Asymptotic Methods for Asset Market Equilibrium Analysis

Kenneth L. Judd*
Hoover Institution
Stanford, CA 94305

Sy-Ming Guu
Department of Industrial Engineering
Yuan-Ze University
Nei-Li, Taoyuan, Taiwan, R.O.C. 32026

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Abstract. General equilibrium analysis is difficult when asset markets are incomplete. We make the simplifying assumption that uncertainty is small and use bifurcation methods to compute Taylor series approximations for asset demand and asset market equilibrium. A computer must be used to derive these approximations since they involve large amounts of algebraic manipulation. To illustrate this method, we apply it to analyzing the allocative, price, and welfare effects of introducing a new derivative security.

1. Introduction

Precise analysis of equilibrium in asset markets is difficult since few cases can be solved exactly for equilibrium prices and volume. Many analyses assume that markets are complete, implying that equilibrium is efficient and equivalent to some social planner's problem. That approach is limited since it ignores transaction costs, taxes, and incompleteness in asset markets. This paper develops bifurcation methods to approximate asset market equilibrium without assuming complete asset markets. We begin from a trivial deterministic case where all assets have the same safe return and use local approximation methods to compute asset market equilibrium when assets have small risk. We compute Taylor series expressing equilibrium asset prices and holdings as a function of preference parameters such as absolute risk aversion, and

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asset return statistics such as mean, variance, and skewness. The formulas completely characterize equilibrium for small risks.

Implementing this approach is straightforward, but involves an enormous amount of algebraic manipulation far beyond the capacity of human hands. Fortunately, desktop computers using symbolic software can execute the necessary algebraic manipulation and compute the series expansions in reasonable time. We use Mathematica, but the computation could be executed by other symbolic languages such as Macsyma and Maple. The asymptotic expansions tell us about the qualitative properties of equilibrium and can be used to compute a numerical approximation to equilibrium of particular problems with a specified nonzero risk. Therefore, the bifurcation approach is computational in two ways: the formulas are qualitative asymptotic approximations derived by computer algebra, and can be used to produce numerical approximations to specific problems. This paper focuses on the qualitative asymptotic results and leaves the numerical applications for future study.

The result is essentially a mean-variance-skewness-etc. theory of asset demand and equilibrium pricing, similar to Samuelson's [22] analysis of asset demand. This approach is also more intuitive than the standard contingent state approach to equilibrium. The incomplete markets paradigm focuses on the difference between the number of contingent states and the number of assets. For example, welfare results in Hart [11], Cass and Citanna [3], and Elul [7] depend on how many assets are missing and the number of agents. It is difficult to interpret such indices of incompleteness since we can count neither the number of contingent states nor the number of different kinds of agents in a real economy. Furthermore, one expects that the impact of asset incompleteness on economic performance is related more to the statistical character of riskiness and the diversity of investor objectives than to the number of states and the number of agents. For example, the number of different agents is a poor measure of agent diversity since an economy with 100 types of investors with different risk aversions close to the mean risk aversion is less diverse than an economy with 10 types of investors with substantially different risk aversions. Similarly, the number of contingent states is at best a poor indicator of the magnitude and character of riskiness. This paper's analysis produces asymptotic formulas depending solely on the moments of asset returns and the differences in utility indices, showing that they, not the number of states, govern the asymptotic properties of equilibrium. Since moments are more easily observed in real markets than the number of contingent states the result is a more practical and intuitive approach to equilibrium analysis of asset markets.

Our approach is intuitive and similar in spirit to standard linearization and comparative static methods from mathematical economics. If fact, the analysis resembles Jones [12] classic analysis of international trade. Linearization methods based on the Implicit Function Theorem (IFT) are important computational tools that allow us to approximate nonlinear relationships with tractable, asymptotically valid approximations. We begin with the no-risk case where we know the equilibrium. We then use that information to compute equilibria for nearby cases of risky economies. However, the IFT does not apply here because the critical Jacobian is singular. In particular, when risk disappears all assets must become perfect substitutes and the portfolios of individuals are indeterminate when risk is zero. We cannot use the IFT if we do not know the equilibrium portfolio in the case of zero risk. Instead, we must apply tools from bifurcation theory to solve our problem. These tools are natural since they are essentially generalizations of L'Hospital's rule. Furthermore, because of the singularity at zero risk, we will need to compute higher-order approximations, not just the familiar first-order terms from linear approximation methods.

The purpose of this paper is to present the key mathematical ideas and illustrate them with basic economic applications. We first apply bifurcation methods to derive approximations of asset demand, refining the similar Samuelson [22] method. We then use these approximations of asset demand to compute approximations of asset market equilibrium. We compute asymptotically valid expressions for equilibrium with different asset combinations, and use them to show how changes in asset availability affects equilibrium.

The bifurcation approach is particularly interesting since it handles the complete and incomplete asset market cases in the same way. This contrasts sharply with the conventional approach where the incomplete asset market case is far more complex than the complete market case (see Magill and Quinzii [21] for a more complete discussion). We can do this because we focus on small risks. Since our analysis makes no assumptions about the span of assets, it is also a method for computing equilibrium in some economies with incomplete asset markets. This is generally a difficult problem because the excess demand function is not continuous. Brown et al. [2] and Schmedders [23] have formulated algorithms for computing equilibria when asset markets are incomplete. Their methods aim to compute equilibrium for any such model. Our method is only valid locally but is much faster since it relies on relatively simple and direct formulas.

The applications presented in this paper are just a small sampling of the possibilities. Guu and Judd [15] applies the results of this paper to compute the optimal derivative asset. Leisen and Judd [19] uses similar methods to price options and determine equilibrium trade in options when they are not priced by arbitrage. We stay with the single good model in this paper so that we can focus on the key mathematical problems. The methods do generalize to the multicommodity models examined in Hart and others, but space limitations force us to leave that for future studies.

Section 2 reviews local approximation theory and previous small noise analyses. Section 3 presents the bifurcation to theorems that generalize the IFT. Section 4 applies the bifurcation theorems to asset demand. Section 5 presents a small noise analysis of an asset market with one risky asset and Section 6 examines a market with one fundamental risky asset plus a derivative asset. Comparisons of these cases allows us to analyze the effects of introducing a derivative asset. Section 7 discusses some computational considerations. Section 8 outlines the approach to more general models. Section 9 concludes.

2. Local Approximation Methods at Nonsingular Points

Local approximation methods are based on a few basic theorems. They begin with Taylor's theorem and the IFT for R^n. We first state the basic theorems in this section, and then present the bifurcation theorems in the next section.

2.1. Taylor Series Approximation. The most basic local approximation is presented in Taylor's Theorem.

Theorem 1. *(Taylor's Theorem for R^n) Let $X \subseteq \mathsf{R}^n$ and p be an interior point of X.*

Suppose $f : X \to \mathsf{R}$ is C^{k+1} in an open neighborhood $\mathcal{N}$ of p. Then, for all $x \in \mathcal{N}$

$$\begin{aligned} f(x) &= f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(p)\ (x_i - p_i) \\ &+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(p)\ (x_i - p_i)\ (x_j - p_j) \\ &\vdots \\ &+ \frac{1}{k!} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(p)\,(x_{i_1} - p_{i_1}) \cdots (x_{i_k} - p_{i_k}) \\ &+ \mathcal{O}\left(\|x - p\|\right)^{k+1} \end{aligned}$$

The Taylor series approximation of $f(x)$ based at p uses derivative information at p to construct a polynomial approximation. The theory only guarantees that this approximation is good near p. While the accuracy of the approximation decays as x moves away from p, this decay is often slow, implying that a finite Taylor series can be a good approximation for x in a large neighborhood of p.

2.2. The Meaning of "Approximation". We often use the phrase "$f(x)$ approximates $g(x)$ for x near p", but the meaning of this phrase is seldom made clear. One trivial sense of the term is that $f(p) = g(p)$. While this is certainly a necessary condition, it is generally too weak to be a useful concept. Approximation usually means at least that $f'(p) = g'(p)$ as well. In this case, we say that "f is a first-order (or linear) approximation to g at $x = p$". In general, "f is an n'th order approximation of g at $x = p$" if and only if

$$\lim_{x \to p} \frac{\| f(x) - g(x) \|}{\| x - p \|^n} = 0$$

This definition says that the error $\|f(x) - g(x)\|$ of the approximation $f(x)$ is asymptotically bounded above by $c\,\|x - p\|^n$ for any constant $c > 0$. Therefore, for any x near p, the approximating function $f(x)$ is very close to $g(x)$. In particular, the degree k Taylor series of a C^{k+1} function is a k'th order approximation since its error is $\mathcal{O}\left(\|x - p\|\right)^{k+1}$. This may seem trivial but this is not always the definition of n'th order approximation used in economics. We state it here for the purpose of precision.

2.3. The Implicit Function Theorem for Analytic Functions. Our analysis will rely on the IFT for analytic functions. It is useful to review some basic facts about analytic functions that will help us understand our results. The following definition for analytic functions is the most helpful of the many equivalent definitions.

Definition 2. *A function $f(x) : \mathsf{R} \to \mathsf{R}$ is analytic at x_0 if and only if there is some nonempty open set $\Omega \subset \mathsf{R}$ such that $x_0 \in \Omega$ and for all $x \in \Omega$, $f(x) = \sum_{i=0}^{\theta} a_i x^i$ and $\sum_{i=0}^{\theta} a_i |x|^i < \infty$ for all $x \in \Omega$.*

Basically, analytic functions are C^∞ and locally equal to the power series created by Taylor series expansions. The key word here is "local". For example, the power series expansion of $\log x$ around $x_0 = 1$ cannot be globally valid since $\log x$ is not defined at $x = 0$. To make this precise, we need the concept of radius of convergence. The next theorem states the key result that the domain of convergence for a power series is a disk.

Theorem 3. *Let $C = \left\{ x | \sum_{i=0}^{\theta} a_i x^i \right\} < \infty$. Then the closure of C, $\overline{C}$, is a disk, and the radius of $\overline{C}$ is called the radius of convergence of $\sum_{i=0}^{\theta} a_i x^i$.*

The focus on analytic functions is essential since some C^∞ functions are not analytic. The best example of this is e^{-1/x^2}. The function e^{-1/x^2} is defined everywhere, even at $x = 0$. Furthermore, it is C^∞ everywhere, even at $x = 0$ where each derivative equals zero. This implies that the Taylor series expansion based at $x_0 = 0$ is the zero function. However, e^{-1/x^2} equals zero just at $x = 0$, not in any neighborhood of $x = 0$. Therefore, e^{-1/x^2} does not equal its Taylor series expansion in any open neighborhood of $x = 0$ and is not analytic at $x = 0$. In general, a C^∞ function is analytic at x_0 if and only if it equals its power series in some nondegenerate neighborhood of x_0.

We have discussed just the univariate case. Analytic functions on R^n are similarly defined; see, for example, Zeidler [26]. The next important tool is the Implicit Function Theorem (IFT) for analytic functions.

Theorem 4. *(Implicit Function Theorem) Let $H(x, y) : \mathsf{R}^n \times \mathsf{R}^m \to \mathsf{R}^m$ be analytic at (x_0, y_0) and assume $H(x_0, y_0) = 0$. If $H_y(x_0, y_0)$ is nonsingular, then there is a unique function $h : \mathsf{R}^n \to \mathsf{R}^m$ such that $h(x)$ is analytic at x_0 and $H(x, h(x)) = 0$ for*

(x, y) in an open neighborhood of (x_0, y_0). Furthermore, the derivatives of h at x_0 can be computed by implicit differentiation of the identity $H(x, h(x)) = 0$.

The IFT states that h can be uniquely defined for x near x_0 by $H(x, h(x)) = 0$ if $H_y(x_0, y_0)$ is not singular and allows us to implicitly compute the derivatives of h. For example, the gradient of h at x_0 is

$$\frac{\partial h}{\partial x}(x_0) = -H_y(x_0, y_0)^{-1} H_x(x_0, y_0)$$

and provides us with the first–order terms of the power series representation for $h(x)$ based at x_0. When we combine Taylor's theorem and the IFT, we have a way to compute a locally valid polynomial[1] approximation of a function $h(x)$ for x near x_0 implicitly defined by $H(x, h(x)) = 0$. There is an IFT for C^∞ functions, but it does not give us a positive radius of convergence for the implied power series. Therefore, we must proceed with an analytic function perspective.

The focus on analytic functions is not restrictive since most functions economists use are locally analytic at points of economic relevance. For example, $\log c$ is a common utility function and is analytic at each positive value of c. Similarly for Cobb-Douglas production functions $k^\alpha \ell^{1-\alpha}$. However, these functions are only locally analytic, implying that different power series representations are valid over different finite intervals. For example, suppose we construct a power series for $u(c) = \log c$ based at $c_0 = 1$. Since $\log c$ is undefined at $c = 0$, the radius of convergence for that power series is at most 1, which in turn implies that that power series is not valid for any $c > 2$. However, the power series based at $c_0 = 2$ is valid for $c \in (0, 4)$. When we use the IFT for analytic functions, we need to be aware of the radii of convergence of the power series we implicitly use and be sure that they are consistent with our application of the IFT.

The power series constructed in the IFT for analytic functions will have a positive radius of convergence, but we know anything about its magnitude in general. This is a drawback in some contexts. This issue is not important in this paper since we

[1] The derivative information could also be used to compute a Padé approximant, or other nonlinear approximation schemes. Judd and Guu (1993) and Judd (1998) examine both approaches. In this paper, we will stay with the conventional Taylor expansions.

examine only the asymptotic properties of models. We will return later to the issue of the range of validity for our formulas.

2.4. Previous Small Noise Analyses. The small noise approach is not new to the economics literature, but the approach we take differs in substance and formalism from previous efforts. One line of previous work is taken by Fleming [8], which was elaborated on by Judd and Guu [14]. Fleming showed how to go from the solution of a deterministic control problem to one with small noise added to the law of motion. Specifically, consider the problem

$$\begin{aligned} \max \qquad & E\left\{\int_0^T e^{-\rho t}\pi(x,u)dt\right\} \\ & dx = f(u,x)dt + \epsilon\sigma(u,x)dz \end{aligned} \tag{1}$$

Fleming approximated the problem in (1) for small ϵ by finding the control law $u = U(x,t)$ of the $\epsilon = 0$ problem and then apply the IFT to Bellman's equation. A key detail was that the control law needed to be unique in the $\epsilon = 0$ case. Judd and Guu implement this approach for infinite horizon problems, and show that the Fleming procedure produces good approximations.

The problem discussed in Fleming, and Judd and Guu was easy since it could be handled by the standard IFT. A less trivial problem was examined in Samuelson [22]. He examined the problem of asset demand when riskiness was small. We will return to that problem below.

A third example of the small noise analysis is Magill's [20] analysis of what is now called real business cycles. Magill showed how to compute linear approximations to (1), use these approximations to compute spectra of the resulting linear model, and proposed that the spectra of these models be compared to empirical data on spectra. Kydland and Prescott [18] focussed on the special case of Magill's method where the law of motion $f(u,x)$ is linear in (u,x), and partially implemented Magill's spectral comparison ideas by comparing variances and covariances of these linear approximations of deterministic models to the business cycle data. This special case of Magill's approach to stochastic dynamic general equilibrium has been important in the Real Business Cycle literature. Gaspar and Judd [10] shows how to compute higher-order expansions around deterministic steady states. Also, the methods in Magill, and Kydland and Prescott were "certainty equivalent approximations", that is, they compute a linear approximation to the deterministic problem, $\epsilon = 0$, and apply it to problems where $\epsilon \neq 0$, whereas Gaspar and Judd [10] computes approximations which includes the effect of ϵ. Similarly, we will compute high-order expansions where ϵ is allowed to vary.

A fourth example that particularly illustrates the importance of using bifurcation theory is Tesar [25]. Tesar used a linear-quadratic approach to evaluate the welfare impact on countries from opening up trade in assets. Some of her numerical examples showed that moving to complete markets would result in a Pareto inferior allocation, a finding that contradicts the first welfare theorem of general equilibrium. Kim and Kim [16] have shown that this approach will often produce incorrect results. These examples illustrate the need for using methods from the mathematical literature instead of relying on ad hoc approximation procedures based loosely on "economic intuition."

This paper illustrates the critical mathematical structure of asset market problems with small risks, and develops the relevant mathematical tools. While the model analyzed below is simple, the basic approach is generally applicable.

3. Bifurcation Methods

Our asset market analysis requires us to approximate an implicitly defined function at a point where the conditions of the IFT do not hold. Fortunately, we will be able to exploit additional structure and arrive at a solution using bifurcation methods. We first present the general theorems and then apply them to some asset problems.

3.1. Bifurcation in R^1.

Suppose that $H(x, \epsilon)$ is C^2 and $x(\epsilon)$ is implicitly defined by $H(x(\epsilon), \epsilon) = 0$. One way to view the equation $H(x, \epsilon) = 0$ is that for each ϵ it defines a collection of x that solves $H(x, \epsilon) = 0$. The number of such x may change as we change ϵ. We next define the concept of a bifurcation point.

Definition 5. *(x_0, ϵ_0) is a bifurcation point of H iff the number of solutions x to $H(x, \epsilon) = 0$ changes as ϵ passes through ϵ_0, and there are two distinct parametric paths, $(X_i(s), E_i(s)),\ i = 1, 2$, such that $H(X_i(s), E_i(s)) = 0$, and $\lim_{s \to 0}(X_i(s), E_i(s)) = (x_0, \epsilon_0),\ i = 1, 2$.*

A trivial example of a bifurcation is $H(x, \epsilon) = \epsilon(x - \epsilon)$ at $(x, \epsilon) = (0, 0)$. If $\epsilon \neq 0$, the unique solution to $H = 0$ is $x(\epsilon) = \epsilon$, but at $\epsilon = 0$ any x solves $H = 0$. There is a bifurcation point at $(x, \epsilon) = (0, 0)$, and the two branches of solutions to $H = 0$ are $X_1(s) = E_1(s) = s$ and $X_1(s) = s$, $E_1(s) = 0$. We cannot apply the IFT to $H(x(\epsilon), \epsilon) = 0$ at $(0, 0)$ directly since the Jacobian of H_x is singular at $(0, 0)$. Suppose that we are interested in the branch $x(\epsilon) = \epsilon$, and not the trivial branch where $\epsilon = 0$ and x is arbitrary. This is natural since we want to know how x changes as ϵ changes, not just the situation at $\epsilon = 0$. Bifurcation theorems help us accomplish this. The case for $x \in \mathsf{R}$ is summarized in the Theorem 6.

Theorem 6. *(Bifurcation Theorem for* R*) Suppose* $H : \mathsf{R} \times \mathsf{R} \to \mathsf{R}$*,* H *is analytic for* (x, ϵ) *in a neighborhood of* $(x_0, 0)$*, and* $H(x, 0) = 0$ *for all* $x \in \mathsf{R}$*. Furthermore, suppose that*

$$H_x(x_0, 0) = 0 = H_\epsilon(x_0, 0), \;\; H_{x\epsilon}(x_0, 0) \neq 0.$$

Then $(x_0, 0)$ *is a bifurcation point and there is an open neighborhood* $\mathcal{N}$ *of* $(x_0, 0)$ *and a function* $h(\epsilon)$*,* $h(\epsilon) \neq 0$ *for* $\epsilon \neq 0$*, such that* h *is analytic and* $H(h(\epsilon), \epsilon) = 0$ *for* $(h(\epsilon), \epsilon) \in \mathcal{N}$*.*

Proof. The strategy to prove this theorem follows the trick of "solving a singularity through division by ϵ" (see Zeidler, 1998, Chapter 8). Define

$$F(x, \epsilon) = \begin{cases} \frac{H(x,\epsilon)}{\epsilon}, & \epsilon \neq 0 \\ \frac{\partial H(x,0)}{\partial \epsilon}, & \epsilon = 0 \end{cases} . \tag{2}$$

Since H is analytic and $H(x, 0) = 0$ for all x, $H(x, \epsilon) = \epsilon F(x, \epsilon)$ and F is analytic in (x, ϵ). Since $0 = H_\epsilon(x_0,0)$, $F(x_0,0) = 0$. Direct computation shows $F_x(x, \epsilon) + \epsilon F_{x\epsilon}(x, \epsilon) = H_{x\epsilon}(x, \epsilon)$, which implies $F_x(x_0, 0) = H_{x\epsilon}(x_0, 0) \neq 0$. Since $F_x(x_0, 0) \neq 0$, we can apply the IFT to F at $(x_0, 0)$. Therefore, there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and an analytic function $h(\epsilon)$, $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, such that $F(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$, which in turn implies $H(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$. ■

In general, Theorem 6 tells us we can compute derivatives through implicit differentiation. In particular, $h'(0)$ and $h''(0)$ are defined by

$$\begin{aligned} h'(0) &= -[F_x(x_0,0)]^{-1} F_\epsilon(x_0,0) = -\frac{1}{2}[H_{x\epsilon}(x_0,0)]^{-1} H_{\epsilon\epsilon}(x_0,0) \\ 3H_{x\epsilon}(x_0, 0)h''(0) &= -[3h'(0)H_{xx\epsilon}(x_0, 0)h'(0) + 3H_{x\epsilon\epsilon}(x_0, 0)h'(0) + H_{\epsilon\epsilon\epsilon}(x_0, 0)] \end{aligned}$$

which implies a unique value for $h'(0)$ and $h''(0)$ as long as $H_{x\epsilon}(x_0, 0) \neq 0$. Notice the sequentially linear character of the problem. One only needs linear operations to compute $h'(0)$, and once we have computed $h'(0)$ the problem of computing $h''(0)$ is also a linear problem. The existence of $h'(0)$, $h''(0)$, and all higher derivatives of h relies solely on the solvability condition $H_{x\epsilon}(x_0,0) \neq 0$ and the existence of the higher-order derivatives of H at the bifurcation point.

Theorem 6 resolves the problem when $H(x, \epsilon) = \epsilon(x - \epsilon) = 0$. In this case, $H(x, 0) = 0$ for all x, $H_x(0, 0) = 0 = H_\epsilon(0, 0)$, but $H_{x\epsilon}(x_0, 0) = 1 \neq 0$. Implicit differentiation shows that $h'(0) = 1$, and that every other derivative of h at $x = 0$ is zero. The example of $H = \epsilon(x - \epsilon)$ seems quite trivial, but our problems will have a similar form and Theorem 6 gives us conditions under which the general problem is really no more complex than this simple example.

Implicit differentiation of $H(x(\epsilon), \epsilon) = 0$ will produce a power series expansion for $x(\epsilon)$ around $\epsilon = 0$, but we know nothing about the radius of convergence of that power series. For example, $H(x, \epsilon) = \epsilon \left(x - (\epsilon + 1)^{1/2}\right) = 0$ has the obvious global solution $x(\epsilon) = (\epsilon + 1)^{1/2}$ but the power series for $(\epsilon + 1)^{1/2}$ around $\epsilon = 0$ is valid only when $-1 < \epsilon < 1$ because there is a singularity at $\epsilon = -1$.[2] Also, in practice, we will only be able to use finite-order Taylor series approximations, which are just the initial segments of the full power series. In general, any such Taylor series approximation will do well for ϵ close to zero, but the quality of the approximation will degrade as ϵ moves away from zero.

We assumed $H_{x\epsilon}(x_0, 0) \neq 0$ in Theorem 6. The division-by-zero trick can be applied to problems with higher-order degeneracies. If $H_{x\epsilon}(x_0,0) = 0$ then $F_x(x_0,0) = 0$, and we cannot apply the IFT to F in the proof. But if $F_\epsilon(x_0,0) = 0$ and $F_{x\epsilon}(x_0,0) \neq 0$ we can apply the bifurcation theorem to F.

3.2. Bifurcation in R^n : The Zero Jacobian Case. The foregoing focussed on one-dimensional functions h. We can also apply these ideas for functions over R^n. The same trick used in Theorem 6 works for Theorem 7; therefore, its proof is

[2] The difficulty in this case could be fixed by a nonlinear change of variables. Appropriate and clever nonlinear change of variables can help with this problem, but we do not pursue that strategy in this paper.

omitted.

Theorem 7. *(Bifurcation Theorem for R^n) Suppose $H : \mathsf{R}^n \times \mathsf{R} \rightarrow \mathsf{R}^n$ is analytic near $(x_0, 0)$, and $H(x, 0) = 0$ for all $x \in \mathsf{R}^n$. Furthermore, suppose that*

$$\begin{aligned} H_x(x_0, 0) &= 0_{n \times n} \quad (3) \\ H_\epsilon(x_0, 0) &= 0_n \quad (4) \\ \det(H_{x\epsilon}(x_0, 0)) &\neq 0 \quad (5) \end{aligned}$$

Then there is an open neighborhood $\mathcal{N}$ of $(x_0, 0)$ and an analytic function $h(\epsilon) : \mathsf{R} \rightarrow \mathsf{R}^n$ such that $h(\epsilon) \neq 0$ for $\epsilon \neq 0$, and $H(h(\epsilon), \epsilon) = 0$ for $(h(\epsilon), \epsilon) \in \mathcal{N}$.

Since Theorem 7 shows that h is analytic, it can be approximated by a multivariate Taylor series. In particular, the first-order derivatives are defined by

$$h'(0) = -\frac{1}{2} H_{x\epsilon}^{-1}(x_0, 0)\ H_{\epsilon\epsilon}(x_0, 0) \quad (6)$$

Theorem 7 assumes $H_x(x_0, 0)$ is a zero matrix. There are generalizations that only assume that $H_x(x_0, 0)$ is singular. We do not present any extensions here since they are substantially more complex to present and are not needed below. See Zeidler or Chow and Hale for more complete treatments of bifurcation problems.

4. Portfolio Demand with Small Risks

The key assumption we exploit is that risks are small. This is motivated not by any claim that actual risks are small, but is reasonable for three reasons. First, this assumption allows us to solve the problem without making any parametric assumptions for either tastes or returns. We derive critical formulas for allocations and welfare in a parameter-free fashion. The results tell us which moments of asset returns are important and which properties of the utility function are important for the case of small risks. Second, the results for small risks may be suggestive of general results. For example, the asymptotic results could provide counterexamples to conjectures since the asymptotic results are asymptotically explicit solutions. Furthermore, any general property of the model will be true for the case of small risks and will be revealed as general properties of our asymptotic solutions. In this paper, we pursue the implications of the small risk assumption, leaving it for later work to see how robust those results.

Third, the period of time in our model is not meant to be an entire life, but rather the period of time between trades. Given modern markets and the presence of many high-volume, low-transaction cost traders, it is reasonable to assume that only a moderate amount of risk is borne between trading periods. A dynamic model is necessary to examine the validity of this point, but we believe that our static analysis will give useful insights and leave dynamic generalizations for future work.

4.1. Demand with Two Assets. We begin by applying the bifurcation approximation methods to asset market demand. Suppose that an investor has W in wealth to invest in two assets. The safe asset, called a bond, yields one dollar per dollar invested, and the risky asset, called stocks or equity, yields Z dollars per dollar invested. There is no savings-consumption decision in this model. Therefore, this is equivalent to making bonds in the second period the numeraire. If an investors has θ shares of stock, final wealth is $Y = (W - \theta) + \theta Z$. We assume that he chooses θ to maximize $E\{u(Y)\}$ for some concave utility function $u(\cdot)$.

Economists have studied this problem by approximating u with a quadratic function and then solving the "approximate" quadratic optimization problem. The bifurcation approach allows us to examine this procedure rigorously and extend it. We first create a continuum of portfolio problems by assuming

$$Z = 1 + \epsilon z + \epsilon^2 \pi \tag{7}$$

where z is a fixed random variable. We assume $E\{z\} = 0$ since we want (7) to decompose Z into its mean, $1 + \epsilon^2 \pi$, and its risky component, ϵz. We also assume $\sigma_z^2 = 1$; this makes ϵ the standard deviation of Z and ϵ^2 its variance in the ϵ problem. Both of these assumptions are just normalizations, implying no loss of generality. At $\epsilon = 0$, Z is degenerate and equal to 1, the payoff of the bond. The scalar π represents the risk premium. More precisely, $\sigma_z^2 = 1$ implies that π is the the price of risk, that is, the risk premium per unit variance. In this demand problem we make the natural assumption that $\pi > 0$ but that is not necessary for the analysis.

Equation (7) scales its terms in a manner consistent with economic theory. We want (7) to represent a continuum of problems connecting a degenerate deterministic problem to problems with nontrivial risk. Note that (7) multiplies z by ϵ and π by ϵ^2. Since the variance of ϵz is $\epsilon^2 \sigma_z^2$, this models the intuition that risk premia are proportional to the variance. The continuum of problems parameterized in (7) all have the same price of risk π. The particular parameterization in (7) may seem to prejudge the results. That will not be a problem since the application of the bifurcation theorems will validate the assumptions implicitly made in (7).[3]

The investor chooses θ to maximize $E\{u(W + \theta(\epsilon z + \epsilon^2 \pi))\}$. The first-order condition for the investor's problem is

$$\epsilon E\{u^{\prime}(W + \theta(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon\pi)\} = 0. \tag{8}$$

The condition (8) states that the future marginal utility of consumption must be orthogonal to the excess return of equity. Let μ be the probability measure for z and (a, b) the (possibly infinite) support. The choice of θ as a function of ϵ is implicitly defined by

$$0 = H(\theta(\epsilon), \epsilon) \equiv \int_a^b u'(W + \theta(\epsilon)(\epsilon z + \epsilon^2 \pi))\,(z + \epsilon\pi)\, d\mu. \tag{9}$$

We want to analyze the solutions of (9) for small ϵ. However, $0 = H(\theta, 0)$ for all θ, because at $\epsilon = 0$ the assets are perfect substitutes. $\theta(0)$ is multivalued since any choice of θ satisfies the first-order condition (9) when $\epsilon = 0$. Furthermore, $0 = H(\theta, 0)$ for all θ implies $0 = H_\theta(\theta, 0)$ for all θ, violating the nonsingularity condition in the IFT. Therefore, we cannot use the IFT to compute a Taylor series for $\theta(\epsilon)$ at $\epsilon = 0$.

The situation is displayed in Figure 1. As ϵ changes, the equilibrium demand for equity, θ, follows a path like ABC or like $DEGF$. Since the asset demand problem is a concave optimization problem there is a unique path of solutions to the first-order conditions whenever $\epsilon \neq 0$. At $\epsilon = 0$, however, the entire $\epsilon = 0$ horizontal axis is also a solution to the equity demand problem. The path ABC crosses the θ axis vertically and represents a pitchfork bifurcation, whereas the path $DEGF$ crosses the θ axis

[3]Pages 518-519 in Judd (1998) show that alternative parameterizations of the form $Z = 1 + \epsilon z + \epsilon^\nu \pi$ for $\nu \neq 2$ lead to singularities which prevent the application of implicit function or bifurcation theorems.

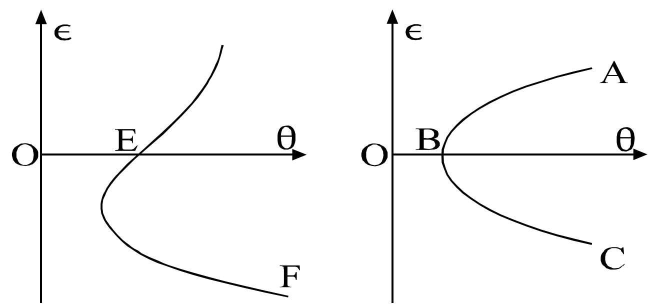

Figure 1: Bifurcation possibilities for asset demand problem

obliquely and represents a transcritical bifurcation. The objective is to first find the bifurcation point, B or E, where the branch of equity demand solutions crosses the trivial branch of solutions to the first-order conditions, and then compute a Taylor series that approximates $\theta(\epsilon)$ along the nontrivial branch.

Computing θ_0. We proceed intuitively to derive a solution which we validate with the Bifurcation Theorem. Since we want to solve for θ as a function of ϵ near 0, we first need to compute $\theta_0 \equiv \lim_{\epsilon\to 0} \theta(\epsilon)$. Implicit differentiation of (9) with respect to ϵ implies

$$0 = H_\theta(\theta(\epsilon), \epsilon)\theta'(\epsilon) + H_\epsilon(\theta(\epsilon), \epsilon). \tag{10}$$

Differentiating $H(\theta, \epsilon)$ with respect to θ and ϵ implies

$$\begin{aligned} H_\epsilon(\theta, \epsilon) &= \int_a^b u''(Y)\,(\theta z + 2\theta\epsilon\pi)\,(z + \epsilon\pi) + u'(Y)\pi \; d\mu \\ H_\theta(\theta, \epsilon) &= \int_a^b u''(Y)\,(z + \epsilon\pi)^2 \epsilon \; d\mu \end{aligned}$$

At $\epsilon = 0$, $H_\theta(\theta, 0) = 0$ for all θ. The derivative $\theta'(0)$ can be well-defined in (10) only if $H_\epsilon(\theta,\, 0) = 0$. Therefore, we look for θ_0 defined by $0 = H_\epsilon(\theta_0,\, 0)$. At $\epsilon = 0$, this reduces to (using the fact that $\int_a^b z^2 \; d\mu = \sigma_z^2 = 1$) $0 = u''(W)\,\theta_0 + u'(W)\pi$, which implies

$$\theta_0 = -\,\frac{u'(W)}{u''(W)}\,\pi \tag{11}$$

This is the simple portfolio rule indicating that θ is the product of risk tolerance and the risk premium per unit variance. If θ_0 is well-defined, then this must be its value.

Theorem 8 states the critical result.

Theorem 8. *Let (11) define θ_0. If $H(\theta, \epsilon)$ is analytic at $(\theta_0, 0)$, then there is an analytic function $\theta(\epsilon)$ that satisfies (9) such that $\theta(0) = \theta_0$ and $\theta(\epsilon) \neq 0$ for $\epsilon \neq 0$.*

Proof. Direct application of the Bifurcation Theorem. ■

The assumption in Theorem 8 that $H(\theta, \epsilon)$ is analytic at θ_0 is not trivially satisfied. $H(\theta, \epsilon)$ is an integral and is analytic if $u(c)$ is analytic over the set of c at which $u'(c)$ is evaluated in the integrand of $H(\theta, \epsilon)$, because the integral of a power series is a power series. If the support of μ is compact and u is analytic at W then $H(\theta, \epsilon)$ is analytic at $(\theta_0, 0)$ since for small ϵ, $u'(c)$ is evaluated only at values of c close to W. However, if μ has infinite support there may be problems because $u'(c)$ in the integrand of (9) is evaluated over an infinite range whenever $\epsilon, \theta \neq 0$. If the radius of convergence for the power series representation of $u'(c)$ based at W is finite, then it will not be valid at some points in the support of μ, rendering the power series approach invalid. This will be the case, for example, if $u(c) = \log c$ and μ is the measure for a log Normal random variable. The radius of convergence of power series approximations of $u(c)$ at $c = W$ is a critical element, as well as the analyticity of the density function of μ. The next corollary presents a sufficient condition for using the bifurcation approach on an open neighborhood $\mathcal{N}$.

Corollary 9. *Define θ_0 as in (11). If $u(c)$ is analytic at $c = W$ and the support of μ is compact, then there is a function $\theta(\epsilon)$ analytic and satisfies (9) on $(-\epsilon_0, \epsilon_0)$ for some $\epsilon_0 > 0$ with $\theta(0) = \theta_0$ and $\theta(\epsilon) \neq 0$ for $\epsilon \neq 0$ in $(-\epsilon_0, \epsilon_0)$.*

In all formulas below, we will assume that the critical functions are locally analytic.

Computing $\theta'(0)$. Equation (11) is not an approximation to the portfolio choice at any particular variance. Instead, θ_0 is the limiting portfolio share as the variance vanishes. We generally need to compute several terms of the Taylor series expansion for $\theta(\epsilon)$

$$\theta(\epsilon) = \theta_0 + \theta'(0)\epsilon + \theta''(0)\frac{\epsilon^2}{2} + \theta'''(0)\frac{\epsilon^3}{6} + ... \tag{12}$$

In particular, the linear approximation is

$$\theta(\epsilon) \doteq \theta(0) + \epsilon\ \theta'(0). \tag{13}$$

To calculate $\theta'(0)$, differentiate (10) with respect to ϵ to find $0 = H_{\theta\theta}\,\theta'\theta' + 2H_{\theta\epsilon}\,\theta' + H_\theta\theta'' + H_{\epsilon\epsilon}$. At $(\theta_0, 0)$, $H_{\epsilon\epsilon} = u'''(W)\theta_0^2\, E\{z^3\}$, $H_{\theta\theta} = 0$, and $H_{\theta\epsilon} = u''(W)$. Therefore,

$$\theta'(0) = -\frac{1}{2} H_{\theta\epsilon}^{-1} H_{\epsilon\epsilon} = -\frac{1}{2}\frac{u'''(W)}{u''(W)} E\{z^3\}\ \theta_0^2. \tag{14}$$

Again, we can use Corollary 9 to establish the existence of the derivatives of H for some random variables.

Equation (14) tells us how the share of wealth invested in equity changes as the riskiness increases. It highlights the importance of the third derivative of utility and the skewness of returns. If the distribution of Z is symmetric, then $E\{z^3\} = 0$, and the constant θ_0 is the linear approximation of $\theta(\epsilon)$ at $\epsilon = 0$. This is also true if $u'''(W) = 0$, such as in the quadratic utility case. The case of $\theta'(0) = 0$ corresponds to a pitchfork bifurcation point like B in Figure 1. However, if the utility function is not quadratic and the risky return is not symmetrically distributed, then $\theta'(0) \neq 0$, and the linear approximation is a nontrivial function of utility curvature and higher moments of the distribution. This indicates that the bifurcation point is transcritical like E in Figure 1.

Dividing both sides of (14) by θ_0 implies

$$\frac{\theta'(0)}{\theta_0} = \frac{1}{2}\frac{u'(R)}{u''(R)}\frac{u'''(R)}{u''(R)}\ \pi\ E\{z^3\} \tag{15}$$

Equation (15) expresses the relative change in equity demand as ϵ increases in terms of skewness, $E\{z^3\}$, the risk premium, π, and utility derivatives. Our formulas would be unintuitive and cumbersome if we expressed them in terms of $u(c)$ and its derivatives. Fortunately, there are some useful utility parameters we can use. Define the functions

$$\begin{aligned} \tau(c) &\equiv -\frac{u'(c)}{u''(c)} \\ \rho(c) &\equiv \frac{\tau^2}{2}\frac{u'''(c)}{u'(c)} = \frac{1}{2}\frac{u'(c)}{u''(c)}\frac{u'''(c)}{u''(c)} \end{aligned}$$

The function $\tau(c)$ is the conventional risk tolerance. The bifurcation point θ_0 equals $\tau(W)\ \pi$, the product of risk tolerance at the deterministic consumption, $\tau(W)$, and the price of risk, π.

The definition of $\rho(c)$ implies that (15) can be expressed as

$$\frac{\theta'(0)}{\theta_0} = \rho(W)\ \pi\ E\{z^3\} \tag{16}$$

This motivates our definition of skew tolerance.[4]

Definition 10. *Skew tolerance at c is*

$$\rho(c) = \frac{1}{2}\frac{u'(c)}{u''(c)}\frac{u'''(c)}{u''(c)}$$

Skew tolerance has ambiguous sign since the sign of u''' is ambiguous. If there is more upside potential than downside risk, then skewness is positive. If $u''' > 0$, an increase in skewness will cause asset demand to increase as riskiness increases. We suspect that investors prefer positively skewed returns, holding mean and variance constant. For example, $u''' > 0$ for the CRRA and CARA families of utility functions. We never assume this, but this case provides us with some intuition for the results. There are many ways to manipulate the expression in (14). We chose our definition of skew tolerance because of the expression in (16) and the intuitive role it plays in critical expressions below.

The linear approximation (13) may not be sufficient. To compute $\theta''(0)$, differentiate (10) with respect to ϵ at $\epsilon = 0$ to find

$$3H_{\theta\epsilon}\theta''(0) = -(3H_{\theta\epsilon\epsilon}\theta'(0) + 3H_{\theta\theta\epsilon}(\theta'(0))^2 + H_{\epsilon\epsilon\epsilon}) \tag{17}$$

Equation (17) is linear in $\theta''(0)$. Since $H_{\theta\epsilon} \neq 0$ at $(\theta_0, 0)$, $\theta''(0)$ exists and is uniquely defined by (17). To express $\theta''(0)$, we define kurtosis tolerance.

Definition 11. *Kurtosis tolerance at c is*

$$\kappa(c) = -\frac{1}{3}\frac{u''''(c)}{u''(c)}\frac{u'(c)}{u''(c)}\frac{u'(c)}{u''(c)}$$

[4]Skew tolerance is obviously related to prudence, as defined in Kimball (1990), but we do not pursue those connections here.

Solving (17) at $\epsilon = 0$ shows that

$$\frac{\theta''(0)}{\theta_0} = \pi^2 \left((6\rho(W) - 2) + 4\rho(W)^2 E\{z^3\}^2 + \kappa(W)E\{z^4\}\right) \tag{18}$$

Equation (18) says that the impact of kurtosis on equity demand is proportional to the square of the price of risk and the kurtosis tolerance.

We could continue this indefinitely if u is locally analytic, an assumption satisfied by standard utility functions. Of course, the terms become increasingly complex. We end here since it illustrates the main ideas and these results are the only ones needed for the applications below. The general procedure is clear. Computing the higher-order terms is straightforward since any particular derivative is the solution to linear equations similar to (17) once we have computed lower-order derivatives.

Samuelson's Method. Samuelson [22] also examined the problem of asset demand with small risks. We now illustrate the relationships between our bifurcation approach and Samuelson's method. Samuelson's method replaced $u(Y)$ with a polynomial approximation based at the deterministic consumption, as in

$$\begin{aligned} u(W + \theta(\epsilon z + \epsilon^2 \pi)) \;\doteq\; & u(W) + \epsilon\theta z u'(W) \\ & + \frac{\epsilon^2}{2}\left(2\theta\pi^2 u'(W) + \theta^2 z^2 u''(W)\right) \\ & + \frac{\epsilon^3}{6}\left(6z\pi^2\theta^2 u''(W) + \theta^3 z^3 u'''(W)\right) + \ldots \end{aligned}$$

When we use the quadratic approximation in the first-order condition (8) we arrive at the equation $0 = (\pi u'(W) + \theta u''(W))\,\epsilon^2 + O(\epsilon^3)$, which, to $O(\epsilon)$, implies $\theta(\epsilon) \doteq -(u'(W)/u''(W))\pi$, our bifurcation point.

However, the Samuelson method differs from ours for higher-order approximations. Samuelson's second-order approximation is computed by using the third-order approximation of $u(Y)$ in the first-order condition (8), implying

$$0 \doteq (\pi u'(W) + \theta u''(W))\,\epsilon^2 + \epsilon^3 \frac{1}{2}\theta^2 E\{z^3\} u'''(W) \tag{19}$$

which is a quadratic equation with solution

$$\theta(\epsilon) \doteq \frac{-u''(W) + \sqrt{u''(W)^2 - 2\pi E\{z^3\}\epsilon u'''(W) u'(W)}}{E\{z^3\} u'''(W)\epsilon} \tag{20}$$

One could arrive at our first-order derivative in equation (14) by differentiating (20) with respect to ϵ at $\epsilon = 0$. The two methods are consistent and of similar complexity for the first-order approximation in a two-asset problem. However, the asymptotic approach we pursue here becomes relatively more efficient as we move to higher-order approximations and to more assets. Samuelson's approach generally requires solving nonlinear equations, as was the case in equation (19). The equations become more difficult to solve, and are impossible to solve exactly beyond the fourth order since there is no closed-form solution for polynomials of degree five and higher. Our bifurcation method uses linear operations to compute asymptotically valid approximations of the function $\theta(\epsilon)$. Therefore, we can easily derive each term and go to an arbitrary order as long as the necessary moments and derivatives exist.

The main reason for pursuing the asymptotic approach is its ability to derive economically interesting results. Equation (20) shows that linear-quadratic approximations would not be as good as higher-order approximations since equation (20) involves the skewness of Z and the third derivative of utility. However, Samuelson conjectured that LQ approximations are probably adequate in actual economic problems. This paper gives examples where the linear-quadratic approximation would be unreliable, and higher-order approximations are necessary to answer critical questions.

4.2. Demand with Three Assets. We applied the R^1 version of the Bifurcation Theorem to the two-asset case. We next analyze the three-asset case to show the generality of the method and illustrate the key multivariate details. Consider again our investor model but with three assets. The bond yields one dollar per dollar invested and risky asset i yields Z_i dollars per dollar invested, for $i = 1, 2$. Let θ_i denote the proportion of wealth invested in risky asset i. Final wealth is $Y = (W - \theta_1 - \theta_2) + \theta_1 Z_1 + \theta_2 Z_2$. The investor chooses θ_i to maximize $E\{u(Y)\}$. To apply the Bifurcation Theorem, we assume that $Z_i = 1 + \epsilon z_i + \epsilon^2 \pi_i$. Without loss of generality, we assume that $E\{z_i\} = 0$. Let $\sigma_i^2 = E\{z_i^2\}$ be the variance of risky asset i's return and $\sigma_{12} = E\{z_1 z_2\}$ the covariance. We assume that the assets are not perfectly correlated; hence, $\sigma_i^2 \sigma_j^2 \neq (\sigma_{ij})^2$.

The first-order condition for risky asset i is $\epsilon E\left\{u'(Y)(\epsilon\pi_i + z_i)\right\} = 0$. The asset demand functions $\theta_i(\epsilon)$ are defined implicitly by $H(\theta_1, \theta_2, \epsilon) : \mathrm{R}^3 \to \mathrm{R}^2$ where $H^i(\theta_1, \theta_2, \epsilon) \equiv E\left\{u'(Y)(\epsilon\pi_i + z_i)\right\}$, $i = 1, 2$. To invoke Theorem 7, we first note that $H_\theta(\theta_1, \theta_2, 0) = 0_{2\times 2}$ for all (θ_1, θ_2). We compute a candidate bifurcation point by solving $H_\epsilon(\theta_1, \theta_2, 0) = 0$. Direct computation shows

$$H_\epsilon(\theta_1, \theta_2, 0) = u'(W)\begin{bmatrix}\pi_1 \\ \pi_2\end{bmatrix} + u''(W)\Sigma\begin{bmatrix}\theta_1 \\ \theta_2\end{bmatrix}$$

where Σ is the variance-covariance matrix of the risky returns (z_1, z_2). The solution of the bifurcation equation $H_\epsilon(\theta_1, \theta_2, 0) = 0$ is

$$\begin{bmatrix}\theta_1(0) \\ \theta_2(0)\end{bmatrix} = -\frac{u'(W)}{u''(W)}\Sigma^{-1}\begin{bmatrix}\pi_1 \\ \pi_2\end{bmatrix}$$

We need to verify the nonsingularity of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$. Direct computation shows that $H_{\theta\epsilon}(\theta_1(0), \theta_2(0), 0) = u''(W)\Sigma$ for all θ_1, θ_2. The determinant of $H_{\theta\epsilon}$ at $(\theta_1(0), \theta_2(0), 0)$ is $u''(W)(\sigma_1^2\sigma_2^2 - (\sigma_{12})^2)$, which is nonzero as long as assets 1 and 2 are not perfectly correlated.

These calculations show that all the conditions in Theorem 7 hold for our model. Hence, the bifurcation theorem for R^2 ensures the existence of analytic functions $\theta_1(\epsilon)$ and $\theta_2(\epsilon)$ which satisfy $H(\theta_1(\epsilon), \theta_2(\epsilon), \epsilon) = 0$ in some neighborhood of $\epsilon = 0$. This procedure can be applied for an arbitrary number of assets. We can also produce higher-order expansions as long as the necessary moments and derivatives exist. We next use these ideas to compute asset market equilibrium.

5. Asset Market Equilibrium with One Risky Asset

We now take our portfolio choice analysis and turn it into an equilibrium analysis[5]. We assume a two-period model, period 0 and period 1, with no consumption in period 0. Agents trade assets in period 0 and consume the asset payoffs in period 1. One bond yields 1 unit of consumption in period 1; the bond serves as our numeraire in period 0. Each share of equity has price p in period 0 and has a random period 1 value

[5] Chiappori et al. (1992) used similar methods to prove the existence of sunspot equilibria near deterministic steady states in overlapping generations models. We go through the details of our application since they are substantially different than the application in Chiappori et al.

of $1+\epsilon z$ units of consumption where z is a random variable with finite moments. We assume $E\{z\}=0$ and $E\{z^2\}=1$. For each value of ϵ we have an asset market with two assets; we call that economy the ϵ-economy.

We assume two types of traders. Type i traders have initial endowments of B_i^e units of the bond and θ_i^e shares of equity. The utility of a type i trader is $u_i(Y_i)$, a concave function, where Y_i is the final wealth and consumption of type i traders. The supply of equity is fixed at the endowment $\theta_1^e+\theta_2^e$. Without loss of generality, we assume $\theta_1^e+\theta_2^e=1$; this implies that z denotes aggregate risk in the aggregate endowment. Let θ_i be the shares of equity and B_i the value of bonds held by trader i after trading in period 0. The final wealth for trader i is $Y_i=\theta_i(1+\epsilon z)+B_i$. Each trader of type i chooses θ_i to maximize his expected utility $E\{u_i(Y_i)\}$, subject to the budget constraint $B_i+\theta_i p=B_i^e+\theta_i^e p$. His first-order condition for θ_i is $E\{u_i^{\prime}(Y_i)(1+\epsilon z-p)\}=0$. Market clearing implies $\theta_1+\theta_2=\theta_1^e+\theta_2^e=1$. Define $\theta=\theta_1$; then $\theta_2=1-\theta$. For each ϵ-economy, we want to find the equilibrium values of θ and p; let $\theta(\epsilon)$ and $p(\epsilon)$ be the equilibrium values of θ and p in the ϵ-economy. The equilibrium values of $\theta(\epsilon)$ and $p(\epsilon)$ must satisfy the equilibrium pair of equations

$$H^i(\theta(\epsilon),p(\epsilon),\epsilon)=E\{u_i^{\prime}(Y_i)(1+\epsilon z-p(\epsilon))\}=0,\ i=1,2 \tag{21}$$

which are implied by the agents' first-order conditions.

Equation (21) implicitly defines $(\theta(\epsilon),p(\epsilon))$. However, the IFT cannot be applied to analyze (21) around $\epsilon=0$. Since the assets are perfect substitutes at $\epsilon=0$, they must trade at the same price; hence, $p(0)=1$. However, $\theta(0)$ is indeterminate because $H(\theta,p,0)=0$, for all θ. The indeterminacy of θ implies that $H_\theta(\theta,1,0)=0$, ruling out application of the IFT.

We want to apply the Bifurcation Theorem, but we cannot apply it to $H(\theta,p,0)$ because $H_\theta(\theta,1,0)\neq 0$. Intuitively, the Bifurcation Theorem presented above requires that both θ and p are indeterminate at $\epsilon=0$. Moreover, we know $p'(0)$ if it exists. Implicit differentiation of $H(\theta(\epsilon),p(\epsilon),\epsilon)$ with respect to ϵ implies

$$H_\theta^i(\theta,p,\epsilon)\theta^{\prime}(\epsilon)+H_p^i(\theta,p,\epsilon)p^{\prime}(\epsilon)+H_\epsilon^i(\theta,p,\epsilon)=0.$$

For each i, $H_\theta^i(\theta,p(0),0)=0$ for all θ since $p(0)=1$. Therefore, if $p(\epsilon)$ is differentiable at $\epsilon = 0$, then

$$0 = H_p^i(\theta, p, 0)p^{'}(0) + H_\epsilon^i(\theta, p, 0) = \left(E\{z\} - p^{'}(0)\right) u_i^{'}(c_i)$$

for $i = 1, 2$, where $c_i = B_i^e + \theta_i^e$ is consumption in the no-risk case. Since $u_i^{'}(c_i)$ is never zero, $p^{'}(0) = E\{z\} = 0$ must hold if $\theta(\epsilon)$ and $p(\epsilon)$ are differentiable at $\epsilon = 0$. Therefore, we have indeterminacy of $\theta(0)$ but there is only a single possible value for both $p(0)$ and $p^{'}(0)$. This prevents us from using Theorem 7 directly since the Jacobian matrix $H^i_{(\theta,p)}$ is not a zero matrix.

This problem is solved by reformulating the problem in terms of the price of risk, not the price of the equity. More precisely, we assume the equity price parameterization

$$p(\epsilon) = 1 - \epsilon^2 \pi(\epsilon) \tag{22}$$

where $\pi(\epsilon)$ is the risk premium in the ϵ-economy. Since $\sigma_z^2 = 1$, ϵ^2 is the variance of risk and $\pi(\epsilon)$ is the risk premium per unit variance. Since we expect the risk premium to depress the price of equity, we use the form in (22).

We have assumed the parameterization in (22) but we have not proved anything yet. We now need to show that this parameterization is consistent with Theorem 7. To check the sufficient conditions in Theorem 7, we reformulate equilibrium as the system of equations

$$0 = \mathcal{H}^i(\theta, \pi, \epsilon) \equiv E\left\{u_i^{'}(Y_i)\,(z - \epsilon\pi)\right\} = 0. \tag{23}$$

where $\mathcal{H}^i(\theta, \pi, \epsilon) = \epsilon^{-1} H^i(\theta, 1 - \epsilon^2\pi, \epsilon)$, $i = 1, 2$. It is clear that (θ, π, ϵ) satisfy (23) if and only if they also satisfy (21).

The parameterization in (22) and the equilibrium characterization in (23) now allow us to apply the Bifurcation Theorem. The functions $\mathcal{H}^i(\theta, \pi, \epsilon)$ have the degeneracy assumed in Theorem 7 since $\mathcal{H}_\theta^i(\theta, \pi, 0) = \mathcal{H}_\pi^i(\theta, \pi, \epsilon) = 0$ for all (θ, π). Intuitively, at $\epsilon = 0$, any portfolio satisfies the first-order conditions since all assets are perfect substitititutes and any price of risk, π, is consistent with equilibrium since the total amount of risk is zero. The Jacobian matrix

$$\mathcal{H}_{(\theta,\pi),\epsilon} = \begin{bmatrix} \mathcal{H}^1_{\theta\epsilon}(\theta(0), \pi(0), 0) & \mathcal{H}^1_{\pi\epsilon}(\theta(0), \pi(0), 0) \\ \mathcal{H}^2_{\theta\epsilon}(\theta(0), \pi(0), 0) & \mathcal{H}^2_{\pi\epsilon}(\theta(0), \pi(0), 0) \end{bmatrix} = \begin{bmatrix} u_1^{''} & -u_1^{'} \\ u_2^{''} & u_2^{'} \end{bmatrix}$$

has determinant $u_1''u_2'+u_1'u_2'' < 0$. Therefore, all the sufficient conditions of Theorem 7 hold, and the Bifurcation Theorem provides a local proof of existence and uniqueness of solutions $(\theta(\epsilon), \pi(\epsilon))$ to (23). Theorem 12 summarizes the result.

Theorem 12. *If $u_i(c)$ is locally analytic for c near $B_i^e + \theta_i^e$, $i = 1, 2$, and $\mathcal{H}(\theta, \pi, \epsilon)$ is locally analytic near a solution (θ_0, π_0) to $\mathcal{H}_\epsilon(\theta, \pi, 0) = 0$, then there is some $\epsilon_0 > 0$ such that for all $\epsilon \in (-\epsilon_0, \epsilon_0)$ there is a unique analytic equilibrium selection $(\theta(\epsilon), \pi(\epsilon))$ such that $\mathcal{H}(\theta(\epsilon), \pi(\epsilon), \epsilon) = 0$.*

The basic approach to using the Bifurcation Theorem is to guess some parameterization for the unknown functions and then use the Bifurcation Theorem to check that it is correct and can produce a locally analytic approximation. Some of the choices we made, particularly the construction of (22) and (23), may appear arbitrary, but their use is validated by the Bifurcation Theorem. Our formulation is economically intuitive. For example, (22) just says that risk premia are proportional to variance. Therefore, application of these ideas to more complex problems is not difficult as long as we remember the intuition behind our construction. There are more complex versions of the Bifurcation theorem which would lead more directly to (22) and (23); see Zeidler [26]. We prefer the approach used here since it is straightforward once one uses economic intuition to arrive at (22) and (23).

Figure 2 displays the geometry of the bifurcation in (23). When $\epsilon = 0$, the entire $\theta - \pi$ plane constitutes an equilibrium. However, for nonzero ϵ we have a locally unique equilibrium. In Figure 2 the curve ABC represents the equilibrium manifold.

We can now proceed to compute asymptotic expressions for $(\theta(\epsilon), \pi(\epsilon))$. Direct computation shows that the bifurcation point $(\theta_0,\ \pi_0)$ for (23) is defined by $\mathcal{H}_\epsilon^i(\theta_0, \pi_0, \epsilon) = 0$, $i = 1, 2$, and satisfies the linear equations:

$$\begin{aligned} -u_1'(c_1)\pi_0 + u_1''(c_1)\theta_0 &= 0 \\ u_2'(c_2)\pi_0 + u_2''(c_2)\theta_0 &= u_2''(c_2) \end{aligned} \tag{24}$$

where $c_i = B_i^e + \theta_i^e$. The linear equations in (24) imply the unique candidate bifurcation point

$$\theta_0 = \frac{\tau_1}{\tau_1 + \tau_2},\ \pi_0 = \frac{1}{\tau_1 + \tau_2} \tag{25}$$

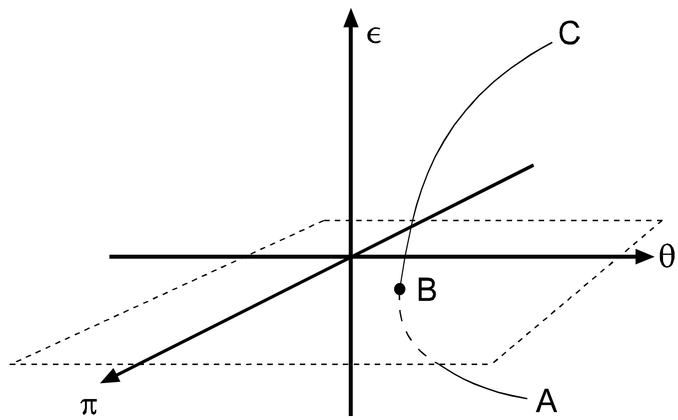

Figure 2: Bifurcation of Equilibrium Correspondance

where τ_i is evaluated at $c_i = B_i^e + \theta_i^e$, consumption in the deterministic limit. These formulas for θ_0 and π_0 are intuitive; the τ_i terms are the individual risk tolerances at $\epsilon = 0$, and the denominator is their sum, which is the social risk tolerance. The results are both very intuitive. The equilibrium risk premium is the inverse of total risk tolerance. Also, the fraction of equity held by investor 1 equals his contribution to social risk tolerance. These solutions resemble the intuitive results from mean-variance models.

The solution in (25) just tells us what the limit portfolio is as variance goes to zero. We want to know what the equilibrium portfolio is for nonzero variance. This requires computing the derivatives $\theta^{\theta}(0)$ and $\pi^{\theta}(0)$. Further implicit differentiations of $\mathcal{H}^i$ yield $(\theta^{\theta}(0), \pi^{\theta}(0))$ and any other higher-order derivative.

Theorem 13. *The first-order derivatives of the equilibrium correspondence* $(\theta(\epsilon),\ \pi(\epsilon))$ *at* $\epsilon = 0$ *are*

$$\theta'(0) = \frac{\tau_1}{\tau_1+\tau_2}\frac{\tau_2}{\tau_1+\tau_2}\frac{\rho_1-\rho_2}{\tau_1+\tau_2}E\left\{z^3\right\} \tag{26}$$

$$\pi'(0) = -\left(\frac{\tau_1}{\tau_1+\tau_2}\rho_1 + \frac{\tau_2}{\tau_1+\tau_2}\rho_2\right)\frac{E\left\{z^3\right\}}{(\tau_1+\tau_2)^2} \tag{27}$$

Therefore, type 1 investors increase their holdings of equity as ϵ *increases if* $(\rho_1 - \rho_2)E\left\{z^3\right\} > 0$, *and the risk premium per unit variance decreases as* ϵ *increases if* $E\left\{z^3\right\} > 0$.

Proof. Apply (6). ■

Theorem 13 gives us our first-order approximation to $\theta(\epsilon) = \theta_0 + \epsilon\theta'(0)$. We need to be clear what this tells us. For example, if $\theta'(0) > 0$ then we know that for all $\epsilon > 0$ sufficiently close to θ_0, $\theta(\epsilon)$ exceeds θ_0, and that $\theta(\epsilon)$ grows at rate $\theta'(0)$. We know this because $\theta(\epsilon)$ is locally analytic, implying that our Taylor series approximations are valid for ϵ sufficiently close to $\epsilon = 0$. This could be reversed for large ϵ with $\theta(\epsilon)$ less than θ_0. But, for sufficiently small ϵ, equations (26) and (27) tell us precisely how $\theta(\epsilon)$ and $\pi(\epsilon)$ behave.

Theorem 13 is economically intuitive. Equation (26) shows that the equity holdings of a type 1 investor are greater than θ_0 if ϵ is small and positive, if skewness, $E\{z^3\}$, is positive, and if his skew tolerance exceeds the skew tolerance of type 2 investors, where we evaluate skew tolerance at the $\epsilon = 0$ allocations. Equation (27) shows that the risk premium will decrease as ϵ increases (and the price of equity relative to bonds will increase) if skewness is positive. The magnitude of the change depends on a weighted sum of the skew tolerances, where the weights are the limit portfolio holdings. Notice that we get these results for any utility function, not just for CRRA utility functions or other families that have $u''' > 0$. The results in Theorem 13 resemble the style of analysis in Jones [12]. Jones examines the impact of changes in endowments on equilibrium, whereas we are examining the change in asset market equilibrium as we move away from the deterministic case. The problems are economically different but the mathematical idea is the same: use implicit function theorems or their generalizations to analyze the impact of small changes in parameters on equilibrium.

The derivatives $\theta'(0)$ or $\pi'(0)$ could be zero. This does not mean that $\theta(\epsilon)$ or $\pi(\epsilon)$ is constant for small ϵ. It just means that the local behavior is governed by higher-order terms in the expansion. For example, if $E\{z^3\} = 0$, then $\theta'(0) = \pi'(0) = 0$ and the local behavior of $\theta(0)$ and $\pi(0)$ is governed by $\theta''(0)$ and $\pi''(0)$, which depend on the kurtosis $E\{z^4\}$ and fourth-order properties of $u(c)$. We do not pursue these higher-order issues in this paper since Theorem 13 is adequate for the analysis below.

6. Asset Market Equilibrium with a Derivative Asset

The previous section examined a market with only a bond and a stock. In this section, we compare markets with different asset spans. In particular, we introduce a new derivative asset into the market and compute asymptotically valid expressions for equilibrium. The results allow us to single out important factors for these expressions.

We assume that the derivative pays ϵy and has price $q(\epsilon)$ in the ϵ-economy. We also assume that $y = f(z)$, which makes y a derivative security, such as an option. We force the payoff of the derivative to be zero when $\epsilon = 0$; hence, $q(0) = 0$. This implies no loss of generality since any portion of the asset's return which is deterministic given ϵ will be equivalent to the bond, adding nothing to the asset span. We assume that the net supply of the derivative is zero since we want to model the introduction of a derivative security. For instance, $y = \max[0, (z - S)]$ represents a call option, and ϵy is the call option $\max[0, \epsilon z - \epsilon S]$ with strike price ϵS. This may initially seem odd, but it is a standard option if $\epsilon = 1$. Also, if $F(z)$ is the cdf of z then the probability of exercise, $F(S)$, is unaffected by ϵ.

We decompose y into components that are spanned by the stock and bond, and a component orthogonal to the stock and bond. We assume

$$y = \overline{y} + \alpha z + \nu \tag{28}$$

where $\overline{y}$ is the mean of y, α is the covariance with z, the risky component of equity, and a nonzero random variable ν, the innovation in y. Therefore, $0 = E\{\nu\} = E\{z\nu\}$. This formulation implicitly assumes that markets are initially incomplete since we assume that ν is not spanned by 1 and z. For example, if z is a random variable with only two possible values, then the stock and bond span the market and there is no $y = f(z)$ such that ν in (28) is not identically equal to zero[6].

We compute the equilibrium holdings and prices of both assets. Let θ_i and B_i be the equity and bond holdings, and let ϕ_i be the units of y held by trader i after trading. The final wealth for trader i is $Y_i = \theta_i(1 + \epsilon z) + B_i + \phi_i \epsilon y$, and his budget constraint is $\theta_i p + B_i + \phi_i q = B_i^e + \theta_i^e p$. When we use the budget constraint to

[6] We could add securities which generate random shocks, such as pure gambling. Since investors are risk averse, there is no demand for such assets. Therefore, we ignore assets with pure noise payoffs.

eliminate B_i, the first-order conditions for θ_i and ϕ_i are

$$\begin{aligned} E\{u_i'(Y_i)(1+\epsilon z - p(\epsilon))\} &= 0,\ i=1,2 \\ E\{u_i'(Y_i)(\epsilon y - q(\epsilon))\} &= 0,\ i=1,2 \end{aligned} \tag{29}$$

Equilibrium is defined by combining the first-order conditions of type 1 and type 2 agents with the market clearing conditions; we shall compute the equilibrium values for θ_i, ϕ_i, p, and q as functions of ϵ in some neighborhood of $\epsilon = 0$. Let θ and ϕ denote θ_1 and ϕ_1; hence $\theta_2 = 1 - \theta$ and $\phi_2 = -\phi$. Similar to the analysis of previous section, $\theta(0)$ and $\phi(0)$ are indeterminate but $p(0) = 1$ and $q(0) = 0$.

We need to determine an appropriate parameterization for this problem, just as we did in the case of equilibrium with one asset. We implicitly differentiate the four first-order conditions in (29) with respect to ϵ, and find that differentiability of q and p at $\epsilon = 0$ requires $[E\{y\} - q'(0)]u_i'((B_i^e + \theta_i^e)) = 0$ and $[E\{z\} - p'(0)]u_i'((B_i^e + \theta_i^e)) = 0$. Therefore, if q and π are well behaved, $q'(0) = \overline{y}$ and $p'(0) = E\{z\} = 0$. We want to solve for θ, ϕ, p,and q as functions of ϵ, at least in some neighborhood of $\epsilon = 0$, and we need $p(0) = 1$, $p'(0) = E\{z\} = 0$ and $q(0) = 0$, $q'(0) = \overline{y}$. We choose the following parameterization:

$$p(\epsilon) = 1 - \epsilon^2\pi(\epsilon),\ q(\epsilon) = \epsilon\overline{y} - \epsilon^2\psi(\epsilon) \tag{30}$$

We next check if the parameterization in (30) is consistent with Theorem 7. The bifurcation point $(\phi_0, \theta_0, \pi_0, \psi_0)$ is computed by solving the system of linear equations

$$\begin{bmatrix} u_1''\sigma_y^2 & u_1''\sigma_{yz} & 0 & -u_1' \\ u_1''\sigma_{yz} & u_1''\sigma_z^2 & -u_1' & 0 \\ u_2''\sigma_{yz} & u_1''\sigma_z^2 & u_2' & 0 \\ u_2''\sigma_y^2 & u_2''\sigma_{yz} & 0 & u_2' \end{bmatrix} \begin{bmatrix} \phi_0 \\ \theta_0 \\ \pi_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ u_2''\sigma_z^2 \\ u_2''\sigma_{yz} \end{bmatrix}$$

which has the unique solution

$$\theta_0 = \frac{\tau_1}{\tau_1+\tau_2},\ \phi_0 = 0,\ \pi_0 = \frac{1}{\tau_1+\tau_2},\ \psi_0 = \frac{\sigma_{yz}}{\tau_1+\tau_2}. \tag{31}$$

The existence of solutions for $\phi(\epsilon)$, $\theta(\epsilon)$, $\pi(\epsilon)$, and $\psi(\epsilon)$ near the bifurcation point is established by applying Theorem 7 at the candidate bifurcation point (31). Furthermore, the first-order derivatives $(\theta'(0),\ \phi'(0),\ \pi'(0),\ \psi'(0))$, the second-order derivatives $(\theta''(0),\ \phi''((0),\ \pi''(0),\ \psi''(0))$, and other derivatives can be obtained by solving linear systems of equations as long as the utility function u is analytic at the deterministic consumption. Since the solutions are cumbersome, we omit them except for the first-order derivatives.

The results follow standard intuition. The equilibrium price of the derivative security is asymptotically equal to

$$q(\epsilon) = \epsilon E\left\{y\right\} - \epsilon^2 \frac{\sigma_{yz}}{\tau_1 + \tau_2} + O(\epsilon^3), \tag{32}$$

which tells us that the derivative y carries a positive risk premium (modelled here as a discount in the price) only if $\sigma_{yz} > 0$, that is, y is positively correlated with aggregate risk z. The limit price and holdings of equity are unaffected by the presence of the derivative, and trading volume for the derivative is zero in the limit.

We see again that a key step is finding an appropriate parameterization of asset prices. There is no precise, generally applicable formula describing how we arrived at the parameterization in (30) which allowed us to apply the Bifurcation Theorem, but the steps we have followed in the one- and two-asset problems are clear. We first compute derivatives of the equilibrium equations and examine them to see if some terms in the Taylor series of the unknown functions are fixed. For example, we found that $q'(0) = \overline{y}$ and $p'(0) = 0$ must be true if there is to be a coherent Taylor expansion. If conventional IFT methods indicate the value of low-order terms in an expansion, we then focus on the next higher-order term. Since $q'(0) = \overline{y}$ and $p'(0) = 0$, we then examined the parameterization in (30) where $\pi(\epsilon)$ and $\psi(\epsilon)$ became the unknown terms which could not be determined by applying the logic of the conventional IFT. We continued this for each unknown function until we reach a point where the terms in its expansion could not be fixed by the IFT. At that point we can apply the Bifurcation Theorem.

6.1. Trading Patterns for the Derivative Asset. We next determine the trading patterns of y. Since $\phi(0) = 0$, the value of $\phi'(0)$ determines the trading patterns for nonzero ϵ. Direct computation produces Theorem 14.

Theorem 14. *Type 1 investors buy the derivative y if and only if* $(\rho_1 - \rho_2)\, Cov(\nu, z^2) >$

0. *In general,*

$$\phi'(0) = \frac{\tau_1 \tau_2 (\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3} \frac{Cov(\nu, z^2)}{E\{\nu^2\}} \tag{33}$$

Recall that $\phi'(0) > 0$ means that trader 1 buys and trader 2 sells the derivative asset y. If type 1 investors have more skew tolerance and y provides the market with a new risk that is positively correlated with the tails of equity returns, then type 1 investors buy y and type 2 investors sell it. If $Cov(\nu, z^2) > 0$, the new asset y adds a type of riskiness that appeals to individuals with relatively high skew tolerance, and type 1(2) agents will buy y if $\rho_1 > \rho_2$ ($\rho_1 < \rho_2$).

If $Cov(\nu, z^2) = 0$ then we would need to examine $\phi''(0)$ to determine who buys the derivative. We do not pursue that here since no financial institution has an interest in introducing a derivative with no first-order volume. We continue to focus on derivatives where $Cov(\nu, z^2) \neq 0$.

6.2. Change in Equity Holdings. The derivative asset y may change investors' holdings of equity. Let $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$ denote the equilibrium holding of equity by type 1 investors without and with the derivative security.[7] At $\epsilon = 0$, $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$ will be the same since all assets will be equivalent. To compare the equilibria across these market structures, we compute the series expansion of both $\theta^b(\epsilon)$ and $\theta^a(\epsilon)$, and then use the difference in their series expansions to express the difference between the two market equilibria. We can do this for any index of market equilibrium. Direct computation shows Theorem 15.

Theorem 15. *Let $\theta^b(\epsilon)$ ($\theta^a(\epsilon)$) denote the equilibrium equity demand of type 1 investors without (with) the derivative y. Then*

$$\theta^a(\epsilon) - \theta^b(\epsilon) = -\frac{\tau_1 \tau_2 (\rho_1 - \rho_2)}{(\tau_1 + \tau_2)^3} \, \alpha \, \frac{Cov(\nu, z^2)}{E\{\nu^2\}} \, \epsilon + O(\epsilon^2) \tag{34}$$

If ν and z are uncorrelated, (34) reduces to zero, implying that the introduction of y has only $O(\epsilon^3)$ effects on the demand for the equity. If $\alpha = Cov(\nu, z) > 0$ then the change in type 1 investors' holding of equity is negatively related to their demand for y since (33) and (34) imply that $\theta^a_z(\epsilon) - \theta^b_z(\epsilon) = -\,\alpha\,\phi'(0) + O(\epsilon^2)$.

[7] Loosely speaking, θ^b is equilibrium equity holding "before" introduction of y and θ^a is holding "after" introduction.

6.3. Price Effects of the Derivative Asset. Our computations show that the equilibrium price for equity remains unchanged up to $O(\epsilon^3)$ in its Taylor expansion. The fourth-order term reveals the dominant effect of the derivative y on the price of equity.

Theorem 16. *Let $P^a(\epsilon)$ ($P^b(\epsilon)$) denote the equilibrium price of equity with (without) the derivative y. The price difference is*

$$P^a(\epsilon) - P^b(\epsilon) = 2\frac{\tau_1\tau_2\left(\rho_1-\rho_2\right)^2}{\left(\tau_1+\tau_2\right)^5}\frac{E\left\{\nu z^2\right\}^2}{E\left\{\nu^2\right\}}\epsilon^4 + O(\epsilon^5) > 0$$

In particular, the equity rises in value and rises more as the derivative is more correlated to the tails of equity returns, and as investors differ more in their skewness tolerance.

Theorem 16 shows the elements that affect the impact of the derivative on stock price. The price change is always positive, but depends on third-order properties of the utility function. The derivative asset y complements equity and allows investors to allocate tail risk independent of other risks. This makes equity more attractive.

Also, the magnitude is proportional to the covariance of the derivative's innovation ν with the extremes of equity returns. If ν is uncorrelated with those extremes then there is no price change to the order ϵ^4. There may be a price effect but it would be an order of magnitude smaller asymptotically.

6.4. Welfare Effects of the Derivative Asset. We next derive the effect of a derivative on the welfare of each trader. Theory tells us that in one-good models such as ours, individual investors may gain or lose utility from adding an asset, but someone must gain. Our solutions will add some precision to those statements.

With the derivatives computed by the bifurcation method, we can study the welfare effect of the derivative y. Precisely, we shall expand the utility functions in terms of ϵ and examine the dominated term. Let $U_i^b(\epsilon)$ and $U_i^a(\epsilon)$ denote trader i's optimal utility levels without and with y. The utility effect can be expressed by $[U_i^a(\epsilon) - U_i^b(\epsilon)]/u_i^{\prime}(B_i^e + \theta_i^e)$, a measure of the welfare change in terms of a consumption equivalent. The following theorem summarizes the result of our perturbation analysis.

Theorem 17. *Let $U_1^a(\epsilon)$ and $U_1^b(\epsilon)$ denote the equilibrium expected utility of type 1 investors with and without the derivative y. Then*

$$\frac{U_1^a(\epsilon) - U_1^b(\epsilon)}{u_1^{0}} = \frac{\tau_1^2 \tau_2^2 \left(\rho_1 - \rho_2\right)^2}{2\left(\tau_1 + \tau_2\right)^6} \left(4\left(\frac{\theta_1^e}{\tau_1} - \frac{\theta_2^e}{\tau_2}\right) + \frac{1}{\tau_1}\right) \frac{E\left\{\nu z^2\right\}^2}{E\left\{\nu^2\right\}} \epsilon^4 + O(\epsilon^5)$$

The second trader's welfare change is symmetrically expressed.

Again, the result corresponds to basic theory. The key term is $4\left(\theta_1^e/\tau_1 - \theta_2^e/\tau_2\right) + \tau_1^{-1}$, which may be positive or negative. The term $\theta_1^e/\tau_1 - \theta_2^e/\tau_2$ is proportional to the amount of equity type one investors sell to type two investors in the limit as ϵ goes to zero. If there is no equity trade asymptotically then the dominant impact on utility is the improved opportunity for risk-sharing provided by the introduction of y. The risk-sharing gain is proportional to τ^{-1}, which is absolute risk aversion, for type i investors. If $\theta_1^e/\tau_1 - \theta_2^e/\tau_2 \neq 0$, the investor type that sells shares also gains from the equity price increase caused by the introduction of the derivative asset. So, one type gains from the price increase and the other loses, but both gain from new risk-sharing opportunities. One of the investors may lose, but not both.

The results in Theorems 14, 15, 16, and 17 demonstrate the importance of higher-order expansions. Linear-quadratic expansions would completely miss all of the effects studied in these theorems since $\rho = 0$ for linear-quadratic utility functions. Approximation methods that only use the first two derivatives of utility functions would incorrectly predict that adding y would have no effect on equilibrium. The advantage of the approach used here is that one need not make a choice about how many derivatives to use since that decision is automatically made by the power series generated by the bifurcation (and the IFT) approach. The mechanical computation of the power series expansions of equilibrium prices and quantities tells us which power of ϵ contains the asymptotically dominant effects, and which derivatives of utility and which moments of returns should be used.

7. Computational Considerations

The analysis above focused on applying the bifurcation method to a simple asset market model. The results were obtained only after much computational effort. Theorem 17 is a good example of why the computer is necessary. Since the effect of the derivative asset y on utility was zero at orders ϵ^2 and ϵ^3, we had to compute the fourth-order Taylor series expansion of utility. Also, equilibrium utility is a function of all four variables determined in equilibrium, the two premia and the two portfolio variables. These four equilibrium variables are locally analytic functions of ϵ. Therefore, Theorem 17 required a fourth-order expansion of a four-dimensional function where each argument is a fourth-order Taylor series in ϵ. This resulted in thousands of intermediate terms. The final result in Theorem 17 is compact since almost all of the intermediate terms disappear when they are evaluated at $\epsilon = 0$. However, the intermediate terms must be kept until that last step. The computations in this paper took only a few minutes using Mathematica on a 400 MHz machine, but would be impossible for us to do without a computer.

This paper used the computer to derive algebraic formulas and theoretical asymptotic results. The computational burden was particularly heavy since we were interested in general formulas expressing the results in terms of elasticities, shares, and prices. The computational costs will rise rapidly as we move to larger problems with more types of investors and/or more assets. However, as we gained experience with the simple model we discovered patterns which we can incorporate into the code to substantially improve performance and make possible examination of more complex models. For example, the definitions of risk tolerance and skew tolerance, and the decomposition in (28) substantially reduced the complexity and length of the formulas. With Mathematica and these simplifications, we can now handle larger problems, such as problems with four investor types and four assets.

The Taylor series expansions for equilibrium price correspondences $p(\epsilon)$ and portfolio allocations $\theta(\epsilon)$ could also be used to arrive at numerical approximations for specific utility functions and asset return distributions. The bifurcation method then reduces to computing the numerical values of all derivatives of the equations defining equilibrium up to the fourth order at $\epsilon = 0$, and then executing numerical linear operations instead of symbolic operations. Since numerical operations are faster and more compact than symbolic operations, computing expansions for specific examples would be far faster. The computer could handle much larger problems if we specify all utility functions and returns.

We would like to know how well these formulas do for nontrivial ϵ. In general, a power series constructed by the IFT for analytic functions will have a positive radius of convergence, but we know nothing about its magnitude in general. However, there is a simple diagnostic which can help. Suppose that $h(x)$ is implicitly defined by $H(x, h(x)) = 0$ and that we construct the degree k Taylor series approximation $h^*(x)$ based at $x = x_0$. If $h^*(x)$ is a good approximation to $h(x)$ then $H(x, h^*(x))$ should be nearly zero. Once we have computed $h^*(x)$, we can evaluate its quality by computing $H(x, h^*(x))$ for various values of x. The behavior of $H(x, h^*(x))$ as x moves away from x_0 will indicate where the approximation can be trusted. Judd and Guu [14] applied this approach to similar approximations of stochastic growth models. We have constructed examples of the asset models studied in this paper for which our Taylor series approximations for $p(\epsilon)$ and $\theta(\epsilon)$ imply very small Euler equation errors. Roughly, we found that the method does well if the disturbance z has compact support, but does poorly if z is log Normal, a finding consistent with the fact that making z a log Normal random variable makes it unlikely that $H(\theta, \epsilon)$ is analytic.

More generally, we could compare the results of our approach for large ϵ with the numerical approach in Schmedders [23]. If our formulas work, then they would produce results faster than Schmedders [23], but our formulas will not work for the large ϵ cases where Schmedders' algorith would work. There could be a partnership between the two approaches with our Taylor-style expansions used to produce an initial guess for Schmedders' algorithm. Further discussion and serious examination of these numerical issues must be left for another paper.

We used Mathematica to compute our results. Space limitations prevent us from presenting and explaining the code here. The reader can obtain the code by sending e-mail to judd@hoover.stanford.edu, or by going to the webpage http://bucky.stanford.edu/ or the Economic Theory webpage for this paper.

8. Generalizations

This paper has examined a few simple problems, but we believe that the same tools can be used to examine a large class of models. We briefly discuss those claims here.

This paper assumed a single good, two types of agents, and only one source of risk. Space limitations prevent us from presenting an analysis for more general cases, but we can outline the general approach. Adding more types of agents and more assets but staying with one good is a direct generalization of the methods above. The equilibrium in our examples were expressed as first-order conditions for each agent with respect to each asset. Adding agents and assets just implies a longer list of first-order conditions but the key elements are unchanged: the deterministic consumption levels are fixed at the endowment, the price of risk, π, and portfolio allocations, θ, are indeterminate in the deterministic model, and we can parameterize θ so that the Bifurcation theorem applies to a system of equations $H(\pi, \theta, \epsilon) = 0$ which include individual first-order conditions and market-clearing conditions.

The generalization to several goods is more complex. Let p be the price vector for goods, π the vector of prices of risk for the assets, and θ the allocation of assets across agents. In GEI models with several goods, equilibrium can be expressed as the solution to a system of equations $H(p, \pi, \theta, \epsilon) = 0$ where the components of H are the agents' first-order conditions over asset and consumption choices plus feasibility conditions. The excess demand for assets may not exist at some prices because of arbitrage; therefore, H may not be continuous. However, theory tells us that equilibrium will generically exist. If we let ϵ parameterize uncertainty then a system $H(p, \pi, \theta, \epsilon) = 0$ would represent equilibrium in the ϵ-economy and implicitly define equilibrium maps $p(\epsilon)$, $\pi(\epsilon)$, and $\theta(\epsilon)$. At $\epsilon = 0$, the economy reduces to a deterministic Arrow-Debreu general equilibrium. There will be trade in the goods in the deterministic limit economy, and goods' prices $p(0)$ will be determined by equilibrium conditions. Asset prices in the deterministic limit, $q(0)$, will also be determined by $p(0)$. The goods prices and asset prices would generically be locally determinate by the standard general equilibrium theory. However, the portfolio decisions $\theta(0)$ will be indeterminate in the $\epsilon = 0$ economy since all assets would be perfect substitutes. If asset prices in general can be represented as $q = q_0 - \epsilon^2 \pi(\epsilon)$, just as in equation (22) for the two-asset case, then the limit prices for risk, $\pi(0)$, measured in terms of excess return per unit variance, will be indeterminate since the level of risk is zero.

The geometrical structure of the GEI problem is illustrated in Figure 3. Let the axis labeled Δ denote the price simplex for goods, and the axis labeled (π, θ) represent the prices of risk and portfolio allocations of the risky assets. As in Figures 1 and 2, the ϵ axis in Figure 3 represents the level of risk. Suppose that the arc ABC

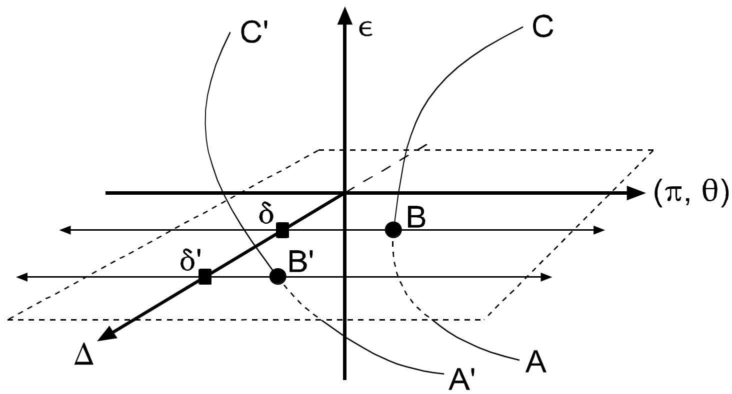

Figure 3: Bifurcation diagrams for general equilibrium problems

describes equilibrium values for p, π, and θ as ϵ changes. When $\epsilon = 0$, the problem reduces to an Arrow-Debreu model and equilibrium fixes goods' prices p at some point, say δ, in Δ, but the price of risk π and portfolio holdings would be indeterminate. Therefore, any point along the line $\overleftrightarrow{\delta B}$ would be an equilibrium. In order to analyze the arc ABC we need to find B. We analyze the Jacobian $H_{(p,\pi,\theta)}$ to find some suitable parameterization for $p(\epsilon)$, $\pi(\epsilon)$, and $\theta(\epsilon)$ such that the Bifurcation Theorem applies and produces B. The parameterization $q(\epsilon) = q_0 - \epsilon^2 \pi(\epsilon)$ corresponds to the robust result that risk premia are related to the variance of risk, indicating that the Bifurcation Theorem should continue to apply. There may be cases where the bifurcation method used above does not apply, but we conjecture that this approach will often succeed since, generically, equilibrium does exist for endowment economies with incomplete asset markets.

The multicommodity case would produce more complex results. For example, there could be a second equilibrium arc, such as $A'B'C'$ which corresponds to a second set of equilibrium prices at δ' for goods in the deterministic economy. That does not present any essential difficulty as long as the local properties of the system of equilibrium equations $H(p, \pi, \theta, \epsilon) = 0$ satisfies the bifurcation theorem. Other complex possibilities may arise, such as multiple equilibrium arcs passing through a bifurcation point B. The bifurcation methods presented in this paper cannot handle such a case, but, fortunately, there are more powerful tools from bifurcation and singularity theory which could handle some of these problems. Presumably, the variety of welfare results in Hart, Elul, and Cass and Citanna, would also arise asymptotically in multigood economies. The key point is that the situation in Figure 3 is conceptually similar to the structure in Figures 1 and 2, and basic tools from bifurcation theory should be able to handle many multicommodity models.

9. Conclusion

We have used bifurcation approximation methods to examine simple asset market problems with small noise. The analysis produces a mean-variance-skewness-etc. theory of asset demand and asset market equilibrium, and found several interesting results. We found that the addition of derivative asset will increase the price of the underlying equity stock. Also, the demand for a derivative asset depends on skewness properties of asset returns and the relative skew tolerance of investors. These results indicate that skewness and skew tolerance will be important determinants of asset innovation in more general contexts and indicate that results from linear-quadratic or mean-variance models are of limited relevance. The approach also shows that, in small noise economies, equilibrium depends on the utility properties of traders and the moments of returns, not on the number of contingent states. The asymptotic approach provides more intuitive results than the usual state-contingent approach.

The mathematical tools are quite general and can be applied to far more complex problems. Zeidler shows that the critical bifurcation theorems hold in Banach spaces. For example, partial differential equations that characterize asset prices in continuous time can also be approximated by examining bifurcations of deterministic cases. The steps in such an application of the bifurcation theorem require the solution of linear partial differential equations.

This paper focussed on qualitative analyses, but the expansions derived here could have value as a numerical method for solving specific cases; we leave that possibility for another study. This paper focussed on applications of bifurcation methods but many of the same points could be made for applications of the IFT. Economists are familiar with comparative statics analysis, such as that in Jones [12], but that is generally limited to first-order expansions. Higher-order approximations could often be used to improve qualitative and quantitative analysis of economic models.

The necessary mathematics for deriving expansions have been known for a long time, but the cumbersome algebra made them impractical until now. Fortunately, the speed of modern computers and the availability of symbolic language software now makes bifurcation methods, and similar perturbation methods, a practical way to address important economic problems.

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