Chapter 15 Notes

Advanced Asymptotic Methods

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Asymptotics

Sometimes the IFT fails because there are multiple solutions to $H(x, f(x))$ at or near (x_0, y_0). We present methods which apply to such cases.

Bifurcation Methods

Suppose $H(h(\epsilon), \epsilon) = 0$

- IFT says
$$h'(0) = -\frac{H_\epsilon(x_0, 0)}{H_x(x_0, 0)}$$
- If $H_x(x_0, 0) = 0$, $h'(0)$ has the form $0/0$ at $x = x_0$.
- l'Hospital's rule implies
$$h'(0) = -\frac{H_{\epsilon\epsilon}(x_0, 0)}{H_{\epsilon x}(x_0, 0)}.$$
which is well-defined if $H_{\epsilon x}(x_0, 0) \neq 0$.

We will use bifurcation theory to extend the IFT.

Definition: ϵ_0 is a *bifurcation point* if the number of solutions to $H(x,\epsilon) = 0$ changes as ϵ passes through ϵ_0.

Theorem 1 *(Bifurcation Theorem on R): Suppose $H : R \times R \to R$ and $H(x,\epsilon) = 0$ for all x if $\epsilon = 0$. Furthermore, suppose that*

$$H_x(x_0, 0) = 0 = H_\epsilon(x_0, 0), \;\; H_{x\epsilon}(x_0, 0) \neq 0 \tag{1}$$

for some $(x_0, 0)$. Then

1. *If $H_{\epsilon\epsilon}(x_0, 0) \neq 0$, there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon), h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that $H(h(\epsilon), \epsilon) = 0$ and locally $H(x,\epsilon)$ is diffeomorphic to $\epsilon(\epsilon - x)$ or $\epsilon(\epsilon + x)$ on $\mathcal{N}$.*
2. *If $H_{\epsilon\epsilon}(x_0, 0) = 0 \neq H_{\epsilon\epsilon\epsilon}(x_0, 0)$, then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and a function $h(\epsilon), h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that $H(h(\epsilon), \epsilon) = 0$ and $H(x,\epsilon)$ is diffeomorphic to $\epsilon^3 - x\epsilon$ or $\epsilon^3 + x\epsilon$ on $\mathcal{N}$.*
3. *In both cases, (x_0, ϵ) is a bifurcation point.*

A multidimensional version constructs a one-dimensional path in (x, y) space which satisfies the implicit relation $H(y, x, \epsilon) = 0$.

Theorem 2 *(Bifurcation Theorem on R^n): Suppose $H : R^m \times R^n \times R \to R^n$ and $H(y_0, x, \epsilon) = 0$ for all x if $\epsilon = 0$. Furthermore, suppose that, for some $(y^0, x^0, 0)$, H is analytic in a neighborhood of $(y^0, x^0, 0)$, $H_x(y^0, x^0, 0) = 0_{n\times n}$, $H_\epsilon(x^0, 0) = 0_n$, and* $\det(H_{x\epsilon}(y^0, x^0, 0)) \neq 0$. *Then, there is an open neighborhood $\mathcal{N}$ of $(y^0, x^0, 0)$ and a locally analytic function $h(\epsilon) \equiv (h^y(\epsilon), h^x(\epsilon))$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that $H(h^y(\epsilon), h^x(\epsilon), \epsilon) = 0$ on $\mathcal{N}$.*

Remark 3 *These theorems are true for the bifurcation point. We generally do not know the location of bifurcation points. We use these bifurcation conditions to locate bifurcation points*

Example: Portfolio Choices for Small Risks

- Simple asset demand model:
  - safe asset yields R per dollar invested
  - risky asset yields Z per dollar invested
  - $Y = W((1-\omega)R + \omega Z)$.
  - Portfolio problem

$$\max_{\omega} E\{u(Y)\}$$

- Small Risk Analysis
  - Parameterize cases

$$Z = R + \epsilon z + \epsilon^2 \pi \tag{2}$$

  - Compute $\omega(\epsilon) \doteq \omega(0) + \epsilon\omega'(0) + \frac{\epsilon^2}{2}\omega''(0)$.around the deterministic case of $\epsilon = 0$.
  - Failure of IFT: at $\epsilon = 0$, $Z = R$, and $\omega(\ \epsilon)$ is indeterminate.
  - But, $\omega(\ \epsilon)$ is determinate for $\epsilon \neq 0$

- Bifurcation analysis
  - The first-order condition for ω

$$0 = E\{u'\left(WR + \omega W(\epsilon z + \epsilon^2 \pi)\right)(z + \epsilon\pi)\} \equiv G(\omega, \epsilon) \tag{3}$$

$$0 = G(\omega, 0), \quad \forall \omega. \tag{4}$$

  - Solve for $\omega(\epsilon) \doteq \omega(0) + \epsilon\omega'(0) + \frac{\epsilon^2}{2}\omega''(0)$. Implicit differentiation implies

$$0 = G_\omega \omega' + G_\epsilon \tag{5}$$

$$\begin{aligned} G_\epsilon &= E\{u''(Y)W(\omega z + 2\omega\epsilon\pi)W(z + \epsilon\pi) + u'(Y)\pi\} & (6) \\ G_\omega &= E\{u''(Y)(z + \epsilon\pi)^2\epsilon\} & (7) \end{aligned}$$

  - At $\epsilon = 0$, $G(\omega, 0) = G_\omega(\omega, 0) = 0$ for all ω.
  - No point $(\omega, 0)$ for application of IFT to (4) to solve for $\omega'(0)$.

- We want $\omega_0 = \lim_{\epsilon\to 0}\omega(\epsilon)$.
  - Bifurcation theorem keys on ω_0 satisfying

$$\begin{aligned} 0 &= G_\epsilon(\omega_0, 0) \\ &= u''(RW)\omega_0\sigma_z^2 W + u'(RW)\pi & (8) \end{aligned}$$

    which implies

$$\omega_0 = -\frac{\pi}{\sigma_z^2}\frac{u'(WR)}{Wu''(WR)} \tag{9}$$

  - (9) is asymptotic portfolio rule
    * same as mean-variance rule
    * ω_0 is product of risk tolerance and the risk premium per unit variance.
    * ω_0 is the limiting portfolio share as the variance vanishes.
    * ω_0 is not first-order approximation.

- To calculate $\omega'(0)$:
  - differentiate (2.4) with respect to ϵ

$$0 = G_{\omega\omega}\omega'\omega' + 2G_{\omega\epsilon}\omega' + G_{\omega}\omega'' + G_{\epsilon\epsilon} \tag{10}$$

  where (without loss of generality, we assume $W = 1$)

$$\begin{aligned} G_{\epsilon\epsilon} = \quad & E\{u'''(Y)(\omega z + 2\omega\epsilon\pi)^2(z+\epsilon\pi) + u''(Y)2\omega\pi(z+\epsilon\pi) \\ & +2u''(Y)(\omega z + 2\omega\epsilon\pi)\pi\} \\ G_{\omega\omega} = \quad & E\{u'''(Y)(z+\epsilon\pi)^3\epsilon\} \\ G_{\omega\epsilon} = \quad & E\{u'''(Y)(\omega z + 2\omega\epsilon\pi)(z+\epsilon\pi)^2\epsilon + u''(Y)(z+\epsilon\pi)2\pi\epsilon \\ & +u''(Y)(z+\epsilon\pi)^2\} \end{aligned}$$

  - At $\epsilon = 0$,

$$\begin{aligned} G_{\epsilon\epsilon} &= u'''(R)\omega_0^2 E\{z^3\} \\ G_{\omega\omega} &= 0 \\ G_{\omega\epsilon} &= u''(R)E\{z^2\} \neq 0 \\ G_{\epsilon\epsilon\epsilon} &\neq 0 \end{aligned}$$

  - Therefore, Theorem 1.1 applies and

$$\omega' = -\frac{1}{2}\frac{u'''(R)}{u''(R)}\frac{E\{z^3\}}{E\{z^2\}}\omega_0^2. \tag{11}$$

  - Equation (11) is a simple formula.
    * $\omega'(0)$ proportional to u'''/u''
    * $\omega'(0)$ proportional to ratio of skewness to variance.
    * If u is quadratic or z is symmetric, ω does not change to a first order.
  - We could continue this and compute more derivatives of $\omega(\epsilon)$ as long as u is sufficiently differentiable.

- Other applications
  - Equilibrium: add other agents, make π endogenous
  - Add assets
  - Produce a mean-variance-skewness-kurtosis-etc. theory of asset markets
  - More intuitive approach to market incompleteness then counting states and assets

Gauge Functions and Asymptotic Expansions

- A system of *gauge functions* is a sequence of functions, $\{\delta_n(\epsilon)\}_{n=1}^{\infty}$, such that

$$\lim_{\epsilon \to 0} \frac{\delta_{n+1}(\epsilon)}{\delta_n(\epsilon)} = 0 \tag{12}$$

- An *asymptotic expansion* of $f(x)$ near $x = 0$ is any expansion $f(0) + \sum_{i=1}^{n} a_i \delta_i(x)$ where, for each $k < n$,

$$\lim_{x \to 0} \frac{f(x) - (f(0) + \sum_{i=1}^{k} a_i \delta_i(x))}{\delta_k(x)} = 0 \tag{13}$$

  The notation

$$f(x) \sim f(0) + \sum_{i=1}^{n} a_i \delta_i(x)$$

  expresses the asymptotic relations in (13).

- In regular perturbations and Taylor series expansions using the parameter ϵ, the sequence of gauge functions is $\delta_k(\epsilon) = \epsilon^k$.

- Some functions have asymptotic expansions which are not Taylor expansions.
  - $e^{x^{1/3}}$ does not have a Taylor expansion around $x = 0$
  - However, $e^{x^{1/3}}$ has an asymptotic expansion

$$e^{x^{1/3}} \sim 1 + x^{1/3} + \frac{1}{2}(x^{1/3})^2 + \frac{1}{6}(x^{1/3})^3 + \cdots \tag{14}$$

The Method of Undetermined Gauges: An Adverse Selection Example

- Recall the RSW model of adverse selection.
  - Assume that the high (low) risk type receives observable income of 1 with probability q (p), $q > p$, and zero otherwise.
  - In the Nash equilibrium, the high risk type consumes q in each state, and the low-risk type consumes x in the good state and y otherwise.
  - x and y are fixed by the incentive compatibility condition

$$u(q) = pu(x) + (1-p)u(y)$$

    and the zero-profit condition,

$$p = (1-p)y + px$$

  - Substitution yields the single equation for x

$$u(q) = pu(x) + (1-p)u\left(\frac{p(1-x)}{1-p}\right) \tag{15}$$

- Perturbation approach.
  - If $q = p$, then $x = y = p = q$ is the unique solution.
  - We seek solutions to (15) for $q = p - \epsilon$ where ϵ is small, modelling situations where the risk differences are small.
  - The solution, x, will depend on ϵ and is implicitly defined by

$$u(p-\epsilon) = pu(x(\epsilon)) + (1-p)u\left(\frac{p(1-x(\epsilon))}{1-p}\right) \tag{16}$$

- Regular perturbation approach.
  - Suppose $x(\epsilon) \sim \sum_{i=1}^{\infty} \alpha_i \epsilon^i$.
  - Taylor series expansion of (16) around $\epsilon = 0$ implies

$$\begin{aligned} u(p) \quad & -u'(p)\epsilon + \tfrac{1}{2}u''(p)\epsilon^2 + \dots \\ & = p(u(p) + u'(p)(\alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots) \\ & \qquad + \tfrac{1}{2}u''(p)(\alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots)^2 + \cdots) \\ & \quad + (1-p)(u(p) + u'(p)(\tfrac{-p}{1-p})(\alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots) \\ & \qquad\qquad + \tfrac{1}{2}(\tfrac{p}{1-p})^2 u''(p)(\alpha_1\epsilon + \alpha_2\epsilon^2 + \cdots)^2 + \cdots) \end{aligned}$$

  - Combining like terms, we find

$$\begin{aligned} u(p) \quad & -u'(p)\epsilon + \tfrac{1}{2}u''(p)\epsilon^2 + \cdots \\ & = u(p)(p + (1-p)) + \epsilon\left(p\alpha_1 u'(p) - p\alpha_1 u'(p)\right) \\ & \quad + \epsilon^2\left(p\alpha_2 u'(p) + \tfrac{1}{2}p\alpha_1 u''(p) - p(\alpha_2 u'(p) + \tfrac{1}{2}p\alpha_1 u''(p)\right) + \cdots \end{aligned}$$

  - This implies the impossible (since $u'(p) \neq 0$)

$$-u'(p)\epsilon + \frac{1}{2}u''(p)\epsilon^2 = 0 + \mathcal{O}(\epsilon^3)$$

- Method of Undetermined Gauges
  - Suppose instead that for some sequence $0 < \nu_1 < \nu_2 < ...$ and a_i, $i = 1, 2, 3, ...,$

$$x(\epsilon) \sim p + \sum_{i=1}^{\infty} \alpha_i \epsilon^{\nu_i} \tag{17}$$

    Regular perturbation assumed the restrictive form $\sum_{i=1}^{\infty} \alpha_i \epsilon^i$
  - Substitute (17) into Taylor expansion of (16), we get

$$\begin{aligned} u(p) \quad & -u'(p)\epsilon + \tfrac{1}{2}u''(p)\epsilon^2 \\ & = p(u(p) + u'(p)(\alpha_1\epsilon^{\nu_1} + \alpha_2\epsilon^{\nu_2} + \cdots) \\ & \quad + \tfrac{1}{2}u''(p)(\alpha_1\epsilon^{\nu_1} + \alpha_2\epsilon^{\nu_2} + \cdots)^2 + \cdots) \\ & \quad + (1-p)(u(p) + u'(p)(-\tfrac{p}{1-p})(\alpha_1\epsilon^{\nu_1} + \alpha_2\epsilon^{\nu_2} + \cdots) \\ & \quad + \tfrac{1}{2}u''(p)(\tfrac{p}{1-p})^2(\alpha_1\epsilon^{\nu_1} + \alpha_2\epsilon^{\nu_2} + \cdots)^2 + \cdots) \end{aligned}$$

  - Combining like terms yields

$$\begin{aligned} -u'(p)\epsilon \quad & + \tfrac{1}{2}u''(p)\epsilon^2 + \cdots \\ & = \tfrac{1}{2}u''(p)p(\alpha_1^2\epsilon^{2\nu_1} + 2\alpha_1\alpha_2\epsilon^{\nu_1+\nu_2} + \alpha_2^2\epsilon^{2\nu_2} + \cdots) \\ & \quad + \tfrac{1}{2}u''(p)\tfrac{p^2}{1-p}(\alpha_1^2\epsilon^{2\nu_1} + 2\alpha_1\alpha_2\epsilon^{\nu_1+\nu_2} + \alpha_2^2\epsilon^{2\nu_2} + \cdots) \end{aligned} \tag{18}$$

  - Since $0 < \nu_1 < \nu_2 < \nu_3 < \cdots$, we know that $2\nu_1 < \nu_1 + \nu_2 < 2\nu_2$, and that $2\nu_2$ is less than the power of any ϵ term which is not in (4.11). Therefore, the dominant term as $\epsilon \to 0$ on the RHS is $\epsilon^{2\nu_1}$, the next dominant is $\epsilon^{\nu_1+\nu_2}$, etc.
  - Matching dominant terms of each side in (18) implies $\epsilon = \epsilon^{2\nu_1}$.and

$$0 = \epsilon(u'(p) + \frac{1}{2}p\alpha_1^2u''(p) + \frac{1}{2}\frac{p^2}{1-p}\alpha_1^2u''(p))$$

    which implies that

$$\alpha_1 = \sqrt{-2\frac{u'(p)}{u''(p)}\frac{1-p}{p}} \tag{19}$$

  - Further values of ν_i and α_i can be determined by examining more terms in the expansion of (16)