Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

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Questions (40 pts)

1. Air flows isentropically through a constant-area duct at $Ma = 0.4$. The stagnation pressure at point 1 is $P_0 = 1.3$ bar. What is the stagnation pressure at a downstream point 2?

   \[
   \text{Since the flow is isentropic, the stagnation pressure remains constant, } P_{01} = P_{02} = 1.3 \text{ bar.}
   \]

2. In addition to the conditions described in the previous question, it is also known that the stagnation density at point 1 is $\rho_0 = 1.0 \text{ kg/m}^3$. What is the static enthalpy at the downstream point 2?

   \[
   \frac{P_2}{P_0} = \left(1 + \frac{\gamma - 1}{2} \frac{Ma^2}{Ma^2}ight)^\frac{\gamma - 1}{\gamma} = 0.89 = \left(\frac{q_2}{q_0}\right)^\gamma = \frac{P_2}{P_0} = \left(\frac{\rho_2}{\rho_0}\right)^\gamma \Rightarrow h_2 = \frac{h_2}{h_0} + \frac{T_2}{T_0} = \left(\frac{P_2}{P_0}\right)^\gamma \Rightarrow \frac{h_2}{h_0} = \frac{3.3812}{5} \Rightarrow \frac{T_2}{T_0} = \frac{440}{500}.
   \]

3. Sketch the streamwise variations of pressure, speed of sound and flow velocity for a) a subsonic flow through a divergent nozzle, b) a supersonic flow through a divergent nozzle, and c) a subsonic flow through a convergent nozzle.

4. A gas with \(\gamma = 1.4\) flows supersonically through a diverging duct. At section 1, the cross-sectional area is $A_1$ and the Mach number is $Ma_1 = 1.6$. At section 2, the cross-sectional area is $A_2 = 5A_1$. Calculate the Mach number $Ma_2$ at section 2.

   \[
   \text{Continuity: } \frac{\dot{m}}{\dot{m}_0} = \text{U/A = Const.} \Rightarrow \frac{\dot{m}}{\dot{m}_0} = \frac{\rho_0 a_0 U}{\rho_0 a_0} \Rightarrow \frac{A_2}{A_1} = \frac{Ma_2 \left[1 + \frac{(\gamma - 1)Ma_2^2}{2} \right]^{\gamma + 1}}{2(\gamma - 1)} = \frac{5}{3} \Rightarrow \text{obtain } \frac{Ma_2}{Ma_1} = \frac{4}{3} \Rightarrow \text{from Table I} \quad \frac{A_2}{A_1} = \frac{5}{3}.
   \]

   \[
   \text{This eq. can be solved by doing } \frac{A_2}{A_1} = \frac{A_2}{A_1} \cdot \frac{A_3}{A_2} = 5.4 \Rightarrow \text{from Table I } \quad \frac{A_2}{A_1} = 6.28 \Rightarrow \frac{Ma_2}{Ma_1} = 3.4.
   \]
**Problem 1 (30 pts)**

Consider the subsonic diffuser shown in the figure where the inlet Mach number $Ma_1 < 1$ and the area ratio $A_2/A_1 > 1$ are known parameters.

![Diagram of subsonic diffuser](image)

a) Derive an expression to compute the static-pressure ratio $P_2/P_1$ as a function of $Ma_1$, $A_2/A_1$ and $\gamma$.

b) Calculate the static-pressure ratio $P_2/P_1$ for $Ma_1 = 0.4$ and $A_2/A_1 = 4$.

c) Prove that the expression

$$\frac{P_0}{P_1} = \left[1 + \frac{(\gamma - 1)}{2} Ma_1^2\right]^{\gamma/(\gamma - 1)}$$

leads to the incompressible Bernoulli equation

$$P_1 + \rho U_1^2/2 = P_0$$

in the incompressible limit $Ma^2 \ll 1$, where $P_0$ is the stagnation pressure, $U_1$ is the flow velocity, and $\rho$ is the density.

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a) From mass conservation:

$$\dot{m} = \frac{\dot{Q} A}{\dot{D} A} = \frac{\dot{Q} \frac{U T}{g_0} A_0}{\dot{D} A_0} A = \text{const} \frac{\dot{Q} \frac{U T}{g_0} Ma_1 \left(\frac{T_1}{T_0}\right)^{\gamma/2} A_1}{\frac{\dot{Q} \frac{U T}{g_0} A_0}{\dot{D} A_0}}$$

And since $s_1 = \frac{p_1}{\rho_0 T_1}$ and $s_2 = \frac{p_2}{\rho_0 T_2}$, then

$$\frac{\dot{Q} \frac{U T}{g_0} \frac{p_2}{\rho_0 T_2}}{\dot{Q} \frac{U T}{g_0} \frac{p_1}{\rho_0 T_1}} \frac{Ma_2}{Ma_1} \left(\frac{T_1}{T_0}\right)^{\gamma/2} \frac{A_2}{A_1} = \Delta.$$

Substituting the relation $\frac{T_2}{T_1} = \frac{T_2}{T_0} \frac{T_0}{T_1} = \left(1 + \frac{(\gamma - 1) Ma_1^2}{2}\right)/\left(1 + \frac{(\gamma - 1) Ma_2^2}{2}\right)$ into Eq (a),

Yields:

$$\frac{p_2}{p_1} \frac{Ma_2}{Ma_1} \left(\frac{1 + \frac{(\gamma - 1) Ma_2^2}{2}}{1 + \frac{(\gamma - 1) Ma_1^2}{2}}\right) \frac{A_2}{A_1} = \Delta,$$

which gives the quadratic equation

$$Ma_2^2 \left(1 + \frac{(\gamma - 1) Ma_2^2}{2}\right) = Ma_1^2 \left(1 + \frac{(\gamma - 1) Ma_1^2}{2}\right) \left(\frac{A_2}{A_1}\right)^2 \left(\frac{p_1}{p_2}\right)^2$$

For $Ma_2$. 

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THE SOLUTION IS

\[(\gamma-1)Ma_2^2 = -1 + \left[1 + \frac{2(\gamma-1)Ma_1^2}{\left(\frac{P_2}{P_1}\right)^2 \left(\frac{A_2}{A_1}\right)^2} \right]^{1/2}\]

SUBSTITUTING THIS EXPRESSION INTO THE ISENTROPIC RELATIONS

\[
\frac{P_2}{P_1} = \frac{P_3}{P_0} \frac{P_0}{P_1} = \left(\frac{1 + \frac{(\gamma-1)Ma_2^2}{2}}{1 + \frac{(\gamma-1)Ma_1^2}{2}}\right)^{\frac{\gamma}{\gamma-1}}
\]

GIVES

\[
\frac{P_2}{P_1} = \left(\frac{1}{2} + \frac{1}{2} \left[1 + 2(\gamma-1)Ma_1^2 \left(1 + \frac{(\gamma-1)Ma_1^2}{2}\right)\right]^{1/2} \cdot \frac{\gamma}{\gamma-1}\right)
\]

\[
1 + \frac{(\gamma-1)Ma_1^2}{2}
\]

WHICH REPRESENTS THE IMPLICIT RELATION BETWEEN \(\frac{P_2}{P_1}\), \(\frac{A_2}{A_1}\) AND \(Ma_1\).

b) FOR \(Ma_1 = 0.4\) AND \(\frac{A_2}{A_1} = 4\), EQUATION (2) CAN BE SOLVED ITERATIVELY TO

GIVE \[\frac{P_2}{P_1} \approx 1.14\]

\[
\frac{P_0}{P_1} = \left(1 + \frac{(\gamma-1)Ma_1^2}{2}\right)^{\frac{\gamma}{\gamma-1}} \quad \text{IS OF THE FORM} \quad (1 + \varepsilon)^b \quad \text{WHERE \(\varepsilon\) CAN BE}
\]

EXPANDED IN TAYLOR SERIES AS \((1 + \varepsilon)^b \sim 1 + b\varepsilon\) FOR \(\varepsilon \ll 1\). IN THIS WAY,

\[
\frac{P_0}{P_1} = \left(1 + \frac{(\gamma-1)Ma_1^2}{2}\right)^{\frac{\gamma}{\gamma-1}} \approx 1 + \frac{\gamma}{2}Ma_1^2 = 1 + \frac{\sqrt{U_1^2}}{2P_1} = 1 + \frac{8U_1^2}{2P_1}
\]

\[\Rightarrow \quad \frac{P_0}{P_1} = \frac{P_0 + \frac{8U_1^2}{2}}{2}
\]

\[\checkmark \quad 10/10\]
Problem 2 (30 pts)

A small hole is perforated on the wall of a chamber of volume $V$ that is initially void of gas. The hole resembles a convergent nozzle of minimum cross-sectional area $A_2 \ll V^{2/3}$. Air from the atmosphere flows through the hole filling up the chamber, as shown in the figure below. The atmosphere is at pressure $P_a$ and temperature $T_a$.

![Diagram of a chamber with a convergent nozzle](image)

a) Show that mass and total internal-energy conservation constraints require that the time evolution of the density $\rho_d$ and pressure $P_d$ of the gas in the chamber are given by

$$\frac{d\rho_d}{dt} = \frac{\dot{m}}{V} \quad (1)$$

and

$$\frac{dP_d}{dt} = \frac{\dot{m} a_a^2}{V} \quad (2)$$

where $\dot{m}$ is the mass flow rate and $a_a$ is the speed of sound in the atmospheric air.

b) Obtain an expression for the time $t^*$ during which the hole remains choked as a function of $\gamma$, $V$, $A_2$, $T_a$ and the gas constant $R_g$.

c) Plot the time evolution of the density $\rho_d/\rho_a$ and pressure $P_d/P_a$ of the gas in the chamber during the interval $0 < t/t^* < 5$ for $\gamma = 1.4$. 

![Image of a control volume with continuity and internal energy equations](image)
b) Initially there is no gas in the chamber ⇒ \( P_d = \rho_d = 0 \) at \( t = 0 \). As gas flows through the hole, the pressure \( P_d \) and density \( \rho_d \) increase according to Eqs (3) and (4). It is however necessary to obtain an expression for the mass flow rate \( \dot{m} \).

At the beginning, the pressure in the chamber is small and the nozzle is choked, with \( \dot{m}_2 = 1 \) and

\[
\dot{m} = \dot{m}_c = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{1}{2(\gamma-1)}
\]

Using (3) in (2) leads to

\[
\frac{dP_d}{dt} = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{1}{2(\gamma-1)}
\]

\( P_d(t) = 0 \)

so that

\[
\frac{P_a}{P_d} \geq \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{V}{\sqrt{\gamma^2 R_0 T_a A_2}}
\]

\( t^* = \sqrt{\frac{2}{\gamma+1}} \sqrt{\frac{V}{\gamma^2 R_0 T_a A_2}} \)

so that \( \frac{P_a}{P_d} \geq \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{V}{\sqrt{\gamma^2 R_0 T_a A_2}} \)

\( t^* \)

(c) For \( 0 < t < t^* \) the nozzle is choked \( (m = m^*) \) and

\[
\frac{dP_d}{dt} = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{1}{2(\gamma-1)}
\]

\( P_d(0) = 0 \)

\( \dot{m}_d \)

For \( t > t^* \) the nozzle ceases to be choked and \( \dot{m} = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{V}{\sqrt{\gamma^2 R_0 T_a A_2}} \)

so that

\[
\frac{dP_d}{dt} = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{1}{2(\gamma-1)} \left( \frac{P_a}{P_d} \right)^{\frac{\gamma+1}{\gamma-1}} \left( \frac{P_a}{P_d} \right)^{\frac{\gamma+1}{\gamma-1}}
\]

\( P_d(t) = \dot{m}_d \)

\( \dot{m}_d \)

Where

\[
\dot{m}_d = \frac{P_a}{\sqrt{\frac{\gamma}{R_0 T_a}}} A_2 \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \frac{1}{2(\gamma-1)} \left( \frac{P_a}{P_d} \right)^{\frac{\gamma+1}{\gamma-1}} \left( \frac{P_a}{P_d} \right)^{\frac{\gamma+1}{\gamma-1}}
\]

The critical values of density and temperature. Normalizing \( \tilde{\rho} = \frac{\dot{m} \rho}{\dot{m}_d} \) and \( \tilde{T} = \frac{T}{T_0} \), Equations (6) and (7) become

\[
\frac{d\tilde{T}}{dt} = \frac{1}{\gamma} \left( \frac{\gamma+1}{2} \right)^{\frac{\gamma+1}{\gamma-1}} \tilde{T} \]

\( \tilde{T}(0) = \tilde{T}(0) = 0 \)

\( \dot{m}_d = \frac{\gamma}{\gamma-1} \frac{\gamma+1}{2} \)

\( \dot{m}_d \)

where \( \dot{m}_d \) is choked flow \( (\dot{m} = \text{const.}) \).
The integration of Eqs. (3) is trivial and yields the linear variations

\[ \tilde{\theta} = \frac{A}{Y} \left( \frac{\gamma + 1}{2} \right)^{-\frac{1}{\gamma - 1}} \tilde{\gamma}, \quad \tilde{P} = \gamma \tilde{\theta} \]

(10).

The integration of Eqs. (9) determines the time evolutions \( \tilde{P}(\tilde{\gamma}) \) and \( \tilde{\theta}(\tilde{\gamma}) \) when the chamber is filled enough such that the flow is not choked anymore. The integration is not straightforward, but a sketch of the solution is given below.