Bifurcation Methods for Asset Market Equilibrium Analysis

Kenneth L. Judd Sy-Ming Guu

Hoover Institution Yuan-Ze University

Asset Market Analysis

- Asset markets allocate risks
- Asset markets are incomplete
- Precise analysis of asset markets is difficult
- New assets only partially complete market
- Hart example: adding an asset may reduce welfare
- Elul (1995), Cass and Citanna (1998):
  - "Any local change in utility is possible if the number of missing assets exceeds the number of agents."
  - Assumes small change in asset market.
  - Does not show what ordinary assets do to welfare.
  - Assumes we count number of agents and missing assets.
  - Welfare depends on magnitude of risks, not number of states.
- Computation
  - Brown-de Marzo-Eaves (1996)
  - Schmedders (1998)

Questions

- What are consequence of adding arbitrary securities?
- How do new assets affect prices of existing securities?
- Which kind of assets should be added?
- Can we compute equilibrium approximations quickly?

Our Model

- Use simple single-period model
- Use Taylor-style expansions to solve for equilibrium of economies with small risk.
  - Disadvantage: results are proven valid only for cases with small risk
  - Advantage: focus on moments, estimable statistics
  - Advantage: more reliable than alternative approaches which are never valid.
  - Conjecture: validity is much greater than the proofs indicate
  - Disadvantage: complex algebraic manipulations
  - Solution: Use computer algebra software
- Style: "Jones (1965) meets Samuelson (1970)"
- Results
  - Completely solves equilibrium for a collection of economies
  - Derives impact on welfare of new assets
  - Derives optimal new asset

Alternative Approaches

- Samuelson (1970) took a polynomial approximation approach, replacing utility function with polynomial approximation, and solve nonlinear equations involving utility function derivatives and return moments.
- Magill (1977): derived linear approximations for dynamic programming problems with small additive shocks
- Kydland-Prescott, McGrattan, and King-Plosser-Rebelo offer *ad hoc* approaches
  - Idea: replace objective and/or first-order conditions and budget constraint with linear (or loglinear) approximations to arrive at an analytically tractable set of equations.
  - Magill (1977) shows general invalidity of this approach
  - This approach is at best a certainty equivalent and first-order
  - Tesar (1995) uses KP-KPR approach and shows that first welfare theorem is false(?!)
  - Kim-Kim (1999) demonstrates general invalidity of KP-KPR approach; offers a new *ad hoc* approach, but not one based on implicit function theory
- Similar Campbell (1993) procedure is also *ad hoc*, not based on IFT or extensions

Mathematics and Mathematical Economics Literature

- Fleming (1971) and Fleming-Souganides (1986) examine similar problems.
- See Judd (1996) for several citations.
- Bifurcation and related IFTs produce theorems as well as high-quality numerical approximations.

Demand with Two Assets

- Simple problem which displays general mathematical difficulty
- Two assets:
  - Safe asset: pay \$1 today, receive R tomorrow
  - Risky asset: pay \$1 today, receive $Z = R + \epsilon z + \epsilon^2 \pi$ tomorrow
  - All proceeds consumed tomorrow
- First-order condition for θ:

$$\begin{aligned} 0 &= E\{u'(R + \theta(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon \pi)\} \\ &\equiv G(\theta, \epsilon) \end{aligned}$$

- Demand function:

$$G(\theta(\epsilon), \epsilon) = 0 \tag{1}$$

- Want to find expansion

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \frac{1}{2}\theta''(0)\epsilon^2 + ... \tag{2}$$

- Question: What is θ_0?
- Problem: θ is indeterminate at $\epsilon = 0$!

Bifurcation Approach

- Problem: $G(\theta, 0) = 0$ for all θ.
  - Demand at $\epsilon = 0$ is not well-defined.
  - However, it is defined at all $\epsilon \neq 0$
- B and E are bifurcation points

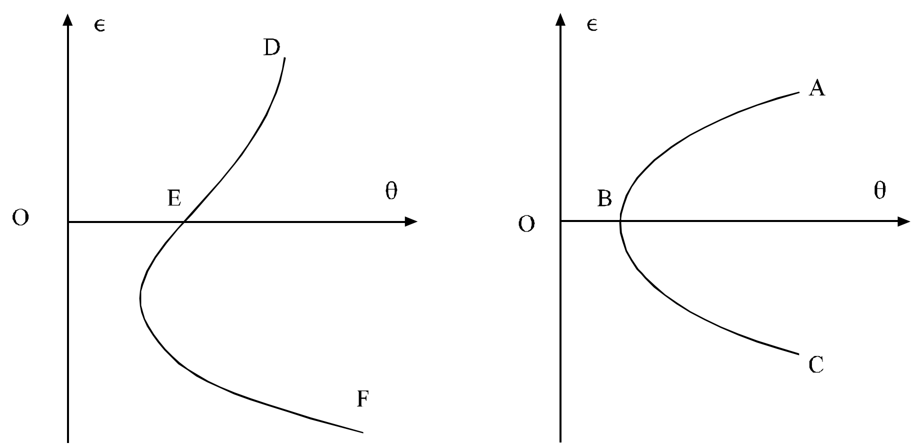

Figure 1: Bifurcation possibilities for asset demand problem

- Want to find expansion

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \frac{1}{2}\theta''(0)\epsilon^2 + ... \tag{3}$$

- Problem 1: Need to find

$$\theta_0 \equiv \lim_{\epsilon \to 0} \theta(\epsilon) \tag{4}$$

- Problem 2: Implicit differentiation of $G(\theta(\epsilon), \epsilon) = 0$ with respect to ϵ implies

$$\begin{aligned} 0 &= G_\theta(\theta(\epsilon), \epsilon)\theta' + G_\epsilon(\theta(\epsilon), \epsilon) \\ \Rightarrow & \ \theta'(0) = -\frac{G_\epsilon(\theta_0, 0)}{G_\theta(\theta_0, 0)} \end{aligned}$$

  but $G_\theta(\theta_0, 0) = 0$ because $G(\theta, 0) = 0$ for all θ!

- Solution to both problems:
  - $\theta'(0)$ is well-defined only if $G_\epsilon(\theta(0), 0) = 0$.
  - Hence, $G_\epsilon(\theta_0, 0) = 0$ must hold and fixes θ_0.
  - Given θ_0, L'Hospital's rule implies

$$\begin{aligned} \theta'(0) &= -\frac{G_\epsilon(\theta_0, 0)}{G_\theta(\theta_0, 0)} \\ &= -\frac{G_{\epsilon\epsilon}(\theta_0, 0)}{G_{\theta\epsilon}(\theta_0, 0)} \end{aligned}$$

    which is well-defined if $G_{\theta\epsilon}(\theta_0, 0) \neq 0$.

Theorem 1 *(Bifurcation Theorem for R) Suppose $G : R \times R \to R$, G is analytic and $G(\theta, 0) = 0$ for all $\theta \in R$. Furthermore, suppose that for some $(\theta_0, 0)$*

$$G_\theta(\theta_0, 0) = 0 = G_\epsilon(\theta_0, 0), \;\; G_{\theta\epsilon}(\theta_0, 0) \neq 0. \tag{5}$$

Then there is an open neighborhood $\mathcal{N}$ of $(\theta_0, 0)$ and a function $\theta(\epsilon)$, $\theta(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that θ is analytic

$$G(\theta(\epsilon), \epsilon) = 0 \text{ for } (\theta(\epsilon), \epsilon) \in \mathcal{N}. \tag{6}$$

and $(\theta_0, 0)$ is a bifurcation point.

Proof. Apply IFT to

$$F(x, \epsilon) = \begin{cases} \frac{G(x,\epsilon)}{\epsilon}, & \epsilon \neq 0 \\ \frac{\partial G(x,0)}{\partial \epsilon}, & \epsilon = 0 \end{cases} . \tag{7}$$

"Division-by-zero trick." ■

Remark 2 *We used L'Hospital's rule in our intuition above. The proof of the Bifurcation Theorem actually proves L'Hospital's rule.*

Bifurcation Method

- Procedure: To find the bifurcation point θ_0 solve

$$G_\epsilon(\theta_0, 0) = 0 \tag{8}$$

and check that

$$G_{\theta\epsilon}(\theta_0, 0) \neq 0 \tag{9}$$

- In asset demand problem

$$\begin{aligned} G_\epsilon(\theta, 0) &= E\left\{u''(R)\theta z^2\right\} + u'(R)\pi = 0 \\ \Longrightarrow & \; \theta_0 = -\frac{u'(R)}{u''(R)}\frac{\pi}{\sigma_z^2} = \tau\,\frac{\pi}{\sigma_z^2} \end{aligned}$$

- Simple asymptotic portfolio rule: θ_0 is product of
  - risk tolerance (a utility parameter)

$$\tau = -\frac{u'(R)}{u''(R)} \tag{10}$$

  - and the price of risk (a statistic of risk)

$$\frac{\pi}{\sigma_z^2} \tag{11}$$

Linear Approximation

- Calculate $\theta'(0)$ via implicit differentiation:

$$\begin{aligned} 0 &= G_{\theta\theta}\,\theta'\theta' + 2G_{\theta\epsilon}\,\theta' + G_{\theta}\theta'' + G_{\epsilon\epsilon} \\ &= 0 \,\cdot\, \theta'\theta' + 2G_{\theta\epsilon}\,\theta' + 0 \,\cdot\, \theta'' + G_{\epsilon\epsilon} \\ \Longrightarrow&\ \theta'(0) = -G_{\theta\epsilon}^{-1} G_{\epsilon\epsilon} \end{aligned}$$

- Solvability again depends on $G_{\theta\epsilon}(\theta_0, 0) \neq 0$
- In asset case $G_{\theta\epsilon}(\theta_0, 0) \neq 0$ and

$$\theta'(0) = -\ \frac{1}{2}\,\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{\sigma_z^2}\ \theta_0^2 \tag{12}$$

- Relative form (a.k.a. $\widehat{\theta}$) is

$$\begin{aligned} \frac{\theta'(0)}{\theta_0} &= \rho\,\frac{\pi}{\sigma_z^2}\,\frac{E\{z^3\}}{\sigma_z^2} \\ \rho &= \frac{1}{2}\frac{u'}{u''}\frac{u'''}{u''}(R) \end{aligned}$$

  - Depends on third-order properties of utility; we call ρ skew tolerance
  - Depends on third moment of return

- Linear approximation is

$$\begin{aligned} \theta(\epsilon) &\doteq \theta_0 + \theta'(0)\,\epsilon \\ &\doteq -\ \frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2} - \frac{1}{2}\,\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{\sigma_z^2}\,\left(\frac{u'(R)}{u''(R)}\,\frac{\pi}{\sigma_z^2}\right)^2\,\epsilon \end{aligned}$$

Higher-order Approximations

- We compute higher-order approximations via implicit differentiation
  - We solve a succession of linear equations, just as in Implicit Function Theorem.
  - Solvability at each stage follows from $G_{\theta\epsilon}(\theta_0, 0) \neq 0$
  - Necessary assumptions:
    * Differentiability of u
    * Finite moments for z
- Second-order approximation is

$$\frac{\theta''(0)}{\theta_0} = \left((6\rho - 2)\, E\{z^2\} \;+\; 4\rho^2 \frac{E\{z^3\}^2}{E\{z^2\}^2} \;+\; \kappa \frac{E\{z^4\}}{E\{z^2\}} \right) \left(\frac{\pi}{\sigma_z^2} \right)^2 \tag{13}$$

where kurtosis tolerance at c is

$$\kappa = -\frac{1}{3} \frac{u'}{u''} \frac{u'}{u''} \frac{u''''}{u''} \tag{14}$$

Multidimensional Bifurcation Theorem

Theorem 3 *(Bifurcation Theorem for R^n) Suppose $H : R^n \times R \to R^n$ is analytic, and $H(x, 0) = 0$ for all $x \in R^n$. Furthermore, suppose that for some $(x_0, 0)$*

$$H_x(x_0, 0) = 0_{n\times n} \tag{15}$$

$$H_\epsilon(x_0, 0) = 0_n \tag{16}$$

$$\det(H_{x\epsilon}(x_0, 0)) \neq 0 \tag{17}$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon) : R \to R^n$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that

$$H(h(\epsilon), \epsilon) = 0 \text{ for } (h(\epsilon), \epsilon) \in \mathcal{N} \tag{18}$$

Furthermore, h is analytic and can be approximated by a Taylor series. In particular, the first-order derivatives equal

$$h'(0) = -\frac{1}{2} H_{x\epsilon}^{-1} H_{\epsilon\epsilon}. \tag{19}$$

Note: There does exist a general theorem for singular $H_x(x_0, 0)$ but we don't need it.

Asset Market Equilibrium with Small Risks

- Safe asset yields R per dollar invested
- Examine a continuum of economies
- Risky asset in ϵ-economy yields $R(1+\epsilon z)$ per share
- Price for risky asset in ϵ-economy is

$$p(\epsilon) = 1 + \epsilon E\{z\} + \epsilon^2 \pi(\epsilon). \tag{20}$$

- Type i endowment: a_i dollars and θ_i^e shares
- Final wealth and consumption is

$$Y_i = \theta_i R(1+\epsilon z) + B_i R. \tag{21}$$

- Market clearing condition:

$$\theta_1 + \theta_2 = 1. \tag{22}$$

- First order condition for θ_i:

$$E\{u_i^{'}(Y_i)(\epsilon z + \epsilon^2 \pi)\} = 0. \tag{23}$$

- Equilibrium is (let $\theta_1 = \theta$, $\theta_2 = 1-\theta$)

$$0 = E\{u_i^{'}(Y_i)(z + \epsilon\pi\},\ i = 1,2 \tag{24}$$

- For each ϵ we have two equations and two unknowns, θ and π.
- At $\epsilon = 0$ all (θ, π) values are equilibria.

- Bifurcation structure applies
  - At $\epsilon = 0$ asset prices are fixed at 1, but π and θ are indeterminate.
  - At $\epsilon \neq 0$ asset prices and allocation are determinate.

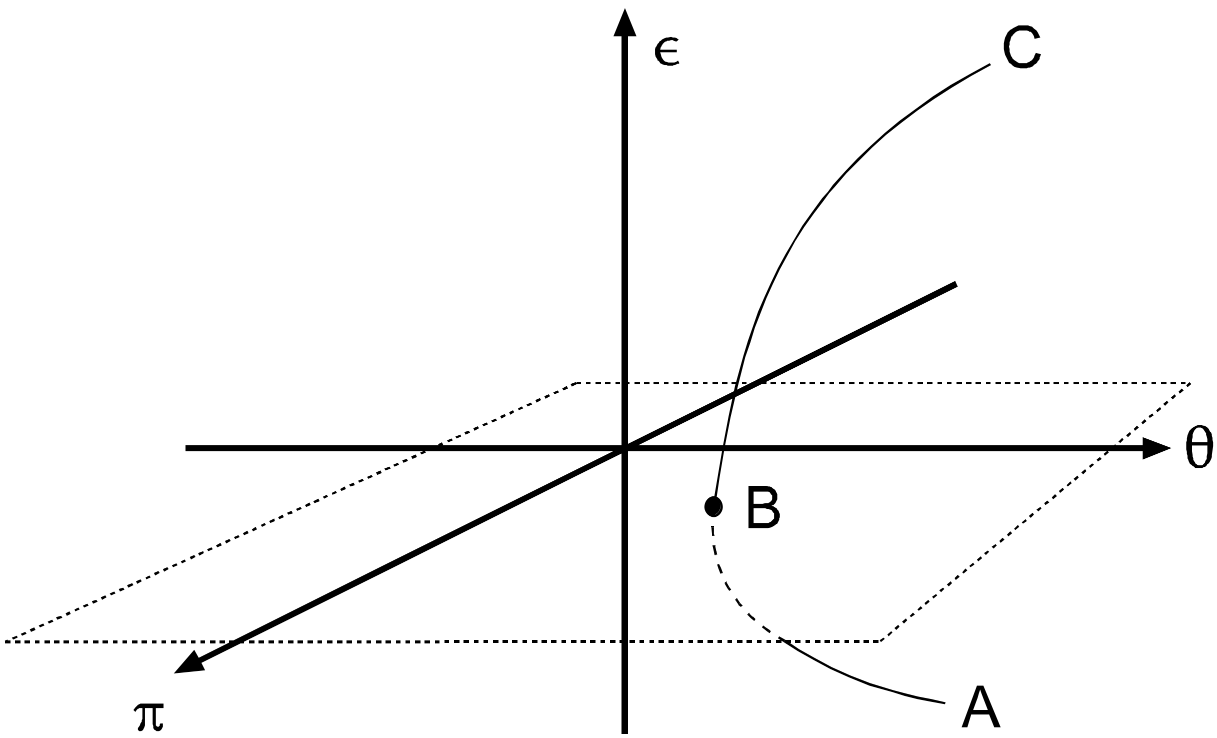

Figure 2:

General Formulas

- Direct computation (assume $E\{z\} = 0$) yields bifurcation point

$$\begin{aligned}
\theta_0 &= \frac{u_1^{'}/u_1^{''}}{u_2^{'}/u_2^{''} + u_1^{'}/u_1^{''}} = \frac{\tau_1}{\tau_1 + \tau_2} \\
\pi_0 &= -\frac{RE\left\{z^2\right\}}{\tau_1 + \tau_2}
\end{aligned}$$

- Direct computation yields linear approximation for small ϵ :

$$\begin{aligned}
\theta'(0) &= R\,\frac{\tau_1}{\tau_1 + \tau_2}\,\frac{\tau_2}{\tau_1 + \tau_2}\,\frac{\rho_1 - \rho_2}{\tau_1 + \tau_2}\,\frac{E\{z^3\}}{\sigma_z^2} \\
\pi'(0) &= \frac{2R^2}{(\tau_1 + \tau_2)^2}\left(\frac{\tau_1}{\tau_1 + \tau_2}\rho_1 + \frac{\tau_2}{\tau_1 + \tau_2}\rho_2\right)\frac{E\{z^3\}}{\sigma_z^2}
\end{aligned}$$

  - Change in θ governed by
    - differences in risk aversion and skew tolerance
    - skewness variance ratio
  - Risk premium $(-\pi)$ decreases with positive skewness and u''', $\rho > 0$

- Direct computation produces arbitrary order approximation as long as u has sufficient derivatives and moments are finite.

- Examples indicate that low-order expansions are quite good approximations for sensible cases: efficient computation.

Comparing Market Structures

- Expected utility, prices, trade, etc. are all functions of ϵ
- For each asset configuration m, we can compute expected utility as a function of ϵ

$$U_{i,m}(\epsilon) = U_{i,m}(0) + U'_{i,m}(0)\epsilon + \frac{1}{2}U''_{i,m}(0)\epsilon^2 + ... \tag{25}$$

- To compare markets m and n, we compute the difference in expansions

$$\Delta_{i,m,n}(\epsilon) = U_{i,m}(\epsilon) - U_{i,n}(\epsilon) \tag{26}$$

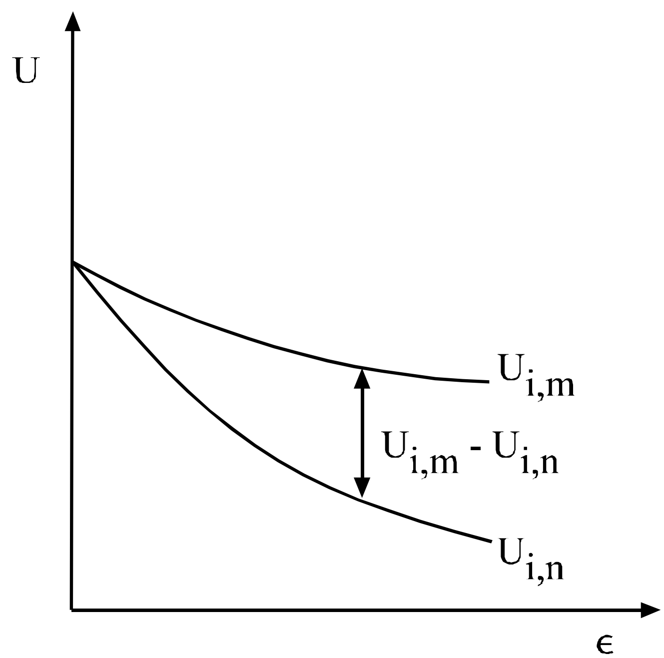

Figure 3: Difference in Asset Market Structures

- Can compute similar expressions for prices, trading volume, etc.

Adding a Derivative Asset

- Suppose we add a new asset, $y = f(z)$; agent 1 buys ϕ units.
- WLOG, $y = \overline{y} + \alpha z + \nu$ where $0 = E\{\nu\} = E\{z\nu\}$

Trading Patterns for the New Asset

- $\phi(\epsilon)$ is trader 1 purchases
- Bifurcation calculation shows $\phi(0) = 0$.
- If $\phi'(0) > 0$, then trader 1 buys and trader 2 sells the new asset.
- Asymptotic results revolve around two indices:
  - Difference in skew tolerance, $\rho_1 - \rho_2$,(utility parameters)
  - Covariance between z^2 and ν, $Cov(\nu, z^2)$

Theorem 4 *Type 1 agents buy y if ν is positively correlated with the tails of z and type 1 skew tolerance exceeds type 2 skew tolerance:*

$$\phi'(0) = R\,\frac{\tau_1\tau_2\,(\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3}\,Cov(\nu, z^2) \tag{27}$$

$$\phi'(0) > 0 \textit{ iff } (\rho_1 - \rho_2)\; Cov(\nu, z^2) > 0 \tag{28}$$

Asset Holding Effects of the New Asset

Theorem 5 *Let $\theta_z^b(\epsilon)$ and $\theta_z^a(\epsilon)$ denote type 1 investors' holding of old asset before and after the new asset being introduced. Then*

$$\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -R\frac{\tau_1\tau_2\,(\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3}\alpha Cov(\nu, z^2)\frac{\epsilon^2}{2} + O(\epsilon^3). \tag{29}$$

Note that

$$\theta_z^a(\epsilon) - \theta_z^b(\epsilon) = -\,\alpha\,\phi'(0) \tag{30}$$

- If $\alpha > 0$, change in equity holdings is opposite of purchases of new asset.
- If $\alpha = 0$, no change in equity holdings to order $O(\epsilon^2)$.

Price Effects of the New Asset

Theorem 6 *Let $P_z^b(\epsilon)$ and $P_z^a(\epsilon)$ denote the equilibrium price of the first risky asset before and after the new asset being introduced.*

$$P_z^a(\epsilon) - P_z^b(\epsilon) = 8R^3 \frac{\tau_1 \tau_2 \left(\rho_1 - \rho_2\right)^2}{\left(\tau_1 + \tau_2\right)^5} \frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}} \frac{\epsilon^4}{2} + O(\epsilon^5) > 0 \tag{31}$$

In particular, the initial risky asset rises in value and rises more as the new asset is more correlated to the tails of the original asset.

- Robust result: *any* nontrivial new asset will increase the price of the old asset.
- Price change depends on third-order properties of the utility function.
- Price change depends on $E\left\{\nu z^2\right\}$
- No third- (or first- or second-) order impact if $E\left\{\nu z^2\right\} = 0$.

Welfare Effects of the New Asset

Theorem 7 *The asymptotic welfare change from adding the asset y for trader 1, measured by the unit-free consumption equivalent, equals*

$$\begin{aligned}\frac{U_1^a(\epsilon)-U_1^b(\epsilon)}{u_1^{'}} &= \frac{R^4\tau_1^2\tau_2^2\left(\rho_1-\rho_2\right)^2}{2\left(\tau_1+\tau_2\right)^5} \\ &\times\left(5\left(\frac{\theta_1^e}{\tau_1}-\frac{\theta_2^e}{\tau_2}\right)+\theta_2^e\right)\frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}}\epsilon^4+O(\epsilon^5)\end{aligned}$$

where θ_i^e is i's endowment of stock. Similarly for the second trader's welfare change, $[U_2^a(\epsilon)-U_2^b(\epsilon)]/u_2^{'}$.

- Welfare effect on individuals is ambiguous
  - Pareto-improving if tastes and endowment are not too dissimilar
    - No difference in skew-tolerance, i.e., $\rho_1-\rho_2=0$, then no welfare change to $O(\epsilon^4)$.
    - Volume (and type 1's purchases) is proportional to

$$\frac{\theta_2^e}{\tau_2}-\frac{\theta_1^e}{\tau_1} \tag{32}$$

  - Loss is possible
    - If type 1 buys much equity then he loses from price increase.
    - Someone must gain – single good case.
  - Skew-tolerance difference magnifies welfare change from other factors.
  - Co-skewness of new asset with old, $E\left\{\nu z^2\right\}$, is in dominant term

- Asymptotic unanimity:
  - If some new asset makes all better off, then any new asset does.
  - If agent i prefers y' to y, then so does $j \neq i$.
- Results depend solely on utility function properties and moments of asset returns, not on number of states.
- Social welfare, measured in wealth terms, changes by the positive quantity

$$\begin{aligned} \Delta SW &= \frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1'} + \frac{U_2^a(\epsilon) - U_2^b(\epsilon)}{u_2'} \\ &= \frac{R^4 \tau_1^2 \tau_2^2 \left(\rho_1 - \rho_2\right)^2}{2\left(\tau_1 + \tau_2\right)^5} \frac{E\left\{\nu z^2\right\}^2}{E\left\{z^2\right\}} \epsilon^4 + O(\epsilon^5) \end{aligned} \tag{33}$$

- Computational notes
  - All derivations were done using Mathematica
  - Intermediate terms in utility expansion numbered over 10,000.
  - If we began with specific utility functions and return distributions, then computational effort would be much smaller, competitive with general methods.

Optimal New Asset: General Approach

- Few results based on special cases, often assuming
  - *Assumed* traded assets were linear combinations of factors, or
  - *Assumed* a new asset is a linear combination of factors
- We make no assumption about returns or utility, and compute asymptotically optimal new asset
- Social Welfare
  - For new asset y and ϵ, type i equilibrium utility is $U_i(y, \epsilon)$.
  - Any social welfare function is, for some weight $\alpha > 0$,

    $$W(y, \epsilon) = \alpha U_1(y, \epsilon) + (1 - \alpha) U_2(y, \epsilon) \tag{34}$$

  - The optimal new asset problem is

    $$\max_y W(y, \epsilon) \tag{35}$$

- Let $Y(\epsilon)$ be the optimal new asset for the ϵ economy.
  - The function $Y(\epsilon)$ is defined by the solution to the first-order condition

    $$0 = W_y(Y(\epsilon), \epsilon) \tag{36}$$

  - At $\epsilon = 0$ all new assets are redundant, implying $0 = W_{yy}(y, 0)$
- Apply bifurcation theorem to find $Y(0)$. Either
  - We want a new asset and $W_{yy\epsilon}(Y(\epsilon), \epsilon)$ is negative definite, or
  - It is optimal to introduce no new asset.

Optimal New Asset

- Find $y = f(z)$ which maximizes

$$\left(E\left\{yz^2\right\}\right)^2 = Cov\left(y, z^2\right)^2 \tag{37}$$

  which is sufficient since the other terms in the change in utility are unrelated to y.

- Normalizations we assumed above
  - Scale y so that $\sigma_y^2 = 1$.
  - Uncorrelated to bonds: $E\{y\} = 0$
  - Uncorrelated to stocks: $E\{yz\} = 0$.

- Optimal asset solves a (degenerate) calculus of variations problem.:

$$\begin{array}{ll} \max_f & \left(\int f(z) z^2 d\mu(z)\right)^2 \\ s.t. & \int f(z) d\mu(z) = 0 \\ & \int z f(z) d\mu(z) = 0 \\ & \int f(z)^2 d\mu(z) = 1, \end{array} \tag{38}$$

- Lagrangian function

$$\mathcal{L} = \left(\int f(z)z^2 d\mu\right)^2 + \lambda_1 \int f(z) d\mu + \lambda_2 \int z f(z) d\mu + \lambda_3 \left(\int f(z)^2 d\mu - 1\right) \tag{39}$$

- Maximize (39) at each z, implying

$$2z^2 \left(\int f(z)z^2 d\mu\right) + \lambda_1 + \lambda_2 z + 2\lambda_3 f(z) = 0. \tag{40}$$

- (40) implies

$$f^*(z) = -\frac{2z^2 \left(\int f(z)z^2 d\mu\right) + \lambda_1 + \lambda_2 z}{2\lambda_3} \tag{41}$$

- $f^*(z)$ is a parabola in z
- All parabolas are equivalent

Theorem 8 *As riskiness goes to zero, any optimal new asset is asymptotically equivalent up to order ϵ^4 to adding $y = z^2$.*

- Optimal new asset (or, collection thereof) is independent of tastes: Asymptotic unanimity
- Optimal second option to introduce is z^3

Theorem 9 *As riskiness goes to zero, any pair of optimal new assets is asymptotically equivalent up to order ϵ^5 to adding $y = z^2$ and $y = z^3$.*

- Conjecture: "Complete the moments to complete the market"

Theorem 10 *In two-type model, one optimal asset achieves the Pareto frontier. As riskiness goes to zero,*

1. *up to order ϵ^4, it is asymptotically equivalent to adding $y = z^2$,*
2. *up to order ϵ^5, it is asymptotically equivalent to adding $y = z^2$ and $y = z^3$,*
3. *up to order ϵ^5, it is asymptotically equivalent to adding some linear combination of $y = z^2$ and $y = z^3$, and*
4. *these equivalences are independent of tastes and returns.*

Other Options

- Call options are not asymptotically efficient
- Optimal options: Example
  - $z \sim U[-1, 1]$
  - Optimal strike price is zero: expected price
- Optimal bundles
  - Assume several assets, z_i, $i = 1, .., n$, i.i.d.
  - one option: $f(z)$
  - Optimal new asset is square of market value of bundle: $\left(\sum_i z_i\right)^2$

Extension: Multiple agents and Multiple goods

- Endowments: $e_{i,\ell} = e^0_{i,\ell} + \epsilon x_{i,\ell}$ units of good ℓ for agent i.
- Assets:
  - price of asset j is $q_j = q^0_j + \epsilon^2 \pi_j(\epsilon)$ of numeraire good
  - return is $Z_{j,\ell} = Z^0_{j,\ell} + \epsilon z_{j,\ell}$ units of good ℓ
- Agent i is endowed with $b_{i,j}$ units of asset j and trades for $\theta_{i,j}$ units.
- Δ is price space for goods, and (π, θ) is space of risk premia and portfolios

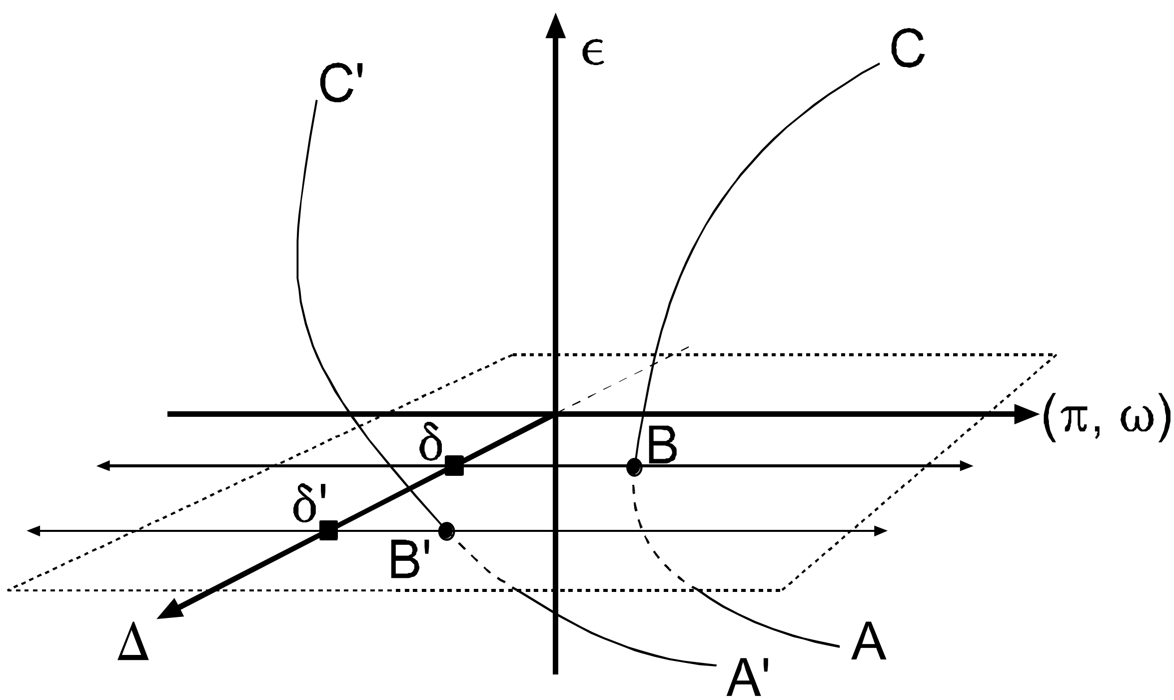

Figure 4: Bifurcation in general model

Other Applications of IFT and Bifurcation Methods

- Arbitrary riskiness, similar utility
- Dynamic trading in options not spanned by stocks and bonds (Leisen-Judd)
- Asymmetric information, close to symmetric information
- Multiple periods in a recursive economy

Conclusions

- Bifurcation methods are natural extensions of IFT for asset market problems
- Bifurcation approach avoids *ad hoc* assumptions usually used for sake of tractability
- Bifurcation approach produces asymptotically valid results, avoiding *ad hoc* approximation schemes.
- Small noise results can be extended via computations in non-small cases
- Results depend on natural conditions on statistics and tastes
- Bifurcation methods have substantial computational value