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Chapter 13 Notes
Regular Perturbations of Simple Systems

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Taylor Series

- Suppose that $f : R^n \to R^m$.
  - Linear approximation at $x = x_0$ is
  $$f(x) \doteq f(x_0) + f_x(x_0)(x - x_0)$$
  - Taylor series approximation of $f(x)$ based at x_0
  $$f(x) \doteq f(x_0) + f_x(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^\top f_{xx}(x_0)(x - x_0) + ...$$
  and computed via repeated differentiation.
  - A function f and its Taylor series at x_0 have equal low-order derivatives at x_0.
  - Taylor series is a numerical approximation technique, useful when f and its derivatives are easily computed at some special point x_0.
- Analytic Functions
  - A function $f : R^n \to R$ is *analytic* at x iff f equals a power series on some open neighborhood of x.
  - If analytic, $f(x)$ equals its infinite-term Taylor series expansion at x
  - We approximate $f(x)$ for x near x_0 with first n terms of Taylor series expansion of f at x_0
- Pade Approximation
  - A *Pade approximation* of f at x_0 is any rational function with same low-order derivatives as f at x_0.
  - Construction: compute n derivatives of f at x_0 and build rational function with same derivatives. (See Chapter 6.)

Implicit Function Theorem

- Suppose $h : R^n \to R^m$ is defined in $H(x, h(x)) = 0$, $H : R^n \times R^m \to R^m$, and $h(x_0) = y_0$.
  - Implicit differentiation shows
  $$H_x(x, h(x)) + H_y(x, h(x))h_x(x) = 0$$
  - At $x = x_0$, this implies
  $$h_x(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$
  if $H_y(x_0, y_0)$ is nonsingular. More simply, we express this as
  $$h_x^0 = -\left(H_y^0\right)^{-1} H_x^0$$
  - Linear approximation for $h(x)$ is
  $$h^L(x) \doteq h(x_0) + h_x(x_0)(x - x_0)$$
- To check on quality, we compute
  $$E = \hat{H}(y, h^L(y))$$
  where $\hat{H}$ is a unit free equivalent of H. If $E < \varepsilon$, then we have an ε-solution.
- If $h^L(y)$ is not satisfactory, compute higher-order terms by repeated differentiation.
  - $D_{xx}H(x, h(x)) = 0$ implies
  $$H_{xx} + 2H_{xy}h_x + H_{yy}h_x h_x + H_y h_{xx} = 0$$
  - At $x = x_0$, this implies
  $$h_{xx}^0 = -\left(H_y^0\right)^{-1} \left(H_{xx}^0 + 2H_{xy}^0 h_x^0 + H_{yy}^0 h_x^0 h_x^0\right)$$
  - Construct the quadratic approximation
  $$h^Q(x) \doteq h(x_0) + h_x^0(x - x_0) + \frac{1}{2}(x - x_0)^\top h_{xx}^0(x - x_0)$$
  and check its quality by computing $E = H(x, h^Q(x))$.

Regular Perturbation: The Basic Idea

- Suppose x is an endogenous variable, ε a parameter
  - Want to find $x(\varepsilon)$ such that $f(x(\varepsilon),\varepsilon)=0$
  - Suppose $x(0)$ known.
- Use Implicit Function Theorem
  - Apply implicit differentiation:

$$f_x(x(\varepsilon),\varepsilon)x'(\varepsilon)+f_\varepsilon(x(\varepsilon),\varepsilon)=0 \tag{13.1.5}$$

  - At $\varepsilon=0$, $x(0)$ is known and (13.1.5) is linear in $x'(0)$ with solution

$$x'(0)=-f_x(x(0),0)^{-1}f_\varepsilon(x(0),0)$$

  - Well-defined only if $f_x\neq 0$, a condition which can be checked at $x=x(0)$.
  - The linear approximation of $x(\varepsilon)$ for ε near zero is

$$x(\varepsilon)\doteq x^L(\varepsilon)\equiv x(0)-f_x(x(0),0)^{-1}f_\varepsilon(x(0),0)\varepsilon \tag{13.1.6}$$

- Can continue for higher-order derivatives of $x(\varepsilon)$.
  - Differentiate (13.1.5) w.r.t. ε

$$f_x x''+f_{xx}(x')^2+2f_{x\varepsilon}x'+f_{\varepsilon\varepsilon}=0 \tag{13.1.7}$$

  - At $\varepsilon=0$, (13.1.7) implies that

$$\begin{aligned}x''(0)=&-f_x(x(0),0)^{-1}\left(f_{xx}(x(0),0)\;(x'(0))^2\right.\\&\left.+2f_{x\varepsilon}(x(0),0)\;x'(0)+f_{\varepsilon\varepsilon}(x(0),0)\right)\end{aligned}$$

  - Quadratic approximation is

$$x(\varepsilon)\doteq x^Q(\varepsilon)\equiv x(0)+\varepsilon x'(0)+\frac{1}{2}\varepsilon^2x''(0) \tag{13.1.8}$$

- Checking a Perturbation Approximation
  - Suppose you want approximation for x at $\varepsilon = 1$.
  - Linear approximation: if $f(x^L(1), 1)$ is small, then accept $x^L(1)$.
  - Otherwise, compute $f(x^Q(1), 1)$ and see if $f(x^Q(1), 1)$ is small.
  - Continue until some approximation, $\widehat{x}(\varepsilon)$, produces a small $f\left(\widehat{x}(1), 1\right)$.
  - Also, go fish for a functional form - e.g., rational function - until you find a good one.

- Tax example
  - p is apple price paid by consumers
  - τ is the tax per apple paid by producers.
  - Equilibrium p given the τ solves

$$0 = D(p) - S(p - \tau)$$

- $P(\tau)$ is defined by the implicit relation

$$0 = D(P(\tau)) - S(P(\tau) - \tau) \tag{13.2.1}$$

  - Let $\eta_S = pS'/S > 0$, and $\eta_D = pD'/D < 0$, at $\tau = 0$, be elasticities
  - Compute $P'(0)$

$$0 = D'P' - S'(P' - 1) \Rightarrow P' = \frac{S'}{S' - D'} = \frac{\eta_S}{\eta_S - \eta_D} \tag{13.2.2}$$

  - Write results in unit-free terms such as elasticities and shares.
  - Linear approximation around $\tau = 0$:

$$P(\tau) \doteq P(0) + \frac{\eta_S}{\eta_S - \eta_D}\tau \tag{13.2.3}$$

- Social surplus

$$SS(\tau) \equiv \int_0^{D(P(\tau))} (D^{-1}(p) - S^{-1}(p))dp + \tau D(P(\tau)) \tag{13.2.4}$$

  - Differentiating (13.2.4) implies that $SS'(0) = 0$.
  - Need to go to second-order terms to find impact of τ on SS.
  - Compute second-order expansion

$$SS(\tau) \doteq SS(0) + 0 \cdot \tau + \frac{1}{2} D(P(0)) \frac{\eta_D \eta_S}{\eta_S - \eta_D} \tau^2 \tag{13.2.5}$$

  - Common rule-of-thumb: the welfare cost of a tax is proportional to τ^2.

- Lesson: Continue expansion until we find the first nonzero term, the *dominant term*

  - First-order, second-order, etc. is meaningless terminology.
    * Define $\psi = \tau^2$, then linear term of expansion of SS in terms of ψ is nonzero
    * Define $\psi = \sqrt{\tau}$, then first nontrivial term is fourth-order in ψ.
  - Notion of nth order is not invariant to nonlinear changes in variables
  - Dominant term notion is invariant to nonlinear COV

- General Perturbation Strategy
  - Find special (likely degenerate, uninteresting) case where one knows solution
    * General relativity theory: begin with case of a universe with zero mass: ε is mass of universe
    * Quantum mechanics: begin with case where electrons do not repel each other: ε is force of repulsion
    * Business cycle analysis: begin with case where there are no shocks: ε is measure of exogenous shocks
  - Use local approximation theory to compute nearby cases
    * Standard implicit function may be applicable
    * Sometimes standard implicit function theorem will not apply; use appropriate bifurcation or singularity method.
  - Check to see if solution is good for problem of interest
    * Use unit-free formulation of problem
    * Go to higher-order terms until error is reduced to acceptable level
    * *Always* check solution for range of validity

Single-Sector, Deterministic Growth

Consider dynamic programming problem

$$\max_{c(t)} \int_0^\infty e^{-\rho t} u(c) dt$$
$$\dot{k} = f(k) - c$$

Ad-Hoc Method: Convert to an LQ problem:

- Popular idea in macroeconomics
  - Compute steady state of deterministic problem
  - Convert nonlinear deterministic problem to LQ problem around steady state
  - Apply LQ methods to the approximate problem
  - Add noise to law of motion to get stochastic approximation
- McGrattan, JBES (1990)
  - Replace $u(c)$ and $f(k)$ with approximations around c^* and k^*
  - Solve linear-quadratic problem

    $$\begin{array}{l} \max_c \int_0^\infty e^{-\rho t} \left(u(c^*) + u'(c^*)(c - c^*) + \frac{1}{2} u''(c^*)(c - c^*)^2 \right) \; dt \\ \text{s.t.} \quad \dot{k} = f(k^*) + f'(k^*)(k^* - k), \end{array}$$

  - Resulting approximate policy function is

    $$C^{McG}(k) = f(k^*) + \rho(k - k^*) \neq C(k^*) + C'(k^*)(k - k^*)$$

    * Local approximate law of motion is $\dot{k} = 0$
    * Add noise to get

      $$dk = 0 \cdot dt + dz$$

    * Approximation is *random walk* when theory says solution is stationary

- Kydland-Prescott
  - Restate problem so that $\dot{k}$ is linear function of state and controls
  - Replace $u(c)$ with quadratic approximation
  - Note 1: such transformation may not be easy
  - Note 2: special case of Magill (JET 1977).
- Lesson
  - Kydland-Prescott, McGrattan provide no mathematical basis for meth
  - Formal calculations based on appropriate IFT should be used.
  - Beware of *ad hoc* methods based on an intuitive story!

Perturbation Method for Dynamic System

- Formalize problem as a system of functional equations

  – Bellman equation:

$$\rho V(k) = \max_c \; u(c) + V'(k)(f(k) - c) \tag{1}$$

  – $C(k)$: policy function defined by

$$\begin{aligned} 0 &= u'(C(k)) - V'(k) \\ \rho V(k) &= u(C(k)) + V'(k)(f(k) - C(k)) \end{aligned} \tag{2}$$

  – Apply envelope theorem to (1) to get

$$\rho V'(k) = V''(k)(f(k) - C(k)) + V'(k)f'(k) \tag{1_k}$$

  – Steady-state equations

$$\begin{aligned} c^* &= f(k^*) \\ 0 &= u'(c^*) - V'(k^*) \\ \rho V(k^*) &= u(c^*) + V'(k^*)(f(k^*) - c^*) \\ \rho V'(k) &= V''(k)(f(k^*) - c^*) + V'(k)f'(k) \end{aligned}$$

  – Steady State: We know k^*, $V(k^*)$, $C(k^*)$, $f'(k^*)$, $V'(k^*)$:

$$\begin{aligned} \rho &= f'(k^*) \\ C(k^*) &= f(k^*) \\ V(k^*) &= \rho^{-1}u(c^*) \\ V'(k^*) &= u'(c^*) \end{aligned}$$

  – Want Taylor expansion:

$$\begin{aligned} C(k) &\doteq C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ... \\ V(k) &\doteq V(k^*) + V'(k^*)(k - k^*) + V''(k^*)(k - k^*)^2/2 + ... \end{aligned}$$

- Linear approximation around a steady state
  - Differentiate $(1_k, 2)$ w.r.t. k:

$$\rho V'' = V'''(f - C) + V''(f' - C') + V''f' + V'f'' \qquad (1_{kk})$$

$$0 = u''C' - V'' \qquad (2_k)$$

  - At the steady state

$$0 = -V''(k^*)C'(k^*) + V''(k^*)f'(k^*) + V'(k^*)f''(k^*) \qquad (1_k^*)$$

  - Substituting (2_k) into (1_k^*) yields

$$0 = -u''(C')^2 + u''C'f' + V'f''$$

  - Two solutions

$$C'(k^*) = \frac{\rho}{2}\left(1 \pm \sqrt{1 + \frac{4u'(C(k^*))f''(k^*)}{u''(C'(k^*))f'(k^*)f'(k^*)}}\right)$$

  - However, we know $C'(k^*) > 0$; hence, take positive solution

- Higher-Order Expansions
  - Conventional perception in macroeconomics: “perturbation methods of order higher than one are considerably more complicated than the traditional linear-quadratic case ...” – Marcet (1994, p. 111)
  - Mathematics literature: No problem (See, e.g., Bensoussan, Fleming, Souganides, etc.)

- Compute $C''(k^*)$ and $V'''(k^*)$.
  - Differentiate $(1_{kk}, 2_k)$:

$$\begin{aligned}\rho V''' &= V''''(f-C) + 2V'''(f'-C') + V''(f''-C'') \qquad (1_{kkk})\\ &\quad + V'''f' + 2V''f'' + V'f''' \\ 0 &= u'''(C')^2 + u''C'' - V''' \qquad (2_{kk})\end{aligned}$$

  - At k^*, (1_{kkk}) reduces to

$$0 = 2V'''(f'-C') + 3V''f'' - V''C'' + V'f''' \qquad (1^*_{kkk})$$

  - Equations $(1^*_{kkk}, 2^*_{kk})$ are *LINEAR* in unknowns $C''(k^*)$ and $V'''(k^*)$:

$$\begin{pmatrix} u'' & -1 \\ V'' & -2(f'-C') \end{pmatrix} \begin{pmatrix} C'' \\ V''' \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Unique solution since determinant $-2u''(f'-C') + V'' < 0$.

- Compute $C^{(n)}(k^*)$ and $V^{(n+1)}(k^*)$.
  - Linear system for order n is, for some A_1 and A_2,

$$\begin{pmatrix} u'' & -1 \\ V'' & -n(f' - C') \end{pmatrix} \begin{pmatrix} C^{(n)} \\ V^{(n+1)} \end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

  - Higher-order terms are produced by solving linear systems
  - The linear system is always determinate since $-nu''(f'-C')+V'' < 0$
- Conclusion:
  - Computing first-order terms involves solving quadratic equations
  - Computing higher-order terms involves solving linear equations
  - Computing higher-order terms is easier than computing the linear term.

Accuracy Measure

Consider the one-period relative Euler equation error:

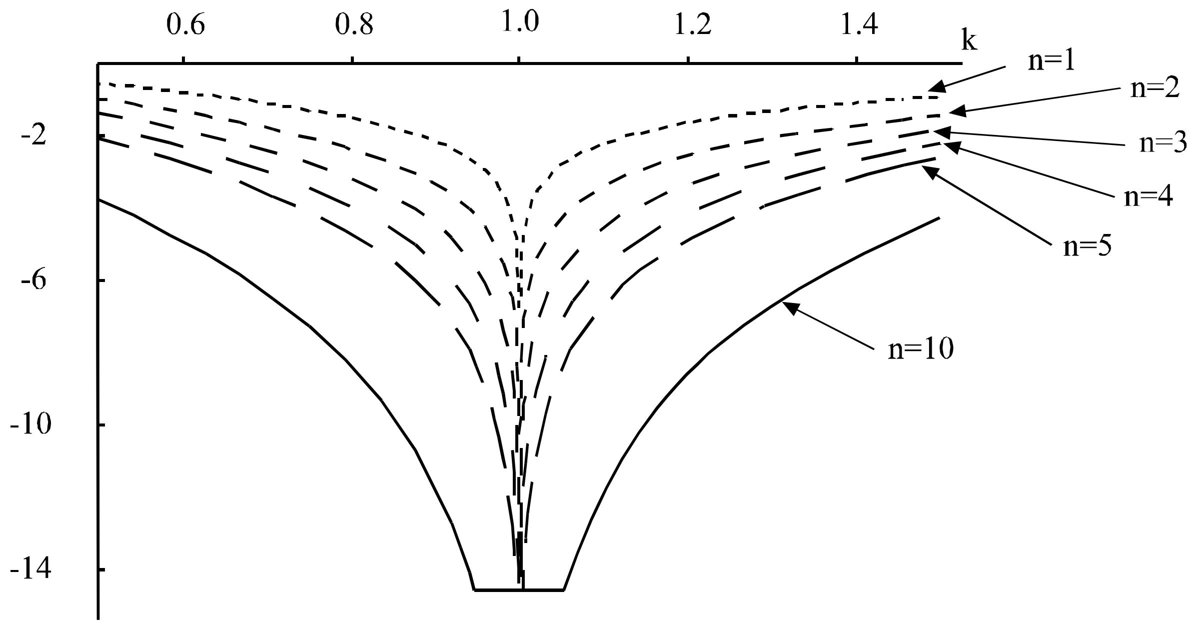
$$E(k) = 1 - \frac{V'(k)}{u'(C(k))}$$

- Equilibrium requires it to be zero.
- $E(k)$ is measure of optimization error
  - 1 is unacceptably large
  - Values such as .00001 is a limit for people.
  - $E(k)$ is unit-free.
- Define the L^p, $1 \leq p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \parallel E(k) \parallel_p$$

- The L^∞ error is the maximum value of $E(k)$.

Global Quality of Asymptotic Approximations



Graph of $\log_{10}|E(k|$

- Linear approximation is very poor even for k close to steady state
- Order 2 is better but still not acceptable for even $k = .9, 1.1$
- Order 10 is excellent for $k \in [.6, 1.4]$

Single-Sector, Stochastic Growth

- Problem

$$\max_c \; E\left\{\int_0^\infty e^{-\rho t}u(c)dt\right\}$$

$$dk = (f(k) - c)dt + \frac{1}{2}\sqrt{\sigma}kdz$$

- Standard Macroeconomics Approach
  - Compute linear rule for deterministic problem near steady state:

$$C^L(k) = C(k^*) + C'(k^*)(k - k^*)$$

  - Use $C^L(k)$ for stochastic problems - assumes certainty equivalance
  - Ignores fact that true $C(k)$ is not certainty equivalent

- Perturbation approach
  - Define the value function (making dependence on σ explicit)

$$V(k,\sigma) = \sup_{c\in\mathcal{F}} E\left\{\int_0^\infty e^{-\rho t}u(c)dt\right\}$$

  - Bellman equation:

$$\rho V(k,\sigma) = \max_c \; u(c) + V_k(k)\;(f(k) - c) + \sigma k^2 V_{kk}(k,\sigma)$$

  - Taylor expansions of C and V:

$$\begin{aligned} C(k,\sigma) \doteq\, & C(k^*,0) + C_k(k^*,0)(k-k^*) + C_\sigma(k^*,0)\sigma \\ & + C_{kk}(k^*,0)(k-k^*)^2/2 + C_{\sigma k}(k^*,0)\sigma(k-k^*) \\ & + C_{\sigma\sigma}(k^*,0)\sigma^2/2 + \ldots \\ V(k,\sigma) \doteq\, & V(k^*,0) + V_k(k^*,0)(k-k^*) + V_\sigma(k^*,0)\sigma \\ & + V_{kk}(k^*,0)(k-k^*)^2/2 + V_{\sigma k}(k^*,0)\sigma(k-k^*) \\ & + V_{\sigma\sigma}(k^*,0)\sigma^2/2 + \ldots \end{aligned}$$

  - True linear approximation involves $C_\sigma(k^*,0)$ term - certainty non-equivalence

- Functional equations
  - Bellman equation
$$0 = -\rho V + u(C) + V_k(f - C) + \sigma k^2 V_{kk} \tag{1}$$
  - First-order condition
$$0 = u'(C) - V_k \tag{2}$$
- Compute V_σ :
  - Differentiating the system (1,2) with respect to σ
$$0 = -\rho V_\sigma + V_{k\sigma}(f - C) + V_{kk}k^2 + \sigma V_{kk\sigma}k^2 \tag{1_σ}$$
$$0 = u''C_\sigma - V_{k\sigma} \tag{2_σ}$$
  - At $k = k^*$ and $\sigma = 0$, we know that $C = f$. Furthermore, $u' = V_k$ always.
  - Therefore, at $k = k^*$ and $\sigma = 0$, (1_σ) reduces to
$$V_\sigma(k^*) = V_{kk}(k^*)k^2\rho^{-1} \tag{1_σ^*}$$
- V_σ proportional to
  - output variance, $\sigma\ [k^*]^2$
  - $V_{kk}(k^*)$, the curvature of the value function for the deterministic problem at its steady state.
  - Intuition: when output variance is $\sigma\ [k^*]^2$, the value function at k^* is reduced, to a first order, by $V_{kk}(k^*)$, the curvature of the value function for the deterministic problem at its steady state.

- Compute C_σ and $V_{k\sigma}$.
  - Differentiating (1_σ) with respect to k yields

$$
\begin{aligned}
0 = & -\rho V_{k\sigma} + V_{kk\sigma}(f - C) + V_{kkk}k^2 + V_{k\sigma}(f' - C') \\
& +\sigma V_{kkk\sigma}k^2 + 2kV_{kk} + 2k\sigma V_{kk\sigma}
\end{aligned}
\qquad (1_{\sigma k})
$$

  - At the steady state, $(1_{\sigma k})$ becomes

$$
0 = -C_k V_{k\sigma} + V_{kkk}k^2 + 2kV_{kk} \qquad (1^*_{\sigma k})
$$

  - Therefore

$$
\begin{aligned}
V_{k\sigma} &= \frac{V_{kkk}k^2 + 2kV_{kk}}{C_k} \\
C_\sigma &= \frac{V_{kkk}k^2 + 2kV_{kk}}{C_k u''}
\end{aligned}
$$

- Note:
  - Only linear equations arise
  - *Even if u is quadratic and f linear,* $C_\sigma \neq 0$!
  - C_σ and $V_{k\sigma}$ depend on
    * third derivative of the value function in the deterministic case
    * steady state value of consumption in the deterministic case
  - We could continue this for higher order expansions.

Deterministic, Discrete-Time Growth

Consider

$$\begin{array}{ll} \max_{c_t} & \sum_{t=0}^{\infty} \beta^t \, u(c_t), \\ \text{s.t.} & k_{t+1} = F(k_t) - c_t. \end{array} \tag{13.7.13}$$

- Policy function, $C(k)$, satisfies

$$u'\left(C(k)\right) = \beta\, u'\left(C\left(F(k) - C(k)\right)\right)\; F'\left(F(k) - C(k)\right). \tag{13.7.14}$$

- At the steady state, k^*:
  - $F(k^*) - C(k^*) = k^*$
  - $u'\left(C(k^*)\right) = \beta\, u'\left(C(k^*)\right)\; F'(k^*)$
  - Steady state condition $1 = \beta F'(k^*)$ uniquely determines k^*.

- Take derivative of (13.7.14) with respect to k:

$$\begin{aligned} u''\left(C\right)\, C' &= \beta u''\left(C\left(F - C\right)\right)\, C'\left(F - C\right)\;\left[F' - C'\right] F'(F - C) \\ &\quad + \beta\, u'\left(C\left(F(k - C)\right)\right)\; F''\left(F - C\right)\;\left[F' - C'\right] \end{aligned} \tag{13.7.15}$$

  - At $k = k^*$, (13.7.15) reduces to (drop all arguments)

$$u''C' = u''C'\left[F' - C'\right] + \beta u'\, F''\left[F' - C'\right]. \tag{13.7.16}$$

  - In (13.7.16) we know all terms at $k = k^*$ except $C'(k^*)$
  - (13.7.16) at $k = k^*$ a quadratic equation in $C'(k^*)$ with the solution

$$C' = \frac{1}{2}\left(1 - F' + \beta\,\frac{u'}{u''}\,F'' + \sqrt{\left(-1 + F' - \beta\,\frac{u'}{u''}\,F''\right)^2 + 4\,\frac{u'}{u''}\,F''}\right) \tag{13.7.17}$$

- Compute $C''(k^*)$:
  - Take another derivative of (13.7.15)

$$\begin{aligned} u''C'' &+ u'''C'C' \\ &= \beta u''' \left(C'F'(1-C')\right)^2 F' + \beta u''C'' \left(F'(1-C')\right)^2 F' \\ &+ 2\beta u''C'F'(1-C')^2 F'' + \beta u'F'''(1-C')^2 + \beta u'F''(-C'') \end{aligned} \tag{13.7.18}$$

  - (13.7.18) at $k = k^*$ is a linear equation in the unknown $C''(k^*)$

Stochastic, Discrete-Time Growth

$$\begin{array}{ll} \max_{c_t} & E\left\{\sum_{t=0}^{\infty} \beta^t\, u(c_t)\right\} \\ \text{s.t.} & k_{t+1} = (1+\varepsilon z)F(k_t - c_t) \end{array} \tag{13.7.19}$$

- New state variable:
  - k_t is capital stock at the beginning of period t
  - consumption comes out of k
  - the remaining capital, $k_t - c_t$, is used in production
  - resulting output is $(1+\varepsilon z)F(k_t - c_t) = k_{t+1}$
  - perturbation parameter is ε, the standard deviation, not the variance.
- Do deterministic perturbation analysis.
  - Solution when $\varepsilon = 0$ is $C(k)$ solving

$$u'\left(C(k)\right) = \beta\, u'\left(C\left(F(k - C(k))\right)\right)\; F'\left(k - C(k)\right). \tag{13.7.20}$$

  - At the steady state, k^*, $F(k^* - C(k^*)) = k^*$, and $1 = \beta\, F'(k^* - C(k^*))$
  - Derivative of (13.7.20) with respect to k implies

$$\begin{aligned} u''\left(C(k)\right)\; C'(k) = & \ \beta u''\left(C\left(F(k - C(k))\right)\right)\; C'\left(F(k - C(k))\right) \\ & \times F'(k - C(k))[1 - C'(k)]\, F'(k - C(k)) \\ & + \beta\, u'\left(C\left(F(k - C(k))\right)\right)\; F''\left(k - C(k)\right)\; [1 - C'(k)] \end{aligned} \tag{13.7.21}$$

– At $k = k^*$, (13.7.21) reduces to (drop all arguments)

$$u''C' = \beta u''C'F'[1-C']\,F' + \beta u'\,F''[1-C']. \tag{13.7.22}$$

with stable solution

$$C' = \frac{1}{2}\left(1-\beta-\beta^2\,\frac{u'}{u''}\,F'' + \sqrt{\left(1-\beta-\beta^2\,\frac{u'}{u''}\,F''\right)^2 + 4\,\frac{u'}{u''}\,\beta^2\,F''}\right)$$

– Take another derivative of (13.7.21) and set $k = k^*$ to find

$$\begin{aligned} u''C'' + u'''C'C' =& \beta u'''\left(C'F'(1-C')\right)^2\,F' + \beta u''C''\left(F'(1-C')\right)^2\,F' \\ &+2\beta u''C'F'(1-C')^2\,F'' + \beta u'F'''(1-C')^2 \\ &+\beta u'F''(-C''), \end{aligned}$$

which is a linear equation in the unknown $C''(k^*)$.

- Stochastic problem:
  - Euler equation is

$$u'\left(C(k)\right) = \beta\, E\left\{u'\left(g(\varepsilon,\, k,\, z)\right)\; R(\varepsilon,\, k, z)\right\}, \tag{13.7.23}$$

    where

$$\begin{aligned} g(\varepsilon, k, z) \; &\equiv C((1+\varepsilon z)F(k - C(k))), \\ R(\varepsilon,\, k,\, z) &\equiv (1+\varepsilon\, z)\, F'\left(k - C(k)\right). \end{aligned} \tag{13.7.24}$$

  - Compute C_ε
    * Differentiate (13.7.24) with respect to ε yields (we drop arguments of F and C)

$$\begin{aligned} g_\varepsilon &= C_\varepsilon + C'\left(zF - (1+\varepsilon z)F'C_\varepsilon\right), \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon\,\left(zF - (1+\varepsilon z)F'C_\varepsilon\right) + C''\left(zF - (1+\varepsilon z)F'C_\varepsilon\right)^2, \\ &\quad + C'\left(-zF'C_\varepsilon\, 2 + (1+\varepsilon z)\, F''C_\varepsilon^2 - (1+\varepsilon z)\, F'C_{\varepsilon\varepsilon}\right). \end{aligned} \tag{13.7.25}$$

    * At $\varepsilon = 0$, (13.7.25) implies that

$$\begin{aligned} g_\varepsilon &= C_\varepsilon + C'(zF - F'C_\varepsilon), \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon(zF - F'C_\varepsilon) + C''(zF - F'C_\varepsilon)^2, \\ &\quad + C'(-2zF'C_\varepsilon + F''C_\varepsilon^2 - F'C_{\varepsilon\varepsilon}). \end{aligned} \tag{13.7.26}$$

* Differentiate (13.7.23) with respect to ε

$$u''C_\varepsilon = \beta\, E\left\{u''g_\varepsilon(1+\varepsilon z)F' + u'F'z - u'(1+\varepsilon z)F''C_\varepsilon\right\} \tag{13.7.27}$$

$$\begin{aligned} u'''C_\varepsilon^2 + u''C_{\varepsilon\varepsilon} = \beta\, E\{ & u'''g_\varepsilon^2(1+\varepsilon z)\, F' + 2u''g_\varepsilon F'z \\ & -2u''g_\varepsilon(1+\varepsilon z)F''C_\varepsilon + u''g_{\varepsilon\varepsilon}(1+\varepsilon z)F' \\ & -2u'zF''C_\varepsilon + u'(1+\varepsilon z)F'''C_\varepsilon^2 - u'(1+\varepsilon z)F''C_{\varepsilon\varepsilon}\} \end{aligned} \tag{13.7.28}$$

* Since $E\{z\} = 0$, (13.7.27) says that $C_\varepsilon = 0$, which in turn implies that

$$\begin{aligned} g_\varepsilon &= C'zF, \\ g_{\varepsilon\varepsilon} &= C_{\varepsilon\varepsilon} + 2C'_\varepsilon\, z\, F + C''(zF)^2 - C'F'C_{\varepsilon\varepsilon}. \end{aligned}$$

– Compute $C_{\varepsilon\varepsilon}$

 * Second-order terms in (13.7.28), we find that at $\varepsilon = 0$,

$$\begin{aligned} u'''C_\varepsilon^2 + u''C_{\varepsilon\varepsilon} = \beta\, E\, \{ & u'''g_\varepsilon^2\, F' + 2u''g_\varepsilon\, F'\, z - 2u''g_\varepsilon\, F''C_\varepsilon \\ & + u''g_{\varepsilon\varepsilon}\, F' - 2u'\, z\, F''C_\varepsilon \\ & + u'F'''C_\varepsilon^2 - u'F''C_{\varepsilon\varepsilon} \} \end{aligned}$$

 * Using the normalization $E\{z^2\} = 1$, we find that

$$\begin{aligned} u''C_{\varepsilon\varepsilon} = \beta\, [& u'''\, C'C'F^2F' + 2u''C'FF' \\ & + u''(C_{\varepsilon\varepsilon} + C''F^2 - C'F'C_{\varepsilon\varepsilon})F' - u'F''C_{\varepsilon\varepsilon}] \end{aligned}$$

 * Solving for $C_{\varepsilon\varepsilon}$ yields

$$C_{\varepsilon\varepsilon} = \frac{u'''\, C'C'F^2 + 2u''C'F + u''C''F^2}{u''C'F' + \beta u'F''}$$

- This exercise demonstrates that perturbation methods can also be applied to the discrete-time stochastic growth model, albeit at somewhat greater cost.

Nonautonomous Perturbations of a Perfect Foresight Model

- We often want to examine nonautonomous perturbations of an autonomous equation.
- Standard perfect foresight model

$$\dot{\lambda} = g^1(\lambda, k, \varepsilon h(t)), \tag{13.4.4a}$$
$$\dot{k} = g^2(\lambda, k, \varepsilon h(t)), \tag{13.4.4b}$$

with the boundary conditions,

$$\lim_{t\to\infty} |k(t)| < \infty, \quad k(0) = k_0(\varepsilon), \tag{13.4.5}$$

where

  - k is predetermined
  - λ is free
  - ε is a scalar parameter
  - $h(t)$ is bounded and eventually constant.

- We assume that there is a unique solution to (13.4.4) for each ε.
- For each value of ε we have a different solution; denote the solutions $\lambda(t, \varepsilon)$ and $k(t, \varepsilon)$
- We often want to know the dynamic evaluation function

$$W(\varepsilon) = \int_0^\infty e^{-\rho t} v(\lambda(t, \varepsilon), k(t, \varepsilon))dt,$$

where $v(\lambda, k)$ is the utility or profit flow expressed as a function of λ and k.

- Differentiation of the system (13.4.4) with respect to ε yields

$$\frac{d}{dt}\begin{pmatrix}\lambda_\varepsilon\\ k_\varepsilon\end{pmatrix} = J\begin{pmatrix}\lambda_\varepsilon\\ k_\varepsilon\end{pmatrix} + \begin{pmatrix}g_3^1(\lambda_0,k_0,0)h(t)\\ g_3^2(\lambda_0,k_0,0)h(t)\end{pmatrix}, \tag{13.4.7}$$

where J is the Jacobian of $G : R^2 \to R^2$ at (λ_0, k_0) where

$$G(\lambda,k) = \begin{pmatrix}g^1(\lambda,k,0)\\ g^2(\lambda,k,0)\end{pmatrix}$$

  – (13,4.7) is a linear differential equation in $\lambda_\varepsilon(t,0)$ and $k_\varepsilon(t,0)$
  – Initial condition $k(0) = k_0(\varepsilon)$ pins down $k_\varepsilon(0,0)$
  – Stability pins down $\lambda_\varepsilon(0,0)$:

$$\begin{aligned}\lambda_\varepsilon(0,0) &= -\frac{(\mu-J_{11})(k_0'(0)+g_3^2H(\mu))}{J_{21}} - g_3^1H(\mu)\\ &= -\frac{J_{12}(k_0'(0)+g_3^2H(\mu))}{\mu-J_{22}} - g_3^1H(\mu).\end{aligned} \tag{13.4.10}$$

  – With $\lambda_\varepsilon(0,0)$, (13.4.7) becomes a nonautonomous linear IVP which can be solved by standard methods
  – Solution for both $k_\varepsilon(t,0)$ and $\lambda_\varepsilon(t,0)$ implies

$$\begin{aligned}\frac{dW}{d\varepsilon} &= \int_0^\infty e^{-\rho t}\left(v_\lambda\ \lambda_\varepsilon(t;0) + v_k\ k_\varepsilon(t;0)\right)dt \qquad (13.4.11)\\ &= \begin{pmatrix}v_\lambda\\ v_k\end{pmatrix}^\top (\rho I - J)^{-1}\\ &\quad\times\begin{pmatrix}g_3^1(H(\rho)-H(\mu)) - \frac{\mu-J_{11}}{J_{21}}\left(g_3^2H(\mu)+k_0'(0)\right)\\ k_0'(0)+g_3^2H(\rho)\end{pmatrix}.\end{aligned}$$

Linearization of Multidimensional Dynamic Systems

- Dynamic multidimensional system

$$\begin{aligned} \dot{x} &= f(x, \lambda, u), \\ \dot{\lambda} &= g(x, \lambda, u), \\ 0 &= h(x, \lambda, u, \mu), \\ x(0) &= x_0, \ \lim_{t\to\infty} |x(t)|, |\lambda(t)|, |\mu(t)| < \infty. \end{aligned} \tag{14.2.1}$$

- In optimal control problems,
  - $x \in R^n$ is the state vector, $f : R^n \times R^n \times R^m \to R^n$ (predetermined variables)
  - $u \in R^m$ are the controls,
  - $\lambda \in R^n$ is the costate vector, $g : R^n \times R^n \times R^m \to R^n$, (free dynamic variables)
  - $\mu \in R^\ell$ are nondynamic variables like Kuhn-Tucker multipliers, $h : R^n \times R^n \times R^m \times R^\ell \to R^{m+\ell}$.
- We assume that the relation $0 = h(x, \lambda, u, \mu)$ can be inverted, yielding functions $\mathcal{U}(x, \lambda)$ and $\mathcal{M}(x, \lambda)$ that satisfy

$$0 = h(x, \lambda, \mathcal{U}(x, \lambda), \mathcal{M}(x, \lambda))$$

- Rewrite (14.2.1) as

$$\begin{aligned} \dot{x} &= f(x, \lambda, \mathcal{U}(x, \lambda)), \\ \dot{\lambda} &= g(x, \lambda, \mathcal{U}(x, \lambda)). \end{aligned} \tag{14.2.2}$$

- A steady state of (14.2.1) is a triple $(x^*, \lambda^*) \equiv Z^*$, u^*, and μ^* such that

$$0 = f(Z^*, u^*) = g(Z^*, u^*) = h(Z^*, u^*, \mu^*)$$

- Assume local uniqueness. Fortunately failures will be revealed in perturbation analysis.

Perturbation

Define a continuum of problems with

$$x(0,\varepsilon) = x^* + \varepsilon \xi_x(0)$$

with solutions $x(t,\varepsilon)$ for x and $\lambda(t,\varepsilon)$ for λ.

- Want to compute $\xi_\lambda(t) \equiv \lambda_\varepsilon(t,0)$ and $\xi_x(t) \equiv x_\varepsilon(t,0)$ to produce linear approximations

$$x(t,\varepsilon) \doteq x(t,0) + \varepsilon x_\varepsilon(t,0)$$
$$\lambda(t,\varepsilon) \doteq \lambda(t,0) + \lambda x_\varepsilon(t,0)$$

- Differentiate (14.2.2) and $x(0,\varepsilon) = x^* + \varepsilon\xi_x(0)$ w.r.t. ε
- Produce a system of linear differential equations for $\xi_x(t)$ and $\xi_\lambda(t)$:

$$\dot{\xi} = \begin{pmatrix} \dot{\xi}_x \\ \dot{\xi}_\lambda \end{pmatrix} = \begin{pmatrix} f_x + f_u \mathcal{U}_x & f_\lambda + f_u \mathcal{U}_\lambda \\ g_x + g_u \mathcal{U}_x & g_\lambda + g_u \mathcal{U}_\lambda \end{pmatrix} \begin{pmatrix} \xi_x \\ \xi_\lambda \end{pmatrix} \qquad (14.2.4)$$
$$\equiv A \begin{pmatrix} \xi_x \\ \xi_\lambda \end{pmatrix}$$

with constant coefficients A.

- Solution to (14.2.3) is $\xi = \xi(0)e^{At}$, where $\xi(0)$ is the initial value of ξ.
- Given a value for $\xi_x(0)$, asymptotic stability fixes the free variables $\xi_\lambda(0)$

$$\lim_{t\to\infty} |\xi(0)\, e^{At}| < \infty \,. \qquad (14.2.4)$$

- Decompose Jordan form:

$$A = N^{-1} D N$$

$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, \quad N^{-1} = \begin{pmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{pmatrix}$$

where D_1 is stable and D_2 unstable.

- Solution to $\xi_\lambda(0)$ is

$$\xi_\lambda(0) = -N_{22}^{-1}\ N_{21}\ \xi_x(0). \tag{14.2.5}$$

This assumes that N is nonsingular. See Anderson et al. for general case.

- In general, $x(0)$, determines $\lambda(0)$ through stability condition.
  - Let $\lambda = \Lambda(x)$ be the general relation.
  - (14.2.5) implies $\Lambda_x(x^*) = -N_{22}^{-1}N_{21}$.
- Recursive formulation wants $U(x)$
  - (14.2.1) defines $u = \mathcal{U}(x, \lambda)$.
  - $U(x) = \mathcal{U}(x, \Lambda(x))$.
  - Since $\Lambda_x(x^{ss}) = -N_{22}^{-1}N_{21}$,

$$\frac{\partial U}{\partial x}(x^*) = \frac{\partial \mathcal{U}}{\partial x}(x^*, \lambda^*) - \frac{\partial \mathcal{U}}{\partial \lambda}(x^*, \lambda^*)\ N_{22}^{-1}\ N_{21}. \tag{14.2.6}$$

- Equilibrium dynamics

$$\dot{x} = f(x, \Lambda(x), U(x)), \tag{14.2.7}$$

  - For x near x^*, (14.2.7) is approximately $\dot{x} = B(x - x^*)$,

$$B = f_x(x^*, \lambda^*, u^*) + f_\lambda(x^*, \lambda^*, u^*)\Lambda_x(x^*) + f_u(x^*, \lambda^*, u^*)U_x(x^*)$$

  - e^{Bt} are the *impulse response functions* for the states x_i.