SURIM 2020 Allen Cahn Project

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Contents

Bass Week 1

Bass 2.3

Proof. $\cup_{i=1}^{\infty}\mathcal{A}_i$ is not necessarily a σ-algebra. Let $X = \mathbb{N}$. Then let $\mathcal{A}_i$ be the collection of all the subsets of $X_i = \{1, 2, \dots, i\} \subset X$ and all of their subsets' complements in X. Clearly $\mathcal{A}_i$ is a σ-algebra for each $i = 1, 2, \dots$ And we have $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$. Now consider $\cup_{i=1}^{\infty}\mathcal{A}_i$. Let $S_i = 2, 4, \dots, 2i$. By the construction of our $\mathcal{A}_i$s, we have $S_i \in \mathcal{A}_{2i}$ for all $i = 1, 2, \dots$ Thus, we have $S_i \in \cup_{i=.1}^{\infty}\mathcal{A}_i$ for each $i \in \mathbb{N}$. However, the union of all such S_i, $\cup_{i=1}^{\infty} S_i = 2\mathbb{N}$ is not contained in $\cup_{i=.1}^{\infty}\mathcal{A}_i$ because no $\mathcal{A}_i$ contains contains $2\mathbb{N}$. □

Bass 2.5

We have $X = f^{-1}(Y) \in \mathcal{B}$. A set in $\mathcal{B}$ looks like $f^{-1}(A)$ for $A \in \mathcal{A}$. Then

$$f^{-1}(A)^c = \{x \in X : f(x) \notin A\} = \{x \in X : f(x) \in A^c\} = f^{-1}(A^c). \tag{1}$$

A countable family of sets in $\mathcal{B}$ looks like $\left\{f^{-1}(A_n)\right\}_1^{\infty}$ for $A_n \in \mathcal{A}$. Then

$$\bigcup_n f^{-1}(A_n) = \{x \in X : f(x) \in A_n \,\text{for some}\, n\} = \{x \in X : f(x) \in \bigcup_n A_n\} = f^{-1}(\bigcup_n A_n).$$

Because $\mathcal{A}$ is a σ-algebra, $A^c, \bigcup_n A_n \in \mathcal{A}$, so $f^{-1}(A^c), \bigcup_n f^{-1}(A_n) \in \mathcal{B}$, so $\mathcal{B}$ is a σ-algebra.

Bass 3.1

Proof. By definition we have $\mu(\varnothing) = 0$. Now we need to prove countable additivity. Let $A_1, \dots \in \mathcal{A}$ be pair-wise disjoint subsets of X. Then define $B_1 = A_1, B_2 = A_1 \cup A_2, \dots, B_n = \cup_{i=1}^{n} A_i, \dots$ Clearly, we have $B_1 \subset B_2 \subset \cdots$. Thus, by finite additivity, we have

$$\begin{aligned}
\mu(\cup_{i=1}^{\infty} B_i) &= \mu(\cup_{i=1}^{\infty} A_i) \\
&= \lim_{i\to\infty} \mu(B_i) \\
&= \lim_{i\to\infty} \mu(\cup_{n=1}^{i} A_n) = \lim_{i\to\infty} \sum_{n=1}^{i} \mu(A_n) \\
&= \sum_{n=1}^{\infty} \mu(A_n).
\end{aligned}$$

□

Bass 3.3

Proof. First, $\mu(\varnothing) = 0$ since $\varnothing$ has countable elements. Next let A_i be a countable collection of pair-wise disjoint subsets of X such that each A_i has countable elements. Then we have $\cup_{i=1}^{\infty} A_i$ has countably many elements as well. Thus, we have

$$\mu(\cup_{i=1}^{\infty} A_i) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

Now, I claim that $\mathcal{A}$ does not contain union of pair-wise disjoint subsets of A_i for which two or more A_i have uncountably many elements. Suppose this is the case, then we will have two sets A_1 and A_2 uncountable with $A_1 \cap A_2 = \varnothing$. Since $A_1, A_2 \in \mathcal{A}$, the collection of subsets of X such that either $A \subset X$ is countable or $A^c \subset X$ is countable, we know that A_1^c, A_2^c are countable subsets of X. Thus, taking the complement of their intersection, we have $A_1^c \cup A_2^c = X$. This cannot happen because X has uncountably many elements whereas $A_1^c \cup A_2^c$ is only countably many. Thus, for any countable pair-wise disjoint collection of subsets of X in $\mathcal{A}$, we can only have at most one uncountable subset of X. And thus, we have if A_j has uncountably many elements, then

$$\mu(\cup_{i=1}^{\infty} A_i) = 1 = \mu(A_j) = 1 = \sum_{i=1}^{\infty} \mu(A_i).$$

as desired. Thus, μ is a measure on $\mathcal{A}$. □

Bass 4.2

Write $A = \bigcup_{n\in\mathbb{Z}} A_n$, where $A_n = A \cap [n, n+1)$ are disjoint. By the convergence of $\sum_{n=-\infty}^{\infty} m(A_n) = m(A) < \infty$, choose N such that $\sum_{|n|>N} m(A_n) < \epsilon$. Because each $m(A_n) < \infty$, choose open $G_n \supset A_n$ and closed $F_n \subset A_n$ with $m(G_n - A_n), m(A_n - F_n) < \epsilon 2^{-n}$.

Define $G = \bigcup_{n\in\mathbb{Z}} G_n$ and $F = \bigcup_{|n|\le N} F$. Notice $F \subset A \subset G$, and as a finite union of closed sets, F is closed (and G is open). Because $G - A \subset \bigcup_{n\in\mathbb{Z}} G_n - A_n$, by monotonicity $m(G-A) \le \sum_{n\in\mathbb{Z}} m(G_n - A_n) < \epsilon \sum_{n\in\mathbb{Z}} 2^{-n} = 3\epsilon$. Similarly $A - F \subset \bigcup_{|n|>N} A_n \cup \bigcup_{|n|\le N} A_n - F_n$, so $m(A-F) \le \sum_{|n|>N} m(A_n) + \sum_{|n|\le N} \epsilon 2^{-n} \le \epsilon + 3\epsilon$. Finally, $m(G-F) \le m(G-A) + m(A-F) < 7\epsilon$, so we're done.

Bass 4.4

$$m(x) = \lim_{n\to\infty} m((x - \frac{1}{n}, x]) = \lim_{n\to\infty} l((x - \frac{1}{n}, x]) = \lim_{n\to\infty} \alpha(x) - \alpha(x - \frac{1}{n}) = \alpha(x) - \alpha(x-).$$

Bass 4.6

Part A

We claim $B = \bigcap_{k\in\mathbb{N}} \bigcup_{n\ge k} A_n$, from which B is measurable because measurable sets form a σ-algebra. Indeed, if x is in finitely many A_n, then there exists N for which $x \notin A_n$ for $n > N$, and so x is not in the set on the right. On the other hand,, if $x \in A_n$ for infinitely many n, then for each $k \in \mathbb{N}$, there exists $n_k \ge k$ with $x \in A_{n_k}$, so x is in the set on the right.

Part B

Write B_k for $\bigcup_{n\ge k} A_n$ and notice that the B_k form a decreasing sequence with intersection B (and as subsets of $[0,1]$ they are all of finite measure). By monotonicity, $m(B_k) = m(\bigcup_{n\ge k} A_n) \ge m(A_k) \ge \delta$, so by continuity $m(B) = \lim_{k\to\infty} m(B_k) \ge \delta$.

Part C

Fix $\epsilon > 0$. By the convergence of the series, choose N such that $\sum_{n=N}^{\infty} m(A_n) < \epsilon$. Then by monotonicity and subadditivity, $m(B) \le m(B_N) \le \sum_{n=N}^{\infty} m(A_n) \le \epsilon$. Because $\epsilon > 0$ was arbitrary, $m(B) = 0$.

Part D

Let $A_n = [0, \frac{1}{n}]$. Then $B = \{0\}$ but $\sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$.

Bass 4.10

First suppose $m(A) < \infty$. Fix $0 < \epsilon < 1$ and choose U open with $A \subset U$ and $m(U) \leq m(A) + \epsilon$. Open subsets of $\mathbb{R}$ are unions of countably many disjoint open intervals, so write $U = \bigcup_n I_n$ for open intervals I_n. Then

$$\begin{aligned} m(A) &= m(A \cap U) \\ &= \sum_n m(A \cap I_n) \\ &\leq (1-\epsilon) \sum_n m(I_n) \\ &= (1-\epsilon)m(U) \leq (1-\epsilon)m(A) + \epsilon(1-\epsilon). \end{aligned} \tag{2}$$

Rearrange to conclude that

$$\epsilon m(A) \leq \epsilon(1-\epsilon) \implies m(A) \leq 1-\epsilon \tag{3}$$

for all $\epsilon < 1$, so $m(A) = 0$.

When $m(A) = \infty$, write $A_n = A \cap [n, n+1)$ for $n \in \mathbb{Z}$. By monotonicity, $m(A_n \cap I) \leq m(A \cap I) \leq (1-\epsilon)m(I)$ for all open intervals I, so by the previous part $m(A_n) = 0$, so $m(A) = m(\bigcup_{n \in \mathbb{Z}} A_n) = 0$.

Bass Week 2

Bass 5.1

Since $\mathbb{Q}$ is dense in R, for every $a \in \mathbb{R}$, there exists an decreasing sequence r_n with $r_n \in \mathbb{Q}$ for every n converging to a. Therefore

$$\{x : f(x) > a\} = \bigcup_{n=1}^{\infty} \{x : f(x) > r\}$$

and thus f is measurable.

Bass 5.4

A is the inverse image of the Borel set $\{0\}$ under the measurable function $\limsup_n f_n - \liminf_n f_n$.

Bass 5.9

For any $a \in \mathbb{R}$, $(g \circ f)^{-1}(a, +\infty) = f^{-1}(g^{-1}((a, +\infty)))$. Since g is continuous, $g^{-1}((a, +\infty))$ is open, and thus a countable union of open intervals. Now since f is Lebesgue measurable, the inverse image of countable union of open intervals is Lebesgue measurable. Therefore $g \circ f$ is Lebesgue measurable.

Suppose g is Borel measurable. The $g^{-1}(a, +\infty)$ is Borel measurable. Since f is Lebesgue measurable, the inverse image under f of a Borel measurable set is Lebesgue measurable. Hence $g \circ f$ is Lebesgue measurable.

If B is Borel and g is Borel measurable (or in particular continuous), then $g^{-1}(B)$ is Borel and so $(g \circ f)^{-1}(B) = f^{-1}(g^{-1}(B))$ is Lebesgue measurable.[1]

It is not true if g is Lebesgue measurable. Let F and A be as in example 5.12. Then F is Borel measurable, and $\chi_{F(A)}$ is Lebesgue measurable (because as a set $F(A)$ is), but their composition $\chi_{F(A)} \circ F$ is not, because $F^{-1}(\chi_{F(A)}^{-1}(\{1\})) = F^{-1}(F(A)) = A$, which is not Lebesgue measurable.

[1] Because open rays generate the Borel sets, a function is Borel measurable iff it pulls back every open ray to a Lebesgue measurable set iff it pulls back every Borel set to a Lebesgue measurable set.

Bass 6.4

Proof. Without loss of generality, assume f non-negative (if not, then let $\int f d\delta_y = \int f^+ d\delta_y - \int f^- d\delta_y$ as defined in the text). First assume f is a simple function such that $f = \sum_{i=1}^M a_i \chi_{A_i}$ for $a_i \geq 0$. If $y \notin \cup_{i=1}^\infty A_i$, then we have $\delta_y(A_i) = 0$ for all $i = 1, \dots, M$. Thus,

$$\int f d\delta_y = \sum_{i=1}^M a_i \delta_y(A_i) = 0 = f(y)$$

as desired. Now, suppose $y \in \cup_{i=1}^N A_{k_i}$ for some $N = 1, \dots, M$. Then we have

$$f(y) = \sum_{i=1}^M a_i \chi_{A_i}(y) = \sum_{j=1}^N a_{k_j}$$

and $\delta(A_i) = \begin{cases} 1 & \text{if } i = k_j, j = 1, \dots, N \\ 0 & \text{otherwise.} \end{cases}$ for $i = 1, \dots, M$. Thus,

$$\int f d\delta_y = \sum_{i=1}^M a_i \delta_y(A_i) = \sum_{j=1}^N a_{k_j} = f(y)$$

as desired.

Now, let f be a non-negative function mapping from $X \to \mathbb{R}$. Let $\epsilon > 0$. By definition of $\int f d\delta_y$, there exists a simple function $s = \sum_{i=1}^n a_i \chi_{A_i}$ such that $0 \leq s \leq f$ and $\int f d\delta_y \leq \int s d\delta_y + \epsilon$. Since $s \leq f$ and $s(y) = \int \delta_y$, we have $\int f d\delta_y \leq s(y) \leq f(y)$. It suffices to prove that $f(y) \geq \int f d\delta_y$. Observe that

$$\int f d\delta_y \leq \int s d\delta_y + \epsilon = s(y) + \epsilon \leq f(y) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\int f d\delta_y \leq f(y)$ as desired. □

Bass 6.5

Proof. First assume f is a simple function such that $f = \sum_{i=1}^M a_i \chi_{A_i}$ for some $A_i \subset X$. Without loss of generality, let us assume that each A_i are pair-wise disjoint (if not, say $A_i \cap A_j \neq \varnothing$, then define $A_i' = A_i - A_j \cap A_i, A_j' = A_j - A_j \cap A_i$ and $\tilde{f} = \sum_{n=1, n\neq i,j}^M a_n \chi_{A_n} + (a_i)\chi_{A_i'} + (a_j)\chi_{A_j'} + (a_i + a_j)\chi_{A_{A_i \cap A_j}}$. One can check that $\tilde{f} = f$ and $A_i', A_j', A_i \cap A_j$ are disjoint subsets of X). Since X is countable, we have each A_i at most countable. Thus, each $A_i = \{a_1^i, a_2^i, \dots\}$ with $a_j^i \in X$. Thus, since A_i are pair-wise disjoint, we have $f(k) = \begin{cases} a_i & \text{if } k = a_j^i \text{ for some } a_j^i \in A_i \\ 0 & \text{otherwise} \end{cases}$ for all $k \in X$. With this formulation, we have

$$\sum_{k=1}^\infty f(k) = \sum_{i=1}^M \left[\sum_{j=1}^{N_i} a_i \right] = \sum_{i=1}^M \left[a_i \sum_{j=1}^{N_i} 1 \right] = \sum_{i=1}^M a_i \mu(A_i) = \int f d\mu$$

with each N_i denoting the number of elements in $A_i, N_i \in \mathbb{N} \cup \{\infty\}$. Thus we have what we want.

Now, let f be a non-negative function mapping from $X \to \mathbb{R}$. Let $\epsilon > 0$. By definition of $\int f d\mu$, there exists a simple function $s = \sum_{i=1}^n a_i \chi_{A_i}$ such that $0 \leq s \leq f$ and $\int f d\mu \leq \int s d\mu + \epsilon$. Since $s \leq f$ and $\sum_{k=1}^\infty s(k) = \int s d\mu$, we have $\int f d\mu \leq \sum_{k=1}^\infty s(k) \leq \sum_{k=1}^\infty f(k)$. It suffices to prove that $\sum_{k=1}^\infty f(k) \geq \int f d\delta_y$. Observe that

$$\int f d\mu \leq \int s d\mu + \epsilon = \sum_{k=1}^\infty s(k) + \epsilon \leq \sum_{k=1}^\infty f(k) + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\int f d\mu \leq \sum_{k=1}^\infty f(k)$ as desired. □

Bass 6.7

If $0 \leq \varphi \leq f$ is simple and $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$ with E_i disjoint, then taking $N = \max_{1 \leq i \leq n} a_i$ gives $\varphi \leq (f \wedge N)$. Thus $\varphi \leq (f \wedge N)$, so $\int \varphi \leq \int (f \wedge N)$. Because $f \wedge N$ is increasing in N, we have $\int \varphi \leq \lim_N \int (f \wedge N)$. Taking supremum over $0 \leq \varphi \leq f$ simple gives $\int f \leq \lim_N \int (f \wedge N)$.

For the reverse inequality, $(f \wedge N) \leq f$ for all N, so $\int (f \wedge N) \leq \int f$ for all N, so $\lim_N \int (f \wedge N) \leq \int f$.

Bass 7.10

Let $F_n = |f_n| - |f_n - f|$. By the triangle inequality, $|F_n| \leq ||f_n| - |f_n - f|| \leq |f_n - f_n + f| = |f|$, and $|f|$ is integrable. On the other hand, $f_n \to f$ a.e., so $F_n \to |f|$ a.e.. By the dominated convergence theorem, $\int F_n \to \int |f|$. But also $\int F_n = \int |f_n| - \int |f_n - f| \to \int |f| - \lim_n \int |f_n - f|$. Because $\int |f| < \infty$, this implies $\int |f_n - f| \to 0$.

Bass 7.16

Integrate by parts and use dominated convergence theorem. Answer is 1.

Proof.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^\infty ne^{-nx} \frac{x^2+1}{x^2+x+1} dx &= \lim_{n\to\infty} \int_0^\infty ne^{-nx} \left(1 - \frac{x}{x^2+x+1}\right) dx \\
&= \lim_{n\to\infty} \left[-e^{-nx} \left(1 - \frac{x}{x^2+x+1}\right)\right]_0^\infty + \lim_{n\to\infty} \int_0^\infty e^{-nx} \frac{x^2-1}{(x^2+x+1)^2} dx \\
&= 1 + \lim_{n\to\infty} \int_0^\infty e^{-nx} \frac{x^2-1}{(x^2+x+1)^2} dx
\end{aligned}$$

Let $f_n(x) = e^{-nx} \frac{x^2-1}{x^2+x+1}$. Then

$$|f_n| = \left| e^{-nx} \frac{x^2-1}{(x^2+x+1)^2} \right| \leq \left|2e^{-nx}(x^2-1)\right| \to 0$$

as $n \to \infty$. This is because that $|x^2+x+1| \geq \frac{3}{2}$ for all $x \in \mathbb{R}$ and the fact that exponential growth will dominate polynomial growth. Thus, we have $f_n \to 0$ for all $x \in (0, \infty)$. Moreover, we also have $|f_n| \leq |2e^{-x}(x^2-1)|$ and $\int_0^\infty |2e^{-x}(x^2-1)| < \infty$. Thus, by dominated convergence theorem, we have

$$\lim_{n\to\infty} \int f_n(x) dx = 0.$$

Thus, $\lim_{n\to\infty} \int_0^\infty ne^{-nx} \frac{x^2+1}{(x^2+x+1)^2} dx = 1$. ☐

Bass 7.26

Part A

This is true for simple functions: if $\varphi = \sum_{i=1}^n a_i \chi_{E_i}$, then $\int \varphi \, \mathrm{d}\mu_n = \sum_{i=1}^n a_i \mu_n(E_i) \to \sum_{i=1}^n a_i \mu(E_i) = \int \varphi \, \mathrm{d}\mu$.

Because $f = f^+ - f^-$, it suffices to prove the result for f^+ and f^-, so we may suppose $f \geq 0$. Because a bounded non-negative measurable function is a uniform limit of simple functions, fix ϵ and pick φ simple such that $|f - \varphi| < \epsilon$ on X. Because φ is simple, $\left|\int \varphi \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu_n\right| < \epsilon$ for n sufficiently large. Then for n sufficiently large,

$$\begin{aligned}
\left|\int f \, \mathrm{d}\mu - \int f \, \mathrm{d}\mu_n\right| &\leq \left|\int f \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu\right| + \left|\int \varphi \, \mathrm{d}\mu - \int \varphi \, \mathrm{d}\mu_n\right| + \left|\int \varphi \, \mathrm{d}\mu_n - \int f \, \mathrm{d}\mu_n\right| \\
&< \epsilon\mu(X) + \epsilon + \epsilon\mu_n(X) \\
&= 3\epsilon.
\end{aligned} \tag{4}$$

Note this proof also works when $\mu_n(X)$ are uniformly bounded.

Part B

This is true for non-negative simple functions by Part A. Because simple functions have finite range, they are bounded, so non-negative simple functions satisfy (2) with equality (by the first part the right side is $\lim \int f \, d\mu_n = \int f \, d\mu$).

The general case now follows. Let $0 \leq \varphi \leq f$ be simple. Then

$$\int \varphi \, d\mu_n \leq \int f \, d\mu_n. \tag{5}$$

Take lim inf in n on both sides:

$$\int \varphi \, d\mu = \liminf_n \int \varphi \, d\mu_n \leq \liminf_n \int f \, d\mu_n. \tag{6}$$

Now take supremum over $\varphi \leq f$ simple to get

$$\int f \, d\mu \leq \liminf_n \int f \, d\mu_n. \tag{7}$$

Alternatively, here is a direct proof for simple functions.

Direct proof for simple functions. Let $f = \sum_{i=1}^{M} a_i \chi_{A_i}$ for non-negative a_i and $A_i \in \mathcal{A}$. Then, define $\tilde{f} = \sum_{i=1}^{M} a_i \mu(A_i) \chi_X$ and $\tilde{f}_n = \sum_{i=1}^{M} a_i \mu_n(A_i) \chi_X$. Since $\mu(X) = \mu_n(X) = 1$, observe that

$$\int f d\mu = \sum_{i=1}^{M} a_i \mu(A_i) = \sum_{i=1}^{M} a_i \mu(A_i)\mu(X) = \sum_{i=1}^{M} a_i \mu(A_i)\mu_n(X) = \int \tilde{f} d\mu = \int \tilde{f} d\mu_n.$$

Similarly, we have

$$\int f d\mu_n = \sum_{i=1}^{M} a_i \mu_n(A_i) = \sum_{i=1}^{M} a_i \mu_n(A_i)\mu(X) = \sum_{i=1}^{M} a_i \mu_n(A_i)\mu_n(X) = \int \tilde{f}_n d\mu = \int \tilde{f}_n d\mu_n.$$

Moreover, we also have $\tilde{f}_n \to \tilde{f}$ for all $x \in X$ since $\mu_n(A) \to \mu(A)$ for all $A \in \mathcal{A}$ and $M < \infty$. Thus, we have $\liminf_n \tilde{f}_{n\to\infty} = \lim_{n\to\infty} \tilde{f}_n = \tilde{f}$. Now, by Fatou's Lemma, we have

$$\int f d\mu = \int \tilde{f} d\mu_n = \int \liminf_{n\to\infty} \tilde{f}_n d\mu_n \leq \liminf_{n\to\infty} \int \tilde{f}_n d\mu_n = \liminf_{n\to\infty} \int f d\mu_n$$

as desired.

Let $\epsilon > 0$. For general non-negative functions f, there exists simple function $0 \leq s \leq f$ such that $\int s d\mu_n \leq \int f d\mu_n$ and $\int f d\mu \leq \int s d\mu + \epsilon$. Thus,

$$\int f d\mu \leq \int s d\mu + \epsilon \leq \liminf_{n\to\infty} \int s d\mu_n + \epsilon \leq \liminf_{n\to\infty} \int f d\mu_n + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we have $\int f d\mu \leq \liminf_{n\to\infty} \int f d\mu_n$. □

Bass 15.2

Let $f \in L^p$ and choose a sequence of simple functions f_n such that $f_n \to f$, $|f_n| \leq |f|$.

If $p = \infty$, then the f_n can be chosen so that $f_n \to f$ uniformly where $f \leq \|f\|_\infty$ (see the construction in Proposition 5.14). Thus $|f - f_n|$ can be made arbitrarily small a.e., so $f_n \to f$ in L^∞.

Now we prove for $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$. Since f measurable, we have a sequence of simple functions $s_n \to f$ point-wise. Thus, we have $|f - s_n|^p \leq ||f| + |s_n||^p \leq 2^p |f|^p$ which is integrable. Thus, by Dominated Convergence Theorem, we have

$$\lim_{n\to\infty} \int |f - s_n|^p = \int \lim_{n\to\infty} |f - s_n|^p = 0$$

as desired.

Bass 15.4

Let $||f||_\infty = M$. Then

$$||f||_p = \Big(\int_0^1 f(x)^p dx\Big)^{\frac{1}{p}} \le \Big(\int_0^1 M^p\Big)^{\frac{1}{p}} = M.$$

So $||f||_p$ is bounded above by $||f||_\infty$. If $r < p$,

$$||f||_r^r = \int_0^1 f(x)^r dx \le (\int_0^1 f(x)^p dx)^{\frac{r}{p}} \int_0^1 1dx = ||f||_p^r,$$

i.e.

$$||f||_r \le ||f||_p$$

For any $\epsilon > 0$, the set $E = \{x | f(x) > M - \epsilon\}$ is of positive measure. Therefore,

$$\Big(\int_0^1 f(x)^p dx\Big)^{\frac{1}{p}} \ge \Big(\int_E (M-\epsilon)^p\Big)^{\frac{1}{p}} = (M-\epsilon) m(E)^{\frac{1}{p}}.$$

As p goes to infinity, $m(E)^{\frac{1}{p}} \to 1$. This shows that $\lim_{p\to\infty} ||f||_p \ge M - \epsilon$ for any ϵ, so $||f||_p$ converges to $||f||_\infty$.

Bass 15.6

We have $x^\alpha \in L^p(0,1)$ for $1 \le p < \frac{1}{\alpha}$ and $x^{-\alpha} \in L^p(1,\infty)$ for $\max(1, \frac{1}{\alpha}) < p$. Then $1 < p < \frac{1}{\alpha} < q < \infty$, x^α is in $L^p(0,1)$ but not $L^q(0,1)$ and $x^{-\alpha}$ is in $L^q(1,\infty)$ but not $L^p(1,\infty)$.

Bass 15.25

1. By definition

$$\begin{aligned} ||Tf||_1 &= \int_X |Tf| \mu(dx) \\ &= \int_X \Big| \int_X K(x,y) f(y) \mu(dy)) \Big| \mu(dx) \\ &\le \int_X \int_X |K(x,y)||f(y)| \mu(dy)\mu(dx) \\ &= \int_X \Big(\int_X |K(x,y)| \mu(dx) \Big) |f(y)| \mu(dy) \\ &\le \int_X M |f(y)| \mu(dy) \\ &= M ||f||_1 \end{aligned}$$

2. Let $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\begin{aligned} ||Tf||_p^p &= \int \Big| \int K(x,y) f(y)\,\mathrm{d}y \Big|^p \mathrm{d}x \\ &\le \int \Big(\int |K(x,y)|\,|f(y)|\,\mathrm{d}y \Big)^p \mathrm{d}x. \end{aligned} \tag{8}$$

Using Holder's inequality $(|K(x,y)|\,|f(y)| = |K(x,y)|^{\frac{1}{q}} (|K(x,y)|^{\frac{1}{p}}\,|f(y)|))$ gives

$$||Tf||_p^p \le \int \Big(M^{\frac{1}{q}} \Big(\int |K(x,y)|\,|f(y)|^p\,\mathrm{d}y \Big)^{\frac{1}{p}} \Big)^p \mathrm{d}x \tag{9}$$

Rearranging and applying Fubini's theorem again gives

$$\begin{aligned}\|Tf\|_p^p &\le M^{\frac{p}{q}} \int\int |K(x,y)|\,|f(y)|^p\,\mathrm{d}y\,\mathrm{d}x \\ &= M^{\frac{p}{q}} \int\int |K(x,y)|\,\mathrm{d}x\,|f(y)|^p\,\mathrm{d}y \\ &\le M^{1+\frac{p}{q}} \int |f(y)|^p\,\mathrm{d}y \\ &= M^{1+\frac{p}{q}}\,\|f\|_p^p,\end{aligned} \tag{10}$$

and taking p-th roots gives

$$\|Tf\|_p \le M^{\frac{1}{p}+\frac{1}{q}}\,\|f\|_p = M\,\|f\|_p\,. \tag{11}$$

Bass Week 3

Bass 16.1

We have

$$\widehat{\chi_{[a,b]}}(u) = \int e^{iux}\chi_{[a,b]}x\,\mathrm{d}x = \int_a^b e^{iux}\,\mathrm{d}x = \frac{e^{ibu}-e^{iau}}{iu}. \tag{12}$$

When $[a,b]=[-n,n]$, this is

$$\frac{1}{u}\frac{e^{inu}-e^{-inu}}{i} = \frac{2}{u}\sin nu. \tag{13}$$

Bass 16.4

Proof. By definition of Fourier transform and directional derivative,

$$\hat{f}_j(u) = \int_{\mathbb{R}^n} e^{iu\cdot x} f_j(x)dx = \int_{\mathbb{R}^n} e^{iu\cdot x}\lim_{h\to 0}\frac{f(x+he_j)-f(x)}{h}dx.$$

Since $|e^{iux}|$ is bounded by 1, we can put it into the limit. Thus,

$$\int_{\mathbb{R}^n} e^{iu\cdot x}\lim_{h\to 0}\frac{f(x+he_j)-f(x)}{h}dx = \int_{\mathbb{R}^n}\lim_{h\to 0}\left[e^{iu\cdot x}\frac{f(x+he_j)-f(x)}{h}\right]dx.$$

Furthermore, since $e^{iu\cdot x}\frac{f(x+he_j)-f(x)}{h}$ is bounded in absolute value by $|f_j(x)|$. Since f_j is integrable by assumption, the dominated convergence theorem applies. Thus, by DCT and proposition 16.1,

$$\begin{aligned}\int_{\mathbb{R}^n}\lim_{h\to 0}\left[e^{iu\cdot x}\frac{f(x+he_j)-f(x)}{h}\right]dx &= \lim_{h\to 0}\frac{1}{h}\left[\int_{\mathbb{R}^n} e^{iu\cdot x}\frac{f(x+he_j)-f(x)}{d}x\right] \\ &= \lim_{h\to 0}\frac{1}{h}\left[e^{-ihu_j}\hat{f}(u)-\hat{f}(u)\right] \\ &= \lim_{h\to 0}\frac{e^{-ihu_j}-1}{h}\hat{f}(u) = -iu_j\hat{f}(u)\end{aligned}$$

as desired. □

Bass 16.5

The product rule and closure of $\mathcal{S}$ under addition, scalar multiplication, and multiplication implies that $x^n f^{(m)}(x)\in\mathcal{S}$. Because $x^2 f\to 0$ (and f is bounded on $[-1,1]$), $f\in L^1$.

We then have $\frac{\mathrm{d}^n}{\mathrm{d}u^n}\hat{f}(u) = \frac{\mathrm{d}^n}{\mathrm{d}u^n}\int e^{iux}f(x)\,\mathrm{d}x = \int e^{iux}(ix)^n f(x)\,\mathrm{d}x = \widehat{(ix)^n f}(u)$, where differentiation under the integral sign can be justified at each step by the dominated convergence theorem (because $x^n f(x) \in \mathcal{S}$ for all n).

Because $f \in \mathcal{S}$, induction on Exercise 16.4 gives $\widehat{f^{(m)}}(x) = (-ix)^m \hat{f}(x)$.

One can check that $\mathcal{S}$ is closed under addition, scalar multiplication, multiplication, and differentiation, by the product rule $x^n f^{(m)}(x) \in \mathcal{S}$, so by the above two paragraphs,

$$\widehat{x^n f^{(m)}}(u) = (-i)^n \frac{\mathrm{d}^n}{\mathrm{d}u^n}\widehat{f^{(m)}}(u) = (-i)^{n+m}\frac{\mathrm{d}^n}{\mathrm{d}u^n}(u^m \hat{f}(u)). \tag{14}$$

By Riemann-Lebesgue, the left side goes to 0 as $|u| \to \infty$, so the left side does too. By the product rule, $f \in \mathcal{S}$ if and only if $\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^m f) \to 0$ for all n, m. Thus $\hat{f} \in \mathcal{S}$.

Bass 16.8

When the right is infinite, we're done.

It suffices to consider the case $a = b = 0$. Indeed, applying the inequality to $g(x) = e^{ibx}f(x+a)$ and applying a change of variables and properties of the Fourier transform, we have

$$\begin{aligned} \frac{\pi}{2}\left(\int |f(x)|^2\,\mathrm{d}x\right) = \frac{\pi}{2}\left(\int |g(x)|\right)^2 &\le \int |xg(x)|^2\,\mathrm{d}x \int |x\hat{g}(x)|^2\,\mathrm{d}x \\ &= \int \left|xe^{ibx}f(x+a)\right|^2\mathrm{d}x \int \left|x\widehat{f_a}(x+b)\right|^2\mathrm{d}x \\ &= \int |xf(x+a)|^2\,\mathrm{d}x \int \left|xe^{-iua}\hat{f}(x+b)\right|^2\mathrm{d}x \\ &= \int |(x-a)f(x)|^2\,\mathrm{d}x \int \left|(x-b)\hat{f}(x)\right|^2\mathrm{d}x. \end{aligned} \tag{15}$$

So now it suffices to prove that

$$\left(\int x^2|f(x)|^2dx\right)\left(\int |f'(u)|^2du\right) \ge \frac{1}{4}\left(\int |f|^2dx\right)^2.$$

To prove this, since $xf(x)$ and $f'(x)$ are in $L^2(\mathbb{R})$, $\lambda xf(x) + f'(x)$ is in $L^2(\mathbb{R})$ for any $\lambda \in$ \$. Thus,

$$\int |\lambda xf(x) + f'(x)|^2dx = \int \lambda^2|xf(x)|^2dx + \int 2|\lambda xf(x)f'(x)|dx + \int |f'(x)|^2dx.$$

Using integration by parts on the second term of right hand side, we get

$$\int 2|\lambda xf(x)f'(x)|dx = \left[\lambda|x||f(x)|^2\right]_{-\infty}^{\infty} - \lambda\int |f(x)|^2dx = -\lambda\int |f(x)|^2dx$$

since $xf(x) \in L^2(\mathbb{R})$. Thus, we have

$$\int |\lambda xf(x) + f'(x)|^2dx = \lambda^2\int |xf(x)|^2dx - \lambda\int |f(x)|^2dx + \int |f'(x)|^2dx \ge 0.$$

Thus, viewing as a quadratic in λ, we must have its discriminant smaller or equal to zero. Thus,

$$\left(\int |f(x)|^2\right)^2 - 4\left(\int |xf(x)|^2dx\right)\left(\int |f'(x)|^2dx\right) \ge 0,$$

which is equivalent of

$$\frac{1}{4}\left(\int |f(x)|^2\right)^2 \ge \left(\int |xf(x)|^2dx\right)\left(\int |f'(x)|^2dx\right),$$

as desired.

Bass 18.3

Let f_n be a Cauchy sequence in $C^k([0,1])$. Then for $\epsilon > 0$, there exists N such that for any $n, m \geq N$, $||f_n - f_m||_{C^k} < \epsilon$, i.e.

$$||f_n - f_m||_\infty + ||f'_n - f'_m||_\infty + \cdots + ||f_n^{(k)} - f_m^{(k)}||_\infty < \epsilon$$

Therefore $f_n^{(j)}$ converges uniformly almost everywhere for all j. Let $f^{(j)}$ be the limit of $f_n^{(j)}$. We need to show that the derivative of $f^{(j)}$ is $f^{(j+1)}$. Let g_j be the derivative of $f^{(j)}$. Then

$$g_j(x) = \lim_{h \to 0} \frac{f^{(j)}(x+h) - f^{(j)}(x)}{h}.$$

On the other hand, $f_n^{(j+1)}$ converges uniformly to $f^{(j+1)}$, so for any $\epsilon > 0$, there is some N such that for all $n \geq N$,

$$||f_n^{(j+1)}(x) - f^{(j+1)}(x)||_\infty < \epsilon.$$

Therefore, for any x and any h,

$$\Big| \int_x^{x+h} f_n^{(j+1)}(t)dt - \int_x^{x+h} f^{(j+1)}(t)dt \Big| \leq \int_x^{x+h} |f_n^{(j+1)}(t) - f^{(j+1)}(t)|dt \leq \epsilon h,$$

since the converges is uniform almost everywhere. This means $\int_x^{x+h} f_n^{(j+1)}(t)dt$ converges to $\int_x^{x+h} f^{(j+1)}(t)dt$.

Therefore

$$\begin{aligned} f^{(j)}(x+h) - f^{(j)}(x) &= \lim_{n\to\infty} (f_n^{(j)}(x+h) - f_n^{(j)}(x)) \\ &= \lim_{n\to\infty} \int_x^{x+h} f_n^{j+1}(t)dt \\ &= \int_x^{x+h} f^{(j+1)}(t)dt \end{aligned}$$

Now divided by h and let h go to zero, we see that $g_j = f^{j+1}$. Hence $C^k([0,1])$ is complete.

Bass 18.4

Let f_n be a Cauchy sequence in $C^\alpha([0,1])$. Because $\|\cdot\|_u \leq \|\cdot\|_{C^\alpha}$, the f_n form a Cauchy sequence in $C([0,1])$, so $f_n \to f$ for some $f \in C([0,1])$. We first show $f_n \to f$ in C^α. Because $f_n \to f$ in $C([0,1])$, it suffices to show we can take n big enough to control the second term. Fix $\epsilon > 0$. Because $\{f_n\}$ is Cauchy, there is N big enough such that for $n, m \geq N$, we have

$$\frac{|f_n(x) - f_n(y) - (f_m(x) - f_m(y))|}{|x-y|^\alpha} < \epsilon \tag{16}$$

for all $x, y \in [0,1]$. Fix n and pass to the limit $m \to \infty$ to get

$$\frac{|f_n(x) - f_n(y) - (f(x) - f(y))|}{|x-y|^\alpha} = \frac{|(f_n - f)(x) - (f_n - f)(y)|}{|x-y|^\alpha} \leq \epsilon \tag{17}$$

for all $x, y \in [0,1]$, so $f_n \to f$ in C^α.

It remains to show $f \in C^\alpha$. The reverse triangle inequality on eq. (17) gives

$$\frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha} - \frac{|f(x) - f(y)|}{|x-y|^\alpha} \leq \epsilon \tag{18}$$

for n sufficiently large. Because $\{f_n\}$ is Cauchy in C^α, it is bounded, so there exists $M > 0$ such that $\frac{|f_n(x)-f_n(y)|}{|x-y|^\alpha} \le \|f_n\|_{C^\alpha} \le M$ for all n. Rearranging and taking $n \to \infty$, we have

$$\frac{|f(x) - f(y)|}{|x-y|^\alpha} \le M + \epsilon, \tag{19}$$

and because $[0,1]$ is compact and (f is continuous), $\sup_{[0,1]} |f| < \infty$, so $f \in C^\alpha$. We conclude that $C^\alpha([0,1])$ is complete.

Bass 18.14

To see that A is closed, suppose $f_n \to f$ in $C([0,1])$ with $f_n \in A$. Fix $\epsilon > 0$ and take n sufficiently large so that $\|f_n - f\|_u < \epsilon$. Then

$$\begin{aligned}\left|\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f - 1\right| &= \left|\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f - \int_0^{\frac{1}{2}} f_n + \int_{\frac{1}{2}}^1 f_n\right| \\ &= \left|\int_0^{\frac{1}{2}} (f - f_n) - \int_{\frac{1}{2}}^1 (f - f_n)\right| \\ &\le \left|\int_0^{\frac{1}{2}} (f - f_n)\right| + \left|\int_{\frac{1}{2}}^1 (f - f_n)\right| \\ &\le \int_0^1 |f - f_n| \\ &< \epsilon.\end{aligned} \tag{20}$$

Because ϵ is arbitrary, we conclude that $f \in A$.

To see that A is convex, let $f, g \in A$ and $t \in [0,1]$. Then

$$\begin{aligned}\int_0^{\frac{1}{2}} tf + (1-t)g - \int_{\frac{1}{2}}^1 (tf + (1-t)g) &= t\left(\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f\right)(1-t)\left(\int_0^{\frac{1}{2}} g - \int_{\frac{1}{2}}^1 g\right) \\ &= t + 1 - t = 1,\end{aligned} \tag{21}$$

so $tf + (1-t)g \in A$.

We claim that $\inf_{f\in A} \|f\| = 1$. Indeed, if $\|f\| < 1$, then

$$\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f < \int_0^{\frac{1}{2}} 1 - \int_{\frac{1}{2}}^1 (-1) = 1, \tag{22}$$

so 1 is a lower bound for $\|f\|$. On the other hand, given $\epsilon > 0$, the continuous (piecewise linear) function with odd symmetry around $x = \frac{1}{2}$ given by

$$f(x) = \begin{cases} 1 + \epsilon & 0 \le x \le \frac{1}{2} - \delta \\ 1 + \epsilon - \frac{1+\epsilon}{\delta}(x - \frac{1}{2} + \delta) & \frac{1}{2} - \delta \le x \le \frac{1}{2} \\ -f(1-x) & \frac{1}{2} \le x \le 1, \end{cases} \tag{23}$$

where

$$\delta = \frac{\epsilon}{(1+\epsilon)} \tag{24}$$

clearly has $\|f\| = 1 + \epsilon$, and we will check (this is easier with a picture) that $f \in A$. Thus $\inf_{f\in A} \|f\| = 1$. By symmetry $\int_0^{\frac{1}{2}} f - \int_{\frac{1}{2}}^1 f = 2\int_0^{\frac{1}{2}} f$, and the graph of f on $[0, \frac{1}{2}]$ is a rectangle with height $1 + \epsilon$ missing a triangle of width δ and height $1 + \epsilon$, and thus has total area $\frac{1}{2}(1+\epsilon) - \frac{1}{2}\delta(1+\epsilon) = \frac{1}{2}(1 + \epsilon - \frac{\epsilon}{1+\epsilon}(1+\epsilon)) = \frac{1}{2}$, so $f \in A$. Now, we need to show that there are no continuous function with norm 1 in set A. Suppose such function, call it g exists. We first show that g must be a constant function, that is, for all $x \in [0,1]$, we mush have $|g(x)| = 1$. Suppose not, since $\|g\|= 1$, we know that $|g(x)| \leq 1$ for all $x \in [0,1]$. Suppose there exists $x \in [0,1]$ such that $|g(x)| = c < 1$. Then by intermediate value theorem, there exists an open interval around x , say $(x-\delta, x+\delta)$ such that for all $y \in (x-\delta, x+\delta)$, $|g(y)| < 1$. Thus, if we evaluate the integral, we get

$$\begin{aligned} \left|\int_0^{\frac{1}{2}} g(x)dx - \int_{\frac{1}{2}}^1 g(x)dx\right| &\leq \left|\int_0^{\frac{1}{2}} g(x)dx\right| + \left|\int_{\frac{1}{2}}^1 g(x)dx\right| \\ &\leq \int_0^1 |g(x)|dx \\ &\leq (\int_0^{x-\delta} + \int_{x+\delta}^1)dx + \int_{x-\delta}^{x+\delta} |g(x)|dx < 1, \end{aligned}$$

Thus, this shows that g is not in A. Therefore, we must have $|g(x)| = 1$ for all $x \in [0,1]$. And since g is continuous, we must have $g \equiv 1$ or -1. But both of which does not satisfy the condition to be in set A, so such g does not exists. This finishes our proof.

Bass 18.15

We claim A_n is closed. Suppose $f_k \to f$ uniformly with $f_k \in A_n$. For each f_k, there is some $x_k \in [0,1]$ such that $\frac{f(x)-f(y)}{|x-y|} \leq n$ for all $y \in [0,1]$. Passing to a subsequence by Bolzano-Weierstrass and relabelling, we can take $x_k \to x_0$ for some $x_0 \in [0,1]$. Fix $\epsilon > 0$. By (a weak version of the converse of) Arzela-Ascoli (I'm doing this to say $f_k(x_k) \to f(x)$), the f_k are uniformly equicontinuous; that is, there exists δ such that $|x-y| < \delta$ implies $|f_k(x) - f_k(y)| < \epsilon$ for all k. Let k be big enough so that $\|f - f_k\| < \epsilon$ (uniform convergence) and $|x_0 - x_k| < \delta$. Then for any fixed y,

$$\begin{aligned} |f(x_0) - f(y)| &\leq |f(x_0) - f_k(x_0)| + |f_k(x_0) - f_k(x_k)| \\ &\quad + |f_k(x_k) - f_k(y)| + |f_k(y) - f(y)| \\ &< 3\epsilon + n\,|x_k - y| \\ &< 3\epsilon + n\,|x_k - x_0| + n\,|x_0 - y| \\ &< (n+3)\epsilon + n\,|x_0 - y|\,. \end{aligned} \tag{25}$$

Because $\epsilon > 0$ was arbitrary, we conclude that $|f(x_0) - f(y)| \leq n\,|x_0 - y|$, so $f \in A_n$.

Because A_n is closed, to show it is nowhere dense it suffices to show that it contains no open interval. We will show that for any $f \in A_n$ and $\epsilon > 0$, there exists $g \in C([0,1])$ with $\|f-g\|_u < \epsilon$. The idea is to make a spiky function that follows the curve of f. Let M be an integer to be specified shortly and $g \in C([0,1])$ be defined as $g(\frac{k}{M}) = (-1)^k$ for $0 \leq k \leq M$ and g be linear on each $[\frac{k}{M}, \frac{k+1}{M}]$. Then $\|\epsilon g\| = \|f - (f+\epsilon g)\| \leq \epsilon$, but $f + \epsilon g \notin A_n$: for any x, there exists y_0 close to x with $|g(x) - g(y_0)| = 2M\,|x - y_0|$,

$$\begin{aligned} |f(x) - f(y_0) + \epsilon g(x) - \epsilon g(y_0)| &\geq \epsilon\,|g(x) - g(y)| - |f(x) - f(y_0)| \\ &\geq (2M\epsilon - n)\,|x - y_0|\,. \end{aligned} \tag{26}$$

Taking $M > \frac{2n}{\epsilon}$ makes the right side strictly greater than $n\,|x - y_0|$. Thus A_n is nowhere dense.

By the Baire category theorem, $C([0,1]) \setminus \bigcup_n A_n$ is non-empty, so there exists $f \in C([0,1])$ such that for all x, there exists $\{y_n\}$ such that $\frac{|f(x)-f(y)|}{|x-y|} > n$. Fix x and suppose the corresponding sequence $\{y_n\}$ is bounded away from x. Then $\frac{|f(x)-f(y_n)|}{|x-y_n|} \leq M$ for some M (because the denominator is bounded away from 0), which yields a contradiction with the left side being greater than n as $n \to \infty$. Thus we have some subsequence $y_{n_k} \to x$, so we cannot have $\lim_{y \to x} \frac{f(x)-f(y_n)}{x-y} < \infty$. That is, f is not differentiable at x.

Evans Chapter 5

Evans 5.1

It is clear that this is a vector space. Homogeneity and the triangle inequality of $\|\cdot\|_{C^{k,\gamma}}$ follow from the respective properties for $|\cdot|$ and $\|\cdot\|_{C(\overline{U})}$. If $\|u\|_{C^{k,\gamma}} = 0$, then $\|u\|_{C(\overline{U})} = 0$, so $u \equiv 0$.

The proof of completeness is essentially the same as for the completeness of $C^{0,\gamma}(\overline{U})$ (done as a Bass exercise). Let $\{u_n\}$ be Cauchy in $C^{k,\gamma}(\overline{U})$. Because $\|D^\alpha u\|_{C(\overline{U})} \le \|u\|_{C^{k,\gamma}(\overline{U})}$ for all $|\alpha| \le k$ and $u \in C^{k,\gamma}$, each $D^\alpha u_n$ is Cauchy in $C(\overline{U})$ for $|\alpha| \le k$, so because this space is complete, each $D^\alpha u_n$ converges uniformly. In particular, let $u_n \to u$ uniformly. Induction (apply to each component) on the fact from single variable analysis that if $f_n \in C^1$, $f_n \to f$ uniformly, and $f_n' \to g$ uniformly, then $f' = g$ allows us to conclude that $D^\alpha u_n \to D^\alpha u$ in $C(\overline{U})$ for each $|\alpha| \le k$. This allows us to take N large enough so that $\|D^\alpha u - D^\alpha u_n\|_{C(\overline{U})} < \epsilon$ for all $|\alpha| \le k$ and $n > N$. Take N possibly larger so that for $n, m > N$, we also have $[D^\alpha u_n - D^\alpha u_m]_{C^{0,\gamma}(\overline{U})} \le \|u_n - u_m\|_{C^{k,\gamma}(\overline{U})} < \epsilon$ for $|\alpha| = k$. Then for $n, m > N$,

$$\frac{|D^\alpha u_m(x) - D^\alpha u_m(x) - (D^\alpha u_n(x) - D^\alpha u_n(y))|}{|x-y|^\gamma} < \epsilon \tag{27}$$

for all $x \ne y \in U$. Passing to the limit $m \to \infty$ gives

$$\frac{|(D^\alpha u - D^\alpha u_n)(x) - (D^\alpha u - D^\alpha u_n)(y)|}{|x-y|^\gamma} < \epsilon \tag{28}$$

for all $x \ne y \in U$, so $[D^\alpha u - D^\alpha u_n]_{C^{0,\gamma}(\overline{U})} < \epsilon$ for $n > N$. Thus for $n > N$,

$$\begin{aligned} \|u - u_n\|_{C^{k,\gamma}} &= \sum_{|\alpha| \le k} \|D^\alpha u - D^\alpha u_n\|_{C(\overline{U})} + \sum_{|\alpha| = k} [D^\alpha u - D^\alpha u_n]_{C^{0,\gamma}(\overline{U})} \\ &< \sum_{|\alpha| \le k} \epsilon + \sum_{|\alpha| = k} \epsilon = C\epsilon. \end{aligned} \tag{29}$$

That is, $u_n \to u$ in $C^{k,\gamma}$. It remains to check that $u \in C^{k,\gamma}$. Because each $D^\alpha u \in C(\overline{U})$, we just need to show that $[D^\alpha u]_{C^{0,\gamma}(\overline{U})} < \infty$ for $|\alpha| = k$. Applying the reverse triangle inequality to eq. (28) gives

$$\frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\gamma} - \frac{|D^\alpha u_n(x) - D^\alpha u_n(y)|}{|x-y|^\gamma} < \epsilon \tag{30}$$

for all $x \ne y \in U$ for n large enough, so taking a supremum over $x \ne y \in U$ gives

$$[D^\alpha u]_{C^{0,\gamma}} < [D^\alpha u_n]_{C^{0,\gamma}} + \epsilon < \infty. \tag{31}$$

Evans 5.4

The same proof should work for $p = \infty$, but the book proves that $W^{1,\infty}(U)$ is Lipschitz functions on U when ∂U is C^1.

Part A

By problem set 0, $L^p(0,1) \subset L^1(0,1)$. If $u \in W^{1,p}(0,1)$ for $1 \le p < \infty$, then u has a weak derivative $v \in L^p \subset L^1$. By a classic theorem, because $v \in L^1([0,1])$, $F(x) := \int_0^x v$ is absolutely continuous.

For any $\varphi \in C_c^\infty(0,1)$, we have

$$\int_0^1 (F-u)\varphi' = \int_0^1 F\varphi' - \int_0^1 u\varphi' = -\int_0^1 v\varphi + \int_0^1 v\varphi = 0, \tag{32}$$

where the second last equality holds by integration by parts (F is absolutely continuous) and the definition of a weak derivative.

We claim$f' = 0$ a.e. implies $f = C$ a.e. From this we are done, because $(F-u)'$ a.e., so we have $u = F + C$ a.e. for some constant C. Thus u agrees a.e. with the absolutely continuous function $F + C$.

Now we prove the claim.

Proof 1, on $(0,1)$. Fix $\xi \in C_c^\infty(0,1)$ and $\eta \in C_c^\infty(0,1)$ with $\int_0^1 \eta = 1$. Define

$$\varphi(x) := \int_0^x \left(\xi(t) - \eta(t) \int_0^1 \xi(s)\,\mathrm{d}s \right) \mathrm{d}t. \tag{33}$$

It is clear that $\varphi \in C^\infty(0,1)$ (because φ' is a linear combination of things in C_c^∞), and also $\varphi(0) = 0$, and

$$\varphi(1) = \int_0^1 \xi(t)\,\mathrm{d}t - \int_0^1 \eta(t)\,\mathrm{d}t \int_0^1 \xi(s)\,\mathrm{d}s = 0 \tag{34}$$

because $\int_0^1 \eta = 1$. Thus $\varphi \in C_c^\infty(0,1)$. Because $f' = 0$ a.e.,

$$0 = -\int_0^1 f'\varphi = \int_0^1 f\varphi' = \int_0^1 f(x) \left(\xi(x) - \eta(x) \int_0^1 \xi(t)\,\mathrm{d}t \right) \mathrm{d}x, \tag{35}$$

which gives

$$\int_0^1 f(x)\xi(x)\,\mathrm{d}x = \int_0^1 f(x)\eta(x)\,\mathrm{d}x \int_0^1 \xi(t)\,\mathrm{d}t, \tag{36}$$

so we conclude that $\int_0^1 (f - C)\xi = 0$, with $C := \int_0^1 f\eta$. Because $\xi \in C_c^\infty(0,1)$ was arbitrary, it must be that $f = C$ a.e. □

Proof 2, on connected domain. For another proof, suppose U is connected with $f \in W^{1,p}(U)$ and $Df = 0$ a.e. Then $f * \eta_\epsilon$ is smooth, so $D(f * \eta_\epsilon) = Df * \eta_\epsilon = 0$, which implies $f * \eta_\epsilon = C_\epsilon$. For any $V \subset\subset U$, $C_\epsilon = f * \eta_\epsilon \to f$ in $L^1(V)$. Then $C_\epsilon \to C$ in $\mathbb{R}$ for some C, because C_ϵ converges and is thus Cauchy in L^1, and the domain is compact, so it is also Cauchy in $\mathbb{R}$. Pick a subsequence $C_{\epsilon_k} \to f$ a.e. Because $|C_{\epsilon_k} - f| \le |C_{\epsilon_k}| + |f|$, which is integrable because the first term is bounded and the domain is compact, by the dominated convergence theorem, $\int_V |f - C| = 0$. Thus $f = C$ a.e. on V, and because this holds for all $V \subset\subset U$, we conclude $f = C$ a.e. on U. □

Part B

Suppose $x \le y$. Using the previous part we have that

$$|u(x) - u(y)| = \Big| \int_x^y v(t)dt \Big| \le \int_x^y |v(t)|dt.$$

Using the Holder's Inequality,

$$\int_x^y |v(t)|dt \le ||v||_p ||1||_q = \Big(\int_x^y |v(t)|^p dt \Big)^{1/p} |x - y|^{1 - \frac{1}{p}}.$$

Since $|v(t)|$ is non-negative,

$$\int_x^y |v(t)|^p dt \le \int_0^1 |v(t)|^p dt.$$

This concludes the proof.

Evans 5.5

Take W such that $V \subset\subset W \subset\subset U$. Consider $\chi_W^\epsilon = \eta_\epsilon * \chi_W$, which are smooth on W_ϵ. We can choose ϵ small enough such that $V \subset W_\epsilon$, then χ_W^ϵ is 1 on V and 0 near ∂U.

Evans 5.7

When $1 < p < \infty$, $|u|^p \in C^1(\overline{U})$, so by Gauss-Green, we have

$$\int_{\partial U} |u|^p \,\mathrm{d}S \le \int_{\partial U} |u|^p \,\boldsymbol{\alpha}\cdot\boldsymbol{\nu}\,\mathrm{d}S \le \int_U \operatorname{div}(|u|^p\,\boldsymbol{\alpha})\,\mathrm{d}x. \tag{37}$$

By the product rule, this is

$$\int_{\partial U} |u|^p \,\mathrm{d}S \le \int_U D\,|u|^p\cdot\boldsymbol{\alpha} + |u|^p \operatorname{div}\boldsymbol{\alpha}\,\mathrm{d}x. \tag{38}$$

The first term can be controlled by Young's inequality $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $a, b \ge 0$ and $\frac{1}{p} + \frac{1}{q} = 1$:

$$|D\,|u|^p| = p\,|u|^{p-1}\,|Du| \le p\left(\frac{p-1}{p}\,|u|^p\right) + \frac{|Du|^p}{p} = C(|u|^p + |Du|^p), \tag{39}$$

so

$$\int_U D\,|u|^p\cdot\boldsymbol{\alpha}\,\mathrm{d}x \le \int_U |\boldsymbol{\alpha}|\,|D\,|u|^p| \le C\int_U |u|^p + |Du|^p\,\mathrm{d}x, \tag{40}$$

because $|\boldsymbol{\alpha}|$ is bounded on U ($\boldsymbol{\alpha}$ is a smooth vector field on U defined on ∂U, and $\overline{U}$ is bounded and thus compact). The second term is bounded by $C\int_U |u|^p\,\mathrm{d}x$ because $|\operatorname{div}\boldsymbol{\alpha}|$ is bounded (as $|\operatorname{div}\boldsymbol{\alpha}| \le C\,|D\boldsymbol{\alpha}|$). Thus

$$\int_{\partial U} |u|^p\,\mathrm{d}S \le C\int_U |Du|^p + |u|^p\,\mathrm{d}x. \tag{41}$$

When $p = 1$, $|u|$ is not C^1, but we can approximate it as $\sqrt{u^2+\epsilon^2}$, which is $C^1(\overline{U})$ because $D^\alpha\sqrt{u^2+\epsilon^2} = \frac{uD^\alpha u}{\sqrt{u^2+\epsilon^2}} \in C(\overline{U})$ for $|\alpha| = 1$. Then by Gauss-Green and the product rule (same inequalities as above),

$$\int_{\partial U} \sqrt{u^2+\epsilon^2}\,\mathrm{d}S \le \int_U D\sqrt{u^2+\epsilon^2}\cdot\boldsymbol{\alpha} + \sqrt{u^2+\epsilon^2}\operatorname{div}\boldsymbol{\alpha}\,\mathrm{d}x. \tag{42}$$

Because $\boldsymbol{\alpha}$ is smooth up to ∂U (same compactness arguments as above), we have

$$\int_{\partial U} \sqrt{u^2+\epsilon^2}\,\mathrm{d}S \le C\int_U \left|D\sqrt{u^2+\epsilon^2}\right| + \left|\sqrt{u^2+\epsilon^2}\right|\mathrm{d}x. \tag{43}$$

As $\epsilon \to 0$, $\left|D^\alpha\sqrt{u^2+\epsilon^2}\right| = \left|\frac{u}{\sqrt{u^2+\epsilon^2}}\right|\,|D^\alpha u|$ increases to $|D^\alpha u|$, and $\sqrt{u^2+\epsilon^2}$ decreases to $|u|$. Because U is bounded (so that $\sqrt{u^2+\epsilon^2}$ is $L^1(U)$ for $\epsilon > 0$), the monotone convergence theorem allows us to pass to the limit as $\epsilon \to 0$ and conclude that

$$\int_{\partial U} |u|\,\mathrm{d}S \le C\int_U |Du| + |u|\,\mathrm{d}x. \tag{44}$$

Evans 5.15

Not done.

First

$$\begin{aligned}(u)_U &= \frac{1}{|U|}\int_U |u| = \frac{1}{|U|}\int_{U-\{u=0\}} |u| \\ &\le \frac{|U-\{u=0\}|^{\frac{1}{2}}}{|U|}\,\|u\|_{L^2(U-\{u=0\})} \le \frac{(|U|-\alpha)^{\frac{1}{2}}}{|U|}\,\|u\|_2\,.\end{aligned} \tag{45}$$

By the reverse triangle inequality in Poincare's inequality,

$$\begin{aligned}C\,\|Du\|_2 \ge \|u-(u)_U\|_2 &\ge \|u\|_2 - \|(u)_U\|_2 \\ &\ge \|u\|_2 - |U|^{\frac{1}{2}}\,\frac{(|U|-\alpha)^{\frac{1}{2}}}{|U|}\,\|u\|_2 \\ &\ge \left(1-\sqrt{1-\frac{\alpha}{|U|}}\right)\|u\|_2\,,\end{aligned} \tag{46}$$

with C depending only on n (because the domain U is fixed). We conclude that

$$\|u\|_2 \le C\,\|Du\|_2\,, \tag{47}$$

with the constant depending only on n and α.

Evans 5.17

Throughout, let $M = \sup|F'|$ be the Lipschitz constant of F and $\varphi \in C^\infty(U)$.

First we verify that $F(u)$ and $F'(u)u_{x_i}$ are in L^p. First suppose $1 \le p < \infty$. We have $F(x) \le F(0) + M\,|x| \le 2\max(F(0), M\,|x|)$ on $\mathbb{R}$, so

$$\int_U |F(u)|^p \le 2\int_U (\max(|F(0)|^p\,, M^p\,|u|^p)) < \infty, \tag{48}$$

because U is finite measure (so the constant function $F(0)$ is $L^p(U)$) and $u \in L^p(U)$. Thus $v = F(u) \in L^p$. Also,

$$\int_U |F'(u)u_{x_i}|^p \le C\int_U |u_{x_i}|^p < \infty \tag{49}$$

because F' is bounded and $u_{x_i} \in L^p(U)$. When $p = \infty$, we have $|F(u)| \le \max(F(0), M\,|u|)$ (again for M the Lipschitz constant of F) and $|F'(u)u_{x_i}| \le M\,|u_{x_i}|$, so both are in $L^\infty(U)$.

Now we verify that $v_{x_i} = F'(u)u_{x_i}$ (in the weak sense). For $1 \le p < \infty$, choose a sequence $\{u^n\} \subset C^\infty(U)$ with $u^n \to u$ in $W^{1,p}(U)$ (possible because U is bounded). Because F is C^1 and u^n are smooth, then we can integrate by parts to get

$$\int_U F(u^n)\varphi_{x_i} = -\int_U F'(u^n)u^n_{x_i}\varphi. \tag{50}$$

We are done if we can pass to the limit as $n \to \infty$. On the left side,

$$\int_U |(F(u^n) - F(u))\varphi_{x_i}| \le \|F(u^n) - F(u)\|_p\,\|\varphi_{x_i}\|_q \le M\,\|\varphi_{x_i}\|_q\,\|u^n - u\|_p \to 0, \tag{51}$$

so

$$\int_U F(u^n)\varphi_{x_i} \to \int_U F(u)\varphi_{x_i}. \tag{52}$$

Now we analyze the right side $\int_U F'(u^n)u^n_{x_i}\varphi$. Choose some subsequence of u^n and refine it to a subsequence converging a.e. to u. Refine to a further subsequence of $u^n_{x_i}$ converging a.e. to u_{x_i}. Then $F'(u^{n_k})u^{n_k}_{x_i} \to F'(u)u_{x_i}$ a.e. Now $\left|F'(u^{n_k})u^{n_k}_{x_i} - F'(u)u_{x_i}\right| \le C(\left|u^{n_k}_{x_i}\right| + |u_{x_i}|) \in L^1(U)$ (constant comes from F' bounded), so by the dominated convergence theorem, $F'(u^{n_k})u^{n_k}_{x_i} \to F'(u)u_{x_i}$ in $L^1(U)$. But because every subsequence of $F'(u^n)u^n_{x_i}$ has a subsequence converging in $L^1(U)$ to $F'(u)u_{x_i}$, the whole sequence $F'(u^n)u^n_{x_i} \to F'(u)u_{x_i}$ in L^1. We conclude that

$$\int_U F'(u^n)u^n_{x_i}\varphi \to \int_U F'(u)u_{x_i}\varphi. \tag{53}$$

WARNING: not sure if this works.

When $p = \infty$, we can't approximate by smooth functions. But the book says that $W^{1,\infty}(U)$ is the space of Lipschitz functions on U (if ∂U is C^1), and these functions are differentiable a.e. (and the weak derivative coincides with the classical derivative) so we can integrate by parts (on the set where u is differentiable) to get

$$\int_U F(u)\varphi_{x_i} = -\int_U F'(u)u_{x_i}\varphi. \tag{54}$$

This proof could probably be extended to the case where U is unbounded, as long as $F(0) = 0$, because the boundedness of U was only used to argue that $F(0)$ is L^p and approximate elements of $W^{1,p}(U)$ by smooth functions, but we can just do that locally (because we are always integrating on $\operatorname{supp}\varphi$).

Evans 5.18

Let $F_\epsilon(t) = \sqrt{t^2+\epsilon^2} - \epsilon$. Then $F_\epsilon{}'(t) = \frac{t}{\sqrt{t^2+\epsilon^2}}$ which is continuous and bounded by 1, so by the chain rule (previous exercise, applies because U is bounded),

$$\int_U F_\epsilon(u)\varphi_{x_i} = -\int_U F_\epsilon{}'(u) u_{x_i}\varphi. \tag{55}$$

We have $F_\epsilon(t) \to |t|$ as $\epsilon \to 0$, and $F_\epsilon(t) \le |t|$ (start with $z^2+\epsilon^2 \le z^2+\epsilon^2+2\epsilon$ and take square roots). Moreover, $F_\epsilon{}'(u) \to \operatorname{sgn} u$ pointwise as $\epsilon \to 0$, and $F_\epsilon{}'(u)u_{x_i} \le |u_{x_i}|$. Because φ is $C_c^\infty(U)$, we can use the dominated convergence theorem to pass to the limit as $\epsilon \to 0$ and obtain

$$\int_U |u|\,\varphi_{x_i} = -\int_U \operatorname{sgn} u u_{x_i}\varphi, \tag{56}$$

so we conclude that $|u| \in W^{1,p}(U)$ with $|u|_{x_i} = u_{x_i}\operatorname{sgn} u$ (a.e.).

1. By part(b), $|u| = u^+ - u^-$, a linear combination of functions in $W^{1,p}$.
2. Consider F_ϵ as given. Then by the chain rule,

$$DF_\epsilon = \begin{cases} \frac{u}{\sqrt{u^2+\epsilon^2}} Du & \text{if } u > 0 \\ 0 & \text{if } u \le 0 \end{cases}$$

   This converges to the proposed Du^+ as ϵ goes to 0. Also, for any $\phi \in C_c^\infty(U)$,

$$\int_U F_\epsilon \phi' = -\int_U DF_\epsilon \phi.$$

   Letting ϵ goes to 0 (to see justification for this, see above), we have

$$\int_U u^+\phi' = -\int_U Du^+\phi.$$

   Same can be done for u^-.

3. Now $Du = Du^+ - Du^-$, and for $u = 0$, $Du^+ = Du^- = 0$ a.e.. Hence $Du = 0$ a.e. on the set $\{u = 0\}$.

Evans 5.19

Note: it seems like we only need $Du^\epsilon \rightharpoonup 0$ in L^2, and not $u^\epsilon \rightharpoonup 0$ in L^2, but everything is proved here.

First we show $\|u^\epsilon\|_{H^1}$ is bounded uniformly in ϵ. Because $\varphi(0) = 0$ and φ' is bounded, $|\varphi(x)| \le C\,|x|$. Then

$$\|u^\epsilon\|_2^2 = \int_U |u^\epsilon|^2 = \int_U \left|\epsilon\varphi(\frac{u}{\epsilon})\right|^2 \le C\int_U |u|^2 = C\,\|u\|_2^2. \tag{57}$$

Also,

$$\|Du^\epsilon\|_2^2 = \int_U |Du^\epsilon|^2 = \int_U \left|\varphi'(u\epsilon^{-1})\right|^2 |Du|^2 \le C\,\|Du\|_2^2, \tag{58}$$

so $\|u^\epsilon\|_{H^1} \le C\,\|u\|_{H^1} \le C$.

By the Riesz representation theorem for Hilbert spaces, we want to show that

$$\langle u^\epsilon, v\rangle_{H^1(U)} \to 0 \tag{59}$$

for each $v \in H^1(U)$. That is,

$$\int_U u^\epsilon v + Du^\epsilon \cdot Dv \to 0. \tag{60}$$

First suppose $v \in C_c^\infty(U)$. For the first term,

$$\int_U u^\epsilon v = \int_U \epsilon\varphi\left(u\epsilon^{-1}\right) v \leq \epsilon \left\|\varphi\right\|_\infty \left\|v\right\|_1 \to 0, \tag{61}$$

because φ is bounded and $v \in C_c^\infty(U)$. For the second term, integrating by parts and applying the above gives

$$\int_U u^\epsilon_{x_i} v_{x_i} = \int_U u^\epsilon v_{x_i x_i} \to 0 \tag{62}$$

for each x_i, so $\int_U Du^\epsilon \cdot Dv \to 0$.

Now let $v \in H^1(U)$. Fix $\delta > 0$ and choose $\psi \in C_c^\infty(U)$ with $\left\|v - \psi\right\|_{H^1} < \delta$. Then take ϵ sufficiently small so that $\left|\int_U u^\epsilon \psi\right| < \delta$. Then

$$\left|\int_U u^\epsilon v\right| \leq \int_U \left|u^\epsilon(v-\psi)\right| + \left|\int_U u^\epsilon \psi\right| < \delta + \left\|u^\epsilon\right\|_2 \left\|v-\psi\right\|_2 < \delta + C\delta \tag{63}$$

Also

$$\left|\int Du^\epsilon \cdot Dv\right| \leq \int_U \left|Du \cdot D(v-\psi)\right| + \left|\int_U Du \cdot D\psi\right| < \delta + \left\|Du^\epsilon\right\|_2 \left\|Dv - D\psi\right\|_2 < C\delta, \tag{64}$$

Because δ was arbitrary, we conclude that $u^\epsilon \rightharpoonup 0$ as $\epsilon \to 0$.

Finally,

$$\left|\int_U \varphi'(u\epsilon^{-1}) Du \cdot Du\right| = \int_U \varphi'(u\epsilon^{-1}) \left|Du\right|^2, \tag{65}$$

where we remove the absolute value bars because φ' is non-negative. Near $x = 0$, $\varphi(x) = x$, so $\varphi'(0) = 1$. Then because the integrand is non-negative,

$$\int_{\{u=0\}} \left|Du\right|^2 = \int_{\{u=0\}} \varphi'(u\epsilon^{-1}) \left|Du\right|^2 \leq \int_U \varphi'(u\epsilon^{-1}) \left|Du\right|^2 \to 0 \tag{66}$$

as $\epsilon \to 0$, but the left side is independent of ϵ, so the left side is equal to 0. Thus $Du = 0$ a.e. on $\{u = 0\}$.

Evans 5.21

We start with

$$1 + |y|^s \leq 1 + (|x| + |y - x|)^s \leq 1 + C(|x|^s + |y - x|^s) \leq C(1 + |x|^s) + C(1 + |y - x|^s), \tag{67}$$

where we used the inequality $a + b \leq C(a^s + b^s)^{\frac{1}{s}}$ for $a, b \geq 0$(this is the statement that $\left\|\cdot\right\|_1 \leq C \left\|\cdot\right\|_s$ in dimension 2 (Holder's) for $s \geq 1$, and for $0 < s < 1$ we can show $\left\|\cdot\right\|_1 \leq \left\|\cdot\right\|_s$). The constant depends only on s.

We want to show $(1 + |y|^s)\widehat{uv} = (1 + |y|^s)\hat{u} * \hat{v} \in L^2$. We have

$$\begin{aligned}
(1 + |y|^s)\hat{u} * \hat{v}(y) &= \int (1 + |y|^s)\hat{u}(x)\hat{v}(y - x)\,\mathrm{d}x \\
&\leq C \int (1 + |x|^s)\hat{u}(x)\hat{v}(y - x)\,\mathrm{d}x + C \int (1 + |y - x|^s)\hat{u}(x)\hat{v}(y - x)\,\mathrm{d}x \\
&= C((1 + |x|)^s \hat{u}) * \hat{v} - C \int (1 + |t|^s)\hat{u}(y - t)\hat{v}(t)\,\mathrm{d}t \\
&= C((1 + |x|)^s \hat{u}) * \hat{v} - C((1 + |x|^s)\hat{v}) * \hat{u},
\end{aligned} \tag{68}$$

where we made the substitution $t = y - x$. Then

$$\begin{aligned}
\left\|(1 + |y|^s)\widehat{uv}\right\|_2 &\leq C \left\|((1 + |x|^s)\hat{u}) * \hat{v}\right\|_2 + C \left\|((1 + |x|^s)\hat{v}) * \hat{u}\right\|_2 \\
&\leq C \left\|(1 + |x|^s)\hat{u}\right\|_2 \left\|\hat{v}\right\|_1 + C \left\|(1 + |x|^s)\hat{v}\right\|_2 \left\|\hat{u}\right\|_1 \\
&\leq C \left\|u\right\|_{H^1} \left\|v\right\|_1 + C \left\|v\right\|_{H^1} \left\|u\right\|_1,
\end{aligned} \tag{69}$$

and we are done because

$$\|f\|_1 = \int |f| \leq \int (1+|x|^s)^{-1}(1+|x|^s)\,|f(x)|\,\mathrm{d}x \leq \left(\int (1+|x|^s)^{-2}\right)^{\frac{1}{2}} \|f\|_{H^1}\,, \tag{70}$$

and the integral converges because $s > \frac{n}{2}$. Thus

$$\|uv\|_{H^1} \leq C\,\|u\|_{H^1}\,\|v\|_{H^1}\,, \tag{71}$$

with the constant depending on n and s.

Evans Chapter 6

Evans 6.1

We compute

$$\begin{aligned}
\operatorname{div}(w^2 D(\frac{u}{w})) &= \sum_{i=1}^n \partial_i(w^2\partial_i(\frac{u}{w})) \\
&= \sum_{i=1}^n \partial_i(w^2\frac{u_i w - u w_i}{w^2}) \\
&= \sum_{i=1}^n u_{ii}w + u_i w_i - u_i w_i - w_{ii}u \\
&= w\Delta u - u\Delta w \\
&= wcu - ucw \\
&= 0.
\end{aligned} \tag{72}$$

Using the product rule, the divergence structure condition says $a\Delta v + Dv\cdot Da = 0$. Substituting $u := va^{\frac{1}{2}}$ into Laplace's equation with potential and using the product rule $\Delta fg = f\Delta g + g\Delta f + 2Df\cdot Dg$, we compute

$$\begin{aligned}
\Delta(va^{\frac{1}{2}}) &= v\Delta a^{\frac{1}{2}} + a^{\frac{1}{2}}\Delta v + 2Da^{\frac{1}{2}}\cdot Dv \\
&= v\Delta a^{\frac{1}{2}} + a^{\frac{1}{2}}\Delta v + a^{-\frac{1}{2}}Da\cdot Dv \\
&= v\Delta a^{\frac{1}{2}} + a^{-\frac{1}{2}}(a\Delta v + Da\cdot Dv) \\
&= v\Delta a^{\frac{1}{2}}.
\end{aligned} \tag{73}$$

If we take $c := a^{-\frac{1}{2}}\Delta a^{\frac{1}{2}}$, then we have $\Delta u = cu$ for $u = va^{\frac{1}{2}}$.

Evans 6.2

Define the bilinear operator

$$B[u,v] = \int_U \sum_{i,j=1}^n a^{ij}u_{x_i}v_{x_j} + cuv\,dx.$$

We check it satisfies the requirements for Lax-Milgram. First, for $u, v \in H_0^1(U)$,

$$|B[u,v]| \leq \sum_{i,j=1}^n \|a^{ij}\|_{L^\infty(U)}\int_U |Du||Dv|\,dx + \|c\|_{L^\infty(U)}\int_U |u||v|\,dx \leq (\sum_{i,j=1}^n \|a^{ij} + \|c\|_{L^\infty(U)})\|\|u\|_{H_0^1(U)}\|v\|_{H_0^1(U)}.$$

Moreover, we have by Poincare's inequality and since L is uniformly elliptic, for all $u \in H_0^1(U)$

$$\theta\int_U |Du|^2\,dx \leq B[u,u] - \int_U cu^2\,dx \leq B[u,u] + \mu\int_U |u|^2\,dx.$$

And by Poincare's inequality,

$$\frac{\theta+\mu}{1+C}\|u\|_{H_0^1(U)}\leq \theta\|Du\|_{L^2(U)}-\mu\|u\|_{L^2(U)}.$$

with C being the constant for Poincare's inequality. Thus, since

$$\frac{\theta+\mu}{1+C}\|u\|_{H_0^1(U)}\leq B[u,u],$$

as long as

$$\frac{\theta+\mu}{1+C}>0 \implies \mu>-\theta$$

then L satisfies Lax Milgram.

Evans 6.3

First suppose $u\in C_c^\infty(U)$. Then

$$\begin{aligned}\|\Delta u\|_{L^2}^2 &= \int\left(\sum_{i=1}^n u_{x_ix_i}\right)^2 = \int\sum_{i,j=1}^n u_{x_ix_i}u_{x_jx_j}\\ &= -\int\sum_{i,j=1}^n u_{x_ix_ix_j}u_{x_j} = \int\sum_{i,j=1}^n u_{x_ix_j}u_{x_ix_j}\\ &= \left\|D^2u\right\|_{L^2}^2.\end{aligned} \tag{74}$$

To extend this result to $u\in H_0^2(U)$, choose $u_n\in C_c^\infty(U)$ with $u_n\to u$ in $H_0^2(U)$ and then approximate:

$$\begin{aligned}\left|\left\|D^2u\right\|_{L^2}-\|\Delta u\|_{L^2}\right| &\leq \left|\left\|D^2u\right\|_{L^2}-\left\|D^2u_n\right\|_{L^2}\right|\\ &\quad+\left|\left\|D^2u_n\right\|_{L^2}-\|\Delta u_n\|_{L^2}\right|+\left|\|\Delta u_n\|_{L^2}-\|\Delta u\|_{L^2}\right|\\ &\leq \left\|D^2u-D^2u_n\right\|_{L^2}+\|\Delta u-\Delta u_n\|_{L^2}.\end{aligned} \tag{75}$$

The first term is controlled because $u_n\to u$ in H_0^2. To control the second term, note that for any $v\in H^2(U)$,

$$\int(\Delta v)^2=\int\sum_{i,j=1}^n v_{x_ix_i}v_{x_jx_j}\leq\frac{1}{2}\int\sum_{i,j=1}^n v_{x_ix_i}^2+v_{x_jx_j}^2\leq C\int\sum_{i=1}^n v_{x_ix_i}^2\leq C\left\|D^2v\right\|_{L^2}. \tag{76}$$

Returning to eq. (75), we have

$$\left|\left\|D^2u\right\|_{L^2}-\|\Delta u\|_{L^2}\right|\leq C\left\|D^2u-D^2u_n\right\|_{L^2}\to 0, \tag{77}$$

so $\left\|D^2u\right\|_{L^2}=\|\Delta u\|_{L^2}$ for all $u\in H_0^2$. Two applications of Poincare's inequality give $\|u\|_{H_0^2}\leq C\left\|D^2u\right\|_{L^2}$ (because both u and u_{x_i} for $1\leq i\leq n$ are in H_0^1). Because $\|\Delta u\|_{L^2}^2=\left\|D^2u\right\|_{L^2}^2\leq\|u\|_{H_0^2}^2$, we conclude that $\|u\|_{H_0^2}$ and $\|\Delta u\|_{L^2}$ (induced by the inner product $(u,v)=\int\Delta u\Delta v$) are equivalent norms on H_0^2. In particular, this inner product makes H_0^2 a Hilbert space.

We have $\left|\int fv\right|\leq\|f\|_{L^2}\|v\|_{L^2}\leq\|f\|_{L^2}\|v\|_{H_0^2}\leq C\|f\|_{L^2}\|\Delta v\|_{L^2}$, so $v\mapsto\int fv$ is a bounded linear functional on the Hilbert space H_0^2 with the inner product $(\Delta\cdot,\Delta\cdot)_{L^2}$.

We are done by the Riesz representation theorem.

Evans 6.4

If $u\in H^1(U)$ is a weak solution to $\int_U Du\cdot Dv\,dx=\int_U fvdx$ for all $v\in H^1(U)$, take $v\equiv 1$, then we get $\int_U fdx=0$. Conversely, assume $\int_U fdx=0$. Then consider the subspace of $H^1(U)$, denoting $A=\{f\in H^1(U):\int_U fdx=0\}$. We claim that A is a real Hilbert space with respect to the inner product $(f,g)=$

$\int_U Df \cdot Dgdx$. First of all, A is closed because the integral operator $l(f) = \int_U fdx$ where $l : H^1(U) \to \mathbb{R}$ is continuous. Thus, $A = l^{-1}(\{0\})$ is closed. Since $H^1(U)$ is complete, we have as A a complete subspace. We next prove that (f, g) is an inner product. Linearity and symmetry are easy to see. For positive-definitness, we have for all $f \in A$ with $f \not\equiv 0$, $(f, f) = \|Df\|_{L^2(U)} > 0$. Moreover, if $(f, f) = 0$, then by Poincare's Inequality, we have

$$\|f\|_{L^2(U)} = \|f - (f)_U f\|_{L^2(U)} \le \|Df\|_{L^2(U)} = 0$$

. Thus, we have $\int_U |f|^2 dx = 0 \implies f \equiv 0$ a.e.. These prove that (f, g) is an inner product. Next, for all $v \in H^1(U)$, the linear operator $l_f(v) = \int_U fvdx$ is bounded since $f \in L^2(U)$:

$$|l_f(v)| = \left| \int_U fvdx \right| \le \|f\|_{L^2(U)} \|v\|_{L^2(U)} \le \|f\|_{L^2(U)} \|v\|_{H^1(U)}.$$

Thus, by Rietz Representation Theorem, there exists an unique $u \in A$ such that

$$(u, v) = \int_U Du \cdot Dvdx = l_f(v) = \int_U fvdx$$

for all $v \in A$. With this, observe that for arbitrary $v \in H^1(U)$, let $\tilde{v} = v - (v)_U$. Then we have $\tilde{v} \in A$. Thus,

$$(u, \tilde{v}) = \int_U Du \cdot D(v - (v)_U)dx = \int_U Du \cdot Dvdx = l_f(\tilde{v}) = \int_U fv - f(v)_U dx = \int_U fvdx$$

since $\int_U fdx = 0$. This completes the proof.

Evans 6.5

Multiplying the PDE by $v \in C^\infty(\overline{U})$ and using Green's formula and the boundary condition shows that

$$\int_U fv = \int_U -v\Delta u = \int_U Du \cdot Dv - \int_{\partial U} v\frac{\partial u}{\partial \nu} = \int_U Du \cdot Dv + \int_{\partial U} uv \tag{78}$$

holds for all $v \in C^\infty(\overline{U})$ if and only if $u \in C^\infty(\overline{U})$ is a solution. An approximation (valid because ∂U is smooth) shows that the identity holds for $v \in H^1(U)$. We thus say $u \in H^1(U)$ is a weak solution to Poisson's equation with Robin boundary conditions if

$$\int_U Du \cdot Dv + \int_{\partial U} uv = \int_U fv \tag{79}$$

for all $v \in H^1(U)$.

We now verify the conditions of Lax-Milgram; because the bilinear form we have is symmetric, this is the same as checking that the norm induced by the inner product that is the left side of the weak formulation is equivalent to the usual one in H^1. We have

$$\begin{aligned}\int_U Du \cdot Dv + \int_{\partial U} uv &\le \|Du\|_{L^2} \|Dv\|_{L^2} + \|Tu\|_{L^2(\partial U)} \|Tv\|_{L^2(\partial U)} \\ &\le C \|u\|_{H^1} \|v\|_{H^1},\end{aligned} \tag{80}$$

using the boundedness of the trace operator.

Coercivity is harder. Suppose that for each k, there exists $u_k \in H^1$ with $\|u_k\|_{H^1}^2 > k(\|Du_k\|_{L^2}^2 + \|Tu_k\|_{L^2(\partial U)}^2)$. By normalizing, we may suppose that $\|u_k\|_{H^1} = 1$ for all k. Then $\|Du_k\|_{L^2}, \|Tu_k\|_{L^2(\partial U)} \to 0$ as $k \to \infty$. Because the u_k form a bounded sequence in H^1, which is compactly embedded in L^2, we may extract a subsequence

$$\begin{aligned} u_{k_j} &\rightharpoonup u \quad \text{in } H^1 \\ u_{k_j} &\to u \quad \text{in } L^2. \end{aligned} \tag{81}$$

By strong convergence in L^2,

$$\|u\|_{L^2}^2 = \lim_{j\to\infty} \|u_{k_j}\|_{L^2}^2 = \lim_{j\to\infty} (\|u_{k_j}\|_{H^1}^2 - \|Du_{k_j}\|_{L^2}^2) = 1 \tag{82}$$

because $\|u_k\|_{H^1} = 1$ for all k and $\|Du_k\| \to 0$ as $k \to \infty$. By weak convergence,

$$\|Du\|_{L^2}^2 = \lim_{j\to\infty} \int Du_{k_j} \cdot Du \le \lim_{j\to\infty} \|Du\|_{L^2} \|Du_{k_j}\|_{L^2}, \tag{83}$$

and similarly

$$\|Tu\|_{L^2(\partial U)}^2 = \lim_{j\to\infty} \int_{\partial U} Tu_{k_j} Tu \le \lim_{j\to\infty} \|Tu\|_{L^2(\partial U)} \|Tu_{k_j}\|_{L^2(\partial U)}, \tag{84}$$

which implies that $\|Du\|_{L^2} = \|Tu\|_{L^2(\partial U)} = 0$. Thus $u \in H_0^1$, so by Poincare's inequality, $\|u\|_{L^2} \le C\|Du\|_{L^2} = 0$, a contradiction with $\|u\|_{L^2} = 1$. We conclude that there exists $C > 0$ with $C\|u\|_{H^1}^2 \le \|Du\|_{L^2}^2 + \|Tu\|_{L^2(\partial U)}^2$

These two estimates show that $((u,v)) := \int_U Du\cdot Dv + \int_{\partial U} uv$ is an inner product on $H^1(U)$, and moreover the induced norm is equivalent to $\|\cdot\|_{H^1(U)}$. The linear functional $v \mapsto \int_U fv$ is bounded in $\|\cdot\|_{H^1}$. Thus by the Riesz representation theorem, there exists a unique u such that $((u,v)) = \int_U Du\cdot Dv + \int_{\partial U} uv = \int_U fv$ for all $v \in H^1$.

Evans 6.6

If we assume u is a classical solution and use Green's formula, then

$$\int fv = -\int v\Delta u = \int Du\cdot Dv - \int_{\partial U} v\frac{\partial u}{\partial \nu} = \int Du\cdot Dv - \int_{\Gamma_1} v\frac{\partial u}{\partial \nu}, \tag{85}$$

where we used the boundary condition on Γ_2.

WARNING: There is an attempted proof of the Poincare inequality when u vanishes on only a subset of the boundary. It seems to work.

To get rid of the second term, define the space of test functions $\mathcal{H} = \{v \in H^1(U) : v = 0 \,\text{on}\, \Gamma_1\}$. Put an inner product on this space $(u,v) = \int_U Du\cdot Dv$. To prove coercivity, we want to claim that a Poincare inequality holds, but we don't have $u = 0$ on all of ∂U for $u \in \mathcal{H}$, just $u = 0$ on Γ_1. Suppose that there is no constant such that $\|u\|_{L^2} \le C\|Du\|_{L^2}$. Then there exists $\{u_k\} \subset \mathcal{H}$, which we can take satisfying $\|u_k\|_{L^2} = 1$, with $\|u_k\|_{L^2} > k\|Du_k\|_{L^2}$. Then u_k is bounded in $H^1 \subset\subset L^2$, so we can extract a subsequence

$$\begin{aligned} u_{k_j} &\rightharpoonup u \quad \text{in}\, H^1 \\ u_{k_j} &\to u \quad \text{in}\, L^2. \end{aligned} \tag{86}$$

By strong convergence in L^2, $\|u\|_{L^2} = 1$ and by weak convergence,

$$\|Du\|_{L^2}^2 = \lim_{j\to\infty} \int Du_{k_j} \cdot Du \le \lim_{j\to\infty} \|Du\|_{L^2} \|Du_{k_j}\|_{L^2}, \tag{87}$$

so $\|Du\|_{L^2} = 0$. Because U is connected, $Du = 0$ implies u is constant on U. We also have

$$\|Tu\|_{L^2(\Gamma_1)}^2 = \lim_{j\to\infty} \int_{\Gamma_1} Tu_{k_j} Tu \le \lim_{j\to\infty} \|Tu\|_{L^2(\Gamma_1)} \|Tu_{k_j}\|_{L^2(\Gamma_1)}, \tag{88}$$

and so $\|Tu\|_{L^2(\Gamma_1)} = 0$, because $u_{k_j} = 0$ on Γ_1, and so $u = 0$ on Γ_1. Because u is constant a.e, its trace is the same constant a.e. on the boundary Because Γ_1 has positive measure inside ∂U (for example it is relatively open in ∂U), we conclude that $u = 0$ on ∂U. This means $u \equiv 0$ on U, a contradiction with $\|u\|_{L^2} = 1$.

We conclude that $\|u\|_{H^1(U)} \le C\|Du\|_{L^2(U)}$, and so $\int_U Du\cdot Dv$ is an inner product inducing a norm equivalent to the usual one on H^1. We conclude by the Riesz representation theorem that for every $f \in L^2$, there is a unique $u \in \mathcal{H}$ for which

$$\int fv = \int Du\cdot Dv - \int_{\Gamma_1} v\frac{\partial u}{\partial \nu} \tag{89}$$

for all $v \in \mathcal{H}$ (in particular the second integral on the right vanishes).

Evans 6.7

WARNING: in general u is not bounded.

Since u is a weak solution, we have that for any $v \in H_0^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \sum_{i=1}^n u_{x_i} v_{x_i} + c(u) v dx = \int_{\mathbb{R}^n} f v dx.$$

Let

$$A = \sum_{i=1}^n \int_{\mathbb{R}^n} u_{x_i} v_{x_i} dx \qquad B = \int_{\mathbb{R}^n} (f - c(u)) v dx.$$

Let $v = -D_k^{-h}(D_k^h(u))$. Then

$$\begin{aligned} A &= \sum_{i=1}^n \int_{\mathbb{R}^n} u_{x_i} v_{x_i} dx \\ &= \sum_{i=1}^n \int_{\mathbb{R}^n} D_k^h(u)_{x_i} (D_k^h(u))_{x_i} dx \\ &= \int_{\mathbb{R}^n} |D_k^h Du|^2 dx. \end{aligned}$$

On the other hand,

$$\begin{aligned} |B| &\le \int_{\mathbb{R}^n} (|f| + |c(u)|)|v| dx \\ &\le \epsilon \int_{\mathbb{R}^n} |v|^2 dx + \frac{1}{4\epsilon} \int_{\mathbb{R}^n} (|f| + |c(u)|)^2 dx \end{aligned}$$

We know that

$$\int_{\mathbb{R}^n} |v|^2 dx \le C \int_{\mathbb{R}^n} |D_k^h(Du)|^2$$

Now choose ϵ so that we obtain

$$\int_{\mathbb{R}^n} |D_k^h Du|^2 dx = A \le \frac{1}{2} \int_{\mathbb{R}^n} |D_k^h Du|^2 dx + C \int_{\mathbb{R}^n} (|f| + |c(u)|)^2 dx.$$

The righthand side is bounded since u has compact support and $c(0) = 0$. Therefore outside of the support of u, $c(u) = 0$. Also, u is bounded, and $c' \ge 0$, so c is bounded above by $c(\sup u)$.

Evans 6.8

We compute

$$\begin{aligned} Lv &= -\sum_{i,j=1}^n a^{ij} \left(|Du|^2 + \lambda u^2 \right)_{x_i x_j} \\ &= -\sum_{i,j=1}^n a^{ij} \left(\sum_{k=1}^n \left(2u_{x_k x_i} u_{x_k x_j} + 2u_{x_k} u_{x_k x_i x_j} \right) + 2\lambda u_{x_i} u_{x_j} + 2\lambda u u_{x_i x_j} \right). \end{aligned} \tag{90}$$

Now the uniform ellipticity condition implies that

$$-\sum_{i,j,k=1}^n a^{ij} u_{x_k x_i} u_{x_k x_j} \le -\sum_{k=1}^n \theta \left| Du_k \right|^2 = -\theta \left| D^2 u \right|^2 \tag{91}$$

and

$$-\sum_{i,j=1}^{n} a^{ij} u_{x_i} u_{x_j} \leq -\theta \left|Du\right|^2 . \tag{92}$$

Differentiating $Lu = 0$ in the x_k direction gives

$$\begin{aligned}
-\sum_{i,j=1}^{n} a^{ij} u_{x_i x_j x_k} &= \sum_{i,j=1}^{n} a^{ij}_{x_k} u_{x_i x_j} \\
&\leq C \left|Du\right| \left|D^2 u\right| \\
&\leq \theta \left|D^2 u\right|^2 + C \left|Du\right|^2 ,
\end{aligned} \tag{93}$$

where we used Cauchy's inequality with $\epsilon = \theta$. The constant C depends only on the coefficients a^{ij}, whose derivatives are bounded. Substituting all of these gives

$$\begin{aligned}
Lv &\leq -\theta \left|D^2 u\right|^2 + 2\lambda u Lu - 2\lambda\theta \left|Du\right|^2 + C \left|Du\right|^2 \\
&\leq (C - 2\lambda\theta) \left|Du\right|^2 ,
\end{aligned} \tag{94}$$

which can be made negative independent of u for λ sufficiently large.

Thus $Lv \leq 0$ for λ sufficiently large. Let $\lambda \geq 1$. Because u is smooth up to the boundary (elliptic regularity), so is v, so by the weak maximum principle, $\max_{\overline{U}} v = \max_{\partial V} v$. Then

$$\begin{aligned}
\|Du\|_{L^\infty(U)} &\leq \|v\|_{L^\infty(U)}^{\frac{1}{2}} \\
&= \|v\|_{L^\infty(\partial U)}^{\frac{1}{2}} \\
&\leq \left(\|Du\|_{L^\infty(\partial U)}^2 + \lambda \|u\|_{L^\infty(\partial U)}^2 \right)^{\frac{1}{2}} \\
&\leq \lambda(\|Du\|_{L^\infty(\partial U)} + \|u\|_{L^\infty(\partial U)}),
\end{aligned} \tag{95}$$

where we used $\|f(u)\|_{L^\infty} = f(\|u\|_{L^\infty})$ for u smooth and f increasing and $\sqrt{a^2+b^2} \leq a + b$ for $a, b \geq 0$.

Evans 6.9

Proof. Since f is bounded, there exists $M \geq 0$ such that $|f(x)| \leq M$ for all $x \in U$. Now, consider $L(u+M\omega)$. Since $L\omega \geq 1$, we have $L(u+M\omega) = f + M \geq 0$. Thus, we can apply weak maximum principle to show that

$$\min_{x\in U} u(x) + M\omega(x) = \min_{x\in \partial U} u(x) + M\omega(x) = 0$$

since $\omega(x^0) = 0, \omega \geq 0$ on ∂U, and $u = 0$ on ∂U. Thus, we have

$$u(x) + M\omega(x) \geq 0 \qquad \text{on } U$$

and

$$u(x^0) + M\omega(x^0) = 0.$$

Thus, we have

$$\partial u(x^0) + M\omega(x^0)\nu \leq 0 \implies \partial u(x^0)\nu \leq -M\partial\omega(x^0)\nu.$$

On the other hand, do the same thing with $L(u - M\omega)$, we get that $L(u - M\omega) \leq 0$. Thus,

$$\min_{x\in U} u(x) - M\omega(x) = \min_{x\in \partial U} u(x) - M\omega(x) = 0$$

. Thus,

$$u(x) - M\omega(x) \leq 0 \qquad \text{on } U$$

and

$$u(x^0) - M\omega(x^0) = 0.$$

The above two implies that

$$\partial u(x^0) - M\omega(x^0)\nu \geq 0 \implies \partial u(x^0)\nu \geq M\partial\omega(x^0)\nu.$$

Moreover, we have

$$pd\omega(x^0)\nu \leq 0$$

because $L\omega \geq 1 > 0$. Thus,

$$\left|\partial u(x^0)\nu\right| \leq M\left|\partial\omega(x^0)\nu\right|.$$

Now, given the tangent space of ∂U at x^0, we know it is $n-2$ dimension. Let $\{v_1, \ldots, v_{n-2}\}$ be a basis of the tangent space, then $\{v_1, \ldots, v_{n-2}, \nu\}$ spans ∂U. However, since $u = 0$ on ∂U, $\partial u v_i = 0$ for all $i = 1, \ldots, n-2$. Thus,

$$|Du(x_0)| = \sqrt{(\partial u v_i)^2 + (\partial u(x^0)\nu)^2} = \left|\partial u(x^0)\nu\right|.$$

Thus, the ineuality follows.

$$|Du|.$$

□

Evans 6.10

If u is a smooth solution, it is in particular a weak solution, so by Exercise 6.4, we have $\int_U Du \cdot Dv = 0$ for all $v \in H^1(U)$. Taking $v = u$ gives $\|Du\|_{L^2}^2 = 0$, so because U is connected and u is smooth, we conclude u is a constant.

Using maximum principle, apply Hopf and SMP.

Evans 6.11

Because u is bounded and φ is smooth, φ has bounded derivatives in the range of u, so $\varphi(u) \in H^1(U)$ with the expected derivative. We then have

$$\begin{aligned}
B[w, v] &= \int_U -\sum_{i,j=1}^n a^{ij}\varphi(u)_{x_i}v_{x_j} \\
&= \int_U \sum_{i,j=1}^n a^{ij}\varphi'(u)u_{x_i}v_{x_j} \\
&= -\int_U \sum_{i,j=1}^n a^{ij}\varphi'(u)_{x_j}u_{x_i}v \\
&= -\int_U \sum_{i,j=1}^n a^{ij}\varphi''(u)u_{x_j}u_{x_i}v \\
&\leq -\int_U \theta\varphi''(u)v\,|Du|^2 \\
&\leq 0.
\end{aligned} \tag{96}$$

There is no boundary term when we integrate by parts because $v \in H_0^1$. The final inequality holds because the integrand is positive ($\varphi'' \geq 0$ because φ is convex, and $v \geq 0$ by assumption).

Evans 6.12

Suppose $u \in C^2(U) \cap C(\bar{U})$ and let $w = \frac{u}{v}$. Let Mw be defined by

$$Mw = \sum_{i,j=1}^n a^{ij}(v^2 w_{x_i})_{x_j} - \sum_{i=1}^n v^2 b^i w_{x_i}.$$

We have that

$$w_{x_i} = \frac{v_{x_i}u - u_{x_i}v}{v^2},$$

and

$$(v^2 w_{x_i})_{x_j} = (v_{x_i}u - u_{x_i}v)_{x_j} = -vu_{x_ix_j} + v_{x_i}u_{x_j} - v_{x_j}u_{x_i} + v_{x_ix_j}u.$$

Therefore

$$\sum_{i,j=1}^{n} a^{ij}(v^2 w_{x_i})_{x_j} = -v(\sum_{i,j=1}^{n} a^{ij}u_{x_ix_j}) + (\sum_{i,j=1}^{n} a^{i,j}v_{x_ix_j})u.$$

Furthermore,

$$\sum_{i=1}^{n} v^2 b^i w_{x_i} = \sum_{i=1}^{n} b^i(v_{x_i}u - u_{x_i}v) = -v(\sum_{i=1}^{n} b^i u_{x_i}) + (\sum_{i=1}^{n} b^i v_{x_i})u.$$

Hence

$$Mw = -v(\sum_{i,j=1}^{n} a^{ij}u_{x_ix_j} - \sum_{i=1}^{n} b^i u_{x_i}) + (\sum_{i,j=1}^{n} a^{i,j}v_{x_ix_j} - \sum_{i=1}^{n} b^i v_{x_i})u$$

Since $Lv \geq 0$, $v > 0$ and $Lu \leq 0$, we see that

$$Mw \leq -cuv + cvu = 0$$

on the set $\{u > 0\}$. Therefore M is a operator with no zeroth-order order term, and $Mw \leq 0$ on the set $\{u > 0\}$. By the weak maximum principle, $w = \frac{u}{v}$ must attain its maximum on the boundary of $\{u > 0\}$, or on the boundary of U. In the first case, $\max \frac{u}{v} = 0$, and in the second case $\max \frac{u}{v} \leq 0$. Hence $\frac{u}{v} \leq 0$ on all of U. Since $v > 0$ it follows that $u \leq 0$ on all of U.

Evans 6.16

- (a)

  Proof. By calculation,

  $$-\Delta w = -\sum_{i=1}^{n}(i\sigma\omega_i)w_{x_j} = \sigma^2 w \sum_{i=1}^{n} \omega_i^2 = \lambda w$$

  as desired. □

- (b)

  Proof. By calculation,

  $$-\Delta\Phi = -\sum_{i=1}^{3}\left[\frac{\frac{i\sigma x_i}{|x|}e^{i\sigma|x|}4\pi|x| - e^{i\sigma|x|}\frac{4\pi x_i}{|x|}}{(4\pi|x|)^2}\right]_{x_i} = -\sum_{i=1}^{3}\left[\Phi(i\sigma\frac{x_i}{|x|} - \frac{x_i}{|x|^2})\right]_{x_i} = \lambda\Phi.$$

  Note: The problem is $-\Delta\Phi = \lambda\Phi + \delta_0$, not sure where does the δ_0 come from. □

- (c)

  Proof. We have $w_r = Dw \cdot \frac{x}{|x|} = i\sigma w(\omega \cdot \frac{x}{|x|})$. Thus,

  $$\lim_{r\to\infty} r(w_r - i\sigma w) = \lim_{r\to\infty} ir\sigma w(\omega \cdot \frac{x}{|x|} - 1) \neq 0$$

  as $(\omega \cdot \frac{x}{|x|} - 1) \neq 0$. On the other hand,

  $$\Phi_r = \Phi(i\sigma - \frac{1}{|x|^2}).$$

  Thus,

  $$\lim_{r\to\infty} r(\Phi_r - i\sigma\Phi) = \lim_{r\to\infty} -\frac{\Phi}{r} = 0$$

  because $|\Phi| \to 0$ as $|x| \to \infty$. □

Evans Chapter 8

Evans 8.1

1. (a)

   Proof. Since $\sin(kx) = \frac{e^{ikx}-e^{-ikx}}{2i}$, by Riemann Lebesgue Lemma, we have since for any $v \in L^2(0,1)$, $v\chi_{[0,1]} \in L^1(\mathbb{R})$, we have

$$\lim_{k\to\infty}\int_{[0,1]} \sin(kx)v\,dx = \lim_{k\to\infty}\int_{\mathbb{R}} \frac{e^{ikx}-e^{-ikx}}{2i} v\chi_{[0,1]}\,dx = 0.$$

   Thus, $u_k \rightharpoonup 0$ in $L^2(0,1)$. □

2. (b)

   First suppose $\varphi \in C_c^\infty(0,1)$. For k large enough, φ is uniformly continuous, so given $\epsilon > 0$, there exists C_j such that $|\varphi - C_j| < \epsilon$ on $[\frac{j}{k}, \frac{j+1}{k}]$ for each $0 \le j \le k-1$. Now

$$\left|\int_0^1 (u_k(x) - (\lambda a + (1-\lambda)b))\varphi\right| = \left|\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} a\varphi + \int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} b\varphi - \int_{\frac{j}{k}}^{\frac{j+1}{k}} \lambda a\varphi - \int_{\frac{j}{k}}^{\frac{j+1}{k}} (1-\lambda)b\right|. \tag{97}$$

   Estimating each summand,

$$\begin{aligned}\left|\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} a\varphi - \int_{\frac{j}{k}}^{\frac{j+1}{k}} \lambda a\varphi\right| &\le \left|\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} a\varphi - \int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} aC_j\right| + \left|\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} aC_j - \int_{\frac{j}{k}}^{\frac{j+1}{k}} \lambda a\varphi\right| \\ &\le |a|\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} |\varphi - C_j| + \left|\int_{\frac{j}{k}}^{\frac{j+1}{k}} \lambda aC_j - \int_{\frac{j}{k}}^{\frac{j+1}{k}} \lambda a\varphi\right| \\ &\le 2\,|a|\,\lambda\frac{\epsilon}{k}.\end{aligned} \tag{98}$$

   and similarly

$$\left|\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} b\varphi - \int_{\frac{j}{k}}^{\frac{j+1}{k}} (1-\lambda)b\right| \le 2\,|a|\,(1-\lambda)\frac{\epsilon}{k}. \tag{99}$$

   Recalling $0 < \lambda < 1$ and summing over $0 \le j \le k-1$, we have

$$\left|\int_0^1 (u_k(x) - (\lambda a + (1-\lambda)b))\varphi\right| \le 2(|a| + |b|)\epsilon. \tag{100}$$

   Now let $\varphi \in L^2(0,1)$ and choose $\varphi_n \in C_c^\infty(0,1)$ with $\varphi_n \to \varphi$ in L^2 (and thus in L^1 because the domain is bounded). Also notice that $|u_k|, |\lambda a + (1-\lambda)b| \le C$. Define $v_k := u_k - (\lambda a + (1-\lambda)b)$. Given $\epsilon > 0$, choose φ_n with $\|\varphi_n - \varphi\|_1 < \epsilon$. Then

$$\left|\int v_k\varphi\right| \le \left|\int v_k(\varphi - \varphi_n)\right| + \left|\int v_k\varphi_n\right| \le C\epsilon + \left|\int v_k\varphi_n\right| \le C\epsilon \tag{101}$$

   for k sufficiently large.

   Take any function $v \in L^2(0,1)$. Then

$$\int_0^1 u_k(x)v(x)dx = \sum_{j=0}^{k-1} a\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx + b\sum_{j=0}^{k-1}\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} v(x)dx.$$

On the other hand, let $u = \lambda a + (1-\lambda)b$. We have

$$(\lambda a + (1-\lambda)b)\int_0^1 v(x)dx = (\lambda a + (1-\lambda)b)\Big(\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx + \sum_{j=0}^{k-1}\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} v(x)dx\Big).$$

Taking the difference,

$$\begin{aligned}
\langle u, v\rangle - \langle u_k, v\rangle &= (\lambda a + b - \lambda b - a)\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx + (\lambda a + b - \lambda b - b)\sum_{j=1}^{k-1}\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} v(x)dx \\
&= (\lambda - 1)(a-b)\sum_{j=1}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx + \lambda(a-b)\sum_{j=0}^{k-1}\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} v(x)dx \\
&= (a-b)\Big(\lambda\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx + \lambda\sum_{j=0}^{k-1}\int_{\frac{j+\lambda}{k}}^{\frac{j+1}{k}} v(x)dx - \sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx\Big) \\
&= (a-b)\Big(\lambda\int_0^1 v(x)dx - \sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx\Big)
\end{aligned}$$

Now for any $\epsilon > 0$, for sufficiently large k we have

$$\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx \in (\frac{\lambda}{k}(v(\frac{j}{k}) - \epsilon), \frac{\lambda}{k}(v(\frac{j}{k}) + \epsilon)).$$

Hence,

$$\sum_{j=0}^{k-1}\int_{\frac{j}{k}}^{\frac{j+\lambda}{k}} v(x)dx \in (\frac{\lambda}{k}\sum_{j=0}^{k-1} v(\frac{j}{k}) - \lambda\epsilon, \frac{\lambda}{k}\sum_{j=0}^{k-1} v(\frac{j}{k}) + \lambda\epsilon).$$

Hence the convergence holds.

Evans 8.4

- (a) By calculation,

$$L_{p_i^k}(D\mathbf{u}, \mathbf{u}, x) = \eta(\mathbf{u})(\mathrm{cof}(D\mathbf{u}))_i^k \qquad k, i = 1, \ldots, n.$$

  by divergence-free rows, we furthermore have

$$-\sum_{i=1}^{n}(L_{p_i^k}(D\mathbf{u}, \mathbf{u}, x))_{x_i} = -\sum_{i=1}^{n}\sum_{j=1}^{n}(\eta_{z^j}(\mathbf{u}))\mathbf{u}^j_{x_i}(\mathrm{cof}(D\mathbf{u}))_i^k = -\sum_{j=1}^{n}\eta_{z^j}(\mathbf{u})\delta_{jk}\det(D\mathbf{u}).$$

  since $\det P\delta_{ij} = \sum_{k=1}^n p_k^i(\mathrm{cof}P)_k^j$ by $\det PI = P(\mathrm{cof}P)^T$. On the other hand,

$$L_{z^k}(D\mathbf{u}, \mathbf{u}, x) = \eta_{z^k}(\mathbf{u})\det D\mathbf{u}.$$

  So $-\sum_{i=1}^n (L_{p_i^k}(D\mathbf{u}, \mathbf{u}, x))_{x_i} + L_{z^k}(D\mathbf{u}, \mathbf{u}, x) = 0$ as desired.

- (b)

  Proof. This is just theorem 1 in 8.1.b □

Evans 8.5

WARNING: First claim does not work.

First, the integral expression is independent of η satisfying the constraints of the problem. To see this, let η_1, η_2 be two such functions. The support of $\eta \circ \mathbf{u}$ is contained in the closed set $W := \mathbf{u}^{-1}(B(x_0, r))$, and applying $\mathbf{u}^{-1}$ to $B(x_0, r) \cap \mathbf{u}(\partial U) = \varnothing$ gives $W \cap \partial U = \varnothing$. Thus we may choose a smooth cutoff function $0 \le \zeta_\epsilon \le 1$ with $\zeta_\epsilon \equiv 1$ on ∂U and $\operatorname{supp} \zeta_\epsilon \subset \{x : d(x, \partial U) < \epsilon d(W, \partial U)\}$. We can moreover arrange that $|D\zeta_\epsilon| \le C\epsilon^{-1}$. Then $\zeta_\epsilon \mathbf{u} = \mathbf{u}$ on ∂U, so by 8.4b,

$$\int_U (\eta_1 - \eta_2)(\mathbf{u}) \det D\mathbf{u} = \int_U (\eta_1 - \eta_2)(\zeta_\epsilon \mathbf{u}) \det D(\zeta_\epsilon \mathbf{u}) \tag{102}$$

I want to do something like pass to the limit with $\int_U (\eta_1 - \eta_2)(\chi_{\partial U} \mathbf{u}) = 0$ because of where η_1, η_2 are supported, but it seems $|\det D\zeta_\epsilon|$ grows too quickly for the limit of the right side to be 0. But assuming the degree is well-defined (independent of η), the rest of the proof works.

The degree is locally constant in x_0 (it is constant on, say, $B(x_0, \frac{r}{2})$, because $\operatorname{supp} \eta \subset B(x, \frac{r}{2})$ for each $x \in B(x_0, \frac{r}{2})$).

Suppose x_0 is a regular value of $\mathbf{u}$; namely, $\det D\mathbf{u}(x) \neq 0$ for each $x \in S := \mathbf{u}^{-1}(\{x_0\})$. The inverse function theorem implies that $\mathbf{u}$ is injective on a neighbourhood of each $x \in S$, and thus S is discrete. Moreover, S is closed because $\mathbf{u}$ is continuous. Thus $S \subset U$ is closed and bounded and thus finite (as a discrete compact set). By the inverse function theorem, choose for each $x_i \in S$ $(1 \le i \le m)$ neighbourhoods V_i such that $\mathbf{u}$ maps V_i diffeomorphically onto $\mathbf{u}(V_i)$, which is a neighbourhood of x_0. If needed, shrink each V_i so that they are pairwise disjoint and $\det D\mathbf{u}$ has constant sign on V_i. Then let $V := \bigcap_{i=1}^m \mathbf{u}(V_i)$ and choose η so that $\int_{\mathbb{R}^n} \eta = 1$ and $\operatorname{supp} \eta \subset V \cap B(x_0, r)$, with r as in the problem statement. By the change of variables formula,

$$\int_{V_i} \eta(\mathbf{u}) \det D\mathbf{u} \, \mathrm{d}x = \operatorname{sgn} \det D\mathbf{u}(x_i) \int_{\mathbf{u}(V_i)} \eta(x) \, \mathrm{d}x = \operatorname{sgn} \det D\mathbf{u}(x_i), \tag{103}$$

where the last equality holds because $\operatorname{supp} \eta \subset \mathbf{u}(V_i)$ and $\int_{\mathbb{R}^n} \eta = 1$. Then

$$\int_U \eta(\mathbf{u}) \det D\mathbf{u} = \sum_{i=1}^m \int_{V_i} \eta(\mathbf{u}) \det D\mathbf{u} = \sum_{i=1}^m \operatorname{sgn} \det D\mathbf{u}(x_i), \tag{104}$$

where the first equality holds because $\eta \circ \mathbf{u}$ is supported in $\{\mathbf{u}(x) \in \bigcap \mathbf{u}(V_i)\} \subset \bigcup V_i$.

If x_0 is not a regular value, pick a sequence of regular values $x_n \to x_0$ by Sard's theorem (regular values are dense). Because the degree is locally constant in x_0, for n large enough, $\deg(\mathbf{u}, x_0) = \deg(\mathbf{u}, x_n)$, and the right side is an integer by above.

Guaraco Problems

Problem 3

Integrate by parts in the Allen-Cahn energy functional:

$$\begin{aligned} \epsilon E_\epsilon(u) &= \int_M \frac{\epsilon^2}{2} |\nabla u|^2 + W(u) \\ &= \int_M -\frac{u}{2} \epsilon^2 \Delta u + W(u) \\ &= \int_M -\frac{1}{2} u(u^3 - u) + \frac{1}{4}(1 - u^2)^2 \\ &= \int_M -\frac{1}{4} u^4 + \frac{1}{4}. \end{aligned} \tag{105}$$

If $u \neq 0$ anywhere, then $\epsilon E_\epsilon(u) < \int_M \frac{1}{4} = \int W(0) = \epsilon E_\epsilon(0)$. That is, 0 maximizes E_ϵ.

Problem 6 (Jared)

I think this requires a fair bit of comfort with geometry (some of which I forgot this morning), so I'll try to go through explicitly.

Guaraco asks you to rescale the metric $g \to \epsilon^{-2} g$ - what effect does this have? Well first, this changes the volume

$$dvol_g \to dvol_{g_\epsilon} \overset{loc}{=}$$

$$\sqrt{|\det g_\epsilon|} dx_1 \wedge \cdots \wedge dx_n = \sqrt{|\det \epsilon^{-2} g|} dx_1 \wedge \cdots \wedge dx_n$$

$$= \sqrt{|\epsilon^{-2n} \det g|} dx_1 \wedge \cdots \wedge dx_n = \epsilon^{-n} \sqrt{|\det g|} dx_1 \wedge \cdots \wedge dx_n$$

the initial expression for $dvol_{g_\epsilon}$ is standard and can be found here for instance. Intuitively, metrics are measuring length, so rescaling by ϵ^{-2} is like changing the scale by ϵ for vectors. This is seen in that our euclidean notion of length is $||v|| = \sqrt{\langle v, v \rangle}$ so we replace the inner product with our metric to get $||v|| = \sqrt{g(v,v)}$ so that $||v||_\epsilon = \sqrt{\epsilon^{-2} g(v,v)} = \epsilon^{-1} ||v||$, i.e. we've rescaled by a factor of ϵ^{-1}. This also manifests in the volume integral, where it's like we've gone from the volume form at scale $r = 1 \to r = \epsilon$ (think of integrating over a ball of $r = 1$ vs. $r = \epsilon$ and trying to connect the two by the diffeomorphism $f_\epsilon(x) = \epsilon x$.

With this, we have that

$$E_\epsilon(u; B_\epsilon(p)) = \int_{\{z \mid d_g(z,p) \leq \epsilon\}} \left(\epsilon g(\nabla u, \nabla u) + \frac{W(u)}{\epsilon} \right) dvol_g$$

$$= \int_{\{z \mid d_{g_\epsilon}(z,p) \leq 1\}} \left(\epsilon^3 g_\epsilon(\nabla u, \nabla u) + \frac{W(u)}{\epsilon} \right) \epsilon^n dvol_{g_\epsilon}$$

Note the labelling of the domain of integration has changed from "points less than ϵ away" (under g) to "points less than 1 away" (under g_ϵ), reflecting the change in metric. However, we're still integrating over the same points on the manifold - just calling them by different names.

As you've shown in problem 1 (or maybe "will show"), we have that

$$\Delta_{g_\epsilon} u = u(u^2 - 1)$$

in what sense does this hold true? Well if initially, we're investigating this problem on $B_\epsilon(p)$, then we compose u with a **chart**, call it φ so that $\varphi : B_1(0) \overset{\cong}{\to} B_\epsilon(p)$ - Thus, on $B_1(0)$ we have

$$(\Delta_{g_\epsilon} u) \circ \varphi(x) = (u(u^2 - 1)) \circ \varphi(x)$$

where I've composed both sides of the equation with our chart map.

In particular, if you write the above out as equations on $B_1(0)$, then you'll get an elliptic PDE (here, use that g is a Riemannian metric), and so Schauder estimates apply. Because this argument is local, we can exchange the distance weighting in the Schauder estimates for a constant and get

$$||u \circ \varphi||_{1,\alpha} \leq K \left(||u \circ \varphi||_{C^0} + ||u(u^2 - 1) \circ \varphi||_{C^0} \right)$$

The point is that there is no chain rule happening in the above because all we've done is composed with a chart map, and so whenever we talk about a derivative, we calculate it with the function $u \circ \varphi$ which is a bonafide function from Euclidean space to $\mathbb{R}$. With this, we get that

$$\sup_{i=1,\ldots,n} \sup_{x \in B_1(0)} |(u \circ \varphi)_i(x)| \leq 2K(\epsilon)$$

by definition/conventions of geometry, we have $(u \circ \varphi)_i = u_i \circ \varphi$, i.e. we can only get values from a function and its derivatives after moving to charts. **Note:** there is ϵ dependency in the coefficients $\{a_{ij}\}$ because we've changed to the metric g_ϵ. From here, I'll suppress composition with φ. If we do an FTC computation in coordinates, we get that

$$\forall z \in B_1(p), \qquad |u(z) - u(p)| = |u(z)| \leq 2K||z - p||_g$$

the distance $||z - p||$ is calculated with respect to the scaled metric, g_ϵ. Now we write $\int_{B_\epsilon(p)}$ as an integral over $B_1(0)$ in "radial coordinates", where r represents the distance from p (our fixed point) to $z \in B_1(0)$ via a geodesic. We then have that

$$E_\epsilon(u; B_\epsilon(p)) \geq \int_{B_{\min(1,(2K(\epsilon))^{-1}}(p)} \frac{W(u)}{\epsilon} \epsilon^n dvol_{g_\epsilon}$$
$$= \epsilon^{n-1} \int_0^{R(\epsilon)} \int_{d(z,p)=r} (1 - u(z)^2)^2 dvol_{g_\epsilon}$$

where $R(\epsilon) = \min(1, (2K(\epsilon))^{-1})$. Now we can bound this below by a radial integral, i.e.

$$E_\epsilon(u; B_\epsilon(p)) \geq \epsilon^{n-1} \int_0^{R(\epsilon)} \mu\{d(z,p) = r\}(1 - (2K(\epsilon)r)^2)^2 dr$$
$$\geq \epsilon^{n-1} c(M, g_\epsilon) \int_0^{R(\epsilon)} r^{n-1}(1 - r^2)^2 dr = c(\epsilon) f(R(\epsilon)) > 0$$

Here, $c(M, g_\epsilon)$ is a constant, universal in p and r and only dependent on the ambient manifold and metric g_ϵ, which acts as a lower bound for $\frac{\mu\{d(z,p)=r\}}{r^{n-1}} \geq c(M, g_\epsilon)$ - under the euclidean metric, this constant is just the prefactor which occurs for the area of an $n-1$-sphere. In $\mathbb{R}^n$, this is independent of the point that the sphere is based at. Because our metric is Riemannian, and so uniformily elliptic (because we're on a closed and hence compact manifold), a similar lower bound on the constant of proportionality should hold.

Note that our lower bound is independent of p and u, but not independent of ϵ. This is okay because in dimensions ≥ 2 we have the trivial upper bound of (under the euclidean metric for simplicity)

$$\int_{B_\epsilon(p)} \frac{W(u)}{\epsilon} \leq C\epsilon^n \frac{1}{\epsilon} = C\epsilon^{n-1}$$

Here, C is the constnat of proportionality which is some combination of Gamma function and π's (see here) and we've bounded $W(u) \leq 1$. The above might be why Gautam got a constant independent of ϵ when doing the computation for the heteroclinic solution on $\mathbb{R}$ - the above bound would just be constant, independent of ϵ for $n = 1$. But when $n \geq 2$, our lower bound must be less than a constant times ϵ^{n-1}, implying some ϵ dependency on c_0 in the problem statement.

Guaraco 6

In $\frac{\mathrm{d}}{\mathrm{d}t} E(u + t\varphi)|_{t=0} = 0$, substitute $\varphi \equiv 1$ (valid because we are on a closed manifold), and get $\int W'(u) = 0$, a contradiction because $0 < |u| < 1$ (so $W'(u)$ is constant sign and non-zero).

Guaraco 7

Guaraco 8

Remark. *Doesn't conclude that it suffices to take* $\epsilon^2\lambda_1 < \frac{1}{2}$.

The argument in the hint (minimizer either 0 or constant sign in interior) was done in full in Otis 2.5a. Compute

$$\begin{aligned} E(\varphi) &< E(0) \\ \int \frac{\epsilon}{2} |D\varphi|^2 + \frac{1}{\epsilon} W(\varphi) &< \int \frac{1}{\epsilon} W(0) \\ \int -\frac{\epsilon^2}{2} \varphi \Delta \varphi + \frac{\varphi^4}{4} - \frac{\varphi^2}{2} + \frac{1}{4} &< \int \frac{1}{4} \\ \epsilon^2 \lambda_1 \int \varphi^2 &< \int \varphi^2 - \frac{1}{2} \int \varphi^4 \\ \epsilon^2 \lambda_1 &< 1 - \frac{1}{2} \frac{\int \varphi^4}{\int \varphi^2}. \end{aligned} \tag{106}$$

Thus for ϵ or λ_1 sufficiently small, the minimizer is non-zero in Ω.

Guaraco 9

Let u_1, u_2 be positive solutions on U to Allen-Cahn with Dirichlet boundary data. Then $-\Delta u_i = -u_i^3 + u_i < u_i$, so $\Delta u_i + u_i > 0$. Write

$$\int \left(\frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2} \right) (u_2^2 - u_1^2) = \int -u_1 \Delta u_1 + \Delta u_2 \frac{u_1^2}{u_2} + \Delta u_1 \frac{u_2^2}{u_1} - u_2 \Delta u_2. \tag{107}$$

Compute the derivative

$$D\left(\frac{u_1^2}{u_2}\right) = 2\frac{u_1}{u_2} Du_1 - \frac{u_1^2}{u_2^2} Du_2, \tag{108}$$

where the right side is L^2, assuming $\frac{u_1}{u_2} \in L^\infty$ (and the same for u_1, u_2 swapped), so $\frac{u_1^2}{u_2} \in H_0^1$. Integrating by parts gives

$$\begin{aligned} \int -u_1 \Delta u_1 + \Delta u_2 \frac{u_1^2}{u_2} &= \int |Du|^2 - Du_2 \cdot \left(2\frac{u_1}{u_2} Du_1 - \frac{u_1^2}{u_2^2} Du_2 \right) \\ &= \int \left| Du - \frac{u_1}{u_2} Du_2 \right|^2 \\ &\geq 0, \end{aligned} \tag{109}$$

and the same bound holds with u_1, u_2 swapped. Thus

$$\begin{aligned} \int \left(\frac{\Delta u_1}{u_1} - \frac{\Delta u_2}{u_2} \right) (u_2^2 - u_1^2) &= \int \left(\frac{u_1^3 - u_1}{u_1} - \frac{u_2^3 - u_2}{u_2} \right) (u_2^2 - u_1^2) \\ &= \int \left(u_1^2 - u_2^2 \right) \left(u_2^2 - u_1^2 \right) \\ &\leq 0, \end{aligned} \tag{110}$$

and thus

$$\int (u_1^2 - u_2^2)^2 = 0, \tag{111}$$

from which we conclude using $u_1, u_2 > 0$ in U that $u_1 = u_2$.

Lemma. $\frac{u_1}{u_2}, \frac{u_2}{u_1} \in L^\infty(U)$.

First, there exist positive constants $0 < c < C$ such that $\partial_\nu u_i < -c$ (by Hopf's lemma), and $-C < \partial_\nu u_i$ (because U is smooth and u_i are smooth on ∂U compact).

Remark. *The idea for the first proof is $\partial_\nu u_i < -c$ means going from the boundary into the domain strictly increases by at least a fixed amount. Then something like $u_i(x - t\nu) > ct$ for t sufficiently small. Some compactness of the boundary should make t uniform in x. Same thing for the upper bound $Ct > u_i(x - t\nu)$.*

First Proof. By a tubular neighbourhood theorem, choose t_0 so small such that each $y \in V := \{x - t\nu : x \in \partial U, 0 \leq t \leq t_0\}$ satisfies $V \subset U$ and $y = x - t\nu(x)$ for a unique $x \in \partial U$. Then let M be the maximum second normal derivative in this tubular neighbourhood; namely, $M := \sup_V \left| \frac{\mathrm{d}^2}{\mathrm{d}s^2} u_i(x - s\nu(x))|_{s=t} \right|$, where the derivative is one-sided ($s \to 0^+$). M exists by compactness of V and smoothness of u_i and is well-defined because of how V was constructed. Now possibly lower t_0 so that $t_0 < \min(\frac{c}{2M}, \frac{C}{M})$. Now for $(x, t) \in \partial U \times [0, t_0]$, the inward normal derivative is bounded above and below by $\frac{c}{2}$ and $2C$ respectively, so $\frac{c}{2} t < u_i(x - t\nu(x)) < 2Ct$ by the mean value theorem. Thus $\frac{c}{4C} < \frac{u_1}{u_2}, \frac{u_2}{u_1} < \frac{4C}{c}$ in V. In the compact set $\overline{U - V}$, u_1, u_2 are smooth and positive, and thus their quotients are bounded. Thus $\frac{u_1}{u_2}, \frac{u_2}{u_1} \in L^\infty(U)$. ☐

This possibly simpler proof works by straightening the boundary.

Second Proof. For any $x \in \partial U$, there exist smooth local coordinates $(y_1, \ldots, y_n)$ and a neighbourhood V of $y(x)$ such that $V \cap \overline{U} = \{z \in V : y_1(x) \geq 0\}$. In these coordinates, the condition $Du \cdot \nu \neq 0$ on ∂U is $\partial_{y_1} u_i \neq 0$ on ∂U, because $\nu(y) = sy_1$ for some constant s. Because $u_i = 0$ on ∂U, $u_i = 0$ in V where $y_1 = 0$, so by the fundamental theorem of calculus,

$$u_i(y_1, \ldots, y_n) = \int_0^1 \frac{\partial u_i}{\partial t}(ty_1, \ldots, y_n)\, \mathrm{d}t = y_1 \int_0^1 \frac{\partial u_i}{\partial y_1}(ty_1, \ldots, y_n)\, \mathrm{d}t. \tag{112}$$

Define $f_i : V \cap \overline{U} \to \mathbb{R}$ to be the integral expression on the right. Then $u_i = y_1 f_i$, where f_i is nonzero on ∂U (because $\partial_{y_1} u_i \neq 0$ on ∂U). Because u_i are smooth, differentiating under the integral sign shows that f_i are smooth. Thus in V, $\frac{u_2}{u_1} = \frac{f_2}{f_1}$, which is smooth on ∂U. Same thing for $\frac{u_1}{u_2}$. Thus $\frac{u_1}{u_2}, \frac{u_2}{u_1}$ are smooth in a neighbourhood of each point of ∂U, and also smooth in the interior because u_1, u_2 are non-zero in U. We conclude that $\frac{u_1}{u_2}, \frac{u_2}{u_1}$ are smooth and thus bounded on $\overline{U}$. □

Guaraco 10

Guaraco 11

If it were not rotationally symmetric, then there would be some hyperplane dividing the ball into two half balls such that u on one half ball is not the reflection of u on the other half ball. Notice even reflection (of either side) produces a new continuous function in $H_0^1(B_R(0))$ positive in the interior of $B_R(0)$. If the energy on one half ball is less than the energy on the other, then reflecting this half creates a function with less energy than u, contradicting the minimality of u. Thus the reflected function has the same energy as u but is distinct from u, contradicting the uniqueness of u.

Thus $u(x)$ depends only on $|x|$. From Exercise 10, $1 - u(x) \leq Ce^{-\sigma \frac{R-|x|}{\epsilon}}$. Now fix K compact and take R_0 large enough so that $K \subset B_{R_0}(0)$. By Schauder estimates, u all its derivatives are uniformly bounded and uniformly equicontinuous, so by Arzela-Ascoli the solutions on $B_R(0)$ converge uniformly along a subsequence to a limit function $\tilde{u}$. For $x \in K$, we have $1 - u(x) \leq Ce^{-\sigma \frac{R-R_0}{\epsilon}}$, so $u \to 1$ uniformly on K as $R \to \infty$.

Guaraco 14

Any $f : S^n \to \mathbb{R}$ can be extended to $\tilde{f} : R^{n+1} - \{0\} \to \mathbb{R}$ by $\tilde{f}(x) = f(x\,|x|^{-1})$. Then (in spherical coordinates) $\nabla_{S^n} f = (0, \nabla \tilde{f}|_{S^n})$, ordering r first. To see this, recall that the gradient of a function in R^n along a point of a submanifold in $\mathbb{R}^n$ is the projection of the gradient at that point to the tangent space of that point; in this case, $\tilde{f}$ has no radial component.

Remark. *This "tangential gradient" thing is not really necessary; one could also notice directly that spherical coordinates odd wrt reflection across an equator (i.e. negation of an angle).*

For ϵ sufficiently small (because the domain is fixed), there exists u_+ positive minimizing energy with Dirichlet boundary conditions on the half-sphere $S^n_+ := S^n \cap \{x_{n+1} > 0\}$. Define S^n_- and u_- analogously.

We show $u_\pm$ is rotationally symmetric. Fix a hyperplane through a half-great circle orthogonal to the equator of $S^n_\pm$. Even reflection across this hyperplane (from either side) produces a new continuous function in $H_0^1(S^n_\pm)$ positive in the interior. The energy of $u_\pm$ on either half must be the same (otherwise even reflection would strictly lower energy on the whole of $S^n_\pm$), so reflecting across the hyperplane produces a function with the same energy as $u_\pm$ on all of $S^n_\pm$ (with the same sign and boundary conditions). By uniqueness, this reflected function must be the original $u_\pm$; because the hyperplane was arbitrary, $u_\pm$ is rotationally symmetric.

Define $\tilde{u}_-$ on S^n_- by odd reflection as $\tilde{u}_-(x', x_{n+1}) = -u_+(x', -x_{n+1})$, where $x' = (x_1, \ldots, x_{n-1})$. Then $\tilde{u}_-$ is negative on S^n_- and satisfies Dirichlet boundary conditions. We claim $\tilde{u}_- = u_-$. Indeed, $E(\tilde{u}_-, S^n_-) = E(u_+, S^n_+) = E_0$. We know $E_0 \geq E(u_-, S^n_-)$ because u_- minimizes energy on S^n_-. If the inequality were strict then the odd reflection of u_- to a (positive with DBC) function on S^n_+ would have strictly lower energy than u^+. Thus $E_0 = E(\tilde{u}_-, S^n_-) = E(u_-, S^n_-)$, and by uniqueness of u_-, we have $u_- = \tilde{u}_-$.

Now let $x \in S^n \cap \{x_{n+1} = 0\}$. Then $\nabla_{S^n} u^-(x) = \nabla u^+(x', -x_{n+1}) = \nabla_{S^n} u^+(x)$. Thus $u_\pm$ and their gradients agree on the equator $\{x_{n+1} = 0\}$, so the glued solution u which is $u_\pm$ on $S^n_\pm$ and 0 on the equator weakly solves Allen-Cahn on the equator and thus on S^n.

Guaraco 15

Guaraco 16

Guaraco 19

An argument like the one in Exercise 14 that the solutions of constant sign in A_t and $D_t^{\pm}$ are rotationally symmetric (this is a product of the rotational symmetry of the domains themselves). Their gradients are thus also rotationally symmetric, and in particular they satisfy homogeneous Neumann boundary conditions. Moreover, the same argument (appeal to uniqueness) shows the solution on A_t is symmetric with respect to even reflection about the equator $\{x_{n+1} = 0\}$; because it is smooth, this solution therefore satisfies a zero Neumann condition on the equator. Of course, these symmetry properties hold if the minimizer is 0.

Fix $\epsilon > 0$ and let $t \in (0, 1)$. We now consider a non-positive Dirichlet solution u_t on A_t; the argument for non-negative solutions on $D_t^{\pm}$ is similar. Let $A_t = A_t^+ \cup A_t^-$, with the sign being the sign of x_{n+1}. Because u_t is symmetric about the equator, we can focus on A_t^+. Let $Du_t \cdot \nu \equiv C_t$ on ∂A_t. For t_0 fixed, we want to show $C_t \to C_{t_0}$. Using $Du_t \cdot \nu \equiv 0$ on the equator,

$$\int_{A_t^+} W'(u_t) = \int_{A_t^+} \epsilon^2 \Delta u_t = \epsilon^2 \int_{\partial A_t^+} Du_t \cdot \nu = \frac{\epsilon^2}{\left|\partial A_t^+\right|} C_t. \tag{113}$$

and evidently $\left|\partial A_t^+\right| \to \left|\partial A_{t_0}^+\right|$ (in $(n-1)$-measure). Extend u_t by 0 (thus continuously) to $S^n \cap \{x_{n+1} \geq 0\}$. By Schauder estimates, u_t are uniformly bounded and uniformly equicontinuous, so by Arzela-Ascoli they converge uniformly along a subsequence on $\overline{A_{t_0}^+}$ to u_{t_0}. Then

$$\begin{aligned} \left| \int_{A_{t_0}^+} W'(u_{t_0}) - \int_{A_t^+} W'(u_t) \right| &\leq \int_{A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| \\ &\quad + \int_{A_t^+ - A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| \\ &\leq \int_{A_{t_0}^+} \left|W'(u_{t_0}) - W'(u_t)\right| + 2\left|A_t^+ - A_{t_0}^+\right|, \end{aligned} \tag{114}$$

where the first term goes to 0 by the uniform convergence of $u_{t_0} \to u_t$ and the second term goes to 0 by the geometry of the domains. In light of eq. (113), we conclude that C_t is continuous in t.

Remark. *What follows assumes that $\lambda_1(A_t), \lambda_1(D_{1-t}^{\pm}) \to \infty$ as $t \to 0$ (small domain, big eigenvalues). Not sure how to prove this (probably need to get into the geometry), but some scaling argument (for example D_t are geodesic disks) might help.*

Fix $\epsilon > 0$ small enough so that the minimizer is non-zero on both $A_{\frac{1}{2}}$ and $D_{\frac{1}{2}}^{\pm}$ (this is possible by Exercise 8). Define $u_t \in C(S^n)$ by gluing the minimizers on A_t and $D_t^{\pm}$. By Exercise 8 and the remark, the minimizer in A_t is 0 for t small—say for $0 < t \leq t_1$—and the minimizer in $D_t^{\pm}$ is 0 for t large, say for $t_2 \leq t < 1$. Take t_1 as large as possible and t_2 as small as possible. Then the minimizer is non-zero on both A_t and $D_t^{\pm}$ if and only if $t_1 < t < t_2$. By our choice of ϵ, $t_1 < \frac{1}{2} < t_2$, so $t_1 < t_2$.

We claim $C_{t_2}(A_{t_2}) > 0$, as otherwise eq. (113) would say $\int_{A_{t_2}} W'(u_{t_2}) \leq 0$, a contradiction with $u_{t_2} < 0$ in A_{t_2}. Similarly $C_{t_1}(D_{t_1}^{\pm}) < 0$. By continuity there is some $t_0 \in (t_1, t_2)$ with

$$Du_{t_0} \cdot \nu_{A_{t_0}}|_{\partial A_{t_0}} = C_{t_0}(A_{t_0}) = -C_{t_0}(D_{t_0}^{\pm}) = -Du_{t_0} \cdot \nu_{D_{t_0}^{\pm}}|_{\partial D_{t_0}^{\pm}}. \tag{115}$$

In particular, because $D_{t_0}^{\pm}$ and A_{t_0} share boundary (with opposite orientation), we conclude that the gradients of the minimizers coincide on ∂A_{t_0}. Thus u_{t_0} solves Allen-Cahn weakly on S^n, and by construction its nodal set is $S^n \cap \{x_{n+1} = \pm t_0\}$.

Otis Chapter 2

Otis 2.1

Part A, Assumed Schauder's estimate works for H^1 functions

Proof. It suffices to show that for any compact set $V \in \mathbb{R}^2$ a critical point is smooth. Let u be a critical point. Then by definition $u \in H^1(V) \cap L^\infty(V)$. We first prove that u is in fact Hölder's continuous with some coefficient $\alpha \in (0,1)$. We use theorem 8.24 in GT's Chapter 8. Since $u \in L^\infty$, let $g(x) = W'(u)$, then $-\Delta u = g(x)$ with

$$\sup_{x \in V} |g(x)| \leq C(\|u\|^3_{L^\infty(V)} + \|u\|_{L^\infty(V)}) \leq C.$$

Thus, $g \in L^\infty$. Thus, the theorem applies and we get that

$$\|u\|_{C^\alpha(V)} \leq C(\|u\|_{L^2(V)} + k) \leq C < \infty.$$

Thus, u is hölder continuous with respect to $\alpha > 0$.

Since $u \in C^\alpha(V)$, and products and sum of hölder continuous functions on a bounded domain is also hölder continuous, $g \in C^\alpha(V)$ as well. Next, we differentiate the Allen-Cahn to obtain that

$$\Delta u_{x_i} - W''(u)u_{x_i} = 0 \tag{116}$$

for each $i = 1, \dots, n$. Thus, $\boldsymbol{Du}$ solves this system of Allen-Cahn. We can again apply theorem 8.24 in GT to 116 with $Lu_{x_i} = \delta u_{x_i} + W''(u)u_{x_i} = 0$. Since $W''(u) \in L^\infty$, the theorem applies and we get that for each u_{x_i}

$$\|u_{x_i}\|_{C^\alpha(V)} \leq C(\|u_{x_i}\|_{L^2(V)} + k) < \infty.$$

Thus, we have

$$\|Du\|_{C^\alpha(V)} < \infty \implies u \in C^{1,\alpha}.$$

Since $Du, u \in L^\infty$, we thus have

$$\|g'(x)\|_{L^\infty} = \|W''(u)Du\|_{L^\infty} \leq \|W''(u)\|_{L^\infty} \|Du\|_{L^\infty} < \infty.$$

Thus, in particular, we have the hölder's norm of $[g]_{\alpha;V} \leq \|g'(x)\|_{L^\infty}$ is bounded. With this, apply the interior Schauder's estimate and get that

$$|u|_{2,\alpha;V} \leq C(|u|_{0;V} + |g|_{0,\alpha;V}) < \infty.$$

Thus, $u \in C^{2,\alpha}(V)$. Now, we do induction on $k = 0, 1, \dots$. i.e. we will show that if $u \in C^{k,\alpha}(V)$, then $u \in C^{k+1,\alpha}(V)$. We have already proved the base case with $k = 0$. Know, assume $C^{k,\alpha}(V)$. Then we have $|D^i u|$ bounded for all $i = 0, \dots, k$. Thus, we would have

$$g^{(k)(x)} \leq \sum_{i=0}^{k} c_i \left\|D^i u\right\|_{L^\infty} < \infty.$$

Thus, $|g|_{k-1.\alpha;V}$ is bounded and we can apply Exercise 6.1 in GT and get that

$$|u|_{k+1,\alpha;V} \leq C(|u|_{0;V} + |g|_{k-1,\alpha;V}) < \infty.$$

Thus, $u \in C^{k+1,\alpha}(V)$. Since $k = 0, 1, 2, \dots$, $u \in C^\infty(V)$, for any compact sets $V \subset M$. □

Part A, Alternate Solution (not assuming anything)

Remark. *This shows $u \in C^\alpha \implies u \in C^\infty$. The idea is to bypass the issue about applying Schauder estimates to functions we don't yet know are in the space by mollifying them first. Technical issues arise because C^α functions cannot be approximated by smooth functions in C^α norm, but this approximation holds in $C^{\alpha-\epsilon}$, which is enough.*

Now if only we could get $u \in C^\alpha$ without using G-T Chapter 8...

Definition. *Let the "little Holder space" $c^{k,\alpha}(U)$ be the set of functions $f \in C^{k,\alpha}(U)$ such that*

$$\lim_{\delta\to 0} \sup_{\substack{x,y\in K\\ 0<|x-y|\le\delta}} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x-y|^\alpha} = 0 \tag{117}$$

for each $|\gamma| = k$ and compact $K \subset U$.

Lemma. *If $0 < \beta < \alpha < 1$, then $C^{k,\alpha}(U) \subset c^{k,\beta}(U)$.*

Just $k = 0$ is proved, but the same argument works for $k \neq 0$.

Proof. Suppose $0 < \beta < \alpha < 1$, $f \in C^{0,\alpha}(U)$, and $K \subset U$ is compact. Then

$$\begin{aligned} \sup_{\substack{x,y\in K\\ 0<|x-y|\le\delta}} \frac{|f(x)-f(y)|}{|x-y|^\beta} &= \sup_{\substack{x,y\in K\\ 0<|x-y|\le\delta}} |x-y|^{\alpha-\beta} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \\ &\le \delta^{\alpha-\beta} |f|_{0,\alpha} \\ &\to 0 \,\text{as}\, \delta \to 0. \end{aligned} \tag{118}$$

□

Now we show that the little Holder space is the "closure of smooth functions in the topology of Holder convergence on compact sets."

Lemma. *Given $f \in C^{k,\alpha}(U)$, we have $f \in c^{k,\alpha}(U)$ if and only if for each compact $K \subset U$ there exists $f_n \in C^\infty(K)$ with $f_n \to f$ in $C^{k,\alpha}(K)$.*

Notice that because compact sets can be covered by finitely many balls, it suffices to replace "for each compact $K \subset U$" with "for each ball $B \subset\subset U$".

Proof. Suppose $f \in C^{k,\alpha}(U)$ and $f_n \to f$ in $C^{k,\alpha}(U)$ with $f_n \in C^\infty(U)$. Fix a ball $B \subset\subset U$ and $\epsilon > 0$. For some n large enough,

$$|(D^\gamma f_n - D^\gamma f)(x) - (D^\gamma f_n - D^\gamma f)(y)| \le \epsilon |x-y|^\alpha \tag{119}$$

for all $|\gamma| = k$ and all $x, y \in B \subset U$. Let $M := \sup_{|\gamma'|=|\gamma|+1} \left|D^{\gamma'} f_n\right|$. Then the reverse triangle inequality and the mean value theorem (B is convex) gives

$$\begin{aligned} |D^\gamma f(x) - D^\gamma f(y)| &\le M|x-y| + \epsilon |x-y|^\alpha \\ |D^\gamma f(x) - D^\gamma f(y)| &\le |x-y|^\alpha (\epsilon + M|x-y|^{1-\alpha}), \end{aligned} \tag{120}$$

so $|D^\gamma f(x) - D^\gamma f(y)| \le 2\epsilon |x-y|^\alpha$ for $|x-y| < (\epsilon M^{-1})^{\frac{1}{1-\alpha}}$. That is, $f \in c^{k,\alpha}(U)$.

Now suppose $f \in c^{k,\alpha}(U)$. Fix $\epsilon > 0$, fix $|\gamma| = k$ a multi-index, and fix a ball $B_R \subset\subset U$. By the definition of the little Holder space, there exists $\delta < R$ such that if $|x-y| < \delta$, then $|D^\gamma f(x) - D^\gamma f(y)| \le \epsilon |x-y|^\alpha$ for all $x \neq y \in B_R$. Let f_t be the mollification of f. We know $f_t \to f$ in C^k, so we just need to control the Holder term. Let t_0 be small enough so that $\|f_t - f\|_{C^k} \le \epsilon\delta^\alpha$ (possible because f is continuous) and $B_{R+t} \subset\subset U$ (ball with same center, different radius) for all $t < t_0$. Then f_t is defined for $t < t_0$. For $t < t_0$ and $|x-y| < \delta$,

$$\begin{aligned} |D^\gamma f_t(x) - D^\gamma f_t(y)| &= \left| \int_{\mathbb{R}^n} \varphi_t(z)(D^\gamma f(x-z) - D^\gamma f(y-z))\,\mathrm{d}z \right| \\ &\le \epsilon |x-y|^\alpha \int_{\mathbb{R}^n} \varphi_t(z) \\ &= \epsilon |x-y|^\alpha, \end{aligned} \tag{121}$$

and so

$$|(D^\gamma f_t - D^\gamma f)(x) - (D^\gamma f_t - D^\gamma f)(y)| \le \epsilon |x-y|^\alpha. \tag{122}$$

If on the other hand $|x-y| \geq \delta$ (and still $t < t_0$),

$$|(D^\gamma f_t - D^\gamma f)(x) - (D^\gamma f_t - D^\gamma f)(y)| \leq 2 \|D^\gamma f_t - D^\gamma f\|_{C_k} \leq \epsilon \delta^\alpha < \epsilon |x-y|^\alpha . \tag{123}$$

Thus for $t < t_0$, we have $|(D^\gamma f_t - D^\gamma f)(x) - (D^\gamma f_t - D^\gamma f)(y)| < \epsilon |x-y|^\alpha$ for all $x, y \in B_R$. Thus f is the limit of smooth functions in $C^{k,\alpha}(B_R)$. ☐

Combining the above two lemmas give the following.

Corollary. *If $f \in C^{k,\alpha}(U)$ and $\beta < \alpha$, then for each compact $K \subset U$, there exist $f_n \in C^\infty(K)$ with $f_n \to f$ in $C^{k,\beta}(K)$.*

Proof. Use $C^{k,\alpha}(U) \subset c^{k,\beta}(U)$ and the characterization of $c^{k,\beta}$. ☐

Lemma. *Let U be a bounded smooth domain, let $V \subset\subset U$, and suppose $f \in C^{k,\alpha}(U)$. If v solves $\Delta v = f$ with $v = 0$ on ∂U, then $v \in C^{k+2,\alpha}(V)$.*

Proof. By the above, there exist $f_n \in C^\infty(\overline{V})$ with $f_n \to f$ in $C^{k,\beta}(\overline{V})$ for $\beta < \alpha < 1$. Let v_n be the smooth solution of $\Delta v_n = f_n$ with Dirichlet boundary conditions. By Schauder estimates,

$$|v_n|_{k+2,\alpha,V} \leq C(|v_n|_{0,U} + |f_n|_{k,\alpha,U}), \tag{124}$$

with C not depending on n. By G-T Theorem 3.7 (proof based on maximum principle, genuinely not cryptic), we have $|v_n|_{0,U} \leq C(\operatorname{diam} U) |f_n|_{0,\alpha,U}$ (because $v_n = 0$ on ∂U). Then the above becomes $|v_n|_{k+2,\alpha,V} \leq C |f_n|_{k,\alpha,U}$. We can choose the f_n so that $|f_n|_{k,\alpha,U} \leq 2 |f|_{k,\alpha,U}$,[2] so that $|v_n|_{k+2,\alpha,V} \leq C |f|_{k,\alpha,U}$.

Because $C^{k+2,\beta}(\overline{V}) \subset\subset C^{k+2,\alpha}(\overline{V})$ (Arzela-Ascoli), the v_n converge along a subsequence in $C^{k+2,\beta}(V)$ to some $\tilde{v}$. Then in V,

$$\begin{aligned} |\Delta v - f|_0 &\leq |\Delta v - \Delta v_n|_0 + |\Delta v_n - f_n|_0 + |f_n - f|_0 \\ &\leq N |v - v_n|_2 + |f_n - f|_0 \\ &\to 0, \end{aligned} \tag{125}$$

so $\Delta \tilde{v} = f$ in V. Moreover, because the v_n are uniformly bounded in $C^{k+2,\alpha}(V)$ and they converge uniformly along with their derivatives, we can actually conclude that $\tilde{v} \in C^{k+2,\alpha}$ (although the convergence is in $C^{k+2,\beta}$).

Now $\Delta(v - \tilde{v}) = 0$ in V, so $v - \tilde{v} \in C^\infty(V)$. But because $\tilde{v} \in C^{k+2,\alpha}(V)$, we conclude $v \in C^{k+2,\alpha}(V)$. ☐

Now if $\Delta u = W'(u)$, take $f := W' \circ u$ and notice f has the same Holder regularity as u on compact sets. Thus on V precompact, $u \in C^{k,\alpha}(V)$ for all k (induction on the above lemma), and so if $u \in C^\alpha$, the induction above begins with $k = 0$, and we can conclude u is smooth.

Part B

Assuming the elliptic regularity of 2.1a, suppose u is a solution to Allen-Cahn (thus it is smooth). The truncation $u\chi_{\{|u| \leq 1\}} + \chi_{\{u > 1\}} - \chi_{\{u < -1\}}$ is continuous, in H^1, and weakly solves Allen-Cahn, so it is smooth. The smoothness is only possible if $|u| \leq 1$.

Otis 2.2

Part A

Let u be a smooth solution to Allen-Cahn. To rule out an infinite-energy solution, it suffices to show that if $u' = 0$ somewhere, then either the solution is finite energy or does not exist for all time, so that by continuity the sign of u' is constrained to be that of $u'(0)$. Then we are done by Part B, because if $u' < 0$, then

[2] The Holder term is the same, as seen in eq. (121), and for $|\gamma| \leq k$, $D^\gamma f_t \to D^\gamma f$ uniformly on U (because $D^\gamma f$ is bounded on U and thus uniformly continuous—see Evans Appendix C.5.7 for details), and so $|D^\gamma f_t|_0 \leq |D^\gamma f_t - D^\gamma f|_0 + |D^\gamma f|_0 \leq 2 |D^\gamma f|_0$ for t small enough.

$(-u)'' = -u^3 + u = (-u)^3 - (-u)$, so $-u$ is a solution with strictly positive derivative and thus has finite energy. By Lemma 2.3, we just need to find a new solution to Allen-Cahn. Recall $u'^2 = \frac{1}{2}(1-u^2)^2 + \lambda$.

Suppose $\lambda = 0$. If $u' = 0$ somewhere, then $u' = 0$ everywhere and $u \equiv \pm 1$, which is finite energy. Otherwise (by IVT) we are in the case of Part B.

Suppose $\lambda > 0$. Then $u' = \pm\sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}$. Because $|u'| = \left|\sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}\right| \geq \sqrt{\lambda} > 0$, we cannot have $u' = 0$, and so we are in the case of Part B.

Suppose $\lambda < 0$. Then

$$\begin{aligned} u'^2 = \frac{1}{2}(1-u^2)^2 + \lambda \geq 0 \\ 1 - u^2 \geq \sqrt{-2\lambda} \\ |u| \leq \sqrt{1-\sqrt{-2\lambda}} \quad \text{and} \quad -\frac{1}{2} \leq \lambda < 0. \end{aligned} \tag{126}$$

We can rule out $\lambda = -\frac{1}{2}$ (it is $u \equiv 0$). Define $C := \sqrt{1-\sqrt{-2\lambda}}$ (we have $0 < C < 1$ for $-\frac{1}{2} < \lambda < 0$). Say $u(0) = -C$. Then $u'(0) = 0$ and $u''(0) = -C^3 + C > 0$ (u'' has the opposite sign as u in $[-1,1]$). Because $u' = \pm\sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}$ and $u'' > 0$ for positive time near 0, $u' > 0$ locally, so locally u' is on the positive branch $u' = +\sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}$. By the argument in Part B, a strictly increasing (local) solution to this IVP exists and gets arbitrarily close to C in finite time, say for $t < \frac{T}{2}$. By the same argument, and because $|u''|$ is even in u and $u''(C) < 0$ (so the negative branch of u' is taken), solve the IVP with $u(0) = C$ to get a strictly decreasing solution from C to $-C$ on $[\frac{T}{2}, T)$. Concatenating these gives a T-periodic function.

Is it a solution to Allen-Cahn? Certainly u is continuous. As $u \to C$, $u' \to 0$, so u' is also continuous as it switches from the positive branch to the negative branch. Also, $u'' = (u')' = \frac{u'}{\pm\sqrt{\frac{1}{2}(1-u)^2+\lambda}} u(u^2-1)$ (where $\pm$ is the sign of u') in this formulation is not defined at $\pm C$, but $\frac{u'}{\pm\sqrt{\frac{1}{2}(1-u)^2+\lambda}} = 1$ on $(0, \frac{T}{2})$, so u'' is continuous at C. Because u'' and $u^3 - u$ are continuous functions that agree except possibly at $\pm C$ (a set of measure zero), they are in fact equal, and this is a global solution to Allen-Cahn.

Part B

Throughout take $|u(0)| \leq 1$.

Recall $u'^2 = \frac{1}{2}(1-u^2)^2 + \lambda$ for some $\lambda \in \mathbb{R}$. If we suppose u is a smooth solution with $u' > 0$ for all time, then $u' = \sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}$. We show $\lambda = 0$. This suffices because we can separate variables in $u' = \frac{1}{\sqrt{2}}(1-u^2)$ to see that the heteroclinic solution is the unique one (given an initial condition).

Suppose $\lambda > 0$. Then $u' \geq \sqrt{\lambda} > 0$, so by a comparison principle (which applies because $\sqrt{\frac{1}{2}(1-u^2)^2 + \lambda}$ is locally Lipschitz), $u(t) \geq \sqrt{\lambda} t$ for $t \geq 0$, contradicting $u \in [-1,1]$.

If $\lambda < 0$, then we must have

$$\begin{aligned} u'^2 = \frac{1}{2}(1-u^2)^2 + \lambda > 0 \\ 1 - u^2 > \sqrt{-2\lambda} \\ |u| < \sqrt{1-\sqrt{-2\lambda}} \quad \text{and} \quad -\frac{1}{2} < \lambda < 0. \end{aligned} \tag{127}$$

Define as in Part A $C = \sqrt{1-\sqrt{-2\lambda}} < 1$. Because $u' > 0$, u increases to C as $t \to \infty$. We claim that $u'' \to 0$ as $t \to \infty$ (a contradiction because $u'' \neq 0$ near $u = C$). By Exercise 2.1 ($|u| \leq 1$), $u''' = u'(3u^2 - 1)$ is bounded, because $|u'| \leq \frac{1}{2}\left|(1-u^2)^2\right| + |\lambda| \leq 1$ and $\left|3u^2 - 1\right| \leq 2$. Because u''' is bounded, u'' is uniformly continuous, so $u' \to 0$ by the following lemma (taking $f = u'$ and $\alpha = C$): if $f \in C^1$ and $f \to \alpha < \infty$ as $t \to \infty$ and f' is uniformly continuous, then $f' \to 0$ as $t \to \infty$.[3]

[3]Thinking of f as $\int f'$, this is basically the intuitive statement that a uniformly continuous function whose integral to infinity converges must vanish at infinity.

We now prove the lemma. Suppose $f' \not\to 0$ as $t \to \infty$. Then choose $\epsilon > 0$ and $\{t_n\}$ increasing to infinity such that $|f(t_n)| \geq \epsilon$ for all n. By uniform continuity of f', choose $\delta > 0$ such that $|f'(t) - f'(t_n)| < \epsilon$ for $|t - t_n| < \delta$. If $t \in [t_n, t_n + \delta]$, then

$$|f'(t)| = |f'(t_n) - f'(t_n) + f'(t)| \geq |f'(t_n)| - |f'(t_n) - f'(t)| \geq \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}. \tag{128}$$

Because f is C^1, we have

$$\left| \int_0^{t_n+\delta} f' - \int_0^{t_n} f' \right| = \left| \int_{t_n}^{t_n+\delta} f' \right| \geq \int_{t_n}^{t_n+\delta} |f'| \geq \frac{\epsilon\delta}{2} > 0. \tag{129}$$

But taking limits and applying the fundamental theorem of calculus gives

$$\lim_{n\to\infty} \left| \int_0^{t_n+\delta} f' - \int_0^{t_n} f' \right| = \lim_{n\to\infty} |f(t_n + \delta) - f(t_n)| = |\alpha - \alpha| = 0, \tag{130}$$

a contradiction.

Otis 2.3

Since $\mathbb{H}(t)$ is a solution, $\mathbb{H}'(t) = \frac{1}{\sqrt{2}}(1 - \mathbb{H}^2)$, and so $\mathbb{H}'(t)^2 = \frac{1}{2}(1 - \mathbb{H}(t)^2)^2 = 2W(\mathbb{H}(t))$. Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \Big(\frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t))\Big) dt &= \int_{-\infty}^{\infty} \Big(\frac{1}{2}\mathbb{H}'(t)^2 + \frac{1}{2}\mathbb{H}'(t)^2\Big) dt \\
&= \int_{-\infty}^{\infty} \mathbb{H}'(t)^2 dt \\
&= \int_{-\infty}^{\infty} \frac{1}{2}(1 - \mathbb{H}(t)^2)^2 dt \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}(1 - \mathbb{H}(t)^2)\frac{1}{\sqrt{2}}(1 - \mathbb{H}(t)^2) dt.
\end{aligned}$$

Now let $s = \mathbb{H}(t)$, then $\frac{ds}{dt} = \mathbb{H}'(t) = \frac{1}{\sqrt{2}}(1 - \mathbb{H}^2)$. So

$$\begin{aligned}
\int_{-\infty}^{\infty} \Big(\frac{1}{2}\mathbb{H}'(t)^2 + W(\mathbb{H}(t))\Big) dt &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}(1 - \mathbb{H}(t)^2)\frac{1}{\sqrt{2}}(1 - \mathbb{H}(t)^2) dt \\
&= \int_{-1}^{1} \frac{1}{\sqrt{2}}(1 - s^2) ds \\
&= \Big[\frac{1}{\sqrt{2}} s - \frac{1}{3\sqrt{2}} s^3\Big]_{-1}^{1} \\
&= \frac{2\sqrt{2}}{3}.
\end{aligned}$$

Otis 2.4

Recalling that $\frac{\mathrm{d}}{\mathrm{d}x} \tanh x = \operatorname{sech}^2 x$, we compute

$$\begin{aligned}
\partial_i u_\epsilon(x) = \mathbb{H}'(\epsilon^{-1}\langle a, x\rangle)\epsilon^{-1} a_i \implies |\nabla u_\epsilon|^2 &= \mathbb{H}'^2(\epsilon^{-1}\langle a, x\rangle)\epsilon^{-2}(a_1^2 + a_2^2) \\
&= \frac{1}{2}\epsilon^{-2} \operatorname{sech}^4\left(\frac{\epsilon^{-1}}{\sqrt{2}}\langle a, x\rangle\right),
\end{aligned} \tag{131}$$

using $|a|^2 = a_1^2 + a_2^2 = 1$ and $\mathbb{H}(x) = \tanh \frac{x}{\sqrt{2}}$. Furthermore,

$$W(u_\epsilon(x)) = \frac{1}{4}(1 - {u_\epsilon}^2)^2 = \frac{1}{4} \operatorname{sech}^4\left(\frac{\epsilon^{-1}}{\sqrt{2}}\langle a, x\rangle\right). \tag{132}$$

Thus

$$\begin{aligned} E_\epsilon(u_\epsilon, B_1(0)) &= \int_{B_1(0)} \frac{1}{4}\epsilon^{-1}\operatorname{sech}^4\left(\frac{\epsilon^{-1}}{\sqrt{2}}\langle a, x\rangle\right) + \frac{1}{4}\epsilon^{-1}\operatorname{sech}^4\left(\frac{\epsilon^{-1}}{\sqrt{2}}\langle a, x\rangle\right) \\ &= \frac{1}{2\epsilon}\int_{B_1(0)} \operatorname{sech}^4\left(\frac{\epsilon^{-1}}{\sqrt{2}}\langle a, x\rangle\right). \end{aligned} \tag{133}$$

Evidently this integral (over the circle) is rotationally symmetric in $a \in \partial B_1(0)$, so we may as well take $a = (1, 0)$ so that $\langle a, x\rangle = x_1$. We approximate this integral (after fixing a) by the energy over the square $[-1, 1]^2$ (which is not rotationally symmetric in a):

$$\begin{aligned} E_\epsilon(u_\epsilon, [-1, 1]^2) &= \frac{1}{2\epsilon}\int_{-1}^{1}\int_{-1}^{1} \operatorname{sech}^4\left((\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= \epsilon^{-1}\int_{-1}^{1} \operatorname{sech}^4\left((\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x. \end{aligned} \tag{134}$$

Because an anti-derivative of $\operatorname{sech}^4 x$ is $\frac{1}{3}\tanh x(2 + \operatorname{sech}^2 x)$ and tanh is odd, this is

$$\begin{aligned} E_\epsilon(u_\epsilon, [-1, 1]^2) &= \epsilon^{-1}[\frac{\sqrt{2}\epsilon}{3}\tanh x(2 + \operatorname{sech}^2 x)]_{-1}^{1} \\ &= \frac{2\sqrt{2}}{3}\tanh\frac{x}{\sqrt{2}\epsilon}(2 + \operatorname{sech}^2\frac{x}{\sqrt{2}\epsilon}). \end{aligned} \tag{135}$$

Now we bound the error $E_\epsilon(u_\epsilon, [-1, 1]^2 - B_1(0))$.

$$\begin{aligned} E_\epsilon(u_\epsilon, [-1, 1]^2 \setminus B_1(0)) &= \frac{1}{2\epsilon}\int_{-1}^{1}\int_{\substack{-1\le x\le -\sqrt{1-y^2} \\ \sqrt{1-y^2}\le x\le 1}} \operatorname{sech}^4\left((\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{1}{\epsilon}\int_{-1}^{1}\int_{\sqrt{1-y^2}}^{1} \operatorname{sech}^4\left((\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \frac{2}{\epsilon}\int_{0}^{1}\int_{1-y^2}^{1} \operatorname{sech}^4\left((\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x\,\mathrm{d}y \\ &\le 4\int_{0}^{1}\int_{1-y^2}^{1} \epsilon^{-1}\exp\left(-(\sqrt{2}\epsilon)^{-1}x\right)\,\mathrm{d}x\,\mathrm{d}y \\ &= 4\int_{0}^{1} \epsilon^{-1}\left[-\sqrt{2}\epsilon\exp\left(-(\sqrt{2}\epsilon)^{-1}x\right)\right]_{1-y^2}^{1}\,\mathrm{d}y \\ &= 4\sqrt{2}\int_{0}^{1} e^{-\frac{1-y^2}{\sqrt{2}\epsilon}} - e^{-\frac{1}{\sqrt{2}\epsilon}}\,\mathrm{d}y \\ &\le 4\sqrt{2}e^{-\frac{1}{\sqrt{2}\epsilon}}\int_{0}^{1} e^{\frac{y^2}{\sqrt{2}\epsilon}}\,\mathrm{d}y \\ &\le 4\sqrt{2}e^{-\frac{1}{\sqrt{2}\epsilon}}\int_{0}^{1} e^{\frac{y}{\sqrt{2}\epsilon}}\,\mathrm{d}y \\ &\le 4\sqrt{2}e^{-\frac{1}{\sqrt{2}\epsilon}}\sqrt{2}\epsilon(e^{\frac{y}{\sqrt{2}\epsilon}} - 1) \\ &\le 8\epsilon. \end{aligned} \tag{136}$$

In the above, we substitute $x \mapsto -x$, use the evenness in y of the outer integral and $t^2 \le t$ on $[0, 1]$, use $\operatorname{sech}^4 x \le \operatorname{sech} x \le 2e^{-|x|}$, and evaluate the inner integral. Thus

$$\begin{aligned} E_\epsilon(u_\epsilon, B_1(0)) &= E_\epsilon(u_\epsilon, [-1, 1]^2) - E_\epsilon(u_\epsilon, [-1, 1]^2 \setminus B_1(0)) \\ &= \frac{2\sqrt{2}}{3}(2 + \operatorname{sech}^2\frac{x}{\sqrt{2}\epsilon})\tanh\frac{x}{\sqrt{2}\epsilon} + O(\epsilon) \\ &\to \frac{4\sqrt{2}}{3}\ \text{as}\ \epsilon \to 0. \end{aligned} \tag{137}$$

Otis 2.5

Part A

Remark. *Some of showing the existence and smoothness of u (using trace for example) is complicated by Ω_R not being smooth. However, it is Lipschitz, so it is probably OK: round off the corners slightly to make the domain smooth (in such a way that it is contained in Ω_R), then run the Arzela-Ascoli argument of Part D on the solutions on the approximating smooth domains to obtain a solution on Ω_R.*

The Allen-Cahn energy functional $E[w] = \int \frac{1}{2}|Dw|^2 + W(w)$ is coercive and convex in Dw, so there exists a minimizer u of E in H_0^1. Moreover, $\int W(u) < \infty$, as $E[0] = C|\Omega_R| < \infty$. In particular, $u \in L^4$, as $\int u^4 = \int -1 + 2u^2 + 4W(u) < \infty$. Thus u is a minimizer over $H_0^1 \cap L^4$. We now show that u weakly solves Allen-Cahn (and is thus smooth by 2.1a), a slight modification of the argument in Evans. Set $i(\tau) = E[u+\tau v]$ for fixed $v \in H_0^1 \cap L^4$. Then for $\tau \neq 0$,

$$\begin{aligned} \frac{i(\tau) - i(0)}{\tau} &= \frac{1}{\tau}\int \left[\frac{1}{2}|Du + \tau Dv|^2 + W(u + \tau v) - \frac{1}{2}|Du|^2 - W(u)\right] \\ &:= \int L^\tau. \end{aligned} \tag{138}$$

Taking a directional derivative, $L^\tau \to Du \cdot Dv + W'(u)v.$ as $\tau \to 0$. Also,

$$\begin{aligned} L^\tau &= \frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{2}|Du + sDv|^2 + W(u + sv)\right] \\ &= \frac{1}{\tau}\int_0^\tau \frac{\mathrm{d}}{\mathrm{d}s}\left[\frac{1}{2}|Du + sDv|^2 + W(u + sv)\right] \\ &= \frac{1}{\tau}\int_0^\tau Du \cdot Dv + s|Dv|^2 + W'(u + sv)v \\ &= Du \cdot Dv + \tau|Dv|^2 + W'(u + \tau v)v, \end{aligned} \tag{139}$$

and as $\tau \to 0$,

$$\begin{aligned} |L^\tau| &\leq C(|Du|^2 + |Dv|^2 + \left|u^3 + v^3 + u^2 v + uv^2 + v^3 - u - v\right||v|) \\ &\leq C(|Du|^2 + |Dv|^2 + \left|u^4\right| + \left|v^4\right| + |u|^2 + |v|^2), \end{aligned} \tag{140}$$

where we used Young's inequality to show in particular that $\left|u^3 v\right| \leq C(|u|^4 + |v|^4)$. And so $L^\tau \in L^1$ because $u, v \in H_0^1 \cap L^4$. Passing to the limit as $\tau \to 0$ in eq. (138) by the dominated convergence theorem shows that $i'(0)$ exists and is equal to $\int Du \cdot Dv + W'(u)v$. Because i has a minimum at 0, we conclude that $i'(0) = 0$, and so u weakly solves Allen-Cahn.

If $u > 1$ somewhere, then u attains an interior maximum on U and thus on $V := \{u > 1\}$, because $u = 0$ on ∂U. On V, we have $\Delta u = W'(u) \geq 0$, so by the maximum principle u is constant on V, which contradicts its continuity. Thus $u \leq 1$, and similarly one shows $-1 \leq u \leq 1$. If $u = 1$ somewhere, then $Lu = -\Delta u + 2u$, so that $v := u - 1$ achieves a non-negative interior maximum and satisfies $Lv = -\Delta u + 2u - 2 = -u^3 + 3u - 2 \leq 0$ (because $u \leq 1$), so we conclude by the maximum principle that $u \equiv 1$, which contradicts the boundary condition, so $u < 1$. Similarly one shows $-1 < u$. Thus $|u| < 1$.

$$\int D|u|Dv = \int Du_+ Dv + \int Du_- Dv = -\int_{\{u \geq 0\}} W'(u)v + \int_{\{u < 0\}} W'(u)v.$$

When $u \geq 0$, $W'(u) \leq 0$, and when $u < 0$, $W'(u) > 0$. So

$$-\int_{\{u \geq 0\}} W'(u)v + \int_{\{u < 0\}} W'(u)v = \int |W'(u)|v = -\int W'(|u|)v.$$

Therefore $|u|$ would be a weak solution, and thus smooth.

Part B

Let $V = \{(x,y)|x, y \geq \epsilon, x^2+y^2 \leq (R-\epsilon)^2\}$. Let ζ be a smooth function that is 1 on V, decreases linearly outward and vanishes on the boundary of Ω_R. Then $D\zeta = W(\zeta) = 0$ on the interior of V. Let $U = \Omega_R - V$. Then U has area

$$m(U) \leq \frac{1}{4}(2\pi R^2 - 2\pi(R-\epsilon)^2) + \epsilon R + \epsilon R = \frac{\pi}{2}\epsilon(2R-\epsilon) + 2\epsilon R = (\pi+2)\epsilon R - \frac{\pi}{2}\epsilon^2.$$

On U, $|\frac{\partial\zeta}{\partial x}|$ is at most $\frac{1}{\epsilon}$. Same is true for $|\frac{\partial \zeta}{\partial y}|$. Therefore

$$\frac{1}{2}|D\zeta|^2 \leq \frac{1}{\epsilon^2}.$$

Hence,

$$\begin{aligned} E_1(\zeta) &= \int_U \frac{1}{2}|D\zeta|^2 + \frac{1}{4}(\zeta^2-1)^2 dx \\ &\leq (\frac{1}{\epsilon^2} + \frac{1}{4})((\pi+2)\epsilon R - \frac{\pi}{2}) \\ &\leq CR \end{aligned}$$

upon choosing some appropriate ϵ. Since u_R is a minimizer, it must has energy less than or equal to ζ.

To construct such a function ζ (independent of the dimension), we use mollifiers. First notice that for $|x| < 1$, if φ is the standard mollifier, then

$$\varphi_{x_i}(x) = -2e^{-\frac{1}{1-|x|^2}} \frac{1}{(1-|x|^2)^2} x_i, \tag{141}$$

so

$$|D\varphi(x)| \leq C\,|x| \leq C \tag{142}$$

as $e^{-\frac{1}{1-|x|^2}} \frac{1}{(1-|x|^2)^2}$ is bounded (in fact we can take $C = 2$). By the chain rule $|D\varphi_\epsilon(x)| \leq C\epsilon^{-(n+1)}$. Now let K be a compact set and define $K_\delta := \{x : d(x,K) < \delta\}$. Then $\varphi_{\frac{\epsilon}{2}} * \chi_{K_{\frac{\epsilon}{2}}}$ has range in $[0,1]$, is 1 on K, and is supported in K_ϵ. Thus $D(\varphi_\epsilon * \chi_K)(x) = 0$ for $x \in K$ and for $x \in K_\epsilon - K$,

$$\begin{aligned} \left|D(\varphi_{\frac{\epsilon}{2}} * \chi_{K_{\frac{\epsilon}{2}}})(x)\right| &= \left|(D\varphi_{\frac{\epsilon}{2}} * \chi_{K_{\frac{\epsilon}{2}}})(x)\right| \\ &\leq \int_{|y|<\frac{\epsilon}{2}} \left|D\varphi_{\frac{\epsilon}{2}}(y)\chi_{K_{\frac{\epsilon}{2}}(x-y)}\right| \mathrm{d}y \\ &\leq C\epsilon^{-(n+1)} \int_{|y|<\frac{\epsilon}{2}} 1 \,\mathrm{d}y \\ &\leq C\epsilon^{-1}. \end{aligned} \tag{143}$$

Nice job! Might be more useful to define things radially and start working with the laplacian/gradient operator in radial coordinates, but this is good

Part C

Let $B = B(0,R)$. If $\tilde{u}$ is what we constructed in part B, define by odd reflections

$$u(x,y) = \begin{cases} \tilde{u}(x,y) & x, y > 0 \\ -\tilde{u}(-x,y) & x < 0 < y \\ \tilde{u}(-x,-y) & x, y < 0 \\ -\tilde{u}(x,-y) & y < 0 < x. \end{cases} \tag{144}$$

Then $u \in C^1$ at the axes except possibly at 0, so $u \in H_0^1(B - \{0\})$, and weakly solves Allen-Cahn on $B \setminus \{0\}$. To see this, let R_i be the intersection of $B - \{0\}$ with the i-th quadrant of $\mathbb{R}^2$ and let $v \in C_c^\infty(B - \{0\})$. We can integrate by parts

$$\begin{aligned} \int_{B-0} Du \cdot Dv + W'(u)v \, \mathrm{d}x &= \sum_{i=1}^4 \int_{R_i} Du \cdot Dv + W'(u)v \\ &= \sum_{i=1}^4 \int_{R_i} (-\Delta u + W'(u))v \\ &= 0, \end{aligned} \tag{145}$$

where the boundary terms vanish because $u = 0$ on the axes and ∂B, and we use the fact that u solves A-C strongly on each R_i. An approximation argument lets us take $v \in H_0^1$ above. By elliptic regularity, u is thus smooth on $B - \{0\}$.

Remark. *This doesn't work over the whole ball because we don't know $u \in C^1$ at 0, so we can't immediately show u solves A-C on the whole ball. If we instead used even reflections to construct u, then u would not be C^1 (jump discontinuity of the derivative at axis), so we couldn't integrate by parts.*

To show this, for $0 < r < 1$ define

$$\zeta_r(x) := \begin{cases} 0 & |x| \le r^2 \\ 2 - \frac{\log|x|}{\log r} & r^2 < |x| < r \\ 1 & |x| > r \end{cases}. \tag{146}$$

Then $0 \le \zeta_r \le 1$ and ζ_r is supported away from the origin and converges pointwise to 1 on $B \setminus \{0\}$ as $r \to 0$. Then for any $v \in C_c^\infty(B)$, $\zeta_r v \in C_c^\infty(B - \{0\})$, so

$$0 = \int Du \cdot D(\zeta_r v) + W'(u)\zeta_r v = \int \zeta_r Du \cdot Dv + vDu \cdot D\zeta_r + W'(u)\zeta_r v. \tag{147}$$

Then

$$|\zeta_r Du \cdot D\varphi| \le \|Du\|_{L^2}^2 + \|Dv\|_{L^2}^2 < \infty \tag{148}$$

and because $|u| < 1$,

$$|W'(u)\zeta_r v| \le C + \|v\|_{L^2}^2 < \infty, \tag{149}$$

and the right sides are in L^1 because the domain is finite. On the other hand,

$$\left| \int vDu \cdot D\zeta_r \right| \le \|v\|_{L^\infty} \|Du\|_{L^2} \|D\zeta_r\|_{L^2} \tag{150}$$

by Holder's inequality, and

$$(\zeta_r)_{x_i} = -\frac{x_i}{|x|^2 \log r}, \tag{151}$$

so

$$\int_{B\setminus\{0\}} |D\zeta_r|^2 = \int_{r^2<|x|<r} \frac{1}{|x|^2 |\log r|^2} \le \frac{C}{|\log r|^2} \int_{r^2}^r \rho^{-1} \, \mathrm{d}\rho \le \frac{C}{|\log r|} \tag{152}$$

which goes to 0 as $r \to 0$. Thus we may pass to the limit by the dominated convergence theorem to obtain

$$\int Du \cdot Dv + W'(u)v = 0 \tag{153}$$

in the entire ball. Thus u solves Allen-Cahn on the whole ball, so it is smooth. Applying the energy estimate to each quadrant shows that $E[u] \le CR$ as well.

Part D

Remark. *This maximum principle argument is a lower-tech proof for 5.6 (in dimension 2 at least) than Schauder estimates. Adjusting the function h should probably let this proof work for any dimension. Same argument works any family of solutions to A-C defined on sufficiently large domains.*

The purpose of the set K is to avoid the non-smoothness of u on open sets containing the boundary (like half-squares centred at a boundary point), because the maximum principle needs $u \in C^2(\overline{U})$. For some nice domains, this compact set K stuff is not really necessary: the only purpose of K is to ensure that the domain of the solution contains a half-square centred at each point in the domain of u. But really what we need is that each point in the domain is contained in some half-square centred at some other point in the domain, and this is true for some nice domains (like squares haha), like all of $\mathbb{R}^n$. Also the size of the half-square being 1 is not really necessary; adjusting h could probably let you choose arbitrary sizes.

Lemma. *If u_R are the solutions above, then $\left|D^k u_R\right| \leq C(k, K)$ on compact sets K.*

Proof. First we prove a pointwise estimate on Du, assuming only that $|u|, |\Delta u| \leq C$ and u is defined well outside (like distance 2) K. For R large enough, $B(0, R)$ contains the half-square S centred at each point of K, so this is OK. Without loss of generality, we suppose $0 \in K$. Take $C \geq 1$ if needed. Let $S = \{|x| < 1, 0 < y < 1\}$ be a half-square and define the functions

$$g(x,y) = \frac{u(x,y) - u(x,-y)}{2} \qquad h(x,y) = C(x^2 + \frac{5}{2}y - \frac{3}{2}y^2) \tag{154}$$

We have $|\Delta g| \leq |\Delta u| \leq 1$ and $\Delta h = -C$, so $\Delta(h \pm g) \leq 0$. Also, $0 = g \leq h$ on ∂S when $y = 0$ and $h \geq C$ on the other three sides of ∂S (where $y \neq 0$), so by the maximum principle $\min_{\overline{S}}(h \pm g) = \min_{\partial S}(h \pm g) \geq 0$. We conclude that $|g| \leq h$ in $\overline{S}$. Now

$$\begin{aligned} \frac{g(0,y)}{y} &= \frac{1}{2} \lim_{y \to 0} \frac{u(0,y) - u(0,-y)}{y} \\ &= \frac{1}{2} \lim_{y \to 0} \left[\frac{u(0,y) - u(0,0)}{y} + \frac{-u(0,0) + u(0,-y)}{-y} \right] \\ &= D_y u(0), \end{aligned} \tag{155}$$

so

$$|u_y(0)| = \lim_{y \to 0} \left| \frac{g(0,y)}{y} \right| \leq \lim_{y \to 0} \frac{h(0,y)}{y} \leq C \tag{156}$$

Similarly one shows $|D_x u(0)| \leq C$. Because the bounds on $u, \Delta u$ are translation invariant and $B(0, R)$ contains the half-squares centred at points in K for R large, this argument shows $|Du_R| < C$ everywhere in K for R large.

Now we induct. Let C denote a constant depending on k. Everything below is done in K. Suppose that for each $|\alpha| \leq k$, we have 1. $|D^\alpha u| \leq C$, 2. $\Delta D^\alpha u$ is a finite sum of products of $W^{(|\beta|)}(u)$ and $D^\beta u$ for $|\beta| \leq k$, Let $|\beta| = k + 1$ with $D^\beta u = D_{x_i} D^\alpha u$. Because W and its derivatives are bounded in $[-1, 1]$, (a) and (b) together with the base case $k = 0$ applied $D^\alpha u$ give $\left|D^\beta u\right| \leq C$. Moreover, $\Delta D^\beta u = D_{x_i} \Delta D^\alpha u$ is a finite sum of products $W^{(|\gamma|)}(u)$ and $D^\gamma u$ for $|\gamma| \leq k + 1$. By induction $\left|D^k u\right| \leq C$ for all k. ☐

We now diagonalize to obtain a subsequential limit function $u_R \to u$. By the above, all derivatives of u_R are bounded uniformly in R. Consider a compact domain, to apply Arzela-Ascoli. By Arzela-Ascoli, find a sequence $\{n_{k,0}\} \in \mathbb{N}$ such that $u_{n_{k,0}}$ has a uniform limit u. Refine to a subsequence $\{n_{k,1}\}$ such that $Du_{n_{k,1}}$ converges uniformly. In general, if all derivatives up to order m of $u_{n_{k,m}}$ converge uniformly, then refine to a subsequence $\{n_{k,m+1}\}$ so that $D^{m+1} u_{n_{k,m+1}}$ converge uniformly. Then all derivatives of $u_{n_{k,k}}$ converge uniformly, and thus in fact to the corresponding derivatives of u. Thus u is smooth, and passing to a pointwise limit in $\Delta u_R = W'(u_R)$ shows that u is a smooth solution to Allen-Cahn on $\mathbb{R}^n$. For the rest of the problem, we can re-index u_R so that $u_R \to u$ uniformly in C^∞_{loc}.

Part E

Now we show $\{u = 0\} = \{xy = 0\}$. Because of the symmetry of u, it suffices to show $u \neq 0$ in the interior of the first quadrant. Let $B = B(x_0, r)$ be a ball compactly contained in the first quadrant. As seen in the example constructed in 5.4, a minimizer on B has energy at most Cr, while $E(0, B) = Cr^2$, so taking r large enough and recalling the maximum principle argument made in Part A (which applies because u has constant sign in a quadrant), we conclude that if u were a minimizer on balls, it would be nonzero in the interior of the first quadrant. We are done if we show u is a minimizer on such balls.

For R large enough, Ω_R compactly contains B. Then if $w = u$ on ∂B, the function v that is u on $\Omega_R - B$ and w on B is in $H_0^1(\Omega_R)$, so by Part A, $E(v, \Omega_R) \geq E(u_R, \Omega_R)$, and $v = u_R$ on $\Omega_R - B$, so $E(v, B) \geq E(u_R, B)$. Now we show this property passes to the limit.

Suppose u does not minimize energy on B. Then, as argued in Part A, there exists a minimizer $w \in H^1(B)$ with $w = u$ on ∂B and $E(w, B) \leq E(u, B) - \delta$ for some $\delta > 0$. Moreover $|w| \leq 1$. Define φ_R the log-cutoff function

$$\varphi_R(x) = \begin{cases} 1 & x \in B(x_0, r - \frac{1}{R}) \\ 2 - \frac{\log(r - |x - x_0|)}{\log R} & B(x_0, r - \frac{1}{R^2}) - B(x_0, r - \frac{1}{R}) \\ 0 & x \in B - B(x_0, r - \frac{1}{R^2}) \end{cases} . \tag{157}$$

We now claim

$$E((1 - \varphi_R)u_R + \varphi_R w, \Omega_R) = E(\chi_{\Omega_R - B} u_R + \chi_B w, \Omega_R) + o(1) \tag{158}$$

as $R \to \infty$. Note that $\chi_{\Omega_R - B} u + \chi_B w \in H^1(\Omega_R)$ because $u = w$ on ∂B. First we estimate the derivatives:

$$\begin{aligned} &\|D((1 - \varphi_R)u_R + \varphi_R w)\|_{L^2(\Omega_R)} - \|D(\chi_{\Omega_R - B} u_R + \chi_B w)\|_{L^2(\Omega_R)} \\ &\leq \|(u_R - w) D\varphi_R\|_{L^2(\Omega_R)} + \|(\chi_B - \varphi_R) Du_R\|_{L^2(\Omega_R)} + \|(\chi_B - \varphi_R) Dw\|_{L^2(\Omega_R)} . \end{aligned} \tag{159}$$

For the second term, the integrand is bounded by $2|Du_R| \leq C$ on B and it is 0 outside of B. The third integrand is bounded by $2|Dw| \in L^2$ on B and 0 outside of B. By the dominated convergence theorem ($\varphi_R \to \chi_B$ a.e.), they both go to 0. For the first term,

$$\begin{aligned} \int_{\Omega_R} |u_R - w|^2 |D\varphi_R|^2 &\leq C \int_{B(x_0, r - \frac{1}{R^2}) - B(x_0, r - \frac{1}{R})} \frac{1}{|x - x_0| (r - |x - x_0|) |\log R|^2} \,\mathrm{d}x \\ &\leq \frac{C}{|\log R|^2} \int_{r - \frac{1}{R}}^{r - \frac{1}{R^2}} \frac{\mathrm{d}\rho}{r - \rho} \\ &= \frac{C}{|\log R|} \to 0. \end{aligned} \tag{160}$$

For the potential term,

$$\begin{aligned} &\int_{\Omega_R} |W((1 - \varphi_R)u_R + \varphi_R w) - W(\chi_{\Omega_R - B} u_R + \chi_B w)| \\ &= \int_B |W((1 - \varphi_R)u_R + \varphi_R w) - W(\chi_{\Omega_R - B} u_R + \chi_B w)| , \end{aligned} \tag{161}$$

and the integrand is bounded by $2W(0)$ because $|u|, |w| \leq 1$. The dominated convergence theorem on the finite domain B and the pointwise convergence of both terms in the integrand to $W(\chi_{\Omega_R - B} u + \chi_B w)$ shows that the difference in potential terms is $o(1)$.

Now we derive a contradiction. Starting from the minimizing property of u_R on Ω_R and applying the above,

$$\begin{aligned} E(u_R, \Omega_R) &\leq E((1 - \varphi_R)u_R, \varphi_R w, \Omega_R) \\ &= E(\chi_{\Omega_R - B} u_R + \chi_B w, \Omega_R) + o(1) \\ &= E(u_R, \Omega_R - B) + E(w, B) + o(1) \\ &= E(u_R, \Omega_R - B) + E(u, B) - \delta + o(1) \\ &= E(u_R, \Omega_R - B) + E(u_R, B) - \delta + o(1) \\ &= E(u_R, \Omega_R) - \delta + o(1), \end{aligned} \tag{162}$$

which gives $\delta \leq o(1)$, a contradiction. Notice that we used $E(u_R, B) = E(u, B)$ (because u_R and its derivatives converge uniformly to those of u on B). Thus u vanishes only on $\{xy = 0\}$.

Otis Chapter 5

Problem 5.4

Note that it suffices to assume u is a minimizer on balls. Pick smooth functions $0 \leq \varphi_1, \varphi_2 \leq 1$ with $|D\varphi_1|, |D\varphi_2| \leq C\epsilon^{-1}$ such that φ_1 is 1 on ∂B_R and 0 on $B_{R-\epsilon}$, and φ_2 is 1 on $B_{R-2\epsilon}$ and supported in $B_{R-\epsilon}$ (see 2.5b for construction). Then $w := u\varphi_1 + \varphi_2$ agrees with u on ∂B_R, so it is admissible in the minimization problem on B_R. Then, noting that $|B_R - B_{R-\epsilon}| = C(R^n - (R-\epsilon)^n) \leq CR^{n-1}\epsilon$ (with the constant depending only on n) and recalling from what was proved in 2.5d that $|Du| \leq C$, we compute

$$\begin{aligned}
E[w] &= \int_{B_R} \frac{1}{2} |Dw|^2 + W(w)\,\mathrm{d}x \\
&\leq C \int_{B_R} |\varphi_1 Du + uD\varphi_1 + D\varphi_2|^2 + W(w)\,\mathrm{d}x \\
&\leq C \int_{B_R - B_{R-\epsilon}} |Du|^2 + |D\varphi_1|^2 + |D\varphi_2|^2\,\mathrm{d}x + \int_{B_R - B_{R-2\epsilon}} W(0)\,\mathrm{d}x \qquad (163) \\
&\leq CR^{n-1}\epsilon + C\int_{B_R - B_{R-\epsilon}} \epsilon^{-2} \\
&\leq \frac{CR^{n-1}}{\epsilon} + CR^{n-1}\epsilon.
\end{aligned}$$

For $R > 1$, take $\epsilon = \frac{1}{2}$ to get $E[w] \leq CR^{n-1}$. For $R < 1$, let $\epsilon = \frac{R^2}{4}$ to get $E[w] \leq CR^n \leq CR^{n-1}$.

Exercise 5.5

- (a)

  Proof. If $\nabla u = 0$ everywhere, then we have u is constant on $\mathbb{R}^n$, which we know only has $0, 1, -1$ as solutions. □

- (b)

  Proof. Using the hint of Exercise 4.2, we have

  $$2|\nabla u|\nabla|\nabla u| = \nabla\left(|\nabla u|^2\right) = 2\sum_{i,j=1}^{n} u_{x_i x_j} u_{x_i} = 2D^2u(\nabla u, \cdot) = 2D^2u\nabla u$$

  where D^2u is the Hessian matrix of u with

  $$(D^2u)_{ij} = u_{x_i x_j}.$$

  Thus, taking the norm and square both sides, we get

  $$4|\nabla u|^2|\nabla|\nabla u||^2 = 4|D^2u\nabla u|^2 \leq 4|D^2u|^2|\nabla u|^2.$$

  Then we cancel $4|\nabla u|^2$ on both side since it's nonzero and get that

  $$|\nabla|\nabla u||^2 \leq |D^2u|^2 \implies |D^2u|^2 - |\nabla|\nabla u||^2 \geq 0.$$

  □

- (c)

Proof. It suffices to show that

$$\left| D(\frac{\nabla u}{|\nabla u|}) \right|^2 = 0.$$

Now we compute.

$$\begin{aligned}
\left| D(\frac{\nabla u}{|\nabla u|}) \right|^2 &= \sum_{i,j=1}^n D(\frac{\nabla u}{|\nabla u|})_{ij}^2 \\
&= \sum_{i,j=1}^n \frac{\left[u_{x_i x_j}|\nabla u| - u_{x_i}|\nabla u|_{x_j}\right]^2}{|\nabla u|^2} \\
&= \frac{1}{|\nabla u|^2} \sum_{i,j=1}^n u_{x_i x_j}^2 |\nabla u|^2 + u_{x_i}^2 (|\nabla u|_{x_j})^2 - 2u_{x_i x_j} u_{x_i} |\nabla u| |\nabla u|_{x_j} \\
&= \frac{1}{|\nabla u|^2} (|D^2 u||\nabla u|^2 + |\nabla u|^2 |\nabla |\nabla u||^2 - \frac{1}{2} |\nabla |\nabla u|^2|^2) \\
&= \frac{1}{|\nabla u|^2} (2|\nabla u|^2 |\nabla |\nabla u||^2 - \frac{1}{2} |2|\nabla u| \nabla |\nabla u||^2) = 0.
\end{aligned}$$

We used the fact that $|\nabla |\nabla u||^2 = |D^2 u|^2$ and chain rules. ☐

Exercise 5.6

Fix u smooth solving Allen-Cahn on $\mathbb{R}^n$ with $|u| < 1$. Then $\Delta u = f$ with $f = W' \circ u$. Fix $x_0 \in \mathbb{R}^n$ and let $B_1 = B(x_0, R)$ and $B_2 = B(x_0, 2R)$. Throughout let C denote a constant depending on n, α, and any extra given parameters. By a first estimate (GT 4.45),

$$|u|_{1,B_1} \le |u|_{1,\alpha,B_1} \le C(\operatorname{diam} B_1)\, |u|'_{1,\alpha,B_1} \le C(R)(|u|_{0,B_2} + |f|_{0,B_2}) \le C(R), \tag{164}$$

where we recall that $|\cdot|'_{k,\alpha,\Omega}$ is equivalent to $|\cdot|_{k,\alpha,\Omega}$, with the proportionality constant depending only on k and $\operatorname{diam}\Omega$. Thus

$$\begin{aligned}
|f|_{0,\alpha,B_2} &= |f|_{0,B_2} + [f]_{0,\alpha,B_2} \\
&\le |f|_{0,B_2} + |Df|_{0,B_2} \\
&\le |W'(u)|_{0,B_2} + |W'' \circ u|_{0,B_2}\, |u|_{1,B_2} \\
&\le C(R).
\end{aligned} \tag{165}$$

Then Schauder estimates (GT 6.1a) say

$$|u|_{k+2,B_1} \le |u|_{k+2,\alpha,B_1} \le C(k,R)(|u|_{0,B_2} + |f|_{k,\alpha,B_2}). \tag{166}$$

With $k = 0$, this is

$$|u|_{2,B_1} \le C(R) \tag{167}$$

by the above. More generally, suppose $|u|_{j,B_1} \le C(k,R)$. for all $j \le k+1$. Expanding out $D^j f$ with the product rule, the above calculation gives

$$|f|_{k,\alpha,B_2} \le |f|_{k,B_2} + |Df|_{k,B_2} \le C(k,R)(1 + |u|_{k+1,B_2}) \le C(k,R), \tag{168}$$

with the k-dependence in the constant coming from derivatives of W and $|u|_{j,B_2}$ for $j \le k+1$. Then by induction the Schauder estimate gives

$$|u|_{k+2,B_1} \le C(k,R) \tag{169}$$

for all k. Now fix R and take a supremum over x_0 to get

$$|u|_{k,\mathbb{R}^n} \le C(k) \tag{170}$$

for all k.

Exercise 5.8

A compact set in $\mathbb{R}^2$ is a compact set in $\mathbb{R}^3$, and thus $u^{\pm\infty}(x_1, x_2)$ is stable. Hence

$$u^{\pm\infty}(x_1, x_2) = \mathbb{H}(a_1 x_1 + a_2 x_2 - b).$$

Since the energy $E_1(\cdot, B_R)$ is radially symmetric, it suffices to let $a_1 = 1$, $a_2 = 0$. We can compute that

$$\frac{1}{2}|Du^{\pm\infty}|^2 = \frac{1}{4}\operatorname{sech}^4\left(\frac{x_1 - b}{\sqrt{2}}\right)$$

and

$$W(u^{\pm\infty}) = \frac{1}{4}\operatorname{sech}^4\left(\frac{x_1 - b}{\sqrt{2}}\right).$$

Therefore,

$$\begin{aligned}
E_1(u^{\pm\infty}, B_R) &= \int_{B_R} \frac{1}{2}|Du^{\pm\infty}|^2 + W(u^{\pm\infty}) \\
&= \frac{1}{2}\int_{B_R} \operatorname{sech}^4\left(\frac{x_1 - b}{\sqrt{2}}\right) \\
&\leq \frac{\sqrt{2}}{6}\int_{-R}^{R}\int_{-R}^{R}\int_{-R}^{R} \operatorname{sech}^4\left(\frac{x_1 - b}{\sqrt{2}}\right) dx_1 dx_2 dx_3 \\
&\leq \frac{\sqrt{2}}{6}\int_{-R}^{R}\int_{-R}^{R} \left[\tanh\left(\frac{x - b}{\sqrt{2}}\right)\left(2 + \operatorname{sech}^2\left(\frac{x - b}{\sqrt{2}}\right)\right)\right]_{-R}^{R} dx_2 dx_3 \\
&\leq \frac{\sqrt{2}}{6}\int_{-R}^{R}\int_{-R}^{R} 6 \ dx_2 dx_3 \\
&= \sqrt{2}R^2
\end{aligned}$$

Next,

$$E_1(u^t, B_R) = \int_{B_R} \frac{1}{2}|Du^t|^2 + \int W(u^t).$$

By dominated convergence theorem (the dominating function being 1),

$$\lim_{t\to\infty}\int_{B_R} W(u^t) = \int W(u^\infty).$$

Moreover,

$$|Du^t(x_1, x_2, x_3)|^2 = \sum_{i=1}^{3} u_{x_i}^2(x_1, x_2, x_3 + t)$$

But the derivatives are uniformly (in t) bounded, so by dominated convergence theorem

$$\lim_{t\to\infty}\int_{B_R} \frac{1}{2}|Du^t|^2 = \int_{B_R} \frac{1}{2}|Du^\infty|^2.$$