_v e) + \mathrm{Ric}(e, \nabla^2_{v,v}e)$$

but we know that the first and last term are 0, so

$$\text{Ric}(\nabla_v e, \nabla_v e) = 0 \implies \nabla_v e = \lambda e$$
$$|e| = 1 \implies \nabla_v e = 0, \qquad \forall v \in T_{x_0}M \implies \nabla e = 0$$

Now as an exercise: If we have $\nabla_{\partial_t} e = \partial_t e = 0$, then the splitting of our space is preserved by the flow. $\square$

Corollary 9.2.1. If (M^3, g) compact and $\text{Ric}_g \geq 0$, but M^3 doesn't admit any metric with Ric > 0, then (M^3, g) is isometric to one of the following:

1. Quotient of $\mathbb{R}^3$
2. Quotient of $S^2 \times \mathbb{R}$ where (S^2, h), $\kappa_h > 0$

Proof: Given (M^3, g), we flow by Ricci flow and get $\{g_t\}$. Then at some point we have nullity and use the previous theorems to either get a splitting (e.g. $S^2 \times \mathbb{R}$) or show that the metric is flat (e.g. $\mathbb{R}^3/\Gamma$). $\square$

Note that the condition of finding a metric with Ric > 0 (or ruling it out) is partially dealt with by Hamilton's theorem.

Theorem 9.3. Let $n = 3$ and $(M^3, \{g_t\}_{t\in[0,T]})$ a Ricci Flow. Suppose that $sec_{g_t} \geq 0$ (i.e. $Rm \geq 0$), and $Rm > 0$ fails at (x_0, t_0) for $t_0 > 0$, then one of the following is true

1. (M, g_t) is flat for all $t \leq t_0$. OR
2. (M, g_t) locally splits off a line

Note that in the latter case, the nullity of Rm is 2 because a basis for the domain of Rm is $\{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}$ and $e = e_1$ is the null direction for Ric.

Proof: In fact, this can be deduced from the last theorem (Ric splitting theorem). It suffices to show that Ric > 0 also fails at (x_0, t_0). Note that $Rm > 0$ fails implies that there exists e_1, e_2, e_3 such that Rm is diagonal under $e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$ and

$$Rm = \begin{pmatrix} \kappa_1 & 0 & 0 \\ 0 & \kappa_2 & 0 \\ 0 & 0 & \kappa_3 \end{pmatrix}, \quad \kappa_1 \leq \kappa_2 \leq \kappa_3$$

and

$$0 = \frac{d}{dt} k_1 = k_1^2 + k_2 k_3$$

where

$$Q(Rm) = \begin{pmatrix} k_1^2 + k_2 k_3 & 0 & \dots \\ 0 & \dots & \dots \\ \dots & \dots & \dots \end{pmatrix}$$

Note that $k_1 = 0$ implies $k_2 k_3 = 0$ so $k_2 = 0$ or $k_3 = 0$, i.e. the nullity of Rm is 2 or 3. In either case, this implies that Ric > 0 fails at (x_0, t_0). $\square$

Theorem 9.4 (Cone Rigidity)**.** Let $n = 3$, $(M^3, \{g_t\}_{t\in[0,T]})$ a RF not necessarily compact. Suppose $\text{Ric}_{g_t} \geq 0$. If (M^3, g_T) is isometric to an open subset of a cone over a Riemannian manifold, then $(M^3, \{g_t\})$ is flat

Proof: Recall a cone is given by $dr^2 + r^2 h$ where (N, h) is a 2D manifold. It's an exercise to show that $\text{Ric}(\partial_r, \partial_r) = 0$. Then the theorem tells us that we're either flat, or we split off a line. Suppose not flat. Then we have

$$g_T = dr^2 + h_{N'} = dr^2 + r^2 h_N$$

where we think of $(N', h_{N'})$ and (N, h_N) as two separate 2D manifolds. Now we write

$$\mathrm{Ric} = \begin{pmatrix} 0 & & \\ & \rho_2 & \\ & & \rho_3 \end{pmatrix}$$

where $\rho_2 \leftrightarrow e_2$, $\rho_3 \leftrightarrow e_3$. Then ρ_2 is constant in r if we split like a line, but also ρ_2 scale like r^{-2} in r if we have a cone splitting. This is a contradiction since r is variable in the cone perspective. Thus we must have $\rho_2 = 0$. Same for ρ_3. Thus Ric $= 0$ and we're flat!

10 Lecture 10: 10-27-22

Today

- More preserved curvature condition in $n = 3$
- Hamilton's Ric > 0 theorem

10.1 More preserved curvature condition in $n = 3$

If $C \subseteq E$ is defined as $\psi^{-1}([0, \infty))$ for some concave function

$$\psi : E \to \mathbb{R}$$

then C is a convex subset. Moreover, the preservation of C under the ODE if and only if for all $e \in E$ with $\psi(e) = 0$, let $e(t)$ satisfy $e(0) = e$ and the ODE

$$\nabla_{\partial_t} u = \phi(u)$$

Then,

$$\frac{d}{dt}\psi(e(t)) \geq 0$$

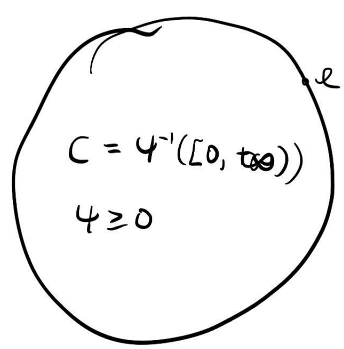

Figure 22

Lemma 10.1. Assume that

$$Rm = \begin{pmatrix} \kappa_1 & & \\ & \kappa_2 & \\ & & \kappa_3 \end{pmatrix}$$

with $\kappa_1 \leq \kappa_2 \leq \kappa_3$. Then

1. $\kappa_1 + \kappa_2 + \kappa_3$ is both concave and convex (because its linear)
2. κ_1 is concave, $\kappa_1(Rm_1) \geq C$, $\kappa_1(Rm_2) \geq C \implies \kappa_1(aRm_1 + bRm_1) \geq C$, for $a, b \geq 0$ and $a + b = 1$
3. κ_3 is convex
4. $\rho_1 = \kappa_1 + \kappa_2$ is concave
5. $\kappa_3 - \kappa_1$ is convex

Remark Here, the underlying space is $\mathbb{R}^3$, so we interpret concave and convex on $\mathbb{R}^3$. Remember that our bundle $E = S_B(\wedge_2\mathbb{R}^3)$

Theorem 10.2 (Pinching condition)**.** For all $\epsilon \in [0, 1/3)$, $n = 3$, $(M^3, \{g_t\}_{t\in[0,T]})$ a Ricci Flow, M^3 compact, then

$$\mathrm{Ric} \geq (\epsilon \cdot R)g$$

is preserved

Remark Note that we've already proved this when $\epsilon = 0$.
Remark

1. Note that
$$\mathrm{trRic} \geq \mathrm{tr}(\epsilon R \cdot g), \quad R \geq 3\epsilon R \implies R \geq 0$$
2. In S^3, we have $\mathrm{Ric} = \frac{1}{3}R \cdot g$, i.e. sharpness for $\epsilon = 1$
3. In a manifold with $\mathrm{Ric} > 0$, there exists an $\epsilon > 0$ such that $\mathrm{Ric} \geq \epsilon R \cdot g$

Proof: We have

$$\mathrm{Ric} = \begin{pmatrix} \kappa_1 + \kappa_2 & & \\ & \kappa_1 + \kappa_3 & \\ & & \kappa_2 + \kappa_3 \end{pmatrix}$$

for $\kappa_1 \leq \kappa_2 \leq \kappa$.

Goal: Write $\mathrm{Ric} \geq \epsilon R g$ as the 0-sublevel set of a concave function ψ and check

$$\frac{d}{dt}\psi(e(t)) \geq 0$$

whenever $\psi(e(0)) = 0$. Note that

$$\begin{aligned} \mathrm{Ric} \geq \epsilon R g &\iff \kappa_1 + \kappa_2 \geq \epsilon(2(\kappa_1 + \kappa_2 + \kappa_3)) \\ &\iff \kappa_1 + \kappa_2 \geq \frac{2\epsilon}{1-\epsilon}\kappa_3 \triangleq \delta\kappa_3, \qquad \delta \in [0, 2) \\ &\iff \kappa_1 + \kappa_2 \geq \delta\kappa_3 \\ &\iff \kappa_1 + \kappa_2 - \delta\kappa_3 \geq 0 \end{aligned}$$

Note that the first line holds since $\kappa_1 + \kappa_2$ is the lowest eigenvalue. And in the last line $\kappa_1 + \kappa_2 - \delta\kappa_3$ is concave.

When "$\psi(e(0)) = 0$", then this corresponds to $\kappa_1 + \kappa_2 = \delta\kappa_3$. Moreover "$\frac{d}{dt}\psi(e(t))\Big|_{t=0}$ corresponds to

$$\frac{d}{dt}\left(\kappa_1 + \kappa_2 - \delta\kappa_3\right) \geq 0$$

Now we use the underlying ODE to compute this, i.e.

$$\nabla_{\partial_t} Rm = \phi(Rm)$$

with $\partial_t \kappa_1 = \kappa_1^2 + \kappa_2\kappa_3$ to get

$$\frac{d}{dt}\left(\kappa_1 + \kappa_2 - \delta\kappa_3\right) = \kappa_1^2 + \kappa_2\kappa_3 + \kappa_2^2 + \kappa_1\kappa_3 - \delta(\kappa_3^2 + \kappa_1\kappa_3)$$

Note that if $\kappa_3 = 0$, then we're done as $\mathrm{Ric} \equiv 0$, so WLOG assume $\kappa_3 \neq 0$ and let

$$\delta = \frac{\kappa_1 + \kappa_2}{\kappa_3}$$

Then with this choice of δ, we have that

$$\kappa_1 + \kappa_3 \geq \delta\kappa_3$$

and also plugging δ into the above we get

$$\kappa_1^2 + \kappa_2\kappa_3 + \kappa_2^2 + \kappa_1\kappa_3 - \delta(\kappa_3^2 + \kappa_1\kappa_3) \geq 0$$

finishing the proof. $\square$

Lemma 10.3. $\forall \epsilon \in (0,1)$, $\exists \delta(\epsilon) > 0$ such that

$$C = \{\frac{\rho_1}{\rho_3} \geq 1 - \frac{1}{\rho_3^{\delta(\epsilon)}},\ \rho_1 \geq \epsilon\rho_3 > 0\}$$

and $\rho_1 \leq \rho_2 \leq \rho_3$ eigenvalues of Ric are convex and preserved by Ricci Flow.

Remark The proof is similar but a bit more involved than the previous lemma, so we'll skip this proof for now.

Theorem 10.4 (Hamilton, $\mathrm{Ric} > 0$)**.** Let (M^3, g) compact, $\mathrm{Ric} > 0$, then M is diffeomorphic to S^3/Γ

Proof: Run Ricci flow for g as the initial condition. Assume T is the maximal existence time. Then

$$T < \infty \ (R_g > 0 \xrightarrow{WMP} R \uparrow \infty \text{ in finite time})$$

By compactness of the manifold, there exists $\epsilon > 0$ such that

$$\frac{\rho_1}{\rho_3} \geq \epsilon$$

at time $t = 0$. After a rescaling, we can find $\delta(\epsilon) > 0$ such that

$$1 - \frac{1}{\rho_3^{\delta(\epsilon)}} \leq \frac{\rho_1}{\rho_3} \leq 1$$

Now our lemma implies that these are also true for all g_t in our RF, $t \in [0, T)$. Now let

$$Q_t = \max_M \rho_3(\cdot, t)$$

This $\to \infty$ as $t \uparrow T$, since we now that scalar curvature blows up.
<u>Claim 1</u>: There exists $C > 0$ such that $\forall \alpha > 0$, there exists $\tilde{\delta} > 0$ such that for any $x \in M$, $t \in [T - \tilde{\delta}, T)$, if

$$\rho_3(x,t) \geq \frac{1}{10} Q_t \geq \frac{1}{100} \max_{M \times [0,t]} \rho_3$$

then

$$\rho(\cdot, t) \in [(1-\alpha)\rho_3(x,t), (1+\alpha)\rho_3(x,t)]$$

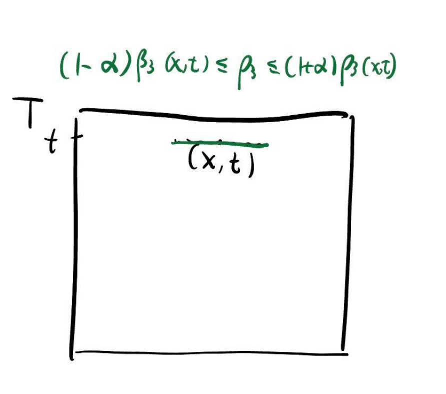

Figure 23

in $B_t(x, c\rho_3^{-1/2}(x,t))$.

Remark Intuitively, this says that "(x,t) almost achieves the max of ρ_3 in $M \times [0,t]$" **Proof:** Let $t_k < T$, $t_k \uparrow T$. Let

$$g_k' = \rho_3(x_k, t_k) g_{t_k}$$

Then $\rho_3 \leq 100$ on g_k' (implies $|Rm| \leq C100$). Now Shi's derivative estimate gives that

$$|\nabla^m Rm| \leq C_m$$

forall $m \in \mathcal{N}$ for g_k'. Now let

$$g_k'' = \exp^*_{x_k, g_k'} g_k'$$

on $T_{x_k}M \cong \mathbb{R}^3$, then via an exercise, we have

$$|\partial^m (g_k'')_{ij}| \leq c_m'$$

on $B(0, 3c)$ for some $c > 0$ and g_k'' satisfies

$$1 - \frac{1}{(\rho_3(x)\rho_3(x_k, t_k))^{\delta(\epsilon)}} \leq \frac{\rho_1(x)}{\rho_3(x)} \leq 1$$

here, $\rho_1(x)$, $\rho_3(x)$ is with respect to g_k'' and $\rho_3(x_k, t_K)$ is with respect to the original setting and g_{t_k}. Thus

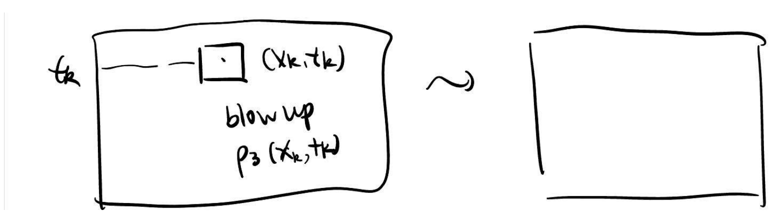

Figure 24

$$\lim_{k \to \infty} g_k'' = g_\infty$$

with convergence in C^∞ on $B(\vec{0}, 2C) \subseteq \mathbb{R}^3$. Moreover

1. $\rho_3(\vec{0}) = 1$

2. $\forall x \in B(\vec{0}, 2C)$ if $\rho_3(x) \neq 0$

$$1 \leq \frac{\rho_1(x)}{\rho_3(x)} \leq 1 \implies \rho_1(x) = \rho_2(x) = \rho_3(x)$$

But if $\rho_3(x) = 0$, then $\rho_1(x) = \rho_2(x) = 0$ by Ric ≥ 0 (before we take the limit, we know Ric > 0, so in the limit we have Ric ≥ 0). This shows that in all cases $\rho_1(x) = \rho_2(x) = \rho_3(x)$, and we aim to show that ρ_i is a constant in x.

Thus

$$\text{Ric} = \lambda g$$

for $\lambda : B(\vec{0}, 2c) \to \mathbb{R}$. By Schur's lemma, λ is a constant. Morever

$$\rho_3(\vec{0}) = 1$$

which implies that $\lambda \neq 0$ and $\rho_1 = \rho_2 = \rho_3$ on $B(\vec{0}, 2C)$. □

Claim 2: $\forall \alpha' > 0$, we can find a point (x, t) such that

$$\rho_3(\cdot, t) \in [(1-\alpha')\rho_3(x,t), (1+\alpha')\rho_3(x,t)] \quad \text{in} B_t(x, 10\pi\rho_3(x,t)^{-1/2})$$

Proof: Repeat Claim 1

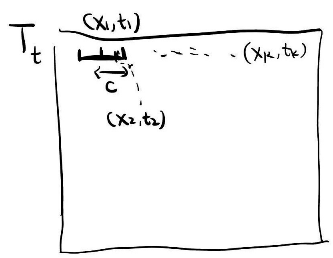

Figure 25

$$\rho_3(\cdot, t) \in [(1-\alpha)\rho_3(x,t), (1+\alpha)\rho_3(x,t) \text{ in } B_t(x, C\rho^{-1/2}(x,t))$$

for $\left[\frac{10\pi}{C}\right] + 1$ times, and we can find (x, t), as long as (x_k, t_k) almost achieves the $\max_{M\times[0,t]} \rho_3$. I.e.

$$\forall k, \qquad \rho_3(x_k, t_k) \geq \frac{1}{10} Q_{t_k} \geq \frac{1}{100} \max_{M\times[0,t_k]} \rho_3$$

$$\rho_3(x_k, t_k) \in [(1-\alpha)^{k-1}\rho_3(x_1,t_1), (1-\alpha)^{k-1}\rho_3(x_1,t_1)]$$

$$k \leq \left[\frac{10\pi}{C}\right] + 1$$

If we choose $\alpha \ll 1$ such that the first line holds for any k. Assume $\alpha' \ll 1$. Then Bonnet-Meyers theorem tells us that

$$diam_t(M) < 4\pi\rho_3^{-1/2}(x,t) \implies B_t(x, 10\pi\rho_3(x,t)^{-1/2})) = M$$

and we know that

$$\rho_3(\cdot, t) \in [(1-\alpha')\rho_3(x,t), (1+\alpha')\rho_3(x,t)]$$

for any other point. Now the differential sphere theorem implies that $M \cong S^3/\Gamma$. □

For posterity, we recall the differential sphere theorem

Theorem 10.5. Let (M^3, g) compact and

$$\kappa_3 \leq (1+\epsilon)\kappa_1$$

$(\kappa_1 \leq \kappa_2 \leq \kappa_3)$, then

$$\implies M \cong S^3/\Gamma$$

11 Lecture 11: 11-1-22

Today

- Curvature estimate of Hamilton's Ric > 0 theorem
- Hamilton-Ivey pinching (3D)
- Preserved curvature condition in $n \geq 3$

11.1 Hamilton's Ric > 0 theorem

Theorem 11.1 (Hamilton)**.** For (M^3, g) compact with $\mathrm{Ric}_g > 0$, then the Ricci flow $(M, \{g_t\}_{t\in[0,T]})$ with T the maximal existence time and $g_0 = g$ satisfies

$$|Rm(\cdot,t)| \leq \frac{C}{T-t} \tag{14}$$

for some $C > 0$.

Remark Equation (14) is called a "Type I singularity" and a "Type II singularity" is when (14) fails to hold.

Proof: Let

$$R_{max}(t) = \max_M R(\cdot,t)$$

<u>Claim:</u> For all $t < T$ that is sufficiently close to T, we have

$$\frac{d}{dt^+} R_{max}^{-1}(t) \leq -C \qquad C > 0$$

Proof: Suppose not, then we can find a sequence $t_k \uparrow T$, $\epsilon_k \to 0$, $\epsilon_k > 0$ so that

$$\frac{d}{dt^+} R_{max}^{-1}(t) \geq -\epsilon_k$$

Suppose $R(x_k, t_k) = R_{max}(t_k)$. Then we showed last time that

$$R^{-1}(x_k, t_k) g_{t_k} \to g_{S^3}$$

smoothly. Recall the ODE for R

$$\frac{d}{dt} R = \Delta R + 2|\mathrm{Ric}|^2$$

and so

$$\frac{d}{dt} R^{-1} = -\frac{\partial_t R}{R^2} = -\frac{1}{R^2}\left(\Delta R + 2|\mathrm{Ric}|^2\right)$$

Note that

$$-\frac{1}{R^2}\left(\Delta R + 2|\mathrm{Ric}|^2\right) = -C$$

on (S^3, g_{S^3}) with $C > 0$. Thus

$$\frac{d}{dt} R^{-1}(x_k, t_k) \leq -\frac{C}{2}$$

for C large, a contradiction. Here, we've noted that $\frac{d}{dt} R^{-1}$ is a scale invariant, i.e.

$$\frac{d}{dt} R_{g_{t_k}}^{-1} = \frac{d}{dt} R_{R(x_k,t_k) g_{t_k}}^{-1}$$

This tells us the claim is true, and now

$$\frac{d}{dt^+} R_{max}^{-1}(t) \leq -C \implies R_{max}(t) \leq \frac{C^{-1}}{T-t}$$

but now scalar curvature bounds norm of the Riemannian tensor up to a constant so

$$|Rm|(t) \leq \frac{C}{T-t}$$

$\square$

Remark To be formal, we have to connect

$$\frac{d}{dt}R^{-1}(x_k, t_k)$$

to

$$\frac{d}{dt}R_{max}(t_k)$$

which aren't the same. But there's a viscosity argument that gives the same bound (see 26) since

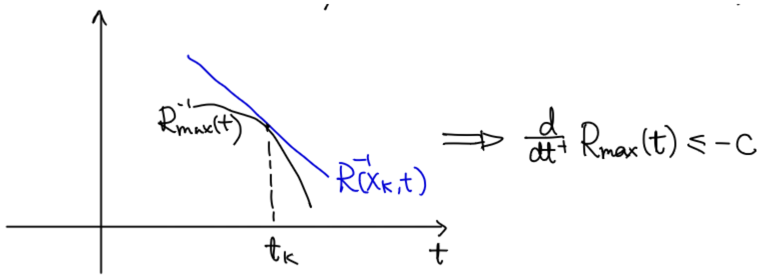

Figure 26

$$R_{max}(t) \geq R(x_k, t)$$

we can show that the appropriate bound on the derivative holds in the correct direction.

Remark We have

$$(T-t)^{-1}g_t \xrightarrow{C^\infty} g_{S^3}$$

11.2 Hamilton-Ivey Pinching

Lemma 11.2. The following subset C_t is convex and preserved by

$$\nabla_{\partial_t} Rm = Q(Rm)$$

for any $t > 0$. Let

$$C_t = \begin{cases} Rm \in S_B(\wedge_2 \mathbb{R}^3) \ : \ R \geq -\frac{3}{2t} \\ \exists X > 0 \ \text{ s.t. } \ \sec \geq -C \text{ and } 2X(\log(2Xt) - 3) \geq R \end{cases}$$

Remark We call C_t the "t^{-1}-positive curvature" subset, i.e. $Rm_g \in C_t$ means that it has "t^{-1}-positive curvature"

Corollary 11.2.1. If $(M^3, \{g_t\}_{t \in [0,T]})$ compact Ricci Flow, assume $Rm_{g_{t_0}} \in C_{t_0}$, then $Rm_{g_t} \in C_t$

Lemma 11.3. Suppose (M^3, g) is t^{-1}-positive for $t > 0$, then $\forall \lambda > 0$, we have that $(M^3, \lambda g)$ is $\lambda^{-1}t^{-1}$-positive, i.e.

$$Rm_g \in C_t \implies Rm_{\lambda g} \in C_{\lambda t}$$

Lemma 11.4. (M^3, g) is T_i^{-1}-positive for a sequence of $T_i \to \infty$ then $\sec_g \geq 0$.

Proof: Fix a point $x_0 \in M$. The first condition of being in C_{T_i} means that

$$R \geq -\frac{3}{2T_i} \to 0$$

The second condition means there exists an x_{T_i} (a constant) such that sec $\geq -x_{T_i}$ and

$$2x_{T_i}(\log(2x_{T_i}T_i) - 3) \leq R(x_0)$$

(here x_0 is a point, not $T_i = 0$). Note that $x_{T_i} \to 0$, else $T_0 \to \infty$ forces

$$2x_{T_i}(\log(2x_{T_i}T_i) - 3) \to \infty$$

a contradiction to the fixed upper bound of $R(x_0)$. This implies that

$$\sec(x_0) \geq \lim_i -x_{T_i} = 0$$

□

Theorem 11.5. $(M^3, \{g_t\}_{t\leq 0})$ an RF implies that $sec \geq 0$ for all $x \in M$, for all $t \leq 0$

Proof: Fix $t_0 \leq 0$, let $T_i \to \infty$ and $g_{i,t} = g_{t-T_i}$, $t \leq T_i$ (27)

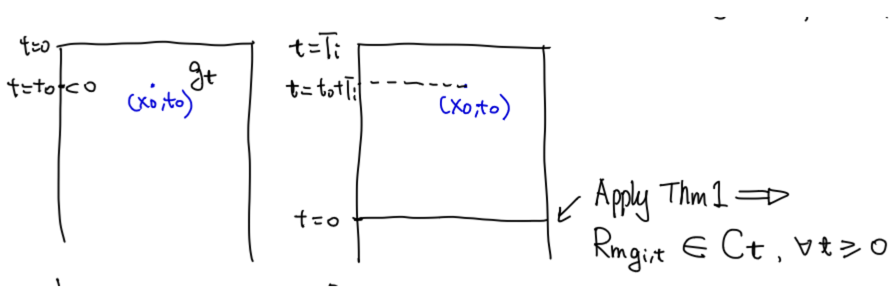

Figure 27

Lemma 11.6. For a Ricci flow $(M^3, \{g_t\}_{t\in[0,T]})$ we have $Rm_{g_t} \in C_t$ for all $t > 0$ (i.e. t^{-1}-positive using our definition of C_t as before)

Proof: Can find $\epsilon_i \to 0$ such that

$$Rm_{g_{\epsilon_i}} \in C_{\epsilon_i}$$

This follows by compactness of M. Now use the strong maximum principle to preserve the properties defined by C_t for all $t > 0$. This finishes the proof. □

Now apply the lemma to $g_{i,T}$ then

$$\begin{aligned} &Rm_{g_{i,t}} \in C_t, \quad \forall t > 0 \\ \implies &Rm_{g_{t-T_i}} \in C_t, \quad \forall t > 0 \\ \implies &Rm_{g_{t_0}} \in C_{t_0+T_i}, \quad (\text{take } t = t_0 + T_i) \end{aligned}$$

This implies that $Rm_{g_{t_0}}$ is $(t_0 + T_i)^{-1}$-positive. Now send $T_i \to \infty$ and get

$$\sec(\cdot, t_0) \geq 0$$

□

Corollary 11.6.1. A closed shrinking solution in 3D must be the shrinking sphere

Proof: Recall that a shrinking soliton generates an ancient Ricci Flow

$$g_t = (-2\lambda t)\phi_t^* g$$

where ϕ_t is a diffeo and $t \in (-\infty, 0]$. Our theorem then gives that $\sec_{g_t} \geq 0$. If Ric > 0, then Hamilton's Ric > 0 theorem tells us that g_t is asymptotically round (i.e. $R_{max}^{-1}(t)g_t \to g_{S^3}$), which implies that g_t itself must be round. This is because we have for t close to T

$$\begin{aligned} (-2\lambda t)^{-1} g_t &\to g_{S^2} \\ \phi_t^* g &\to g_{S^3}, \quad t \uparrow T \\ g &\to g_{S^3}, \quad t \uparrow T \end{aligned}$$

the last metric, g, is constant and $g = g_{S^3}$.

If Ric > 0 fails at a certain point, then

$$M \cong (S^2, h) \times \mathbb{R}/\Gamma \quad \text{or} \quad M \cong T^3/\Gamma$$

which implies that the diameter stays bounded away fro m0 as $t \uparrow 0$ (i.e. $\text{diam}_{g_t} \geq C > 0$ for all t). THis is a contradiction by the definition of the flow

$$g_t = (-2\lambda t)\phi_t^* g \implies \text{diam}(g_t) = \text{diam}(g) \cdot (-2\lambda t)^{1/2} \to 0$$

so we must be in the first case, i.e. the shrinking sphere. (Here we note that ϕ_t is an isometry so the diameter with respect to g is the same as that with respect to $\phi_t^*(g)$)

11.3 Preserved curvature conditions for $n \geq 3$

Here we make a table

name	definition	properties
$Rm \geq 0$	$\lambda_1(Rm) \geq 0,\ \lambda_1 \leq \lambda_2 \leq \dots$	$\implies$ sec ≥ 0
2-non-negative curvature	$\lambda_1(Rm) + \lambda_2(Rm) \geq 0$	$\implies$ Ric ≥ 0 ($n \leq 3$, equiv to Ric ≥ 0)
weakly PIC$_2$	$M \times \mathbb{R}^2$ is weakly PIC	$\implies$ sec ≥ 0
weakly PIC$_1$	$M \times \mathbb{R}$ is weakly PIC	
weakly PIC	$\forall \{e_i\}$ o.n.b a 4-frame s.t. $R_{1331} + R_{1441} + R_{2332} + R_{2442} + 2R_{1234} \geq 0$	

where PIC = "Positive isotopic curvature". Note that weakly PIC$_2$ $\implies$ weakly PIC$_1$ $\implies$ weakly PIC. Note that every surface is weakly PIC, but not weakly PIC$_2$

12 Lecture 12: 11-3-22

Today

- Generalization of WMP
- Geometric Compactness theorem

12.1 Generalization of WMP

Theorem 12.1 (Shi, Short-time Existence)**.** Let (M^n, g) complete, $|Rm| \leq C$, then $\exists$ a Ricci Flow, $\{g_t\}_{[0,T]}$ with $g_0 = g$ and $T = T(C)$

Remark We won't prove this but note that the maximal time can be bounded above by a function dependent on the curvature bound.

We also have that "almost non-negative" curvature is preserved by Ricci Flow.

Theorem 12.2 (Simon-Topping)**.** For (M^3, g), $\mathrm{Ric}_{g_0} \geq -1$ and $\mathrm{vol}(B_{g_0}(x,1)) \geq v_0 > 0$, $\forall x \in M$. Then there exists a Ricci flow $\{g_t\}_{t\in[0,T]}$ such that $\mathrm{Ric}_{g_t} \geq -C$ where $\tau(v_0)C(v_0) > 0$

here C is the curvature bound and only depends on v_0. Moreover, $\tau(v_0)$ is some multiplicative constant.

Theorem 12.3 (Bamler, Cabezas-Rivas, Wilking)**.** Let C be one of the following

$$\begin{aligned} C_1 &= \{Rm\ :\ Rm \geq 0\} \\ C_2 &= \{Rm\ :\ \lambda_1(Rm) + \lambda_2(Rm) \geq 0\} \\ C_3 &= \{Rm\ :\ \text{weakly PIC}_1\} \\ C_4 &= \{Rm\ :\ \text{weakly PIC}_2\} \end{aligned}$$

Let (M^n, g_0) complete and $Rm_{g_0} + Id \in C$ and $\mathrm{vol}(B(x,1)) \geq v_0$, $\forall x$. Assume moreover that (M^3, g_0) is compact (or complete with bounded curvature) if $C = C_2$ or $C = C_3$, then $\exists \{g_t\}_{t\in[0,T]}$ and $Rm + C \cdot Id \in C$, and $\tau(v_0)C(v_0) > 0$.

Remark Here, we think of $Rm : \wedge_2\mathbb{R}^n \to \wedge_2\mathbb{R}^n$ and $Id : \wedge_2\mathbb{R}^n \to \wedge_2\mathbb{R}^n$ so that their sum makes sense.

Theorem 12.4 (L.)**.** Theorem 12.3 holds without assuming anything when $C = C_2,\ C_3$

Remark In the above theorem, non-collapsing (i.e. volume bound) is important. Yi constructs a counter example of a shrinking sphere bundle (see 28) In this example, $\mathrm{Ric} \geq -\epsilon$

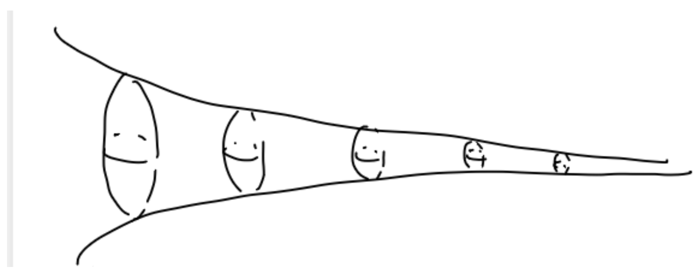

Figure 28

Conjecture 12.4.1. For (M^3, g) complete, $\mathrm{Ric} \geq 0$, then $\exists \{g_t\}_{t\in[0,T]}$ a complete Ricci Flow such that $g_0 = g$

Theorem 12.5 (L.)**.** The conjecture is true modulo completeness assertion

Remark Yi says he idea is to run Singular Ricci Flow and then use the $\mathrm{Ric} \geq 0$ assumption to prevent the formations of singularities. Once the flow exists, $\mathrm{Ric} \geq 0$ is preserved.

Corollary 12.5.1 (A gap theorem)**.** Let $C = C_1,\ C_2,\ C_3,\ C_4$. For all $D > 0$, $v_0 > 0$, there exists $\epsilon(D, v_0) > 0$ such that if (M^n, g) closed, $\mathrm{diam}(M) \leq D$, $\mathrm{vol}(M) \geq v_0$, $Rm + \epsilon Id \in C$, then M admists a metric $\tilde{g}$ such that $Rm_{\tilde{g}} \in C$

Proof: Suppose not, then there exists $\{(M_k, g_k)\}$ and $\epsilon_k \to 0$ such that $Rm_{g_k} + \epsilon_k Id \in C$, but M_k does not have a metric such that $Rm \in C$. The above theorems imply that $\exists g_{k,t}$ a Ricci Flow for $t \in [0,T]$. Moreover, $|Rm|_{g_{k,t}} \leq \frac{C(v_0)}{t}$ (also obtained in out theorems), so that

$$(M_k, g_{k,t}) \xrightarrow{\text{Cheeger-Gromov-Hamilton}} (M_\infty, g_{\infty,t}) \qquad t \in (0, \tau]$$

Moreover

$$Rm_{g_{k,t}} + C\epsilon_k Id \in C \implies Rm_{g_{\infty,t}} \in C$$

so M_∞ has a metric such that $Rm \in C$. Because M_k is diffeomorphic to M_∞ for large k we see that we've found such a metric in C on M_k, a contradiction.

12.2 Geometric Compactness Theorem

We define **Gromov-Hausdorff distance**. Let (Z, d_z), metric space and $X_1, X_2 \subseteq Z$, then the hausdorff distance between them is

$$d_H(X_1, X_2) = \inf\{r > 0 \mid B_r(X_1) \supseteq X_2,\ B_r(X_2) \supseteq X_1\}$$

There's a remark that this can be thought of as a min-max characterization

$$d_H(X_1, X_2) = \inf_{p \in X_1} \sup_{q \in X_2} d(p, q)$$

or something similar.

Now let (X_1, d_1), (X_2, d_2) be two metric spaces. Then

$$d_{GH}(X_1, X_2) = \inf_{\substack{\varphi_i : X_i \to Z \text{ isometric embedding} \\ \text{from } X_i \text{ to a metric space } Z}} d_H(\varphi_1(X_1), \varphi_2(X_2))$$

Now let

$$\mathbb{M} = \text{isometry class of all compact separable metric spaces}$$

Theorem 12.6. $(\mathcal{M}, d_{GH})$ is a separable and complete metric space.

Proof: We go through the metric space requirements

1. We show that $d_{GH}(X, Y) = 0 \implies (X, d_x) \cong (Y, d_Y)$. To see this, we have that

$$d_H^Z(X, Y) \leq i^{-1} \to 0$$

for all i. This means there exists $I_i(x) \in Y$ such that $d(x, I_i(x)) \leq i^{-1}$, and there exists $J_i(y) \in X$ such that $d(y, J_i(y)) \leq i^{-1}$. This tells us that

$$d(I_i(x_1), I_i(x_2)) \leq d(x_1, x_2) + 2i^{-1}$$

for all $x_1, x_2 \in X$. Similar for J and Y (see 29) In particular, $d(x, J_i(I_i(x)) \leq 2i^{-1}$. By a diagonalization

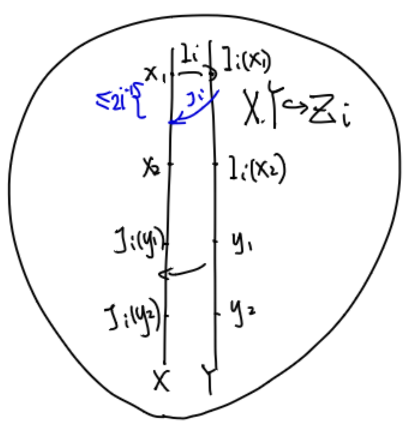

Figure 29

argument, for a dense countable subset $A \subseteq X$ (using separability of X) such that

$$I_i \to I : A \to Y$$

How to do this? For each $x_i \in A$, consider $I_i(x_j) \in Y$. Using compactness of Y (without thinking about the ambient $Y \subseteq Z_i$), we can take (along a subsequence) $I_i(x_j) \to I(x_j) \in Y$. Thus I is defined for A. Now if we extend $I : X \to Y$, distance decreasing extension (and the same for J), we have that

$$d(x, J(I(x)) = 0$$

i.e. I is an isometry with $J = I^{-1}$.

2. Triangle inequality - exercise
3. Completeness - Let $\{X_i\}$ cauchy. We find metric spaces $\{Z_{i,i+1}\}$ such that X_i, X_{i+1} isometrically embedd in $Z_{i,i+1}$. Now we glue together $Z_{i-1,i}$ and $Z_{i,i+1}$ along for all i to get a limiting space Z that isometrically contains all X_i. In Z, we actually have hausdorff convergence, and completeness under the hausdorff distance gives us a space in the limit.
4. Separable - let
$$S = \{(X, d) \in M \mid |X| < \infty, \qquad d_x \text{ takes rational values}\}$$
This is clearly countable, and to see density, we take any X compact, then approximate X by an ϵ-net. Using compactness we get a finite cover.

12.2.1 Examples of G-H convergence

- Let (M, g) riemannian manifold. Let X_i be an approximating ϵ_i-net of M (finite sets!). Then this converges in the GH sense back to (M, g). Note that this works even when (M, g) is smooth, so smoothness (or lack of it) not really preserved by GH (see 30)

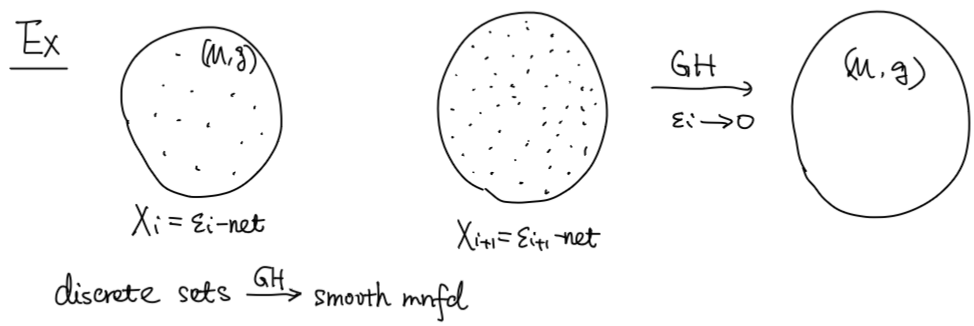

Figure 30

- Let
$$X_i = S^1(1) \times S^1(1/i) \xrightarrow{GH, i\to\infty} S^1(1)$$
This is collapsing (see 31)
- Consider $S^3/\mathbb{Z}_k$ with the induced standard metric on S^3. Then there exists
$$\tau : S^3/\mathbb{Z}_k \to S^2$$
$$\tau^{-1}(p) \text{ has length } \frac{1}{2k} \to 0$$
$$S^3/\mathbb{Z}_k \xrightarrow{GH} S^2, \quad k \to \infty$$
So the topology can totally change under GH convergence, because the first fundamental grouops are all different

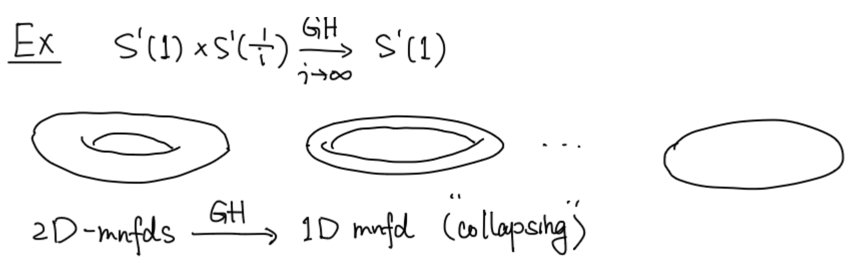

Figure 31

- Consider the berger sphere $S^3(\epsilon) \to S^2$. Then

$$S^3(i^{-1}) \xrightarrow{GH} S^2$$

13 Lecture 13: 11-10-22

Today:

- Pointed Gromov-Hausdorff convergence
- Smooth Cheeger-Gromov Convergence

Recall for $D > 0$, $\overline{N} : \mathbb{R}^+ \to \mathbb{N}$, set

$$\mathbb{M}(D, \overline{N}) = \{(X, d) \in \mathcal{M} \mid \operatorname{diam}(X, d) \leq D, \; N^{(X,d)}(r) \leq \overline{N}(r), \quad \forall r\}$$

where $N^{(X,d)}$ is the minimal number of $\{r-\text{net}\}$. Formally, $N^{(X,d)}$ - have $\{x_1, \ldots, x_N\}$ such that $\cup_i B(x_i, r) \supseteq X$, then $\{x_i\}$ is an "r-net" and $N^{(X,d)}(r)$ is the minimal such N for given r

Theorem 13.1. $\mathbb{M}(D, \overline{N})$ is compact w.r.t d_{GH}, i.e. closed and totally bounded

Corollary 13.1.1. $\forall n \in \mathbb{N}$, D, $k > 0$. Then

$$\{(M^n, d_g) \mid \operatorname{diam}_g(M) \leq D, \;\; \operatorname{Ric} \geq -kg\}$$

(where M is compact and g is riemannian metric) is precompact in $\mathbb{M}$.

Note that we may not have a smooth object in the limit.
Proof: Choose $\{x_1, \ldots, x_n\} \subseteq M$ to be a maximal set such that $B(x_i, r/2)$ are pairwise disjoint. Then

$$M = \cup_{i=1}^n B(x_i, r)$$

Then

$$N \leq \frac{\operatorname{Vol}(M, g)}{\min_{1 \leq i \leq N} \operatorname{Vol}(B(x_i, r/2))} = \frac{\operatorname{Vol}(B(x_j, 0))}{\operatorname{Vol}(B(x_j, r/2))} \leq C(n, D, k, r)$$

assuming that the minimum ball volume is achieved at j for some j. The last inequality follows by a volume comparison theorem. Here, we note that the constant does not depend on the manifold itself, but rather the lower bound for Ricci. Now our theorem gives precompactness. $\square$

Definition 13.2. A metric space (X, d) is a "length space" if

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma : [0, 1] \to X, \qquad \gamma \in C^0, \qquad \gamma(0) = x, \;\; \gamma(1) = y\}$$

where the length of a continuous curve is the sup of the partitions. I.e.

$$\ell(\gamma) = \sup_{P \in \mathcal{P}} \sum_{t_i \in P} d(\gamma(t_i), \gamma(t_{i+1}))$$

where $\mathcal{P}$ is the collection of all partitions of $[0, 1]$.

Theorem 13.3. (X, d) is a length space if and only if $\forall x, y \in X$, $\forall \epsilon > 0$, there exists $z \in X$, $z \neq x$, such that

$$d(x, z) \leq \frac{1}{2} d(x, y) + \epsilon$$

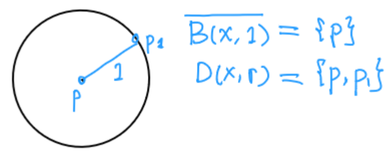

Figure 32

Examples: Let X be the unit circle of radius 1 union the origin (see 32) This is because $d(x,0) = 1$ and we cannot find a continuous curve from the origin to the unit circle. On the other side of the theorem, let x be the origin, then $d(x,z) = 1$ for all $z \neq x$ and so the above would give

$$1 \leq \frac{1}{2} + \epsilon$$

which is false for ϵ small.

Let $X = \mathbb{R}^2 \backslash \{(0,0)\}$ - then this is a length space.

Lemma 13.4. If (X,d) is a length space, then

$$\overline{B(x,r)} = D(x,r)$$

where

$$B(x,r) = \{y \in X,\ d(x,y) < r\}$$
$$D(x,r) = \{y \in X,\ d(x,y) \leq r\}$$

Ex: $\overline{B(x,1)} = \overline{\{0\}} = \{0\}$, and $D(x,1) = X$ for $X = S^1 \backslash \{0\}$.

Definition 13.5. Let (X_i, d_i, x_i) pointed complete, metric length space, $i \leq \infty$. Suppose all the bounded closed subsets are compact. We write

$$(X_i, d_i, x_i) \xrightarrow{PGH} (X_\infty, d_\infty, x_\infty)$$

if $\forall r > 0$

$$(D(x_i, r), d_i) \xrightarrow{GH} (D(x_\infty, r), d_\infty), \qquad i \to \infty$$

RemarkHere, we assume that X_∞ exists and is a length space.

RemarkSimilarly we can find a correspondence (Z, d_Z) such that $(\{\varphi_i\}_{i=1}^\infty, Z)$ and $\varphi_i : X_i \to Z$ isometric embedding such that for all r

$$\varphi_i(D(x_i, r)) \xrightarrow{H} \varphi_\infty(D(x_\infty, r))$$

see (33)

Theorem 13.6. Let (X_i, d_i, x_i) be a pointed complete length space such that bounded subsets are compact. Suppose $\exists \{r_k\} \to \infty$ such that

$$(D(x_i, r_k), d_i) \xrightarrow{GH} (X_{\infty,k}, d_{\infty,k})$$

then

$$(X_{\infty,k}, d_{\infty,k}) \overset{isometric, embed}{\hookrightarrow} (X_{\infty,k'}, d_{\infty,k'}) \qquad \forall k \leq k'$$

and there exists $(X_\infty, d_\infty, x_\infty)$ such that

$$(X_i, d_i, x_i) \xrightarrow{PGH} (X_\infty, d_\infty, x_\infty)$$

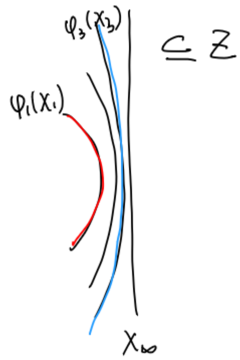

Figure 33

Remark This gives a sufficient condition for getting PGH.

Example. Let $X = \bigvee_{i=1}^{\infty}[0,1]_i$. Then X is bounded, closed, but not compact. Here

$$d_X(A,B) = d_X(A,0) + d_X(B,0)$$

if A, B are not on the same interval (see)

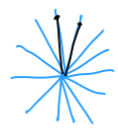

Figure 34

Corollary 13.6.1. $\forall k \in \mathbb{R}$, $\forall n \in \mathbb{N}$

$$\{(M^n, d_g, p) \mid (M,g) \text{ complete }, \quad \text{Ric} \geq -kg\}$$

is precompact in the space of pointed, complete length space whose bounded closed subsets are all compact

Proof: Follows from the compact case of this theorem. $\square$

RemarkThis corollary holds true when replacing k by a function $k(r)$, $r = d(\cdot, p)$.

Example. Let (M^n, g, p) and $\{\lambda_i\} \to \infty$. Let $(M^n, \lambda_i^2 g, p) \to (M^n, d_{\lambda_i^2 g}, p)$ a complete length space (bounded subsets are compact). And now

$$(M^n, d_{\lambda_i^2 g}, p) \xrightarrow{PGH} (T_p M^n, g_{euc}, p)$$

Similarly, (35) if we have a manifold with a cone point and we consider the sequence

$$(M^n, d_{\lambda_i^2 g}, p) \xrightarrow{PGH} (C^n, dr^2 + cr^2 dg_{N^{n-1}}, p)$$

i.e. we get convergence to a cone. Note that the tangent cones at a point are not necessarily metric cones, and not necessarily unique (i.e. may depend on $\{\lambda_i\}$).

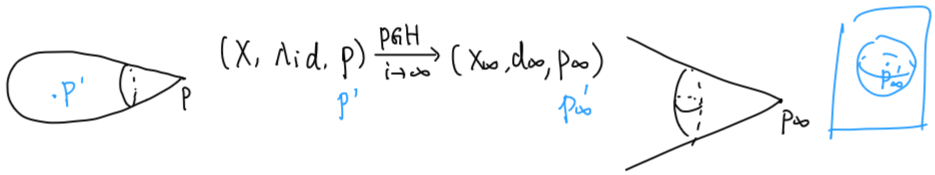

Figure 35

Example. Cigar soliton (call it (M,g)). If we choose $\{p_i\}$ arbitrary sequence of points

$$(M,g,p_i) \xrightarrow{PGH} \begin{cases} (M,g,p^1) & \{p_i\} \text{ bounded} \\ (\mathbb{R}\times S^1, g_{\mathbb{R}\times S^1}) & \{p_i\} \to \infty \end{cases}$$

see 46

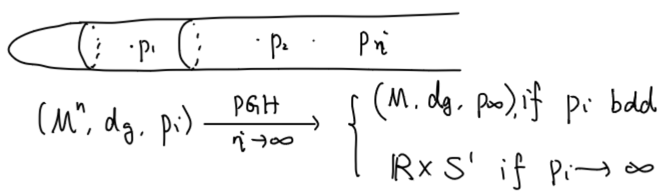

Figure 36

13.1 Smooth Cheeger-Gromov Convergence

Theorem 13.7. Let $(M_i^n, d_{g_i}, p_i) \xrightarrow{PGH,Z,i\to\infty} (X_\infty, g_\infty, p_\infty)$ where each (M_i, d_{g_i}) is a complete RM and (X_∞, g_∞) is a complete length space. Then we say convergence is smooth at $q_\infty \in X_\infty$ if $\exists \{q_i\} \in X_i$ with

$$q_i \xrightarrow{Z} q_\infty$$

where Z is the larger ambient space where this whole correspondence occures. Moreover, there exists $r, V > 0$, $C_m > 0$ such that

1. $Vol(B(q_i, r)) \geq C > 0$
2. $|\nabla^m Rm| \leq C_m$ in $B(q_i, r)$

Furthermore, if we define

$$R^* = \{q_\infty \in X_\infty \;:\; \text{convergence } q_i \xrightarrow{Z} q_\infty \text{ is smooth at } q_\infty\}$$

Then R^* is n-dimensional and has a smooth RM, (R^*, g^*) such that $(R^*, d_{g^*}) \overset{Id}{\hookrightarrow} (X_\infty, d_\infty)$ is a local isometry

Remark Consider $\{M_i\}$ a sequence of tori getting progressively more pinched which converges in a PGH sense to a sphere with two points touching (i.e. fully pinched torus at a point), . Then R^* is everything except the point where we pinch. But

$$(R^*, d_{g^*}) \hookrightarrow (X_\infty, d_\infty)$$

is **not** an isometry. Topologically

$$R^* \cong (0,1) \times S^1$$

see 37

Remark Without the first condition (volume preservation), we can construct the following perverse example of

$$S^1(1) \times S^1(\epsilon) \xrightarrow{S}{}^1$$

and $R^* = \emptyset$.

Without the second condition (bounded curvature), we can let $\{M_i\}$ be a sequence of wedges smoothed out at the vertex, converging to a wedge/cone. In this case

$$R^* = X_\infty \backslash \{\text{cone point}\}$$

also see 37

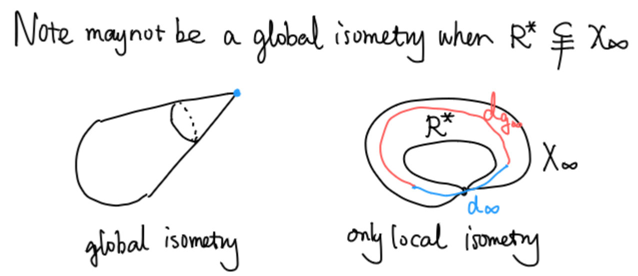

Figure 37

14 Lecture 14: 11-15-22

14.1 Smooth Cheeger-Gromov Convergence

Recall from last time:

Theorem 14.1. Assume $\{(M_i^n, d_{g_i}, p_i)\} \xrightarrow{PGH, Z,\ i\to\infty} (X_\infty, d_\infty, p_\infty)$ and

$$R^* = \{\text{smooth points}\} \ \text{ on } B_{g_i}(q_i, r)$$

then

1. $R^* \subseteq X$ is open, and there exists a riemannian metric g_∞ such that $(R^*, g_\infty) \overset{id}{\hookrightarrow} (X_\infty, d_\infty)$ is a local isometry.

2. There exists an open subset $U_1 \subseteq \cdots \subseteq U_n \subseteq R^*$, such that $\bigcap_{i=1}^\infty U_i = R^*$ and a diffeo

$$\psi_i : U_i \to V_i \subseteq M_i$$

   such that

   (a) $\psi_i^* g_i \xrightarrow{C^\infty_{loc}, i\to\infty} g_\infty$ on R^*

   (b) $\psi_i^* \xrightarrow{Z,\ i\to\infty} id$

Recall that $q_\infty \in R^*$ is smooth there exists $\{q_i \in M_i\}$ such that

$$q_i \xrightarrow{Z} q_\infty$$

and

$$\text{Vol}(B_{g_i}(q_i, r)) \geq V > 0 \tag{15}$$
$$|\nabla^m Rm| \leq C_m \tag{16}$$

Proof: First, for 1., note that the Gromov theorem (We haven't proved this) and equation (15) and (16) give that

$$\mathrm{inj}(q_i) \geq c > 0$$

i.e. the injectivity radius is bounded from below. Thus

$$\underbrace{(D(q_i, C/2), d_i)}_{\text{topological ball}} \xrightarrow{GH, i\to\infty} (D(q_\infty, c/2), d_\infty))$$

Then there exists local coordinates near $q_i \in M$ and

$$\vec{x}_i : B(q_i, \gamma_0) \to \mathbb{R}^n$$
$$q_i \to 0$$

such that

1. $B(\vec{0}, r_1) \subseteq \vec{x}_i(B(q_i, r_0))$
2. $g_{ij} = g_{ij,\mathrm{st}} dx_i^s dx_j^t$

see 38

Recall the three types of coordinates

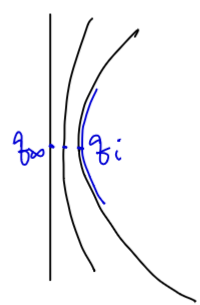

Figure 38

1. Exponential coordinates
$$C'_m = C'_m(C_0, C_1, \ldots, C_m)$$
where $\{C_i\}$ are our curvature bounds
2. Distance coordinates give
$$C'_m = C'_m(C_0, C_1, \ldots, C_{m-1})$$
(Here Yi draws a picture explaining this, essentially you have a base point q_i, and then you fix n points $x_1, \ldots, x_n$ in the ambient space, and we map
$$q \mapsto (d(q, x_1), \ldots, d(q, x_n))$$
)
3. Harmonic coordinates gives
$$||C'_m||_\alpha = C'_m(C_0, C_1, \ldots, C_{m-1})$$

Arzela-Ascoli and our lemma give that

$$g_{i,st} \xrightarrow{C^\infty} g_{\infty,st} \text{ on } B(\vec{0}, \frac{r_1}{2})$$

Recall

$$(D(q_i, C/2), d_i) \xrightarrow{GH, i\to\infty} (D(q_\infty, c/2), d_\infty))$$

then GH limit uniqueness implies that

$$(D(q_\infty, r_2), d_\infty) \overset{isom}{\cong} (D(q_\infty, r_2), d_{g_\infty})$$

□

we can find maps $\psi_i : D(q_\infty, r_2) \to M_i$ diffeos onto the image such that (a) and (b) in our initial theorem

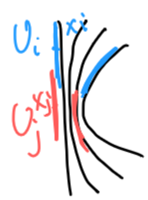

Figure 39

statement (see initial theorem conditions in 2.) are true on $D(q_\infty, r_2)$. This proves 1. (see 40) in our theorem.

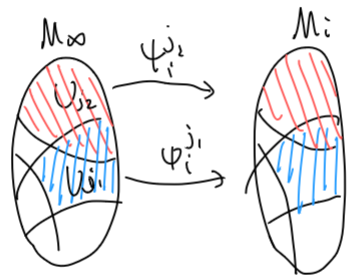

Figure 40

Proof: of 2. First, we can find $\{x_1, x_2, \dots\} \subseteq R^*$ with $\{U^j\}$ neighborhoods of x_k. Form a locally finite cover of R^* and there exists

$$\psi_i^j : U^j \to V_i^j \subseteq M_i$$

such that (a) and (b) are true on U^j. Now let $X_i^j : V_i^j \to U^j$ be the inverse of ψ_i^j (see 41)

We now claim that

$$X_i^{j_2} \circ \psi_i^{j_1} \xrightarrow{C^\infty_{loc}, i\to\infty} id \text{ on } U^{j_1} \cap U^{j_2}$$

$$(X_i^{j_2} \circ \psi_i^{j_1})^* g_\infty \xrightarrow{C^\infty_{loc}, i\to\infty} g_\infty$$

hint: $\psi_i^{j_1}, \psi_i^{j_2}$ are almost isometries. Same with $X_i^{j_1}, X_i^{j_2}$. Then

$$\psi_i^j \xrightarrow{Z} Id$$

$$X_i^j \xrightarrow{Z} Id$$

$$\implies X_i^{j_2} \circ \psi_i^{j_1} \xrightarrow{Z} Id$$

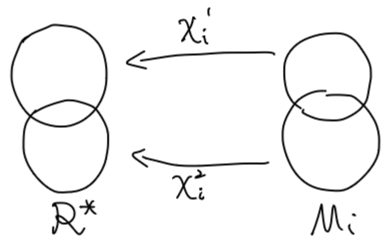

Figure 41

Proof: Next: "glue-up" the maps $X_i^j : V_i^j \to U^j$ for a fixed i. Let $\{\eta^j\}_{j=1}^{\infty}$ be a partition of unity surbordinate to $\{U^j\}_{j=1}^{\infty}$.

Claim: There exist smooth maps $\sigma_k : [0,1]^k \times \Delta_k \to R^*$, where Δ_k is a diagonal neighborhood of $(R^*)^k = R^* \times \cdots \times R^*$ and the diagonal is just $\{(x, \ldots, x) \mid x \in R^*\}$, such that

$$\sigma_k(s_1, \ldots, s_k, x, \ldots, x) = x \tag{17}$$
$$\sigma_k(0, \ldots, 1, \ldots, 0, x_1, \ldots, x_j, \ldots, x_k) = x_j \tag{18}$$
$$\sigma_k(s_1, \ldots, s_{k-i}, 0, \ldots, 0, x_1, \ldots, x_k) = \sigma_{k-i}(s_1, \ldots, s_{k-1}, x_1, \ldots, x_{k-i}) \tag{19}$$

Note that for $k = 2$, σ_2 is the "mid point" of any two nearby points. Now let

$$\hat{\chi}_i(x) = \sigma_N(\eta_1(\chi_i^1(x)), \ldots, \eta_N(x_i^N(x)), x_i^1(x), \ldots, x_i^N(x))$$

where N is an integer such that

$$\eta_k(x_i^k(x)) = 0 \qquad \forall k \geq N$$

can check that $\hat{\chi}_i$ are diffeos and

$$\hat{\psi}_i = \hat{\chi}_i^{-1}$$

and $\hat{\psi}_i$ satisfy (a) and (b) from the theorem assumption. $\square$

Definition 14.2 (C^{∞}-CG convergence)**.** We say $(M, g_i) \xrightarrow{CG} (M_\infty, g_\infty)$ if there exists open $U_1 \subseteq \ldots U_n \subseteq M_\infty$, and

$$\bigcap_{i=1}^{\infty} U_i = M_\infty$$

and diffeos

$$\psi_i : U_i \to V_i \overset{\text{open}}{\subseteq} M_i$$

such that

$$\psi_i^* g_i \xrightarrow{C^\infty_{loc}} g_\infty$$
$$\psi_i^{-1}(p_i) \to p_\infty$$

So 2. from the theorem: $(M_n, g_n, p_n) \xrightarrow{CG} (R^*, g_\infty, p_\infty$

Corollary 14.2.1. Let (M_i^n, g_i, p_i) complete RM and if $\exists r > 0$ and for all $D > 0$ such that

1. $|\nabla^m Rm| \leq C_m(D)$ on $B(p_i, D)$
2. $\mathrm{Vol}(B(p_i, r)) \geq V > 0$ (non-collapsing)

implies that there exists a subsequence such that

$$(M_i, g_i, p_i) \xrightarrow{CG} (M_\infty, g_\infty, p_\infty)$$

14.2 Compactness of RF/Smooth Cheeger-Gromov-Hamilton Convergence

Setup: $(M_i, (g_{i,t})_{t\in[-T_i^-,T_i^+]}, p_i)$ pointed complete Ricci Flows, $T_i^- < 0$, $T_i^+ > 0$ and assume

$$[-T_i^-, T_i^+] \to \hat{I}_\infty$$

then let $I_\infty = \hat{I}_\infty \backslash \{\text{left end point}\}$, e.g. $\hat{I}_\infty = [0,1]$, then $I_\infty = (0,1]$ and

$$(M_i, g_{i,0}, p_i) \xrightarrow{PGH,Z,i\to\infty} (X_\infty, d_\infty, p_\infty) \tag{20}$$

and $R^* \subset X_\infty$ as the subset of smooth points of (20). Let

$$R^{**} = \Big\{ q_\infty \in R^* \mid \exists q_i \xrightarrow{Z} q_\infty \text{ s.t. } \forall [-\hat{T}^-, \hat{T}^+] \subseteq I_\infty,\ \exists C, r > 0$$
$$\text{s.t. } |Rm| \le C \text{ on } B_{g_{i,0}}(q_i, r) \times [-\hat{T}^-, \hat{T}^+] \Big\}$$

for large i.

Claim: $R^{**} \subseteq R^*$ (see 42)

Proof: Shi's estimate gies that if $|Rm| \le C$ for a small time interval locally at a point in X_∞, then

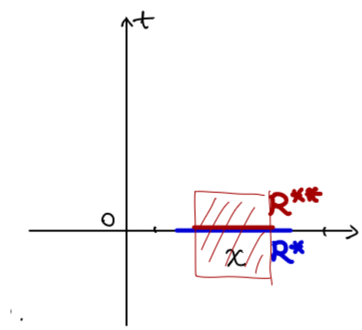

Figure 42

$$|\nabla^m Rm| \le C_m$$

Theorem 14.3. Let $\psi_i : R^* \supseteq U_i \to V_i \subseteq M_i$ and $\cup_{i=1}^\infty U_i = R^*$ be diffeos of

$$(M_i, g_{i,0}, p_i) \xrightarrow{CG} (R^*, g_\infty, p_\infty)$$

such that

$$\psi_i^* g_{i,0} \xrightarrow{C^\infty} g_\infty$$

and

$$\psi_i \xrightarrow{Z} id \qquad i \to \infty$$

then after passing to a subsequence, we have

$$\psi_i^* g_{i,t} \xrightarrow{C^\infty} g_{\infty,t}$$

a smooth ricci flow on R^{**} with $t \in I_\infty$ (where $g_{\infty,0} = g_\infty$)

Proof: Take

$$\tilde{g}_{i,t} = \psi_i^*(g_{i,t})$$

on $R^{**} \cap U_i$. And Shi's derivative estimate tells us that $\tilde{g}_{i,t}$ has bounded derivative up to any order on any compact subset of I_∞. The Arzela-Ascoli lemma now tells us that

$$\tilde{g}_{i,t} \xrightarrow{C^\infty} g_{\infty,t} \qquad \text{on} \quad R^{**} \quad \text{as } i \to \infty$$

Moreover, $\tilde{g}_{i,t}$ satisfies Ricci Flow, so does $g_{\infty,t}$. $\square$

15 Lecture 15: 11-17-22

Today

- Blow-up analysis
- Solitons

15.1 Application of compactness in blow-ups

Let $(M, \{g_t\}_{t\in[0,T)})$ compact RF, $T < \infty$ maximal existence time. Choose $(x_i, t_i) \in M \times [0, T]$, $t_i \uparrow T$ and

$$\max_{M\times[0,t_i]} |Rm| \le C \cdot |Rm|(x_i, t_i) := CQ_i \to \infty$$

where C is independent of i. Let

$$g_{i,t} = Q_i g_{Q_i^{-1}t+t_i}, \qquad t \in [-Q_i t_i, 0]$$

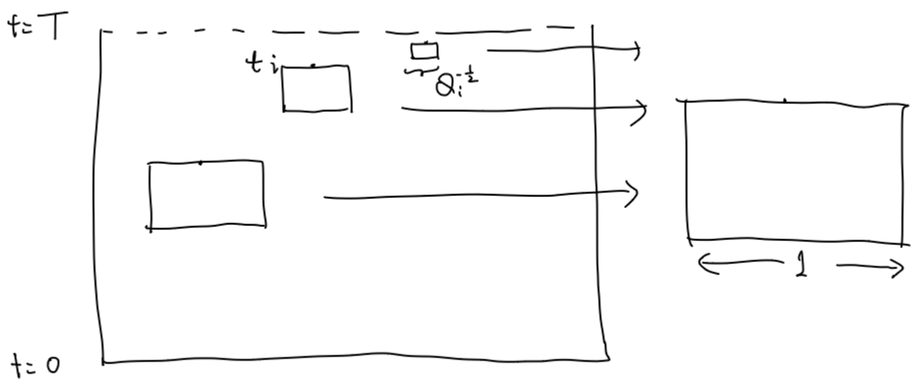

Figure 43

Assume for some $r > 0$, $v > 0$

$$\mathrm{Vol}(B_{g_{i,t}}(x_i, r)) \ge vr^3$$

(We will confirm this later using Perlman's no-local collapsing theorem). So applying the convergence theorem, we have

$$(M, g_{i,t}, x_i) \xrightarrow{CGH} (M_\infty, g_{\infty,t}, x_\infty) \qquad \text{RF, complete } t \in (-\infty, 0]$$

Example. Consider two separate sequences, one converging to the sphere, and one converging to the Bryant soliton. These are called the "neck-pinch" and the "degenerate neck-pinch" (see 44)

Theorem 15.1 (Perelman-Brendle)**.** The only possible singularity models for 3D compact Ricci Flows are S^3/Γ, $S^2 \times \mathbb{R}$, $S^2 \times \mathbb{R}/\mathbb{Z}_2$, and the Bryant soliton.

15.2 Solitions (Revisit)

Definition 15.2. We say that a triple (M, g, X) is a soliton if

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g - \frac{\lambda}{2} g = 0, \qquad \lambda \in \mathbb{R}$$

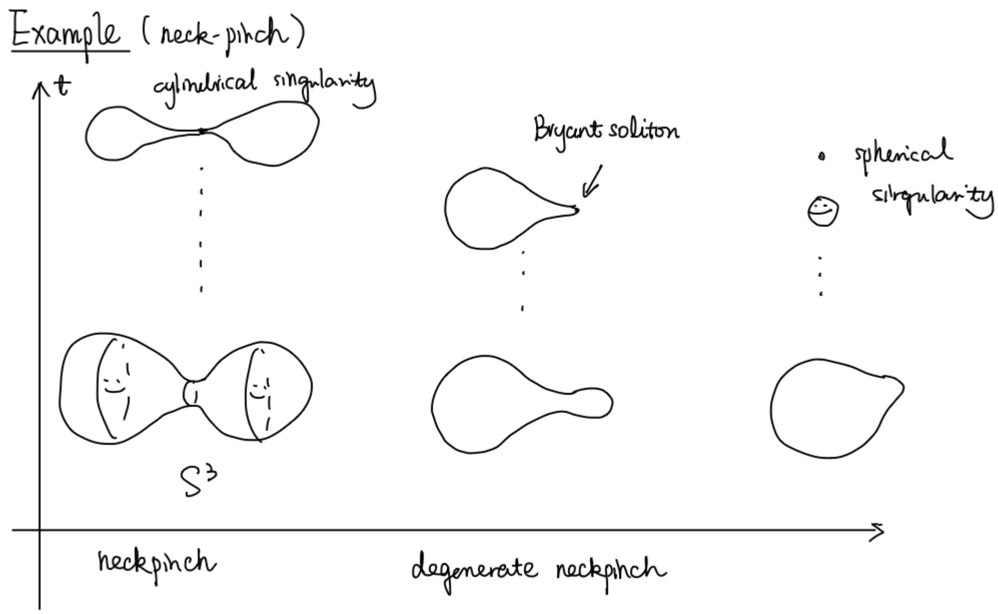

Figure 44

if $X = \nabla f$ then this is a gradiaent solution and the above becomes

$$\mathrm{Ric} + \nabla^2 f - \frac{\lambda}{2} g = 0$$

Moreover

$$\begin{aligned} \text{shrinking } \lambda > 0 \\ \text{steady } \lambda = 0 \\ \text{expanding } \lambda < 0 \end{aligned}$$

Now consider the diffeo

$$\phi_t := \text{flow of } \begin{cases} -\frac{1}{\lambda t} & t < 0, \qquad \text{when } \lambda > 0 \\ X & t \in \mathbb{R}, \qquad \lambda = 0 \\ -\frac{1}{\lambda t} X, & t > 0, \qquad \lambda < 0 \end{cases}$$

Let

$$g_t = \begin{cases} -\lambda t \phi_t^* g & t < 0, \quad \lambda > 0 \\ \phi_t^* g & t \in \mathbb{R}, \quad \lambda = 0 \\ -\lambda t \phi_t^* g, & t > 0, \quad \lambda < 0 \end{cases}$$

Theorem 15.3. $\{g_t\}$ is a Ricci Flow.

Proof: Assume $\lambda > 0$, then

$$\begin{aligned} \frac{d}{dt}(-\lambda t)\phi_t^* g &= (-\lambda)\phi_t^* g - \lambda t \partial_t \phi_t^* g \\ &= (-\lambda)\phi_t^* g + \phi_t^*(\mathcal{L}_X g) \\ &= \phi_t^*(-\lambda g + \mathcal{L}_X g) \\ &= \phi_t^*(-2\mathrm{Ric}) \\ &= -2\mathrm{Ric}(\phi_t^* g) \\ &= -2\mathrm{Ric}(g_t) \end{aligned}$$

Example. All Einstein manifolds, i.e. $\mathrm{Ric} = \frac{\lambda}{2} g$ and let X be a killing field, then

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \mathrm{Ric} = \frac{\lambda}{2} g$$

Example. Gaussian gradient soliton: $(\mathbb{R}^n,\ g_{euc},\ f = \frac{\lambda}{4}|x|^2)$ euclidean coordinates. Note that

$$\nabla^2 |x|^2 = 2 g_{euc}$$

Check

$$\begin{aligned}\mathrm{Ric} + \nabla^2 f - \frac{\lambda}{2} g_{euc} &= \nabla^2 f - \frac{\lambda}{2} g_{eucl} \\ &= \frac{\lambda}{4} 2 g_{euc} - \frac{\lambda}{2} g_{euc} \\ &= 0\end{aligned}$$

see 45

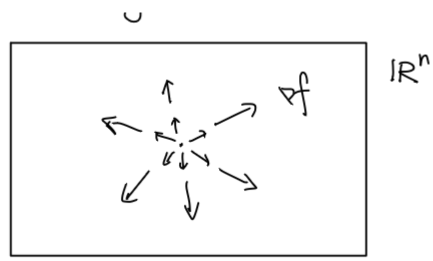

Figure 45

Example. Hamilton-Cigar soliton - on $\mathbb{R}^2$, with $g = dr^2 + h(r)^2 d\theta^2$, and f. h and f have explicit formulae, but not stated here. This is a steady gradient soliton with $h(r) < \infty$ as $r \to \infty$. Here $k > 0$, i.e. positive gauss curvature everywhere. Moreover for $\{p_i\}$ a sequence with $r \to \infty$

$$(M, g, p_i) \xrightarrow{CG} \mathbb{R} \times S^1$$

In 2D, steady solitons are either

- flat
- Cigar soliton

due to Hamilton, Seseum, . . . see 46

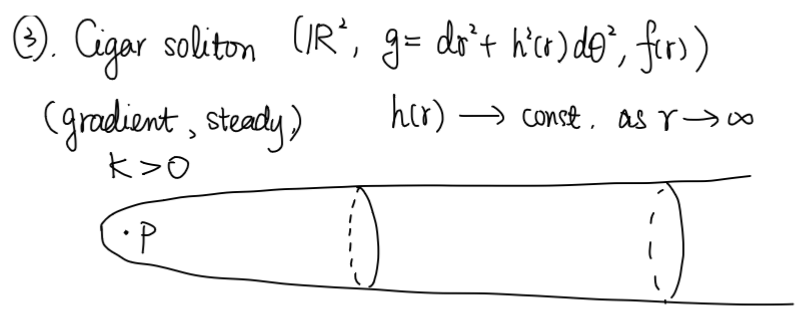

Figure 46

Example. In $n \geq 3$, the rotationally symmetric soliton is called the Bryant soliton (due to R. Bryant). We work on $\mathbb{R}^n$ and

$$g = dr^2 + h(r)^2 g_{S^{n-1}}$$

and an f. see 47

Both h and f unique. This is not like a cigar soliton and the graph of the Bryant soliton is like a polynomial

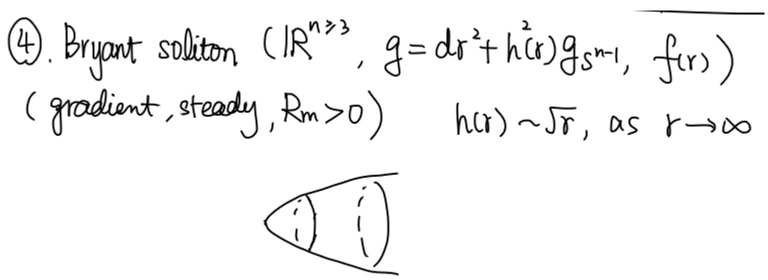

Figure 47

of degree two. If you measure the diameter of the sphere at distance r, we have that $h(r) \sim \sqrt{r}$. Moreover, $Rm > 0$ everywhere, and also for $\{p_i\}$ a sequence tending to $r = \infty$

$$(M, g, p_i) \xrightarrow{CG} \mathbb{R}^n$$

we can also get nice convergence if we rescale

$$(M, R(p_i)g, p_i) \xrightarrow{CG} S^{n-1} \times \mathbb{R}$$

Finally, if we take $\{\lambda_i\} \to 0$ and take the distinguished parabolic point, p, on the Bryant soliton, we have that

$$(M, \lambda_i g, p) \xrightarrow{GH} \mathbb{R}^+ = [0, \infty)$$

Example. In generalized cylinders

$$(S^k \times \mathbb{R}^{n-k}, g = g_{S^k} + g_{\mathbb{R}^{n-k}}, f = \frac{\lambda}{4}|\overline{x}|^2)$$

where $\overline{x}$ is just $n-k$ coordinates on $\mathbb{R}^{n-k}$ see 48

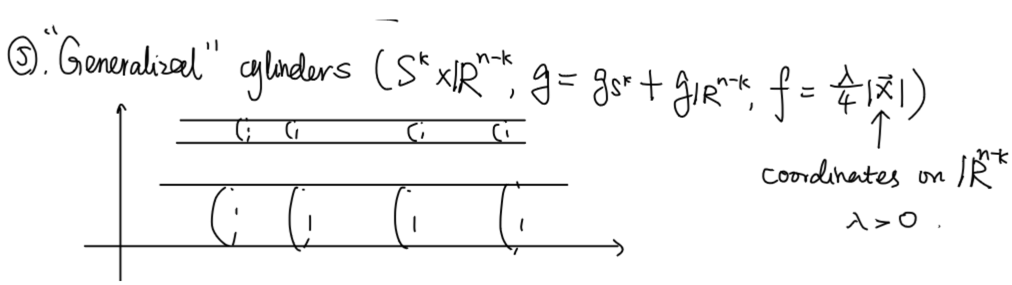

Figure 48

In 2D, shrinking soliton converges to S^2/Γ, $\mathbb{R}^2$ (Gaussian shrinking).
In 3D, shrinking soliton converges to S^3/Γ, $S^2 \times \mathbb{R}/\Gamma$, and $\mathbb{R}^3$ (Gauss).
We have yet to classify steady solitons, though Hamilton conjectured that there is at least one more besides the 3D Bryant soliton and $\mathbb{R} \times$ Cigar. see 49

Recently Yi found a family of steady solitons called "flying wings", the difference is that

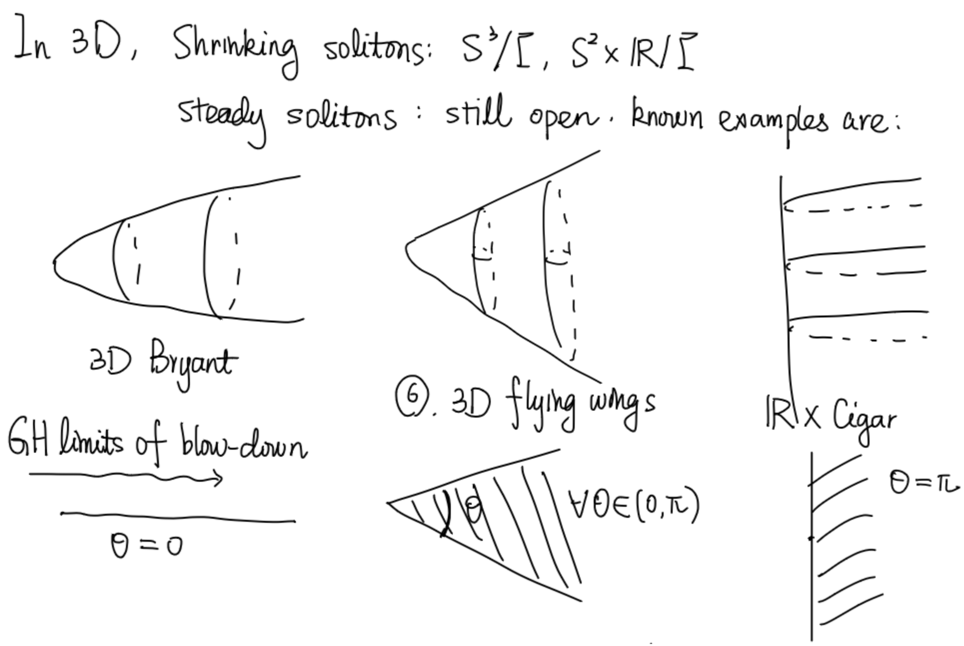

Figure 49

$$\text{3D Bryant soliton} \xrightarrow{\text{Blow down, GH}} \mathbb{R}^+$$
$$\text{Flying Wing} \xrightarrow{\text{Blow down, GH}} \text{(cone)}$$
$$\mathbb{R} \times \text{cigar} \xrightarrow{\text{Blow down, GH}} \mathbb{R} \times \mathbb{R}^+$$

Conjecture 15.3.1. Are these all the 3D steady gradient solitons?

We've shown the existence of at least one more with the flying wings. Moreover this example has $\mathbb{Z}_2 \times O(2)$ symmetry, but all 3D steady gradient solitons have $O(2)$ symmetry. Uniqueness is still open

Example. Danielle: $\forall(N^{n-1}, h)$ with $Rm > Id$, $\exists!(M^n, g, f)$ expanding with $Rm > 0$ and $R(p) = 1$, $\nabla f(p) = 0$ such that (M, g) asymptotic to $C(N)$, the metric cone over N, $dr^2 + r^2 h$, see 50

We have

$$(M, \lambda_i g, p) \xrightarrow{GH} C(N)$$

Moreover, let $\{g_t\}$ be a RF associated to g, i.e.

$$g_t = (-\lambda t)\phi_t^* g$$

Then

$$(M, g_t, p) \xrightarrow{GH,\, t\downarrow 0} (C(N), p)$$

As an aside: Metric comparison geometry say that for (M, g, p) with $\sec \geq 0$, $\lambda_i \to 0$, then

$$(M, \lambda_i g, p) \xrightarrow{GH} (P(X), p_\infty)$$

where $P(X)$ is a cone. Here X is the class of geodesic rays on M, which admits a metric

Example. Due to Yi, in $n \geq 4$, there is a family of $\mathbb{Z}_2 \times O(n-1)$-symmetric steady gradient solitons for $Rm > 0$, see 51

Let

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-2} > \lambda_{n-1} = \lambda_n$$

Let $\alpha = \frac{\lambda_2}{\lambda_1}$ Here, if we take $p_i \to 0$ then

$$(M, R(p_i)g, p_i) \to \mathbb{R} \times \text{Bry}^{n-1} \text{ or } \mathbb{R}^2 \times S^{n-2}$$

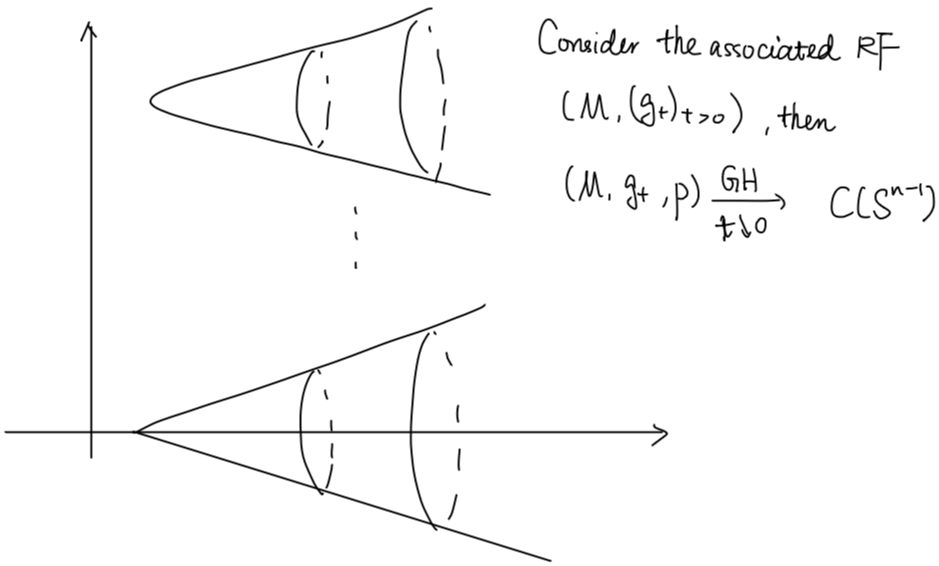

Figure 50

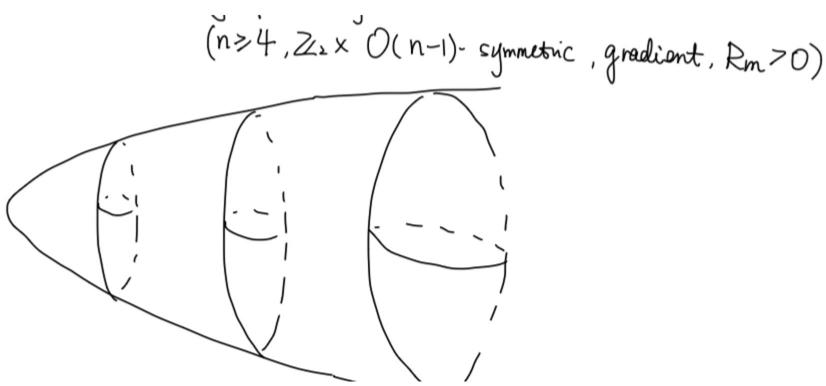

Figure 51

Question: What is $(M, \lambda_i g, p)$ as $\lambda_i \to 0$ and $i \to \infty$? If $\alpha = 1$, we get the bryant soliton. If $\alpha = 0$, we get the $\mathbb{R} \times \mathrm{Bry}^{n-1}$. What about other values of α?

16 Lecture 16: 11-29-22

Today we'll discuss heat flows and conjugate heat flows

16.1 $\mathcal{F}$-functional and λ-invariants

We try to view the ricci flow as a gradient flow of some functional. Let $\{g_t\}_{t \in I}$ be a Ricci flow on M, compact. Let $u, v \in C^2(M \times I)$. Recall that the heat equation is given by

$$\Box u := (\partial_t - \Delta_{g_t} u) = 0$$

We can define the **conjugate heat equation** by

$$\Box^* u = (-\partial_t - \Delta_{g_t} + R_{g_t})u = 0$$

Note that if $u, v \in C_c^2(M \times [t_1, t_2])$ (i.e. $u(\cdot, t),\ v(\cdot, t)$ vanish on ∂M), then

$$\begin{aligned}\int (\Box u) \cdot v dg_t - \int u \Box^* v dg_t &= \int (\partial_t - \Delta) u \cdot v - \int u(-\partial_t - \Delta + R) v \\ &= \int \left[\partial_t(uv) - (\Delta u) v + u(\Delta v) - Ruv\right] dg_t \\ &= \int \partial_t(uv) - Ruv) \\ &= \partial_t \int uv dg_t\end{aligned} \tag{21}$$

where the third line follows from integration by parts, and the fourth line follows because the derivative of the volume form with respect to the metric is the scalar curvature. We phrase this as the following theorem

Theorem 16.1. For $u, v \in C_c(M \times I)$, we have

$$\partial_t \int_M uv dg_t = \int (\Box u) v dg_t - \int u(\Box^* v) dg_t$$

We can also integrate the above and get

$$\int uv dg_t \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \int_M \left[(\Box u) v dg_t - u(\Box^* v) dg_t\right]$$

16.2 Heat Kernels

In order to understand the heat equation, we also want to understand its underlying kernel. Define

$$K(x, t; y, s) > 0, \qquad x, y \in M, \qquad s < t$$

such that

$$\begin{gathered}\Box_{(x,t)} K(x, t; y, s) = 0 \\ \lim_{t \downarrow s} k(\cdot, t; y, s) = \delta_{(y,s)}\end{gathered}$$

see 52

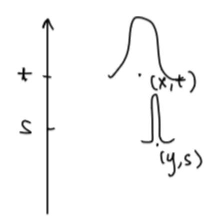

Figure 52

Proposition 3 (Reproduction Formula)**.** If $\Box u = 0$, then for $s < t$, we have

$$u(x, t) = \int_M K(x, t; y, s) u(y, s) dg_s$$

One could view this as a defining property of the heat kernel.

Similarly, we can define a kernel for the conjugate heat equation, i.e. a function

$$K^*(x, t; y, s) > 0$$

such that

$$\Box^*_{(y,s)} K^*(x,t;y,s) = 0$$
$$\lim_{s \uparrow t} K^*(x,t;y,s) = \delta_{(x,t)}$$

see 53 Then the corresponding reproduction formula is

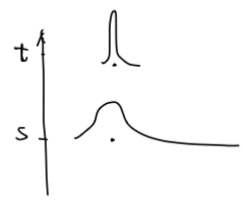

Figure 53

Proposition 4 (Reproduction Formula)**.** If $\Box^* v = 0$, then for $s < t$

$$v(y,s) = \int_M K^*(x,t;y,s) v(x,t) dx$$

Finally we have

Lemma 16.2.

$$K^*(x,t;y,s) = K(x,t;y,s)$$

Proof: Consider

$$F(\tau) = \int K^*(x,t;z,\tau) K(z,\tau;y,s) d_\tau z \qquad \tau \in (s,t)$$

Then recall (21). Applying this to the two heat kernels, we see that $F(\tau)$ is constant in τ. Moreover

$$\lim_{\tau \uparrow t} F(\tau) = K(x,t;y,s) \lim_{\tau \downarrow s} F(\tau) = K^*(x,t;y,s)$$

which finishes the proof. □

Corollary 16.2.1 (Reproduction Formula)**.** If $t_1 < t_2 < t_3$, then

$$K(x_3,t_3;x_1,t_1) = \int_M K(x_3,t_3;\cdot,t_2) K(\cdot,t_2;x_1,t_1) dg_{t_2}$$

Proof: Again by (21), the intergal on the RHS is independent of t_2. Letting $t_2 \downarrow t_1$ or $t_2 \uparrow t_3$ along with the defining properties of the heat kernel give the result. See 54 □

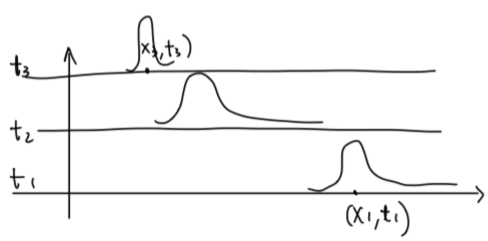

Figure 54

We will now discuss

- $\mathcal{F}$-functional, $\mathcal{W}$-functional, monotonicity
- λ, μ-invariants
- No local collapsing theorem

16.3 $\mathcal{F}$-functional

Let M a smooth manifold, closed. Define

$$\mathcal{F}(g,f) = \int_M (R + |\nabla f|^2)e^{-f}dV, \qquad dV = dVol_g$$

i.e. $f \in C^\infty(M)$ and g is a riemannian metric on M. Let

$$h := Dg$$

(i.e. $h = \dot{g}$ - for a smooth variation of metrics $\{g_t\}$). And similarly $v = Df$ (Also an infinitesimal variation) Then

Theorem 16.3. We have

$$D\mathcal{F}_{(g,f)}(h,v) = \int_M \left[-\langle h, \mathrm{Ric} + \nabla^2 f\rangle + \left(\frac{\mathrm{tr}(h)}{2} - v\right)(2\Delta f - |\nabla f|^2 + R)\right] e^{-f}dV$$

Proof: From hereon we'll label the LHS $D\mathcal{F}$. Recall that

$$D(-2\mathrm{Ric})(h)_{jk} = \Delta_L h_{jk} + \nabla_j\nabla_k \mathrm{tr}(h) + \nabla_j(\delta h)_k + \nabla_k(\delta h)_j$$

So that

$$DR = -\Delta(\mathrm{tr}(h)) - \langle \mathrm{Ric}, h\rangle + \delta^2 h$$

where δ is the divergence. Similarly

$$\begin{aligned}
D|\nabla f|^2 &= -h(\nabla f, \nabla f) + 2\langle \nabla f, \nabla v\rangle \\
D(dV) &= \frac{\mathrm{tr}(h)}{2}dV \\
D(e^{-f}dV) &= \left(\frac{\mathrm{tr}(h)}{2} - v\right)e^{-f}dV \\
\implies D\mathcal{F} &= \int_M \left[-\Delta\mathrm{tr}(h) - \langle \mathrm{Ric}, h\rangle + \delta^2 h - h(\nabla f, \nabla f) + 2\langle \nabla f, \nabla v\rangle + (R + |\nabla f|^2)\cdot\left(\frac{\mathrm{tr}h}{2} - v\right)\right]e^{-f}dV
\end{aligned}$$

where again ($v = Df$). Moving all the derivatives off of the variations h and v and integrating by parts, we get

$$\int_M -\Delta\mathrm{tr}(h)e^{-f}dV = \int_M -\mathrm{tr}(h)\Delta e^{-f}dV = \int_m -\mathrm{tr}(h)\left(|\nabla f|^2 - \Delta f\right)e^{-f}dV$$

$$\begin{aligned}
\int_M \delta^2 h e^{-f}dV = -\int_M \langle \delta h, \nabla e^{-f}\rangle &= \int_M \langle h, \delta^*\nabla e^{-f}\rangle dV \\
&= \int_M \langle h, \nabla^2 e^{-f}\rangle = \int_M \langle h, df\otimes df - \nabla^2 f\rangle e^{-f}dV \\
\int_M 2\langle \nabla f, \nabla v\rangle e^{-f}dV &= \int_M 2\langle -\nabla e^{-f}, \nabla v\rangle dV = \int_M 2\Delta e^{-f}v dV \\
&= 2\int_M v(|\nabla f|^2 - \Delta f)e^{-f}dV
\end{aligned}$$

This implies the theorem. $\square$

We can get rid of $\left(\frac{\mathrm{tr}(h)}{2} - v\right)(2\Delta f - |\nabla f|^2 + R)$ by requiring that

$$\frac{\mathrm{tr}(h)}{2} - v = 0$$

In this case, we also see that

$$D(e^{-f}dV) = \left(\frac{\mathrm{tr}(h)}{2} - v\right) dV = 0$$

i.e. its a constant measure, and hence

$$D\mathcal{F} = \int -\langle h, \mathrm{Ric} + \nabla^2 f\rangle e^{-f} dV$$

Can now define an L^2-product on the space of symmetric 2-tensors by

$$\langle\langle h_1, h_2\rangle\rangle_g := \frac{1}{2}\int_M \langle h_1, h_2\rangle dm$$

for m a fixed measure. The gradient flow of $\mathcal{F}$ is then

$$\begin{aligned}\partial_t g_t &= -2(\mathrm{Ric} + \nabla^2 f)\\ \partial_t f_t &= -R - \Delta f\end{aligned}$$

and hence

$$\frac{d}{dt}\mathcal{F}(g_t, f_t) = \int -|\mathrm{Ric} + \nabla^2 f|^2 e^{-f} dV \leq 0$$

Now let $\tilde{g}_t = \phi_t^*(g_t)$ and $\tilde{f}_t = \phi_t^* f_t$ for ϕ_t defined to be the flow of ∇f. Then

$$\begin{aligned}\partial_t \tilde{g}_t &= -2\mathrm{Ric}(\tilde{g}_t)\\ \partial_t \tilde{f}_t &= -\tilde{R} - \Delta\tilde{f} + |\nabla\tilde{f}|^2\end{aligned} \tag{22}$$

We then see that

$$\mathcal{F}(\tilde{g}_t, \tilde{f}_t) = F(g_t, f_t)$$

and

$$e^{-\tilde{f}}d\tilde{V} = \phi_t^*(e^{-f}dV)$$

is no longer constant, but the integral is!

$$\frac{d}{dt}\int e^{-\tilde{f}}d\tilde{V} = \frac{d}{dt}\int \phi_t^*(e^{-f}dV) = \frac{d}{dt}\int e^{-f}dV = 0$$

just by diffeomorphism invariance of integrals. This means that from (22), we get

$$\partial_t g_t = -2\mathrm{Ric} \tag{23}$$

$$\partial_t f = -R - \Delta f + |\nabla f|^2 \tag{24}$$

$$\implies \partial_t e^{-f} = -\Delta e^{-f} + Re^{-f} \tag{25}$$

where the last equation is the conjugate heat equation, and linear. Thus we can solve for f **backwards in time**

Example. Consider $(\mathbb{R}^n, g_{eucl})$ and $g_t = g_{eucl}$ for all $t \in [0, t_0)$. Let $\tau = t_0 0t$. Let

$$\begin{aligned}f_t &= \frac{|x|^2}{4(t_0 0t)} + \frac{n}{2}\ln(4\pi(t_0 - t))\\ \implies e^{-f_t} &= (4\pi(t_0 - t))^{-n/2} e^{-|x|^2/4(t_0 - t)}\end{aligned}$$

This is a conjugate heat kernal starting at $(0 \in \mathbb{R}^n, t_0 \in (0,\infty))$. Moreover e^{-f_t} satisfies (25) and

$$\int_{\mathbb{R}^n} e^{-f_t} dV = 1$$

Finally

$$\mathcal{F}(g_t, f_t) = \int (R + |\nabla f|^2) e^{-f} dV = (4\pi\tau)^{-n/2} \int \frac{|x|^2}{4\tau^2} e^{-|x|^2} 4\tau dV = \frac{n}{2\tau}$$

16.4 λ-invariant

Given a manifold, we can define

$$\lambda(g) := \inf_{\substack{f \in C^\infty(M) \\ \int e^{-f} dV = 1}} \mathcal{F}(g, f)$$

Letting $\phi = e^{-f/2}$, this is equivalent to

$$\lambda(g) = \int_{\substack{\phi \in C^\infty(M) \\ \int \phi^2 = 1}} \int 4|\nabla \phi|^2 + R\phi^2 dV$$

16.4.1 Existence of minimizer and regularity

There exists a smooth $\phi > 0$ a minimizer of

$$\int 4|\nabla \phi|^2 + R\phi^2$$

and

$$-4\Delta\phi + R\phi = \lambda\phi$$

where λ is the smallest eignevalue of $-4\Delta + R$. Moreover, this occurs if and only if

$$2\Delta f - |\nabla f|^2 + R = \lambda$$

for λ a constant in M. Taking the gradient of the above and applying the 2nd Bianchi identity, we get

$$\operatorname{div}\left[(\mathrm{Ric} + \nabla^2 f)e^{-f}\right] = \operatorname{div}\left[-\frac{1}{2}\nabla\lambda\right] = 0 \tag{26}$$

Check:

$$\delta\lambda_g(h) = \int_M -\langle h, \mathrm{Ric} + \nabla^2 f\rangle e^{-f} dV$$

we define

$$\langle\langle h, h, \rangle\rangle_g = \frac{1}{2}\int_M \langle h, h\rangle \phi^2 dV_g$$

then (26) gives

$$\langle \nabla\lambda, \operatorname{div}^* X\rangle = \langle \operatorname{div}(\nabla\lambda), X\rangle = \langle 0, X\rangle = 0$$

So the gradient of λ is orthogonal to

$$\mathrm{Imdiv}^* = \{\mathcal{L}_X g \ : \ X \text{ v.f.}\}$$

so our ricci flow is a "gradient flow" of λ when projected on $M/\mathrm{Diff}(M)$.

Remark In finite dimensional manifolds,a periodic gradient flow (called a **steady breather**) must be a fixed point (i.e. **evolves by diffeomorphisms**)

Theorem 16.4. If $\{g_t\}_{t\in[0,T]}$ is a RF, the $\lambda(g_t)$ is non-decreasing in t

Proof: WLOG, show that $\lambda(g(\tau)) \geq \lambda(g(0))$ for all τ. Let $f(\tau)$ be a smooth function such that $\phi = e^{-f(\tau)/2}$ is a minimizer of $\lambda(g_\tau)$. Solve the conjugate heat equation

$$-\partial_t u - \Delta u + Ru = 0$$
$$u(\tau) = e^{-f(\tau)}$$

Claim: $u(t) > 0$ everywhere.

Proof: It follows immediately from convolution with the conjugate heat kernal OR: take v to be a test function such that $v > 0$ and $\text{supp}(v) \subseteq B_0(x, r)$ for r small. Solve

$$\partial_t v = \Delta v$$
$$v(0) = v$$

to get $v(t) > 0$ everywhere. Then immediately, we have

$$0 < \int u(\tau)v(\tau)dV_\tau = \int u(0)v(0)dV_0$$

I believe this follows from (21). Now taking $v(0)$ arbitrary we see that $u(x, 0) > 0$ everywhere. $\square$

NOw our claim shows that there exists $f(t)$ for $t \in [0, T]$ such that $e^{-f(t)} = u(t)$, and so $\mathcal{F}(g_t, f(t))$ is non-decreasing i.e.

$$\lambda(g_0) \leq \mathcal{F}(g_0, f(0)) \leq \mathcal{F}(g_\tau, f(\tau)) = \lambda(g_\tau)$$

ending the proof. $\square$

Corollary 16.4.1. A compact steady breather is a steady gradient soliton (and hence is Ricci Flat)

Recall that a steady breather means we have $\{g_t\}$ a RF and $\exists t_2 > t_1$ and ϕ a diffeo such that $g(t_2) = \phi^* g(t_1)$. Hence the ricci flow is periodic after rescaling appropriately.

Proof: Tracing through the previous proof, we find $\mathcal{F}(g_t, f(t)) \equiv c$ a constant. Thus

$$0 = \frac{d}{dt}\mathcal{F}(g_t, f(t)) = 2\int_M |\text{Ric} + \nabla^2 f|^2 e^{-f} dV \implies \text{Ric} + \nabla^2 f = 0$$

which is our criterion for a gradient soliton.

17 Lecture 17: 12-1-22

Today we discuss the $\mathcal{W}$ and μ functionals.

17.1 $\mathcal{W}$-functional

Define

$$\mathcal{W} : M \times C^\infty(M) \times \mathbb{R}^+ \to \mathbb{R}$$
$$\mathcal{W}(g, f, \tau) = \int_M [\tau(|\nabla f|^2 + R) + f - n](4\pi\tau)^{-n/2} e^{-f} dV$$

Then we see that it satisfies

$$\mathcal{W}(\lambda g, f, \lambda\tau) = \mathcal{W}(g, f, \tau) \quad \text{scaling invariance}$$
$$\mathcal{W}(\phi^* g, \phi^* f, \tau) = \mathcal{W}(g, f, \tau) \quad \text{diffeo invariance}$$

Let $Dg = h$, Df, $D\tau$, all be infinitesimal variations in time. And assume that

$$\frac{\mathrm{tr}h}{2} - Df - \frac{nD\tau}{2\tau} = 0 \tag{27}$$

Then we have that

$$D\mathcal{W}_{(g,f,\tau)}(hDf, D\tau) = \int_M \left[D\tau(R + |\nabla f|^2 - \tau\langle h, \mathrm{Ric} + \nabla^2 f\rangle + Df\right] (4\pi\tau)^{-n/2} e^{-f} dV$$

Consider the coupled flow

$$\begin{aligned} \partial_t g &= -2(\mathrm{Ric} + \nabla^2 f) \\ \partial_t f &= -\Delta f - R + \frac{n}{2\tau} \iff (27) \\ \partial_t \tau &= -1 \end{aligned}$$

Then

$$\begin{aligned} \frac{d\mathcal{W}}{dt}(g_t, f_t, \tau) &= \int_M \left[-\left(R + |\nabla f|^2\right) + 2\tau|\mathrm{Ric} + \nabla^2 f|^2 - \Delta f - R + \frac{n}{2\tau}\right] (4\pi\tau)^{-n/2} e^{-f} dV \\ &= \int_M \left[-2(R + \Delta f) + 2\tau|\mathrm{Ric} + \nabla^2 f|^2 + \frac{n}{2\tau}\right] (4\pi\tau)^{-n/2} e^{-f} dV \\ &= \int_M 2\tau|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau} f|^2 (4\pi\tau)^{-n/2} e^{-f} dV \end{aligned}$$

Replacing g, f with the pull backs $\phi_t^* g$, $\phi_t^* f$, for ϕ_t the flow of ∇f, we get

$$\begin{aligned} \partial_t g &= -2\mathrm{Ric} \\ \partial_t f &= -\Delta f + |\nabla f|^2 - R + \frac{n}{2\tau} \implies \int_M (4\pi\tau)^{-n/2} e^{-f} dV = 1 \\ \partial_\tau \tau &= -1 \end{aligned}$$

17.2 μ-functional

We define

$$\begin{aligned} \mu(g,\tau) &= \inf_f \{\mathcal{W}(g,f,\tau) \ : \ \int (4\pi\tau)^{-n/2} e^{-f} dV = 1\} > -\infty \\ &\overset{\phi = e^{-f/2}}{=} \inf_\phi \{\int_M \left[\tau(4|\nabla\phi|^2 + R\phi^2) - 2\phi^2 \log\phi^2 - n\phi^2\right] dV \ : \ \int (4\pi\tau)^{-n/2} \phi^2 = 1\} \end{aligned}$$

There exists a $\phi > 0$, minimizer and smooth such that

$$\tau(-4\Delta + R)\phi = 2\phi\log(\phi) + (\mu(g,\tau) + n)\phi$$

so there exists an f such that $\phi = e^{-f/2}$ and f smooth.

Theorem 17.1. Let $(M, \{g_t\})$ a RF. M compact, let t_0 arbitrary, then $\mu(g_t, t_0 - t)$ is non-increasing and continuous in t, $t \in (-\infty, t_0)$

Proof: Let $f(t_2)$ be the minimizer of $\mu(g_{t_2}, t_0 - t_2)$, then solve

$$\Box^* (4\pi\tau)^{-n/2} e^{-f} = 0$$

to get f_t. Then

$$\mathcal{W}(g_t, f_t, t_0 - \tau)$$

is non-decreasing and

$$\mu(g_{t_1}, t_0 - t_1) \leq \mathcal{W}(g_{t_1}, f_{t_1}, t_0 - t_1) \leq \mathcal{W}(g_{t_2}, f_{t_2}, t_0 - t_2) = \mu(g_{t_2}, t_0 - t_2)$$

where the first inequality holds by definition of inf. $\square$

Corollary 17.1.1. A shrinking breather (compact) is a shrinking gradient soliton.

Recall that a shrinking breather means $\exists \phi$ a diffeo with $c \in (0,1)$ such that

$$g(t_2) = c\phi^*(g(t_1))$$

for some $t_2 > t_1$. This relation gives periodic ricci flow modulo rescalings (i.e. there exists t_i for all $i \geq 3$ such that a similar relation to the above holds).

Proof: WLOG assume $t_2 = ct_1 < 0$, i.e. perform a time shift, see 55 Let $f(t_2)$ be a minimizer of $\mu(g_{t_2}, -t_2)$

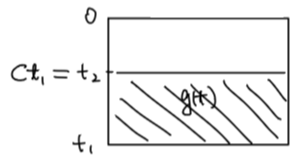

Figure 55

and solve

$$\Box^*(4\pi\tau)^{-n/2}e^{-f} = 0$$

(i.e. heat equation, this is what $\Box^*$ represents) to get f_t and

$$\begin{aligned}\mu(g_{t_1}, t_1) \leq \mathcal{W}(g_{t_1}, f_{t_1}, -t_1) &\leq \mathcal{W}(g_{t_2}, f_{t_2}, -t_2)\\ &= \mu(g_{t_2}, -t_2) = \mu(c\phi^* g_{t_1}, -ct_1)\\ &= \mu(\phi^* g_{t_1}, -t_1) = \mu(g_{t_1}, -t_1)\end{aligned}$$

where the equalities in the third line follow since the μ-functional is rescaling invariant and also diffeomorphism invariant. This implies that

$$\mathcal{W}(g_t, f_t, -t)$$

is constant, i.e.

$$\frac{d}{dt}\mathcal{W} = 0 \implies \int_M 2\tau|\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau}|^2(4\pi\tau)^{-n/2}e^{-f}dV = 0$$

This implies that

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{2\tau}g = 0$$

which is exactly the equation for a gradient soliton.

17.3 Example: The gaussian shrinker

The gaussian shrinker on $(\mathbb{R}^n, g_{eucl})$ with $g(t) = g_{eucl}$ and $t \in (-\infty, 0)$. Then

$$\begin{aligned}&g_t = g_{eucl}\\ &f_t = \frac{|x|^2}{4\tau} \iff (4\pi\tau)^{-n/2}e^{-f} \quad \text{is the conjugate heat kernal}\\ &\tau = -t\end{aligned}$$

see 56

Then

$$\tau(|\nabla f|^2 + R) + f - n = \tau \cdot \frac{|x|^2}{4\tau^2} + \frac{|x|^2}{4\tau} - n = \frac{|x|^2}{2\tau} - n$$

Then

$$\mathcal{W}(g_t, f_t, -t) = \int_{\mathbb{R}^n}\left(\frac{|x|^2}{2\tau} - n\right)(4\pi\tau)^{-n/2}e^{-f}dV = 0$$

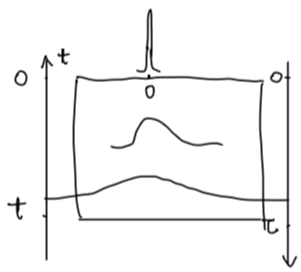

Figure 56

17.4 Example: $\mathbb{R}^2/\mathbb{Z}_k$

Let $g_t = g_{eucl}$ for $t \in [0, t_0)$ and $\tau = t_0 - t$ and $\mathbb{Z}_k$ acting by rotating by $2\pi/k$ around 0. This turns $\mathbb{R}^2$ into a cone. Let

$$(4\pi\tau)^{-n/2}e^{-f} = (4\pi\tau)^{-n/2}e^{-|x|^2/(4\tau)}k$$

then $(4\pi\tau)^{-n/2}e^{-f}$ a conjugate heat kernal, implies that

$$f = \frac{|x|^2}{4\tau} - \ln(k)$$

This gives that

$$\mathcal{W}(g_t, f_t, t_0 - t) = -\ln(k)$$

and the above tends to $-\infty$ as $k \to \infty$. Note that as $k \to \infty$, the cone we have collapses since we quotient out by $\mathbb{Z}_k$

17.5 No local collapsing theorem

Definition 17.2. Let (M, g_t) for $t \in [0, T)$ be a Ricci flow. It is locally collapsing at T if $\exists \{t_k\} \to T$ and $p_k \in M$ and $r_k > 0$ with

$$\sup_k \frac{r_k^2}{t_k} < \infty$$

and

$$|Rm|(g(t_k)) \leq r_k^{-2} \quad \text{on} \quad B_{t_k}(p_k, r_k)$$

but

$$\lim_{k \to \infty} \frac{\text{vol}(B_{t_k}(p_k, r_k))}{r_k^n} = 0$$

Definition 17.3. We say that a manifold (M, g) is k-noncollapsed on the scale of ρ if $\forall x \in M, \ \ \forall r \leq \rho$, we have

$$|Rm| \leq r^{-2} \quad \text{in} \quad B_g(x, r) \quad \implies \text{Vol}(B_g(x, r)) \geq kr^n$$

Example. Consider the Cigar soliton, which we know asymptotically looks like $\mathbb{R} \times S^1$. It is k-noncollapsed on scale 1, but not k-noncollapsed on all scales. Then

- It is k-non-collapsed on scale 1
- But not k-non-collapsed on all scales - This means there exists no k so that the cigar can be k-non-collapsed at all scales

Intuitively, the second is because asymptotically the cigar looks like a cylinder, which converges to a ray at larger scales and this contradicts the quadratic volume growth

Definition 17.4. We say (M, g) is non-collapsed if there exists a $k > 0$ such that (M, g) is k-non-collpased at all scales. Otherwise (M, g) is collapsed.

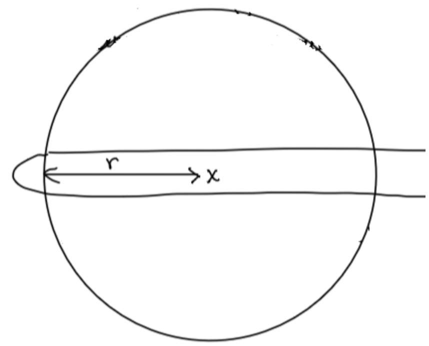

Figure 57

Example. Using the above, the cigar soliton is collapsed. Note that $|Rm| \leq r^{-2}$ in $B(x,r)$, but $\mathrm{Vol}(B(x,r)) = O(r) \ll r^3$ when $r \gg 1$, so collapsed. See 57

Example. Consider $\mathbb{R} \times S^2$. THis is actually non-collapsed because $|Rm| \equiv 1$ and so the curvature condition $|Rm| \leq r^{-2}$ is not satisfied on for $r > 1$, so the implication is vacuously true. For $r < 1$, we have cubic volume growth and can find a concrete $\kappa > 0$. See 58

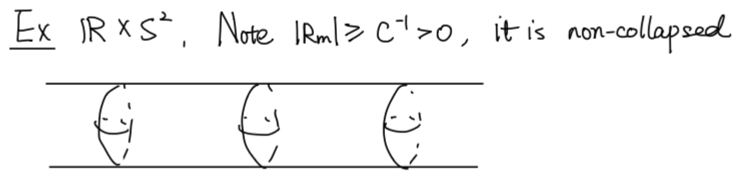

Figure 58

Example. Bryant solution is non-collapsed. See 59

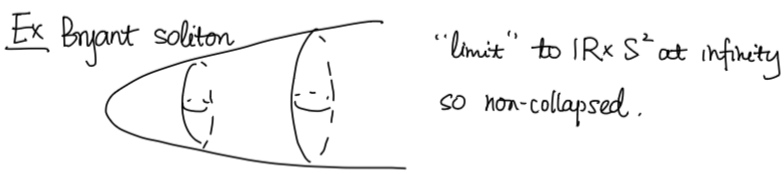

Figure 59

Example. Flying wings - is collapsed because of the geometry at the vertex. See 60

Theorem 17.5. If M is a closed manifold and $T < \infty$, then $g(t)$ is not locally collapsing at T, i.e.

$$\sup_k r_k^2 < \infty, \qquad \lim_{k \to \infty} \frac{\mathrm{Vol}(B(p_k, r_k))}{r_k^n} = 0$$

never happens.

The theorem means that (M, g_t) is k-non-collapsed on some scale ρ - note that k, ρ may depend on g_0 and T.

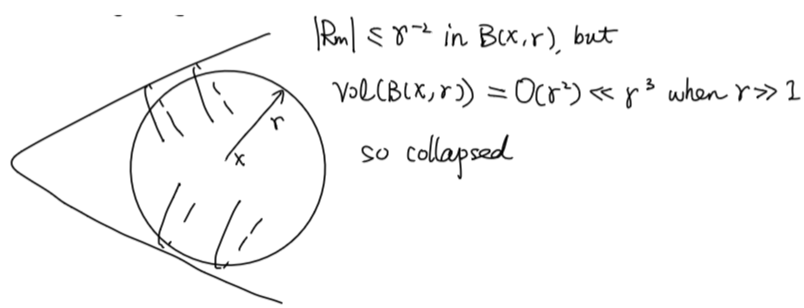

Figure 60

18 Lecture 18: 12-6-22

Theorem 18.1 (Perelman)**.** If M is closed and $T < \infty$, then $g(t)$ is not locally collapsing at T

Proof: Let $\phi = e^{-f}$, then

$$\mathcal{W}(g, f, \tau) = (4\pi\tau)^{-n/2} \int_M 4\tau|\nabla\phi|^2 + (\tau R - 2\ln(\phi)-)\phi^2 dV$$

Suppose theorem not true, then $\exists \{p_k\} \in M$, $r_k > 0$ such that $\sup_k r_k < \infty$ such that

$$|Rm| \leq r_k^{-2} \quad \text{on} \quad B(p_k, r_k) \quad \text{but} \quad \frac{\text{Vol}(B(p_k, r_k))}{r_k^n} \to 0 \quad \text{as } k \to \infty$$

We will find ϕ_k (and hence f_k, $\phi_k = e^{-f_k/2}$) such that $\mathcal{W}(g_{t_k}, f_k, r_k^2) \to -\infty$ as $k \to \infty$ and

$$\mu(g_0, t_k + r_k^2) \leq \mu(g_{t_k}, r_k^2) \leq \mathcal{W}(g_{t_k}, f_k, r_k^2) \to -\infty$$

However $\mu(g_0, t_k + r_k^2) \to -\infty$ is impossible (this will be done in an independent exercise, but as long as the metric is fixed, then $\mu(g_0, r)$ is bounded). This will give a contradiction. We now verify that $\mathcal{W}(g_{t_k}, f_k, r_k^2) \to -\infty$

$$\phi_k = e^{c_k/2}\varphi\left(\frac{d(p_k, \cdot)}{r_k}\right)$$

where φ is a cut off on the half line that is 1 on $[0, 1/2]$ and decays to 0 on $[1/2, 1]$ see 61
Choose c_k so that

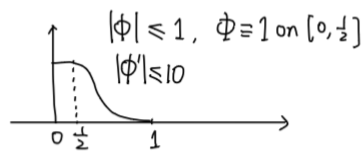

Figure 61

$$\int_M (4\pi r_k^2)^{-n/2}\phi_k^2 = 1$$

We now use Jensen's inequality, which we recall as

Proposition 5. Let (M^n, g), $\varphi : \mathbb{R} \to \mathbb{R}$ convex, $\mu \in L^1(M, dV)$, then

$$\frac{1}{\text{Vol}(M)} \int_M \varphi(u) dV = \fint \varphi(u) dV \geq \varphi\left(\fint u dV\right)$$

Now assume that $(4\pi r_k^2)^{-n/2} = 1$. Then omit "k" for a moment and let

$$B := B(p_k, r_k), \qquad B_{1/2} = B(p_k, r_k/2), \qquad \tau = r_k^2$$

apply Jensen's inequality on $M = B$ with $\varphi = u^2 \ln(u^2)$ (which is convex). This gives

$$\begin{aligned}
&\frac{1}{\mathrm{Vol}(B)} \int_B \phi^2 \ln(\phi^2) \geq \left(\frac{1}{\mathrm{Vol}(B)} \int_B \phi^2 dV\right) \ln\left(\frac{1}{\mathrm{Vol}(B)} \int_B \phi^2 dV\right) \\
&\frac{1}{\mathrm{Vol}(B)} \int_B \phi^2 \ln(\phi^2) \geq \frac{1}{\mathrm{Vol}(B)} \ln\left(\frac{1}{\ln(\mathrm{Vol}(B))}\right) \\
&\implies -\int_B \phi^2 \ln(\phi^2) \leq \ln(\mathrm{Vol}(B))
\end{aligned}$$

This tells us that

$$\int_M (\tau R - 2\ln(\phi) - n)\phi^2 dV \leq c_0 + \ln(\mathrm{Vol}(B))$$

so it suffices to estimate

$$\int 4\tau |\nabla \phi|^2$$

in our initial integral. First we know that

$$|\nabla \phi| \leq c_0 e^{-c/2}$$

this is a consequence of the definition of ϕ_k (remember we're dropping the "k" subindex). This tells us that

$$\int_M 4\tau |\nabla\phi|^2 dV \leq c_0 \mathrm{Vol}(B) e^{-c} \leq c_0 \frac{\mathrm{Vol}(B)}{\mathrm{Vol}(B_{1/2})} \overset{\text{Volume Comparison}}{\leq} c_0$$

Note that $\phi = e^{-c/2}$ on $B_{1/2}$ we have

$$e^{-c}\mathrm{Vol}(B_{1/2}) = \int_{B_{1/2}} \phi^2 dV \leq \int_B \phi^2 dV = 1$$

restoring the subscript "k" and rescaling, this gives

$$\mathcal{W}(g_{t_k}, f_k, r_k^2) \leq c_0 + \ln\left(\frac{\mathrm{Vol}(B(p_k, r_k)}{r_k^n}\right) \to -\infty$$

since

$$\frac{\mathrm{Vol}(B(p_k, r_k)}{r_k^n} \to 0$$

as $k \to \infty$.

18.1 Nash Entropy

Let (M, g_t) a RF, compact. Fix $(x_0, t_0) \in M \times I$ and $\tau = t_0 - t$. Define

$$dV_\tau = dV_{x_0,t_0;\cdot,t} = k(x_0, t_0; \cdot t) dg_t = (4\pi\tau)^{-n/2} e^{-f} dg_t$$

is a probability measure i.e.

$$\int_M dV_\tau = 1$$

and

$$\square^* k(x_0, t_0; \cdot, t) = 0 \iff -\partial_t f = \Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}$$

Denote

$$\mathcal{W}(\tau) = \mathcal{W}_{x_0,t_0}(\tau) = \mathcal{W}(g_{t_0-\tau}, f_{t_0-\tau}, \tau)$$

then is called the **pointed Nash entropy** (often abbreviated $\mathbb{N}(\tau)$)

$$\mathbb{N}_{x_0,t_0}(\tau) := \int_M \left(f_{t_0-\tau} - \frac{n}{2}\right) dV_\tau$$

Theorem 18.2.

$$\frac{d}{d\tau}(\tau\mathbb{N}(\tau)) = \mathcal{W}(\tau) \leq 0$$
$$\frac{d^2}{d\tau^2}(\tau\mathbb{N}(\tau)) \leq 0$$
$$\frac{d}{d\tau}(\mathbb{N}(\tau)) \leq 0$$
$$0 \geq \mathbb{N}(\tau) \geq \mathcal{W}(\tau)$$

Proof: Note that if we prove the first equality, the second inequality follows by monotonicity of $\mathcal{W}$. We compute

$$\begin{aligned}\frac{d}{d\tau}\mathbb{N}(\tau) &= -\frac{d}{d\tau}\left(f - \frac{n}{2}\right)dV_\tau = -\frac{d}{d\tau}\int_M f dV_\tau = -\int_M \Box f dV_\tau \\ &= \int_M \left(2\Delta f - |\nabla f|^2 + R - \frac{n}{2\tau}\right)dV_\tau = \int_M \left((|\nabla f|^2 + R) - \frac{n}{2\tau}\right)dV_\tau\end{aligned}$$

Note that the last equality in the first line follows from (21). Note this doesn't give non-positivity, but we'll show that somehow else. Now the first equation comes from

$$\begin{aligned}\frac{d}{d\tau}(\tau\mathbb{N}(\tau)) &= N(\tau) + \tau\frac{d}{d\tau}\mathbb{N}(\tau) = \int_M f dV_\tau - \frac{n}{2} + \int_M \tau(|\nabla f|^2 + R) - \frac{n}{2}dV_\tau \\ &= \mathcal{W}(\tau)\end{aligned}$$

This gives the first equation.

Its an exercise to show that

$$\lim_{\tau\to 0}\mathbb{N}(\tau) = \lim_{\tau\to 0}\mathcal{W}(\tau) = 0$$

see 62

The picture follows in part from the convexity of $\tau\mathbb{N}(\tau)$. Now with this, we use the first equation to

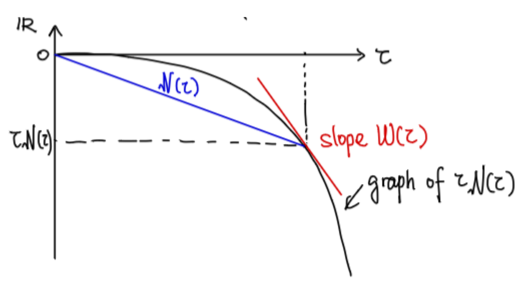

Figure 62

get

$$0 \geq \mathbb{N}(\tau) \geq \mathcal{W}(\tau)$$
$$\implies \frac{d}{d\tau}\mathbb{N}(\tau) \leq 0$$

A rigorous proof can be done but the picture suffices for the idea. This finishes our proof of all statements. $\square$

Theorem 18.3 (No-local collapsing, Bamler)**.** Let $(M, \{g_t\})$ a RF and $[t - r^2, t] \subseteq I$, then

$$R(\cdot, r) \leq r^{-2} \quad \text{on} \quad B_t(x, r) \implies \frac{\mathrm{Vol}(B_t(x,r))}{r^n} \geq c_n e^{N_{x_0,t_0}(r^2)}$$

Remark $R \leq r^{-2}$ is indispensable: because $\mathbb{N}_{(x_0,t_0)}(\tau) \geq -c_0$ holds for any τ on the shrinking sphere, shrinking cylinder, and Bryant soliton. However $\frac{\mathrm{Vol}(B_t(x,r))}{r^n} \to 0$ as opposed to being bounded. This is precisely because the scalar curvature bound does not hold. see 63

Intuitively,

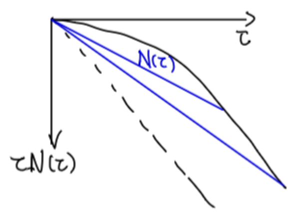

Figure 63

$$\begin{aligned} \text{Nash entropy of RF} &\overset{\text{compare}}{\leftrightarrow} \text{Volume growth ration in } (M^n, g), \text{ Ric} \geq 0 \\ \mathbb{N}_{x_0,t_0}(r^2) &\leftrightarrow \ln(V(x_0,r)) \\ \text{Theorem} &\leftrightarrow \frac{\mathrm{Vol}(B(x_0,r))}{r^n} = e^{\ln(V(x_0,r))} \end{aligned}$$

<u>Thm</u>: gradient estimate on $\mathbb{N}_{x_0,t_0}(\tau) \leftrightarrow V(x_2,r_2) \leq CV(x_1,r_1), \qquad C = C(d(x_1,x_2), r_1, r_2)$

18.2 H_n-center

Definition 18.4. Let (X,d) a metric space, and $P = \{\text{probability measure on } (X,d)\}$. hen for all $\mu_1, \mu_2 \in P$, we define

$$\mathrm{Var}(\mu_1,\mu_2) = \int_X \int_X d^2(x_1,x_2) d\mu_1(x_1) d\mu_2(x_2)$$

This is not exactly a distance function but

$$\sqrt{\mathrm{Var}(\mu_1,\mu_3)} \leq \sqrt{\mathrm{Var}(\mu_1,\mu_2)} + \sqrt{\mathrm{Var}(\mu_2,\mu_3)}$$

Theorem 18.5 (Bamler, 2021)**.** Let $(M, \{g_t\})$ a RF, compact, and ν_1, ν_2 satisfy $\Box^* \nu_i = 0$, $\nu_i \geq 0$, and $\int \nu_i dg_\tau = 1$. Set

$$d\mu_{i,\tau} = \nu_{i,\tau} dg_\tau \in P$$

then

$$\frac{d}{dt}\mathrm{Var}(\mu_{1,t}, \mu_{2,t}) \geq -H_n$$

for H_n some dimensional constant

We'll omit the proof for now

Corollary 18.5.1. $(M, \{g_t\})$ a RF, compact, $s < t \in I$, $x_1, x_2 \in M$. Then

$$\mathrm{Var}(\nu_{x_1,t;s}, \nu_{x_2,t;s}) \leq d_t^2(x_1,x_2) + H_n(t-2)$$

this is called the "distance distortion estimate" on a RF.

Let

$$\mathrm{Var}(\mu_1,\mu_1) =: \mathrm{Var}(\mu_1)$$

then we have

$$H_n(t-s) \geq \mathrm{Var}(\nu_{x_1,t,s}) = \int_M \mathrm{Var}(\nu_{x_1,t;s}, \delta_z) dV_{x_1,t;s}(z)$$

$$\int_M \mathrm{Var}(\nu_{x_1,t;s}, \delta_z) dV_{x_1,t;s}(z) \leq H_n(t-s)$$

which forces equality everywhere.

Definition 18.6. (z, s) is called an H_n-center of (x, t) if $s \leq t$ and

$$\mathrm{Var}(\nu_{x,t;s}, \delta_z) \leq H_n(t-s)$$

Theorem 18.7. Let $(M, \{g_t\})$ a RF. Then $\forall x \in M$, if $R \geq -r^{-2}$ on $M \times [t-r^2, t]$ and $(z, t-r^2)$ is an H_n-center of (x, t), then

$$\frac{\mathrm{Vol}(B_{t-r^2}(z, \sqrt{2H_n}r)}{r^n} \geq c_n e^{N_{x,t}(r^2)}$$

Intuitively

$$d_t(x, z_t) \cong t$$
$$R(z_t) \sim t^{-1}$$

19 Lecture 19: 12-8-22

Today is the last class. Yi is giving an overview of the modern theory of Ricci Flow, particularly pertaining to the work of Bamler.

Conjecture 19.0.1 (Folklore)**.** For a general RF, "Most" singularities are gradient shrinking solitons

Note that the bryant soliton is not a gradient shrinking soliton, but if we take a blow up sequence of $(M, R(p_i)g, p_i)$ which converges to $\mathbb{R} \times S^2$, we see that even in the limit the blow up is a gradient shrinking soliton.

Similarly, with the dumbbell, if we rescale about the pinched point we get $\mathbb{R} \times S^2$, which is again a gradient shrinking soliton.

Finally if we take $M = S^3$ or something close to S^3 with the standard metric, then if we run RF and rescale appropriately we'll get S^3 round in the limit, which is also a gradient shrinking soliton since $\mathrm{Ric} = \lambda g$ on S^3.

Example (Appleton)**.** There exists a RF in $n = 4$ whose blow up limits are Eguchi-Hanson metric on TS^2, and $\mathrm{Ric} \equiv 0$, and asymptotically equivalent to $\mathbb{R}^4/\mathbb{Z}_2$ (a cone on $\mathbb{RP}^3$), a shrinking soliton, see **??**

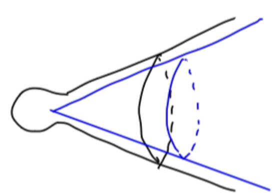

Figure 64

Example (Stolaski)**.** There exist ricci flows in $n \geq 13$ whose gradient shrinking solitons blow up limits are Ricci-flat cones.

These examples tell us that we modify the folklore conjecture to

Conjecture 19.0.2 (Folklore, modified)**.** "Most" singularities are gradient shrinking solitons (smooth) or Ricci Flat cones

Recall: If (M_i^n, g_i, x_i) is a sequence of RM with $\mathrm{Ric} \geq -\lambda g$, assume $B_{g_i}(x_i, r) \geq v > 0$ (non-collapsin), then passing to a subsequence, there exists (X, d, x_∞) a complete length space such that

$$(M_i^n, d_{g_i}, x_i) \xrightarrow{PGH} (X, d, x_\infty)$$

Moreover, (M_i^n, g_i, x_i) are Einstein-manifolds with $\mathrm{Ric} = \lambda_i g_i$, $|\lambda_i| \leq 1$. Then (Cheeger, Colding, Tian, Naber) there exists a decomposition $X = R \cup S$ such that

1. R is an open manifold and $\exists g_\infty$ a smooth Einstein metric,

$$(X, d) = \text{completion of } (R, d_{g_\infty})$$

2. (codimension 4 conjecture) $\dim_M S \leq n - 4$ (Minkowski dimension!), where S is the singular set
3. Any tangent cone at any point of X is a metric cone
4. There is a filtration $S^0 \subset S^1 \subset \cdots \subset S^{n-4} = S$ such that $\dim_M S^k \leq k$ and

$$S^k = \{\text{points in } S \text{ whose tangent cone cannot split off a } \mathbb{R}^{k+1}\text{-factor}\}$$

This impies that if a point has a tangent cone splitting off a $\mathbb{R}^{n-3}$ factor then $x \in \mathbb{R}$.

Theorem 19.1 (Bamler, 2020)**.** Given $(M_i^n, \{g_{i,t}\}_{t\in(-T_i,0]}, (x_i, 0))$ a RF then by passing to a subsequence assume

$$(M_i, \{g_{i,t}, \nu_{x_i,0;t}) \xrightarrow{\mathcal{F}, C.,\ i\to\infty} \big(\chi, (\nu_{x;t})_{t\in[-T_\infty,0)}\big)$$

where

$$\nu_{x_i,0;t} = K(x_i, 0, t)$$

where we have a conjugate heat kernel on the RHS. Also assume that

$$\mathcal{N}_{x_i,0}(\tau_0) \geq -Y_0 \quad \text{for some } \tau_0,\, Y_0 \;\; \text{(non-collapsed)}$$

then we have $X = R \cup S$ where

$$R = \{p \in X \mid \text{convergence is smooth}\}, \qquad S = X \backslash R$$

and

1. $\exists$ a smooth Ricci flow spacetime structure on R
2. $\dim_M S \leq (n+2) - 4$
3. Any "tangent flow" at any point of X is a gradient shrinking soliton or Ric-flat cone (this is an analogue of the Einstein metric case)
4. There is a filtrain $S^0 \subset S^1 \subset \cdots \subset S^{n-2} = S$ such that $\dim_M S^k \leq k$ and

$$S^k = \{\text{points in } S \text{ whose tangent flow cannot split off a } \mathbb{R}^{k-1} \text{ factor}\}$$

Fix a metric space (X, d), complete separable, $\mu_1, \mu_2 \in P(X)$

Definition 19.2. We define the **1-Wasserstein distance** to be

$$d_{w_1}(\mu_1, \mu_2) = \sup_{\substack{f: X\to\mathbb{R} \\ f:\text{1-lip}}} \left(\int_X f d\mu_1 - \int f d\mu_2 \right) = \inf_{\text{q is any coupling of } \mu_1 \times \mu_2} \int_{X\times X} d(x_1, x_2) dq(x_1, x_2)$$

where the equality holds by the Kantorovich-Rabinstein theorem. If (X_1, d_1), (X_2, d_2), $\mu_1 \in P(X_1)$, $\mu_2 \in P(X_2)$, $q \in P(X_1 \times X_2$ is a coupling if

$$(\mathrm{proj}_{x_i})_* q = \mu_i, \qquad i = 1, 2$$

(e.g. $q = \mu_1 \otimes \mu_2$).

The root of this is in optimal transport - and we'll now apply this to Ricci flow.

Example.
- $d_{w_1}(\delta_{x_1}, \delta_{x_2}) = d(x_1, x_2)$
- $d_{w_1}(\mu_1, \mu_2) \leq \sqrt{\text{Var}(\mu_1, \mu_2)} \leq d_{w_1}(\mu_1, \mu_2) + \sqrt{\text{Var}(\mu_1)} + \sqrt{\text{Var}(\mu_2)}$

Theorem 19.3. $(P(X), d_{w_1})$ is a complete metric space.

Definition 19.4. Let (X, d) a metric space and $\mu \in P(X)$. (X, d, μ) is a called a metric measure space if

$$(X_1, d_1, \mu_1) \overset{iso}{\cong} (X_2, d_2, \mu_2) \qquad \text{if } \exists \phi : X_1 \to X_2 \text{ is an isometry}$$

We can also define

$$d_{GW_1}((X_1, d_1, \mu_1), (X_2, d_2, \mu_2)) = \inf_{\substack{\varphi_i : (X_i, d_i) \to (Z, d) \\ \text{iso embedding}}} d_{w_1}((\varphi_1)_*(\mu_1), (\varphi_2)_*(\mu_2))$$

Theorem 19.5. If

$$d_{GW_1}((X_1, d_1, \mu_1), (X_2, d_2, \mu_2)) = 0$$

then they are "isometric", i.e.

$$(\text{supp}(\mu_1), d_1\Big|_{\text{supp}(\mu_1)}, \mu_1) \cong (\text{supp}(\mu_2), d_2\Big|_{\text{supp}(\mu_2)}, \mu_2)$$

We also have

Theorem 19.6. Let $\mathcal{M} = \{(X, d, \mu) \mid \text{supp}(\mu) = X\}/\sim$ is a complete metric space (i.e. mod out by isometry)

Definition 19.7. A metric flow $(\chi, t, \{d_t\}_{t \in I}, (\nu_{x;s})_{x \in X}, s \in I, s \leq t(x))$ where

- $t \in \chi \to I \subset R$
- d_t is metric on $\chi_t = t^{-1}(\chi)$
- $\nu_{x;s} \in P(X_s)$

satisfying

1. (X_t, d_t) is a complete and separable metric space
2. $\nu_{x;t(x)} = \delta_x$ and $\forall t_1 < t_2 < t_3$, $x \in \chi_{t_3}$ with

$$\nu_{x;t_1}(Z) = \int_{X_{t_2}} \nu_{y;t_1}(z) d\nu_{x;t_2}(y)$$

   (Reproduction formula) see 65

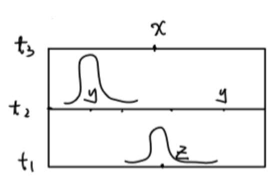

Figure 65

3. All the heat flows satisfy certain "gradient estimates"

19.1 Turn a RF to a metric flow

Now we can turn a RF into a metric flow. We have $(M, \{g_t\}_{t\in I})$ a RF, compact. Define

$$X = M \times I, \qquad d_t = d_{g_t}$$

an $\nu_{(x,t);s} = K(x,t;\cdot,s)$ is the conjugate heat flow starting from (x,t) and

$$K(x,t_3;z,t_1) = \int K(x,t_3;y,t_2)k(y,t_2;z,t_1)d_{t_2}y$$

where $\Box^* K = 0$. We now need to choose a base point to fully convert our Ricci flow to a metric flow.

A metric flow pair $(X, \{\mu_t\}_{t\in I})$ is a metric flow which is equipped with a conjugate heat flow.

Definition 19.8. Let

$$\mathbb{F}_I = \{X, \{\mu_t\}_I\}/\text{isometry}$$

where

$$(X^i, \{\mu^i_t\}_{t\in I}, \quad i = 1,2, \quad \text{two metric flow pairs}$$

and

$$d_{\mathbb{F}}((\chi^1,\{\mu^1_t\}),(\chi^2,\{\mu^2_t\})) = \inf_{r>0}\{r \mid \exists\{q_t\}_{t\in I\setminus E} \text{ coupling } \mu^1_t,\ \mu^2_t \text{ such that}$$

$$|E| \le r^2 \text{ and } \int_{X^1_t\times X^2_t} d^{Z_s}_{w_1}((\varphi^1_s)_*(\nu_{x_1;s}),(\varphi^2_s)_*(\nu_{x_2;s}))dq_t(x_1,x_2) \le r\}$$

and define

$$\mathcal{C} := \{(Z_t, d^Z_t), \{\varphi^i_t\}_{t\in I}\}$$

where φ^i_t is an isometric embedding from $(X^i_t, d^i_t) \to (Z_t, d^Z_t)$. Moreover, all the heat flows satisfy a graident estimate

$$d_{\mathbb{F}}((X_1,\mu^1_t),(X_2,\mu^2_t)) = \inf_e d^e_F((X_1,\mu^1_t),(X_2,\mu^2_t))$$

Example. Let $(M^n, \{g_t\}_{t<0})$ Bryant soliton. χ is metric flow. X^λ is parabolic rescaling by λ. If $\lambda_i \xrightarrow{i\to\infty} 0$, then

1. $(M, \lambda_i d_{g_t}, x) \xrightarrow{PGH, i\to\infty} R_+$
2. $(X^{\lambda_i}, (\nu_{x;t})_{t\le 0}) \xrightarrow{\mathcal{F}} \mathbb{R}\times S^2$ in the metric flow sense

This is an example of getting our known results about ricci flows and blow ups, but in the language of metric flows

(Unfortunately we ran out of class time)