*Numerical Methods in Economics*
MIT Press, 1998

# Notes for Chapter 8
# Monte Carlo and Simulation Methods

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# Monte Carlo and Simulation Methods

- Gaussian, monomial, and Newton-Cotes formulas
  - use predetermined nodes
  - aim at high accuracy
  - need many nodes
- Sampling methods
  - Generate a sequence of points
  - Short sequence produces rough approximation
  - Longer sequences produce better approximations
- Monte Carlo sampling methods
  - Use law of large numbers "intuition"
  - Order $N^{-1/2}$ convergence
  - Use probability theory to prove theorems
  - Use number-theoretic methods to generate deterministic sequences which appear random

# Monte Carlo Integration

- Probability theory
  - If $X_i$ are i.i.d. r.v.'s, density $q(x)$, and support $[0,1]$, then

$$\bar{X} \equiv \frac{1}{N}\sum_{i=1}^{N} X_i$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} X_i = \int_0^1 xq(x)\,dx, \ a.s.$$

$$\text{var}\left(\frac{1}{N}\sum_{i=1}^{N} X_i\right) = \frac{\sigma_x^2}{N}$$

  - If $\sigma_x^2$ is not known *a priori*, an unbiased estimator is

$$\hat{\sigma}_x^2 \equiv (N-1)^{-1}\sum_{i=1}^{N}(X_i - \bar{X})^2$$

- LLN suggests Monte Carlo quadrature:
  - If $X \sim U[0,1]$, then

$$I_f = \int_0^1 f(x)\,dx = E\{f(X)\}$$

  - The *crude Monte Carlo* method makes $N$ draws from $U[0,1]$, $\{x_i\}_{i=1}^N$, and defines

$$\hat{I}_f \equiv \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

$$\hat{\sigma}_f^2 = (N-1)^{-1}\sum_{i=1}^{N}\left(f(x_i) - \hat{I}_f\right)^2$$

- $\hat{I}_f$ is a statistical estimate of $\int_0^1 f(x)\,dx$
  - $\hat{I}_f$ is an unbiased estimate of $\int_0^1 f(x)\,dx$
  - The variance of the $\hat{I}_f$ estimate is

$$\sigma^2_{\hat{I}_f} = N^{-1} \int_0^1 \left(f(x) - I_f\right)^2 \, dx = N^{-1}\sigma_f^2$$

# Variance Reduction Techniques

- Monte Carlo estimates have high variance; need to reduce variance
- Antithetic Variates
  - Induce negative correlation in $f(x)$ values
  - Form the estimate

$$\hat{I}_f = \frac{1}{2N} \sum_{i=1}^{N} \left( f(x_i) + f(1 - x_i) \right).$$

  - If $f$ is monotone, $I_f$, has smaller variance than crude estimate
- Control Variates
  - Suppose $\varphi$ is similar to $f$ but easily integrated.
  - The identity $\int f = \int \varphi + \int (f - \varphi)$ reduces the problem to
    - a Monte Carlo integration of $\int (f - \varphi)$
    - plus the known integral $\int \varphi$.
  - If $cov\,(f, \varphi)$ is large, variance is reduced

- Importance Sampling
  - Sample $f(x)$ where its value is most important
  - If $p(x) > 0$, and $\int_0^1 p(x)\,dx = 1$, then

$$I = \int_0^1 f(x)\,dx = \int_0^1 \frac{f(x)}{p(x)}\,p(x)\,dx$$

  - If $x_i$ is drawn with density $p(x)$, then

$$\hat{I} = \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

    is an unbiased estimate of $I$, and variance of $\hat{I}$ is

$$\sigma^2_{\hat{I}} = \frac{1}{N}\left(\int_0^1 \frac{f(x)^2}{p(x)}\,dx - \left(\int_0^1 f(x)\,dx\right)^2\right).$$

  - If $f(x) > 0$ and $p(x) = f(x)/\int_0^1 f(x)$, then $f(x) = I\,p(x)$ and $\sigma^2_{\hat{I}} = 0$.
  - Add constant $B$ to make $f(x)$ positive
  - Aim: find a $p(x)$ similar to $f(x)$
  - Thin tails problem
    - In $\sigma^2_{\hat{I}}$ formula, key term is $f(x)^2/p(x)$
    - if $f(x)^2/p(x)$ is large when $p(x)$ is small, variance is large.
    - Normal density often has thin tails problem

# Pseudorandom Number Generation

- Random numbers are seldom used
  - Possible methods
    - Flipping coins
    - Geiger counters measuring radioactivity
  - Disadvantages
    - Expensive given RA salaries
    - RA's complain about radiation risk
- Monte Carlo propagandists
  - Use deterministic sequences
  - Act as if they are random sequences
- Pseudorandom numbers are used instead
  - They are deterministic sequences, $X_{k+1} = f(X_k, X_{k-1}, X_{k-2}, ...)$
  - They pass *some* randomness tests, such as
    - Unbiasedness

$$\frac{1}{N}\sum_{k=1}^{N} X_k = \mu \equiv E\{X\}$$

    - Zero serial correlation

$$0 = \sum_{k=1}^{N} (X_k - \mu)\, X_{k+1}$$

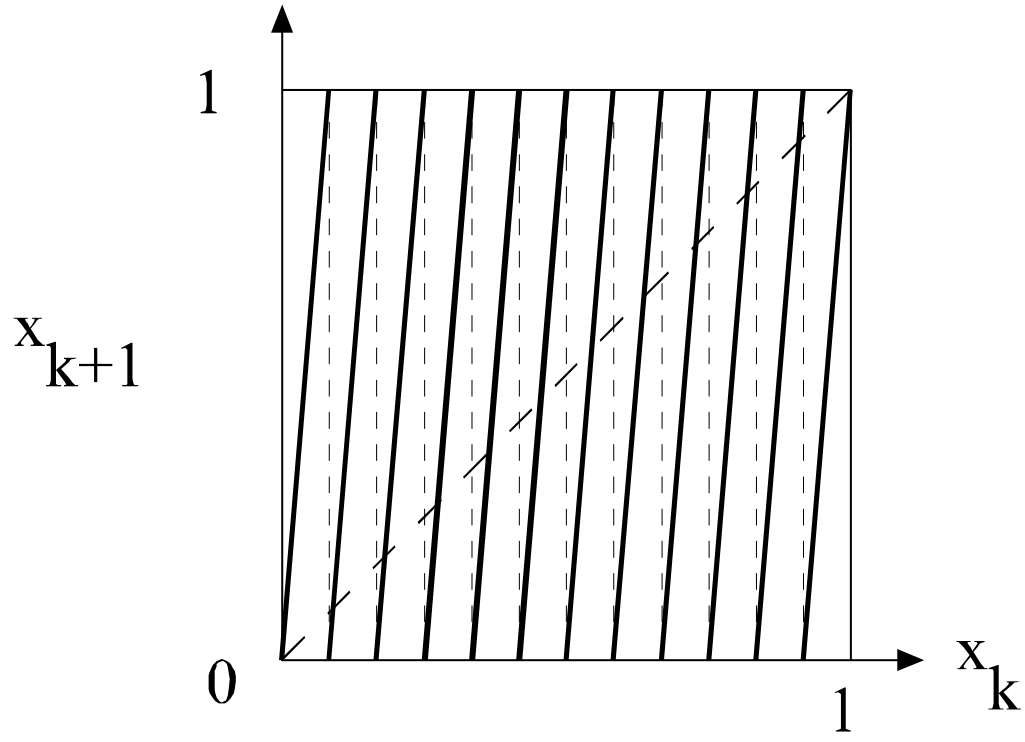
    - Runs tests
    - Lehmer: "each term is unpredictable to the *uninitiated* and .. digits pass a *certain number* of tests traditional with *statisticians*."
  - They *fail* Brock-Dechert-Scheinkman tests for randomness

# Uniform Random Number Generation

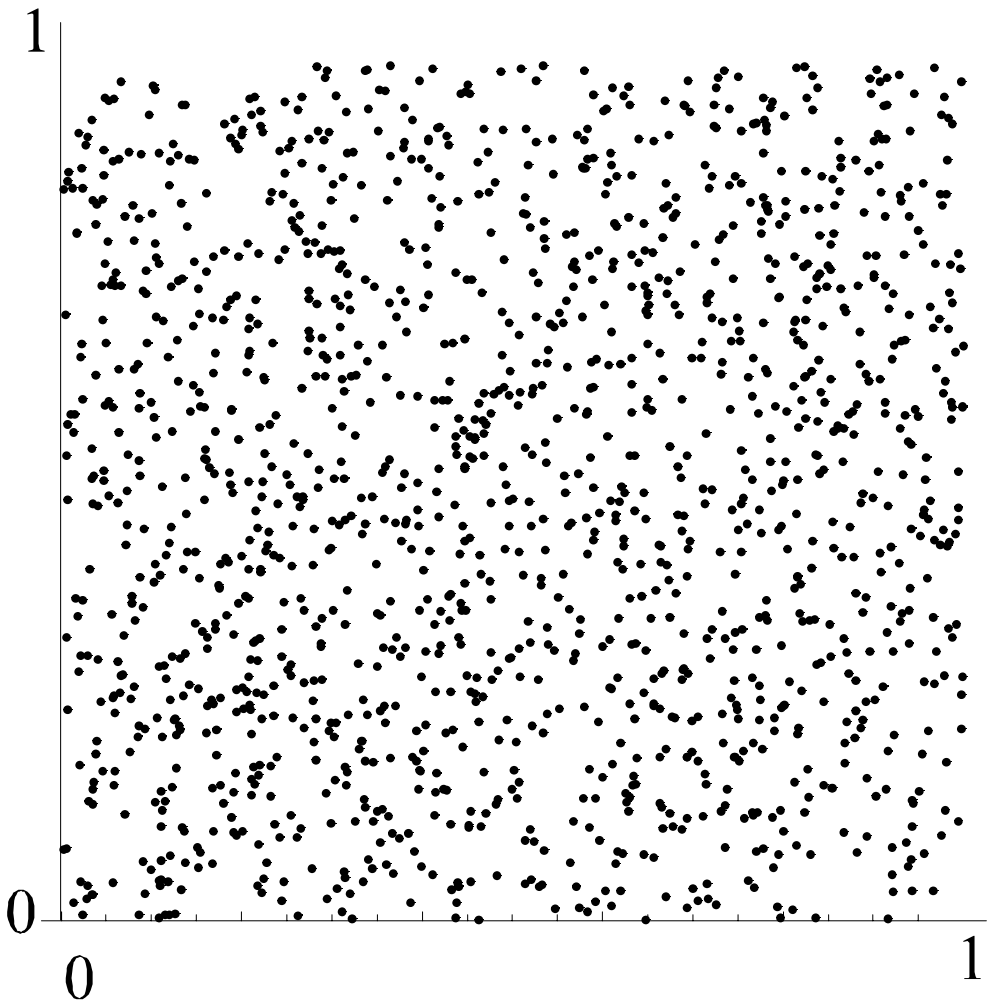
- Linear congruential method (LCM):

$$X_{k+1} = (aX_k + c) \mod m \tag{8.1.1}$$

  - Will generate pseudorandom sequence if parameters chosen well
  - Will eventually cycle; chose parameters to get long cycle
  - $Y_n \equiv (X_{2n+1}, X_{2n+2})$ is a pseudorandom two-dimensional set of points. Similar for $R^d$
  - LCM generators have fallen into disfavor since they lie on a finite set of hyperplanes.




Linear congruential method function




1500 Points generated by LCM

- Nonlinear schemes:
  - MPRNG: an example of LCM plus "random" shifts.
  - $X_{k+1} = f(X_k) \bmod m$
  - Fibonacci generator $X_k = (X_{k-1} + X_{k-2}) \bmod . m$. This sequence has a number of poor properties. In particular, if $X_{k-1}$ and $X_{k-2}$ are small relative to $m$, so will be $X_k$.
  - The Fibonacci-like scheme

$$X_k = (X_{k-24} \cdot X_{k-55}) \; mod 2^{32} \tag{8.1.2}$$

    has a period $\sim 10^{25}$ and passes many randomness tests.

# Nonuniform Random Number Generation

- Need to generate nonuniform random numbers
- Inversion:
  - Suppose $X$ has distribution $F(x)$
  - Then $F^{-1}(U)$ has distribution $F(x)$
  - To approximate $X$, we compute $y_k = F^{-1}(x_k)$ where $x_k$ is a uniform pseudorandom sequence
- Normal random variables: A special method
  - Suppose $U_1$ and $U_2 \sim U[0,1]$
  - Then $X_1$, $X_2 \sim N(0,1)$ are independent where

$$\begin{aligned} X_1 &= \cos(2\pi U_1)\sqrt{-2\ln U_2}\,, \\ X_2 &= \sin(2\pi U_1)\sqrt{-2\ln U_2}\,, \end{aligned} \tag{8.1.3}$$

# Stochastic Approximation

- Consider

$$\min_x E_Z\{g(x, Z)\}, \tag{8.4.1}$$

  where $Z$ is a random variable.

- Conventional methods are impractical for empirical problems, such as

$$\min_\beta E_Z\{g(\beta, X, Z)\}$$

  where $\beta$ are parameters, $X$ is data, and $Z$ is random

  - Too costly - curse of dimensionality
  - High-accuracy methods are not necessary in empirical problems since $X$ data is noisy.

- Econometricians frequently fix $S$, and minimize $\sum_{z\in S} g(\beta, X, z)$.

- *Stochastic approximation* is designed to deal with such problems.

  - Begin with initial guess $x^1$.
  - Draw $z^1$
  - $g_x(x^1, z^1)$ is an unbiased estimate of the gradient $f_x(x^1)$
  - Steepest descent method would change guess by $-\lambda_1 f_x(x^1)$ for some $\lambda_1 > 0$.
  - The *stochastic gradient method* executes the iteration

$$x^{k+1} = x^k - \lambda_k g_x(x^k,\ z^k), \tag{8.4.2}$$

    where $\{z^k\}$ is a sequence of i.i.d. draws from $Z$ and $\lambda_k$ is a changing step size.

**Theorem 1** *Suppose that $f$ is $C^2$. If $\lambda_k \to 0$, $\sum_{k=1}^{\infty} \lambda_k = \infty$, and $\sum_{k=1}^{\infty} \lambda_k^2 < \infty$, then the sequence $x^k$ generated by the stochastic gradient method, (8.4.2), confinded to $U$ will almost surely have a subsequence that converges to a point either on $\partial U$ or at a (possibly local) minimum of $f$.*

- Example:
  - $\min_{x \in [0,1]} E\{(Z - x)^2\}$, $Z \sim U[0, 1]$, with solution $x = 0.5$
  - Let $\lambda_k = 1/k$
  - (8.4.2) becomes

$$x_{k+1} = x_k + \frac{2}{k}(z_k - x_k), \tag{8.4.3}$$

Table 8.1: Statistics of (4.3) for 25 Runs

| Iterate | Average $x_k$ | Standard Deviation |
|---|---|---|
| 1 | .375 | .298 |
| 10 | .508 | .143 |
| 100 | .487 | .029 |
| 200 | .499 | .026 |
| 500 | .496 | .144 |
| 1000 | .501 | .010 |

# Standard Optimization Methods with Simulated Objectives

- Consider optimization problem:

$$\min_{x \in U} E\{g(x, Z)\} = f(x) \tag{8.5.1}$$

  for some random variable $Z$.

  - For many problems of the form in (5.1), the objective $f(x)$ and its derivatives can be computed only with nontrivial error.
  - When solving problems of the form (8.5.1) we need to determine how well we need to approximate the integral.
  - Stochastic approximation was one way to solve (8.5.1). We will now consider standard optimization approaches that use simulation ideas.

- Idea: take a sample of $Z$ of size $N$, and replace $E\{g(x, Z)\}$ in (8.5.1) with its sample mean $\frac{1}{N}\sum_{i=1}^{N} g(x, Z_i)$.

- For example, suppose that we want to solve

$$\min_{x \in [0,1]} E\{(Z - x)^2\}, \tag{8.5.2}$$

  where $Z \sim U[0, 1]$.

- To solve (8.5.2), we take, say, three draws from $U[0, 1]$; suppose they are 0.10, 0.73, and 0.49. We then minimize the sample average of $(Z - x)^2$,

$$\min_{x \in [0,1]} \frac{1}{3}((0.10 - x)^2 + (0.73 - x)^2 + (0.49 - x)^2). \tag{8.5.3}$$

  The solution to (8.5.3) is 0.43, a rough approximation of the true solution to (8.5.2) of 0.5.

- Simple portfolio problem. $u(c) = -e^{-c}$. safe asset has total return $R = 1.01$, and the risky asset has return $Z \sim N(\mu, \sigma^2)$ with $\mu = 1.06$ and $\sigma^2 = 0.04$. The portfolio problem reduces to

$$\max_{\omega} \; -E\{e^{-((1-\omega)R+\omega Z)}\}. \tag{8.5.4}$$

- Optimal $\omega$, denoted $\omega^*$, and equals 1.25.

- The Monte Carlo approach to solve (8.5.4) is to use Monte Carlo integration to evaluate the integral objective.

  – Take $N$ draws of $Z \sim N(\mu, \sigma^2)$, and replace (8.5.4) by

$$\max_{\omega} -\frac{1}{N}\sum_{i=1}^{N} e^{-((1-\omega)R+\omega Z_i)}. \tag{8.5.5}$$

  – Solution to (8.5.5) is $\hat{\omega}^*$; hopefully approximates $\omega^*$.

  – Quality of this procedure depends on $N$

Table 8.2: Portfolio Choice via Monte Carlo

| | $N^{-1}\sum_{i=1}^{N} u(c_i)$ | | $\widehat{\omega}^*$ | |
|---|---|---|---|---|
| $N$ | mean | Standard deviation | mean | Standard deviation |
| 100 | -1.039440 | .021362 | 1.2496 | .4885 |
| 1000 | -1.042647 | .007995 | 1.2870 | .1714 |
| 10,000 | -1.041274 | .002582 | 1.2505 | .0536 |

- Note: error in computing $\omega^*$ is much larger, ten to twenty times larger, than the error in computing an expectation.