Numerical Methods in Economics

MIT Press, 1998

Chapter 10 Notes
Finite-Difference Methods

Kenneth L. Judd
Hoover Institution

October 28, 2002

Classification of Ordinary Differential Equations

- A *first-order ordinary differential equation* (ODE) has the form

$$\frac{dy}{dx} = f(y, x), \tag{10.1.1}$$

  where $f : R^{n+1} \rightarrow R^n$ and the unknown is $y(x) : [a, b] \subset R \rightarrow R^n$.

  – When $n = 1$, we have a single differential equation

  – If $n > 1$, (10.1.1) is a system of differential equations.

- Need boundary conditions to fix unknown function $y(x)$.

  – *Initial value problem* (IVP): impose $y(x_0) = y_0$ for some $x_0 \in [a, b]$

  – *Two-point boundary value problem:* impose n conditions

$$g_i(y(a)) = 0, \quad i = 1, \cdots, n', \tag{10.1.2}$$

$$g_i(y(b)) = 0, \quad i = n' + 1, \cdots, n,$$

  where $g : R^n \rightarrow R^n$.

  – General BVP: impose

$$g_i(y(x_i)) = 0 \tag{10.1.3}$$

  for a set of points, x_i, $a \leq x_i \leq b$, $\quad 1 \leq i \leq n$.

- All problems have form (10.1.2). For example, replace

$$\frac{d^2y}{dx^2} = g(\frac{dy}{dx}, y, x)$$

  with

$$\frac{dy}{dx} = z, \quad \frac{dz}{dx} = g(z, y, x),$$

- IVP and BVP definitions apply to discrete-time systems.

Finite-Difference Methods for IVPs

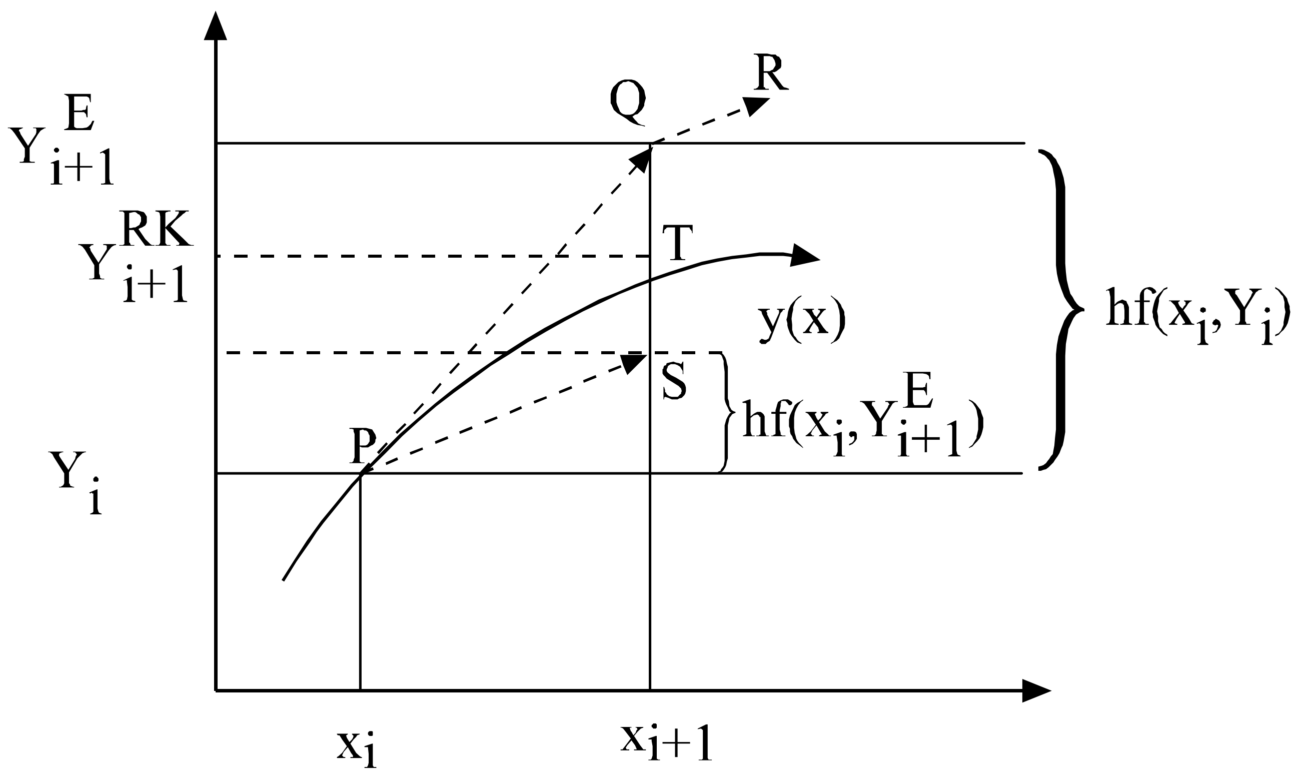
- Consider the IVP
$$y' = f(x, y), \quad y(x_0) = y_0. \tag{10.3.1}$$
- Specify a grid for x, $x_i = x_0 + ih$, $i = 0, 1, \cdots, N$
- Objective: find Y_i which approximates $y(x_i)$.
- Construct a difference equation on the grid
  - An explicit scheme $Y_{i+1} = F(Y_i, Y_{i-1}, \cdots, x_{i+1}, x_i, \cdots)$,
  - or an implicit scheme $Y_{i+1} = F(Y_{i+1}, Y_i, Y_{i-1}, \cdots, x_{i+1}, x_i, \cdots)$
- Y_0 is fixed by the initial condition, $Y_0 = y(x_0) = y_0$.
- Solve difference scheme, and hope that $Y_i \doteq y(x_i)$
- Find scheme using as few grid points as possible

Euler Method

- Algorithm:

$$Y_{i+1} = Y_i + hf(x_i, Y_i) \qquad (10.3.2)$$
$$Y_0 = y(x_0) = y_0$$

- Geometry of Euler's method
  - Current iterate is $P = (x_i, Y_i)$
  - $y(x)$ is the true solution
  - At $P, y'(x_i)$ is the tangent vector $\vec{PQ}$.
  - Euler's method follows $\vec{PQ}$ until $x = x_{i+1}$ at Q.
  - Euler estimate of $y(x_{i+1})$ is Y^E_{i+1}.



- Convergence Theorem:

Theorem 1 *Suppose that the solution to $y'(x) = f(x, y(x))$, $y(0) = y_0$, is C^3 on $[a, b]$, that f is C^2, and that f_y and f_{yy} are bounded for all y and $a \leq x \leq b$. Then the error of the Euler scheme with step size h is $\mathcal{O}(h)$; that is, it can be expressed as*

$$y(x_i) - Y_i = D(x_i)h + \mathcal{O}(h^2)$$

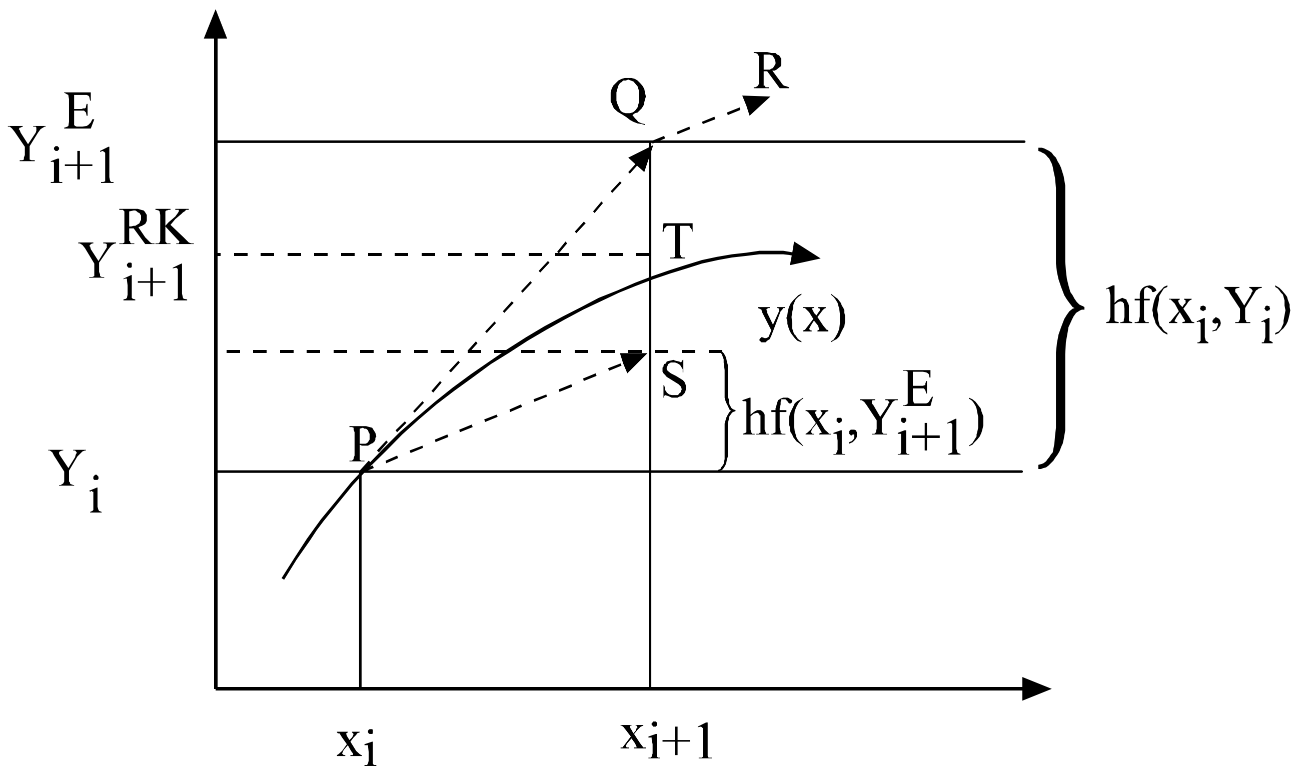
where $D(x)$ is bounded on $[a, b]$ and solves the differential equation

$$D'(x) = f_y(x, y(x))\, D(x) + \frac{1}{2}y''(x), \quad D(x_0) = 0$$

Runge-Kutta Methods

- First-order Runge-Kutta (RK1)
  - Euler estimate of $y(x_{i+1})$ is Y_{i+1}^E.
  - Slope of vector field at (x_i, Y_{i+1}^E) is $f(x_i, Y_{i+1}^E)$, not $f(x_i, Y_i^E)$ as assumed by Euler
  - Slope at (x_i, Y_{i+1}^E) says Y_{i+1} should be $Y_i + h\ f(x_i, Y_{i+1}^E)$, point S in figure 10.1
  - RK1 takes average of these two approximations:

$$Y_{i+1} = Y_i + \frac{h}{2}\left[f(x_i, Y_i) + f(x_{i+1}, Y_i + hf(x_i, Y_i))\right] \qquad (10.3.9)$$



- RK1 converges at rate h^2
- RK1 evaluates f twice per step

- First-order Runge-Kutta (RK1)

$$\begin{aligned} z_1 &= f(x_i, Y_i), \\ z_2 &= f(x_i + \tfrac{1}{2}h,\ Y_i + \tfrac{1}{2}hz_1) \\ z_3 &= f(x_i + \tfrac{1}{2}h,\ Y_i + \tfrac{1}{2}hz_2) \\ z_4 &= f(x_i + h,\ Y_i + hz_3) \\ Y_{i+1} &= Y_i + \tfrac{h}{6}[z_1 + 2z_2 + 2z_3 + z_4] \end{aligned} \tag{10.3.10}$$

- RK4 converges at rate h^4
- RK4 evaluates f four times per step

Systems of Differential Equations

$$\begin{aligned} y_1'(x) &= f_1(x, y_1, y_2, \cdots y_n) \\ &\vdots \\ y_n'(x) &= f_n(x, y_1, y_2, \cdots, y_n) \end{aligned} \tag{10.3.11}$$

- Euler:

$$Y_\ell^{i+1} = Y_\ell^i + h f_\ell\,(x_i, Y_1^i, \cdots, Y_n^i),\ \ \ell = 1, \cdots, n \tag{10.3.12}$$

- RK1:

$$Y^{i+1} = Y^i + \frac{h}{2}\ \left[f(x_i, Y^i) + f(x_{i+1}, Y^i + h f(x_i, Y^i))\right] \tag{10.3.14}$$

- RK4:

$$\begin{aligned} z^1 &= f(x_i, Y^i) \\ z^2 &= f(x_i + \tfrac{1}{2}h,\ Y^i + \tfrac{1}{2}hz^1) \\ z^3 &= f(x_i + \tfrac{1}{2}h,\ Y^i + \tfrac{1}{2}hz^2) \\ z^4 &= f(x_i + h,\ Y^i + hz^3) \\ Y^{i+1} &= Y^i + \tfrac{h}{6}[z^1 + 2z^2 + 2z^3 + z^4] \end{aligned} \tag{10.3.15}$$

Spence Signaling Equilibrium

- Education signalling model implies the nonlinear equation

$$N'(y) = \frac{N(y)^{-1} - \alpha N(y) y^{\alpha - 1}}{y^{\alpha}} \tag{10.4.3}$$

with initial condition

$$N(y_m) = n_m \tag{10.4.4}$$

and closed-form solution

$$N(y) = y^{-\alpha} \frac{(2(y^{1+\alpha} + D)}{1 + \alpha)^{1/2}}, \; D = \frac{1+\alpha}{2} \left(\frac{n_m}{y_m^{-\alpha}} \right)^2 - y_m^{1+\alpha} \tag{10.4.5}$$

- Numerical results:

Table 10.1: Signalling Model Errors

	Euler			RK1			RK4		
h:	.01	.001	.0001	.01	.001	.0001	.1	.01	.001
$y - y_m$									
0.1	3(-2)	1(-3)	1(-4)	1(-3)	1(-4)	1(-6)	3(-2)	3(-3)	2(-6)
0.2	2(-2)	1(-3)	1(-4)	1(-3)	5(-4)	1(-6)	2(-2)	2(-3)	1(-6)
0.4	1(-2)	7(-4)	7(-5)	4(-4)	3(-4)	4(-7)	1(-2)	1(-3)	6(-7)
1.0	6(-3)	4(-4)	4(-5)	2(-4)	1(-4)	2(-7)	4(-3)	4(-4)	3(-7)
2.0	4(-3)	3(-4)	3(-5)	1(-4)	7(-5)	1(-7)	2(-3)	2(-4)	1(-7)
10.0	1(-3)	1(-4)	1(-5)	2(-5)	2(-6)	0(-7)	6(-4)	6(-5)	0(-7)
time:	.11	1.15	9.17	.16	1.49	14.4	.02	.27	2.91

Boundary Value Problems for ODEs: Shooting

- Consider the BVP

$$\begin{aligned} \dot{x} &= f(x, y, t), \\ \dot{y} &= g(x, y, t), \\ & x(0) = x^0, \quad y(T) = y^T \\ & x \in R^n, \ y \in R^m \end{aligned} \tag{10.5.1}$$

- For any guess $y(0) = y^0$, solve IVP in (10.5.2)

$$\begin{aligned} \dot{x} &= f(x, y, t) \\ \dot{y} &= g(x, y, t) \\ x(0) &= x^0, \quad y(0) = y^0 \end{aligned} \tag{10.5.2}$$

  – Let $Y(T, y^0)$ denote the resulting value of $y(T)$
  – $Y(T, y^0)$ depends on y^0
  – Find correct value of y^0 by solving the nonlinear equation $Y(T, y^0) = y^T$

- Programming strategy

  – write procedure which computes $Y(T, y^0) - y^T$ given input y^0.
  – send that routine to a nonlinear equation solver to solve $Y(T, y^0) - y^T = 0$.

Life-Cycle Model of Consumption and Labor Supply

- Simple life-cycle model:

$$\begin{aligned} \max_c \quad & \int_0^T e^{-\rho t} u(c)\, dt, \\ \dot{A} \quad & = f(A) + w(t) - c(t) \\ A(0) \quad & = A(T) = 0 \end{aligned} \tag{10.6.10}$$

  - $u(c)$ is concave utility function over consumption c
  - $w(t)$ is wage rate at t
  - $A(t)$ is assets at time t
  - $f(A)$ is return on invested assets.

- Hamiltonian is $H = u(c) + \lambda(f(A) + w(t) - c)$.
  - Costate equation: $\dot{\lambda} = \rho\lambda - \lambda f'(A)$.
  - First-order condition: $0 = u'(c) - \lambda$, implying consumption function $c = C(\lambda)$.
  - The final system:

$$\begin{aligned} \dot{A} & = f(A) + w - C(\lambda) \\ \dot{\lambda} & = \lambda(\rho - f'(A)) \end{aligned} \tag{10.6.11}$$

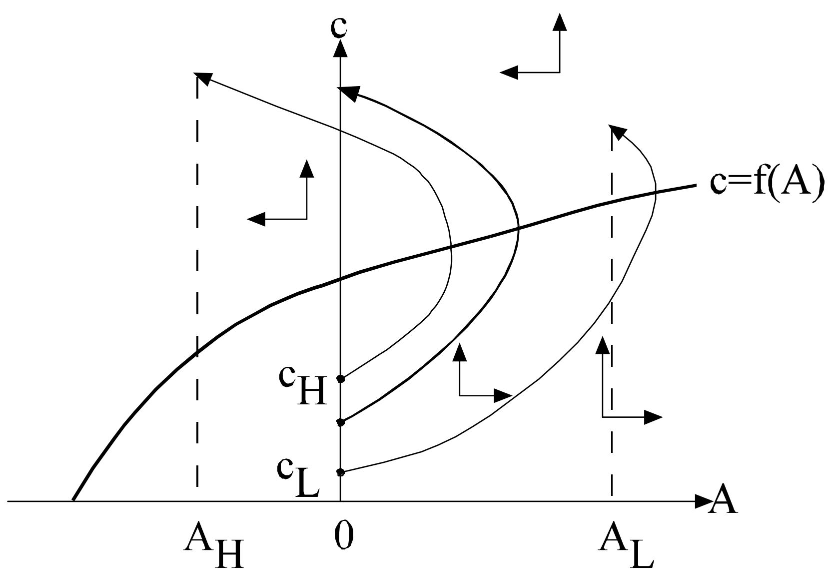
    with the boundary conditions

$$A(0) = A(T) = 0 \tag{10.6.12}$$

- Convert to a system for observable variables.
  - $u'(c) = \lambda$ implies that (10.6.11) can be replaced by

$$\begin{aligned} \dot{c} &= -\frac{u'(c)}{u''(c)}(f'(A) - \rho) \\ \dot{A} &= f(A) + w - c \end{aligned} \tag{10.6.13}$$

  - Phase diagram:



- Shooting: Consider implications of different $c(0)$.
  - If $A(T) < 0$ if $c(0) = c_H$, but $A(T) > 0$ if $c(0) = c_L$, then correct $c(0)$ lies between c_L and c_H.
  - Find true $c(0)$ by using the bisection method presented in algorithm 5.1.
  - In general, if $A(T; c_0)$ is terminal wealth for initial consumption c_0, then find c_0 by solving nonlinear equation $A(T; c_0) = 0$.

Optimal Growth

- Optimal control problem

$$\begin{array}{ll} \max_c & \int_0^\infty e^{-\rho t}\, u(c)\, dt \\ \text{s.t.} & \dot{k} = f(k) - c \\ & k(0) = k_0 \end{array} \tag{10.7.2}$$

  - k is the capital stock
  - c consumption
  - $f(k)$ the aggregate net production function

- $c(t)$ and $k(t)$ satisfy

$$\begin{array}{ll} \dot{c} & = \frac{u'(c)}{u''(c)}\ (\rho - f'(k)) \\ \dot{k} & = f(k) - c \end{array} \tag{10.7.3}$$

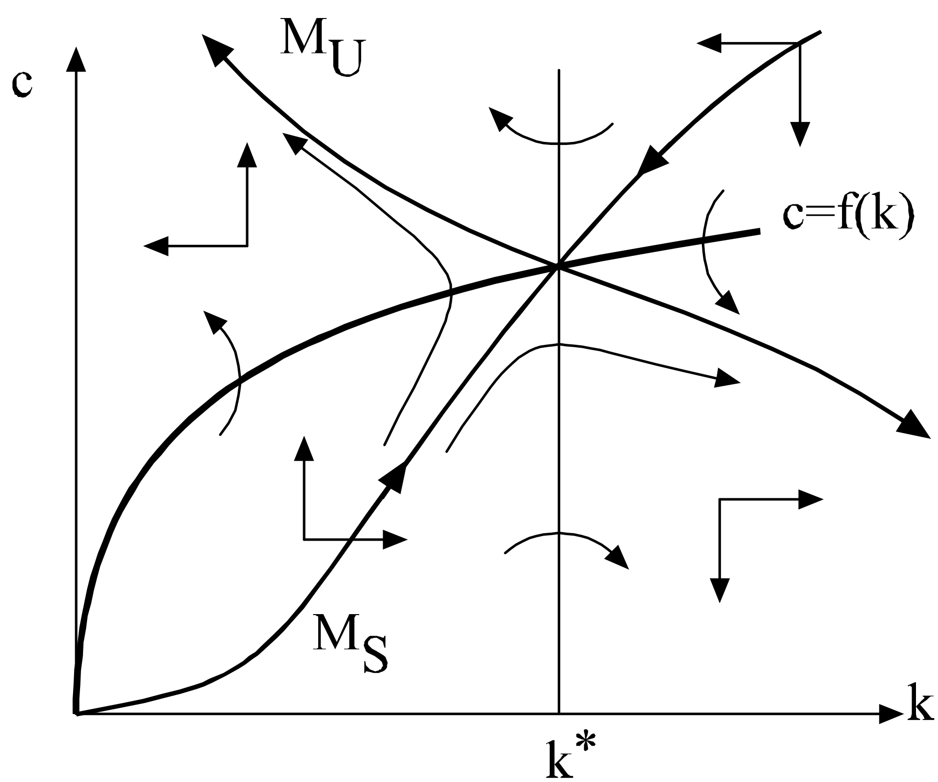
  and boundary conditions are

$$k(0) = k_0,\ 0 < \lim_{t\to\infty} \mid k(t) \mid < \infty$$

- Forward system:
  - We want to compute M_S, but shooting is numerically unstable: even if one starts on M_S, roundoff error pushes solution away from M_S
  - Computing M_U would be easy since it attracts deviations
  - If we computed

$$\begin{aligned} \dot{c} &= \frac{u'(c)}{u''(c)} \ (\rho - f'(k)) \\ \dot{k} &= f(k) - c \end{aligned} \tag{10.7.3}$$

    with initial condition near steady state, we would move along one of the branches of M_U



- Reversed system:
  - M_U in this system is numerically stable and is our consumption function
  - Find M_U by solving

$$\begin{aligned}\dot{c} &= \tfrac{u'(c)}{u''(c)}\ (\rho - f'(k)) \\ \dot{k} &= f(k) - c\end{aligned}$$

    with initial conditions close to steady state.

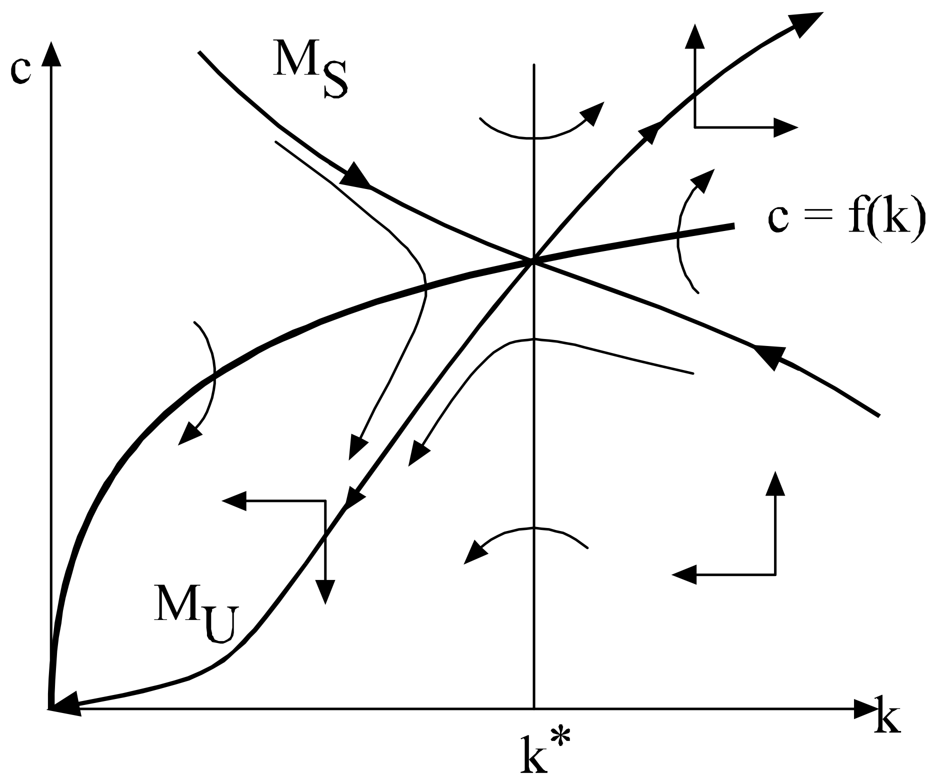


Table 10.2: Optimal Growth with Reverse Shooting

k	c	Errors		
		$h = .1$	$h = .01$	$h = .001$
0.2	0.10272	0.00034	3.1(-8)	3.1(-12)
0.5	0.1478	0.000025	3.5(-9)	4.1(-13)
0.9	0.19069	-0.001	-3.5(-8)	1.8(-13)
1.	0.2	0.	0.	0.
1.1	0.20893	-0.00086	-5.3(-8)	-1.2(-12)
1.5	0.24179	-0.000034	-1.8(-9)	-2.1(-14)
2.	0.2784	-9.8(-6)	-5.(-10)	3.6(-15)
2.5	0.31178	-5.(-6)	-2.6(-10)	-1.3(-14)
2.8	0.33068	-3.8(-6)	-1.9(-10)	-1.2(-14)

- Consumption function $C(k)$ satisfies the differential equation

$$C'(k) = \frac{\dot{c}}{\dot{k}} = \frac{u'(C(k))}{u''(C(k))} \frac{\rho - f'(k)}{f(k) - C(k)} \tag{10.7.5}$$

  which also implies

$$C'(k)(f(k) - C(k)) - \frac{u'(C(k))}{u''(C(k))}(\rho - f'(k)) = 0. \tag{10.7.6}$$

- We first compute $C'(k)$ at k^*.
  - At the steady state, (10.7.5) reduces to $0/0$.
  - L'Hôpital's rule gives a quadratic equation in $C'(k^*)$.

$$C'(k^*) = -\frac{f''(k^*))}{f'(k^*) - C'(k^*)} \frac{u'(C(k^*)}{u''(C(k^*))},$$

  - Since $C'(k) > 0$

$$C'(k^*) = \frac{f'(k^*)}{2} \left(1 + \sqrt{1 + 4\frac{u'(c^*)}{u''(c^*)} \frac{f''(k^*)}{f'(k^*)f'(k^*)}}\right). \tag{10.7.7}$$

- With reverse shooting, $C(k)$ can be calculated by solving two IVPs.
  - For $k > k^*$ we begin with $k = k^* + h$, using the approximate initial condition

$$C(k^* + h) \doteq c^* + hC'(k^*),$$

    and apply an IVP method to (10.7.5) to compute $C(k)$ for $k = k^* + nh$, $n = 2, 3, 4, ..$
  - For $k < k^*$ the initial condition is

$$C(k^* - h) \doteq c^* - hC'(k^*),$$

    and apply an IVP method to (10.7.5) to compute $C(k)$ for $k = k^* - nh$, $n = 2, 3, 4, ..$

- Use data to express $C(k)$ in a simple form.
  - Points $\{(c_i, k_i) \mid i = -N, \cdots, N\}$ from reverse shooting are near graph of $c = C(k)$.
  - Use the least squares approach

$$\min_{\beta} \sum_{i=-N}^{N} \left(c_i - \sum_{j=0}^{m} \beta_j \varphi_j(k_i) \right)^2$$

    to approximate $C(k)$ with $\widehat{C}(k) \equiv \Sigma_{j=0}^{m} \beta_j \varphi_j(k)$ for some basis functions $\varphi_j(k)$.

Integral Equations

- The general integral equation can be written

$$0 = \varphi\left(g(x),\ f(x),\ \int_a^b H(x, z, f(z))\, dz\right), \tag{10.8.1}$$

where $f(x)$ is the unknown function, and $g(x)$ and $H(x, z, y)$ are known functions.

- A *linear Fredholm equation of the first kind* has the form

$$f(x) = \int_a^b K(x, z)\, f(z)\, dz. \tag{10.8.2}$$

where $K(x, z)$ is called the *kernel* of the integral equation.

- A *linear Fredholm equation of the second kind* has the form

$$f(x) = g(x) + \int_a^b K(x, z)\, f(z)\, dz. \tag{10.8.3}$$

where $f(a)$ is the initial condition

- We need to approximate $\int_a^b K(x, z)\, f(z)\, dz$ in (10.8.2) and (10.8.3)
  - For each x this is an integral over z with approximation

$$\int_a^b K(x, z)\, f(z)\, dz \doteq \sum_{j=1}^{n} \frac{\omega_j K(x, x_j)\, f(x_j)}{w(x_j)}. \tag{10.8.6}$$

  - At the $x = x_i$ (10.8.2) implies n linear equations for the $f(x_i)$:

$$f(x_i) = g(x_i) + \sum_{j=1}^{n} \frac{\omega_j\, K(x_i, x_j)\, f(x_j)}{w(x_j)},\quad i = 1, \cdots, n, \tag{10.8.7}$$

  - Quadrature formula reduces estimating the f to estimating f at the quadrature nodes.

- Integral equation methods differ in their choice of the weights and nodes.
  - Set $w(x) = 1$, $x_i = a + (i - 1)(b - a)/(n - 1)$, and choose the ω_i according to a Newton-Cotes formula

$$f(x_i) = g(x_i) + \sum_{j=1}^{n} \omega_j\, K(x_i, x_j)\, f(x_j),\ \ i = 1, \cdots, n. \qquad (10.8.8)$$

  - An alternative is the Gauss-Legendre set of nodes and weights, which also assumes $w(x) = 1$
  - Sometimes an efficient choice will suggest itself. For example, if $K(x, z) = H(x, z)e^{-z^2}$, $-a = b = \infty$, and $H(x, z)$ is polynomial-like in z, then a Gauss-Hermite approximation of the integral is suggested.

- (10.8.9) produces values for $f(x)$ at the x_i, $i = 1, \cdots, n$. To get a function for all $x \in [a, b]$, form $\hat{f}(x)$

$$\hat{f}(x) = g(x) + \sum_{j=1}^{n} \frac{\omega_j K(x, x_j) f(x_j)}{w(x_j)}. \qquad (10.8.10)$$

called the *Nystrom extension.*

Markov Chains: Approximations and Ergodic Distributions

- Consider a Markov process

$$\Pr\{x_{t+1} \leq x | x_t\} = F(x|x_t)$$

where F has a density $f(x|x_t)$.

- We want the ergodic distribution

$$H(x) = \lim_{T\to\infty} P\{x_T \leq x | x_0\}.$$

- If $H(x)$ has a density $h(x)$, then $h(x)$ satisfies the integral equation

$$h(x) = \int h(z)\, f(x|z)\, dz. \tag{10.8.11}$$

- Since $H(\cdot)$ is a distribution, $h(\cdot)$ also must satisfy $\int h(z)\, dz = 1$.

- To solve (10.8.11), pick a weighting function $w(x)$, appropriate quadrature rule with weights ω_i and nodes x_i, and form the linear system of equations

$$h(x_i) = \int h(z)\, f(x|z)\, dz \doteq \sum_{j=1}^{n} \frac{\omega_j h(x_j)\, f(x_i|x_j)}{w(x_j)}. \tag{10.8.12}$$

- (10.8.11) is linear and homogeneous in the unknown $h(x_i)$ values.

- To eliminate indeterminacy, replace one of the equations with probability condition

$$\sum_{j=1}^{n} \frac{\omega_j\, h(x_j)}{w(x_j)} = 1, \tag{10.8.13}$$

- (10.8.12) has a suggestive interpretation (Tauchen and Hussey (1991))
  - (10.8.12) examines only the value of h on the x_i grid.
  - Define

$$\begin{aligned}\pi(x_k|x) &= \frac{f(x_k|x)}{s(x)w(x_k)}\omega_k \\ s(x) &= \sum_{i=1}^{n} \frac{f(x_i|x)}{w(x_i)}\omega_i\end{aligned}$$

  - For each x, $\pi(x_k|x)$ is a probability measure over the x_k.
  - If we restrict $\pi(x_k|x)$ to the x_i grid, we have a Markov chain on the x_i.
  - These probabilities approximate the conditional expectation operator:

$$E\{g(x,y)|x\} = \int g(x,y)f(y|x)dy \doteq \sum_{k=1}^{n} g(x,x_k)\pi(x_k|x).$$

- (10.8.12) approximates the continuous-space chain
  - Commonly used in macroeconomic contexts
  - Most use a small number of points; probably should use more than two or three.