*Numerical Methods in Economics*

MIT Press, 1998

# Chapter 17 Notes
# Solving Rational Expectations Models

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November 20, 2002

# The Lucas Asset Pricing Model

- Model:
  - A single asset paying dividends, $y_{t+1}$:
$$y_{t+1} = \rho y_t + \varepsilon_{t+1}, \tag{17.1.1}$$
    where $\varepsilon_t$ is an i.i.d. innovation process.
  - Representative agent consumes only dividends and has utility function $E\left\{\sum_{t=0}^{\infty} \beta^t u(c_t)\right\}$.
  - $p_t$ price of one unit of asset at time $t$
  - Supply is normalized to be one.
  - Equilibrium price process must satisfy
$$u'(y_t)p_t = \beta E\left\{u'(y_{t+1})\left(y_{t+1} + p_{t+1}\right) \mid y_t\right\}. \tag{17.1.2}$$
- Recursive solution
  - Let $p(y)$ be the ex-dividend price of a share when the dividend is $y$
  - Lucas (1978) shows that $p(y)$ solves
$$u'(y_t)p(y_t) = \beta E\left\{u'(y_{t+1})\left(y_{t+1} + p(y_{t+1})\right) \mid y_t\right\}. \tag{17.1.3}$$
  - Rewrite (17.1.3) as
$$p(y_t) = \beta E\left\{\frac{u'(y_{t+1})}{u'(y_t)}\, y_{t+1}\right| \left. y_t\right\} + \beta E\left\{\frac{u'(y_{t+1})}{u'(y_t)}\, p(y_{t+1})\right| \left. y_t\right\}. \tag{17.1.4}$$
  - (17.1.4) is a linear Fredholm integral equation
$$p(y) = g(y) + \beta \int K(y,z)\, p(z)\, dz, \tag{17.1.5}$$
    where
$$\begin{aligned} g(y) &= \beta \int \tfrac{u'(z)}{u'(y)}\, z\, q(z|y)\, dz, \\ K(y,z) &= \tfrac{u'(z)}{u'(y)}\, q(z|y), \end{aligned}$$
    and $q(z|y)$ is the density of the time $t+1$ dividend conditional on the time $t$ dividend.

## Solution by Simulation

- Present value arguments show that $p(y)$ is

$$p(y_0) = (u'(y_0))^{-1} E\left\{ \sum_{t=1}^{\infty} \beta^t u'(y_t) \middle| y_0 \right\}. \tag{17.1.6}$$

- Sampling scheme:
  - Draw a sequence of $T$ i.i.d. innovations, $\varepsilon_t$
  - Form dividend sequence $y_t, t = 1, \cdots T$, implied by the process (17.1.1) with $y_0$ given.
  - Compute the sum

$$P(y_0; \varepsilon) = (u'(y_0))^{-1} \sum_{t=1}^{T} \beta^t u'(y_t). \tag{17.1.7}$$

  - Repeat this for several draws of $\varepsilon \in R^T$; denote the $j$'th sequence as $\varepsilon^j$, $j = 1, ..., m$.
  - Each $\varepsilon^j$ sequence implies a different dividend sequence and a different value for $P(y_0; \varepsilon^j)$.
  - Estimate of $p(y_0)$, denoted $\hat{p}(y_0)$, is the average of the $P(y_0; \varepsilon)$:

$$\hat{p}(y_0) = \frac{1}{m} \sum_{j=1}^{m} P(y_0; \varepsilon^j). \tag{17.1.8}$$

  - Repeat (17.1.8) for $N$ values of $y_0$, denoted $y_0^i$, $i = 1, ..., N$

- Simulation methods
  - Dominated by other methods if $y$ process has low dimension
  - Are appropriate if desired accuracy is moderate and dividend process is complicated. For example, if

$$y_t = a_0 + a_1 y_{t-1} + a_2 y_{t-2} + \cdots + a_\ell y_{t-\ell} + \varepsilon_t,$$

process is $\ell$-dimensional and would challenge other methods.

# Lucas Model: Discrete-State Approximation

- Tauchen (1991) recognized that (17.1.3) is a linear Fredholm integral equation of the second kind, and applied the procedures similar to those described in Section 10.8.
- Use Gauss-Hermite quadrature nodes $y_i$ and weights $\omega_i$.
- Prices $p\,(y_i)$, satisfy the linear equation

$$p\,(y_i) = g\,(y_i) + \beta \sum_{j=1}^{n} p\,(y_j)\, \frac{u'\,(y_j)}{u'\,(y_i)} \omega_j\, \frac{q(y_j \mid y_i)}{w\,(y_j)}. \qquad (17.1.9)$$

  We solve (17.1.9) to compute approximations to $p\,(y)$ for $y_i \in Y$; call the solutions $\hat{p}_i$. The procedure so far just approximates $p\,(y)$ on $Y$.

- To approximate $p$ globally
  - Use Nystrom extension, equation (10.8.10), to get

    $$\hat{p}\,(y) = g\,(y) + \beta \sum_{j=1}^{n} \hat{p}_j\, \frac{u'\,(y_j)}{u'\,(y)} \omega_j\, \frac{q(y_j \mid y)}{w\,(y_j)}, \qquad (17.1.10)$$

    which is defined for all $y$.
  - This is an example of projection method with Gauss-Hermite polynomials and orthogonal collocation

# Example: Stochastic Dynamic General Equilibrium

- Model

$$\max_{c_t} E\left\{\sum_{t=1}^{\infty} \beta^t u(c_t)\right\}$$
$$k_{t+1} = \theta_t f(k_t) - c_t$$
$$\ln \theta_{t+1} = \rho \ln \theta_t + \varepsilon_t$$

- Euler equation

$$u'(c_t) = \beta E\{u'(c_{t+1})\theta_{t+1} f'(k_{t+1}) | \theta_t\}$$

  – Consumption is determined by recursive function

$$c_t = C(k_t, \theta_t)$$

  – $C(k, \theta)$ satisfies functional equation

$$0 = u'(C(k,\theta)) - \beta E\left\{u'\left(C\left(\theta f(k) - C(k,\theta), \tilde{\theta}\right)\right) \times \tilde{\theta} f'(\theta f(k) - C(k,\theta)) \mid \theta\right\}$$

- Transform Euler equation into the more linear form

$$\begin{aligned} 0 &= C(k,\theta) - (u')^{-1}\left(\beta E\left\{u'\left(C(\theta f(k) - C(k,\theta), \tilde{\theta})\right) \times \tilde{\theta} f'\left(\theta f(k) - C(k,\theta)\right) \mid \theta\right\}\right) \\ &\equiv \mathcal{N}(C)(k,\theta) \end{aligned}$$

- Approximate policy function

$$\widehat{C}(k,\theta;\ \mathbf{a}) = \sum_{i=1}^{n_k} \sum_{j=1}^{n_\theta} a_{ij}\psi_{ij}(k,\theta)$$

$$\psi_{ij}(k,\theta) \equiv T_{i-1}\left(2\frac{k-k_m}{k_M-k_m}-1\right)T_{j-1}\left(2\frac{\theta-\theta_m}{\theta_M-\theta_m}-1\right)$$

- Define integrand of expectations

$$I(k,\theta,\ \mathbf{a},z) = u'\left(\widehat{C}\left(\theta f(k) - \widehat{C}(k,\theta;\ \mathbf{a}), e^{\sigma z}\theta^\rho,\ \mathbf{a}\right)\right) \times e^{\sigma z}\theta^\rho f'\left(\theta f(k) - \widehat{C}(k,\theta;\ \mathbf{a})\right)\pi^{-\frac{1}{2}}$$

- $\mathcal{N}\left(\widehat{C}\left(\cdot,\cdot;\ \mathbf{a}\right)\right)(k,\theta)$ becomes

$$\widehat{C}(k,\theta;\ \mathbf{a}) - (u')^{-1}\left(\beta\int_{-\infty}^{\infty} I(k,\theta;\ \mathbf{a},z)\ \frac{e^{-z^2/2}}{\sqrt{2\pi}}\ dz\right)$$

- Use Gauss-Hermite quadrature over $z$:

$$\int_{-\infty}^{\infty} I(k,\theta,\ \mathbf{a},z)\ \frac{e^{-z^2/2}}{\sqrt{2}}\ dz \doteq \sum_{j=1}^{m_z} I\left(k,\theta,\ \mathbf{a},\sqrt{2}z_j\right)\omega_j$$

where $\omega_j, z_j$ are Gauss-Hermite quadrature weights and points.

- The computable residual function is

$$R(k,\theta;\ \mathbf{a}) = \widehat{C}(k,\theta;\ \mathbf{a}) - (u')^{-1}\left(\beta\sum_{j=1}^{m_z} I\left(k,\theta,\ \mathbf{a},\sqrt{2}z_j\right)w_j\right) \equiv \widehat{\mathcal{N}}\left(\widehat{C}(\cdot,\cdot;\ \mathbf{a})\right)(k,\theta).$$

- Fitting Criteria:
  - Collocation:
    * Choose $n_k$ capital stocks, $\{k_i\}_{i=1}^{n_k}$, and $n_\theta$ productivity levels, $\{\theta_i\}_{j=}^{n_\theta}$
    * Find $\mathbf{a}$ such that

$$R(k_i, \theta_j;\ \mathbf{a}) = 0,\ i = 1, \cdots, n_k,\ j = 1, \cdots, n_\theta$$

  - Galerkin:
    * Compute the $n_k n_\theta$ projections

$$P_{ij}(\ \mathbf{a}) \equiv \int_{k_m}^{k_M} \int_{\theta_m}^{\theta_M} R(k, \theta;\ \mathbf{a}) \psi_{ij}(k, \theta) d\theta dk$$

    * Approximate projections by Gauss-Chebyshev quadrature

$$\hat{P}_{ij}(\mathbf{a}) \equiv \sum_{\ell_k=1}^{m_k} \sum_{\ell_\theta=1}^{m_\theta} R(k_i, \theta_j;\ \mathbf{a}) \psi_{ij}(k_{\ell_k}, \theta_{\ell_\theta}),$$

where

$$k_{\ell_\theta} = k_m + \frac{1}{2}(k_M - k_m)\left(z_{\ell_k}^{m_k} + 1\right),\ \ell_k = 1, \ldots, m_k$$

$$\theta_{\ell_\theta} = \theta_m + \frac{1}{2}(\theta_M - \theta_m)\left(z_{\ell_\theta}^{m_\theta} + 1\right),\ \ell_\theta = 1, \ldots, m_\theta$$

$$z_\ell^n \equiv \cos\left(\frac{(2i-1)\pi}{2n}\right),\ \ell = 1, \ldots, n$$

    * Coefficients, $\mathbf{a}$, are fixed by the system

$$\hat{P}_{ij}(\mathbf{a}) = 0,\ i = 1, \cdots, n_k,\ j = 1, \cdots, n_\theta$$

- Bounded Rationality Accuracy Measure
  - Consider the computable Euler equation error

$$E(k,\theta) = \frac{\widehat{\mathcal{N}}(\widehat{C}(\cdot,\cdot;\ \mathbf{a}))(k,\theta)}{\widehat{C}(k,\theta;\ \mathbf{a})}$$

    where $\widehat{\mathcal{N}}$ uses some integration formula for $E\{\cdot\}$; need not be the same as used in computing $R(k,\theta;\mathbf{a})$. In fact, should use better one.
  - Define the $L^p$, $1 \leq p < \infty$, *bounded rationality accuracy* to be

$$\log_{10} \parallel E(k) \parallel_p$$

- Verify solution: Accept solution to projection equations, $\mathbf{a}$, only if it passes tests
  - Check stability
    - For example, there should be positive savings at low $k$, high $\theta$
    - Could simulate capital stock process implied by $\widehat{C}(k,\theta;\ \mathbf{a})$ to see if it has a stationary distribution
  - Check Euler equation errors
    - $E(k,\theta)$ should be moderate for most $(k,\theta)$ points in $[k_m, k_M] \times [\theta_m, \theta_M]$
    - $E(k,\theta)$ should be small for most $(k,\theta)$ points frequently visited
  - If $\widehat{C}(k,\theta;\ \mathbf{a})$ does not pass these tests, go back and use higher values for $n_k$ and $n_\theta$, and increase $m_k$, and $m_\theta$

- Numerical Results
  - Basis: Chebyshev polynomials
  - Initial guess: Linear rule through deterministic steady state and zero.
  - $k \in [.333, 2.000]$
  - Method: Collocation and Galerkin.
  - Newton's method solved projection equations, $P_i(\mathbf{a}) = 0$, for $\mathbf{a}$.
  - Machine: Compaq 386/20 (old, but relative speeds are still valid)
  - Speed: Stochastic case: under two minutes for a 60 parameter fit.
  - Errors: 2% for 6 parameter fit, .1% for 60 parameter fit – about a penny loss per $10,000 dollar expenditure
  - Orth. poly. + orthog. collocation + Gaussian quad. + Newton outperforms naive methods by factor of 10 or greater; exceeded Monte Carlo methods by factor of 100+.
  - $\widehat{C}(k, \theta;\ \mathbf{a})$ is an $\varepsilon$-equilibrium with small $\varepsilon$ – a bounded rationality interpretation.

Table 17.1: $\log_{10}$ Euler Equation Errors

| | | | $\| E \|_\infty$ | $\| E \|_1$ | $\| E \|_\infty$ | $\| E \|_1$ |
|---|---|---|---|---|---|---|
| $\gamma$ | $\rho$ | $\sigma$ | (2, 2, 2, 2)* | | (4, 3, 4, 3) | |
| −15.00 | 0.80 | 0.01 | −2.13 | −2.80 | −3.00 | −3.83 |
| −15.00 | 0.80 | 0.04 | −1.89 | −2.54 | −2.44 | −2.87 |
| −15.00 | 0.30 | 0.04 | −2.13 | −2.80 | −2.97 | −3.83 |
| − 0.10 | 0.80 | 0.01 | −0.01 | −1.22 | −1.68 | −2.65 |
| − 0.10 | 0.80 | 0.04 | 0.01 | −1.19 | −1.48 | −2.22 |
| − 0.10 | 0.30 | 0.04 | 0.18 | −1.22 | −1.63 | −2.65 |
| | | | (7, 5, 7, 5) | | (7, 5, 20, 12) | |
| −15.00 | 0.80 | 0.01 | −4.28 | −5.19 | −4.43 | −5.18 |
| −15.00 | 0.80 | 0.04 | −3.36 | −4.00 | −3.30 | −3.95 |
| −15.00 | 0.30 | 0.04 | −4.24 | −5.19 | −4.38 | −5.18 |
| − 0.10 | 0.80 | 0.01 | −3.40 | −4.37 | −3.47 | −4.39 |
| − 0.10 | 0.80 | 0.04 | −2.50 | −3.22 | −2.60 | −3.17 |
| − 0.10 | 0.30 | 0.04 | −3.43 | −4.37 | −3.49 | −4.39 |
| | | | (10, 6, 10, 6) | | (10, 6, 25, 15) | |
| −15.00 | 0.80 | 0.01 | −5.48 | −6.43 | −5.61 | −6.42 |
| −15.00 | 0.80 | 0.04 | −3.81 | −4.38 | −3.88 | −4.37 |
| −15.00 | 0.30 | 0.04 | −5.45 | −6.43 | −5.57 | −6.42 |
| −0.10 | 0.80 | 0.01 | −5.09 | −6.12 | −5.17 | −6.15 |
| −0.10 | 0.80 | 0.04 | −2.99 | −3.68 | −3.09 | −3.64 |
| −0.10 | 0.30 | 0.04 | −5.17 | −6.12 | −5.23 | −6.14 |

*$(n_k,\ n_\theta,\ m_k,\ m_\theta)$

Table 17.2: Alternative Implementations

$n_k = 7, n_\theta = 5, m_k = 7, m_\theta = 5$

| $\gamma$ | $\rho$ | $\sigma$ | G[a] error[e] | G[a] time | P[b] error | P[b] time | U[c] error | U[c] time | UP[d] error | UP[d] time |
|---|---|---|---|---|---|---|---|---|---|---|
| −15 | .8 | .04 | −3.18 | 1:15 | −2.13 | :40 | −3.06 | 1:05 | −2.19 | :44 |
| | .3 | .01 | −4.35 | :11 | −4.35 | :52 | −4.07 | :08 | −4.07 | 1:47 |
| −.9 | .8 | .04 | −3.43 | :05 | −3.43 | :19 | −3.42 | :08 | −3.42 | :39 |
| | .3 | .01 | −4.03 | :07 | −4.03 | :30 | −3.76 | :07 | −3.76 | 1:10 |
| −.1 | .8 | .04 | −2.50 | :07 | −2.50 | :41 | −2.52 | :06 | −2.52 | :42 |
| | .3 | .01 | −3.42 | :08 | −3.42 | 1:30 | −3.18 | :07 | −3.18 | :24 |

$n_k = 10, n_\theta = 6, m_k = 25, m_\theta = 15$

| $\gamma$ | $\rho$ | $\sigma$ | G[a] error[e] | G[a] time | P[b] error | P[b] time | U[c] error | U[c] time | UP[d] error | UP[d] time |
|---|---|---|---|---|---|---|---|---|---|---|
| −15 | .8 | .04 | −3.87 | 4:20 | −3.90 | 24:44 | −3.90 | 3:41 | −3.36 | 42:15 |
| | .3 | .01 | −5.68 | 2:19 | −5.14 | 11:31 | −5.49 | 2:14 | −5.30 | 8:06 |
| −.9 | .8 | .04 | −4.00 | 1:31 | −4.00 | 5:17 | −4.01 | 1:31 | −4.01 | 5:02 |
| | .3 | .01 | −5.40 | 1:23 | −4.63 | 7:13 | −5.25 | 1:20 | −5.13 | 6:01 |
| −.1 | .8 | .04 | −3.09 | 1:31 | −3.09 | 9:16 | −3.10 | 1:32 | −3.07 | 12:01 |
| | .3 | .01 | −5.27 | 1:32 | −4.02 | 7:25 | −5.09 | 1:27 | −3.27 | 8:32 |

[a]Chebyshev polynomial basis, Chebyshev zeroes used in evaluating fit

[b]Ordinary polynomial basis, Chebyshev zeroes used in evaluating fit

[c]Chebyshev polynomial basis, uniform grid points

[d]Ordinary polynomial basis, uniform grid points

[e]error measure is $\| E(k) \|_\infty$

Table 17.3: Tensor Product vs. Complete Polynomials[a]

| | | | Tensor Product | | | Complete Polynomials | | |
|---|---|---|---|---|---|---|---|---|
| $\gamma$ | $\rho$ | $\sigma$ | $n=3$ | $n=6$ | $n=10$ | $n=3$ | $n=6$ | $n=10$ |
| $-15.0$ | .8 | .04 | $-2.34$[b] | $-3.26$ | $-3.48$ | $-1.89$ | $-3.10$ | $-4.06$ |
| | | | :01[c] | :13 | 14:21 | :03 | :07 | 1:09 |
| $-.9$ | .3 | .10 | $-2.19$ | $-3.60$ | $-5.27$ | $-2.14$ | $-3.55$ | $-5.22$ |
| | | | :01 | :08 | 1:21 | :01 | :05 | :32 |
| $-.1$ | .3 | .01 | $-1.00$ | $-2.84$ | $-5.21$ | $-0.99$ | $-2.83$ | $-5.17$ |
| | | | :01 | :08 | 1:24 | :01 | :05 | :35 |

[a] Tensor product cases used orthogonal collocation with $n_k = n_\theta = m_k = m_\theta = n$ to identify the $n^2$ free parameters. Complete polynomial cases used Galerkin projections to identify the $1+n+n(n+1)/2$ free parameters.

[b] $\log_{10} \| E \|_\infty$

[c] Computation time expressed in minutes :seconds.

- General Observations:
  - Tensor product of degree $n$ takes more time, but achieves higher accuracy
  - For a specific level of accuracy, complete polynomial method is faster

# Examples: Multiagent Dynamic General Equilibrium

- Model:
  - $n$ types of agents, utility functions, $u_i(c)$, $i = 1, 2, ..., n$,
  - Common discount factor $\beta$.
  - Equity is the only asset
  - $c_i = C_i(k)$, wealth distribution is $k = (k_1, k_2, ..., k_n)$
- Approximate $c_i = \widehat{C}_i(k, \theta; \mathbf{a})$.
- Euler equation for type $i = 1, 2, ..., n$

$$\begin{aligned} R_i(k, \theta, C) &= u_i'(C_i(k, \theta)) - \beta\, E\left\{u'(C_i(Y(k, \theta) - C(k, \theta), \tilde{\theta}))\right. \\ &\left. \times F_k(Y(k, \theta) - C(k, \theta), \tilde{\theta}) \mid \theta\right\} \end{aligned}$$

where

$$\begin{aligned} Y_i(k, \theta) &= k_i F_1(\overline{k}, \theta) + w(\overline{k}, \theta),\ \ i = 1, .., n \\ w(\overline{k}, \theta) &= F(\overline{k}, \theta) - \overline{k} F_1(\overline{k}, \theta) \\ \overline{k} &\equiv \sum_i k_i. \end{aligned}$$

- Approximate residual function for agents of type $i = 1, 2, ..., n$

$$\begin{aligned}\widehat{R}_i(k, \theta, \widehat{C}(\cdot; \mathbf{a})) &= \widehat{C}_i(k, \theta; \mathbf{a}) - (u_i')^{-1}\left(\beta\, \widehat{E}\left\{u_i'\left(c^+\right) F_k\left(k_\cdot^+, \tilde{\theta}\right) \mid \theta\right\}\right) \\ & \quad c_i^+ \equiv \widehat{C}_i(y^+, \tilde{\theta}; \mathbf{a}) \\ & \quad k^+ \equiv Y(k, \theta; \mathbf{a}) - \widehat{C}(k, \theta; a)\end{aligned}$$

  where $\widehat{E}$ is a numerical approximation of the integral. Use product Gaussian quadrature

- Identifying projections are

$$P_{ij}(\mathbf{a}) \equiv \int_{\theta_m}^{\theta_M} \int_{k_m}^{k_M} \cdots \int_{k_m}^{k_M} \widehat{R}_i(k, \theta, \widehat{C}(\cdot; \mathbf{a}))\, \psi_j(k, \theta)\, w(k, \theta)\, dk_1 \cdots dk_n d\theta$$

  where $i = 1, ..., n$, and $j = 1, ..., m$.

- Let $\widehat{P}(\mathbf{a})$ denote a numerical integration approximation of $P(\mathbf{a})$; we will use product Gaussian quadrature

- Solution chooses $\mathbf{a}$ so that $\widehat{P}(\mathbf{a}) = 0$.

# Representation: Tensor vs. Complete Polynomials

- Tensor method:

$$\widehat{C}_i(k,\theta;\mathbf{a}) = \sum_{j_1=0}^{n_k} \cdots \sum_{j_n=0}^{n_k} \sum_{\ell=0}^{n_\theta} a^i_{j_1 \cdots j_n \ell}\, \varphi_{i_1}(k_1) \cdots \varphi_{i_n}(k_n)\, \psi_\ell(\theta), \quad i = 1, ..., n$$

  where $\varphi_i(k_j)$ $(\psi_\ell(\theta))$ is a degree $i-1$ $(\ell-1)$polynomial in $k_j$ $(\theta)$ from some orthogonal family.

- Complete polynomial method

$$C_i(k,\theta;\mathbf{a}) = \sum_{\substack{0 \le j_1 + \cdots + j_n + \ell \le d \\ 0 \le j_i, \ell \le d}} a^i_{j_1 \cdots j_n \ell}\, \varphi_{j_1}(k_1) \ldots \varphi_{j_n}(k_n)\, \psi_\ell(\theta)$$

- Number of unknown coefficients are far smaller in complete poly case, but not as flexible.

## Solution Methods

- Successive Approximation: at grid of $(k, \theta)$ points (e.g., Chebyshev zeroes) and given iteration $j$ for $\mathbf{a}$ (denoted $\mathbf{a}^j$), $\widehat{C}_i(k, \theta; \mathbf{a}^j)$, generate data

$$\begin{aligned}\widehat{C}_i(k, \theta; \mathbf{a}^{j+1}) &= (u')^{-1}\left(\beta\, \widehat{E}\, \left\{u'\left(\widehat{C}_i\left(Y(k,\theta) - \widehat{C}_i(k, \theta; \mathbf{a}), \tilde{\theta}; \mathbf{a}\right)\right)\right.\right. \\ &\quad \left.\left. \times F_k\left(Y(k,\theta) - \widehat{C}_i(k, \theta; \mathbf{a}^j), \tilde{\theta}\right) \mid \theta\right\}\right.\end{aligned} \tag{1}$$

  and set coefficients $\mathbf{a}^{j+1}$ through interpolation or regression

- Time Iteration: same procedure except not generate data for $\widehat{C}_i(k, \theta; \mathbf{a}^{j+1})$ by solving

$$\begin{aligned}\widehat{C}_i(k, \theta; \mathbf{a}^{j+1}) &= (u')^{-1}\left(\beta\, \widehat{E}\, \left\{u'\left(\widehat{C}_i\left(Y(k,\theta) - \widehat{C}_i(k, \theta; \mathbf{a}^{j+1}), \tilde{\theta}; \mathbf{a}^j\right)\right)\right.\right. \\ &\quad \left.\left. \times F_k\left(Y(k,\theta) - \widehat{C}_i(k, \theta; \mathbf{a}^{j+1}), \tilde{\theta}\right) \mid \theta\right\}\right.\end{aligned} \tag{2}$$

- Newton's Method: just solve nonlinear equations $\widehat{P}(\mathbf{a}) = 0$

Table 5: Time and Accuracy Comparisons

| agents | $\gamma$ | deg | basis | num. coef's | Newt's Method time | acc'cy | Succ.Approx: time | accuracy |
|---|---|---|---|---|---|---|---|---|
| 1 | -2 | 1 | t | 4 | :0.05 | -2.7 | :0.2 | -2.7 |
| | | | c | 3 | :0.06 | -2.6 | :0.4 | -2.6 |
| | | 2 | t | 9 | :0.22 | -3.4 | :01 | -3.4 |
| | | | c | 6 | :0.17 | -3.3 | :01 | -3.3 |
| | | 3 | t | 16 | :0.71 | -4.1 | :01 | -4.1 |
| | | | c | 10 | :0.49 | -4.0 | :02 | -4.0 |
| | | 4 | t | 25 | :02 | -4.8 | :02 | -4.9 |
| | | | c | 30 | :0.99 | -4.7 | :03 | -4.6 |
| 2 | -1 | 1 | t | 16 | :0.66 | -3.1 | :01 | -3.1 |
| | -2 | | c | 6 | :0.38 | -2.7 | :01 | -2.7 |
| | | 2 | t | 54 | :07 | -4.1 | :08 | -4.1 |
| | | | c | 20 | :02 | -3.4 | :06 | -3.4 |
| | | 3 | t | 128 | 1:22 | -5.0 | :33 | -4.5 |
| | | | c | 40 | :11 | -4.1 | :21 | -4.1 |
| | | 4 | t | 250 | 12:34 | -5.9 | 1:48 | -4.5 |
| | | | c | 70 | :45 | -4.8 | :56 | -4.7 |

Note: "inf" means infeasible. "$h$ hrs $n : m.l$" means "$h$ hours $n$ minutes, $m.l$ seconds".

Table 5: Time and Accuracy Comparisons (Continued)

| | | | | num | Newt's Method: | | Succ. Approx.: | |
|---|---|---|---|---|---|---|---|---|
| agents | $\gamma$ | deg | basis | coef's | time | accuracy | time | accuracy |
| 3 | -1 | 1 | t | 48 | :07 | -3.4 | :07 | -3.4 |
| | -2 | | c | 15 | 1.48 | -2.8 | :05 | -2.8 |
| | -3 | 2 | t | 243 | 7:07 | -4.6 | 2:11 | -4.5 |
| | | | c | 63 | :21 | -3.6 | :36 | -3.6 |
| | | 3 | t | 768 | inf | inf | 19:57 | -4.6 |
| | | | c | 105 | 4:05 | -4.3 | 3:09 | -4.3 |
| | | 4 | t | 1875 | inf | inf | 1 hr 56 | -4.6 |
| | | | c | 210 | 46:58 | -4.9 | 12:45 | -4.8 |
| 4 | -.5 | 1 | t | 128 | 1:09 | -3.5 | :33 | -3.5 |
| | -1 | | c | 24 | :5.10 | -2.9 | :13 | -2.9 |
| | -2 | 2 | t | 972 | inf | inf | 24:57 | -4.6 |
| | -3 | | c | 84 | 2:47 | -3.7 | 3:04 | -3.7 |
| | | 3 | t | 4096 | inf | inf | 7 hr 13 | -4.6 |
| | | | c | 224 | 52:11 | -4.4 | 26:01 | -4.4 |
| 5 | -.5 | 1 | t | 320 | 8:52 | -3.6 | 2:48 | -3.6 |
| | -1 | | c | 35 | :17.90 | -3.0 | :38 | -3.0 |
| | -2 | 2 | t | 3645 | inf | inf | 5 hr 16 | -4.6 |
| | -3 | | c | 140 | 12:18 | -3.8 | 10:18 | -3.8 |
| | -4 | 3 | t | 20,480 | inf | inf | inf | inf |
| | | | c | 420 | 13 hr | -4.5 | 3 hr 27 | -4.5 |

Note: "inf" means infeasible. "$h$ hrs $n : m.l$" means "$h$ hours $n$ minutes, $m.l$ seconds".

## Table 7: Final Comparisons

| Method: | Basis: | Solution Method: | Advantages: | Disadvantages: |
|---|---|---|---|---|
| Taylor Series | Complete | Eigenvalues, linear eq'ns | Fast | Local validity |
| Projection methods | Tensor or complete | Newton | Quadratic conv. | Infeasible for large problems |
| | Tensor or complete | Successive approx. | Easy Iterations | possible nonconv. |

# Solving Asymmetric Information Asset Models

- General Problem
  - Different agents have different information
  - Question: how much information is revealed by information?
- Grossman (1976)
  - Find all information revealed by trading
  - Finds no incentive to acquire information
  - Assumed special functional forms
  - Assumed and limited type of assets
- More generally
  - Equilibrium is fully revealing if prices are continuous and states finite
  - Equilibrium is often not fully revealing
  - Need more general models which are tractable

- A Gamma-Gaussian Model
  - Investors have $W$ to invest in two assets
  - Safe asset – $R$
  - Risky asset – $\log \tilde{Z}$ is $N(m, w)$.
  - $m \sim N(\mu_m, \sigma_m^2)$ and $w \sim \Gamma(\alpha, \beta)$ (If the variance of $w$ is zero, we have Grossman's model.)
  - Type $i$ informed traders know $y_i \sim N(m, w)$ plus Gaussian noise, $i = 1, 2, 3$.
  - $\omega^i$ is number of shares held by type $i = 1, 2, 3$
  - First-order conditions: Agent $i$, $i = 1, 2, 3$, knows $p$ and $y_i$. His FOC for $\omega^i(p, y_i)$ is

$$0 = E_{y,Z} \left\{ u_i'(C_i(y, Z))(Z - p(y) R) \mid p, y_i \right\} \qquad (17.3.2)$$
$$C_i(y, Z) \equiv (W - \omega_i(p(y), y_i) p(y)) R + \omega_i(p(y), y_i) Z$$

- Equilibrium:
  - $\omega^i(p, y_i)$ satisfying FOC
  - Market-clearing: $p(y)$ satisfying

$$1 = \sum_{i=1,2,3} \omega^i (p(y), y_i))$$

  for all states $y$.

Numerical implementation of the conditional expectation:

- Definition of conditional expectation:

$$Z(X) = E\{Y \mid X\}$$

if and only if for all continuous functions $\phi$

$$\begin{aligned} 0 &= E\{(Z(X) - Y)\phi(X)\} \\ &= \int (Z(X(\omega)) - Y(\omega))\phi(X(\omega))d\omega \end{aligned}$$

- The definition replaces the conditional expectation with an infinite number of unconditional expectation conditions.
- Numerically: We accept $\widehat{Z}(X)$ as an approximation to $Z(X)$ if

$$\begin{aligned} 0 &= E\left\{(\widehat{Z}(X) - Y)\phi_i(X)\right\} \\ &= \int (\widehat{Z}(X(\omega)) - Y(\omega))\phi_i(X(\omega))d\omega, \ i = 1, ..., n \end{aligned}$$

for a finite number of $\phi_i(\cdot)$ functions.

# Numerical Approach

- Parameterize unknown functions
  - $H_i(\cdot)$ denotes the degree $i$ Hermite polynomial
  - price function:

$$p(y_1, y_2, y_3) = \sum_{\substack{0 \le j+k+l \le N_p \\ 0 \le j,k,\ell \le N_p}} a_{jk\ell} H_j(y_1) H_k(y_2) H_\ell(y_3)$$

  - stock demand for a type $i$ investor:

$$\omega_i(p, y_i) = \sum_{0 \le j+k \le N_\theta} b^i_{jk} H_j(p) H_k(y_i), \ i = 1, 2, 3$$

- Goal: determine the $a_{jk\ell}$ and $b^i_{jk}$ coefficients for various $N_\omega$ and $N_p$.
- The first-order-condition for a type $i$ investor
  - Theoretical

$$E_{y,Z} \left\{ U'(\tilde{c}_i)(\tilde{Z} - pR) \mid y_i, p \right\} = 0, \quad i = 1, 2, 3.$$

  - Numerical approximation

$$E_{y,Z} \left\{ U'(\tilde{c}_i)(\tilde{Z} - p(y) R) H_j(p(y)) H_k(y_i) \right\} = 0, \qquad (3)$$
$$j, k \ge 0, j + k \le N_\theta.$$

    * The $(i, j)$ condition says that the (excess return)$\times U'(c_i)$ is uncorrelated with $H_j(p(y)) H_k(y_i)$.
    * Eq'ns in (3) are integrals over $y_1, y_2, y_3, z, m, w$ - six dimensions

- Market clearing
  - Theoretical equilibrium condition: for all states $y$

$$1 = \sum_{i=1,2,3} \omega^i \left(p\left(y\right), y_i\right)\right)$$

  - Numerical approximations are

$$E_y \left\{ \left( \sum_{i=1}^{3} \omega_i(p\left(y\right), y_i) - 1 \right) H_j(y_1) H_k(y_2) H_\ell(y_3) \right\} = 0, \qquad (4)$$
$$j, k, l \geq 0, j + k + l \leq N_p$$

- Both are approximations
  - First-order conditions are satisfied only on average, not in any particular state.
  - Market-clearing does not hold in each state
  - Hope: magnitudes of errors are small and solution is close to true equilibrium.

- Numerical Results
  - Four- and Five-digit accuracy on models where we know results
  - Euler equation errors of 1$ per thousand for all of our models
  - Computation time less than 15 minutes on current machines
- Discretization Comments
  - Cannot discretize state space: generic full revelation
  - Must discretize in spectral domain
- Extensions
  - Endogenous information acquisition
  - Other assets - options
  - Two-period model with first-period volume information used in second period

## Summary of Projection Method

- Can be used for problems with unknown functions
- Uses approximation ideas
- Utilizes standard optimization and nonlinear equation solving software
- Can exploit a priori information about problem
- Flexible: users choose from a variety of approximation, integration, and nonlinear equation-solving methods

Table 17.4: Projection Method Menu

| Approximation | Integration | Projections | Equation Solver |
|---|---|---|---|
| Piecewise Linear | Newton-Cotes | Galerkin | Newton |
| Polynomials | Gaussian Rules | Collocation | Powell |
| Splines | Monte Carlo | M. of Moments | Fixed-pt. iteration |
| Neural Networks | Quasi-M.C. | Subdomain | Time iteration |
| Rational Functions | Monomial Rules | | Homotopy |
| Problem Specific | Asymptotics | | |

- Unifies literature: Previous work can be classified and compared

| Choices | | | |
|---|---|---|---|
| **Authors** | **Approximation** | **Integration** | **Sol'n Method** |
| Gustafson(1959) | piecewise linear | Newt.-Cotes | S.A.-time it. |
| Wright-W.(1982,4) | poly. (of cond. exp.) | Newt.-Cotes | S.A.-time it. |
| Miranda-H.(1986) | polynomials | Newt.-Cotes | S.A.-learning |
| Coleman(1990) | finite element | Gaussian | S.A.-time it. |
| den Haan-M.(1990) | poly. (of cond. exp.) | Sim. M.C. | S.A.-learning |
| Judd(1992) | orthogonal poly. | Gaussian | Newton |