Computational Methods for Dynamic Equilibria with Heterogeneous Agents

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Abstract

We survey recent developments in computational methods for solving dynamic models with several agents. We examine both advances in solving perfect foresight models and time homogeneous dynamic stochastic models. We present the ideas behind both projection and perturbation methods. Computational methods have been particularly useful in solving models with incomplete asset markets, so we use them to highlight the general computational challenges we face in dynamic models.

Contents

1 Introduction

Computational methods have become increasingly important in the analysis of dynamic general equilibrium problems. These methods are being used, for example, to study the incidence of tax and monetary policies in dynamic models of growth, commodity storage in various models of agricultural commodity markets, and price formation in dynamic models of asset markets. Many early computational methods relied primarily on intuitive economic tatonnement stories, and produced moderately successful algorithms. Even when these methods worked, there were usually slow. Furthermore, as we know from general equilibrium theory, tatonnement methods may not converge even with good initial guesses. In the past decade the computational literature has made more use of formal mathematical tools from numerical analysis and perturbation theory. This has resulted in more powerful algorithms which can attack increasingly complex problems. These developments are particularly important when we try to solve models with several agents. This essay reviews the key ideas used in recent work, gives some examples of their advantages, and indicates the directions future work will likely take.

It is particularly appropriate that the 2000 World Congress of the Econometric Society include a survey of recent computational literature since computational methodology is inherently an important part of what is broadly called "econometrics." Ragnar Frisch, in his editorial in the initial issue of Econometrica, defined econometrics as the "unification of the theoretical-quantitative and the empirical-quantitative approach to economic problems." He said:

> This emphasis on the quantitative aspect of economic problems has a profound significance. Economic life is a complex network of relationships operating in all directions. Therefore, so long as we confine ourselves to statements in general terms about one economic factor having an effect on some other factor, almost any sort of relationship may be selected, postulated as a law, and explained by a plausible argument. Thus, there exist a real danger of advancing statements and conclusions which-although true as tendencies in a very restricted sense-are nevertheless thoroughly inadequate, or even misleading if offered as an explanation of the situation. To use an extreme illustration, they may be just as deceptive as to say that when a man tries to row a boat forward, the boat will be driven backward because of the pressure exerted by his feet. The rowboat situation is not, of course, explained by finding out that there exists a pressure in one direction or another, but only by comparing the relative magnitudes of a number of pressures and counter-pressures. It is this comparison of magnitudes that gives a real significance to the analysis. Many, if not most, of the situations we have to face in economics are of just this sort.

Dynamic general equilibrium problems are excellent examples of problems with "a complex network of relationships operating in all directions." Recent work on computational methods for dynamic models shows the value of uniting economic theory and mathematics to create a quantitative analysis of economic problems, and this essay reviews some of these developments. Frisch also said that "[Mathematics] is indispensable in a great many cases. Many of the essential things in the new setting of the problems are so complex that it is impossible to discuss them safely and consistently without the use of mathematics." We will also see examples where intuitive ad hoc schemes lacking a proper mathematical foundation can give unreliable answers to economic questions, but that, fortunately, there are sound methods from numerical analysis which can be used instead.

In this paper we address the problem of computing equilibria of dynamic economic models with special attention to the problems which arise when there are several agents. Some of these techniques are straightforward generalizations of methods applied to representative agent models, but heterogeneous agent models often present new problems requiring new techniques. We stress that the focus of this survey are the recent advances in computational methods; we do not attempt to survey all applications in the applied dynamic general equilibrium literature[1]. Dynamic models with heterogeneous agents are inherently difficult to exposit precisely and compactly. Therefore, we use simpler models to illustrate many computational concepts and then indicate how they have been applied more generally to heterogenous-agent models.

Most dynamic problems in economics have structure that numerical methods can exploit. Some problems are time homogeneous and some have time dependencies; we will distinguish between the two cases since computational methods differ. Our discussion is organized as follows. Section 2 discusses recent developments in solving perfect foresight models. They are extensively used in the applied macroeconomics literature to examine problems in large models where calendar time enters in the analysis because of, for example, unanticipated and partially anticipated policy or productivity shocks. Many economic problems, such as real business cycle models, have a time homogeneous character. Section 3 presents simple examples of time homogeneous models that we will use in our exposition. The projection method from the numerical analysis literature is a powerful tool for solving dynamic models. The framework of the projection method allows us to analyze most algorithms used to solve stationary dynamic economic models and suggests many new, potentially more powerful, algorithms.

[1] One problem which we had in surveying the literature is the incomplete manner in which computational issues are often presented. Authors (sometimes because of editors) often treat computational issues in a casual fashion, declining to cite relevant work or reveal the details of their method. It is often difficult to get relevant computer code in an understandable form. This makes it difficult to ascertain the computational contribution of many papers. Our survey focussed on papers which do reveal to the reader basic details about the computational methods they use. Many other papers were ignored because any comments would have been largely speculative.

Section 4 presents the details of projection methods for solving functional equations. In Section 5 we provide some details on how to apply projection methods for stationary dynamic economic models. Section 6 describes an infinite-horizon model with finitely many agents and incomplete markets, and is an example of asset pricing problems that have received much attention in the past decade. It is difficult to compute equilibria in these models, and the various methods used in this literature illustrate the evolution of computational methodology in economics. Section 7 presents the first methods developed for such models yielding a rough approximation of the equilibria. Section 8 presents methods for solving the dynamic incomplete asset market model that combine continuous approximations of pricing and trading strategies, new methods from the computable general equilibrium literature on solving models with incomplete asset markets, and projection methods. Section 9 presents an example illustrating the computational difficulties that arise naturally in asset market models. Section 10 presents some recent solution methods for problems where agents, such as governments, have market power. Section **??** presents some initial work on numerical methods for solving problems with asymmetric information. Some models are too large for projection methods to solve, but some large models can be analyzed using perturbation methods. Section 11 presents perturbative methods for solving dynamic economic models. Section 12 concludes with some comments on where the literature seems to be going.

2 Perfect Foresight Models

Perfect foresight models are often used to analyze dynamic economic questions, and were the first models for which numerical methods were developed. The typical model has a relatively simple dynamic structure. Let $x_t \in \mathsf{R}^n$ be a list of time t values for economic variables such as consumption, labor supply, capital stock, output, wages, etc., and $z_t \in \mathsf{R}^m$ a list of exogenous variables, such as productivity levels, tax rates, and monetary growth rates, at time t. Perfect foresight models have the form

$$\begin{aligned} g(t, X, Z) &= 0, \ t = 0, 1, 2, ... && (1) \\ x_{0,i} &= \overline{x}_{0,i}, \ i = 1, 2, ..., n_I && (2) \\ x_t & \quad \text{bounded} && (3) \end{aligned}$$

where

$$\begin{aligned} X &\equiv (x_0, x_1, x_2, ..., x_s, ...) \\ Z &\equiv (z_0, z_1, z_2, ..., z_s, ...) \end{aligned}$$

and $g(t, X, Z) : \mathsf{R} \times \mathsf{R}^{n \times \infty} \times \mathsf{R}^{m \times \infty} \to \mathsf{R}^n$ is a collection of n functions representing equilibrium. The equations in (1) include Euler equations, market-clearing conditions, and any other equations in the definition of equilibrium. Some economic variables have fixed predetermined values at $t = 0$ represented by the $n_I < n$ conditions in (2). Boundedness conditions in (3) provide additional conditions which tie down equilibrium. We need to find a bounded sequence of values for x_t satisfying all the equations in (1,2).

A good example is the optimal growth problem

$$\begin{array}{ll} \max\limits_{c_t} & \sum\limits_{t=0}^{\infty} \beta^t u(c_t) \\ s.t. & k_{t+1} = F(k_t) - c_t \\ & k_0 = \overline{k}_0 \end{array} \tag{4}$$

The solution to (4) satisfies the Euler equation $u'(c_t) = \beta u'(c_{t+1}) F'(k_{t+1})$. In the notation of (1,2), we define $x \equiv (c, k)$ (there are no exogenous variables) and express the solution to (4) as

$$\begin{array}{rcl} g_1(t, X) & \equiv & u'(c_t) - \beta u'(c_{t+1}) F'(k_{t+1}) = 0, \; t = 0, 1, 2, ... \\ g_2(t, X) & \equiv & k_{t+1} - F(k_t) + c_t = 0, \; t = 1, 2, ... \\ k_0 & = & \overline{k}_0 \end{array} \tag{5}$$

The capital stock has a predetermined value at $t = 0$. We shall use (5) below as an example. (5) is a problem with one type of agent and one good, but the approach can be used to analyze more general models. When we have heterogeneous agents, multiple goods, multiple sectors, and/or multiple countries, equilibrium consists of Euler equations for each type of agent for each decision variable, market-clearing conditions for each market, and any other equilibrium conditions. These are all stacked into the list $g(t, X, Z)$ in (1). Perfect foresight models are used to examine stochastic problems by allowing the z_t to represent shocks and then solving (1,2) for many possible realizations of Z[2].

The first large, rational expectations models were perfect foresight models of the form in (1,2,3). The Fair-Taylor (1983) method was the first one developed for such models. More recently, economists have applied methods from the mathematical literature on solving large systems of equations, and have applied projection methods to perfect foresight models. This section reviews and compares some of methods proposed for solving perfect foresight models.

2.1 General Considerations

Perfect foresight models are essentially nonlinear equations in R^{∞}. The forward-looking aspect of dynamic general equilibrium analysis creates links between current and future economic variables, and generates an infinite system of nonlinear equations with an infinite number of unknowns. Under some conditions,

[2]See Fair and Taylor (1990) for an example of this approach to solving stochastic rational expectations models.

there will be a locally unique solution. For example, (5) has a unique solution for any k_0. All methods we discuss assume local uniqueness.

These models are often Arrow-Debreu general equilibrium problems but their large size makes conventional computational general equilibrium procedures like Scarf's algorithm and homotopy procedures impractical. Any solution method must reduce the problem in some way. Most methods use *domain truncation* to reduce the problem to a finite-horizon problem. That is, they solve the truncated problem

$$\begin{aligned} g(t, x_0, x_1, ..., x_T, x^*, x^*, ..., Z) &= 0, \ t = 0, 1, 2, ..., T & (6) \\ x_{0,i} &= \overline{x}_{0,i}, \ i = 1, 2, ..., n_I & (7) \end{aligned}$$

where x^* is the steady state value of x. Some components of x_T are also fixed at their long-run values to make the number of equations in (6,7) equal to the number of unknowns. Domain truncation reduces (1,2,3) to a system of nT nonlinear equations in nT unknowns. There is no boundedness equation in (6,7) since (6) imposes $x_t = x^*$ for $t > T$. Since T must be large in order to be an acceptable approximation for the total dynamic process, we still cannot use conventional methods.

There is always the question of what T should be. Any method should try alternative values for T and accept a solution only when the choice of T does not substantially affect the solution. There are difficulties with this approach. For example, Kehoe has shown that the solution can be very sensitive to the choice of T, settling down only for very large values of T.

2.2 Gauss-Jacobi and Gauss-Seidel methods

Perfect foresight models of the form

$$g(t, x_t, x_{t+1}, Z) = 0 \tag{8}$$

are solved by using methods from the literature on solving large systems of equations. Fair and Taylor (1983) introduced an intuitive approach. They begin with an initial guess $X^0 = (x_1, x_2, ..., x_T, x^*, x^*, ...)$, which incorporates the domain truncation approach. Then they use the time t equation $g(t, x_t, x_{t+1}, Z) = 0$ to compute a new guess for x_t given the initial guess for x^0_{t+1}. In general, the $i+1$'st guess for X, denoted X^{i+1}, is constructed componentwise by solving

$$g(t, x_t^{i+1}, x_{t+1}^i, Z) = 0, \ t = 1, 2, ... \tag{9}$$

for x_t^{i+1}, the time t component of X^{i+1}. Their scheme is a block Gauss-Jacobi scheme since only elements of X^i are used to compute X^{i+1}. Solving for x_t^{i+1} given x_{t+1}^i in (9) is also a nonlinear equation, but it is generally of moderate size and solvable by conventional schemes, such as Newton or Gauss-Seidel. They also suggest that one try different truncation times T until changes in T create small changes in the solution. The Fair-Taylor scheme is reliable but tends to be slow because of the Gauss-Jacobi structure. The slow speed makes it difficult to solve with high accuracy since tight accuracy targets would require too many iterations. Also, Gauss-Jacobi schemes may not converge even if one begins with a good initial guess.

Since T is typically large, we need to develop special methods. Fortunately, we can apply methods from the literature on solving large systems (see, e.g., Kelley (1995), Saad (1996), Young (1971)). Some schemes reorder the equations in order to accelerate convergence. Some examples of this approach are Hall (1985), Fisher et al. (1986), Fisher (1992), and Hughes Hallett and Piscitelli (1998). Convergence of such methods depends on the order of the equations and is linear at best. The advantages are their simplicity and small memory requirements. However, they may not converge even after using various strategies including reordering of equations and damping factors.

2.3 Newton-Style Methods

More recently, some authors have used Newton and related methods to solve dynamic economic problems. Newton's method for solving the system of equations $g(x) = 0$ is the iteration $x^{k+1} = x^k - J(x^k)^{-1} g(x^k)$. Newton's method converges rapidly if the initial guess x^0 is good. Unfortunately, Newton's method is impractical for general large systems because the Jacobian of a system of a system of n equations has n^2 derivatives, an impractically large amount of computation if n is large. However, Newton methods can be applied to models with the simple lag structure in (8) because the Jacobian for perfect foresight problems is sparse, that is, most elements are zero.

The L-B-J algorithm (see Boucekkine, (JEDC, 1995), and Juillard et al (JEDC, 1998)) takes notice of special structure in many perfect foresight models and exploits it to apply Newton's method. Since the time t equation $g(t, x_t, x_{t+1}, Z) = 0$ involves only x_t and x_{t+1}, each row in a Jacobian involves only a small fraction of all the unknowns. Let $g_i(t, x_t, x_{t+1})$ denote $\partial g(t, x_t, x_{t+1})/\partial x_i$. The Jacobian of (6,7) is

$$J(x) = \begin{pmatrix} g_1(1, x_1, x_2) & g_2(1, x_1, x_2) & 0 & \cdots \\ 0 & g_2(2, x_2, x_3) & g_3(2, x_2, x_3) & \cdots \\ 0 & 0 & g_3(3, x_3, x_4) & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \end{pmatrix}$$

and is a sparse matrix for large n. Specifically, $J(x)$ is nearly diagonal here in the sense that all nonzero elements of $J(x)$ are within n columns of the diagonal even though there are nT columns in $J(x)$.

This fact can be used in a Newton approach. More precisely, iteration $k+1$ of Newton's method solves

$$\begin{aligned} J(x^k)\Delta &= -g(x^k) \\ x^{k+1} &= x^k + \Delta \end{aligned} \tag{10}$$

Since $J(x)$ is sparse, one can use sparse matrix methods to solve the linear equation $J(x^k)\Delta = -g(x^k)$ for the Newton step Δ. Juillard et al.(1998) examined a Newton strategy exploiting this sparseness and was able to solve large problems faster than Gaussian methods. such as the Fair-Taylor method, and do so with high accuracy.

Solving (10) can be difficult if x and T are large even if the Jacobian $J(x)$ is sparse. Gilli and Pauletto (1998) economize on this by using Krylov methods to compute an approximate solution to (10). An approximate solution is adequate since the important is to arrive at some Δ which takes the iteration in the right direction. Krylov methods find an approximate solution by projecting (10) into a smaller dimension and solving the projected problem. Gilli and Paulletto (JEDC, 1998) report significant gains in algorithm speed over sparse Newton methods.

Some Gaussian methods have an economic motivation, often turning on learning ideas. For example, one way to interpret Fair-Taylor is to say that agents compute their actions given expectations, but then those computed actions form the next set of expectations. While the story-telling approach to solving dynamic economic models has some intuitive appeal, it produces algorithms that converge linearly if at all. While Newton's method and other methods from the numerical analysis literature have no obvious economic "story," they bring the possibility of more rapid convergence and more accurate solutions.

2.4 Parametric Path Method

The parametric path approach proposed in Judd (1999) employs a substantially different strategy to solve (1,2). Instead of treating each component of $X = (x_0, x_1, ...)$ as independent, it uses information about how the true value of x_t evolves over time. For example, the sequence 1,2,1,2,.. is not likely to represent a quarterly series for the capital stock or even aggregate consumption. Capital stock sequences will be relatively smooth since the capital stock cannot change quickly. Consumption sequences are also likely to be smooth when agents have concave utility functions. This feature of the solutions is not exploited by standard methods since they treat each distinct x_t separately. Instead, our intuition says that the sequence $(x_0, x_1, ...)$ should be a smooth function of time t. This insight allows us to dramatically reduce (1,2) to a much smaller system to which we can apply methods that could not be used directly on (1,2).

The key idea behind the parametric path method can be illustrated in its application to (5). Theory tells us that the capital sequence which solves (5) converges asymptotically to the steady state at linear rate λ where λ is the stable eigenvalue of the linearization of (5) around the steady state capital stock k^{ss}. We also know that the time path of capital is "smooth" and that convergence is asymptotically monotone. This, together with the initial condition $k(0) = k_0$, suggests the parameterization

$$K(t;a) = \left(k_0 + \sum_{j=1}^{m} a_j t^j\right) e^{-\lambda t} + k^{ss}\left(1 - e^{-\lambda t}\right) \tag{11}$$

There are two key features of (11). First, for any coefficients a, $K(t;a)$ converges to k^{ss} because of the exponential decay term $e^{-\lambda t}$. Second, $k_0 = K(0;a)$ for any a. Therefore, (11) automatically satisfies both the initial conditions and the boundedness condition for any coefficients a. These facts allow us to focus on finding an $a \in \mathsf{R}^m$ that produces a good approximate solution to (5) without getting sidetracked by convergence problems.

The system (5) is equivalent to the second-order difference equation

$$u'\left(F(k(t)) - k(t+1)\right) = \beta u'\left(F(k(t+1)) - k(t+2)\right) F'(k(t+1))$$

We want to approximate $k(t)$ with $K(t;a)$ for some a. Therefore, the parametric path method defines the residual function

$$R(t;a) = u'\left(F(K(t;a)) - K(t+1;a)\right) - \beta u'\left(F(K(t+1;a)) - K(t+2;a)\right) F'(K(t+1;a))$$

and searches for an $a \in \mathsf{R}^m$ which makes $R(t;a)$ close to being close to zero for all t. Note that $R(t;a)$ is well-defined for any real value of t, not just the integers, since $k(t)$ is defined for all t in (11). Since $K(t;a) \to k^{ss}$ as $t \to \infty$, $R(t;a) \to 0$ as $t \to \infty$. Therefore, the Euler equation is satisfied asymptotically for any a, allowing us to focus on making $R(t;a)$ small at finite values of t. To identify the coefficients a, we define the set of projection formulas

$$P_j(a) = \sum_{t=0}^{\infty} R(t;a) t^i e^{-\lambda t} dt,\ i = 0, 1, ... \tag{12}$$

The summation in (12) is infinite, but by combining orthogonal polynomial theory and appropriate changes of variables, Judd (1999) derives good choices of weights ω_i and times t_i such that

$$\widehat{P}_j(a) = \sum_{i=0}^{N} \omega_i R(t_i;a) t_i^j,\ j = 0, 1, ...$$

is a good approximation of (12). for some weights ω_i and times , and then uses Newton's method to find coefficients $a \in \mathsf{R}^m$ that solve the system

$$\widehat{P}_j(a) = 0,\ j = 1, .., m$$

This is a simple example of the projection method described below detail.

There has been steady progress made in solving large perfect foresight systems, and we expect progress to continue. The new developments have a common approach. They all exploit the dynamic structure of the problem more extensively than did the Fair-Taylor procedure, and they also bring appropriate methods from the numerical analysis literature on solving large problems.

3 Time Homogeneous Dynamic Economic Problems

Dynamic economic models often take a stationary form. Equilibrium of stationary problems can be expressed in feedback rules, expressing the free endogenous variables, such as prices, consumption, labor supply, etc., as functions of the predetermined variables, such as capital stocks and lagged productivity levels. They also often involve uncertainty about productivity, policy, or other exogenous economic factors. These models take the form

$$\begin{aligned} 0 &= E\left\{g(x_t, y_t, x_{t+1}, y_{t+1}, z_{t+1}) | x_t\right\} \\ x_{t+1} &= F(x_t, y_t, z_t) \end{aligned}$$

where x_t is a vector of variables that are pre-determined at the beginning of period t, y_t are the free variables, and z_t are shocks to the system. The function $F(x, y, z)$ is the law of motion for the predetermined variables, and g is list of equilibrium conditions such as Euler equations. The objective is to find some equilibrium rule, $Y(x)$, expressing the value of the free variables in terms of the state x such that

$$E\left\{g(x, Y(x), F(x, Y(x), z), Y(F(x, Y(x), z)) | x\right\} = 0 \tag{13}$$

holds for all values of x. The equilibrium rule $Y(x)$ expresses variables such as consumption, prices, and labor supply as functions of the state variables in x.

For example, the stochastic version of (4), investigated in the Taylor–Uhlig symposium (1990) and in Judd (1992), is

$$\begin{array}{rc} \max_c & E\left\{\Sigma_{t=0}^{\infty} \beta^t \ u(c_t)\right\} \\ & k_{t+1} = F(k_t, \theta_t) - c_t \\ & \ln \theta_{t+1} = \rho \ln \theta_t + \epsilon_{t+1} \end{array} \tag{14}$$

where k_t is the beginning-of-period capital stock, θ_t is a productivity parameter (the productivity shocks $\epsilon_t \sim N(0, \sigma^2)$ are independent), and $F(k, \theta)$ is the gross production function. In this problem, both k and θ are needed for a sufficient description of the state. Hence, consumption is a function, $C(k, \theta)$, of both k and θ, and the Euler equation is

$$u'\left(C(k, \theta)\right) = \beta\, E\left\{u'(C(F(k, \theta) - C(k, \theta), \tilde{\theta}))F_k(F(k, \theta) - C(k, \theta), \, \tilde{\theta}) \mid \theta\right\} \tag{15}$$

The case of two agents in a competitive economy can be similarly analyzed. Suppose that type i agents have utility

$$E\left\{\sum_{t=0}^{\infty}\beta^t u_i\left(c_{i,t}\right)\right\}$$

and budget constraint $k_{i,t+1} = R_t k_{i,t} + w_t - c_{i,t}$ where k_i is the amount of capital stock owned by the representative agents of type i, R_t is the random return from capital, and w is the wage from the supply of one unit of labor. Here the state variable is the capital stock owned by each type as well as the productivity level. In this case, equilibrium consumption of type i agents is a function of the distribution of wealth; let $C_i(k_1, k_2, \theta)$ be the consumption of type i agents when the wealth distribution is $k = (k_1, k_2)$ and the productivity level is θ. We assume that equity is the only asset which can be held; the more general case is examined in a later section. The equilibrium is defined by the collection of Euler equations

$$\begin{aligned}
u_i'(C^i(k_1^+, k_2^+, \theta)) &= \beta\, E\,\{u'(C^i(k_1^+, k_2^+, \tilde{\theta}))R(k^+, \tilde{\theta}) \mid \theta\}, \quad i = 1, 2 \qquad (16)\\
k_i^+ &= Y^i(k, \theta) - C^i(k, \theta), \quad i = 1, 2\\
Y^i(k, \theta) &= k_i R(k, \theta) + w(k, \theta), \quad i = 1, 2\\
R(k, \theta) &= F'(k_1 + k_2)\\
w(k, \theta) &= F(k_1 + k_2) - (k_1 + k_2)F'(k_1 + k_2)
\end{aligned}$$

where $Y(k, \theta) \in \mathbb{R}^2$ is the distribution of income in a period with initial capital stock distribution k and productivity θ, and wages are $w(k, \theta)$. This example is a simple one but it illustrates the basic features of models with heterogeneous agents. In particular, consumption and other decisions depend on the distribution of income across agents. This example has only one asset. More generally, we would like to examine the case where there are multiple assets. In that case, the state variable is even larger since the distribution of holdings of each asset may be important.

This paper will discuss various numerical methods for solving such models that have been proposed in recent years. We will discuss both perturbation and projection methods.

4 General Projection Algorithm

At first, focussing on the functional equations of the sort in (16) appears to be difficult since we have expressed equilibrium in terms of an unknown decision rule, an infinite-dimensional problem. Numerical rational expectations methods, beginning with Gustafson (1958), focus on finite-dimensional approximations of policy functions and other important functions, and then implement some sort of iterative procedure to find a finite-dimensional approximation that nearly solves the functional equations defining equilibrium.

For example, solutions to (14) typically use approximations of the form

$$\widehat{C}(k,\theta) = \sum_{i=0}^{n} a_i \varphi_i(k,\theta)$$

where the φ_i comprise a basis for the space of functions we examine, and focus on finding good choices for the a coefficients. This is true of both perturbation and projection methods. This section first presents an extended example of the projection method, and then describes projection methods in a general context. We discuss perturbation methods in a later section.

Most methods used to solve dynamic economic models are examples of what are called projection methods in the mathematics literature.[3] Suppose that economic analysis shows that equilibrium can be expressed as an operator equation

$$\mathcal{N}(f) = 0$$

where f is a function $f : D \subset \mathbb{R}^N \rightarrow \mathbb{R}^M$, $\mathcal{N}$ is an operator $\mathcal{N} : B_1 \rightarrow B_2$, and the B_i are function spaces. The unknown function f express prices, consumption decisions, and similar economic quantities as a function of state variable x. Typically $\mathcal{N}$ expresses equilibrium conditions such as Euler equations and consists of a composition of algebraic operations, differential and integral operators, and functional compositions, and is frequently nonlinear. We shall show how to implement the canonical projection technique in a step-by-step fashion.

The first step is to decide how to represent approximate solutions. We will assume here that we build the approximation $\hat{f}$ from linear combinations of simple functions, but nonlinear representations are also possible. We also need a concept of when two functions are close. Therefore, the first step is to choose a basis and an appropriate concept of distance.

Step 1: Choose bases, $\Phi_j = \{\varphi_i\}_{i=1}^{\infty}$, and inner products, $< \cdot, \cdot >_j$, over B_j, $j = 1, 2$.

There are many criteria that the basis and inner product should satisfy. The basis Φ_1 should be "rich"; in particular, it should be complete in B_1. We will generally use inner products of the form

$$\langle f(x)\, ,\, g(x) \rangle_1 \equiv \int_D f(x) g(x) w(x)\, dx$$

[3] The term "projection method" is a catchall term in the mathematical literature which includes "method of weighted residuals," "finite element," "Galerkin," "least squares," "Rayleigh-Ritz," and other methods.

for some weighting function $w(x) \geq 0$. Computational considerations also play a role in choosing a basis. The φ_i should be simple to compute. They should be similar in size to avoid scaling problems. Practical success requires a basis where only a few elements will do the job. This requires that the basis elements should "look something like" the solution. In particular, we should use smooth functions to approximate smooth functions, but use splines to approximate functions that may have kinks or other extreme local behavior. Orthogonal bases will enhance efficiency and accuracy. Because of its special properties, a generally useful choice is the Chebyshev polynomial family. If, on the other hand, one has a basis that is known to efficiently approximate the solution, one should use that instead or combine it with a standard orthogonal family. A good, problem–specific, choice of basis can substantially improve algorithmic performance over the generic approximation methods discussed above.

Next, we decide how many basis elements to use and how to implement $\mathcal{N}$.

Step 2: Choose a degree of approximation n, a computable approximation $\hat{\mathcal{N}}$ of N, and a collection of n test functions from B_2, $p_i : D \to \mathsf{R}^M$, $i = 1, \cdots, n$. Define $\hat{f}(x) \equiv \Sigma_{i=1}^{n} a_i \, \varphi_i(x)$

The φ_i should increase in "complexity" and "nonlinearity" as i increases. The best choice of n cannot be determined *a priori*. Generally, the only correct choice is $n = \infty$. Larger n should yield better approximations, but one is most interested in the smallest n that yields an acceptable approximation. One initially begins with small n and increases n until some diagnostic indicates that little is gained by continuing. Similar issues arise in choosing $\hat{\mathcal{N}}$. Sometimes we can take $\hat{\mathcal{N}} = \mathcal{N}$, but more generally some approximation is necessary. The test functions p_i are used to identify the unknown coefficients a.

Step 1 lays down the topological structure of our approximation and Step 2 fixes the flexibility of the approximation. Once we have made these basic decisions, we begin our search for an approximate solution to the problem. Since the true solution f satisfies $\mathcal{N}(f) = 0$, we will search for some $\hat{f}$ that makes $\hat{\mathcal{N}}(\hat{f})$ "nearly" equal to the zero function. Since $\hat{f}$ is parameterized by a, the problem reduces to finding a coefficient vector a that makes $\hat{\mathcal{N}}(\hat{f})$ nearly zero. This search for a is the focus of Steps 3–5.

Step 3: For a guess a, compute the approximation, $\hat{f} \equiv \Sigma_{i=1}^{n} a_i \, \varphi_i(x)$, and the residual function,

$$R(x\,;\,a) \equiv (\hat{\mathcal{N}}(\hat{f}))\,(x).$$

The first guess of a should reflect some initial knowledge about the solution. After the initial guess, further guesses are generated in Steps 4 and 5, where we see how we use the inner product, $<\cdot, \cdot>_2$, defined in the space B_2, to define what "near" means.

Step 4: For each guess of a, compute the n projections,

$$P_i(a) \equiv \langle R(\cdot\,;\,a)\,,p_i(\cdot)\rangle_2, i = 1, \cdots, n.$$

or the L^2 norm $\langle R(x;\,a),\ R(x;\,a)\,\rangle$.

Step 4 reduces the original infinite-dimensional problem to a finite dimensional problem. Step 5 finishes the job.

Step 5: By making a series of guesses over a and iterating over steps 3 and 4, find a value for a that sets the n projections equal to zero or minimizes the L^2 norm of $R(x;a)$.

There are many ways to implement the ideas in steps 3-5. First, the *least-squares* approach chooses a to minimize the "weighted sum of squared residuals":

$$\min_a \qquad \langle R(x;\,a),\ R(x;\,a)\,\rangle.$$

This replaces an infinite-dimensional operator equation with a nonlinear minimization problem in R^n. This method often performs poorly since there may be local minima that are not global minima, and the objective may be poorly conditioned. Least-squares methods are easy to implement and can use the excellent optimization software packages available.

While the least-squares method is a direct approach to making $R(x;\,a)$ small, most projection techniques find approximations by fixing n projections and choosing a to make the projection of the residual function in each of those n directions zero. Formally, these methods find a such that $\langle R(x;a),\, p_i(x)\rangle_2 = 0$ for some specified collection of functions, p_i. Different choices of the p_i defines different implementations of the projection method.

One such technique is the *Galerkin method*. A Galerkin method uses the first n elements of the basis for the projection directions, and a is chosen to solve the equations:

$$P_i(a) \equiv \langle R(x;\,a),\ \varphi_i(x)\,\rangle = 0, \qquad i = 1,\,\cdots,\,n$$

Notice that here we have reduced the problem of solving a functional equation to solving a finite set of finite–dimensional nonlinear equations.

There are obviously many ways to implement the projection idea. A *collocation method* takes n points from the domain D, $\{x_i\}_{i=1}^n$, and chooses a to solve

$$R(x_i\,;\,a) = 0, \qquad i = 1,\,\cdots,\,n$$

This is a projection approach since $R(x_i\,;a)$ equals the projection of $R(x;a)$ against the Dirac delta function at x_i, $\delta(x - x_i)$. *Orthogonal collocation* chooses the collocation points in a special way. The chosen x_i are the zeros of the n'th basis element, where the basis elements are orthogonal with respect to the inner product. The Chebyshev Interpolation Theorem says that the zeroes of Chebyshev polynomials are particularly good choices. The performance of Chebyshev collocation is often surprisingly good.

Choosing the projection conditions is a critical decision since the major computational task is the computation of those projections. The collocation method is fastest in this regard since it only uses the value of R at n points. More generally, we generally require numerical quadrature techniques to compute the inner products in $P(a)$. A typical quadrature formula approximates $\int_a^b f(x)\, g(x)\, dx$ with a finite sum $\sum_{i=1}^{n} \omega_i f(x_i)$ where the x_i are the quadrature nodes and the ω_i are the weights. Since these formulas also evaluate $R(x; a\ )$ at just a finite number of points, x_i, quadrature-based projection techniques are essentially weighted collocation methods, but may be better since they use information at more points.

Step 5 determines a through either a minimization algorithm (in the least-squares approach) or a nonlinear equation solver applied to the system $P(a) = 0$. Many alternatives exist, including successive approximation, Newton's method, and homotopy methods, all of which have been used in the economics applications of the projection method.

The projection method is a general approach for numerical solution of functional equations which arise in economic analysis. This section has presented the general framework. We next illustrate its application to some specific applications in dynamic economics.

5 Projection Methods for Time Homogeneous Models

The various approaches to solving rational expectations models differ in three basic ways; first, in the choice of finite-dimensional approximations to functions, second, in the way the expectation in (16) is computed, and, third, in the method used to find an approximate solution. The work discussed below touches on two of the three critical elements — the method used to approximate equilibrium policy and pricing functions, and the method for solving the identifying conditions. In this section we focus on various combinations of approximation and solution methods that appear to be promising in the context of large multiagent rational expectations models.

5.1 Approximating Equilibrium Functions

The first key step in solving problems with heterogeneous agents is approximating the decision rules and pricing functions in an economical fashion. For example, to solve (16) we need to approximate two functions, $C^1(k_1, k_2, \theta)$ and

$C^2(k_1, k_2, \theta)$, each of which is a function of two variables. One obvious possibility is to use polynomials. For example, we could set

$$C^i(k, \theta; a) \doteq \sum_{j_1=0}^{J_1} \sum_{j_2=0}^{J_2} \sum_{m=0}^{M} a^i_{j_1 j_2 m} \; k_1^{j_1} \; k_2^{j_2} \; \theta^m, \quad i = 1, 2 \tag{17}$$

However, ordinary polynomials are not advisable since conditioning problems (similar to multicollinearity problems in regression) make it difficult to identify the a coefficients. Judd (1992) and Gaspar and Judd (1997) instead advocated the use of orthogonal polynomials, resulting in approximations of the form

$$C^i(k, \theta; a) \doteq \sum_{j_1=0}^{J_1} \sum_{j_2=0}^{J_2} \sum_{m=0}^{M} a^i_{j_1 j_2 m} \;\; \varphi_{j_1}(k_1) \; \varphi_{i_2}(k_2) \; \psi_m(\theta), \quad i = 1, 2$$

where $\varphi_i(.)$ $(\psi_m(.))$ is a degree $i-1$ $(m-1)$ polynomial from some appropriate orthogonal family. For example, Chebyshev polynomials are natural to use in the k dimensions since k is expected to stay in some compact domain, but Hermite polynomials should be used for the θ dimension since θ is a Normal random variable. One could also use splines to approximate equilibrium policy functions when the number of types is small. This is discussed more extensively below in the discussion of incomplete asset markets.

For problems with several kinds of agents, forms like (17) suffer from a curse of dimensionality. To counter that Judd and Gaspar (1997) advocate the use of complete polynomials. The key fact about complete polynomials is that one eliminates from (17) terms of high total power. In particular, a degree d approximation would use approximations of the form

$$C^i(k, \theta; a) \doteq \sum_{\substack{0 \le j_1 + j_2 + m \le d \\ 0 \le j_1, j_2, m \le d}} a^i_{j_1 j_2 m} \; k_1^{j_1} \; k_2^{j_2} \; \theta^m, \quad i = 1, 2$$

In general, if there are n agents and we wanted to use a multivariate orthogonal polynomial approximation we would use

$$C^i(k, \theta; a) \doteq \sum_{\substack{0 \le j_1 + \cdots + j_n + \ell \le d \\ 0 \le j_i, \ell \le d}} a^i_{j_1 \ldots j_n \ell} \; \varphi_{j_1}(k_1) \ldots \varphi_{j_n}(k_n) \; \psi_\ell(\theta)$$

Further simplification is possible if symmetry properties are present. For example, suppose there are three types of agents with identical preferences but different wealth. Then, type 1 agents do not care if type 2 agents are poor and type 3 agents are rich or if the reverse is true, but type 1 agents do care about the distribution. This symmetry condition imposes further conditions on the a coefficients, further reducing the number of unknowns. Specifically, the approximate consumption function for type 1 agents would take the form

$$C^1(k, \theta; a) \doteq \sum_{\substack{0 \leq i+j+\ell \leq d \\ 0 \leq i,j,\ell \leq d}} a_{j_1 \ldots j_n \ell}\ \varphi_i(k_1)\varphi_j(k_2, ..., k_n)\ \psi_\ell(\theta) \tag{18}$$

where each $\varphi_j(k_2, ..., k_n)$ is a symmetric polynomial in $(k_2, ..., k_n)$ of total degree j. The symmetric polynomials have a particular structure, built up from a few basic symmetric polynomials. For example, the degree 1 symmetric polynomial in $(x, y, .., z)$ is $x + y + ... + z$, the degree 2 symmetric polynomials in $(x, y, .., z)$ are linear combinations of $x^2 + y^2 + ... + z^2$ and $(x + y + ... + z)^2$, and degree 3 symmetric polynomials in $(x, y, .., z)$ are linear combinations of $x^3 + y^3 + ... + z^3$, $x^2y + x^2z + ... + y^2z + ...$, and $(x + y + ... + z)^3$. Consumption of type m agents is defined using the same coefficients used in (18) but reversing the roles of k_1 and k_m, resulting in the consumption function

$$C^m(k, \theta; a) \doteq \sum_{\substack{0 \leq i+j+\ell \leq d \\ 0 \leq i,j,\ell \leq d}} a_{j_1 \ldots j_n \ell}\ \varphi_i(k_m)\varphi_j(k_1, .., k_{m-1}, k_{m+1}, ..., k_n)\ \psi_\ell(\theta)$$

Krusell and Smith (1997) take this process one step further. They examine a model with a continuum of agents. At first, it would seem impossible to use the approach in (18), but the key insight is to us moments. They assume that the consumption rule for any agent depends on his wealth and the moments of the distribution of wealth. This dependence on moments is a further extension of the idea of using symmetry to reduce the complexity of the approximation used for the consumption function. This is clearly seen from the definition of moments. For example, the mean capital stock is $\sum_i k_i$, which is the degree 1 symmetric polynomial in the k_i. The variance is a linear combination of $\sum_i k_i^2$ and $(\sum_i k_i)^2$ which are the degree two symmetric polynomials. The theory of complete polynomials tell us that the complete degree two approximation would consist of a linear combination of the mean, the variance, and the square of the mean. It also says that a third-order complete representation would involve the mean cubed, the third moment, and the skewness. Using their moment approach, Krusell and Smith are able to analyze how the distribution of wealth interacts with idiosyncratic and systematic risks in a Real Business Cycle model. Surprisingly, they find that a scheme using a few moments produces an excellent approximation of aggregate fluctuations.

den Haan (1997) examined a similar problem where equilibrium depends on the distribution of wealth, but takes a different approach. He approximates the distribution function with some functional family with coefficients b and then assumes that an individual's consumption depends on his wealth and the coefficients b that describe the distribution of wealth. This is more general than Krusell and Smith since one way to parameterize a distribution is through its moments. Even if one just focuses on the moments, there is always some implicit mapping between the moments and the distribution being used. den Haan's approach makes that mapping explicit.

A key element of any algorithm is the manner in which equilibrium decision rules and pricing functions are approximated. We want to use a method that has few unknown parameters but is flexible, capable of approximating equilibrium with small errors. Recent papers have shown that it is important to exploit known features of equilibrium, such as symmetry, since they can drastically reduce the number of free parameters without creating unreasonable approximation error.

5.2 Solving For the Unknown Coefficients

The next critical choices are creating identifying conditions for the coefficients a and then solving form a. We first create some projection conditions $P(a)$, which amount to a finite number of finitistic conditions on the coefficients a. For example, in the case of (16), we define the residual function to be the Euler equation errors, as in

$$R(k,\theta;a) = u_i'(C^i(k,\theta;a)) - \beta\, E\left\{u'(C^i(k^+,\tilde{\theta};a))R(k^+,\tilde{\theta}) \mid \theta\right\},\ \ i=1,2 \quad (19)$$

Due to the presence of the expectation operator in (19), we need to form the approximate residual function for agent i,

$$\widehat{R}^i(k,\theta;a) \quad = \quad u_i'(C^i(k,\theta;a)) - \beta\, \widehat{E}\left\{u_i'(C^i(k^+,\tilde{\theta};a))F_k(k^+,\tilde{\theta}) \mid \theta\right\},\ i=1,2$$

where $\widehat{E}$ represents some numerical approximation of the enclosed integral. This approximation can be Monte Carlo integration, Newton-Cotes integration, or a Gaussian integration formula. Judd and Gaspar (1997) uses Gauss-Hermite, but notes that a variety of integration methods, such as monomial rules and good lattice points may be better in the multidimensional context.

With the approximate residual function defined, we define the identifying conditions. Define the projections

$$P_{ij}(a) \equiv \int_{\theta_m}^{\theta_M}\int_{k_m}^{k_M}\int_{k_m}^{k_M} \widehat{R}^i(k,\theta;a)\,\psi_j(k,\theta)\,w(k,\theta)\,dk_1dk_2d\theta,\ i=1,2$$

where the $\psi_j(k,\theta)$ are distinct functions. The projection conditions themselves can be simple orthogonal conditions computed using Monte Carlo methods, or they can be conditions motivated by orthogonal polynomial theory and Gaussian quadrature.

Once we have specified the projection, or identifying conditions, $P(a)$, we need to choose a method for solving $P(a) = 0$. There are several ways to solve for the projection conditions. Newton's method[4] treats the conditions

[4]It is well-known that one should not apply the original Newton's method. It is more advisable to use an implementation of Powell's hybrid method, such as that contained in the MINPACK collection. One could also use the more advanced TENSOLVE package.

$P(a) = 0$ as a system of nonlinear equations and solves for a by repeated linear approximations. Newton's method is locally quadratically convergent, but each step uses $O(n^3)$ time because it computes a Jacobian. Some refinements economize on this by approximating the Jacobian, but the computational cost per step is still a problem.

Two other procedures are motivated by economic intuition. Time iteration also uses the Euler equation to solve for $\widehat{C}^i(k,\theta)$ but instead uses the equation

$$\begin{aligned} \widehat{C}^{i,j+1}(k,\theta) &= (u')^{-1}(\beta\,\widehat{E}\,\{u_i'(\widehat{C}^{i,j}(Y(k,\theta)-\widehat{C}^{i,j+1}(k,\theta),\tilde{\theta})) \\ &\quad \times F_k(Y(k,\theta)-\widehat{C}^{i,j}(k,\theta),\tilde{\theta}) \mid \theta\}) \end{aligned} \tag{20}$$

to generate the necessary data. For a fixed (k,θ) vector, (20) is a nonlinear equation in $\widehat{C}^{i,j+1}(k,\theta)$. Solving (20) for several choices of (k,θ) generates values for $\widehat{C}^{i,j+1}(k,\theta)$, information which is then used to compute the coefficients for $\widehat{C}^{i,j+1}(k,\theta)$. Time iteration corresponds to solving the corresponding dynamic program problem backwards in time. Successive approximation methods proceed more directly, using less computation per step. Specifically, successive approximation takes the policy functions computed in iteration j, $\widehat{C}^{i,j}$, and applies the computation

$$\begin{aligned} \widehat{C}^{i,j+1}(k,\theta) &= (u')^{-1}(\beta\,\widehat{E}\,\{u'(\widehat{C}^{i,j}(Y(k,\theta)-\widehat{C}^{j}(k,\theta),\tilde{\theta})) \\ &\quad \times F_k(Y(k,\theta)-\widehat{C}^{i,j}(k,\theta),\tilde{\theta}) \mid \theta\}) \end{aligned} \tag{21}$$

at a finite number of points (k,θ) to produce $\widehat{C}^{i,j+1}(k,\theta)$ data sufficient to fix the unknown coefficients of $\widehat{C}^{i,j+1}$.

Both successive approximation and time iteration are only linearly convergent. Since $\widehat{C}^{i,j+1}(k,\theta)$ is expressed directly in terms of the right hand side of (21), the computation cost is smaller for successive approximation. Successive approximation was used in the rational expectations by Miranda and Helmburger (1988) who observed that it was an efficient method for computation. It can also be motivated by learning arguments in Marcet and Sargent (1989). Successive approximation is often quite stable, converging to the equilibrium. Judd (1998, pages 557-8) presents a stability analysis of an example of successive approximations and shows that it is locally convergent except for some extreme choices of tastes and technology. However, the computational demands of each iteration are only $O(n^2)$. Time iteration is more reliable but generally slower than successive approximation when the latter converges. Time iteration was used by Gustafson, and in the Wright and Williams work and theory indicates that it will be much slower than Newton's method for small problems and slower than the successive approximation results below.

The time iteration method is also used by Rios-Rull (1999), who solves for individual value functions as well as policy functions in recursive equilibria. Sometimes equilibrium is best expressed in terms of individual value functions. See Rios-Rull (1999) for a detailed presentation of that approach. Of course, Newton's method or similar nonlinear equation methods could also be used to solve problems formulated in terms of value functions since the equilibrium is approximated by a nonlinear set of conditions on the coefficients of the parameterization of the value and policy functions. That approach is taken when solving dynamic games, a topic discussed below, but we are unaware of it being applied to recursive equilibrium problems.

6 Incomplete Asset Markets with Infinitely-lived Agents

The incomplete asset market model with infinitely-lived agents is one that has been analyzed using computational methods in several recent papers. We will review that literature in some detail to highlight the critical numerical issues that arise in asset market problems, and to give some more detail on the general ideas discussed in previous sections.

6.1 A Model of Incomplete Asset Markets

Consider a Lucas (1978) economy with heterogeneous agents, a single commodity and incomplete asset markets. There are H infinitely-lived investors. At each period $t = 0, 1, \cdots$ investor h receives a stochastic labor income e_t^h. In addition there is a Lucas tree (which we will refer to as the stock) with stochastic dividends d_t. At $t = 0$ each agent h owns a fraction of the tree $s_{-1}^h \geq 0$, so that aggregate endowments (output) at each time t equals $\sum_{h=1}^{H} e_t^h + d_t$. All uncertainty can be described by a time-homogeneous finite-state Markov process. Let $Y = \{1, 2, \ldots, S\}$ denote the exogenous states, and y_t be the time t value. Individual labor endowments $e^h : Y \to \mathbb{R}_{++}$ and the dividends $d : Y \to \mathbb{R}_+$ depend on the current state y alone.

Each agent h maximizes the expected utility function

$$U_h(c) = E\left\{\sum_{t=0}^{\infty} \beta^t u_h(c_t, y)\right\}$$

over possible infinite consumption streams c. We assume that the utility functions $u_h(., y) : \mathbb{R}_{++} \to \mathbb{R}$ are strictly monotone, C^2, strictly concave, and satisfy $\lim_{x\to 0} u_c(x, y) = \infty$. We also assume that the discount factor $\beta \in (0, 1)$ is the same for all agents, and that agents have common beliefs about the transition matrix for the exogenous states.

Agents trade two securities in order to smooth their consumption across time and states. They can trade shares of equity denoting ownership in the Lucas tree and a one-period bond in every time period. One bond at time t delivers one unit of the consumption good at time $t+1$ for any y_{t+1}. Bonds are in zero net supply. Markets are incomplete if $S > 2$ and perfect risk-sharing will generally be impossible. Let b^h denote an agent's bond holding, s^h his stock holding, q^s the price of the stock, and q^b the price of the bond. At each time t agent h faces the budget constraint

$$c_t^h = e^h(y_t) + b_{t-1}^h + s_{t-1}^h(q_t^s + d(y_t)) - b_t^h q_t^b - s_t^h q_t^s.$$

In addition we assume the short-sale constraints

$$b_t^h \geq K_b^h \text{ and } s_t^h \geq K_s^h, \, \forall h = 1, ..., H$$

These last constraints play an important role in equilibrium and present special challenges for any computational strategy.

6.2 Recursive Equilibria

It is well known that the model always has a competitive equilibrium, i.e. their exist prices and allocations such that all markets clear and all agents maximize utility subject to their budget restrictions (see e.g. Magill and Qunizii (1996)). However, in order to compute an equilibrium for an infinite horizon model it is necessary to focus on recursive equilibria. Recursive equilibria are dynamically simple, expressing prices, trades, and consumption as a time-invariant function of a finite number of state variables. In this problem, the state variables include the exogenous states y and the agents' portfolios. This problem is more difficult than Lucas (1978) where everything depended solely on the exogenous state. Because of agent heterogeneity, the state space includes the portfolios because the distribution of wealth will influence equilibrium prices. For the incomplete asset model it is standard to assume that the exogenous income and dividend state $y \in Y$ together with the agents' portfolio holdings $\Theta := (b^h, s^h)_{h=1}^H$ constitute a sufficient state space. We proceed under this assumption and return below to discussing its validity.

We denote the endogenous state space of all possible portfolio holdings of all agents by Z. Due to the short-sale constraints the set Z is compact[5]. Furthermore, we assume that the recursive equilibrium can be described by a continuous policy function $f^h = (f^{hb}, f^{hs}) : Y \times Z \to Z$ which determines agents' optimal portfolio choice given portfolio holdings and the income state of the current period, and by a continuous price function $g = (g^b, g^s) : Y \times Z \to \mathbb{R}_{++}^2$ which maps the current state into the asset prices q^b and q^s.

The equilibrium functions f and g are defined by the following requirements.

(RE1) Market clearing:

$$\sum_{h=1}^{H} f^{hb}(y, \Theta) = 0, \, \sum_{h=1}^{H} f^{hs}(y, \Theta) = 1 \quad \forall y \in Y, \Theta \in Z.$$

[5] In fact, short sale constraints imply that $Z = \prod_{h=1}^{H} \left([K_b^h, -\sum_{i \neq h} K_b^i] \times [K_s^h, 1 - \sum_{i \neq h} K_s^i] \right)$.

(RE2) Consumption choices are consistent with wealth and asset trades for all $y \in Y$ and all $\Theta \in Z$:

$$c^h = c^h(y, \Theta) = e^h(y) + b^h + s^h(g(y, \Theta) + d(y)) - f(y, \Theta) \cdot g(y, \Theta).$$

(RE3) Choices are optimal; hence, for any two subsequent exogenous states y and y_+ and all $\Theta \in Z$ consumption satisfies

$$\begin{aligned}
c_+^h \quad &: \quad = c^h(y_+, f(y, \Theta)) \\
q_+^s \quad &: \quad = g^s(y_+, f(y, \Theta)) \\
\lambda^{hb} \quad &: \quad = u_h'(c) g^b(y, \Theta) - \beta E(u_h'(c_+^h)) \geq 0 \\
\lambda^{hs} \quad &: \quad = u_h'(c) g^s(y, \Theta) - \beta E(u_h'(c_+^h)(q_+^s + d(y_+))) \geq 0 \\
\lambda^{hs}(f^b(y, \Theta) - K_b^h) \quad &= \quad 0 \\
\lambda^{hb}(f^s(y, \Theta) - K_s^h) \quad &= \quad 0
\end{aligned}$$

The macroeconomic literature often assumes stationary growth for endowments. If all agents have identical constant relative risk aversion utility, (RE1)-(RE3) can be rewritten in terms of consumption/wealth ratios, transforming the nonstationary growth problem into a problem confined to a compact set of ratios..

6.3 Kuhn-Tucker Conditions as a System of Equations

Due to the short-sale constraints the agents face utility maximization problems with inequality constraints resulting in first-order conditions of optimality which include shadow prices and inequalities, see equations (RE3). The inequalities greatly increase the difficulty of computing an equilibrium; in the dynamic general equilibrium literature three different approaches have been used to tackle this problem.

The discrete state-space approach of Heaton and Lucas (1996) captures the short-sale constraints by allowing agents to hold only pre-specified portfolios which automatically satisfy the constraints. While this approach is very easy it has some disadvantages as we will describe below . Marcet and Singleton (1999) use an combinatorial approach examining many possible combinations of binding constraints. They solve the model as if all agents face no constraints on their portfolio transactions. If an agent violates one of the constraints, then they check all possible combinations of portfolios (each agent being constrained in each asset) until they find a solution to the system of equations and inequalities. Judd et al. (1999) transform the collection of equations and inequalities into a nonlinear system of equations via a simple trick (see Garcia and Zangwill (1981)) which can be solved using a nonlinear-equation routine.

Let l be a natural number and $\alpha^{ha} \in \mathbb{R}$ for $h = 1, 2$ and $a \in \{b, s\}$. Note the following relations:

$$(\max\{0, \alpha^{ha}\})^l = \begin{cases} (\alpha^{ha})^l & \text{if } \alpha^{ha} > 0 \\ 0 & \text{if } \alpha^{ha} \leq 0 \end{cases}$$

$$(\max\{0, -\alpha^{ha}\})^l = \begin{cases} 0 & \text{if } \alpha^{ha} > 0 \\ |\alpha^{ha}|^l & \text{if } \alpha^{ha} \leq 0 \end{cases}$$

Moreover,

$$(\max\{0, \alpha^{ha}\})^l \geq 0, (\max\{0, -\alpha^{ha}\})^l \geq 0, \text{ and } (\max\{0, \alpha^{ha}\})^l \cdot (\max\{0, \alpha^{ha}\})^l = 0.$$

We define

$$\lambda^{ha} = (\max\{0, \alpha^{ha}\})^l \text{ and } f^a(y, \Theta) - K_a^h = (\max\{0, -\alpha^{ha}\})^l$$

which allows us to state first-order conditions of optimality as a system of equations which is equivalent to (RE3):

$$\begin{aligned} -g^b(y, \Theta)u_h'(c) + \beta_h E_t(u_h'(c_+^h))) + (\max\{0, \alpha^{hb}\})^l &= 0 \\ -f^b(y, \Theta) + K_b^h + (\max\{0, -\alpha^{hb}\})^l &= 0 \\ -g^s(y, \Theta)u_h'(c) + \beta_h E_t \left\{(q_+^s + d(y_+))u_h'(c_+)\right\} + (\max\{0, \alpha^{hs}\})^l &= 0 \\ -f^s(y, \Theta) + K_s^h + (\max\{0, -\alpha^{hs}\})^l &= 0 \end{aligned}$$

We have transformed the system of optimality conditions into a system of 4 equations for every agent with the variables θ^h and $\alpha^h = (\alpha^{hb}, \alpha^{hs})$.

6.4 Bounded Portfolio Space

The computational challenge will be to approximate the equilibrium functions f and g. At this point, it is important to stress that the short sale constraints are essential for the computations. From a economic modeling perspective short-sale constraints are an undesirable part of the model. The bounds on shorts sales have to be chosen exogenously and since in reality explicit short sale constraints rarely exist, this choice cannot be guided by data. Although economic agents do face trading restrictions and debt constraints there are often no legal limits on short-positions in individual securities.

From a theoretical point of view, in order to close the model, one must rule out Ponzi schemes, i.e. the possibility of an infinite accumulation of debt. However, the standard approach (see e.g. Levine and Zame (1996) is to impose an implicit debt constraint,

$$\sup_t \|q_t^b b_t^h + q_t^s s_t^h\| < \infty, \ \forall h. \tag{22}$$

Since this constraint merely requires that agents must always be able to pay back their debt in finite time this is clearly not an unrealistic restriction.

For a model with only a single bond implicit debt constraint can often be reformulated as a short-sale constraint. Zhang (1997a, 1997b) develops algorithms to compute equilibria in models with a single asset and implicit debt constraints.

Unfortunately, his approach does not generalize to models with more than one asset. Since the price of the Lucas tree is endogenous, the state space Z cannot be reduced to the set of possible wealth distributions. Any algorithm has to approximate the equilibrium functions over the distribution of portfolio-holdings. For any initial distribution of portfolio holdings in Z, one has to ensure that agents' new equilibrium choice also lies in Z. However, when there is more than one asset, this is generally not possible without explicitly restricting agents' choices to lie in Z. Whenever an agent faces a bad idiosyncratic shock, he will try to reduce the value of his portfolio, $q_b b^h + q_s s^h$. In models which are calibrated to yearly data, an implicit debt constraints (22) will often imply that the agents can borrow up to 10 times his average endowments (see Zhang (1997a)). After sufficiently many bad shocks an agent's actual debt will approach this value. Since in addition to the idiosyncratic shock there is uncertainty about the stocks dividends, the agents' equilibrium (unrestricted) portfolio holdings will then consist of a very short position in one asset and a very large long position in the other security. Agents can inflate their portfolios without any bounds without violating the above debt constraint.

This is a familiar problem in general equilibrium models with incomplete asset markets and real assets. In extreme cases, $s^h + b^h$ can become unbounded. Without short-sale constraints, equilibria do not always exist because demand functions are not everywhere continuous.

For all practical purposes it is therefore crucial to ex ante fix a bounded set of admissible portfolio holdings for all agents. The easiest way to obtain a bounded set of portfolios is to impose a short-sale constraint. As we have done in our model, short sales can be constrained through a priori specified fixed exogenous lower bounds on the portfolio variables. In equilibrium, when all financial markets are required to clear, all agents' portfolios are also bounded above resulting in a compact set of admissible portfolios for the entire economy. Note that we define the bounds on short sales as agent dependent since it is certainly realistic to assume that an agent's income influences how much he can borrow.

We show below that in many cases, when there are two assets, short-sale constraints will be frequently binding. We must keep this fact in mind when interpreting the results of our simulations.

6.5 Existence of Equilibrium and Computational Errors

Unfortunately, recursive equilibria do not always exist – Kubler and Schmedders (2000) construct some counterexamples. However, these examples crucially rely on multiplicity of competitive equilibria and for many realistically calibrated economies where competitive equilibria are unique recursive equilibria will exist. However, their are no known conditions on the fundamentals which ensure existence. It is necessary to develop methods which give the user some confidence that the computed approximations of the equilibrium functions are indeed very close to a true equilibrium.

When approximating policy and price functions one has to deal with various kinds of computational error and it is impossible to determine the equilibrium functions exactly. This fact of life leads to the central question how large an error is acceptable and when to stop an approximation algorithm. The typical procedure used to solve our model is of iterative nature and terminates when a stopping rule is satisfied. Such stopping rules do not specify when the approximate equilibrium prices are close to the true equilibrium prices but instead when the difference between consecutive approximations is small. The method is then thought to have stabilized around an approximate solution. At this point, however, it remains unclear how close the computed prices and portfolios are to the true equilibrium prices and portfolios and if we actually have an approximate description of a recursive equilibrium. There is an obvious need for a close evaluation of the computed solutions.

Ideally, one wants to derive error bounds or accuracy estimates of the computed solutions. While bounds like this exist for finite dimensional problems (see e.g. Blum et al. (1998)) and for numerical dynamic programming (see Santos and Vigo-Aguiar (1998)) there exist no comparable theories for the general equilibrium model under consideration.

A popular approach to verify the quality of a solution is to compute the maximum relative errors in the agents' first-order conditions. For the case where short-sale constraints are not binding and the related shadow prices are zero, conditions (RE3) imply that the maximum relative errors are given by

$$\max_{\theta} \left\| \frac{\beta E(u_h'(c_+^h)) - u_h'(c) g^b(y,\Theta)}{u_h'(c) g^b(y,\Theta)} \right\|$$

and

$$\max_{\theta} \left\| \frac{\beta E((q_+^s + d(y)) u_h'(c_+^h)) - u_h'(c) g^s(y,\Theta)}{u_h'(c) g^s(y,\Theta)} \right\|$$

Unfortunately, low errors in agents' Euler equations do not give any indication of how close we are to an equilibrium. This well-known fact has various interpretations in the literature. Judd (1992) argues that it is not sensible to expect infinite precision from agents and that therefore the computed prices and allocations are likely to be a good description of the actual economic outcome.

For this line of reasoning it is important to show how small the errors actually are. Without knowing the actual solution this is not unambiguously possible. Judd (1992) suggests to evaluate the Euler equations at the computed prices and allocation and compute the wealth equivalent of the Euler equation residual when projected in directions not used to compute the approximation. A small error here would be consistent with the interpretation of an approximate equilibrium in the sense that agents are close to rational.

Heaton and Lucas (1996) and Telmer (1993) use a slightly different error criterion. For each agent, they compute the asset prices which support the agents' computed decision. When there are two agents, they therefore get prices q_1^s, q_1^b for the first agent and q_2^s, q_2^b for the second agent. They then report $\left(q_1^s - q_2^s\right)/q_1^s$ and $\left(q_1^b - q_2^b\right)/q_1^b$.

When the short-sale constraint is not binding, these two formulations are very similar since

$$\frac{q_2 u_1'(c) - \beta E(u_1'(c_+))}{q_2 u_1'(c)} = \frac{(q_1 - q_2)u_1'(c) - \beta E(u_1'(c_+))}{q_2 u_1'(c)} = \frac{q_1 - q_2}{q_2} - 1$$

The errors in both agents' Euler equations can presumably be decreased by setting $q^* = (q_1 + q_2)/2$, the errors reported in Heaton and Lucas and Telmer are therefore likely to slightly overstate the errors in the Euler equation. In most cases, however, the difference will be negligible.

7 Approximating Asset Market Equilibria Over a Discretized State Space

A crucial feature of every approximation method is the way it deals with the endogenous state space Z. This set Z is uncountable and it is therefore impossible to explicitly compute the equilibrium functions at every point in Z. Existing algorithms to approximate equilibria can be divided into two classes. This section discusses algorithms in the first class that discretize the set Z and allow only finitely many values of the endogenous state variable. Other algorithms using polynomial approximations are discussed later.

Telmer (1993), Lucas (1994) and Heaton and Lucas (1996) are examples of papers which approximate recursive equilibria by discretizing the endogenous state space Z, that is, agents' portfolio holdings can take only values in a prespecified finite set. We describe the basic ideas of these methods in the context of Lucas' (1994) two-investor two-asset model.

(D1) Under the assumption that there are only two investors, market clearing implies that the endogenous state space reduces to $[K_b^1, -K_b^2] \times [K_s^1, -K_s^2]$. In each set of the product choose a finite number of N points. Thus, the continuous two-dimensional state space has been collapsed to N^2 points.

(D2) The agents must choose their portfolios to always be exactly one of these N^2 points. Put differently, given an "old" portfolio in the discrete state space and a new exogenous shock $y \in Y$ the agents must choose a "new" portfolio that also lies in that discrete set. Therefore, the equilibrium policy function $f^h, h = 1, 2$, can be represented as S collections of N^2 pairs of portfolio points in the discrete set, one collection for every exogenous state $y \in Y$. Similarly, the price functions g are also only a collection of $S \cdot N^2$ points.

(D3) The algorithm searches for asset prices and an allocation of assets which comes as close as possible to agents' optimality. The algorithm considers only those portfolio combinations of the agents that satisfy the market-clearing equation. Due to the discretization of the state space it is impossible that all agents' decisions are optimal. The goal of the algorithms must be to find functions g and f which minimize the errors in the agents' Euler equations or equivalently minimize the relative difference between supporting prices.

The discrete methods have the advantage that due to their simplification of the equilibrium solution problem they are numerically stable and easy to implement. For applications with only a single state variable (e.g. versions of the model where either the bond market or the stock market are shut down) they perform very well. However, we explain below that for models with more than one endogenous state variable these methods are generally too slow or of far too low a precision to be effective tools.

The distinguishing feature of discrete methods is how they solve the agents' Euler equations, that is, how they perform Step (D3) of the basic approach setup. We discuss this point now in more detail.

7.1 A Single Security: No Equity

Telmer (1993) considers a greatly simplified version of our model and assumes that equity is not tradable, instead agents are only active in the bond market. The state space then simplifies to the one-dimensional interval $[K_b^1, -K_b^2]$ which can be easily discretized choosing N points of the interval.

In order to minimize the error in the Euler equations Telmer (1993) uses a Gauss-Seidel method. Recall that every Euler equation concerns two subsequent time periods and therefore portfolio terms for three time periods. For the third of these portfolios ("tomorrow's") and the corresponding prices some decision rule as a function of the second ("today's") portfolio is assumed. For a given first portfolio ("yesterday's") the algorithm now minimizes the Euler equation error by choosing the second portfolio. This ("today's") portfolio impacts both terms in the Euler equation. Today's portfolio is computed for every point in the state space. The error-minimizing portfolios and prices at every point in the state space lead to new updated decision rules. These new rules are then used for tomorrow's decision rules in the next iteration. This iterative process continues until the difference between two subsequent iterates (in an appropriate norm) are tiny. Actually, the difference in the portfolio functions will be exactly zero due to the discrete-valued nature of the functionals.

For this case of only one bond, Telmer can discretize the endogenous state space into 150 points per interval length of 0.1, he uses a total of more than 1000 points. Since he only considers 3 exogenous shocks, the problem remains feasible. With such a fine discretization the resulting errors turn out to be very low – for in the interior of the interval of possible bond-holdings he reports maximum pricing errors of 0.0001 percent. Even though the errors are likely to be substantially higher at the boundary, for models with a one-dimensional state space discrete methods generally achieve very good approximations. However, many interesting economic questions require a higher dimensional state space.

7.2 Two Assets and the Curse of Dimensionality

Lucas (1994) and Heaton and Lucas (1996) use a discrete state-space to approximate equilibria for the full model. They employ an auctioneer algorithm - which in essence is very similar to a Walrasian tatonnement process. Starting with a (approximating) policy function $\hat{f}_i$, the algorithm computes the supporting asset prices for both agents (which will be different). For that agent whose supporting stock price is higher, the stock holding is increased, for that agent whose supporting bond price is higher, the bond holding is increased. The amended portfolio holdings are then used for the new policy function $\hat{f}_{i+1}$. Iteration continues until the difference between the implied prices becomes sufficiently small, or cannot be improved further.

In principal, by employing a fine enough discretization, they could achieve a similar precision as Telmer in his simple setting. However, despite the speed of modern computers it is still not nearly feasible to allow enough discrete points for the endogenous state space. Heaton and Lucas (1996) allow for $30 \cdot 30$ different holdings. This coarse discretization results in high errors. They report average (not maximum) errors of up to 0.4 percent. For the purposes of many economic insights this might well be sufficient. In particular, the purpose of their paper is to investigate how missing asset markets might help to explain the equity premium puzzle, i.e. the first moment of asset returns. It is very unlikely that the true equilibrium is so far away from their approximation that it has an influence on average returns (in fact, Judd et al. (1999a) repeat their calculations with a different algorithm and come to the same conclusions).

However, for many other applications which investigate welfare effects or higher moments of security prices, discrete methods are of limited use. It seems that running times would increase drastically if one tried to reduce these errors. The main problem lies in the fact that for a discretized state space the system of Euler equations cannot be solved by efficient algorithms, such as Newton's method, which are designed for smooth systems. Instead they must be solved by some search procedure such as in Lucas' and Telmer's papers. For more than one endogenous state variable, it therefore seems worthwhile to develop a alternative approach.

8 Continuous Endogenous State Space: A Spline Collocation Algorithm

Some algorithms approximate equilibria with two assets over a continuous endogenous state space. The central theme of these algorithms is to approximate the policy functions f and g not only at finitely many points but instead on the entire state space $Y \times Z$. For such a continuous approximation a family of polynomials is used to approximate these functions. In this section we review the spline collocation method used in Judd et al. (1999) Below we examine the method used in Marcet and Singleton (1999)

There are obviously two crucial issues one has to face when developing an algorithm to approximate recursive equilibria. First, one has to find a scheme to approximate the true equilibrium functions f and g. Generally, they the approximating functions will be determined by a finite number of parameters. The second step must then be to solve for these unknown parameters.

Judd et al. (1999) use cubic splines to approximate the equilibrium functions and compute the spline coefficients using collocation methods. They solve the collocation equations with an iterative approach. We will briefly describe their algorithm and use a simple example to show where the difficulties lie in approximating and in solving for the equilibrium functions.

The main steps of the algorithms are as follows:

(C1) The equilibrium functions f and g are approximated by piecewise polynomial functions. They can therefore be parameterized by a finite number of coefficients.

(C2) In each set of the endogenous state space $[K_b^1, -K_b^2] \times [K_s^1, -K_s^2]$ choose a finite number of $N \times N$ points, called *collocation points*.

(C3) The algorithm searches for coefficients of the approximating functions which ensure that at the collocation points, the Euler equations and market clearing conditions hold.

8.1 Representing the Equilibrium Functions

The equilibrium functions are defined on an uncountable set and so we cannot specify all their values exactly. Instead it is only feasible to approximate the functions f and g using a fairly small number of parameters. As we show below, these functions are likely to have extremely high curvature in certain regions. In fact, if the short-sale constraints are modeled explicitly, these equilibrium functions fail to be C^1. A global polynomial approximation (using e.g. so-called orthogonal polynomials) therefore does not work unless one is willing to allow for large computational errors. The most sensible approach turns out to approximate the equilibrium function in a piecewise fashion by finitely parameterized functions $\hat{f}, \hat{g}$ using relatively few parameters. We use piecewise cubic polynomials (cubic splines) to approximate them.

One-dimensional cubic splines are easily defined. Given m points of a real valued function $(x_i, y_i)_{i=1}^m$ a cubic spline $s(x)$ is defined by the requirement that $s(x_i) = y_i$ for all $i = 1, ..., m$, that in each interval $[x_i, x_{i+1}]$ the function s is a cubic polynomial and that s is C^2 on $[x_1, x_m]$. By representing them as a linear combination of a collection of so called B-splines, the splines can be used for approximating higher dimensional functions. See de Boor (1978) for a thorough description of splines and their approximating properties.

Note that by approximating the equilibrium functions via piecewise linear combinations of polynomials the equilibrium computation has been transformed from finding function values on an uncountable set to finding finitely many weights for the appropriate linear combinations. Put differently, the problem has been reduced from computing infinitely many values to finding a reasonable finite number of parameters.

8.2 A Time-Iteration Algorithm

Using the collocation method to finding approximating functions $\hat{f}$ and $\hat{g}$ the crucial problem has now become to solve for the spline coefficients. In order to obtain sufficient accuracy, the number of unknown coefficients turns out to be rather large. For the example below we have 9600 unknown coefficients. This large number of parameters makes it difficult to solve for them directly – we apply an iterative approach instead.

The basic idea of this iterative approach can be viewed as a continuous generalization of Telmer's (1993) approach, see Section 2.1. The basic intuition is that at each iteration i, we take next period's policy functions as given and compute at every collocation point this period's portfolio holdings and prices which satisfy the Euler equation. Given functions $\hat{f}_i$ and $\hat{g}_i$ we obtain $\hat{f}_{i+1}$ and $\hat{g}_{i+1}$ by interpolating the computed portfolio holdings and prices at the collocation points. Recursive infinite-horizon equilibria are approximated by finite-horizon equilibria as the number of periods becomes very large. For discount factors β close to one the number of iterations needed to obtain a satisfactory approximation lies around 200. It is therefore important, that in each iteration, an efficient way is found to solve the Euler equations – we come back to this problem below in the context of our example

The algorithm can be summarized as follows:

Time Iteration Spline Collocation Algorithm

Step 0:	Select a set G of collocation points and a starting point $\hat{f}_0, \hat{g}_0$.
Step 1:	Given functions $\hat{f}_i, \hat{g}_i$, $\forall \theta \in G$, $\forall y \in Y$ compute portfolios and prices solving the Euler equations and market-clearing conditions.
Step 2:	Compute the new approximations $\hat{f}_{i+1}, \hat{g}_{i+1}$ via interpolation.
Step 3:	Check stopping criterion: If the errors are sufficiently small, then go to Step 4. Otherwise increase i by 1 and go to Step 1.
Step 4:	The algorithm terminates. Set $\hat{f} = \hat{f}_{i+1}, \hat{g} = \hat{g}_{i+1}$, these are the approximate equilibrium functions.

9 The Main Computational Challenges and an Example

As it turns out, approximating equilibria for models with more than one asset is a difficult task. In this section we use a simple example to illustrate the following two problems.

1. Short-sale constraints are frequently binding, resulting in non-smooth policy functions. Global polynomial approximation schemes cannot be used to approximate these functions.

2. Even in regions of the state space where constraints are not binding, the system of equalities describing equilibrium is extremely ill-conditioned. Without an extremely good starting point, Newton-methods cannot find a solution.

There are two investors with identical CRRA utility, $u_h(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with coefficient of relative risk aversion of $\gamma = 1.5$ and a discount factor $\beta = 0.96$. There are 4 exogenous states, idiosyncratic shocks to labor income are

$$e^1 = \begin{pmatrix} 3 \\ 3 \\ 7 \\ 7 \end{pmatrix}, \; e^2(s) = 10 - e^1(s)$$

and the stochastic dividends – the only source of aggregate uncertainty in our simple example – are given by $d = (2.5, 2, 2.5, 2)'$.

The transition matrix Π is chosen to ensure that idiosyncratic shocks are very persistent. This seems to be a stylized fact and, as we will see below, it causes substantial computational difficulties. Π is given by

$$\Pi = \begin{pmatrix} 0.4 & 0.4 & 0.1 & 0.1 \\ 0.4 & 0.4 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.4 & 0.4 \end{pmatrix}.$$

9.1 Short-Sale Constraints

As explained in Section 6.4, short-sale constraints are an essential part of our model. We set the short sale constraints to be $K_b^h = -2.5$ and $K_s^h = -0.1$ for both agents $h = 1, 2$. Figure 1 depicts investor 1's computed equilibrium demand for bonds, $\hat{f}_y^{1b}$, for the exogenous shock of $y = 1$ for our example. For large regions of the endogenous state space the short-sale constraint on the bond is binding. The economic interpretation of this is straightforward: Investors cannot use the stock to insure against bad idiosyncratic shocks ex ante. Given a bad shock the only possibility to smooth consumption is therefore to borrow. Such can be either achieved by selling the stock or the bond. However, for most levels of stock holdings, the investor prefers to sell the bond and does so up to the short-sale constraints. Although the given parametrization is a very stylized example, this phenomenon is very likely to occur in models with realistically calibrated persistent idiosyncratic shocks.

(FIGURE 1 ABOUT HERE)

The main computational problem with short-sale constraints is the fact that the resulting policy functions f are no longer smooth. Clearly, the plotted $\hat{f}_1^{1b}$ does not approximate a smooth function. At those values of last period's portfolio holdings where the short-sale constraint becomes binding, the policy function is not differentiable. This fact shows why it is impossible to approximate these functions globally with polynomials. Even with splines a good approximation is only possible if the number of collocation points is very high – in this example we chose 20×20 collocation points. Of course, by the definition of cubic splines, the approximating function is still C^2 while the true function is not. In fact, one can see in Figure 1 that the approximating function is slightly 'hump-shaped' close to the 'theoretical non-differentiability'. This hump leads to substantial errors in the Euler equations. While the resulting average errors are very low (around 0.0001 percent), the maximum error in the Euler equation lies around 0.3 percent. Furthermore, if only 225 collocation points are used the maximum error jumps up to 0.9 percent.

For many applications, particularly if there are more than two endogenous states, it is impractical to have 20 collocation points in each dimension. Therefore it is useful to consider variations of the model which lead to less extreme shapes of the policy functions.

9.1.1 Transaction Costs

One easy way to avoid extreme trading is to assume that trading the stock and the bond is costly. In fact, it might even be realistic to assume that there is a real cost in acquiring financial assets. The following specification of transaction costs is from Heaton and Lucas (1996, Section IV D).

At each date t an agent h pays transaction costs of $\omega(\theta_{t-1}^h, \theta_t^h)$. We assume that ω has the functional form

$$\omega(\theta_{t-1}, \theta_t) = \tau^b {b_t}^2 + \tau^s (q_t^s (s_t - s_{t-1}))^2,$$

where τ^b, τ^s are constants. The assumption of strictly convex costs is unrealistic but it is needed to ensure that agents face a differentiable and convex programming problem. We set $\tau^s = 0.0001$ and $\tau^b = 0.0001$. Note that these costs are so small that they are not likely to have a huge effect on agents' welfare. Trading 2.5 units of the bond (the biggest possible amount) will cost 0.000625 percent of aggregate endowments, trading 1.2 units of the stock (1.2 times the entire tree) will cost 0.00144 percent of aggregate endowments.

Costs this low are likely to have small effects on equilibrium prices. However, they substantially affect equilibrium trades. Figure 2 is the analogue of Figure 1 for the case of small transaction costs. The policy function is now much better behaved. Not surprisingly, the errors are much smaller. Even with $10 \cdot 10$ collocation points maximum errors lie around 0.0001 percent.

(FIGURE 2 ABOUT HERE)

Figures 3 and 4 show $\hat{g}^b$ and $\hat{g}^s$ for the case with transaction costs. They are visually indistinguishable from the case of no transaction costs, indicating that transaction costs have negligible effects on asset prices. In fact, in simulations first and second moments of returns to out to be within 0.01 percent of each other.

(FIGURES 3 AND 4 ABOUT HERE)

9.1.2 Penalties on Portfolios

An alternative approach which allows us to drop the assumption of short-sale constraints altogether is to assume that agents are allowed to hold portfolios of any size but get penalized for large portfolio holdings. The intuition behind such a model assumption is that there are costs associated with large short positions and in a simplification we model them as penalties to agents' utilities; when these penalties get sufficiently large, agents avoid extreme positions. The advantage of utility penalties on large short positions is that this restriction does not constitute an a priori exogenous constraint on short sales. Instead, the penalties lead to endogenous avoidance of short sales depending on how much agents desire large short positions.

We use a penalty function of the form

$$\rho^h(s,b) = \kappa^b \min(0, b - L^{hb})^4 + \kappa^s \min(0, s - L^{hs})^4$$

where $\kappa^a \geq 0, a \in \{b, s\}$ and $L^{ha} \leq 0$. Note, there is no punishment for large long positions. If κ^a is sufficiently large the penalty function almost acts like a hard short-sale constraint on the corresponding asset $a \in \{b, s\}$. For a more general description of the model it suffices that ρ is a convex function satisfying $\rho(b, s) \to \infty$ as $|(b, s)| \to \infty$. The portfolio penalties lead our agents to have utility functions over consumption and portfolio holdings of the form

$$V_h(c, b, s) = U_h(c) - E\left\{\sum_{t=0}^{\infty} \beta^t \rho^h(b_t, s_t)\right\}.$$

The main problem with this approach is that the portfolio penalties ρ^h have to be chosen a priori, to guarantee that the resulting policy-function maps into the endogenous state space Z. The exact choice of these penalties influence the computed equilibrium prices substantially.

9.2 Ill-conditioned systems

The system of Euler and market-clearing equations which define a recursive equilibrium, (RE1)-(RE3), tends to be numerically very unstable. We measure numerical stability of a system by computing is condition number. The condition number of an invertible real $n \times n$ matrix A is defined as $\kappa(A) = \|A\| \|A^{-1}\|$where $\|A\|$ is the operator norm,

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|}$$

It is a useful measure of a matrix being nearly singular and used to measure the relative error of solutions to the linear problem $Ax = b$ (see Judd (1999) or Blum et al. (1998)).

A large condition number implies that a system is sensitive to small changes and difficult to solve. Newton's method iterates on $x_{k+1} = x_k(Df(x_k))^{-1} f(x_k)$ and a large condition number of the Jacobian $Df(x_k)$ implies that Newton's method cannot be used if one does not have a starting point very close to the solution. On normal computers with machine precision around 10^{-15} condition numbers above 10^{10} are considered unacceptable since it indicates that an input or round-off error of ϵ leads to an output error of $10^{10}\epsilon$.

In models the assets have strongly correlated returns the choices are nearly indeterminate and the condition number of the Euler equations will be very large. In dynamic models with infinitely lived agents and transitory shocks, the choice between stock and bonds is usually uniquely determined but of second-order importance to the investor. Since shocks are transitory, a bad idiosyncratic shock can generally be smoothed out by borrowing. Agents are therefore primarily interested in their total debt-level $s_t^h q_t^s + b_t^h q_t^b$. In regions where the short sale constraint is not binding, they are nearly indifferent between stock and bond. In our example, the condition number of the equilibrium system at the solution lies around 10^9, in some regions of the state space it even reaches 10^{10}. This causes substantial numerical difficulties.

9.2.1 Transaction Costs

If, in our example, we impose a substantial trading cost for the stock, the conditioning of the system improves drastically. For example, if we set $\tau^s = 0.05$, the condition number goes down to around 10^6. However, a large transaction cost clearly also has a substantial impact on trading and prices. An alternative is to impose transaction costs on both bonds and stocks. With the above specification of $\tau^s = 0.0001$ and $\tau^b = 0.0001$ the condition number of the system goes down to 10^7.

9.2.2 Solving the Euler Equations with Homotopy Methods

In many applications one is interested in equilibria which result without any restrictions on transactions. In this case, special care is needed to solve the Euler equations. If we do not have a good starting point, Newton-method-based algorithms for solving (2) are not likely to perform well because they are not globally convergent and because the system of equations is not well-conditioned for many values of the endogenous variables. In this case we have to use homotopy methods to solve (2). The key insight for solving system (2) is that it is similar to the equilibrium conditions of the well-known General Equilibrium Model with Incomplete Markets (GEI Model). Therefore, in order to solve system (2) we can apply – with some modifications – algorithms which have been developed for the GEI Model (in particular Schmedders (1998)).

The main idea of homotopy methods is to deform a system of equations into an simple system that can be easily solved. Then this easy system is continuously transformed back into the original system. Beginning with the known solution of the easy system a path of solutions to the encountered intermediate systems is followed leading eventually to a solution of the given system of equations. Eaves and Schmedders (1999) give an intuitive description of the homotopy principle addressing many issues in the framework of simple economic examples.

In the context of problems involving endogenous portfolio choices it is often useful to set up the easy system such that the agents are forced to hold a prespecified portfolio. Such a system can be easily achieved by adding a portfolio penalty term to the agents' utility functions. In the easy system this penalty has maximal force inducing the agents to hold the specified portfolio. As the penalty is relaxed and driven to zero a path of portfolio choices leads to the agents' equilibrium portfolios.

9.3 Economics and Computations

The example we consider in this paper is obviously highly stylized and oversimplified. However, Heaton and Lucas (1996) carefully choose parameters for the model to match annual US data. When interpreting the results of their simulations, Heaton and Lucas (1996) claim that short-sale constraints are rarely binding in their equilibrium approximation. They claim that the constraints are a technical artifact without any economic consequences.

Judd et al. (1999) show that this is not due to the parameterization but that due to their computational procedure. Indeed, without transaction costs, the algorithm described above approximates an equilibrium in which the short-sale constraints are often binding and which is very similar to the equilibrium computed above.

The reason why in Heaton and Lucas (1996) the short-sale constraints rarely bind even without transaction costs is simple. Allowing only for fairly large discrete jumps of agents' portfolio holdings implicitly introduces a transaction cost. This transaction cost has a particularly severe impact on the agents behavior when they are fairly poor. That is, they hold large short-positions in the assets and would like to trade only minimal amounts of the assets. This, however, is prohibited by the discrete portfolio space in which agents have to trade according to the numerical procedure. Therefore, the agents hardly trade at all anymore long before they hit the short-sale constraints.

As mentioned above, with respect to pricing implications of the model, these additional transaction costs are not significant. Many of the computational problems in this model are caused by the fact that small errors in trading strategies have almost no impact on welfares or prices. However, it is generally not possible to prove this formally and one might feel uncomfortable adopting a computational strategy which does not try to approximate the true trading strategies but relies on some form of transaction costs to make the algorithm more efficient.

9.4 A Parameterized Expectations Algorithm Approach

Marcet and Singleton (1999[6]) used Marcet's version of the parameterized expectations algorithm to solve a model with equity, a bond, and idiosyncratic shocks to income. Let W_i be the wealth of type i agents, p the ex-dividend price of equity, d the dividend, and y the current state of aggregate and idiosyncratic shocks. They focus on the expectations functions

$$\begin{aligned} \varphi_i(W_{1.t}, W_{2.t}, y_t) &= E\left\{u_i'(c_{i,t+1})|W_{1,t}, W_{2.t}, y_t\right\},\ i = 1,2 \\ \varphi_{2+i}(W_{1.t}, W_{2.t}, y_t) &= E\left\{u_i'(c_{i.t+1})(p_{t+1} + d_{t+1})|W_{1,t}, W_{2.t}, y_t\right\},\ i = 1,2 \end{aligned}$$

[6] It should be noted that Marcet and Singleton (1999) was essentially the same as the 1991 working paper version.

for next period's marginal utility of consumption and stocks. They use the Euler equations for bond and equity investment to fix consumption and asset holdings at time t in terms of the $\varphi(W_1, W_2, z)$ functions. This parameterization has the advantage, as first demonstrated in Wright and Williams' (1982a, 1982b, 1984) application of this approach to commodity markets, that conditional expectations functions will be smoother than consumption and pricing functions since borrowing constraints may produce kinks in the relationship between consumption and wealth. This procedure worked very well in Wright and Williams even though they had to model nonnegativity constraints on storage, similar to the borrowing constraints modelled in Marcet and Singleton. Therefore, it is natural to use the smoothing idea of Wright and Williams in this context also.

Marcet and Singleton approximate the φ functions with low-order exponential polynomials of wealth and the exogenous shocks of the form

$$\psi_j(\beta, W_1, W_2, y) = \exp(\beta_{j,1} + (\beta_{j,2}, \beta_{j,3}, \beta_{j,4}) \log y + \beta_{j,5} W_1 + \beta_{j,6} W_2) \quad (23)$$

They combined simulation methods and a successive approximation method to fix the β coefficients in (23). More precisely, they make a guess for the unknown coefficients, they simulate the process with those parameters. They use the data generated by the simulation to estimate the Euler equation errors implied by the candidate parameterization, and then adjust the parameterization using learning ideas of Marcet and Sargent (1989). This is repeated until the parameterization has converged.

Marcet and Singleton report that their algorithm had problems converging. This is not surprising in light of the ill-conditioning in the Euler equations and the nature of equilibrium displayed in Figures 1 and 2. The ill-conditioning implies that small changes in the coefficients will produce large changes in the Euler equation errors, making it difficult for any nonlinear equation procedure to converge, particularly the first-order, successive approximation scheme used in Marcet and Singleton. Second, the graphs in Figures 1 and 2 show that the equilibrium properties are not going to be well-approximated by low-order polynomials.

10 Dynamic Models with Strategic Power

Previous sections discussed only problems of competitive markets. There are many dynamic problems where some agents do not take the actions of others as fixed, and instead realize that their choices will affect the future behavior of other agents. Outside of some linear-quadratic cases, these models typically do not have closed-form solutions. Numerical methods are therefore crucial to analyzing these problems.

There were some efforts to solve strategic problems in the 1980's. Examples include Wright and Williams (1982a) analysis of oil policy, and Kotlikoff, Shoven, and Spivak's (1986) analysis of strategic bequests. These papers used intuitive polynomial approximation methods. Kotlikoff et al. solved a dynamic bargaining problem by approximating the value function with low order polynomials. The approximation was generated by computing solving the value function at 80 values and then using regression to approximate the value functions. They allowed their iterative schemes to continue until economic variables agreed in the first two significant digits. Since the scheme is linearly convergent, this indicates that the accuracy was somewhat less than two digits.

Recently the more formal approach suggested by projection methods from the mathematics literature described in Judd (1992) have been used to solve strategic problems, in particular problems about time consistency of government policy. Rui and Miranda (1996) use a projection method to solve a game between countries using commodity storage policies to affect prices. Ha and Sibert (1997) also use projection methods to solve games of tax policy between open economies. They used projection methods with orthogonal polynomials, and had little difficulty in producing stable and reliable algorithms. Vedenov and Miranda (2000) use collocation methods to solve dynamic duopoly models. The Euler equation errors of their solutions were very small, on the order of 1 part in 10,000, indicating high accuracy in their solutions.

The success of these three recent papers indicate that much more complex dynamic games can be reliably and quickly solved numerically.

11 Asymptotic Methods

Perturbation methods and other asymptotic methods are often used in economics, and recent developments in mathematics are beginning to be applied in economics. They are similar to projection methods in that they try to compute a polynomial or similar approximation but they use different information. Perturbation and asymptotic methods use implicit function theorems. They are quite different from projection methods in terms of the underlying mathematics and the computer software needed to use them. We examine both regular perturbation methods for rational expectations models and an example of a problem where bifurcation methods can solve a problem with a singularity.

11.1 Regular Perturbations and Rational Expectations Models

We illustrate regular perturbation in the context of the basic rational expectations equations in a simple optimal growth model and then in the more general context of (13). Consider first the simple stochastic optimal growth problems

$$\begin{array}{ll} \max_{c_t} & \sum_{t=0}^{\infty} \beta^t \, u(c_t) \\ \text{s.t.} & k_{t+1} = F(k_t - c_t)(1 + \epsilon z_t) \end{array} \tag{24}$$

where the z_t are i.i.d. with unit variance, and ϵ is a parameter expressing the standard deviation. The solution of the deterministic case, $\epsilon = 0$, can be expressed as a policy function, $C(k)$, satisfying the Euler equation

$$u'\left(C(k)\right) = \beta\, u'\left(C\left(F(k - C(k))\right)\right)\, F'\left(k - C(k)\right).$$

Standard linearization methods produce $C'(k)$. However, we are not satisfied with the linear approximation. We want more terms in the Taylor series expansion

$$C(k) = C(k^*) + C'(k^*)(k - k^*) + C''(k^*)(k - k^*)^2/2 + ...$$

Successive differentiations of (24) produce higher-order derivatives of $C(k)$ at $k = k^*$. For example, the second derivative of (24) together with the steady-state condition $k = k^*$ implies that $C''(k^*)$ satisfies the linear equation

$$\begin{aligned} u''C'' + u'''C'C' = \;& \beta u''' \left(C'F'(1 - C')\right)^2\, F' + \beta u''C'' \left(F'(1 - C')\right)^2\, F' \\ & +2\beta u''C'F'(1 - C')^2 F'' + \beta u' F'''(1 - C')^2 \\ & +\beta u' F''(-C'') \end{aligned}$$

where all functions are evaluated at the steady state value of their arguments. Linear operations combined with successive differentiations of (24) produce all higher-order derivatives.

The solution in the general case is a policy function, $C(k, \epsilon)$, which expresses consumption as a function of the capital stock k as well as the standard deviation ϵ. $C(k, \epsilon)$ satisfies the Euler equation

$$u'\left(C(k)\right) = \beta\, E\left\{u'\left(g(\epsilon,\, k,\, z)\right)\; R(\epsilon,\, k, z)\right\} \tag{25}$$

where

$$\begin{aligned} g(\epsilon, k, z) \quad & \equiv C((1 + \epsilon z)F(k - C(k))) \\ R(\epsilon,\, k,\, z) \quad & \equiv (1 + \epsilon\, z)\, F'\left(k - C(k)\right) \end{aligned}$$

Differentiation of (25) shows that

$$\begin{aligned} C_\epsilon \;&=\; 0 \\ C_{\epsilon\epsilon} \;&=\; \frac{u'''\, C'C'F^2 + 2u''C'F + u''C''F^2}{u''C'F' + \beta u'F''} \end{aligned}$$

where all the derivatives of u and F are evaluated at the steady-state values of c and k. This can be continued to compute higher-order derivatives as long as u and F have the necessary derivatives.

This approach can also be used to analyze multidimensional problems. In general, if there are several endogenous variables, $Y(x)$, which are functions of a multidimensional state variable x then we can compute the steady state, compute the linearization $Y(x)$ through standard eigenvalue decomposition methods from linear rational expectations, and then proceed as above to compute the higher-order derivatives of Y. To use a perturbation method we express (13) in the form

$$\begin{aligned} 0 &= E\{g(x_t, y_t, x_{t+1}, y_{t+1}, \epsilon)|x_t\} \\ x_{t+1} &= F(x_t, y_t, \epsilon z_t) \end{aligned} \tag{26}$$

where ϵ is a scaling parameter for the disturbance terms z. If the components of z_t have unit variance then ϵ is the standard deviation. Different values for ϵ represent economies with different disturbances. The key observation is that we often know much about the $\epsilon = 0$ economy since (26) reduces to a deterministic problem. We build on that fact by using implicit function theorems.

The objective is to find some equilibrium rule, $\widehat{Y}(x, \epsilon)$, such that

$$E\left\{g(x, \widehat{Y}(x, \epsilon), F(x, \widehat{Y}(x, \epsilon), \epsilon z), \widehat{Y}(F(x, \widehat{Y}(x, \epsilon), \epsilon z), \epsilon)|x\right\} \doteq 0$$

Perturbation methods aim to approximate $Y(x, \epsilon)$ with a polynomial, just as projection methods do[7]. However, perturbation methods fix the unknown coefficients by computing the derivatives of $Y(x, \epsilon)$ at some value of (x, ϵ) where we know exactly $Y(x, \epsilon)$. Perturbation methods begin with the deterministic steady state, which is the solution to

$$\begin{aligned} g(x^*, y^*, x^*, y^*, 0) &= 0 \\ x^* &= F(x^*, y^*, 0) \end{aligned}$$

The objective is to find the derivatives of $Y(x, \epsilon)$ with respect to x and ϵ at the deterministic steady state, and use that information to construct Taylor series approximations of $Y(x, \epsilon)$, such as

$$Y(x, \epsilon) \doteq y^* + Y_x(x^*, 0)(x - x^*) + Y_\epsilon(x^*, 0)\epsilon + (x - x^*)' Y_{xx}(x^*, 0)(x - x^*) + ...$$

The second step in perturbation methods is to compute the linear terms of the approximation $Y_x(x^*, 0)$. Standard linearization methods show that the coefficients $Y_x(x^*, 0)$ are the solution, $y = Y_x x$, to the linear rational expectations model

$$g_1 x_t + g_2 y_t + g_3 x_{t+1} + g_4 y_{t+1} = 0 \tag{27}$$

where all the gradients of g in (27) are evaluated at the deterministic steady state. Anderson et al. (1996) and Anderson, Evans, Sargent, and McGrattan (1996) survey methods for solving such models. This is the difficult step computationally, but can be handled by conventional methods.

[7] Perturbation methods can be used more generally to construct approximations which are nonlinear in their coefficients, but this is seldom done in economics. See Judd and Guu (1997) for an example of where Pade' approximations are generated from perturbation data and significantly outperform standard Taylor series expansions.

Computing the higher-order derivatives, such as $Y_{xx}(x^*,0)$, $Y_\epsilon(x^*,0)$, $Y_{x\epsilon}(x^*,0)$, etc., is actually easier than computing $Y_x(x^*,0)$ since they are solutions to linear algebraic equations. Judd and Guu (1993) show how to use perturbation methods to solve simple one sector optimal growth problems, and show that the results are very accurate in that the Euler equation errors are small. Gaspar and Judd (1997) exposit the details of applying perturbation methods to optimal control problems.

Zadrozny and Chen (2000) and Collard, Feve, and Juilliard (2000) extends the perturbation method to general rational expectations models. Zadrozny and Chen proposes using Kronecker product and vectorization notation instead of the tensor notation used in Gaspar and Judd. Both Zadrozny and Chen, and Collard, Feve, and Juilliard combines perturbation and projection methods to compute the deviations from certainty equivalence.

Perturbation methods are the only methods which can handle problems with large dimension. Projection methods will suffer from various curses of dimensionality. The number of unknown coefficients in the policy and value functions become large, the number of nonlinear equations used to identify the unknown coefficients becomes large, and the conditional expectations are multidimensional integrals. The combination of those two factors make it difficult to solve multiagent problems similar to (16) for more than a few agents. Judd and Gaspar find that problems with five agents and one asset are tractable but their computations indicate that that is close to the limit of existing technology. Perturbation methods have the advantage of reducing the integral calculations to moments of the random variables and linear operations. Therefore, large problems become more feasible.

11.2 Bifurcation Methods for Small Noise Portfolio Problems

The problems above began with a unique steady state for the deterministic version. Many interesting problems with heterogeneous agents lack a unique steady state, implying that the techniques discussed above do not apply directly. One such case is where there are multiple assets traded among agents with different tastes for risk. Suppose, for example, that there is trade in both a risky equity asset and a safe bond. In the deterministic steady state all assets must have the same returns, implying that investors are indifferent among various assets. Hence, there is not a unique steady state holding of assets even though the equilibrium holding of assets may be unique whenever there is positive amounts of risk.

There have been some attempts to use simple linear quadratic approximation methods like the perturbation methods described above. For example, Tesar (1995) used a linear-quadratic approach to evaluate the utility impact on countries of opening up trade in bond. Each country had a stochastic endowment and the issue was how much risk sharing could be accomplished only through trade in a bond. This was a model with only one asset. Some of her examples showed that moving from trading in one bond, a case of incomplete asset markets, to complete markets would result in a Pareto inferior allocation, a finding which contradicts the first welfare theorem of general equilibrium. Kim and Kim (1999) have shown that the *ad hoc* linear quadratic approach will often produce incorrect results. These examples illustrate the need for using methods from the mathematical literature instead of relying on *ad hoc* approximation procedures. Again, we see the value of Frisch's observation that mathematics is necessary for safe and consistent analyses.

More typically we would like to solve models with multiple assets. Judd (1996, 1998) and Judd and Guu (2000) describe bifurcation methods, and show that the basic portfolio demand problem is a good example where bifurcation methods can be used. Suppose that an investor has W in wealth to invest in two assets. The safe asset yields R per dollar invested and the risky asset yields Z per dollar invested. If a proportion ω of his wealth is invested in the risky asset, final wealth is $Y = W((1-\omega)R + \omega Z)$. We assume that he chooses ω to maximize $E\{u(Y)\}$ for some concave, von Neumann-Morgenstern utility function $u(\cdot)$.

We want to "linearize" around the deterministic case. To do this we parameterize the problem in terms of a scaling parameter ϵ and compute a Taylor series expansion for asset demand around the case of $\epsilon = 0$. The first problem we encounter is that if we eliminate risk by replacing Z with its mean, $\bar{Z}$, the resulting problem is unbounded if $R \neq \bar{Z}$ and indeterminate if $R = \bar{Z}$. Since the former case is untenable, we opt for the latter. We create a continuum of portfolio problems by assuming

$$Z = R + \epsilon z + \epsilon^2 \pi, \tag{28}$$

where $E\{z\} = 0$. At $\epsilon = 0$, Z is degenerate and equal to R. We assume that $\pi > 0$ since risky assets pay a premium. Note that we multiply z by ϵ and π by ϵ^2. Since the variance of ϵz is $\epsilon^2\,\sigma_z^2$, this models the standard result in finance that risk premia are roughly proportional to variance.

We now investigate the collection of portfolio problems indexed by ϵ in (28). The first-order condition for ω, after dividing by ϵW, is, for all ϵ, equivalent to

$$0 = E\{u'\left(WR + \omega W(\epsilon z + \epsilon^2 \pi)\right)\,(z + 2\epsilon\pi)\} \equiv\, G(\omega, \epsilon). \tag{29}$$

Equation (29) defines the solution to the asset demand problem even when $\epsilon = 0$. We know from concavity of $u(c)$ that there is a unique solution to (29) for ω if $\epsilon \neq 0$. However, at $\epsilon = 0$, ω can be anything since the two assets are perfect substitutes. The indeterminacy of ω at $\epsilon = 0$ follows from the fact that $0 = G(\omega, 0)$ for all ω.

We want to solve for $\omega(\epsilon)$ as a Taylor series in ϵ. If there is such a series, implicit differentiation implies

$$0 = G_\omega\,\omega' + G_\epsilon \tag{30}$$

where

$$\begin{aligned} G_\epsilon &= E\{u''(Y)\,W(\omega z + 2\omega\epsilon\pi)\,W\,(z+2\epsilon\pi) + u'(Y)\pi\} \\ G_\omega &= E\{u''(Y)\,(z+\epsilon\pi)^2\epsilon\} \end{aligned}$$

At $\epsilon = 0$, $G_\omega = 0$ for all ω. This implies that at no point $(\omega, 0)$ can we apply the implicit function theorem to (30) to solve for $\omega'(0)$. Moreover we do not know $\lim_{\epsilon\to 0}\omega(\epsilon)$. However, let's proceed as if we can apply the implicit function theorem. Then (30) can be written as

$$\omega' = -\frac{G_\epsilon}{G_\omega}$$

This looks bad since $G_\omega = 0$ at $\epsilon = 0$ until we remember L'Hospital's rule. Suppose that we found a point ω_0 satisfying

$$0 = G_\epsilon(\omega_0,\, 0)$$

Then L'Hospital's rule says that

$$\omega' = -\frac{G_{\epsilon\epsilon}}{G_{\omega\epsilon}}$$

which is well-defined as long as $G_{\omega\epsilon} \neq 0$. So, let's proceed in this way. At $\epsilon = 0$, the second derivative of (29) reduces to $0 = u''(RW)\,\omega_0\sigma_z^2 W + u'(RW)\pi$ which implies that

$$\omega_0 = -\,\frac{\pi}{\sigma_z^2}\,\frac{u'(WR)}{Wu''(WR)}. \tag{31}$$

Now that we have found a candidate bifurcation point, we can continue to derive the Taylor series. The formula (31) is the simple portfolio rule from linear-quadratic analysis, indicating that ω is the product of risk tolerance and the risk premium per unit variance. However, ω_0 is not an approximation to the portfolio choice at any particular variance. Instead, ω_0 is the limiting portfolio share as the variance vanishes. If we want the linear and quadratic approximations of $\omega(\epsilon)$ at $\epsilon = 0$, we must go further, since the quadratic approximation to $\omega(\epsilon)$ is $\omega(\epsilon) \doteq \omega(0) + \epsilon\omega'(0) + \frac{\epsilon^2}{2}\,\omega''(0)$.

To calculate $\omega'(0)$ and $\omega''(0)$, we need to do two more rounds of implicit differentiation. If we differentiate (29) twice with respect to ϵ, we find that

$$0 = G_{\omega\omega}\,\omega'\omega' + 2G_{\omega\epsilon}\,\omega' + G_\omega\omega'' + G_{\epsilon\epsilon},$$

where (without loss of generality, we assume that $W = 1$)

$$\begin{aligned} G_{\epsilon\epsilon} = \;& E\{u'''(Y)\,(\omega z + 2\omega\epsilon\pi)^2\,(z+\epsilon\pi) + u''(Y)\,2\omega\pi(z+\epsilon\pi) \\ & +2u''(Y)\,(\omega z + 2\omega\epsilon\pi)\pi\}, \\ G_{\omega\omega} = \;& E\{u'''(Y)\,(z+\epsilon\pi)^3\,\epsilon\}, \\ G_{\omega\epsilon} = \;& E\{u'''(Y)\,(\omega z + 2\omega\epsilon\pi)\,(z+\epsilon\pi)^2\,\epsilon + u''(Y)\,(z+\epsilon\pi)\,2\pi\epsilon \\ & +u''(Y)\,(z+\epsilon\pi)^2\}. \end{aligned}$$

At $\epsilon = 0$, $G_{\epsilon\epsilon} = u'''(R)\omega_0^2\, E\{z^3\}$, $G_{\omega\omega} = 0$, and $G_{\omega\epsilon} = u''(R)\, E\{z^2\} \neq 0$. Therefore the L'Hospital's rule applies and

$$\omega' = -\frac{1}{2}\frac{u'''(R)}{u''(R)} \quad \frac{E\{z^3\}}{E\{z^2\}}\,\omega_0^2. \tag{32}$$

Equation (32) is a simple formula. It shows that as riskiness increases, the change in ω depends on $u^{'''}/u^{''}$ and the ratio of skewness to variance. If u is quadratic or z is symmetric, ω does not change to a first order. We could continue this and compute more derivatives of $\omega(\epsilon)$ as long as u is sufficiently differentiable.

We end the development of this example here. However, it is clear much more can be done. Judd and Guu (2000) develops this approach using formal tools from bifurcation theory and applies it to problems of asset demand with several assets, asset equilibrium with incomplete asset markets, and problems of optimal asset innovation. Since the key bifurcation theorems are also true in Banach spaces, these methods can presumably be used to approximate equilibria in stationary dynamic models.

12 Conclusion

The work in the past decade indicates that there is steady progress in computing equilibria in markets with several agents. The key tools have been exploitation of approximation theory and of methods for solving large systems of equations. These efforts have shown that it is now tractable to solve some dynamic markets with complete or incomplete asset markets, and even problems with strategic interactions, such as time inconsistency problems.

Further progress is likely since economists have just begun to exploit the full range of available numerical tools. For example, economists are just beginning to make use of methods which combine symbolic and numerical methods. Also, more advanced numerical and symbolic methods make higher-dimensional problems more tractable.

References

[1] Anderson, Evan, Lars P. Hansen, and Ellen McGrattan. "Mechanics of Forming and Estimating Dynamic Linear Economies," in *Handbook of Computational Economics: Volume I*, edited by Hans M. Amman, David A. Kendrick, and John Rust. Amsterdam; New York and Oxford: Elsevier Science, North-Holland, 1996.

[2] Bensoussan, Alain. *Perturbation Methods in Optimal Control*, John Wiley and Sons, 1988.

[3] Blanchard, Olivier Jean and Charles M. Kahn.1980. "The Solution of Linear Difference Models under Rational Expectations" *Econometrica* **48**:1305–1311.

[4] Blum, L., F. Cucker, M. Shub and S. Smale, *Complexity and Real Computation*, Springer Verlag, (1998).

[5] Boucekkine, R. "An Alternative Methodology for Solving Nonlinear Forward-Looking Models." *Journal of Economic Dynamics and Control* **19** (1995.), 711-734.

[6] Brock, William A., and L. J. Mirman. 1972. "Optimal Economic Growth and Uncertainty: The Discounted Case," *Journal of Economic Theory* **4**: 479–513.

[7] Christiano, Lawrence J. 1990. "Solving the Stochastic Growth Model by Linear-Quadratic Approximation and by Value-Function Iteration," *Journal of Business and Economic Statistics*, **8**:23–26.

[8] W. J. Coleman, II. 1990. "Solving the Stochastic Growth Model by Policy Function Iteration," *Journal of Business and Economic Statistics*, **8**:27–29.

[9] Daniel, J. W. 1976. "Splines and Efficiency in Dynamic Programming," *Journal of Mathematical Analysis and Applications* **54**:402–407.

[10] deBoor, C., *A Practical Guide to Splines*, Springer Verlag, (1978).

[11] Duffie, D., J. Geanakoplos, A. Mas-Colell, and A. McLennan, "Stationary Markov Equilibria," *Econometrica* 62 (1994), 745–781.

[12] Eaves, B.C. and K. Schmedders, "General Equilibrium Models and Homotopy Methods," *Journal of Economic Dynamics and Control* 23 (1999) 1249–1279.

[13] Fair, R.C., and Taylor, J.B., 1983. "Solution and maximum likelihood estimation of dynamic nonlinear rational expectations models." *Econometrica* **51** (4), 1169- 1185.

[14] Fair, R.C., and John B. Taylor. "Full Information Estimation and Stochastic Simulation of Models with Rational Expectations," *Journal ofApplied Econometrics* **5**, (4, 1990): 381-392.

[15] Fleming, Wendell. 1971. "Stochastic Control for Small Noise Intensities," *SIAM Journal of Control* **9** (No. 3, August, 1971).

[16] Fisher, P.G., and Hughes Hallett, A.J., 1987. "The convergence characteristics of iterative techniques for solving econometric models." *Oxford Bulletin of Economics and Statistics* 49, 231- 244.

[17] Fisher, P.G., and Hughes Hallett, A.J., 1988a. "Iterative techniques for solving simultaneous equation systems. a view from the economics literature." *Journal of Computational and Applied Mathematics* 24, 241- 255.

[18] Gilli, M, and Pauletto, G, 1997. "Sparse direct methods for model simulation. *Journal of Economic Dynamics & Control* **21**, 1093- 1111.

[19] Gilli, M., and Pauletto, G., 1998. "Nonstationary iterative methods for solving models with forward looking variables." *Journal of Economic Dynamics and Control*, this issue.

[20] Robert L. Gustafson, "Carryover Levels for Grains: A Method for Determining Amounts That are Optimal Under Specified Conditions," USDA Technical Bulletin 1178, 1958.

[21] den Haan, Wouter, and Albert Marcet. 1990. "Solving the Stochastic Growth Model by Parameterizing Expectations," *Journal of Business and Economic Statistics*, **8**:31–34

[22] den Haan, W., "Solving Dynamic Models with Aggregate Shocks and Heterogeneous Agents, *Macroeconomic Dynamics* **1** (1997), 355–386.

[23] Garcia, C. and W. Zangwill, *Pathways to Solutions, Fixed Points, and Equilibria*, Prentice Hall, Englewood Cliffs (1981).

[24] Ha, Jiming, Anne Sibert. "Strategic Capital Taxation in Large Open Economies with Mobile Capital," *International Tax and Public Finance* **4** (1997), 243-62.

[25] Hall, Robert. 1971. "The Dynamic Effects of Fiscal Policy in an Economy with Foresight," *Review of Economic Studies* **38**:229-244.

[26] Heaton, J. and D.J. Lucas, "Evaluating the Effects of Incomplete Markets on Risk Sharing and Asset Pricing," *Journal of Political Economy* **104** (1996), 443–487.

[27] Hernandez, A. and M.S.Santos, "Competitive Equilibria for Infinite-Horizon Economies with Incomplete Markets," *Journal of Economic Theory* **71** (1996), 102–130.

[28] Hughes Hallett, A., and Piscitelli, L., 1998. "A new convergence theorem for successive overrelaxation iterations," *Computational Economics*.

[29] Hughes Hallett, A.J., and Piscitelli, L., 1998. "Simple reordering techniques for expanding the convergence radius of first order iterative techniques." *Journal of Economic Dynamics & Control*.

[30] Judd, Kenneth L. 1985. "Short-run Analysis of Fiscal Policy in a Simple Perfect Foresight Model", *Journal of Political Economy* **93**: 298-319.

[31] Judd, Kenneth L. 1992. "Projection Methods for Solving Aggregate Growth Models", *Journal of Economic Theory* **58:**410–452.

[32] Judd, Kenneth L. *Numerical Methods in Economics* , MIT Press, 1998.

[33] Judd, Kenneth L. "Parametric Path Method for Solving Perfect Foresight Models", http://bucky.stanford.edu/cef97/sessions2.htm, 1999.

[34] Judd, Kenneth L. 1996. "Approximation, Perturbation, and Projection Methods in Economic Analysis," in *Handbook of Computational Economics*, edited by Hans Amman, David Kendrick, and John Rust, North Holland.

[35] Judd, Kenneth L., and Jess Gaspar. "Perturbation Methods for Discrete-Time Dynamic Deterministic Models," Hoover Institution, mimeo, 1997.

[36] Judd, Kenneth L., and Sy-Ming Guu. 1993. "Perturbation Solution Methods for Economic Growth Models," in *Economic and Financial Modeling with Mathematica*, Hal Varian, ed., Springer-Verlag: New York.

[37] Judd, Kenneth L., and Sy-Ming Guu. "Asymptotic Methods for Aggregate Growth Models," forthcoming in *Journal of Economic Dynamics and Control*.

[38] Judd, K.L., F. Kubler and K. Schmedders, "Computing Equilibria in Infinite Horizon Finance Economies I: The Case of One Asset," *Journal of Economic Dynamics and Control* 24 (2000) 1047–1078.

[39] Judd, K.L., F. Kubler and K. Schmedders, "The Impact of Portfolio Constraints in Infinite-Horizon Incomplete-Markets Models," in *The Theory of Markets,* eds. P.J.J. Herings, A.J.J. Talman, and G. van der Laan, North-Holland, Amsterdam (1999).

[40] Judd, K.L., F.Kubler and K.Schmedders, "A Solution Method for Incomplete Asset Markets with Heterogeneous Agents," working paper (1999).

[41] Juillard, M., 1996. "DYNARE: A program for the resolution and simulation of dynamic models with forward variables through the use of a relaxation algorithm." CEPREMAP Working Paper, No. 9602, Paris, France.

[42] Juillard, M., Laxton, D., McAdam, P., and Pioro, H., 1998. "An algorithm competition: first-order iterations versus Newton-based techniques." *Journal of Economic Dynamics & Control*.

[43] Kotlikoff, Lawrence, John Shoven, and Avi Spivak. "The Effect of Annuity Insurance on Savings and Inequality," *Journal of Labor Economics* **4** (3, part 2, 1986), S183–S207.

[44] Krusell, P. and A.A. Smith, Jr., "Income and Wealth Heterogeneity, Portfolio Choice, and Equilibrium Asset Returns," *Macroeconomic Dynamics* 1 (1997), 387–422.

[45] Kubler, F. and K. Schmedders, "Recursive Equilibria with Incomplete Financial Markets," to appear in Macroeconomic Dynamics (2000).

[46] Kydland, Finn E., and Edward C. Prescott. 1982. "Time to Build and Aggregate Fluctuations," *Econometrica* **50**: 1345-1370.

[47] Laffargue, J.P., 1990. "Resolution d'un mode'le macroeconomique avec anticipations rationnelles." *Annales d'Economie et Statistique* 17, 97- 119.

[48] Laitner, John. 1981. "The Stability of Steady States in Perfect Foresight Models," *Econometrica* **49**:319-333.

[49] Laitner, John. 1984. "Transition Time Paths for Overlapping-Generations Models," *Journal of Economic Dynamics and Control* **7**:111-129.

[50] Levine, D. and W.Zame, "Debt constraint and equilibrium in infinite horizon economies with incomplete markets," *Journal of Mathematical Economics* **26** (1996), 103–131.

[51] Lucas, D.J., "Asset pricing with undiversifiable income risk and short-sale constraints. Deepening the equity premium puzzle," *Journal of Monetary Economics* **34** (1994), 325–241.

[52] Lucas, R.E.Jr., "Asset Prices in an Exchange Economy, *Econometrica* **46** (1978), 1429–1445.

[53] Magill, M. and M. Quinzii, "Incomplete markets over an infinite horizon: Long-lived securities and speculative bubbles," *Journal of Mathematical Economics* **26** (1996), 133–170.

[54] Magill, J. P. Michael. 1977. "A Local Analysis of N-Sector Capital Accumulation under Uncertainty," *Journal of Economic Theory*, **15**: 211–219.

[55] Marcet, Albert. "Simulation Analysis of Dynamic Stochastic Models: Applications to Theory and Estimation," in C. Sims (ed.) *Advances in Econometrics Sixth World Congress*, Vol II., 91–118, Cambridge University Press.

[56] Marcet, Albert, and Thomas Sargent. "Convergence of Least Squares Learning Mechanisms in Self-referential Linear Stochastic Models," *Journal of Economic Theory* **48** (1989): 337-68.

[57] McGrattan, Ellen R. 1990. "Solving The Stochastic Growth Model by Linear-Quadratic Approximation." *Journal of Business and Economic Statistics* **8**: 41–43.

[58] Miranda, Mario J., and Peter G. Helmberger. 1988. "The Effects of Commodity Price Stabiliization Programs." *American Economic Review* **78**:46–58.

[59] Pauletto, G, 1997. *Computational Solution of Large-Scale Macroeconometric Models. Series: Advances in Computational Economics.* Kluwer Academic Publishers, Dordrecht.

[60] Prescott, Edward C., and R. Mehra. 1978. "Recursive Competitive Equilibrium: The Case of Homogeneous Agents," *Econometrica* **48**:1365–1379.

[61] Rios-Rull. "Computation of Equilibria in Heterogenous Agent Models". in Computational Methods for the Study of Dynamic Economics: An Introduction," edited by Ramon Marimon and Andrew Scott, 1999: Oxford University Press.

[62] Rui, Xiongwen and Mario J. Miranda. "Solving Nonlinear Dynamic Games via Orthogonal Collocation: An Application to Strategic Commodity Buffer Stock Policy." *Annals of Operations Research* **68** (1996), 89-108.

[63] Saad, Y., 1996. *Iterative Methods for Sparse Linear Systems*. PWS Publishing Comp., Boston, MA.9

[64] Saad, Y, and Schultz, M, 1986. "GMRES: a generalized minimal residual algorithm for solving nonsymetric linear systems." *SIAM Journal for Scientific and Statistical Computing* 7, 856- 869.

[65] Santos, M.S. and J. Vigo-Aguiar, Analysis of a numerical dynamic programming algorithm applied to economic models, *Econometrica* 66 (1998), 409–426.

[66] Schmedders, K., "Computing Equilibria in the General Equilibrium Model with Incomplete Asset Markets," *Journal of Economic Dynamics and Control* 22 (1998), 1375–1401.

[67] Stokey, Nancy L., and Robert E. Lucas, Jr. *Recursive Methods in Economic Dynamics*, Harvard University Press: Cambridge. 1989.

[68] Taylor, John B., and Harald Uhlig. 1990. "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods," *Journal of Business and Economic Statistics*, **8**:1–18.

[69] Telmer, C.I. "Asset-Pricing Puzzles in Incomplete Markets, *Journal of Finance* 48 (1993), 1803–1832.

[70] Watson, L.T., S.C. Billups, and A.P. Morgan, "HOMPACK: A Suite of Codes for Globally Convergent Homotopy Algorithm," *ACM Transactions on Mathematical Software* 13 (1987), 281–310.

[71] Wright, Brian D., and Jeffrey C. Williams. "The Roles of Public and Private Storage in Managing Oil Import Disruptions," *Bell Journal of Economics* 13 (1982a): 341-53.

[72] Wright, Brian D., and Jeffrey C. Williams. "The Economic Role of Commodity Storage," *Economic Journal* **92** (1982b), pages 596-614.

[73] Wright, Brian D., and Jeffrey C. Williams. 1984. "The Welfare Effects of the Introduction of Storage," *Quarterly Journal of Economics* **99**:169-182.

[74] Williams, Jeffrey, and Brian Wright, *Storage and Commodity Markets*, Cambridge University Press, January, 1991.

[75] Young, D.M., 1971. *Iterative Solution of Large Linear Systems*. Academic Press, New York.

[76] Zhang, H.H., "Endogenous Borrowing Constraint with Incomplete Markets," *Journal of Finance* (1997a), 2187–2209.

[77] Zhang, H.H., "Endogenous Short-Sale Constraint, Stock Prices and Output Cycles," *Macroeconomic Dynamics* 1 (1997b), 228–254.

Figures

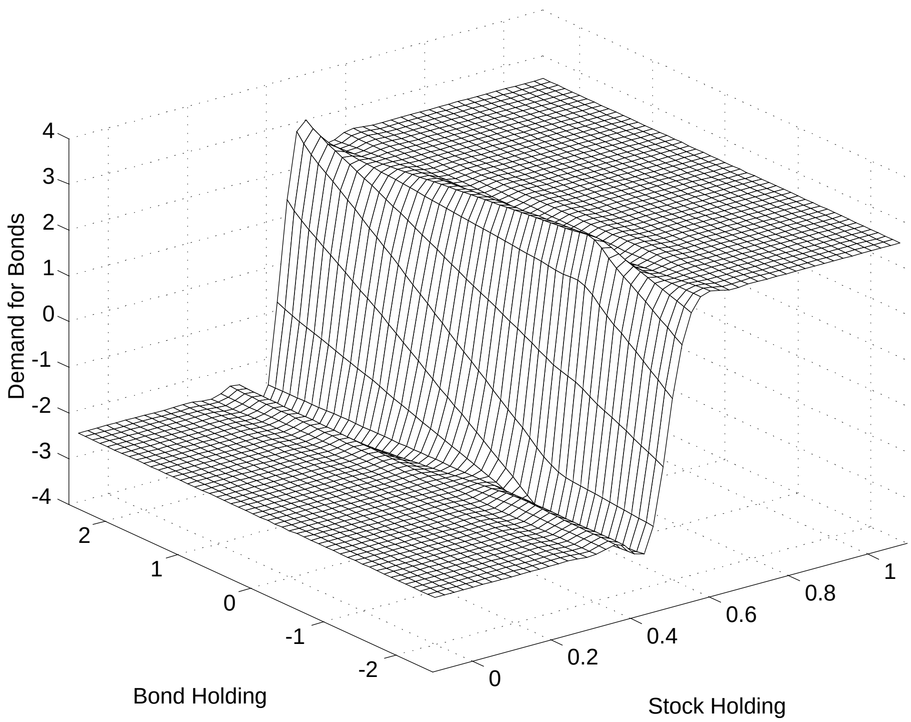

Figure 1: Bond Demand - No Transaction Costs

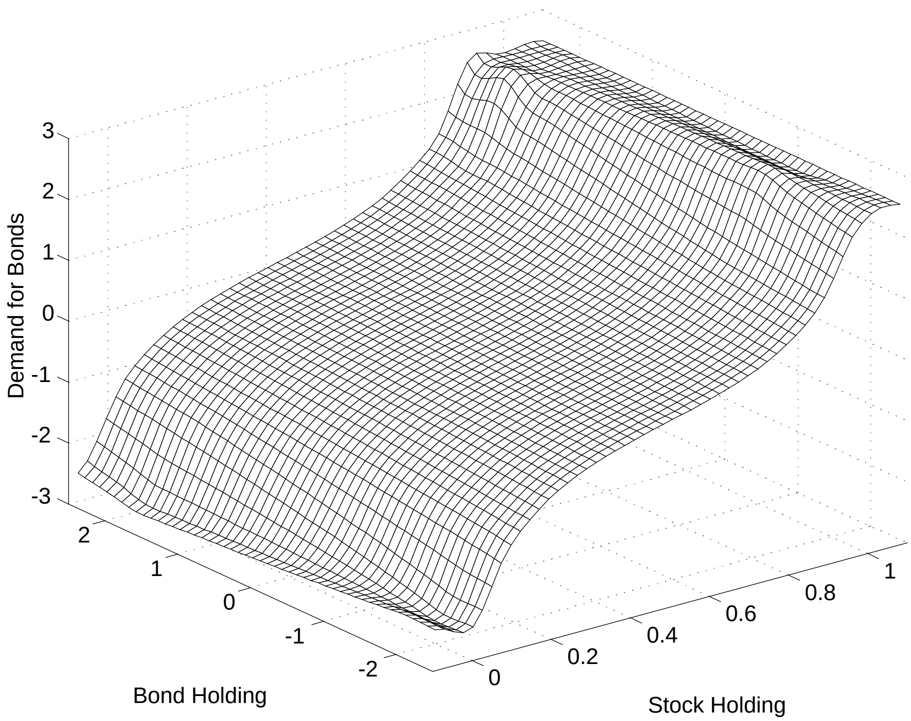

Figure 2: Bond Demand - With Transaction Costs

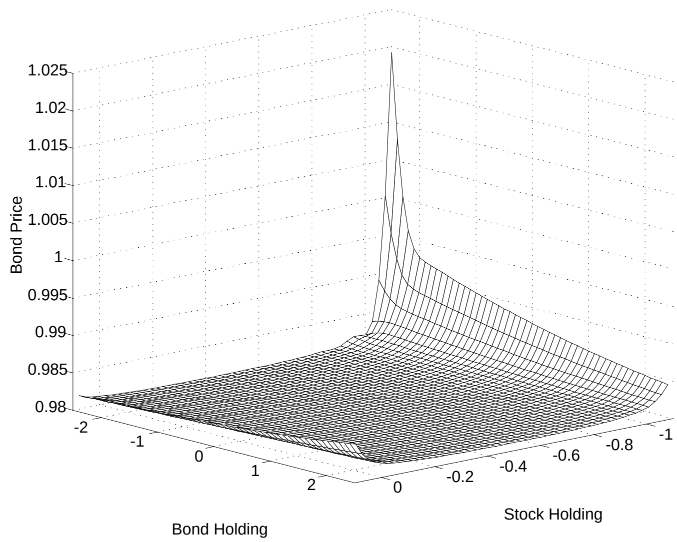

Figure 3: Bond Price Manifold

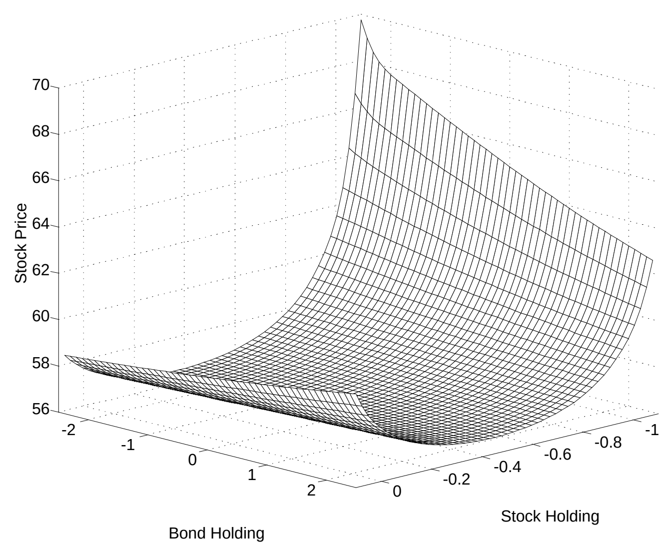

Figure 4: Stock Price Manifold