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POSITIVITY OF CHERN CLASSES OF SCHUBERT CELLS AND VARIETIES

JUNE HUH

Abstract

We show that the Chern-Schwartz-MacPherson class of a Schubert cell in a Grassmannian is represented by a reduced and irreducible subvariety in each degree. This gives an affirmative answer to a positivity conjecture of Aluffi and Mihalcea.

1. Introduction

The classical *Schubert varieties* in the Grassmannian of d-planes in a vector space E are among the most studied singular varieties in algebraic geometry. The subject of this paper is the study of *Chern classes* of Schubert cells and varieties.

There is a good theory of Chern classes for singular or noncomplete complex algebraic varieties. If X° is a locally closed subset of a complete variety X, then the *Chern-Schwartz-MacPherson class* of X° is an element in the Chow group

$$c_{SM}(X^\circ) \in A_*(X),$$

which agrees with the total homology Chern class of the tangent bundle of X if X is smooth and $X = X^\circ$. The Chern-Schwartz-MacPherson class satisfies good functorial properties which, together with the normalization for smooth and complete varieties, uniquely determine it. Basic properties of the Chern-Schwartz-MacPherson class are recalled in Section 2.1.

If $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0)$ is a partition, then there is a corresponding Schubert variety $\mathbb{S}(\underline{\alpha})$ in the Grassmannian of d-planes in E, parametrizing d-planes which satisfy incidence conditions with a flag of subspaces determined by $\underline{\alpha}$. See Section 2.2 for our notational conventions. The Schubert variety is

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a disjoint union of Schubert cells

$$\mathbb{S}(\underline{\alpha}) = \coprod_{\underline{\beta} \leq \underline{\alpha}} \mathbb{S}(\underline{\beta})^\circ,$$

where the union is over all $\underline{\beta} = (\beta_1 \geq \beta_2 \geq \cdots \geq \beta_d \geq 0)$ which satisfy $\beta_i \leq \alpha_i$ for all i. Since each Schubert cell $\mathbb{S}(\underline{\beta})^\circ$ is isomorphic to an affine space, the Chow group of $\mathbb{S}(\underline{\alpha})$ is freely generated by the classes of the closures $\big[\mathbb{S}(\underline{\beta})\big]$. Therefore we may write

$$c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big) = \sum_{\underline{\beta} \leq \underline{\alpha}} \gamma_{\underline{\alpha},\underline{\beta}} \big[\mathbb{S}(\underline{\beta})\big] \in A_*\big(\mathbb{S}(\underline{\alpha})\big)$$

for uniquely determined coefficients $\gamma_{\underline{\alpha},\underline{\beta}} \in \mathbb{Z}$.

Various explicit formulas for these coefficients are obtained in [AM09]. One of the formulas says that $\gamma_{\underline{\alpha},\underline{\beta}}$ is the sum of the binomial determinants

$$\gamma_{\underline{\alpha},\underline{\beta}} = \sum_L \det \left[\binom{\alpha_i - l_{i,i+1} - l_{i,i+2} - \cdots - l_{i,d}}{\beta_j + i - j + l_{1,i} + l_{2,i} + \cdots + l_{i-1,i} - l_{i,i+1} - l_{i,i+2} - \cdots - l_{i,d}} \right]_{1 \leq i,j \leq d}$$

where the sum is over all strictly upper triangular nonnegative integral matrices $L = [l_{p,q}]_{1 \leq p < q \leq d}$ such that

$$0 \leq l_{p,p+1} + l_{p,p+2} + \cdots + l_{p,d} \leq \alpha_{p+1} \quad \text{for} \quad 1 \leq p < d.$$

For example, $\gamma_{(3 \geq 2 \geq 1),(2 \geq 0 \geq 0)}$ is the sum of the determinants of the matrices

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

That is,

$$\gamma_{(3 \geq 2 \geq 1),(2 \geq 0 \geq 0)} = 3 + 2 + 2 + (-1) + 2 + 0 + 0 + 2 + 0 + 1 + 0 + 0 = 11.$$

Based on substantial computer calculations, Aluffi and Mihalcea conjectured that all $\gamma_{\underline{\alpha},\underline{\beta}}$ are nonnegative [AM09, Conjecture 1].

Conjecture 1. *For all $\underline{\beta} \leq \underline{\alpha}$, the coefficient $\gamma_{\underline{\alpha},\underline{\beta}}$ is nonnegative.*

When $d = 2$, the classical Lindström-Gessel-Viennot lemma shows that $\gamma_{\underline{\alpha},\underline{\beta}}$ is the number of certain nonintersecting lattice paths joining pairs of points in the plane, and hence nonnegative [AM09, Theorem 4.5].

The following is the main result of this paper. Fix a nonnegative integer $k \leq \dim \mathbb{S}(\underline{\alpha})$, and write $c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big)_k$ for the k-dimensional component of $c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big)$ in $A_k\big(\mathbb{S}(\underline{\alpha})\big)$.

Theorem 2. *There is a nonempty reduced and irreducible k-dimensional subvariety $Z(\underline{\alpha})$ of $\mathbb{S}(\underline{\alpha})$ such that*

$$c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big)_k = \big[Z(\underline{\alpha})\big] \in A_k\big(\mathbb{S}(\underline{\alpha})\big).$$

For an explicit description of the subvariety $Z(\underline{\alpha})$, see Theorem 15. The proof of Theorem 2 is based on an explicit description of the Chern class of a vector bundle at the level of cycles. This vector bundle lives on a carefully chosen desingularization of $\mathbb{S}(\underline{\alpha})$, and it is not globally generated in general.

Since any 0-dimensional subvariety is a point, the assertion of Theorem 2 when $k = 0$ is just

$$\chi\big(\mathbb{S}(\underline{\alpha})^\circ\big) = \int_{\mathbb{S}(\underline{\alpha})} c_{SM}\big(\mathbb{S}(\underline{\alpha})^\circ\big) = 1.$$

In general, homology classes representable by a reduced and irreducible subvariety have significantly stronger properties than those representable by an effective cycle. These stronger properties are sometimes of interest in applications [Huh12a, Huh15]. Unfortunately, little seems to be known about homology classes of subvarieties of a Grassmannian. For the case of curves and multiples of Schubert varieties, however, see [Bry10, Cos11, CR13, Hon05, Hon07, Per02].

It is known that the cone of effective cycles in $A_k\big(\mathbb{S}(\underline{\alpha})\big) \otimes \mathbb{Q}$ is a polyhedral cone generated by the classes of k-dimensional $\mathbb{S}(\underline{\beta})$ with $\underline{\beta} \le \underline{\alpha}$ [FMSS95]. Therefore Theorem 2 gives an affirmative answer to Conjecture 1.

Corollary 3. *For all $\underline{\beta} \le \underline{\alpha}$, the coefficient $\gamma_{\underline{\alpha},\underline{\beta}}$ is nonnegative.*

Corollary 3 was previously known for all $\underline{\alpha}$ when $d = 2$ [AM09] or $d = 3$ [Mih07], and for all $\underline{\beta} \le \underline{\alpha}$ such that the codimension of $\mathbb{S}(\underline{\beta})$ in $\mathbb{S}(\underline{\alpha})$ is at most 4 [Str11].

It also follows from Theorem 2 that the Chern-Schwartz-MacPherson class of the Schubert variety

$$c_{SM}\big(\mathbb{S}(\underline{\alpha})\big) = \sum_{\underline{\beta} \le \underline{\alpha}} c_{SM}\big(\mathbb{S}(\underline{\beta})^\circ\big)$$

is represented by an effective cycle. This weaker version of positivity was obtained in [Jon10, Theorem 6.5] for a certain infinite class of partitions $\underline{\alpha}$ using Zelevinsky's small resolution.

Finding a positive combinatorial formula for $\gamma_{\underline{\alpha},\underline{\beta}}$ remains a very interesting problem. As mentioned before, $\gamma_{\underline{\alpha},\underline{\beta}}$ is the number of certain nonintersecting lattice paths joining pairs of points in the plane when $d = 2$. A similar positive combinatorial formula is known for $d = 3$ [Mih07, Corollary 3.10]. The reader will find useful discussions and numerical tables of $\gamma_{\underline{\alpha},\underline{\beta}}$ in [AM09, Mih07, Jon07, Jon10, Str11, Web12].

2. Preliminaries

2.1. Here we briefly recall the basic properties of the Chern-Schwartz-MacPherson class. More details can be found in [Alu05, Ken90, Mac74, Sch05].

Let X be a complete complex algebraic variety. The group of constructible functions on X is the free abelian group $C(X)$ generated by functions of the form

$$\mathbf{1}_W = \begin{cases} 1, & x \in W, \\ 0, & x \notin W, \end{cases}$$

where W is a closed subvariety of X. If $f : X \longrightarrow Y$ is a morphism between complete varieties, then the push-forward f_* is defined to be the homomorphism

$$f_* : C(X) \longrightarrow C(Y), \qquad \mathbf{1}_W \longmapsto \Big(y \longmapsto \chi\big(f^{-1}(y) \cap W\big)\Big)$$

where χ stands for the topological Euler characteristic. This defines a functor C from the category of complete varieties to the category of abelian groups.

Definition 4. The *Chern-Schwartz-MacPherson class* is the unique natural transformation

$$c_{SM} : C \longrightarrow A_*$$

such that

$$c_{SM}(\mathbf{1}_X) = c(T_X) \cap [X] \in A_*(X)$$

if X is a smooth and complete variety with the tangent bundle T_X. When X° is a locally closed subset of X, we write

$$c_{SM}(X^\circ) := c_{SM}(\mathbf{1}_{X^\circ}).$$

The functoriality of c_{SM} says that, for any $f : X \longrightarrow Y$ as above, we have the commutative diagram

$$\begin{array}{ccc} C(X) & \xrightarrow{c_{SM}} & A_*(X) \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_*} \\ C(Y) & \xrightarrow[c_{SM}]{} & A_*(Y). \end{array}$$

The uniqueness of c_{SM} follows from the functoriality, the resolution of singularities, and the normalization for smooth and complete varieties. The existence of c_{SM}, which was once a conjecture of Deligne and Grothendieck, was proved by MacPherson in [Mac74]. The Chern-Schwartz-MacPherson class satisfies the inclusion-exclusion formula

$$c_{SM}(\mathbf{1}_{U_1 \cup U_2}) = c_{SM}(\mathbf{1}_{U_1}) + c_{SM}(\mathbf{1}_{U_2}) - c_{SM}(\mathbf{1}_{U_1 \cap U_2})$$

and captures the topological Euler characteristic as its degree

$$\chi(U) = \int c_{SM}(\mathbf{1}_U).$$

Here U, U_1, U_2 can be any constructible subset of a complete variety. For a construction of c_{SM} with an emphasis on noncomplete varieties, see [Alu06a, Alu06b].

2.2. We define the Schubert variety $\mathbb{S}(\underline{\alpha})$ corresponding to a partition $\underline{\alpha}$ in the Grassmannian of d-planes $\mathrm{Gr}_d(E)$. Schubert varieties will only appear in the last section of this paper.

Our notation for Schubert varieties is consistent with that of [AM09]. In the study of homology Chern classes, this 'homological' notation has advantages over the more common 'cohomological' notation.

Let E be a complex vector space with an ordered basis $e_1, \ldots, e_{n+d}$, and take F_k to be the subspace spanned by the first k vectors in this basis.

Definition 5. Let $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0)$ be a partition with $n \geq \alpha_1$.

(1) The *Schubert variety* corresponding to $\underline{\alpha}$ is the subvariety

$$\mathbb{S}(\underline{\alpha}) := \Big\{ V \mid \dim(V \cap F_{\alpha_{d+1-i}+i}) \geq i \text{ for } i = 1, \ldots, d \Big\} \subseteq \mathrm{Gr}_d(E).$$

(2) The *Schubert cell* corresponding to $\underline{\alpha}$ is the open subset of $\mathbb{S}(\underline{\alpha})$

$$\mathbb{S}(\underline{\alpha})^\circ := \Big\{ V \mid \dim(V \cap F_{\alpha_{d+1-i}+i}) = i, \\ \dim(V \cap F_{\alpha_{d+1-i}+i-1}) = i - 1 \text{ for } i = 1, \ldots, d \Big\}.$$

We summarize the main properties of Schubert cells and varieties:

- Writing $\underline{\beta} \leq \underline{\alpha}$ for the ordering $\beta_i \leq \alpha_i$ for all i, we have

$$\mathbb{S}(\underline{\alpha})^\circ = \mathbb{S}(\underline{\alpha}) \setminus \bigcup_{\underline{\beta} < \underline{\alpha}} \mathbb{S}(\underline{\beta}).$$

- The Schubert cell $\mathbb{S}(\underline{\alpha})^\circ$ is isomorphic to the affine space $\mathbb{C}^{\alpha_1 + \cdots + \alpha_d}$.
- The Schubert cell $\mathbb{S}(\underline{\alpha})^\circ$ is an orbit under the natural action of B on $\mathrm{Gr}_d(E)$.

Here B is the subgroup of the general linear group of E which consists of all invertible upper triangular matrices with respect to the ordered basis $e_1, \ldots, e_{n+d}$. The reader will find details in [AM09, Bri05, Ful97].

3. Chern classes of almost homogeneous varieties

In this section, B is a connected affine algebraic group with the Lie algebra $\mathfrak{b}$.

3.1. Suppose B acts on an irreducible projective variety Y with an open dense orbit Y°. We say that Y is *almost homogeneous* with respect to the action of B. For example, Y can be the Schubert variety $\mathbb{S}(\underline{\alpha})$ of the previous section.

Definition 6. A *B-finite log-resolution* of Y is a proper B-equivariant map $\pi : X \longrightarrow Y$ such that

(1) X is smooth and has finitely many B-orbits,
(2) $\pi^{-1}(Y^\circ) \longrightarrow Y^\circ$ is an isomorphism, and
(3) the complement of $\pi^{-1}(Y^\circ)$ in X is a divisor with normal crossings.

The main result of this section is the following sufficient condition for the Chern-Schwartz-MacPherson class of an almost homogeneous B-variety to be effective.

Theorem 7. *Suppose Y has a B-finite log-resolution. Then there are subvarieties $Z_1, \ldots, Z_p$ of Y and nonnegative integers $n_1, \ldots, n_p$ such that*

$$c_{SM}(Y^\circ) = \sum_{i=1}^{p} n_i [Z_i] \in A_*(Y).$$

In short, the Chern-Schwartz-MacPherson class of Y° is represented by an effective cycle on Y if Y has a B-finite resolution. An explicit description of the subvarieties Z_i can be found in Corollary 13.

When Y is the Schubert variety $\mathbb{S}(\underline{\alpha})$, the conclusion of Theorem 7 is much weaker than that of Theorem 2. However, the main construction which leads to the proof of Theorem 7 will be essential in the proof of Theorem 2.

The rest of this section is devoted to the proof of Theorem 7.

3.2. As preparation, we recall the basic results on algebraic group actions and algebraic vector fields. General references are [MO67] and [Ram64].

Suppose B acts on a smooth and irreducible projective variety X. There is an algebraic group homomorphism from B to the connected automorphism group

$$L : B \longrightarrow \mathrm{Aut}^\circ(X), \qquad b \longmapsto \big(x \longmapsto b \cdot x\big).$$

The differential of L at the identity is the *Lie homomorphism* between the Lie algebras

$$\mathfrak{b} \longrightarrow \Gamma(X, T_X).$$

Explicitly, the Lie homomorphism maps $\xi \in \mathfrak{b}$ to the corresponding fundamental vector field

$$x \longmapsto \frac{d}{dt}\Big|_{t=0} \Big(\exp(-t\xi) \cdot x \Big).$$

If we define the B-action on the vector fields on X by

$$\Big(x \longmapsto v(x)\Big) \longmapsto \Big(x \longmapsto d(b \cdot -)v(b^{-1} \cdot x)\Big),$$

then the Lie homomorphism is B-equivariant with respect to the adjoint action of B on $\mathfrak{b}$. Evaluating the Lie homomorphism, we have the homomorphism between the B-linearized vector bundles

$$\mathscr{L}_X : \mathfrak{b}_X \longrightarrow T_X,$$

where $\mathfrak{b}_X$ is the trivial vector bundle on X modeled on $\mathfrak{b}$.

3.3. Let S be an orbit of the B-action on X, and write ι for the inclusion $S \longrightarrow X$. A choice of a base point $x_0 \in S$ defines the orbit map

$$B \longrightarrow S, \qquad b \longmapsto b \cdot x_0.$$

This identifies S with B/H, where H is the isotropy group B_{x_0}. The Lie homomorphism

$$\mathfrak{b} \longrightarrow \Gamma(S, T_S)$$

gives the B-linearized vector bundle homomorphism

$$\mathscr{L}_S : \mathfrak{b}_S \longrightarrow T_S,$$

and $\mathscr{L}_S$ fits into the commutative diagram

$$\begin{array}{ccc} \mathfrak{b}_S & \xrightarrow{\mathscr{L}_S} & T_S \\ {\scriptstyle \mathscr{L}_X|_S}\downarrow & \swarrow{\scriptstyle \iota_*} & \\ T_X|_S. & & \end{array}$$

Over the base point x_0, $\mathscr{L}_S$ can be identified with the surjective linear map

$$\mathfrak{b} \longrightarrow \mathfrak{b}/\mathfrak{h},$$

where $\mathfrak{h}$ is the Lie algebra of H. Since S is homogeneous, $\mathscr{L}_S$ is surjective over every point of S, and $\ker(\mathscr{L}_S)$ is a vector bundle over S.

Definition 8. The *bundle of isotropy Lie algebras* over S is the locally closed subset

$$\Sigma_S := \mathbb{P}\big(\ker(\mathscr{L}_S)\big) \subseteq X \times \mathbb{P}(\mathfrak{b}).$$

Note that Σ_S is a smooth and irreducible closed subset of $S \times \mathbb{P}(\mathfrak{b})$. We denote the two projections by

$$\begin{array}{ccc} & \Sigma_S & \\ {\scriptstyle \mathrm{pr}_{1,S}} \swarrow & & \searrow {\scriptstyle \mathrm{pr}_{2,S}} \\ S & & \mathbb{P}(\mathfrak{b}). \end{array}$$

If we write $\mathfrak{b}_x$ for the Lie algebra of the isotropy group B_x, then

$$\Sigma_S = \Big\{ (x, \xi) \mid x \in S \text{ and } \xi \in \mathfrak{b}_x \Big\}.$$

The dimension of Σ_S is equal to the dimension of $\mathbb{P}(\mathfrak{b})$, independently of the dimension of S.

3.4. Let D be a simple normal crossing divisor on X. The *logarithmic tangent sheaf* of (X, D) is the subsheaf of the tangent sheaf

$$\mathcal{T}_X(-\log D) \subseteq \mathcal{T}_X$$

consisting of those derivations which preserve the ideal sheaf $\mathcal{O}_X(-D)$. Since D is a divisor with simple normal crossings, the logarithmic tangent sheaf is locally free of rank equal to the dimension of X. General references on logarithmic tangent sheaves are [Del70] and [Sai80].

We write $T_X(-\log D)$ for the logarithmic tangent bundle, the vector bundle corresponding to the logarithmic tangent sheaf. The following equality follows from a construction of the Chern-Schwartz-MacPherson class [Alu06a, Alu06b].

Theorem 9. *We have*

$$c_{SM}(\mathbf{1}_{X \setminus D}) = c\big(T_X(-\log D)\big) \cap \big[X\big] \in A_*(X).$$

For precursors, see [Alu99, GP02] and also Schwartz's construction of the Chern-Schwartz-MacPherson class [BSS09, Sch65a, Sch65b]. Our goal is to show that X has enough logarithmic vector fields to make the right-hand side of Theorem 9 effective when D is B-invariant and X has finitely many B-orbits.

Suppose from now on that D is invariant under the action of B. This implies that the Lie homomorphism of Section 3.2 factors through

$$\mathscr{L} : \mathfrak{b} \longrightarrow \Gamma\big(X, T_X(-\log D)\big).$$

Evaluating the sections, we have the homomorphism between B-linearized vector bundles

$$\mathscr{L}_{X,D} : \mathfrak{b}_X \longrightarrow T_X(-\log D).$$

We denote the induced linear map between the fibers over $x \in X$ by

$$\mathscr{L}_{X,D,x} : \mathfrak{b} \longrightarrow T_{X,x}(-\log D).$$

Definition 10. The *variety of critical points* of (X, D) is the closed subset

$$\mathfrak{X} := \Big\{(x, \xi) \mid \mathscr{L}_{X,D,x}(\xi) = 0\Big\} \subseteq X \times \mathbb{P}(\mathfrak{b}).$$

We denote the two projections by

$$\begin{array}{ccc} & \mathfrak{X} & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ X & & \mathbb{P}(\mathfrak{b}). \end{array}$$

The first projection, $\mathrm{pr}_1 : \mathfrak{X} \longrightarrow X$, may not be a projective bundle, but the restriction $\mathrm{pr}_1^{-1}(S) \longrightarrow S$ is a projective bundle for each B-orbit S in X. These projective bundles have different ranks in general.

Remark 11. When $\mathscr{L}_{X,D}$ is surjective, the pair (X, D) is said to be *log-homogeneous* under the action of B [Bri07]. In this case, $\mathfrak{X}$ is the projectivization of the vector bundle denoted by R_X in [Bri09, Section 2].

For log-homogeneous varieties, the conclusion of Theorem 7 is a standard fact [Ful98, Example 12.1.7]. However, in our main case of interest, (X, D) is rarely log-homogeneous under B. In fact, if (X, D) is log-homogeneous under a *solvable* affine algebraic group B, then X should be a toric variety of a maximal torus $T \subseteq B$ [Bri07, Theorem 3.2.1].

We refer to [BJ08, BK05, Kir06, Kir07] for studies of Chern classes of the logarithmic tangent bundle of log-homogeneous varieties.

3.5. Define $X_0 := X$, $X_1 := D$, and a sequence of closed subsets

$$X_0 \supsetneq X_1 \supsetneq X_2 \supsetneq X_3 \supsetneq \cdots \quad \text{where} \quad X_{i+1} := \mathrm{Sing}(X_i) \quad \text{for} \quad i \geq 1.$$

We introduce two decompositions of X into smooth locally closed subsets, the orbit decomposition $\mathcal{S}_{\mathrm{orb}}$ and the singular decomposition $\mathcal{S}_{\mathrm{sing}}$:

$$\begin{aligned} \mathcal{S}_{\mathrm{orb}} &:= \big\{S \mid S \text{ is a } B\text{-orbit in } X\big\}, \\ \mathcal{S}_{\mathrm{sing}} &:= \big\{S \mid S \text{ is a connected component of some } X_i \setminus X_{i+1}\big\}. \end{aligned}$$

Since B is connected and D is invariant under the action of B, the orbit decomposition refines the singular decomposition. We write the variety of critical points as a disjoint union by taking inverse images over the B-orbits in X:

$$\mathfrak{X} = \coprod_{S \in \mathcal{S}_{\mathrm{orb}}} \mathfrak{X}_S \quad \text{where} \quad \mathfrak{X}_S := \mathrm{pr}_1^{-1}(S).$$

As in Section 3.3, we denote the bundle of isotropy Lie algebras over S by Σ_S.

Lemma 12. *$\mathfrak{X}_S$ is a closed subset of Σ_S for each B-orbit S in X.*

Proof. Let S' be the unique element of $\mathcal{S}_{\text{sing}}$ containing S. Any section of $T_X(-\log D)$ preserves the ideal sheaf of S' and defines a derivation of $\mathcal{O}_{S'}$. Denote the corresponding vector bundle homomorphism over S' by

$$\varphi : T_X(-\log D)|_{S'} \longrightarrow T_{S'}.$$

Note that the restriction of φ to S fits into the commutative diagram

$$\begin{array}{ccc} & & 0 \\ & & \big\downarrow \\ \mathfrak{b}_S & \xrightarrow{\ \mathscr{L}_S\ } & T_S \\ {\scriptstyle \mathscr{L}_{X,D}|_S}\big\downarrow & & \big\downarrow{\scriptstyle \iota_*} \\ T_X(-\log D)|_S & \xrightarrow[\ \varphi|_S\ ]{} & T_{S'}|_S. \end{array}$$

Here $\mathscr{L}_S$ is the vector bundle homomorphism of Section 3.3, $\mathscr{L}_{X,D}|_S$ is the restriction to S of the vector bundle homomorphism of Section 3.4, and ι_* is the differential of the inclusion $\iota : S \to S'$. Since ι_* is injective, $\mathscr{L}_{X,D,x}(\xi) = 0$ implies $\mathscr{L}_{S,x}(\xi) = 0$ for any $x \in S$ and $\xi \in \mathfrak{b}$. □

3.6.

Proof of Theorem 7. Choose a B-finite log-resolution $\pi : X \longrightarrow Y$ and define $X^\circ := \pi^{-1}(Y^\circ)$. By the functoriality of the Chern-Schwartz-MacPherson class, we have

$$\pi_* c_{SM}(X^\circ) = c_{SM}(Y^\circ) \in A_*(Y).$$

Since any effective cycle pushes forward to an effective cycle, it is enough to prove that $c_{SM}(X^\circ)$ is represented by an effective cycle on X.

Let D be the boundary divisor $X \setminus X^\circ$, and let k be a nonnegative integer less than $\dim X$. Our aim is to show that the k-th Chern class

$$c_{SM}(X^\circ)_k = c_{\dim X - k}\big(T_X(-\log D)\big) \cap [X] \in A_k(X)$$

is represented by an effective k-cycle.

We recall from Section 3.4 the variety of critical points $\mathfrak{X}$ and the two projections

$$\begin{array}{ccccc} & & \mathfrak{X} & & \\ & {\scriptstyle \mathrm{pr}_1}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_2} & \\ X & & & & \mathbb{P}(\mathfrak{b}). \end{array}$$

By Lemma 12, we have

$$\mathfrak{X} = \coprod_{S \in \mathcal{S}_{\text{orb}}} \mathfrak{X}_S \subseteq \coprod_{S \in \mathcal{S}_{\text{orb}}} \Sigma_S.$$

Note that each Σ_S is irreducible of dimension equal to that of $\mathbb{P}(\mathfrak{b})$. Since X has finitely many B-orbits, this shows that each irreducible component of $\mathfrak{X}$ has dimension at most $\dim \mathbb{P}(\mathfrak{b})$.

Let Λ be a $(k+1)$-dimensional subspace of $\mathfrak{b}$. If Λ is spanned by $\xi_0, \dots, \xi_k$, then the $(\dim X - k)$-th Chern class of $T_X(-\log D)$ is represented by a cycle supported on the locus

$$\mathfrak{D}_k(\Lambda) := \Big\{ x \in X \mid \mathscr{L}(\xi_0), \dots, \mathscr{L}(\xi_k) \text{ are linearly dependent at } x \Big\},$$

where $\mathscr{L} : \mathfrak{b} \longrightarrow \Gamma\big(X, T_X(-\log D)\big)$ is the Lie homomorphism. See [Ful98, Chapter 14]. As a scheme, $\mathfrak{D}_k(\Lambda)$ is defined by $(k+1)$-minors of the matrices for the vector bundle homomorphism

$$\Lambda_X \longrightarrow T_X(-\log D)$$

obtained by restricting $\mathscr{L}_{X,D}$. Set-theoretically,

$$\mathfrak{D}_k(\Lambda) = \mathrm{pr}_1\Big(\mathrm{pr}_2^{-1}\big(\mathbb{P}(\Lambda)\big)\Big).$$

We recall the following facts on degeneracy loci from [Ful98, Theorem 14.4]:

(1) Each irreducible component of $\mathfrak{D}_k(\Lambda)$ has dimension at least k.
(2) If all the irreducible components of $\mathfrak{D}_k(\Lambda)$ have dimension k, then the Chern class

$$c_{\dim X - k}\big(T_X(-\log D)\big) \cap [X] \in A_k(X)$$

is represented by a positive cycle supported on $\mathfrak{D}_k(\Lambda)$.

Therefore it is enough to show that all the irreducible components of $\mathfrak{D}_k(\Lambda)$ have dimension at most k for a suitable choice of Λ.

In fact, all the irreducible components of $\mathrm{pr}_2^{-1}\big(\mathbb{P}(\Lambda)\big)$ have dimension at most k for a sufficiently general choice of Λ. This is a general fact on maps of the form

$$\mathfrak{X} \longrightarrow \mathbb{P}^n,$$

where all the irreducible components of $\mathfrak{X}$ have dimension $\leq n$. One may argue by induction on n, where in the induction step one chooses a hyperplane of $\mathbb{P}^n$ which does not contain the image of any irreducible component of $\mathfrak{X}$. □

Since each irreducible component of the degeneracy locus $\mathfrak{D}_k(\Lambda)$ has dimension at least k, the above argument shows that each component of $\mathfrak{D}_k(\Lambda)$ has dimension exactly k for a sufficiently general Λ. Each of these components of $\mathfrak{D}_k(\Lambda)$ is projected from an irreducible component of $\mathfrak{X}$ of maximum possible dimension, and this component of $\mathfrak{X}$ is the closure of $\mathfrak{X}_S$ for some

B-orbit S such that $\mathfrak{X}_S = \Sigma_S$ and $\dim S \geq k$. For later use, we record here this refined conclusion of our analysis on the diagram

$$\begin{array}{ccc} & \mathfrak{X} & \\ {}^{\mathrm{pr}_1}\swarrow & & \searrow^{\mathrm{pr}_2} \\ X & & \mathbb{P}(\mathfrak{b}). \end{array}$$

Corollary 13. *For a $(k+1)$-dimensional subspace $\Lambda \subseteq \mathfrak{b}$, let $\mathfrak{D}_k(\Lambda)$ be the degeneracy locus*

$$\mathfrak{D}_k(\Lambda) = \mathrm{pr}_1\Big(\mathrm{pr}_2^{-1}\big(\mathbb{P}(\Lambda)\big)\Big).$$

Then the following hold for a sufficiently general subspace $\Lambda \subseteq \mathfrak{b}$:

(1) Each irreducible component of $\mathfrak{D}_k(\Lambda)$ has the expected dimension k.
(2) Each irreducible component of $\mathfrak{D}_k(\Lambda)$ is the closure of a subvariety of a B-orbit S such that $\mathfrak{X}_S = \Sigma_S$ and $\dim S \geq k$.
(3) The k-th Chern class of X° can be written as a nonnegative linear combination

$$c_{SM}(X^\circ)_k = \sum_i m_i[\mathcal{Z}_i] \in A_k(X),$$

where the $\mathcal{Z}_i$ are the irreducible components of $\mathfrak{D}_k(\Lambda)$.

We express (2) *by saying that the irreducible component of $\mathfrak{D}_k(\Lambda)$ is* generically supported *on S.*

Applying Corollary 13 to the B-finite resolution $\pi : X \longrightarrow Y$, we see that the k-th Chern class of Y° can be written as a nonnegative linear combination

$$c_{SM}(Y^\circ)_k = \sum_i n_i[Z_i] \in A_k(Y),$$

where the Z_i are the k-dimensional irreducible components of $\pi\big(\mathfrak{D}_k(\Lambda)\big)$.

Note that there is at least one B-orbit S with $\mathfrak{X}_S = \Sigma_S$ and $\dim S \geq k$, the open dense orbit $S = X^\circ$. Any irreducible component of $\mathfrak{D}_k(\Lambda)$ generically supported on X° will be called *standard*. All the other irreducible components are *exceptional*.

4. Irreducibility

In this section, we specialize to the case when B is a Borel subgroup of a connected reductive group G. We make use of the following consequence of the strengthened assumption:

- The centralizer of a maximal torus in B is the maximal torus.

Since the union of Cartan subgroups of B contains an open dense subset, it follows that

- the set of semisimple elements of B contains an open dense subset of B and
- the set of semisimple elements of $\mathfrak{b}$ contains an open dense subset of $\mathfrak{b}$.

We will use [Bor91] as a general reference. For Cartan subgroups and Cartan subalgebras, see [TY05, Chapter 29].

Let P be a parabolic subgroup of G containing B, and let Y be the closure of a B-orbit Y° in G/P.

4.1. An element $\xi \in \mathfrak{b}$ is said to be *regular* if its centralizer is a Cartan subalgebra of $\mathfrak{b}$. The set of regular elements is open and dense in $\mathfrak{b}$.

Definition 14. A *regular log-resolution* of Y is a proper map $\pi : X \longrightarrow Y$ such that

(1) $\pi : X \longrightarrow Y$ is a B-finite log-resolution of Y and
(2) the isotropy Lie algebra $\mathfrak{b}_x$ contains a regular element of $\mathfrak{b}$ for each $x \in X$.

Of course, it is enough to require the second condition for any one point from each B-orbit of X.

The following is the main result of this section. Fix a nonnegative integer $k \leq \dim Y$, and write $c_{SM}(Y^\circ)_k$ for the k-dimensional component of $c_{SM}(Y^\circ)$.

Theorem 15. *Suppose Y has a regular log-resolution. Then there is a nonempty reduced and irreducible k-dimensional subvariety Z of Y such that*

$$c_{SM}(\mathbf{1}_{Y^\circ})_k = [Z] \in A_k(Y).$$

The subvariety Z can be chosen to be the closure in Y of the locus

$$Z^\circ(\Lambda) = \Big\{ y \in Y^\circ \mid \Lambda \cap \mathfrak{b}_y \neq 0 \Big\},$$

where Λ is a sufficiently general $(k+1)$-dimensional subspace of $\mathfrak{b}$.

We will see in Section 5 that the classical Schubert variety $\mathbb{S}(\underline{\alpha})$ has a regular log-resolution. The rest of this section is devoted to the proof of Theorem 15.

4.2. Let S be a homogeneous B-space. Recall from Section 3.3 the bundle of isotropy Lie algebras

$$\Sigma_S = \Big\{ (x, \xi) \mid \xi \in \mathfrak{b}_x \Big\} \subseteq S \times \mathbb{P}(\mathfrak{b}).$$

We choose a base point x_0 and identify S with B/H, where H is the isotropy group B_{x_0} with the Lie algebra $\mathfrak{h}$. The *rank* of an affine algebraic group is the dimension of a maximal torus.

Lemma 16. *If* $\operatorname{rank}(B) = \operatorname{rank}(H)$*, then*

$$\operatorname{pr}_{2,S} : \Sigma_S \longrightarrow \mathbb{P}(\mathfrak{b}), \qquad (x, \xi) \longmapsto \xi$$

is a dominant morphism.

Proof. The set of semisimple elements in $\mathfrak{b}$ contains an open dense subset of $\mathfrak{b}$ in our setting. We find a point in Σ_S which maps to the class of a given nonzero semisimple element ξ in $\mathbb{P}(\mathfrak{b})$.

Since ξ is semisimple, ξ is tangent to a torus [Bor91, Proposition 11.8]. We may assume that this torus T_1 is a maximal torus of B.

Let T_2 be a maximal torus of H. Then T_2 is a maximal torus of B because $\operatorname{rank}(B) = \operatorname{rank}(H)$. Since any two maximal tori of B are conjugate, there is an element $b \in B$ such that $T_1 = bT_2b^{-1}$. We have

$$\xi \in \mathfrak{t}_1 = \operatorname{Ad}(b) \cdot \mathfrak{t}_2 \subseteq \operatorname{Ad}(b) \cdot \mathfrak{h} = \mathfrak{b}_{b \cdot x_0}.$$

Therefore $b \cdot x_0$ gives a point in the fiber of ξ. □

4.3.

Remark 17. The results of this subsection are not needed for the proof of Theorem 15 if Y is the classical Schubert variety $\mathbb{S}(\underline{\alpha})$.

Let Λ be a $(k+1)$-dimensional subspace of $\mathfrak{b}$, and let Λ_r be the set of regular elements of $\mathfrak{b}$ in Λ. Define

$$D_k(\Lambda) := \big\{ x \in S \mid \Lambda \cap \mathfrak{b}_x \neq 0 \big\} \quad \text{and} \quad D_k(\Lambda_r) := \big\{ x \in S \mid \Lambda_r \cap \mathfrak{b}_x \neq 0 \big\}.$$

In terms of the diagram

$$\begin{array}{ccc} & \Sigma_S & \\ {\scriptstyle \operatorname{pr}_{1,S}} \swarrow & & \searrow {\scriptstyle \operatorname{pr}_{2,S}} \\ S & & \mathbb{P}(\mathfrak{b}), \end{array}$$

we have

$$D_k(\Lambda) = \operatorname{pr}_{1,S}\Big(\operatorname{pr}_{2,S}^{-1}\big(\mathbb{P}(\Lambda)\big)\Big) \quad \text{and} \quad D_k(\Lambda_r) = \operatorname{pr}_{1,S}\Big(\operatorname{pr}_{2,S}^{-1}\big(\mathbb{P}(\Lambda_r)\big)\Big).$$

Since $\dim \Sigma_S = \dim \mathbb{P}(\mathfrak{b})$, $D_k(\Lambda)$ is either empty or of pure dimension k for a sufficiently general Λ.

Lemma 18. *Suppose* $\mathfrak{h}$ *contains a regular element of* $\mathfrak{b}$*. Then* $D_k(\Lambda_r)$ *contains an open dense subset of* $D_k(\Lambda)$ *for a sufficiently general* $\Lambda \subseteq \mathfrak{b}$*.*

Proof. Note that

$$\operatorname{pr}_{2,S}(\Sigma_S) = \bigcup_{x \in S} \mathbb{P}(\mathfrak{b}_x).$$

The closure of this set is an irreducible subvariety of $\mathbb{P}(\mathfrak{b})$, say V. Let $U \subseteq V$ be the open subset of (the classes of) regular elements in V. This set U is nonempty by our assumption on $\mathfrak{h}$, and hence U is dense in V.

(1) $\dim V \leq \operatorname{codim}(\Lambda \subseteq \mathfrak{b})$: In this case, for a sufficiently general Λ,

$$V \cap \mathbb{P}(\Lambda) = U \cap \mathbb{P}(\Lambda).$$

Therefore $\mathrm{pr}_{2,S}^{-1}\big(U \cap \mathbb{P}(\Lambda)\big) = \mathrm{pr}_{2,S}^{-1}\big(\mathbb{P}(\Lambda)\big)$.

(2) $\dim V > \operatorname{codim}(\Lambda \subseteq \mathfrak{b})$: In this case, $\mathrm{pr}_{2,S}^{-1}\big(\mathbb{P}(\Lambda)\big)$ is irreducible for a sufficiently general Λ by Bertini's theorem [Laz04, Theorem 3.3.1]. Therefore $\mathrm{pr}_{2,S}^{-1}\big(U \cap \mathbb{P}(\Lambda)\big)$ is open and dense in $\mathrm{pr}_{2,S}^{-1}\big(\mathbb{P}(\Lambda)\big)$.

In either case, we see that $D_k(\Lambda_r)$ contains an open dense subset of $D_k(\Lambda)$. □

Let p be a B-equivariant morphism between homogeneous B-spaces

$$p : S \simeq B/H \longrightarrow B/K, \qquad H \subseteq K \subseteq B.$$

The following lemma can be found in [Kir07, Lemma 3.1].

Lemma 19. *If* $\mathfrak{h}$ *contains a regular element of* $\mathfrak{b}$ *and* $\operatorname{rank}(H) < \operatorname{rank}(K)$, *then*

$$\dim D_k(\Lambda) > \dim p\big(D_k(\Lambda)\big)$$

for a sufficiently general $\Lambda \subseteq \mathfrak{b}$.

Proof. By Lemma 18, $D_k(\Lambda_r)$ contains an open dense subset D° of $D_k(\Lambda)$. It is enough to show that

$$\dim\Big(D_k(\Lambda) \cap p^{-1}\big(p(x)\big)\Big) > 0 \quad \text{for all } x \in D^\circ.$$

Let x be a point in D°. Since regular elements are semisimple in our setting, there is a nonzero semisimple element ξ in $\Lambda \cap \mathfrak{b}_x \subseteq \mathfrak{b}_{p(x)}$. Choose a maximal torus T of $B_{p(x)}$ tangent to ξ [Bor91, Proposition 11.8].

The maximal torus T is contained in the centralizer of ξ because global and infinitesimal centralizers correspond [Bor91, Section 9.1]. Therefore, for any $t \in T$,

$$\xi = \mathrm{Ad}(t) \cdot \xi \in \Lambda \cap \mathfrak{b}_{t \cdot x} \neq 0.$$

This shows that

$$T \cdot x \subseteq D_k(\Lambda).$$

Since T is contained in $B_{p(x)}$, we have

$$T \cdot x \subseteq D_k(\Lambda) \cap p\big(p^{-1}(x)\big).$$

We check that $T \cdot x$ has a positive dimension. If otherwise, $T \cdot x = x$ because $T \cdot x$ is connected. Therefore $T \subseteq B_x$, and this contradicts the assumption that $\operatorname{rank}(H) < \operatorname{rank}(K)$. □

4.4. We begin the proof of Theorem 15. Choose a regular log-resolution $\pi : X \longrightarrow Y$ and set

$$X^\circ := \pi^{-1}(Y^\circ), \qquad D := X \setminus X^\circ.$$

By the functoriality, we have

$$\pi_* c_{SM}(X^\circ) = c_{SM}(Y^\circ) \in A_*(Y).$$

Let $\Lambda \subseteq \mathfrak{b}$ be a $(k+1)$-dimensional subspace, and let $\mathfrak{D}_k(\Lambda)$ be the degeneracy locus constructed in Section 3.6. The main properties of $\mathfrak{D}_k(\Lambda)$ are summarized in Corollary 13.

Recall that an irreducible component of $\mathfrak{D}_k(\Lambda)$ is said to be *standard* if it is generically supported on X°. All the other irreducible components are *exceptional*.

Lemma 20. *For a sufficiently general Λ and a positive k, there is exactly one standard component of $\mathfrak{D}_k(\Lambda)$, and this component is generically reduced.*

Proof. Over the open subset X°, the logarithmic tangent bundle agrees with the usual tangent bundle. Therefore

$$\mathfrak{X}_{X^\circ} = \Sigma_{X^\circ}.$$

First we show that $\mathfrak{D}_k(\Lambda) \cap X^\circ$ is irreducible. Since X° has a point fixed by a maximal torus of B, Lemma 16 says that

$$\mathrm{pr}_{2,X^\circ} : \Sigma_{X^\circ} \longrightarrow \mathbb{P}(\mathfrak{b})$$

is a dominant morphism. Therefore Bertini's theorem applies to pr_{2,X° and positive-dimensional linear subspaces of $\mathbb{P}(\mathfrak{b})$ [Laz04, Theorem 3.3.1]. It follows that

$$\mathfrak{D}_k(\Lambda) \cap X^\circ = \mathrm{pr}_{1,X^\circ}\Big(\mathrm{pr}_{2,X^\circ}^{-1}\big(\mathbb{P}(\Lambda)\big)\Big)$$

is irreducible for a sufficiently general Λ.

Next we show that $\mathfrak{D}_k(\Lambda) \cap X^\circ$ is reduced. The tangent bundle of X° is generated by global sections from $\mathfrak{b}$, and hence there is a morphism to the Grassmannian

$$\Psi : X^\circ \longrightarrow \mathrm{Gr}_d(\mathfrak{b}), \qquad x \longmapsto \mathfrak{b}_x \quad \text{where} \quad d = \dim B - \dim X.$$

As a scheme, $\mathfrak{D}_k(\Lambda) \cap X^\circ$ is the pull-back of the Schubert variety

$$\big\{\mathfrak{a} \mid \mathfrak{a} \text{ is a } d\text{-dimensional subspace of } \mathfrak{b} \text{ such that } \mathfrak{a} \cap \Lambda \neq 0\big\} \subseteq \mathrm{Gr}_d(\mathfrak{b}).$$

Therefore $\mathfrak{D}_k(\Lambda) \cap X^\circ$ is reduced for a sufficiently general Λ by Kleiman's transversality theorem [Kle74, Remark 7]. □

In fact, $\mathfrak{D}_k(\Lambda)$ has no embedded components for a sufficiently general Λ (being a degeneracy locus of the expected dimension k), but we will not need this. When Y is the Schubert variety $\mathbb{S}(\underline{\alpha})$, the reduced image in $\mathbb{S}(\underline{\alpha})$ of the unique standard component of $\mathfrak{D}_k(\Lambda)$ will be the subvariety $Z(\underline{\alpha})$ of Theorem 2.

Proof of Theorem 15. When k is positive, there is exactly one standard component by Lemma 20. Write π_* for the push-forward

$$\pi_* : A_*(X) \longrightarrow A_*(Y).$$

Our goal is to show that $\pi_*[\mathfrak{E}] = 0$ for all exceptional components $\mathfrak{E}$ of $\mathfrak{D}_k(\Lambda)$, for a sufficiently general Λ.

For this we consider the case when $k = 0$. Recall from Corollary 13 that $\mathfrak{D}_0(\Lambda)$ consists of a finite set of points, each contained in a B-orbit S such that $\mathfrak{X}_S = \Sigma_S$, for a sufficiently general Λ. By the last assertion of Corollary 13, the number of points in $\mathfrak{D}_0(\Lambda)$ should be equal to

$$\chi(X^\circ) = \int_X c_{SM}(X^\circ) = \sum_S \deg\Big(\mathrm{pr}_{2,S} : \Sigma_S \longrightarrow \mathbb{P}(\mathfrak{b})\Big) = 1,$$

where the sum is over all orbits such that $\mathfrak{X}_S = \Sigma_S$. Together with Lemma 16, the formula shows that every such orbit, except one, is of the form

$$S \simeq B/H, \qquad \mathrm{rank}(B) > \mathrm{rank}(H).$$

This one exception should be X° because X° contains a point fixed by a maximal torus of B.

Return to the case when k is positive. Let S be an orbit with $\mathfrak{X}_S = \Sigma_S$, and suppose that S is different from X°. Consider the B-equivariant map

$$\pi|_S : S \simeq B/H \longrightarrow \pi(S), \qquad \mathrm{rank}(B) > \mathrm{rank}(H).$$

The image of S contains a point fixed by a maximal torus of B because it is a B-orbit in G/P. Therefore $\pi(S)$ is of the form

$$\pi(S) \simeq B/K, \qquad \mathrm{rank}(B) = \mathrm{rank}(K).$$

Since π is a regular log-resolution, this shows that Lemma 19 applies to $\pi|_S$. The degeneracy locus $D_k(\Lambda)$ of Lemma 19 is precisely the intersection $S \cap \mathfrak{D}_k(\Lambda)$ in our case because $\mathfrak{X}_S = \Sigma_S$. The conclusion is that

$$\dim \mathfrak{E} > \dim \pi(\mathfrak{E})$$

for any irreducible component $\mathfrak{E}$ of $\mathfrak{D}_k(\Lambda)$ generically supported on S.

Therefore $\pi_*[\mathfrak{E}] = 0$ for all exceptional components $\mathfrak{E}$, for a sufficiently general Λ. □

5. A regular resolution of a classical Schubert variety

In this section, E is a vector space with an ordered basis $e_1, \ldots, e_{n+d}$, G is the general linear group of E, and B is the subgroup of G which consists of all invertible upper triangular matrices with respect to the ordered basis of E.

5.1. We recall the known resolution of singularities of the classical Schubert variety $\mathbb{S}(\underline{\alpha})$ which is regular in the sense of Definition 14. Theorem 2 therefore can be deduced from Theorem 15.

Let $\underline{\alpha} = (\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_d \geq 0)$, and let $\mathbb{S}(\underline{\alpha}) \subseteq \mathrm{Gr}_d(E)$ be the Schubert variety defined with respect to the complete flag

$$F_\bullet = \Big(F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n+d}\Big) \quad \text{where} \quad F_k := \mathrm{span}(e_1, \ldots, e_k).$$

Definition 21. $\mathbb{V}(\underline{\alpha})$ is the subvariety

$$\begin{aligned}\mathbb{V}(\underline{\alpha}) := &\Big\{V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_d \mid V_i \subseteq F_{\alpha_{d+1-i}+i}\Big\} \\ &\subseteq \mathrm{Gr}_1(E) \times \mathrm{Gr}_2(E) \times \cdots \times \mathrm{Gr}_d(E).\end{aligned}$$

The restriction to $\mathbb{V}(\underline{\alpha})$ of the projection to $\mathrm{Gr}_d(E)$ will be written

$$\pi_{\underline{\alpha}} : \mathbb{V}(\underline{\alpha}) \longrightarrow \mathbb{S}(\underline{\alpha}).$$

The projection $\pi_{\underline{\alpha}}$ maps $\mathbb{V}(\underline{\alpha})$ into $\mathbb{S}(\underline{\alpha})$ because $V_i \subseteq V_d \cap F_{\alpha_{d+1-i}+i}$ for all i.

We note that $\pi_{\underline{\alpha}}$ is the resolution used in [KL74] to obtain the determinantal formula for the classes of Schubert schemes. This resolution was also used in [AM09] to compute the Chern-Schwartz-MacPherson class of $\mathbb{S}(\underline{\alpha})^\circ$. All the properties of $\pi_{\underline{\alpha}}$ we need can be found in [AM09, Section 2]. However, one simple but important point for us was not emphasized in the nonembedded description of $\mathbb{V}(\underline{\alpha})$ in [AM09] as a tower of projective bundles: $\mathbb{V}(\underline{\alpha})$ is a subvariety of the partial flag variety

$$\mathrm{Fl}_{1,\ldots,d}(E) \subseteq \mathrm{Gr}_1(E) \times \mathrm{Gr}_2(E) \times \cdots \times \mathrm{Gr}_d(E),$$

and $\mathbb{V}(\underline{\alpha})$ is invariant under the diagonal action of B. It follows that

(1) $\mathbb{V}(\underline{\alpha})$ has finitely many B-orbits and
(2) every B-orbit of $\mathbb{V}(\underline{\alpha})$ contains a point fixed by a maximal torus of B.

The above properties imply that $\pi_{\underline{\alpha}}$ is a regular log-resolution of $\mathbb{S}(\underline{\alpha})$ in the sense of Definition 14.

Remark 22. We note that the Bott-Samelson variety of [Dem74, Han73] will not have finitely many B-orbits in general. It would be interesting to know which Schubert varieties in flag varieties (do not) admit a regular or B-finite log-resolution.

5.2. For the sake of completeness, we give an argument here that $\pi_{\underline{\alpha}}$ is a regular log-resolution of singularities of $\mathbb{S}(\underline{\alpha})$.

Proposition 23. *$\pi_{\underline{\alpha}}$ is a regular log-resolution of $\mathbb{S}(\underline{\alpha})$. That is,*

(1) $\pi_{\underline{\alpha}}$ is proper and B-equivariant,
(2) $\pi_{\underline{\alpha}}^{-1}\big(\mathbb{S}(\underline{\alpha})^\circ\big) \longrightarrow \mathbb{S}(\underline{\alpha})^\circ$ is an isomorphism,
(3) $\mathbb{V}(\underline{\alpha})$ is smooth and has finitely many B-orbits,
(4) the complement of $\pi_{\underline{\alpha}}^{-1}\big(\mathbb{S}(\underline{\alpha})^\circ\big)$ in $\mathbb{V}(\underline{\alpha})$ is a divisor with normal crossings, and
(5) the isotropy Lie algebra $\mathfrak{b}_x$ contains a regular element of $\mathfrak{b}$ for each $x \in \mathbb{V}(\underline{\alpha})$.

Proof. We start by justifying (2). Note that $\pi_{\underline{\alpha}}$ has a section over the Schubert cell

$$s_{\underline{\alpha}} : \mathbb{S}(\underline{\alpha})^\circ \longrightarrow \pi_{\underline{\alpha}}^{-1}\big(\mathbb{S}(\underline{\alpha})^\circ\big), \qquad V \longmapsto V \cap \Big(F_{\alpha_d+1} \subsetneq F_{\alpha_{d-1}+2} \subsetneq \cdots \subsetneq F_{\alpha_1+d}\Big).$$

The statement

$$s_{\underline{\alpha}} \circ \pi_{\underline{\alpha}}\big|_{\pi_{\underline{\alpha}}^{-1}\left(\mathbb{S}(\underline{\alpha})^\circ\right)} = \mathrm{id}_{\pi_{\underline{\alpha}}^{-1}\left(\mathbb{S}(\underline{\alpha})^\circ\right)}$$

is equivalent to the assertion that

$$V_i = V_d \cap F_{\alpha_{d+1-i}+i}$$

for all i and for all $V_\bullet \in \mathbb{V}(\underline{\alpha})$ with $V_d \in \mathbb{S}(\underline{\alpha})$. This is clear because V_i is contained in the right-hand side and the dimensions of both sides are the same. Therefore

$$\pi_{\underline{\alpha}}^{-1}\big(\mathbb{S}(\underline{\alpha})^\circ\big) \longrightarrow \mathbb{S}(\underline{\alpha})^\circ$$

is an isomorphism, proving (2).

We prove (3) by induction on the number of entries of $\underline{\alpha}$. Define

$$\underline{\widetilde{\alpha}} := (\alpha_2 \geq \alpha_3 \geq \cdots \geq \alpha_d \geq 0)$$

and consider the corresponding subvariety

$$\mathbb{V}(\underline{\widetilde{\alpha}}) \subseteq \mathrm{Gr}_1(E) \times \mathrm{Gr}_2(E) \times \cdots \times \mathrm{Gr}_{d-1}(E).$$

Restricting the projection map which forgets the last coordinate, we have

$$\mathrm{pr}_{\hat{d}} : \mathbb{V}(\underline{\alpha}) \longrightarrow \mathbb{V}(\underline{\widetilde{\alpha}}).$$

Let $\mathscr{F}_\bullet$ be the flag of trivial vector bundles over $\mathbb{V}(\underline{\widetilde{\alpha}})$ modeled on the flag of subspaces $F_\bullet$. Then we may identify $\mathrm{pr}_{\hat{d}}$ with the projective bundle

$$\mathbb{P}(\mathscr{F}_{\alpha_1+d}/\mathscr{V}_{d-1}) \longrightarrow \mathbb{V}(\underline{\widetilde{\alpha}}),$$

where $\mathscr{V}_{d-1}$ is the pull-back of the tautological bundle from the projection $\mathbb{V}(\underline{\widetilde{\alpha}}) \longrightarrow \mathrm{Gr}_{d-1}(E)$. This shows by induction that $\mathbb{V}(\underline{\alpha})$ is smooth. The fact that $\mathbb{V}(\underline{\alpha})$ has finitely many B-orbits is implied by the Bruhat decomposition of G.

Item (4) can also be proved by the same induction. Let $\underline{\widetilde{\alpha}}$ be as above, and set

$$D_{\text{old}} := \mathbb{V}(\underline{\widetilde{\alpha}}) \setminus \pi_{\underline{\widetilde{\alpha}}}^{-1}\big(\mathbb{S}(\underline{\widetilde{\alpha}})^\circ\big).$$

We may suppose that D_{old} is a divisor in $\mathbb{V}(\underline{\widetilde{\alpha}})$ with normal crossings. The key observation is that

$$\mathbb{V}(\underline{\alpha}) \setminus \pi_{\underline{\alpha}}^{-1}\big(\mathbb{S}(\underline{\alpha})^\circ\big) = \mathrm{pr}_{\hat{d}}^{-1}(D_{\text{old}}) \cup D_{\text{new}},$$

where D_{new} is the smooth and irreducible divisor

$$D_{\text{new}} := \mathbb{P}(\mathscr{F}_{\alpha_1+d-1}/\mathscr{V}_{d-1}) \subseteq \mathbb{P}(\mathscr{F}_{\alpha_1+d}/\mathscr{V}_{d-1}) = \mathbb{V}(\underline{\alpha}).$$

The assertion that $\mathrm{pr}_{\hat{d}}^{-1}(D_{\text{old}}) \cup D_{\text{new}}$ has normal crossings can be checked locally. Covering $\mathbb{V}(\underline{\alpha})$ with open subsets of the form $\mathrm{pr}_{\hat{d}}^{-1}(U)$, where U is an open subset of $\mathbb{V}(\underline{\widetilde{\alpha}})$ over which the vector bundle $\mathscr{V}_{d-1}$ is trivial, the assertion becomes clear.

Item (5) is a consequence of the fact that each B-orbit of $\mathbb{V}(\underline{\alpha})$ contains a point fixed by a maximal torus of B. It follows that every point of $\mathbb{V}(\underline{\alpha})$ is fixed by a maximal torus of B. Therefore all the isotropy Lie algebras contain a Cartan subalgebra of $\mathfrak{b}$, whose general member is a regular element of $\mathfrak{b}$. □

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Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

Current address: Institute for Advanced Study and Princeton University, Princeton, New Jersey 08540

E-mail address: huh@princeton.edu