STANFORD 2016

J. URZAY

ME 355: COMPRESSIBLE FLOWS

I. FUNDAMENTAL ASPECTS OF COMPRESSIBLE FLOWS

1. CONSERVATION EQUATIONS

THE TYPES OF FLOWS WE ARE CONCERNED ABOUT IN THIS CLASS ARE THOSE IN WHICH MOLECULAR DIFFUSION OF MOMENTUM AND ENERGY IS NEGLIGIBLE. IN THIS WAY, THE EFFECTS INTRODUCED BY VISCOSITY AND HEAT CONDUCTION ARE VANISHINGLY SMALL. THIS LIMIT IS SATISFIED WHEN THE CHARACTERISTIC REYNOLDS NUMBER, $Re_L = \frac{\rho U L}{\mu}$, ($\rho$ = DENSITY, U = VELOCITY, μ = VISCOSITY, L = LENGTH) IS MUCH LARGER THAN UNITY,

$$\boxed{Re_L = \frac{\rho U L}{\mu} \gg 1} \quad . \quad (1)$$

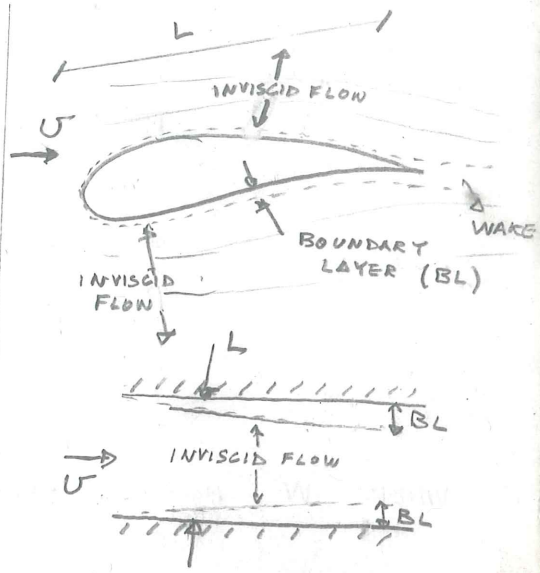


FIG 1: EXAMPLES OF HIGH-REYNOLDS NUMBER FLOWS, (a) FLOW AROUND AN AIRFOIL, (b) FLOW IN A CHANNEL. SEE ALSO REVERSE OF THIS PAGE.

IT IS IMPORTANT TO NOTE THAT VISCOUS AND HEAT-CONDUCTION EFFECTS ARE, HOWEVER, IMPORTANT IN:

- THIN BOUNDARY LAYERS ALONG SOLID SURFACES
- WAKES DOWNSTREAM OF SOLID BODIES
- THIN LAYERS IN THE INTERIOR OF THE FLOW, SUCH AS SHOCK WAVES.

OUTSIDE THESE REGIONS THE FLOW MAY BE CONSIDERED AS INVISCID OR IDEAL IN THE ABSENCE OF TURBULENCE. SUCH IDEAL FLOWS ARE STREAMLINED AND DO NOT UNDERGO ENERGY LOSSES BY FRICTION OR OTHER DISSIPATION MECHANISMS.

THE CONTINUUM LIMIT

ALL THE DESCRIPTIONS GIVEN BELOW ARE BASED ON A FIELD FORMULATION OF THE FLUID VARIABLES, SAY $\phi(\vec{x},t)$, WHERE ϕ AND ITS GRADIENTS ARE CONTINUOUS IN SPACE. THIS IS A VALID APPROXIMATION WHEN THE MEAN FREE PATH BETWEEN MOLECULAR COLLISIONS λ IS MUCH SMALLER THAN THE CHARACTERISTIC LENGTH OF THE BODY L, OR EQUIVALENTLY, WHEN

$$\boxed{Kn = \frac{\lambda}{L} \ll 1} \quad (2)$$

WHERE Kn IS THE KNUDSEN NUMBER.

CONSIDER A GAS OF MOLECULES WITH DIAMETER d. A GIVEN MOLECULE "A" WILL THEN SUFFER A COLLISION WHENEVER THE CENTER OF "A" APPROACHES THE DISTANCE d FROM THE CENTER OF ANY OTHER MOLECULE (SEE FIG. 2). ASSUMING THAT THE MOLECULE MOVES AT MEAN SPEED $\bar{c}$, IT SWEEPS OUT A VOLUME $\pi d^2 \bar{c}$ PER UNIT TIME. IF n IS THE NUMBER OF MOLECULES PER UNIT VOLUME, THEN $\omega = \pi d^2 \bar{c} n$ IS THE THE NUMBER OF COLLISIONS PER UNIT TIME AND

$$\boxed{\lambda = \frac{\bar{c}}{\omega} = \frac{1}{\sqrt{2}\pi n d^2}}$$ IS THE MEAN FREE PATH

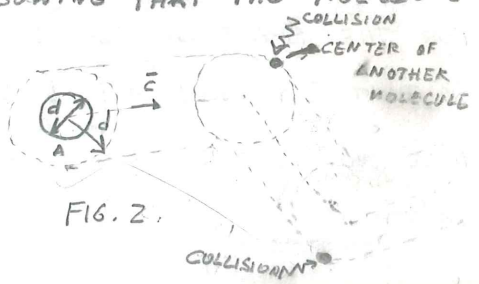


FIG. 2.

THE FACTOR $\sqrt{2}$ IN THE DENOMINATOR ARISES FROM THE FACT THAT $\bar{c}$ IS NOT THE CORRECT SPEED TO CONSIDER, BUT RATHER $\sqrt{2}\bar{c}$ WHICH IS THE MEAN RELATIVE SPEED OF THE MOLECULES.

SINCE $n = \rho/m$, WHERE ρ IS THE GAS DENSITY AND m IS THE MASS OF 1 MOLECULE, THEN λ CAN BE EXPRESSED AS $\lambda = \dfrac{m}{\sqrt{2}\pi d^2 \rho}$. FOR INSTANCE, FOR AIR AT SEA LEVEL:

$T = 298\ K$

$P = 101325\ Pa$

$m \sim \dfrac{W}{N_A} = \dfrac{28.9\cdot 10^{-3}\ kg/mol}{6.022\cdot 10^{23}\ \frac{molecules}{mol}} = 4.79\cdot 10^{-26}\ \frac{kg}{molecule}$

$d \sim d_{N_2} \sim 390\cdot 10^{-12}\ m$

$R_g = \dfrac{R^0}{W} = \dfrac{8.31\ J/molK}{28.9\cdot 10^{-3}\ kg/mol} = 286.8\ J/kgK$

$\rho = \dfrac{P}{R_g T} = 1.18\ kg/m^3 \rightarrow n = \rho/m = 2.4\cdot 10^{25}\ \frac{molecules}{m^3}$

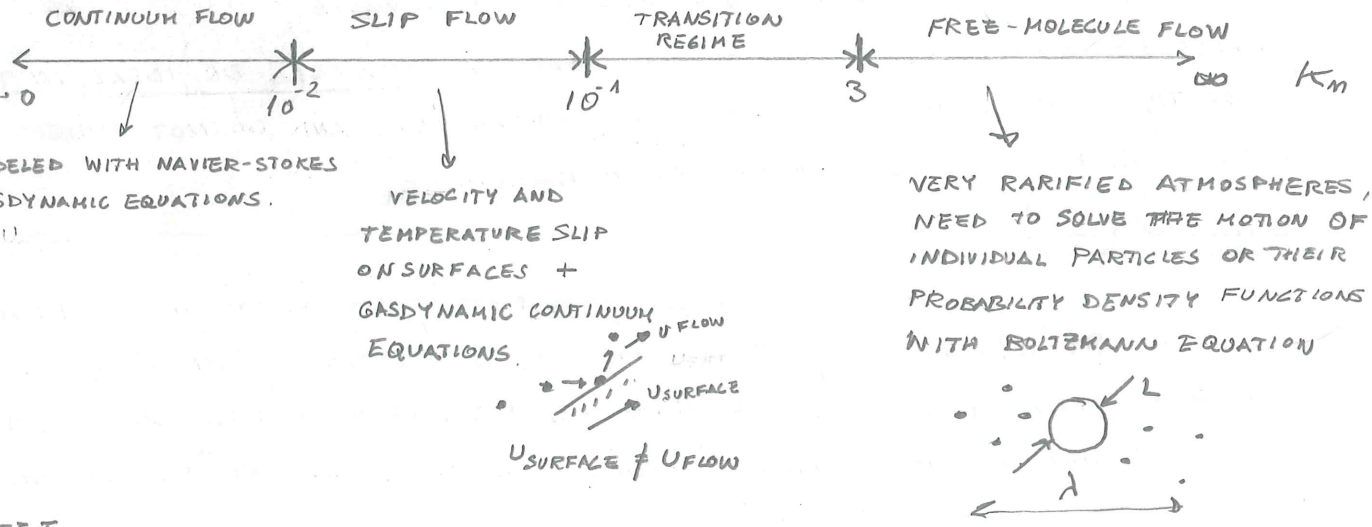
$\lambda \approx 60\cdot 10^{-9}\ m$

$\Theta \approx 10^{10}\ \frac{COLLISIONS}{s}$

$n_0^{-1/3} \sim 3.4\ 10^{-9}\ m$ MEAN INTERMOLECULAR DISTANCE

WHERE W = MOLECULAR WEIGHT OF AIR, N_A = AVOGADRO'S NUMBER, R^0 = UNIVERSAL GAS CONSTANT.

THEREFORE, FOR MACROSCOPIC BODIES OF LENGTH L, $\lambda \ll L$ AND THE CONDITION (2) FOR CONTINUUM FLOW IS TYPICALLY SATISFIED. DIFFERENT FLOW REGIMES EMERGE DEPENDING ON THE VALUE OF THE PARAMETER $Kn = \lambda/L$:



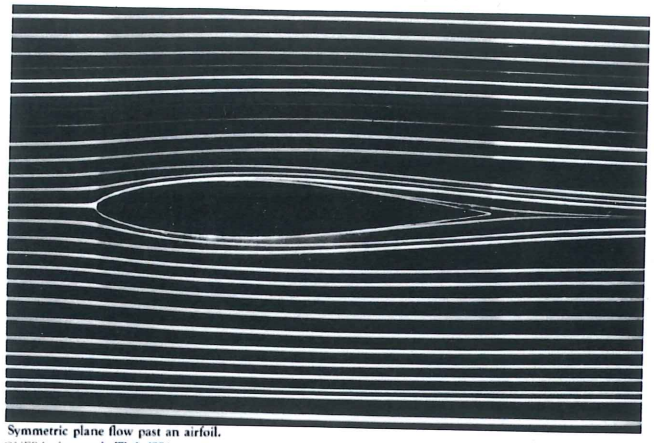
* NOTE I

ALTITUDE OVER THE EARTH'S SURFACE h

SINCE $\dfrac{\rho}{\rho_A} \sim e^{-h/\alpha}$, WITH $\alpha \sim 7200\ m$, THEN $\lambda \sim \lambda_A e^{h/\alpha}$ ($\rho_A \sim 1.18\ kg/m^3$, $\lambda_A \sim 66\cdot 10^{-9}\ m$)

AND $Kn = \dfrac{\lambda_A}{L} e^{h/\alpha} \approx 1$ WHEN $h \sim \alpha \ln\left(\dfrac{L}{\lambda_A}\right)$:

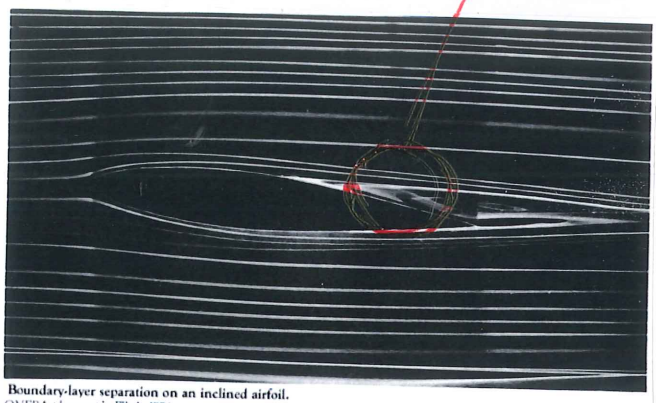
FOR THE SPACE SHUTTLE ORBITER, THE RADIUS OF THE NOSE CONE IS $L/2 \sim 0.7926\ m$ SO THAT $h = 7200 \ln\left(\dfrac{0.7926}{60\cdot 10^{-9}}\right) \approx 123\ km$. FOR $h > 123\ km$ THE FLOW OVER THE NOSE CONE OF THE SPACE SHUTTLE IS RARIFIED AND CANNOT BE COMPUTED WITH CONTINUUM ASSUMPTIONS.



Symmetric plane flow past an airfoil.
ONERA photograph, Werlé 1974

STREAMLINED, INVISCID FLOWS ARE THE SUBJECT OF THIS COURSE.

SEPARATION EFFECTS WILL NOT BE STUDIED HERE.



Boundary-layer separation on an inclined airfoil.
ONERA photograph, Werlé 1974

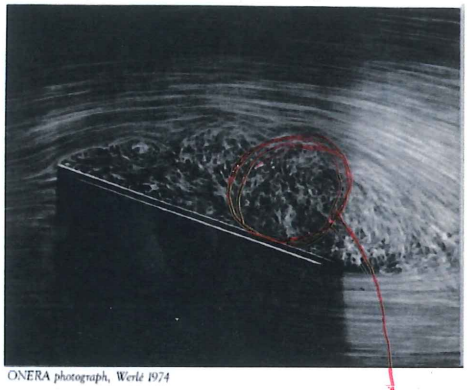
Laminar boundary-layer reattachment over an inclined flat plate



ONERA photograph, Werlé 1974

SEPARATION AND REATTACHMENT EFFECTS WILL NOT BE CONSIDERED HERE.

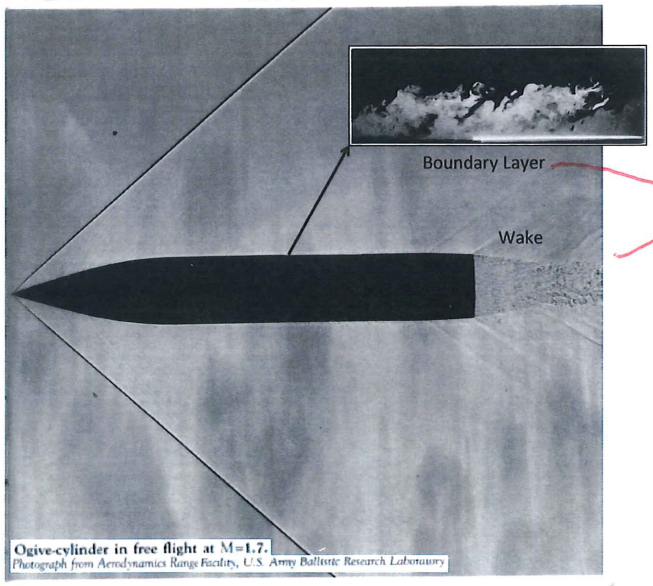
Global separation over an inclined flat plate



ONERA photograph, Werlé 1974

TURBULENT FLOWS DUE TO SEPARATION WILL NOT BE STUDIED HERE

THIS COURSE ADDRESSES THE SLENDER, INVISCID FLOW OUTSIDE BOUNDARY LAYERS AND WAKES.



Ogive-cylinder in free flight at M=1.7.
Photograph from Aerodynamics Range Facility, U.S. Army Ballistic Research Laboratory

VISCOUS EFFECTS IN BOUNDARY LAYERS AND WAKES WILL NOT BE STUDIED HERE.

TYPICAL EXAMPLES OF HIGH-REYNOLDS NUMBER FLOWS AROUND SLENDER BODIES.

* NOTE II : FROM KINETIC THEORY : $\mu \sim \frac{1}{2}\rho \bar{c} \lambda$ DYNAMIC VISCOSITY

THEREFORE $Kn \sim \frac{\lambda}{L} \sim \frac{\mu}{\rho \bar{c} L} = \frac{\mu}{\rho U L}\frac{U}{\bar{c}} \sim \frac{Ma}{Re_L}$, WHERE $Ma = \frac{U}{(a)}$ IS THE MACH NUMBER

$(a) \rightarrow$ SPEED OF SOUND

$\Rightarrow$ HIGH Ma AND LOW $Re \rightarrow$ LARGE $Kn \rightarrow$ RAREFIED FLOWS (TYPICAL OF ATMOSPHERIC REENTRY)

EQUATIONS OF MOTION

AS MENTIONED ABOVE, AT $Re_L \gg 1$ THE INFLUENCES OF VISCOSITY AND HEAT CONDUCTION BECOME NEGLIGIBLE. IN THIS LIMIT, THE MOMENTUM CONSERVATION EQUATION BECOMES

$$\rho \frac{\partial \vec{U}}{\partial t} + \rho \vec{U} \cdot \nabla \vec{U} = -\nabla P \qquad (3)$$

WHERE ρ = DENSITY, $\vec{U}$ = VELOCITY, P = PRESSURE. IN ADDITION TO EQ (3), THE MASS CONSERVATION EQUATION

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{U}) = 0 \qquad (4)$$

IS REQUIRED.

THE SIMPLIFICATION MADE FOR ARRIVING TO EQ. (3) MAY BE EXPLAINED AS FOLLOWS. CONSIDER THE GENERAL CASE OF A FLOW OVER A BODY OF LENGTH L IMMERSED IN A FLUID THAT MOVES AT VELOCITY U FAR AWAY, WHICH HAS A DENSITY ρ AND VISCOSITY μ. THE GENERAL FORM OF THE CONSERVATION OF MOMENTUM IS



$$\underbrace{\rho \frac{\partial \vec{U}}{\partial t}}_{\text{LOCAL ACCELERATION}} + \underbrace{\rho \vec{U} \cdot \nabla \vec{U}}_{\text{CONVECTIVE ACCELERATION}} = \underbrace{-\nabla P}_{\text{PRESSURE GRADIENT}} + \underbrace{\nabla \cdot \bar{\bar{\tau}}}_{\text{VISCOUS FORCES}} + \underbrace{\rho \vec{g}}_{\text{GRAVITATIONAL FORCE}},$$

"ORDERS OF MAGNITUDE" $\Rightarrow$

?	?	?	?	?
$\rho_0 \frac{U}{t_0}$	$\rho_0 \frac{U^2}{L}$	$\rho_0 \frac{U^2}{L}$	$\mu \frac{U}{L^2}$	$\rho_0 g$

WHERE t_0 IS THE CHARACTERISTIC TIME SCALE OF VARIATION OF THE VELOCITY FIELD (FOR INSTANCE, IF THE FREE STREAM VELOCITY WAS OSCILLATING AT A FREQUENCY $f_0 \sim \frac{1}{t_0}$).

BY DEFINING THE NON-DIMENSIONAL VARIABLES

$$\rho^* = \frac{\rho}{\rho_0} \quad \vec{U}^* = \frac{\vec{U}}{U}, \quad t^* = \frac{t}{t_0}, \quad \vec{x}^* = \frac{\vec{x}}{L}, \quad P^* = \frac{P}{\rho_0 U^2}, \quad \bar{\bar{\tau}}^* = \frac{\bar{\bar{\tau}}}{\mu U / L}, \quad \vec{g}^* = \frac{\vec{g}}{g},$$

THEN THE ABOVE EQUATION BECOMES

$$St\, \rho^* \frac{\partial \vec{U}^*}{\partial t^*} + \rho^* \vec{U}^* \cdot \nabla \vec{U}^* = -\nabla P^* + \frac{\nabla \cdot \bar{\bar{\tau}}^*}{Re_L} + \frac{\rho^* \vec{g}^*}{Fr},$$

NOTE $St \ll 1 \Rightarrow$ STEADY FLOW

WHERE $St = \frac{L}{U t_0}$ # STROUHAL

$Re_L = \frac{\rho_0 U L}{\mu}$ # REYNOLDS

$Fr = U^2/gL$ # FROUDE

IN THE LIMITS EQ (3) IS RECOVERED:

- $Fr \gg 1$ (NEGLIGIBLE GRAVITATIONAL ACCEL. COMPARED TO CONVECTION)
- $Re_L \gg 1$ (NEGLIGIBLE VISCOUS FORCES COMPARED TO FLUID INERTIA)
- $St \sim O(1)$

ENTROPY

* FIRST LAW OF THERMODYNAMICS: $\left[de = dq - P d\left(\frac{1}{\rho}\right) \right]$, (WORK)

$\frac{1}{\rho} = v$ SPECIFIC VOLUME

e = SPECIFIC INTERNAL ENERGY (PU MASS)

q = HEAT: $dq > 0$ (INTO THE SYSTEM); $dq < 0$ (INTO THE SURROUNDINGS)

EXAMPLE:



SURROUNDINGS

IF ρ = CONST, THEN $dq > 0$ (HEAT FLOWS INTO THE SYSTEM) AND $de > 0$ (THE INTERNAL ENERGY OF THE SYSTEM INCREASES)

* SPECIFIC HEAT CAPACITIES: "THERMODYNAMIC COEFFICIENTS"

- AT CONSTANT PRESSURE: $C_p = \left.\frac{dq}{dT}\right|_p$
- AT CONSTANT VOLUME (DENSITY): $C_v = \left.\frac{dq}{dT}\right|_\rho$

* IDEAL GAS EQUATION: $\boxed{P = \rho R_g T}$ (5), $R_g = \frac{R^0}{W}$ (GAS CONSTANT); $R^0 \to 8.31$ J/mol K (UNIVERSAL GAS CONSTANT); $W \to$ MOLECULAR WEIGHT

⇒ VALID FOR MONOATOMIC GASES AT LOW P AND HIGH T, SUCH THAT THE ATTRACTION BETWEEN MOLECULES ARE NEGLIGIBLE.

ADDITIONALLY, FOR AN IDEAL GAS, THE INTERNAL ENERGY e IS SOLELY A FUNCTION OF THE TEMPERATURE ⇒ $e = e(T)$, AND THEREFORE $\left.\frac{\partial e}{\partial \rho}\right)_T = 0$, OR EQUIVALENTLY, $de = \left.\frac{\partial e}{\partial T}\right)_\rho dT$.

SUBSTITUTING THIS RESULT INTO THE FIRST LAW:

$$\left.\frac{\partial e}{\partial T}\right)_\rho dT = dq - P d\left(\frac{1}{\rho}\right),$$

AND SIMPLIFYING TO CONSTANT ρ: $\left.\frac{dq}{dT}\right)_\rho = C_v = \left.\frac{\partial e}{\partial T}\right)_\rho \Rightarrow \boxed{e = c_v T}$ (6)

(ASSUMED "CALORICALLY PERFECT GAS" C_v = CONST TOO HERE)

INTERNAL ENERGY OF AN IDEAL GAS

SIMILARLY, AT CONSTANT P: $\underbrace{\left.\frac{\partial e}{\partial T}\right)_\rho}_{C_v} = \underbrace{\left.\frac{dq}{dT}\right)_P}_{C_p} + \underbrace{\frac{P}{\rho^2}\frac{d\rho}{dT}}_{\frac{R_g T}{\rho}\left(\frac{-P}{R_g T^2}\right)} \Rightarrow \boxed{C_p - C_v = R_g}$ (7)

$\frac{R_g}{C_v} = \gamma - 1$, $\frac{R_g}{C_p} = \frac{\gamma - 1}{\gamma}$

$\boxed{\gamma = \frac{C_p}{C_v} = 1 + \frac{R_g}{C_v} = \frac{1}{1 - R_g/C_p}}$

ADIABATIC COEFFICIENT

UNLESS STATED OTHERWISE, THE REST OF THIS COURSE WILL CONSIDER ONLY IDEAL AND CALORICALLY-PERFECT GASES.

FOR MONOATOMIC GASES (A_r, He, Ne): $C_v = \frac{3}{2} R_g$, $C_p = \frac{5}{2} R_g$, $\gamma = \frac{5}{3}$

FOR DIATOMIC GASES (N_2, O_2): $C_v = \frac{5}{2} R_g$, $C_p = \frac{7}{2} R_g$, $\gamma = \frac{7}{5}$

FOR AIR: $\gamma = 1.4$, $C_p = 1$ kJ/kg K

OBTAINED FROM THE KINETIC THEORY OF GASES. (OUTSIDE THE SCOPE OF THIS CLASS)

* ADIABATIC PROCESSES

USING THE EQUATIONS ABOVE, THE FIRST LAW CAN BE WRITTEN AS

$$dq = c_v dT - \frac{P}{\rho^2} d\rho = c_p dT - \frac{dP}{\rho}$$

FOR AN ADIABATIC PROCESS $dq = 0$, THEREFORE

$$\left.\begin{aligned} c_v dT &= \frac{p}{\rho^2} d\rho \\ c_p dT &= \frac{dp}{\rho} \end{aligned}\right\} \quad \frac{c_p}{c_v} = \frac{\rho}{p}\frac{dp}{d\rho} = \gamma \Rightarrow \frac{dp}{d\rho} = \frac{\gamma p}{\rho}, \quad \int_{p_0}^{p} \frac{dp}{p} = \int_{\rho_0}^{\rho} \frac{\gamma\, d\rho}{\rho}$$

$$\Rightarrow \boxed{\frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} = \text{CONST}} \quad (8)$$

ADIABATIC TRAJECTORY.

* ENTROPY EQUATION

$$ds = \frac{dq}{T} \Rightarrow \quad s_2 - s_1 = \int_1^2 \frac{dq}{T}$$

ENTROPY (WHEN THE PROCESS IS REVERSIBLE)

SUBSTITUTING THE EXPRESSIONS FOR dq IN AN IDEAL GAS:

$$\left.\begin{aligned} ds &= c_v \frac{dT}{T} - R_g \frac{d\rho}{\rho} \Rightarrow s - s_0 = c_v \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{\rho}{\rho_0}\right) \\ ds &= c_p \frac{dT}{T} - R_g \frac{dp}{p} \Rightarrow s - s_0 = c_p \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{p}{p_0}\right) \end{aligned}\right.$$

NOTE THAT $s = s_0$ IMPLIES $\gamma \ln\left(\frac{\rho}{\rho_0}\right) = \ln\left(\frac{p}{p_0}\right) \Rightarrow \frac{p}{\rho^\gamma} = \frac{p_0}{\rho_0^\gamma} = \text{CONST}$ IS ALSO AN ISENTROPIC TRAJECTORY

IN GENERAL: $s - s_0 = c_v \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{\rho}{\rho_0}\right) = c_p \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{p}{p_0}\right)$

OR EQUIVALENTLY: $$\boxed{s - s_0 = c_v \ln\left(\frac{p}{\rho^\gamma} \Big/ \frac{p_0}{\rho_0^\gamma}\right)} \quad (9)$$

A TRANSPORT EQUATION FOR THE ENTROPY CAN BE OBTAINED BY COMBINING THE CONSERVATION EQUATION FOR THE INTERNAL ENERGY

$$\rho \frac{\partial e}{\partial t} + \rho \vec{u} \nabla e = -p \nabla \cdot \vec{u} + \underbrace{\mathcal{E}}_{\text{VISCOUS DISSIPATION}} + \underbrace{Q}_{\text{HEAT ADDITION}} + \underbrace{\nabla \cdot (k \nabla T)}_{\text{HEAT CONDUCTION}}$$

AND USING THE FIRST LAW $T ds = de - \frac{p}{\rho^2} d\rho$, WHICH GIVES

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = \left(\frac{De}{Dt} - \frac{p}{\rho^2}\frac{D\rho}{Dt}\right)\frac{1}{T} \Rightarrow \frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = \overbrace{\frac{\nabla \cdot (k \nabla T)}{\rho T} + \frac{\mathcal{E}}{\rho T} + \frac{Q}{\rho T}}^{\text{ENTROPY SOURCES}}$$

FROM IDEAL GAS EQUATION; FROM CONTINUITY $\frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{u}$

HEAT CONDUCTION | VISCOUS DISSIPATION | HEAT ADDITION

$\sim$ ORDERS OF MAGNITUDE: $\quad \frac{k \Delta T}{\rho_0 T_0 L^2} \quad \frac{\mu U^2}{L^2 \rho_0 T_0} \quad \frac{Q}{\rho_0 T_0}$

IT CAN BE SHOWN THAT THE ENTROPY VARIATIONS CAUSED BY HEAT CONDUCTION AND VISCOUS DISSIPATION ARE NEGLIGIBLE AT HIGH REYNOLDS NUMBERS:

$$\frac{\text{HEAT-CONDUCTION SOURCE OF } s}{\text{CONVECTIVE TRANSPORT OF } s} = \frac{\frac{k \Delta T}{\rho_0 T_0 L^2}}{\frac{U \Delta s}{L}} \sim \underbrace{\frac{1}{Re_L Pr}}_{\ll 1} \underbrace{\left(\frac{\Delta T}{T_0}\right)}_{\sim O(1)} \frac{c_p}{\Delta s} \sim 1 \iff \text{ONLY IF } \frac{\Delta s}{c_p} \ll 1$$

WITH $\nu = \frac{\mu}{\rho}$, $\kappa = \frac{k}{\rho c_p}$: $\quad \boxed{Pr = \frac{\mu c_p}{k} = \frac{\nu}{\kappa}}$ # PRANDTL NUMBER (10)

($Pr \approx 0.7$ AIR)

AND

$$\frac{\text{VISCOUS-DISSIPATION SOURCE OF } s}{\text{CONVECTIVE TRANSPORT OF } s} = \frac{\frac{\mu U^2}{L^2 \rho_0 T_0}}{\frac{U \Delta s}{c}} \sim \underbrace{\frac{U^2}{c_p T_0}}_{\sim 2} \underbrace{\frac{1}{Re_L}}_{\sim \Gamma_{\Gamma 1}} \frac{c_p}{\Delta s} \sim 1 \quad \longleftrightarrow \quad \frac{\Delta s}{c_p} \ll 1$$

$$O(Ma^2) \sim O(1)$$

MACH #

IN SUMMARY, AT HIGH REYNOLDS NUMBERS THE ENTROPY EQUATION BECOMES

$$\frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = \frac{Q}{\rho T} .$$

IF THERE IS NO HEAT ADDITION, $Q = 0$, AND $\frac{Ds}{Dt} = 0$

THE EULER EQUATIONS

BASED ON THE SIMPLIFICATIONS MADE ABOVE (I.E. HIGH REYNOLDS NUMBERS, LOW KNUDSEN NUMBERS, IDEAL GAS), THE SET OF CONSERVATION EQUATIONS CONSIDERED HERE ARE

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{u}) = 0 \qquad \text{MASS CONSERVATION} \qquad (11)$$

$$\rho \frac{\partial \vec{u}}{\partial t} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla P \qquad \text{MOMENTUM CONSERVATION} \qquad (12)$$

$$\frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = \frac{Q}{\rho T} \qquad \text{ENTROPY CONSERVATION} \qquad (13)$$

ALONG WITH THE EQUATION OF STATE FOR AN IDEAL GAS

$$P = \rho R_g T \qquad (14)$$

AND THE DEFINITION OF ENTROPY FOR AN IDEAL GAS

$$s = c_v \ln\left(P/\rho^\gamma\right) + s_0 . \qquad (15)$$

EQUATION (15) CAN BE REPLACED BY ANY OF THE FOLLOWING EQUATIONS:

$$\rho \frac{\partial e}{\partial t} + \rho \vec{u} \cdot \nabla e = -P \nabla \cdot \vec{u} + Q \qquad \text{INTERNAL ENERGY CONSERVATION} \qquad (16)$$

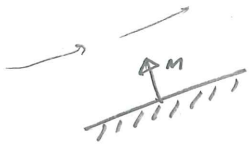
$$\rho \frac{\partial h}{\partial t} + \rho \vec{u} \cdot \nabla h = \frac{\partial P}{\partial t} + \vec{u} \cdot \nabla P + Q \qquad \text{ENTHALPY CONSERVATION} \qquad (17)$$

$$\rho \frac{\partial}{\partial t}\left(e + \frac{|\vec{u}|^2}{2}\right) + \rho \vec{u} \cdot \nabla \left(e + \frac{|\vec{u}|^2}{2}\right) = -\nabla \cdot (P \vec{u}) + Q \qquad \text{TOTAL ENERGY CONSERVATION} \qquad (18)$$

$$\rho \frac{\partial}{\partial t}\left(h + \frac{|\vec{u}|^2}{2}\right) + \rho \vec{u} \cdot \nabla \left(h + \frac{|\vec{u}|^2}{2}\right) = \frac{\partial P}{\partial t} + Q \qquad \text{TOTAL ENTHALPY CONSERVATION} \qquad (19)$$

WHERE $h = c_p T$ IS THE ENTHALPY AND $e = c_v T$ THE INTERNAL ENERGY,

EQUATIONS (11) - (13) ARE INTEGRATED SUBJECTED TO INITIAL CONDITIONS AND BOUNDARY CONDITIONS. FOR THE LATTER, NOTE THAT THE EQUATIONS HAVE LOST THE PARABOLIC CHARACTER PROVIDED BY THE MISSING VISCOUS TERMS, AND ARE NOW ELLIPTIC OR HYPERBOLIC (DEPENDING ON THE MACH NUMBER). THEREFORE SOME OF THE ORIGINAL BOUNDARY CONDITIONS, SUCH AS NON-SLIP VELOCITY CONDITIONS ON SOLID SURFACES, CANNOT BE IMPOSED. ON THE OTHER HAND, WHEN THE FLUID IS SURROUNDED BY AN IMPERMEABLE SOLID, THE CONDITIONS TO BE APPLIED ON THE WALLS ARE THOSE OF <u>NO PENETRATION</u>:

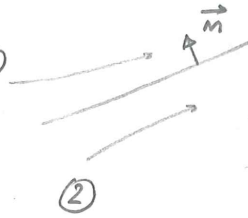


$\vec{U} \cdot \vec{m} = 0$ (ZERO NORMAL VELOCITY TO THE SURFACE).

↳ IMPLIES THAT $\vec{U}$ IS PARALLEL TO THE WALL

↳ DOES NOT IMPLY THAT $\vec{U} = 0$

SIMILARLY, WHEN THE FLUID IS BOUNDED BY ANOTHER IMMISCIBLE FLUID, THE CONDITIONS TO BE APPLIED ARE THAT THE VELOCITY NORMAL TO THE INTERFACE IS THE NORMAL VELOCITY OF THE INTERFACE, AND THE PRESSURES ARE THE SAME ON BOTH SIDES OF THE INTERFACE.



$P_1 = P_2$

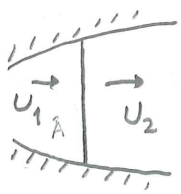
$\frac{Df}{Dt} = 0$ (i.e. $\vec{U}_1 \cdot \vec{m} = \vec{U}_2 \cdot \vec{m}$)

↳ IT DOES NOT IMPLY THAT $\vec{U}_1 = \vec{U}_2$

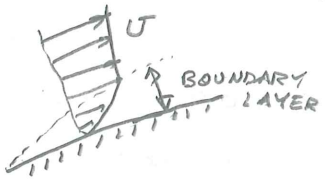
↳ IT IMPLIES THAT $\vec{U}_1$ AND $\vec{U}_2$ ARE PARALLEL.

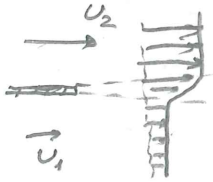
THERE IS NO GENERAL SOLUTION TO THE EULER EQUATIONS THAT IS CONTINUOUS AND ITS DERIVATIVES ARE CONTINUOUS. AS SHOWN BELOW, THE EULER EQUATIONS GIVE RISE TO FLOW DISCONTINUITIES IN THE FORM OF SHOCK WAVES AND TANGENTIAL DISCONTINUITIES, IN WHICH THE SOLUTION AND ITS DERIVATIVES ARE NOT CONTINUOUS.

VISCOUS AND HEAT-CONDUCTION EFFECTS, NOT CONSIDERED IN THE EULER EQUATIONS, COME INTO PLAY IN THIN BOUNDARY LAYERS ALONG SOLID SURFACES, IN SHEAR LAYERS BETWEEN TWO FLUID STREAMS, OR INSIDE SHOCK WAVES. HOWEVER, IT IS REMARKABLE THAT THE OUTER INVISCID STREAM, CAN BE COMPUTED FROM THE EULER EQUATIONS WITHOUT DETAILS OF WHAT HAPPENS IN THOSE THIN LAYERS.*



SHOCK WAVE





* NOTE THAT TURBULENCE AND BOUNDARY LAYER SEPARATION EFFECTS WILL NOT BE CONSIDERED HERE.

SPEED OF SOUND

THE SPEED OF SOUND IS DEFINED AS

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}, \qquad (20)$$

WHICH, FOR AN IDEAL GAS, BECOMES

$$\boxed{a = \left(\frac{\gamma p}{\rho}\right)^{1/2} = \sqrt{\gamma R_g T} = \sqrt{(\gamma-1)h}} \qquad (21)$$

$a = 346$ m/s FOR $\gamma = 1.4$, $W = 28.9$ kg/mol, $R_g = \frac{R^0}{W}$, $T = 298$ K

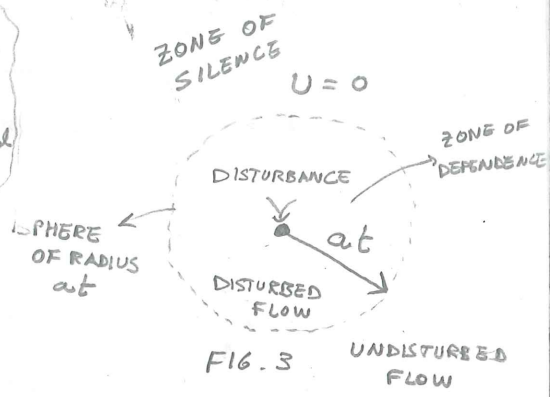
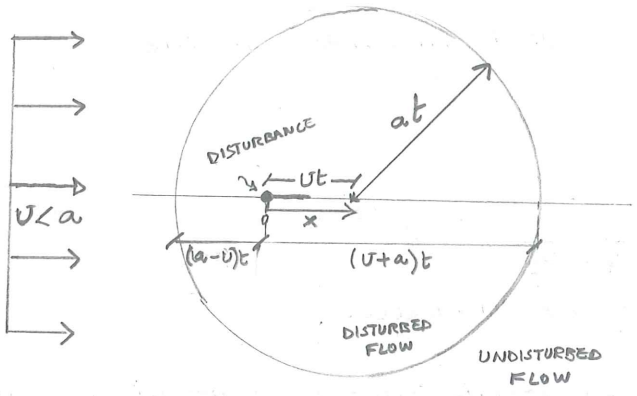


FIG. 3

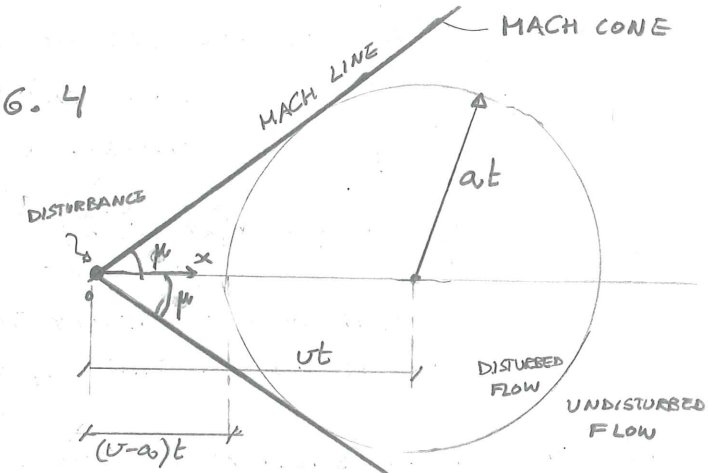
IN PARTICULAR, THE SPEED OF SOUND IS THE PROPAGATION VELOCITY OF DISTURBANCES GENERATED IN A FLUID AT REST (THIS WILL BE SHOWN LATER WHEN DESCRIBING WAVE MOTION IN COMPRESSIBLE FLOWS). THEREFORE, A DISTURBANCE IN A FLUID AT REST WILL BE PROPAGATED AT A DISTANCE at AFTER A TIME t (SEE FIG 3). IN A FLUID MOVING WITH VELOCITY U, A DISTURBANCE PRODUCED AT THE ORIGIN AT $t=0$ WILL BE AT $t>0$ ON A SPHERE OF RADIUS at CENTERED AT A DISTANCE Ut FROM THE ORIGIN (SEE FIG. 4a).

FIG. 4



(a) DISTURBANCE PROPAGATION IN A SUBSONIC FLOW

$M_a = U/a < 1$



(b) DISTURBANCE PROPAGATION IN A SUPERSONIC FLOW

$M_a = U/a > 1$

A STREAM IS CALLED SUBSONIC WHEN ITS VELOCITY U IS SMALLER THAN a, OR EQUIVALENTLY, WHEN THE MACH NUMBER

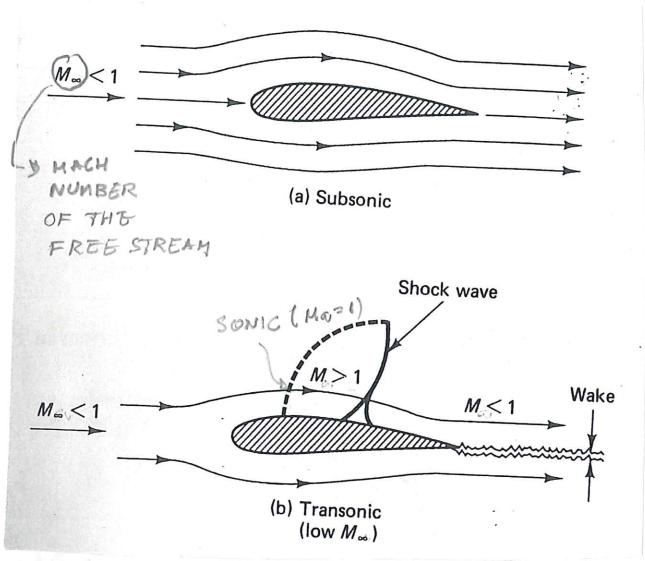
$$\boxed{M_a = U/a \quad \text{\# MACH NUMBER}} \qquad (22)$$

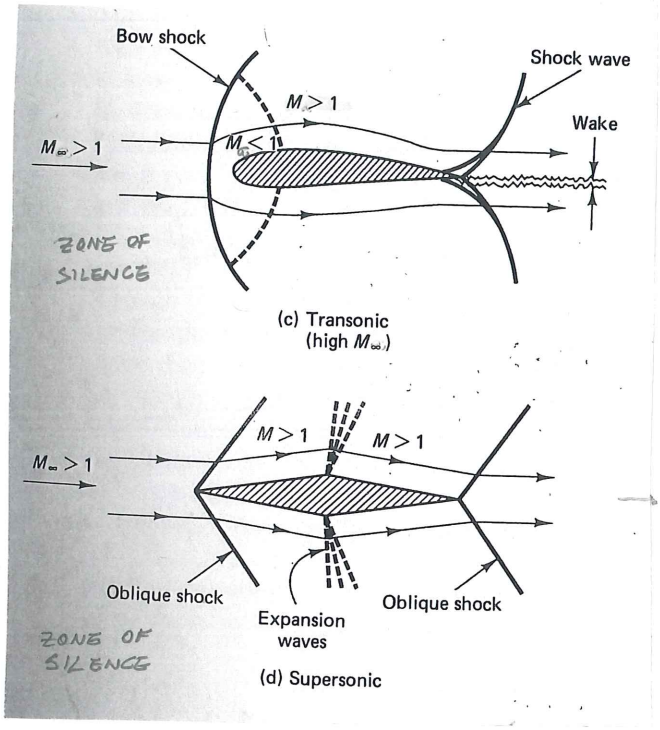
IS SMALLER THAN ONE, $M_a < 1$, IN SUCH A WAY THAT THE EFFECT OF A DISTURBANCE PRODUCED AT $x=0$ AT $t=0$ IS NOTICED AT A DISTANCE $(a-U)t$ UPSTREAM FROM THE ORIGIN AT TIME t (SEE FIG. 4a).

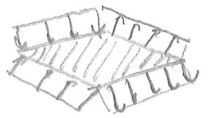
A STREAM IS CALLED SUPERSONIC WHEN $M_a > 1$, IN SUCH A WAY THAT THE EFFECT OF THE DISTURBANCE CANNOT BE NOTICED UPSTREAM FROM THE POINT WHERE IT IS PRODUCED. THE ZONE INFLUENCED BY THE EFFECTS OF THE DISTURBANCE IS THE INTERIOR OF A CONE (I.E. THE MACH CONE) OF →

OF SEMIANGLE $\mu = \arcsin\left(\frac{at}{ut}\right) = \arcsin\left(\frac{1}{Ma}\right)$ (23), WHICH IS CALLED "THE MACH ANGLE" (SEE FIG. 4b).

AS THE MACH NUMBER INCREASES FROM SUBSONIC TO SUPERSONIC, CONSIDERABLE CHANGES IN THE FLOW FIELDS OCCUR THAT HEAVILY INFLUENCE THE AERODYNAMICS OF STREAMLINED BODIES, AS ILLUSTRATED IN FIG. 5.







NO NET LIFT BUT WAVE DRAG

FIG. 5

AT SMALL Ma, THE PRESENCE OF THE BODY IS NOTICED FAR UPSTREAM, SO THAT THE STREAMLINES ADJUST SMOOTHLY GIVING RISE TO SLOW VARIATIONS OF THE FLOW VARIABLES (FIG 5a). AS THE MACH NUMBER INCREASES, THE FLOW EXPANSION ON THE LATERAL SURFACES LEADS TO SONIC ($Ma=1$) AND SUPERSONIC CONDITIONS, WITH A SHOCK WAVE BEING FORMED THAT CREATES A SUDDEN JUMP IN PRESSURE AND VELOCITY. THESE ARE TRANSONIC CONDITIONS (FIG 5b). AT SUPERSONIC SPEEDS, A DETACHED SHOCK WAVE (OR BOW SHOCK) IS CREATED UPSTREAM FROM THE BODY THAT SLOWS DOWN AND PRESSURIZES THE FLOW. THE LATERAL CURVATURE OF THE BODY THEN ACCELERATES THE FLOW AND MAKES IT SUPERSONIC AGAIN. UPON MERGING WITH THE FLOW STREAM ON THE OTHER SIDE, A TRAILING-EDGE SHOCK WAVE DECELERATES AND TURNS THE FLOW IN THE STREAMWISE DIRECTION (FIG 5c). AT EVEN HIGHER SPEEDS, THE LEADING SHOCK SURROUNDS THE BODY MUCH CLOSER AND MAY BECOME ATTACHED TO IT IF A WEDGE-TYPE OF GEOMETRY IS UTILIZED (FIG. 5d). NOTE THAT IN SUPERSONIC REGIMES THE BODY MOVES FASTER THAN THE SPEED OF SOUND AND ITS EFFECT IS NOT NOTICED UPSTREAM.

THE DESIGN OF AIRFOILS, AND EVEN ENTIRE AIRCRAFTS, IS FUNDAMENTALLY DIFFERENT DEPENDING ON WHETHER IT FLIES SUBSONICALLY OR SUPERSONICALLY. FOR SUBSONIC AIRCRAFTS, A SMOOTH SHAPE IS GENERALLY SELECTED FOR ENABLING STREAMLINED FLOWS WITHOUT SEPARATION IN ORDER TO INCREASE LIFT AND MINIMIZE DRAG (FIG. 6a). FOR SUPERSONIC AIRCRAFT, THE DESIGN MUST BE MODIFIED TO DEAL WITH SHOCK WAVES. FOR THESE, A THIN SHARP BODY WITH A POINTED NOSE IS PREFERRED IN ORDER TO MINIMIZE THE SHOCK STRENGTH NEAR THE FRONT OF THE AIRCRAFT (I.E. TO MAKE IT INCREASINGLY OBLIQUE TO THE FREE STREAM), THEREBY MINIMIZING THE DRAG (FIG. 6b).

AIRBUS A380
Ma ~ 0.85



BOEING 777
Ma ~ 0.83



GULFSTREAM G50
Ma ~ 0.92



LOCKHEED AC130
Ma ~ 0.4



(a) SUBSONIC AIRCRAFTS



CONCORDE
Ma ~ 2.1



MIG 21
Ma ~ 2.0



LOCKHEED SR-71
Ma ~ 3.0



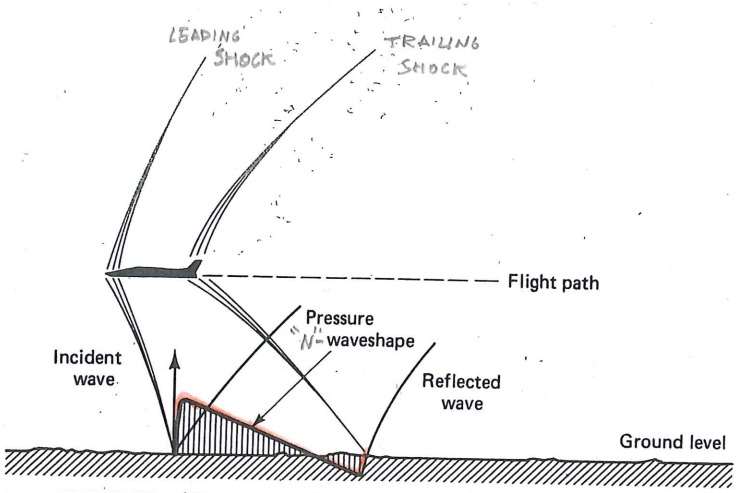
MD F-18
Ma ~ 1.8

(b) SUPERSONIC AIRCRAFTS

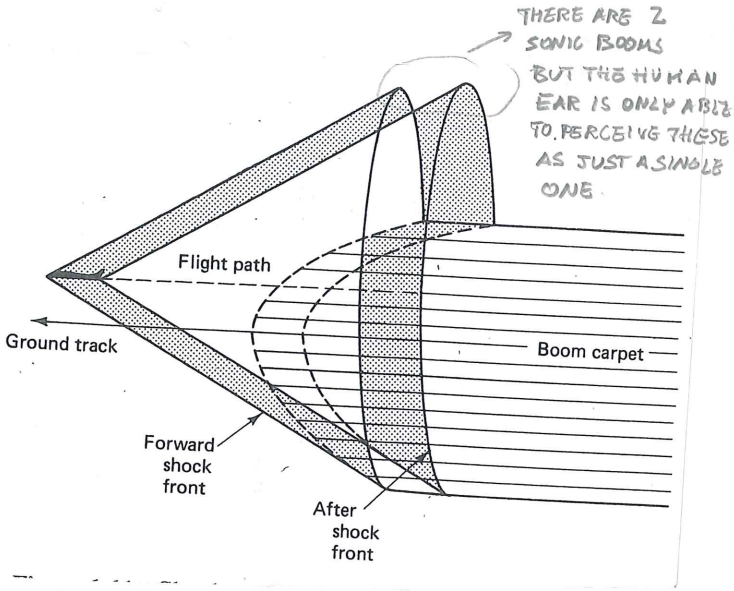
FIG 6.

SONIC BOOM

WHEN AN AIRPLANE FLIES AT SPEEDS EXCEEDING THE LOCAL SPEED OF SOUND GENERATES PRESSURE DISTURBANCES IN THE FORM OF SHOCK WAVES AND EXPANSION FANS (TO BE STUDIED LATER IN THE COURSE). THESE PRESSURE VARIATIONS PROPAGATE TO THE GROUND AS AUDIBLE EFFECTS KNOWN AS "SONIC BOOM" (SEE FIG 7a).



(a) "N"-PRESSURE WAVE CAUSED ON THE GROUND BY A SUPERSONIC AIRCRAFT.



(b) SHOCK CONES PRODUCED BY A SUPERSONIC AIRCRAFT.

FIG. 7.

THE SONIC BOOM MOVES WITH THE AIRCRAFT DURING FLIGHT AND CREATES AN "N"-SHAPED PRESSURE WAVE ON THE GROUND AS A RESULT OF THE SHOCK WAVES. THE NARROW PATH AFFECTED ON THE GROUND IS PARABOLIC IN SHAPE AND ALSO MOVES WITH THE AIRCRAFT (SEE FIG. 7b). THE CHARACTERISTIC OVERPRESSURES CAUSED BY SONIC BOOMS ARE JUST OF THE ORDER OF 40 Pa (SR-71 AT $M_a = 3$) AND 90 Pa (CONCORDE AT $M_a = 2$), BUT THESE TRANSLATE INTO HUNDREDS OF DECIBELS OF SOUND PRESSURE LEVEL.

IT IS IMPORTANT TO RECOGNIZE THAT THE SPEED OF SOUND a VARIES WITH ALTITUDE (SEE FIG. 8). A SONIC BOOM REACHES THE GROUND WHEN THE AIRCRAFT SPEED IS LARGER THAN THE SPEED OF SOUND AT GROUND LEVEL, WHICH IN MOST PRACTICAL CASES (FLIGHT BELOW THE STRATOPAUSE) REQUIRES MACH NUMBERS LARGER THAN UNITY BASED ON THE SPEED OF SOUND AT THE AIRCRAFT ALTITUDE.

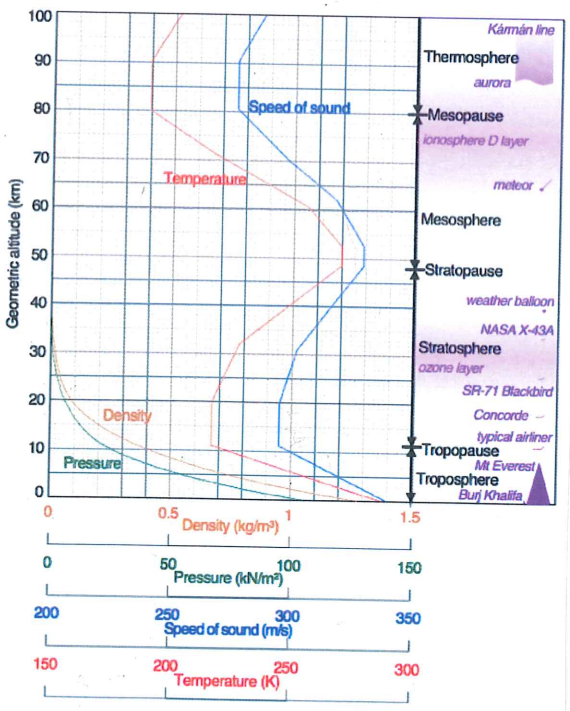


FIG. 8. VARIATIONS OF ρ, P, T AND a IN THE ATMOSPHERE.

NOTE: SOUND PRESSURE LEVEL:
$SPL = 20 \log_{10}\left(\frac{P'}{P_{ref}}\right)$ IN dB. $P_{ref} = 20\,\mu Pa$

ISENTROPIC FLOWS

IN INVISCID FLOWS WHERE THERE IS NO HEAT ADDITION, THE ENTROPY EQUATION (13) SIMPLIFIES TO

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{u}\cdot\nabla s = 0 ,$$

WHICH CAN BE INTEGRATED TO GIVE $s = s_p$, WHERE s_p IS THE INITIAL ENTROPY OF A FLUID PARTICLE. THIS MOTION IS REFERRED TO AS "ISENTROPIC". HOWEVER IT DOES NOT REQUIRE THAT THE ENTROPY BE UNIFORM IN SPACE. FOR INSTANCE, IN STEADY FLOWS, THE EQUATION ABOVE BECOMES

$$\vec{u}\cdot\nabla s = 0 ,$$

WHICH CORRESPONDS TO AN ISENTROPIC FLOW IN WHICH THE ENTROPY IS CONSTANT ALONG STREAMLINES (SEE FIG. 9). IF, IN ADDITION, ALL PARTICLES HAVE THE

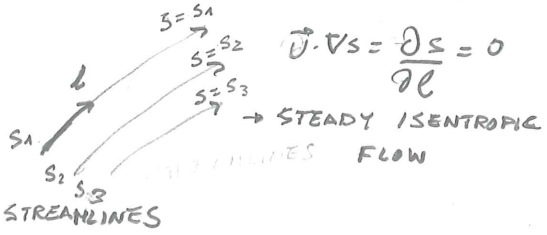


FIG. 9

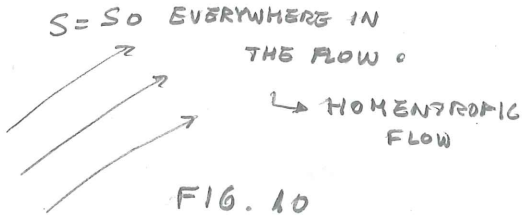


FIG. 10

SAME ENTROPY, THEN $\nabla s = 0$, AND THE FLOW IS REFERRED TO AS "HOMENTROPIC". IN THIS LIMIT, THE ENTROPY IS UNIFORM EVEN THOUGH IT COULD BE A FUNCTION OF TIME, $s = s_0(t)$. AN ISENTROPIC MOTION WITH UNIFORM INITIAL ENTROPY FOR ALL PARTICLES IS ALSO AN HOMENTROPIC FLOW. IN PRACTICE, THE TERM "ISENTROPIC FLOW" WILL BE USED HERE TO INDICATE FLOWS IN WHICH THE ENTROPY IS CONSTANT ALONG STREAMLINES.

THE BERNOUILLI EQUATION FOR COMPRESSIBLE FLOWS

A FUNDAMENTAL RELATION CAN BE DERIVED BY SIMULTANEOUSLY CONSIDERING MOMENTUM AND ENERGY CONSERVATION IN (INVISCID) COMPRESSIBLE FLOWS. THIS RELATION IS CALLED THE BERNOUILLI EQUATION, AND ITS DERIVATION IS GIVEN BELOW.

CONSIDER FIRST THE MOMENTUM EQUATION (12),

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p , \qquad (24)$$

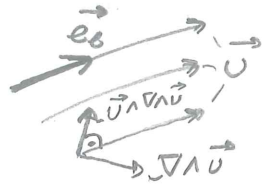
AND ASSUME STEADY FLOW $\left(\frac{\partial}{\partial t} = 0\right)$. NOTING THAT THE CONVECTIVE TERM CAN BE EXACTLY REWRITTEN AS

$$\vec{u}\cdot\nabla\vec{u} = \nabla\left(\frac{|\vec{u}|^2}{2}\right) - \vec{u}\wedge\nabla\wedge\vec{u} ,$$

→

THEN MULTIPLICATION BY AN UNIT VECTOR $\vec{e}_\ell$ TANGENT TO THE STREAMLINES LEADS TO

$$\vec{e}_\ell \cdot (\vec{v} \cdot \nabla \vec{v}) = \vec{e}_\ell \cdot \nabla \left(\frac{|\vec{v}|^2}{2} \right) - \vec{e}_\ell \cdot (\vec{v} \wedge \nabla \wedge \vec{v}) .$$



THEREFORE, MULTIPLICATION OF EQ. (24) BY THE UNIT VECTOR $\vec{e}_\ell$ GIVES

$$\frac{\partial}{\partial \ell} \left(\frac{|\vec{v}|^2}{2} \right) + \frac{1}{\rho} \frac{\partial p}{\partial \ell} = 0 , \quad (25)$$

WHERE $\frac{\partial}{\partial \ell} = \vec{e}_\ell \cdot \nabla$ IS THE DIRECTIONAL DERIVATIVE. IN <u>INCOMPRESSIBLE FLOWS</u>, ρ = CONST. AND (25) BECOMES

$$\frac{\partial}{\partial \ell} \left(\frac{p}{\rho} + \frac{|\vec{v}|^2}{2} \right) = 0 \Rightarrow \frac{p}{\rho} + \frac{|\vec{v}|^2}{2} = C , \quad (26)$$

ONLY FOR INCOMPRESSIBLE FLOWS

WHERE C IS A CONSTANT THAT IN PRINCIPLE DEPENDS ON THE STREAMLINE.

NOTE, HOWEVER, THAT IN <u>COMPRESSIBLE FLOWS</u>, THE CHARACTERISTIC VARIATIONS OF THE DENSITY ARE OF THE SAME ORDER AS THE DENSITY ITSELF, $\Delta\rho / \rho_0 = O(1)$, AND EQ. (26) IS NOT VALID. CONVERSELY, ONE CAN USE IN (25) THE FIRST PRINCIPLE OF THERMODYNAMICS PARTICULARIZED FOR <u>ISENTROPIC PROCESSES</u>,

$$\cancel{T \frac{\partial s}{\partial \ell}}^{\,0} = \frac{\partial h}{\partial \ell} - \frac{1}{\rho} \frac{\partial p}{\partial \ell} \Rightarrow \frac{\partial h}{\partial \ell} = \frac{1}{\rho} \frac{\partial p}{\partial \ell} , \quad \text{WITH } h = c_p T$$

WHICH YIELDS THE BERNOULLI EQUATION FOR <u>COMPRESSIBLE FLOWS</u>:

$$\frac{\partial}{\partial \ell} \left(\frac{|\vec{v}|^2}{2} + h \right) = 0 \Rightarrow \boxed{h + \frac{|\vec{v}|^2}{2} = C} \quad (27) .$$

COMPRESSIBILITY

COMPRESSIBLE FLOWS ARE CHARACTERIZED BY MACH NUMBERS LARGER THAN 0.3. IN COMPRESSIBLE FLOWS, THE PRESSURE VARIATIONS ARE OF THE SAME ORDER AS THE KINETIC ENERGY, AS EASILY OBSERVED FROM EQ. (24):

$$\rho \vec{v} \cdot \nabla \vec{v} \sim -\nabla p \Rightarrow \Delta p \sim \rho_0 U^2$$

$$\widetilde{O(\rho_0 U^2/L)} \quad \widetilde{O(\Delta p / L)}$$

WITH Δp BEING COMPARABLE TO THE PRESSURE ITSELF SINCE

$$\frac{\Delta p}{p_0} \sim \frac{\rho_0 U^2}{p_0} \sim Ma^2 \sim O(1) .$$

FURTHERMORE, THE GAS EQUATION OF STATE (5) SUGGESTS THAT THE RELATIVE CHANGES IN TEMPERATURE AND DENSITY ARE ALSO OF ORDER UNITY.

$$P = \rho R_g T \;\Rightarrow\; \Delta P \approx \rho_0 R_g \Delta T + R_g T_0 \Delta\rho \;\Rightarrow\; \frac{\Delta P}{P_0} \approx \frac{\Delta T}{T_0} + \frac{\Delta\rho}{\rho_0} \sim \mathcal{O}(1)$$

AND THEREFORE $\frac{\Delta P}{P_0} \sim \frac{\Delta T}{T_0} \sim \frac{\Delta\rho}{\rho_0} \sim M_a^2 \sim \mathcal{O}(1)$ //

WITH $\frac{\Delta P}{P_0} \sim \gamma \frac{\Delta\rho}{\rho_0}$ FOR ISENTROPIC FLOWS

THIS IS IN CONTRAST TO INCOMPRESSIBLE FLOWS, WHERE THE DENSITY IS CONSTANT AND THE PRESSURE VARIATIONS ARE ALWAYS SMALL, COMPARED TO THE PRESSURE ITSELF.

↳ AND OF ORDER $M_a^2 \ll 1$

STAGNATION AND STATIC QUANTITIES

THE BERNOULLI EQUATION (27) CAN BE REWRITTEN AS

$$h + \frac{|\vec{u}|^2}{2} = h_0 \;\Rightarrow\; c_p T + \frac{|\vec{u}|^2}{2} = c_p T_0 \quad \text{OR} \quad \frac{\gamma}{\gamma-1}\frac{P}{\rho} + \frac{|\vec{u}|^2}{2} = \frac{\gamma P_0}{(\gamma-1)\rho_0},$$

WHERE USE OF THE DEFINITION OF ENTHALPY FOR AN IDEAL GAS $h = c_p T = \frac{\gamma}{\gamma-1}\frac{P}{\rho}$ HAS BEEN MADE. HERE, h_0, T_0, ρ_0 AND P_0 REPRESENT STAGNATION QUANTITIES (ALSO CALLED "TOTAL" QUANTITIES). THEY CORRESPOND TO THE THERMODYNAMIC STATE OF A GAS ISENTROPICALLY BROUGHT TO REST FROM ITS INITIAL VELOCITY $\vec{u}$. NOTE THAT h_0, T_0, ρ_0 AND P_0 ARE RELATED AMONG THEMSELVES AND TO OTHER VARIABLES SUCH AS THE STAGNATION ENERGY AND SPEED OF SOUND:

$$h_0 = c_p T_0 = \frac{\gamma}{\gamma-1}\frac{P_0}{\rho_0} = \gamma e_0 = \frac{a_0^2}{\gamma-1}.$$

IN GENERAL, HOWEVER, JUST TWO OF THESE VARIABLES SUFFICE TO COMPLETELY CHARACTERIZE THE STAGNATION THERMODYNAMIC STATE OF THE MIXTURE.

CONVERSELY, THE VARIABLES h, T, P, ρ, a AND e ARE CALLED "STATIC" IN THAT THEY DO NOT ACCOUNT FOR THE KINETIC ENERGY OF THE FLOW. IT IS STRAIGHTFORWARD TO SEE THAT

$$h < h_0,\; P < P_0,\; T < T_0,\; \rho < \rho_0,\; a < a_0 \text{ AND } e < e_0,$$

WHICH ILLUSTRATES THAT IN COMPRESSIBLE FLOWS PART OF THE INITIAL THERMAL ENERGY CAN BE TRANSFORMED INTO KINETIC ENERGY. IN FACT, THE THERMAL AND KINETIC ENERGIES BEHAVE AS RESERVOIRS OF ENERGY THAT CAN BE TRANSFERRED OR TRANSFORMED IN EITHER TYPE DEPENDING ON THE FLOW CHARACTERISTICS, WITH THE VARIATIONS OF BOTH TYPES BEING OF THE SAME ORDER: $\frac{\Delta(|\vec{u}|^2/2)}{\Delta e} \sim M_a^2 \sim 1$ //

IT IS INSTRUCTIVE TO REWRITE THE BERNOULLI EQ. (27) IN A MORE COMPACT FORM AS

$$h + \frac{|\vec{U}|^2}{2} = h_0 \Rightarrow \boxed{\frac{T_0}{T} = 1 + \frac{|\vec{U}|^2}{2c_pT} = 1 + \frac{\gamma-1}{2} Ma^2} \quad (28)$$

COMBINING THIS RELATION WITH THE ISENTROPIC RELATIONS

$$\frac{T_0}{T} = \left(\frac{\rho_0}{\rho}\right)^{\gamma-1} = \left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}$$

THE EXPRESSIONS

$$\boxed{\frac{h_0}{h} = \frac{T_0}{T} = \frac{e_0}{e} = \left(\frac{a_0}{a}\right)^2 = \left(\frac{\rho_0}{\rho}\right)^{\gamma-1} = \left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \left(\frac{\gamma-1}{2}\right) Ma^2} \quad (29)$$

FOR INSTANCE, AT $Ma = 1$ THIS FACTOR IS 1.2 (20% REDUCTION IN T WITH RESPECT TO T_0)

ARE OBTAINED, WHICH RELATE THE STAGNATION AND STATIC QUANTITIES WITH THE LOCAL MACH NUMBER $Ma = |\vec{U}|/a$. EXPRESSION (29) IS TABULATED IN TABLE I.

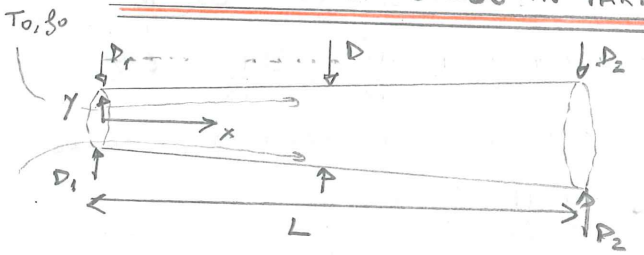
IN SUMMARY, THE CONSERVATION OF STAGNATION QUANTITIES STATED BY EQ. (29) ONLY HOLDS FOR STEADY $\left(\frac{\partial}{\partial t} = 0\right)$ AND ISENTROPIC $\left(\frac{\partial s}{\partial e} = 0\right)$ FLOWS.

CRITICAL QUANTITIES

THE CRITICAL QUANTITIES ARE DEFINED AS THE ONES THAT WOULD BE OBTAINED BY ISENTROPICALLY ACCELERATING THE FLOW FROM REST TO SONIC ($Ma = 1$) CONDITIONS. DENOTING THEM BY ASTERISKS, THE CRITICAL QUANTITIES CAN BE COMPUTED BY SETTING $Ma = 1$ IN EQ. (29), WHICH GIVES:

$$\boxed{\frac{h^*}{h_0} = \frac{T^*}{T_0} = \frac{e^*}{e_0} = \left(\frac{a^*}{a_0}\right)^2 = \left(\frac{\rho^*}{\rho_0}\right)^{\gamma-1} = \left(\frac{p^*}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{2}{\gamma+1}} \quad (30).$$

IDEAL MOTION OF GASES IN VARIABLE-AREA DUCTS.



$\vec{x} = (x, y)$, $\vec{U} = (u, v)$

$T_0, \rho_0 \equiv$ STAGNATION TEMPERATURE AND DENSITY.

$a_0 = \sqrt{\gamma R_g T_0}$ STAGNATION SPEED OF SOUND

CONSIDER A VARIABLE AREA DUCT SUCH THAT

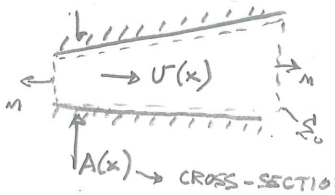
$D_2 - D_1 \sim D \ll L \Rightarrow D/L \ll 1$ (SLOWLY AREA-VARYING DUCT)

THEN, FROM EQ. OF MASS CONSERVATION:

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \Rightarrow \boxed{\frac{V}{U} \sim \frac{D}{L} \ll 1}$$

$\rho_0 U/L$ $\quad$ $\rho_0 V/D$

TRANSVERSE VELOCITY IS NEGLIGIBLE COMPARED WITH THE LONGITUDINAL ONE



$\Rightarrow$ MASS CONSERVATION: $\int_{\Sigma_0} \rho \vec{v}\cdot\vec{n}\, d\Sigma = 0 \Rightarrow \dot{m} = \rho U A = \text{CONST.}$

MASS-FLOW RATE

SINCE THE FLOW IS ISENTROPIC:

$$(31)\quad \dot{m} = \rho U A = \frac{\rho}{\rho_0}\rho_0 \frac{U}{a}\frac{a}{a_0} A = \rho_0 a_0 A\, Ma \left[1 + \frac{(\gamma-1)}{2}Ma^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

FROM THE ISENTROPIC RELATIONS (29): $\frac{a}{a_0} = \left(1 + \frac{(\gamma-1)}{2}Ma^2\right)^{-1/2}$, $\frac{\rho}{\rho_0} = \left(1 + \frac{(\gamma-1)}{2}Ma^2\right)^{\frac{-1}{\gamma-1}}$

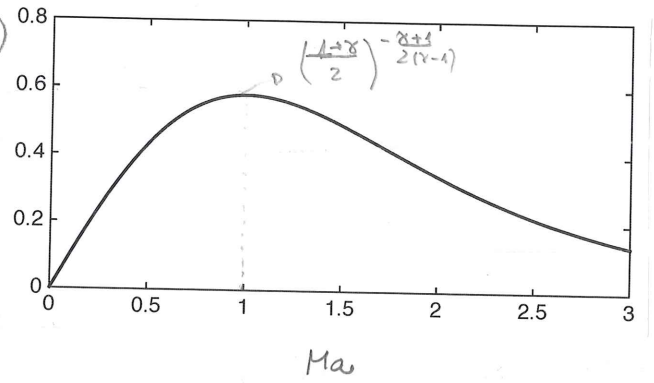
* NOTE THAT IF $A = A(x)$, THEN $U = U(x)$, $Ma = Ma(x)$, $T = T(x)$, $\rho = \rho(x)$, $P = P(x)$ AND SO ON, BUT $S = S_0$ BECAUSE OF THE ISENTROPIC ASSUMPTION.

* IT IS ALSO INTERESTING TO NOTE THAT THE FUNCTION

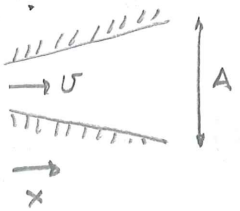
$$F(Ma) = Ma\left[1 + \frac{(\gamma-1)}{2}Ma^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} \longrightarrow$$

HAS A GLOBAL MAXIMUM $\left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$ AT $Ma = 1$.

THIS HAS PROFOUND CONSEQUENCES ON THE DYNAMICS OF HIGH SPEED FLOWS IN VARIABLE-AREA DUCTS, AS DESCRIBED BELOW.



EFFECTS OF VARIABLE AREA IN SUBSONIC AND SUPERSONIC FLOWS



SINCE $\dot{m} = \rho U A = \text{const.} \Rightarrow d\dot{m} = \left(\frac{\partial \dot{m}}{\partial \rho}\right)_{UA} d\rho + \left(\frac{\partial \dot{m}}{\partial U}\right)_{\rho A} dU + \left(\frac{\partial \dot{m}}{\partial A}\right)_{U\rho} dA = 0$

THEN $UA\,d\rho + \rho A\,dU + \rho U\,dA = 0 \Rightarrow \frac{1}{\rho}\frac{d\rho}{dx} + \frac{1}{U}\frac{dU}{dx} + \frac{1}{A}\frac{dA}{dx} = 0$. (32)

ALSO, FROM THE ENTROPY RELATION $S = S_0 \Rightarrow \frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} \Rightarrow \frac{d\rho}{dx} = \frac{1}{a^2}\frac{dP}{dx}$ (33)

AND FROM BERNOUILLI EQ. (27): $h_0 = h + \frac{U^2}{2} \Rightarrow \frac{dh}{dx} = -U\frac{dU}{dx}$ (34)

FINALLY, FROM THE FIRST PRINCIPLE APPLIED TO AN ISENTROPIC SYSTEM:

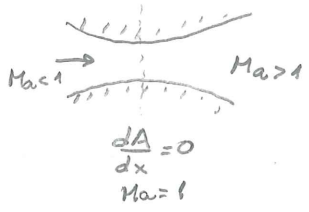
$$\cancel{T\,ds} = dh - \frac{dP}{\rho} \Rightarrow \frac{dh}{dx} = \frac{1}{\rho}\frac{dP}{dx} \quad (35).$$

SUBSTITUTING (35) INTO (34): $\frac{1}{\rho}\frac{dP}{dx} = -U\frac{dU}{dx}$, (36)

SUBSTITUTING (36) INTO (33): $\frac{d\rho}{dx} = -\frac{\rho U}{a^2}\frac{dU}{dx}$ (37)

SUBSTITUTING (37) INTO (32): $-\frac{U}{a^2}\frac{dU}{dx} + \frac{1}{U}\frac{dU}{dx} = -\frac{1}{A}\frac{dA}{dx} \Rightarrow \boxed{\frac{(1-Ma^2)}{U}\frac{dU}{dx} = -\frac{1}{A}\frac{dA}{dx}}$ (38)

EQUATION (38) STATES THAT SONIC CONDITIONS ($Ma = 1$) IMPLY AN AREA MINIMUM $\frac{dA}{dx} = 0$. THIS WILL BE SEEN BELOW TO INDICATE THAT SONIC CONDITIONS IN CONVERGENT-DIVERGENT NOZZLES MUST BE NECCESSARILY REACHED IN THE THROAT IN STEADY-STATE OPERATING CONDITIONS.



A QUANTITY DENOMINATED CRITICAL AREA A^* CAN BE OBTAINED BY SUBSTITUTING $Ma = 1$ INTO EQ. (30) FOR $\dot{m}$:

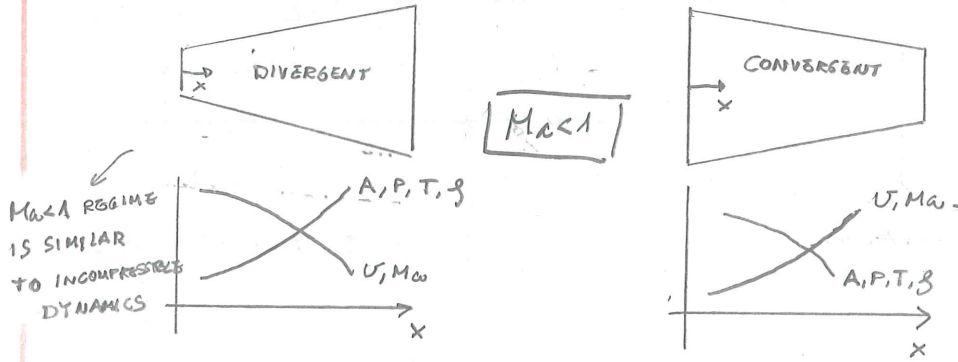
$$\dot{m} = \rho_0 a_0 A Ma \left[1 + \frac{(\gamma-1)}{2} Ma^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} = \rho_0 a_0 A^* \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \boxed{A^* = \frac{\dot{m}}{\rho_0 a_0} \left(\frac{1+\gamma}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}} \quad (39), \quad \text{WITH} \quad \boxed{\frac{A}{A^*} = \frac{1}{Ma}\left[\frac{2}{\gamma+1}\left(1 + \frac{\gamma-1}{2} Ma^2\right)\right]^{\frac{\gamma+1}{2(\gamma-1)}}}$$

AT $A = A^*$, THE CRITICAL QUANTITIES GIVEN BY EQ. (30) ARE REACHED.

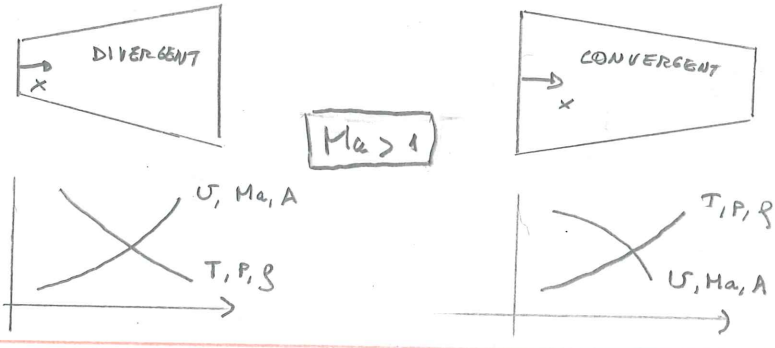
→ THIS FUNCTION IS TABULATED IN TABLE I

* NOTE THAT IN SUBSONIC FLOWS, AN INCREASE IN AREA ($\frac{dA}{dx} > 0$) IMPLIES AN DECREASE IN VELOCITY ($\frac{du}{dx} < 0$, SEE EQ. (38)), AN INCREASE IN DENSITY ($\frac{d\rho}{dx} > 0$, SEE EQ. (37)) AN INCREASE IN PRESSURE ($\frac{dp}{dx} > 0$, SEE EQ. (36)), AN INCROASE IN TEMPERATURE ($\frac{dT}{dx} > 0$, SEE EQ. (35)), AND A DECREASE IN MACH NUMBER ($\frac{dMa}{dx} < 0$, SEE EQ. (31))



NOTE THAT IT IS IMPOSSIBLE TO REACH $Ma > 1$ WITH JUST A CONVERGENT NOZZLE STARTING FROM SUBSONIC CONDITIONS

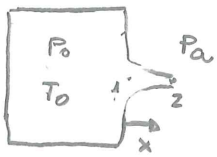
FIG 9. SUBSONIC (UPPER PANEL) AND SUPERSONIC (LOWER PANEL) FLOWS IN VARIABLE-AREA DUCTS



⇒ SUPERSONIC FLOWS ARE ACCELERATED BY DIVERGENT NOZZLES AND PRESSURIZED BY CONVERGENT ONES

EXAMPLE: FLOW IN A CONVERGENT NOZZLE ($Ma < 1$)

CONSIDER A CONVERGENT NOZZLE ATTACHED TO A LARGE RESERVOIR FILLED WITH GAS AT TEMPERATURE T_0 AND PRESSURE P_0. THE GAS ESCAPES THROUGH THE NOZZLE. ASSUME ISENTROPIC FLOW AND THAT THE NOZZLE HAS AN AREA $A(x)$, WITH $A^{1/2}/L \ll 1$.



P_a = AMBIENT PRESSURE

2 : EXIT PLANE

SINCE THE FLOW IS ISENTROPIC: $\frac{T_0}{T} = \left(\frac{P_0}{P}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{\rho_0}{\rho}\right)^{\gamma-1} = 1 + \frac{\gamma-1}{2} Ma^2$

AND THEREFORE $\dot{m} = \rho U A = \rho_0 a_0 A Ma \left(1 + \frac{(\gamma-1)}{2} Ma^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$,

WHERE $\rho_0 a_0 = \frac{P_0}{R_g T_0} \sqrt{\gamma R_g T_0} = P_0 \sqrt{\frac{\gamma}{R_g T_0}}$.

TO OBTAIN THE MASS FLOW RATE, THE PRESSURE AT THE EXIT IS IMPOSED TO BE THE AMBIENT PRESSURE, $P_2 = P_a$, AND THEREFORE

$$\left(\frac{P_0}{P_a}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{(\gamma-1)}{2} Ma_2^2 \Rightarrow Ma_2 = \left[\frac{2}{\gamma-1}\left[\left(\frac{P_0}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\right]^{1/2}$$ MACH # AT THE OUTLET

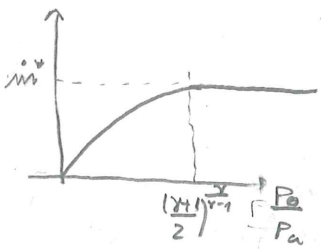
SUBSTITUTING THIS IN THE MASS FLOW-RATE EXPRESSION:

$$\dot{m} = P_0 \sqrt{\frac{\gamma}{R_g T_0}} A_2 \left(\frac{P_0}{P_a}\right)^{-\frac{\gamma+1}{2\gamma}} \left[\left(\frac{P_0}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\frac{2}{\gamma-1}\right)^{1/2}$$ MASS FLOW RATE

NOTES:

i) IN GENERAL, $T_2 \neq T_a$ AND $\rho_2 \neq \rho_a$. THESE CONDITIONS CANNOT BE IMPOSED, AND IN GENERAL THE TEMPERATURE AND DENSITY OF A SUBSONIC JET ARE NOT THOSE OF THE AMBIENT. THE EQUALIZATION OF THESE QUANTITIES REQUIRES SLOW AERODYNAMIC MIXING PROCESSES. THIS IS IN CONTRAST TO THE PRESSURE, WHICH RAPIDLY ATTAINS VALUES CLOSE TO P_a BECAUSE OF THE ACTION OF ACOUSTIC WAVES.

ii) FOR $\frac{P_0}{P_a} \geq \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}} = 1.893$ ($\gamma = 1.4$), THE NOZZLE IS CHOKED, IN THAT $Ma_2 = 1$ (SONIC CONDITIONS ARE REACHED AT THE OUTLET) AND $\dot{m} = \dot{m}^* = P_0 \sqrt{\frac{\gamma}{R_g T_0}} A_2 \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$ IS THE MAXIMUM FLOW RATE.



iii) FOR $\frac{P_0}{P_a} > \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$, THE EXIT PRESSURE P_2 IS LARGER THAN P_a

2. SHOCKS AND EXPANSION WAVES

SURFACES OF DISCONTINUITY

THE EULER EQUATIONS SUPPORT SOLUTIONS THAT CAN DISPLAY DISCONTINUITIES IN THE AERO-THERMODYNAMIC VARIABLES OR THEIR DERIVATIVES. THE REGIONS OVER WHICH THESE DISCONTINUITIES OCCUR ARE THIN COMPARED TO FLUID-MECHANICAL SCALES AND ARE CALLED "SURFACES OF DISCONTINUITY". THIS TERM ENGLOBES SHEAR LAYERS, WAKES, MIXING LAYERS, BOUNDARY LAYERS, SHOCK WAVES AND MACH SURFACES. IN GENERAL, IN NON-STEADY GAS FLOW THE SURFACES OF DISCONTINUITY DO NOT REMAIN FIXED, ALTHOUGH THEIR PROPAGATION VELOCITY IS UNRELATED TO THE VELOCITY OF THE GAS.

CONSIDER A GENERAL SURFACE OF DISCONTINUITY AND A REFERENCE FRAME ATTACHED TO IT.

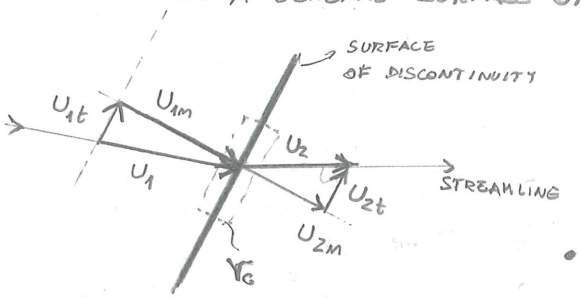


FIG. 10

THE INTEGRATION OF MASS (11), MOMENTUM (12) AND TOTAL ENTHALPY (14) IN THE CONTROL VOLUME V_c (SEE FIG. 10) GIVES THE FOLLOWING JUMP CONDITIONS

- $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \Rightarrow \frac{d}{dt}\int_{V_c} \rho \, dV + \int_{S_c} \rho \vec{v} \cdot d\vec{S} = 0$

$$\Rightarrow \boxed{\rho_1 U_{1m} = \rho_2 U_{2m} = \dot{m}'' = \text{const}} \quad (39)$$

CONTINUITY OF MASS FLUX PU AREA $\dot{m}'' = \text{CONST}$

- $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla P \Rightarrow \frac{d}{dt}\int_{V_c} \rho \vec{v} \, dV + \int_{S_c} \rho \vec{v}\vec{v} \cdot d\vec{S} = -\int_{S_c} P \, d\vec{S}$

CONTINUITY OF MOMENTUM FLUX

$$\Rightarrow \boxed{\rho_1 U_{1m}^2 + P_1 = \rho_2 U_{2m}^2 + P_2} \quad (40)$$ IN THE NORMAL DIRECTION, AND

$$\Rightarrow \boxed{\rho_1 U_{1m} U_{1t} = \rho_2 U_{2m} U_{2t}} \quad (41)$$ IN THE TANGENTIAL DIRECTION.

- $\rho \frac{\partial}{\partial t}\left(h + \frac{|\vec{v}|^2}{2}\right) + \rho \vec{v} \cdot \nabla\left(h + \frac{|\vec{v}|^2}{2}\right) = \frac{\partial P}{\partial t} + \dot{Q} \Rightarrow \int_{S_c} \rho \vec{v} \cdot \left(h + \frac{|\vec{v}|^2}{2}\right) d\vec{S} = 0$

CONTINUITY OF TOTAL ENTHALPY

$$\Rightarrow \boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}} \quad (42).$$

EQUATIONS (39) - (42) REPRESENT A COMPLETE SET OF BOUNDARY CONDITIONS AT A SURFACE OF DISCONTINUITY. FROM THEM, TWO TYPES OF SURFACE OF DISCONTINUITY CAN BE DEDUCED:

- TANGENTIAL DISCONTINUITIES: THESE DO NOT HAVE MASS FLUX THROUGH THE SURFACE ($\dot{m} = 0$) THEN $\boxed{U_{1m} = U_{2m} = 0}$ AND $\boxed{P_1 = P_2}$. NOTE, HOWEVER, THAT $\rho_1 \neq \rho_2$, $T_1 \neq T_2$ AND $U_{1t} \neq U_{2t}$. AN EXAMPLE OF A TANGENTIAL DISCONTINUITY IS A SHEAR LAYER IN AN EXHAUST JET. (FIG 11). NOTE THAT AT FINITE REYNOLDS NUMBERS, MIXING PROCESSES OCCUR THAT PRODUCE ENTRAINMENT INTO THE SHEAR LAYER AND THERMAL EQUILIBRATION, BUT THESE WILL NOT BE CONSIDERED HERE.

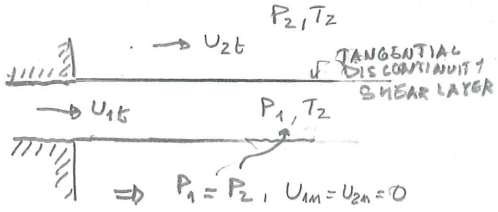
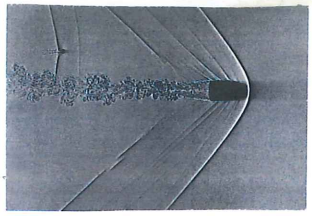
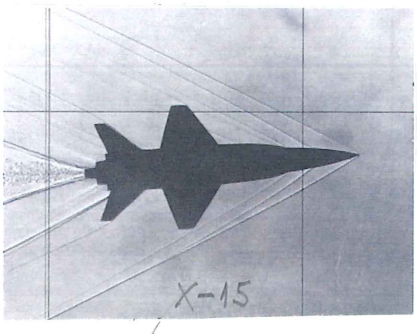


FIG. 11

– NORMAL DISCONTINUITIES : IN THESE THE MASS FLUX IS NOT ZERO, $\dot{m}'' \neq 0$, AND THEREFORE $U_{1n} \neq 0$ AND $U_{2n} \neq 0$. HOWEVER, THE COMBINATION OF (39) AND (41) LEADS TO $\boxed{U_{1t} = U_{2t}}$ BY WHICH THE TANGENTIAL VELOCITY IS CONTINUOUS. A DISCONTINUITY OF THIS TYPE IS A SHOCK WAVE.



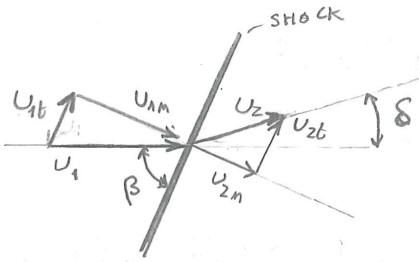


THE JUMP CONDITIONS ACROSS A SHOCKWAVE ARE:

$$\begin{aligned} \rho_1 U_{1n} &= \rho_2 U_{2n} \\ \rho_1 U_{1n}^2 + P_1 &= \rho_2 U_{2n}^2 + P_2 \\ U_{1t} &= U_{2t} \\ h_1 + \frac{U_1^2}{2} &= h_2 + \frac{U_2^2}{2} \end{aligned} \qquad (43)$$

NOTE THAT THE STAGNATION OR TOTAL ENTHALPY IS CONSERVED ACCROSS A SHOCK WAVE, ALTHOUGH THE STAGNATION PRESSURE AND DENSITY ARE NOT.

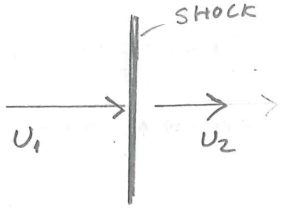
SHOCK WAVES



(OBLIQUE)

δ = STREAMLINE DEFLECTION ANGLE

β = ANGLE OF INCIDENCE



(NORMAL)

FIG. 11

$\beta = \frac{\pi}{2}$

CONSIDER FIRST THE CASE OF NORMAL SHOCK WAVES (THE OBLIQUE CASE IS ADDRESSED FURTHER BELOW). FOR NORMAL SHOCK WAVES, EQS. (43) BECOME

$$\left.\begin{aligned} \rho_1 U_1 &= \rho_2 U_2 = \dot{m}'' \\ \rho_1 U_1^2 + P_1 &= \rho_2 U_2^2 + P_2 \\ h_1 + \frac{U_1^2}{2} &= h_2 + \frac{U_2^2}{2} \end{aligned}\right\}$$

SUBSTITUTING THE FIRST INTO THE SECOND:

$$\frac{\dot{m}''^2}{\rho_1} + P_1 = \frac{\dot{m}''^2}{\rho_2} + P_2 \Rightarrow \boxed{\frac{P_2 - P_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} = -\dot{m}''^2} \qquad (44)$$

"RAYLEIGH'S LINE"

FROM THE THIRD ONE: $h_2 - h_1 = \frac{U_1^2}{2} - \frac{U_2^2}{2} = \frac{U_1^2}{2}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right) = \frac{\dot{m}''^2}{2\rho_1^2}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right)$,

AND USING (44): $\boxed{h_2 - h_1 = \frac{1}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)(P_2 - P_1)} \qquad (45)$ "HUGONIOT EQUATION"

EQUATIONS (44) AND (45) ARE FUNDAMENTAL TO THE TREATMENT OF SHOCK WAVES.

IN PARTICULAR, EQUATION (44) STATES THAT SOLUTIONS OF THE TYPES

$P_2 > P_1$ AND $\rho_2 > \rho_1$ (COMPRESSION WAVE)

OR

$P_1 > P_2$ AND $\rho_1 > \rho_2$ (DISCONTINUOUS EXPANSION WAVE)

ARE POSSIBLE SINCE $\dot{m}'' > 0$, BUT IT IS SHOWN BELOW THAT ONLY THE FORMER CASE CAN OCCUR.

SIMILAR TO THE OTHER AEROTHERMODYNAMIC QUANTITIES, THE ENTROPY IS ALSO DISCONTINUOUS IN A SHOCK WAVE. BECAUSE OF THE LAW OF ENTROPY INCREASE, THE ENTROPY OF A GAS CAN ONLY INCREASE DURING ITS MOTION. THEREFORE, THE ENTROPY S_2 OF THE GAS THAT HAS PASSED THROUGH THE SHOCK WAVE MUST EXCEED ITS INITIAL ENTROPY S_1, $S_2 > S_1$.

RECALLING EXPRESSION (15) FOR THE ENTROPY OF A PERFECT GAS:

THE FLOW ACROSS A SHOCK IS NOT ISENTROPIC!

$$S_2 - S_1 = C_v \ln\left[\frac{P_2}{P_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right] \overset{S_2>S_1}{\Rightarrow} \frac{P_2}{\rho_2^{\gamma}} > \frac{P_1}{\rho_1^{\gamma}} \Rightarrow \frac{P_2}{P_1} > \left(\frac{\rho_2}{\rho_1}\right)^{\gamma} \quad (46)$$

THE ENTROPY-INCREASE CONSTRAINT (46), TOGETHER WITH THE HUGONIOT EQUATION (45), SHOW THAT ONLY COMPRESSION WAVES ($P_2 > P_1$, $\rho_2 > \rho_1$) CAN EXIST IN (NON-REACTING*) NORMAL DISCONTINUITIES. TO SEE THIS, REWRITE EQ (45) IN TERMS OF $\mathcal{P} = \frac{P_2}{P_1}$ AND $\mathcal{V} = \frac{\rho_1}{\rho_2}$,

$$\boxed{\mathcal{P} = \frac{\frac{\gamma+1}{\gamma-1} - \mathcal{V}}{\frac{\gamma+1}{\gamma-1}\mathcal{V} - 1}} \quad (47)$$

HUGONIOT EQUATION IN DIMENSIONLESS FORM.

IT IS EASY TO SEE THAT THE HUGONIOT CURVE IS BELOW THE ISENTROPIC ($\mathcal{P} = 1/\mathcal{V}^{\gamma}$) FOR $\mathcal{P} < 1$ AND $\mathcal{V} > 1$, AND THEREFORE DESCRIBES THERMODYNAMIC PROCESSES FORBIDDEN BY THE 2ND PRINCIPLE. THE COMPLETE SPACE OF SOLUTIONS OF THE RANKINE-HUGONIOT EQUATIONS IS SUMMARIZED IN THE CHART BELOW.

$\rho_1, u_1 \rightarrow \rho_2, u_2$

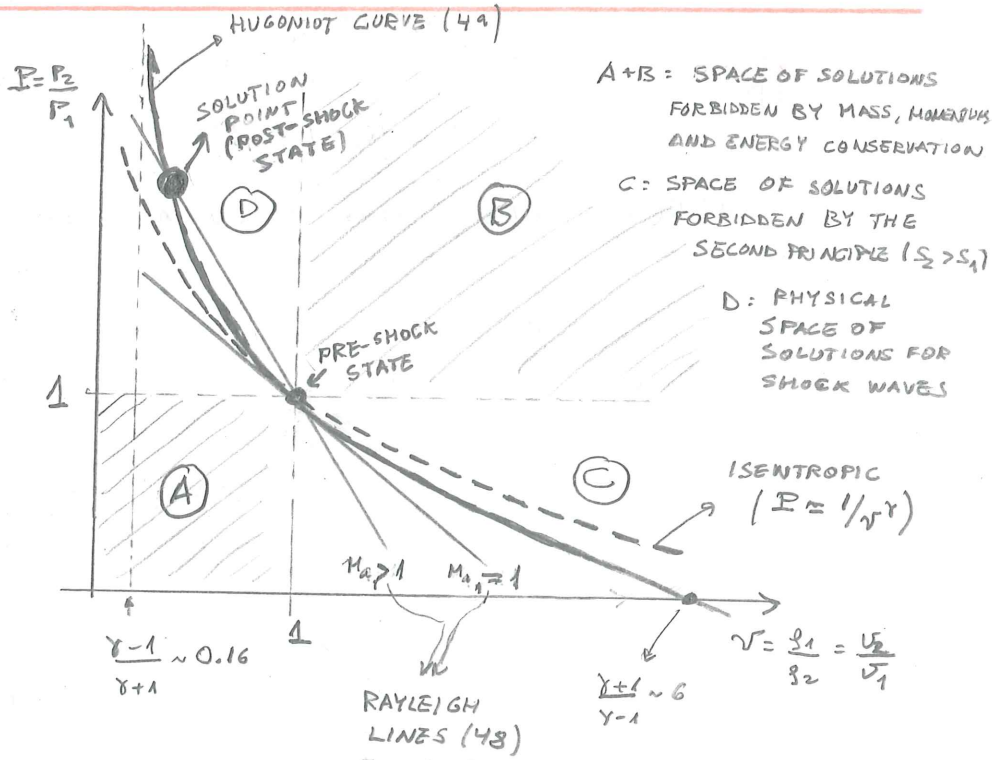
RANKINE-HUGONIOT EQS. FOR NORMAL SHOCK WAVES:

$$\mathcal{P} = \frac{P_2}{P_1}, \quad \mathcal{V} = \frac{\rho_1}{\rho_2} = \frac{u_2}{u_1}, \quad \mu = \frac{(\rho_1 u_1)^2}{P_1 \rho_1} = \frac{u_1^2}{P_1/\rho_1} = \gamma M_{a_1}^2$$

NOT VISCOSITY!

$$\Rightarrow \boxed{\frac{\mathcal{P}-1}{\mathcal{V}-1} = -\mu} \quad (48)$$

$$\boxed{\mathcal{P} = \frac{\frac{\gamma+1}{\gamma-1} - \mathcal{V}}{\frac{\gamma+1}{\gamma-1}\mathcal{V} - 1}} \quad (49)$$



A+B: SPACE OF SOLUTIONS FORBIDDEN BY MASS, MOMENTUM AND ENERGY CONSERVATION

C: SPACE OF SOLUTIONS FORBIDDEN BY THE SECOND PRINCIPLE ($S_2 > S_1$)

D: PHYSICAL SPACE OF SOLUTIONS FOR SHOCK WAVES

THE UPSTREAM/DOWNSTREAM STATES OF THE GAS ARE DETERMINED BY THE INTERSECTION POINT BETWEEN THE RAYLEIGH LINE AND THE HUGONIOT CURVE.

THE SHOCK JUMP CONDITIONS AS A FUNCTION OF THE UPSTREAM MACH NUMBER ARE OBTAINED BY SUBSTITUTING (49) INTO (48), WHICH GIVES

$$\frac{1}{\mathcal{V}} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{a_1}^2}{2+(\gamma-1)M_{a_1}^2} \qquad (50)$$

SUBSTITUTING (50) INTO (48) AGAIN, THE JUMP CONDITION FOR THE PRESSURE

$$\mathcal{P} = \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(M_{a_1}^2 - 1\right) \qquad (51)$$

IS OBTAINED. SIMILARLY, USING THE EQUATION OF STATE:

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{\left[2\gamma M_{a_1}^2 - (\gamma-1)\right]\left[2+(\gamma-1)M_{a_1}^2\right]}{(\gamma+1)^2 M_{a_1}^2} \qquad (52)$$

THE JUMP CONDITION FOR THE VELOCITY IS OBTAINED USING THE EQUATION OF CONTINUITY

$$\frac{U_2}{U_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)M_{a_1}^2}{(\gamma+1)M_{a_1}^2} \qquad (53)$$

THE MACH NUMBER DOWNSTREAM OF THE SHOCK WAVE IS

$$M_{a_2}^2 = \left(\frac{U_2}{a_2}\right)^2 = \left(\frac{U_1}{a_1}\frac{U_2}{U_1}\frac{a_1}{a_2}\right)^2 = M_{a_1}^2\left(\frac{U_2}{U_1}\right)^2\left(\frac{T_1}{T_2}\right) = \frac{2+(\gamma-1)M_{a_1}^2}{2\gamma M_{a_1}^2 + 1 - \gamma} \qquad (54)$$

WHILE THE JUMP IN THE STAGNATION PRESSURE AND DENSITY IS

$$\frac{P_{02}}{P_{01}} = \frac{\rho_{02}}{\rho_{01}} = \left(\frac{1+\frac{(\gamma-1)}{2}M_{a_2}^2}{1+\frac{(\gamma-1)}{2}M_{a_1}^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{P_2}{P_1} \qquad (55)$$

EQUATIONS (50) - (55) ARE PLOTTED BELOW AS A FUNCTION OF M_{a_1}

NOTE THAT

$$\frac{\rho_{02}}{\rho_{01}} = \frac{\rho_{02}}{\rho_2}\frac{\rho_2}{\rho_1}\frac{\rho_1}{\rho_{01}}$$

$$\left(1+\frac{(\gamma-1)}{2}M_{a_2}^2\right)^{\frac{1}{\gamma-1}}$$

$$\left(1+\frac{(\gamma-1)}{2}M_{a_1}^2\right)^{\frac{-1}{1-\gamma}}$$

$$1+\frac{\gamma-1}{2}M_{a_2}^2$$

AND

$$\frac{\rho_2}{\rho_1} = \frac{P_2}{P_1}\frac{T_1}{T_{01}}\frac{T_{01}}{T_2}$$

$$\frac{1}{1+\frac{(\gamma-1)}{2}M_{a_1}^2}$$

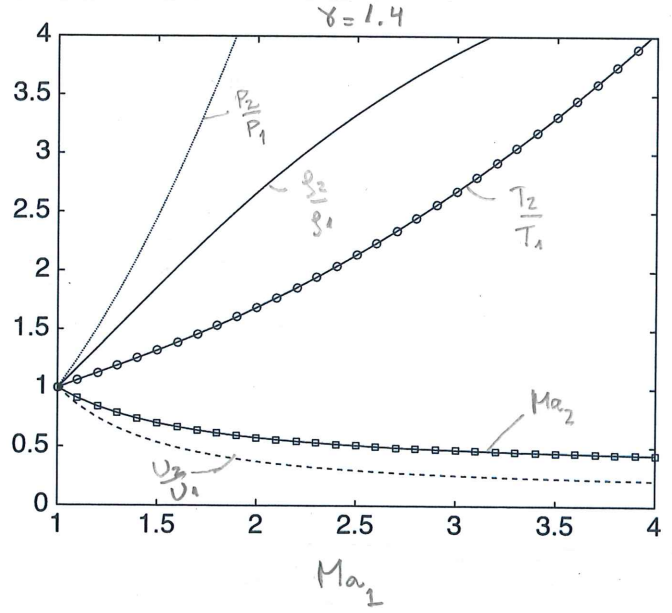


FIG. 12

NOTE:

$\frac{P_2}{P_1} > 1$ $\quad$ $\frac{P_{02}}{P_{01}} < 1$

$\frac{\rho_2}{\rho_1} > 1$ $\quad$ $\frac{\rho_{02}}{\rho_{01}} < 1$

$\frac{T_2}{T_1} > 1$ $\quad$ $\frac{T_{02}}{T_{01}} = 1$

$\frac{U_2}{U_1} < 1$ $\quad$ $\frac{S_2}{S_1} > 1$

$M_{a_2} < 1$

→ THE FLOW BEHIND A NORMAL SHOCK WAVE (IN THE SHOCK FRAME) IS SUBSONIC.

CONSIDER AGAIN FIG. 11, THIS TIME FOR OBLIQUE SHOCKS ($\beta \neq \frac{\pi}{2}$).

FROM THE TRIANGLES OF VELOCITIES IT IS EASY TO VERIFY THAT

$$\left.\begin{aligned} U_{1m} &= U_1 \sin\beta \\ U_{2m} &= U_2 \cos\left(\frac{\pi}{2}-\beta+\delta\right) = U_2 \sin\left(\beta-\delta\right) \end{aligned}\right\} \quad \left.\begin{aligned} U_{1t} &= U_1 \cos\beta \\ U_{2t} &= U_2 \cos\left(\beta-\delta\right) \end{aligned}\right. \tag{56}$$

FOR THE NORMAL AND TANGENTIAL COMPONENTS OF THE VELOCITY. SIMILARLY,

$$\boxed{\begin{aligned} Ma_{1m} &= \frac{U_{1m}}{a_1} = Ma_1 \sin\beta \\ Ma_{2m} &= \frac{U_{2m}}{a_2} = Ma_2 \sin\left(\beta-\delta\right) \end{aligned}} \tag{57}$$

FOR THE MACH NUMBERS OF THE NORMAL FLOW. NOTE THAT THE ENERGY EQUATION

$$h_1 + \frac{U_1^2}{2} = h_2 + \frac{U_2^2}{2}$$

SIMPLIFIES TO

$$h_1 + \frac{U_{1m}^2}{2} = h_2 + \frac{U_{2m}^2}{2}$$

BECAUSE OF CONTINUITY OF THE TANGENTIAL VELOCITIES. IN THIS WAY, THE CONSERVATION EQUATIONS FOR OBLIQUE SHOCK WAVES ARE

$$\begin{aligned} \rho_1 U_{1m} &= \rho_2 U_{2m} \\ \rho_1 U_{1m}^2 + P_1 &= \rho_2 U_{2m}^2 + P_2 \\ h_1 + \frac{U_{1m}^2}{2} &= h_2 + \frac{U_{2m}^2}{2} \end{aligned}$$

WHICH ARE IDENTICAL TO THE ONES FOR NORMAL SHOCKS IF U_1 IS REPLACED BY U_{1m}. THEREFORE, THE RANKINE-HUGONIOT EQUATIONS (48)-(49) AND THE JUMP CONDITIONS (50)-(55) CAN ALSO BE USED FOR OBLIQUE SHOCKS IF Ma_1 IS REPLACED BY M_{1m}:

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,Ma_{1m}^2}{2+(\gamma-1)\,Ma_{1m}^2}, \qquad \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_{1m}^2-1\right),$$

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{\left[2\gamma Ma_{1m}^2-(\gamma-1)\right]\left[2+(\gamma-1)Ma_{1m}^2\right]}{(\gamma+1)^2 Ma_{1m}^2}$$

$$\frac{U_{2m}}{U_{1m}} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)Ma_{1m}^2}{(\gamma+1)Ma_{1m}^2}, \qquad Ma_{2m}^2 = \frac{2+(\gamma-1)Ma_{1m}^2}{2\gamma Ma_{1m}^2+1-\gamma},$$

$$\frac{P_{02}}{P_{01}} = \frac{\rho_{02}}{\rho_{01}} = \left(\frac{1+\frac{(\gamma-1)}{2}Ma_2^2}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right)^{\frac{\gamma}{\gamma-1}} \frac{P_2}{P_1} \tag{58}$$

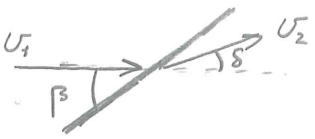
JUMP CONDITIONS FOR OBLIQUE SHOCK WAVES

NOTE P_{02} GIVES P_{01} THE SAME RESULT IF Ma_{1m} AND Ma_{2m} ARE USED INSTEAD OF Ma_1 AND Ma_2

A RELATION BETWEEN THE INCIDENCE ANGLE β AND THE ANGLE OF STREAMLINE DEFLECTION δ CAN BE OBTAINED BY USING THE GEOMETRICAL RELATION

$$\frac{U_{2n}}{U_{1n}} = \frac{U_{2n}/U_{2t}}{U_{1n}/U_{1t}} = \frac{\tan(\beta-\delta)}{\tan\beta}$$

$U_{2n}/U_{2t} = U_{1n}/U_{1t}$ ↓ CONTINUITY OF U_t



ALONG WITH THE JUMP CONDITION IN (58) FOR U_{2n}/U_{1n}, WHICH GIVES

$$\frac{\tan(\beta-\delta)}{\tan\beta} = \frac{2+(\gamma-1)M_{a_1}^2\sin^2\beta}{(\gamma+1)M_{a_1}^2\sin^2\beta} \Rightarrow \boxed{\tan\delta = \frac{2\cot\beta\,(M_{a_1}^2\sin^2\beta - 1)}{2+M_{a_1}^2\left[\gamma+\cos(2\beta)\right]}} \quad (59)$$

SINCE THE CONTOURS OF EQ. (59) ARE GIVEN SCHEMATICALLY IN FIG. 13

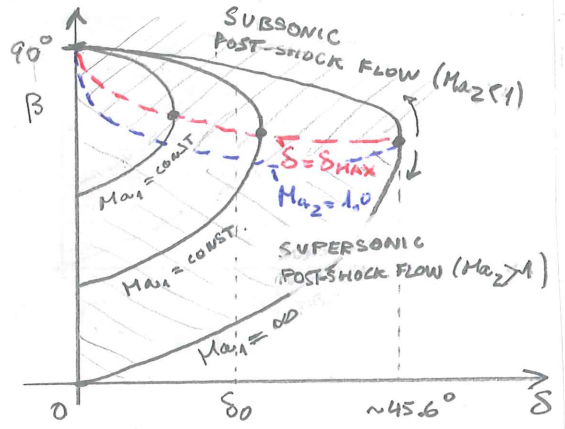
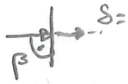


FIG. 13

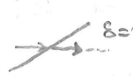
COMMENTS:

- FOR A GIVEN DEFLECTION ANGLE $\delta = \delta_0$ AND A MACH NUMBER M_{a_1}, THERE EXIST TWO SOLUTIONS FOR β: ONE FOR β LARGE (STRONG SHOCK WAVE), AND ANOTHER ONE FOR SMALLER β (WEAK SHOCKWAVE). STRONG SHOCK WAVES ARE CHARACTERIZED BY $M_{a_2} < 1$, (FLOW IS SUBSONIC DOWNSTREAM)

- NO DEFLECTION ($\delta = 0$) OCCURS IN TWO CONDITIONS:



i) IF $\beta = 90°$ ($\cot\beta = 0$) ⇒ NORMAL SHOCK WAVE ($M_{a_2} < 1$)



ii) IF $M_{a_1}^2\sin^2\beta = 1$; OR EQUIVALENTLY, $\sin\beta = \pm 1/M_{a_1} = \sin\mu$ (MACH ANGLE)

⇒ THIS IS A COMPRESSION MACH WAVE (WEAK SHOCKWAVE). SEE FIG. 4b.

NOTE THAT $M_{a_1}^2\sin^2\beta = 1$ IMPLIES $M_{a_{1n}} = 1$ ⇒ SONIC NORMAL FLOW, SHOCK OF VANISHING STRENGTH

- CONSIDER THE WEDGE IN FIG. 14. FOR $\delta > \delta_{MAX}$ THERE IS NO SOLUTION WITH A SHOCK ATTACHED TO THE SHOCK. INSTEAD, THE SHOCK MUST BE DETACHED FROM THE WEDGE, WITH THE SOLUTION DEPENDING ON THE GEOMETRY AND BOUNDARY CONDITIONS.

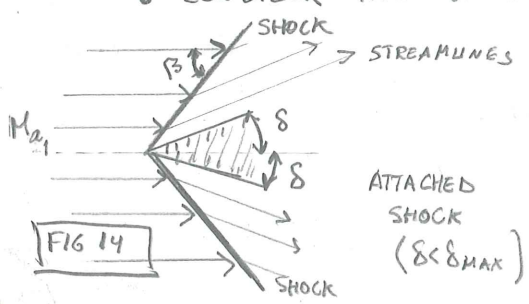
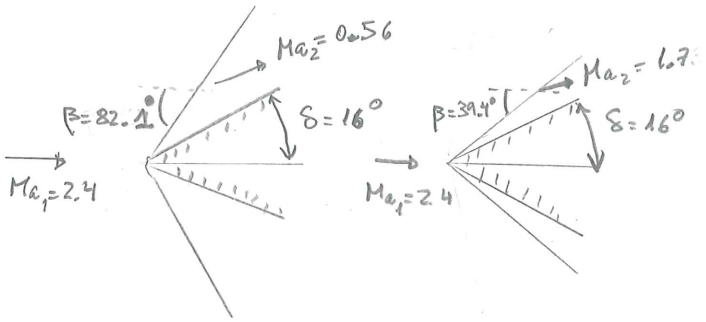


FIG 14



- AS MENTIONED ABOVE, FOR A GIVEN $\delta < \delta_{MAX}$ THERE ARE 2 SOLUTIONS FOR β, ONE REPRESENTS A STRONG SHOCK WAVE ($Ma_2 < 1$), AND THE OTHER IS A WEAK SHOCK (TYPICALLY $Ma_2 > 1$). STRONG.

EXAMPLES:



STRONG SHOCK SOLUTION

$\frac{P_2}{P_1} = 6.3$

WEAK SHOCK SOLUTION

$\frac{P_2}{P_1} = 2.52$

THE STRONG SHOCK SOLUTION YIELDS A LARGER BACKPRESSURE $\frac{P_2}{P_1}$ THAN THE WEAK SHOCK ONE. IN EXTERNAL AERODYNAMICS, STRONG SHOCKS ARE RARELY OBSERVED EXCEPT FOR $\delta > \delta_{MAX}$, WHERE SEGMENTS OF STRONG SHOCKS CAN BE SEEN NEAR THE WEDGE NOSE WITH SUBSONIC FLOW DOWNSTREAM (SEE FIG. 14). FOR ATTACHED SHOCKS, WEAK SHOCKS ARE TYPICALLY OBSERVED DUE TO THE TENDENCY OF THE BACKPRESSURE TO BECOME EQUALIZED IN OPEN ENVIRONMENTS WITH THE AMBIENT PRESSURE. THIS MIGHT NOT BE SO IN INTERNAL FLOWS, WHERE A HIGHER BACK PRESSURE COULD BE SUSTAINED.

WEAK SHOCKS

WEAK SHOCKS ARE CHARACTERIZED BY HAVING A VANISHING STRENGTH, $Ma_1 \sim 1 + \varepsilon$ WITH $\varepsilon \ll 1$. INTRODUCING THIS APPROXIMATION IN THE JUMP CONDITIONS (50), (51) AND (53) YIELDS

$$\frac{\rho_2}{\rho_1} = 1 + \frac{2}{\gamma+1}\varepsilon, \quad \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\varepsilon, \quad \text{AND} \quad \frac{U_2}{U_1} = 1 - \frac{2}{\gamma+1}\varepsilon$$

TO LEADING ORDER IN ε. NOTE THAT THE ENTROPY VARIATION IS OF HIGHER ORDER IN ε,

$$s_2 - s_1 = c_v \ln\left[\frac{P_2}{P_1}\left(\frac{\rho_2}{\rho_1}\right)^{-\gamma}\right] \sim c_v \frac{2}{3}\frac{\gamma(\gamma-1)}{(1+\gamma)^2}\varepsilon^3,$$

THEREFORE WEAK SHOCKS ARE ESSENTIALLY ISENTROPIC COMPRESSION WAVES OF VANISHING STRENGTH. WEAK OBLIQUE SHOCKS ARE ALSO CHARACTERIZED BY A VANISHING DEFLECTION ANGLE. THIS CAN BE SEEN BY REARRANGING EQ. (59) AS

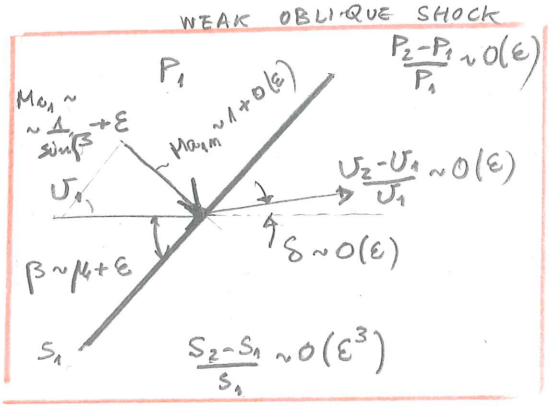
$$Ma_1^2 \sin^2\beta - 1 = \frac{(\gamma+1)}{2} Ma_1^2 \frac{\sin\beta\cos\delta}{\cos(\beta-\delta)}$$, WHICH FOR $\delta \to 0$ BECOMES

$$Ma_1^2 \sin^2\beta - 1 \simeq \left(\frac{\gamma+1}{2} Ma_1^2 \tan\beta\right)\delta \quad (60) \qquad (\text{I.E. } \delta \sim O(\varepsilon))$$

NOTE THAT WHEN $\delta \to 0$, THE INCIDENCE ANGLE β APPROACHES THE MACH ANGLE : $Ma_1^2 \sin^2\beta = 1 \Rightarrow \sin\beta \sim \sin\mu = \frac{1}{Ma_1}$, MEANING THAT WEAK SHOCKS ARE MACH LINES IN THE FIRST APPROXIMATION. THIS CAN BE EXPRESSED EQUIVALENTLY AS

$$\tan\beta \simeq \tan\mu = \frac{1}{\sqrt{Ma_1^2 - 1}}$$

WEAK OBLIQUE SHOCK



WHICH CAN BE SUBSTITUTED IN (60) TO GIVE

$$Ma_1^2 \sin^2\beta - 1 \simeq \frac{\gamma+1}{2} \frac{Ma_1^2}{\sqrt{Ma_1^2-1}} \delta \quad (61)$$

SUBSTITUTING THIS RELATION INTO THE JUMP CONDITION (58) FOR P_2/P_1 :

$$\frac{P_2 - P_1}{P_1} = \frac{2\gamma}{\gamma+1}\left(Ma_1^2 \sin^2\beta - 1\right) \simeq \frac{\gamma Ma_1^2}{\sqrt{Ma_1^2-1}} \delta \quad (62)$$

WHICH SHOWS THAT THE STRENGTH OF THE WAVE IS PROPORTIONAL TO THE DEFLECTION ANGLE δ. SIMILARLY, IT CAN BE SHOWN THAT THE VARIATION IN ENTROPY IS OF HIGHER ORDER, $\Delta s/c_v \sim O(\delta^3)$.

OBLIQUE WEAK SHOCKS ARISE, FOR INSTANCE, IN GRADUAL SUPERSONIC COMPRESSION, SUCH AS THE RAMP FLOW SHOWN IN FIG 14. THERE, THE TURNING OF THE FLOW IS ACHIEVED THROUGH LARGE NUMBER OF WEAK OBLIQUE SHOCKS. AWAY FROM THE WALL, THE WEAK OBLIQUE SHOCKS MAY COALESCE AND FORM A STRONG SHOCK. THUS, IF THE COMPRESSION IS ACHIEVED THROUGH A LARGE NUMBER OF WEAK SHOCKS, THE ENTROPY INCREASE CAN BE REDUCED IN COMPARISON WITH A SINGLE STRONG SHOCK.

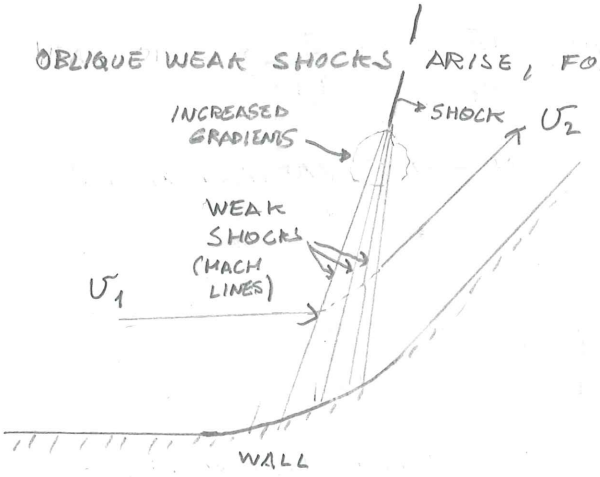
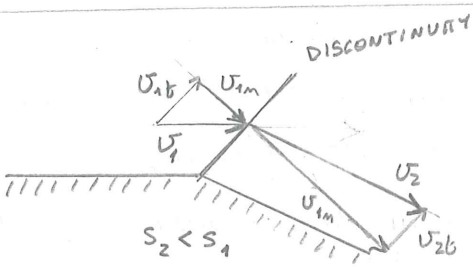


FIG:14. SMOOTH CONTINUOUS COMPRESSION

PRANDTL-MEYER EXPANSION FANS

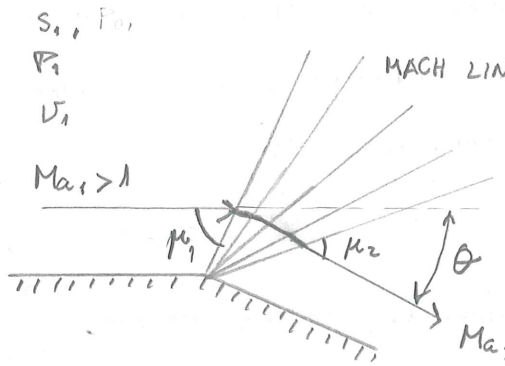
THE EXPANSION OF A SUPERSONIC FLOW CAN BE ACHIEVED THROUGH A CONVEX CORNER. AS STATED PREVIOUSLY, THERE CANNOT BE SHARP DISCONTINUITIES IN THE FORM OF EXPANSION WAVES BECAUSE OF THE 2ND PRINCIPLE OF THERMODYNAMICS. CONVERSELY, EXPANSION WAVES ARE ISENTROPIC AND OCCUPY A FINITE-THICKNESS REGION OF THE FLOW FIELD. IN CONTRAST TO THE SUPERSONIC WEAK COMPRESSION WAVES IN FIG. 14,

→ NOTE THAT THE STAGNATION PROPERTIES ARE CONSERVED ACROSS!



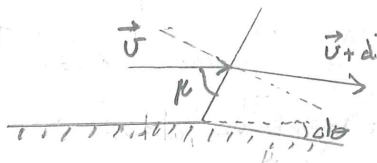
(a) SHARP EXPANSION WAVE NOT POSSIBLE ON THERMODYNAMIC GROUNDS (REGION C IN THE RANKINE-HUGONIOT DIAGRAM.

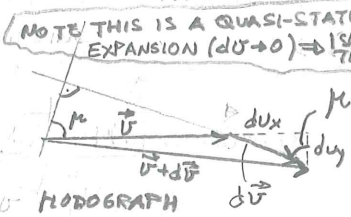
HERE THE MACH LINES ARE DIVERGENTLY EMANATING FROM THE CONVEX CORNER, AND CONSEQUENTLY THERE IS A TENDENCY TO DECREASE GRADIENTS, THE FLOW PARAMETERS ARE CONSTANT ALONG EACH MACH LINE.



(b) EXPANSION FAN:

IN ORDER TO ANALYZE THE FLOW THROUGH THE EXPANSION FAN, CONSIDER THE ELEMENTARY VARIATION OF THE TURNING ANGLE θ ACROSS A MACH LINE

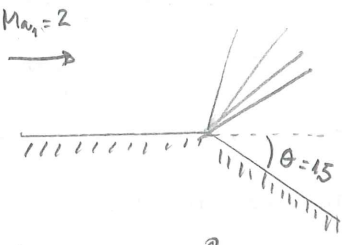




$$du_x \approx dU \quad du_y \approx U\,d\theta \quad \Big\} \quad \tan\mu = \frac{du_x}{du_y} = \frac{dU}{U\,d\theta} \Rightarrow d\theta = \sqrt{Ma^2-1}\,\frac{dU}{U}$$

$$\Rightarrow \text{THEREFORE } \theta = \int_{Ma_1}^{Ma_2} \sqrt{Ma^2-1}\,\frac{dU}{U} = \nu(Ma_2) - \nu(Ma_1)$$

EXAMPLE.



$$\frac{P_2}{P_1} = \frac{1+\frac{\gamma-1}{2}Ma_2^2}{1+\frac{\gamma-1}{2}Ma_1^2} = 0.41$$

$\nu(Ma_1) = 26.38°$

$\nu(Ma_2) = 41.38° = \nu(Ma_1) + \theta$ → $Ma_2 = 2.6$

IN ORDER TO EVALUATE THIS INTEGRAL, CONSIDER THAT $Ma = U/a$ AND $\frac{a_0^2}{a^2} = 1 + \frac{(\gamma-1)}{2}Ma^2$ (ISENTROPIC FLOW), AND THEREFORE

$$\frac{dU}{U} = \frac{dMa}{Ma} + \frac{da}{a} = \frac{dMa}{Ma}\left(\frac{1}{1+\frac{\gamma-1}{2}Ma^2}\right)$$

THE FUNCTION $\nu(Ma)$ (THE PRANDTL-MEYER FUNCTION) IS THEN GIVEN BY

$$\nu(Ma) = \int_1^{Ma} \frac{\sqrt{Ma^2-1}}{1+\frac{(\gamma-1)}{2}Ma^2}\frac{dMa}{Ma} = \sqrt{\frac{\gamma+1}{\gamma-1}}\arctan\sqrt{\frac{\gamma-1}{\gamma+1}(Ma^2-1)} - \arctan\sqrt{Ma^2-1}, \quad (63)$$

"PRANDTL-MEYER FUNCTION" ⇒ TABULATED IN TABLE III

WITH $\theta = \nu(Ma_2) - \nu(Ma_1)$

SUPERSONIC WAVE DRAG AND COMPRESSION LIFT

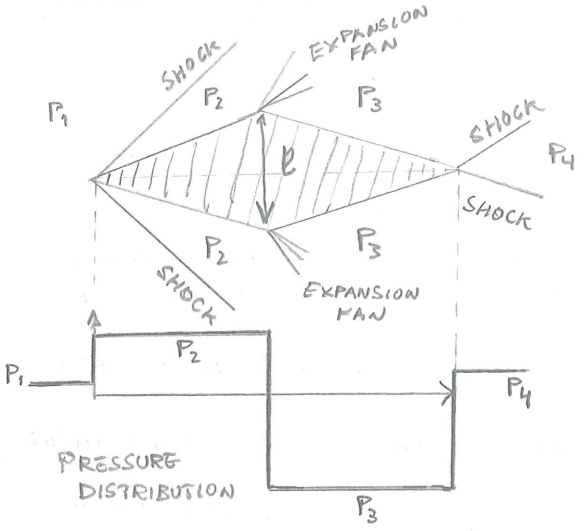


FIG 15: SUPERSONIC WAVE DRAG.

SUPERSONIC WAVE DRAG IS A COMPONENT OF THE AERODYNAMIC DRAG ON AIRCRAFT WINGS AND FUSELAGES OF PROJECTILES DUE TO THE PRESENCE OF SHOCK WAVES (AT SUPERSONIC AND TRANSONIC FLIGHT SPEEDS). NOTE THAT SUPERSONIC WAVE DRAG IS INDEPENDENT OF VISCOUS EFFECTS, AND IT IS DIFFERENT FROM THE FRICTIONAL DRAG AND SEPARATION DRAG THAT ARE ASSOCIATED WITH BOUNDARY LAYERS IN A VISCOUS FLUID.

CONSIDER THE DIAMOND-SECTION AIRFOIL IN FIG. 15. THE LEADING SHOCK COMPRESSES THE FLOW TO THE PRESSURE P_2, THE EXPANSION AT THE SHOULDER EXPANDS IT TO THE PRESSURE P_3, AND THE TRAILING SHOCK RECOMPRESSES IT TO NEARLY THE FREE STREAM VALUE P_4. THE SUPERSONIC WAVE DRAG IS CAUSED BY THE OVERPRESSURE ON THE FORWARD FACES AND THE UNDERPRESSURE ON THE REARWARD FACES. THE SUPERSONIC WAVE DRAG PER UNIT SPAN IS

$$D = (P_2 - P_3)\ell$$

WHERE ℓ IS THE THICKNESS OF THE AIRFOIL AT SHE SHOULDER. IT WILL BE SHOWN LATER THAT THE WAVE DRAG IS HIGHLY SENSITIVE TO THE RATE OF CHANGE OF THE CROSS SECTIONAL AREA OF THE BODY. FOR THAT REASON THE TOTAL CROSS-SECTIONAL AREA DISTRIBUTION OF HIGH SPEED AIRPLANES TENDS TO BE SMOOTH. (SEE FIG. 16)



FUSELAGE INDENTATION TO CONSERVE CROSS SECTIONAL AREA ↓ WHITCOMB'S AREA RULE

FIG. 16: F-102 DELTA-DAGGER SUPERSONIC AIRCRAFT.

THE EMERGENCE OF SUPERSONIC WAVE DRAG TENDS TO HAVE DRAMATIC CONSEQUENCES IN TRANSONIC REGIMES. (SEE FIGS. 17 AND 18).

UNEXPECTED EXTRA ENGINE POWER WAS REQUIRED TO OVERCOME THE DRAG IN THE EARLY ATTEMPTS TO BREAK THE SOUND BARRIER. THE TRANSONICITY AND WAVE DRAG ALSO BROUGHT ABOUT SEVERE PROBLEMS OF LOSS OF AIRPLANE CONTROL AND MANEUVERABILITY.

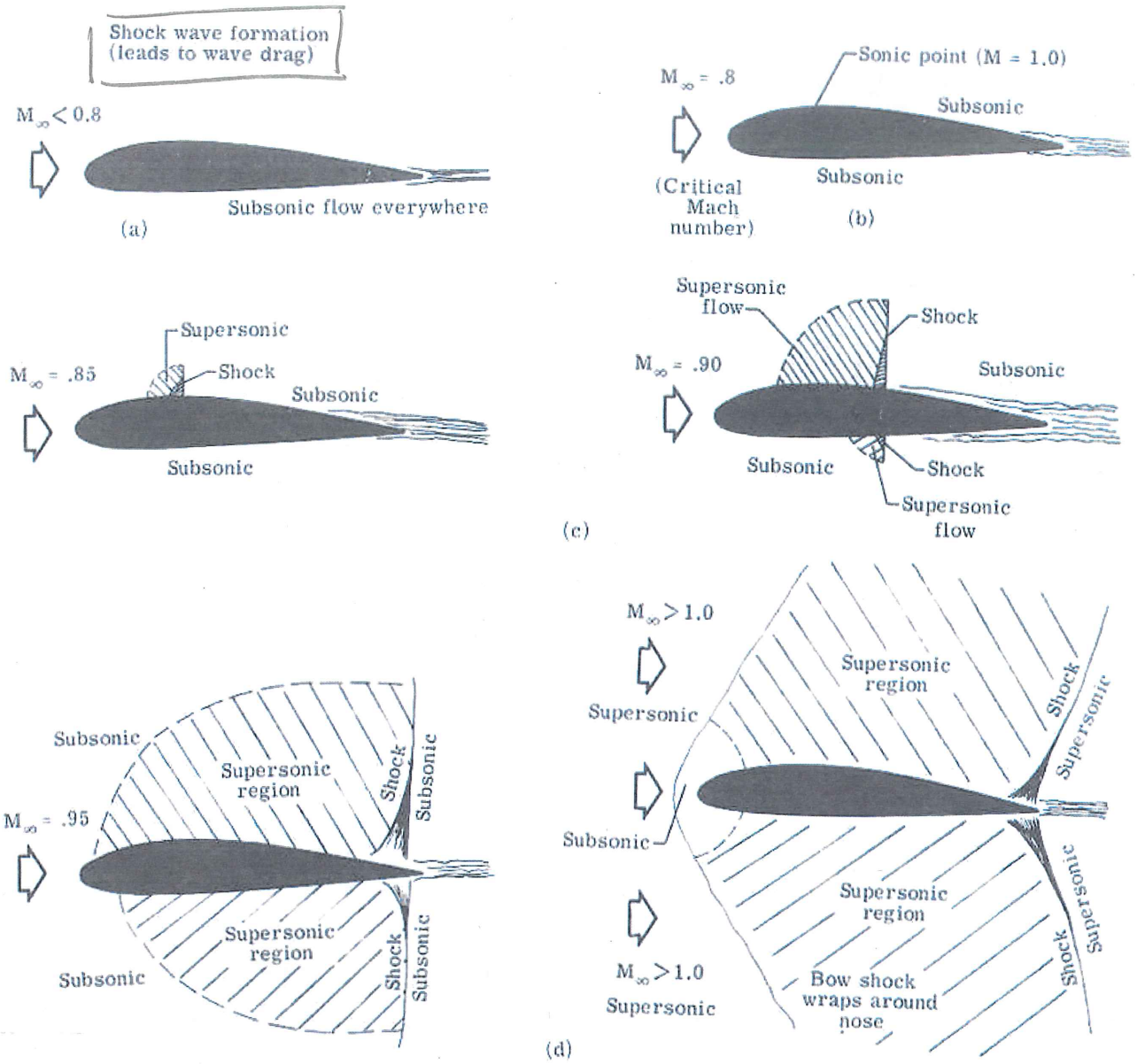


FIG. 17 : AIRFOIL IN TRANSONIC FLIGHT.

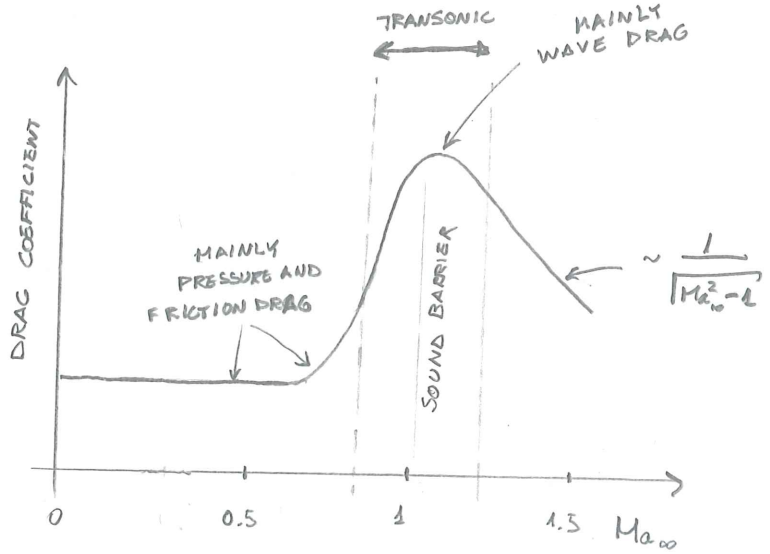


FIG. 18 : TYPICAL DRAG COEFFICIENT DEPENDENCE ON FREE-STREAM MACH NUMBER.

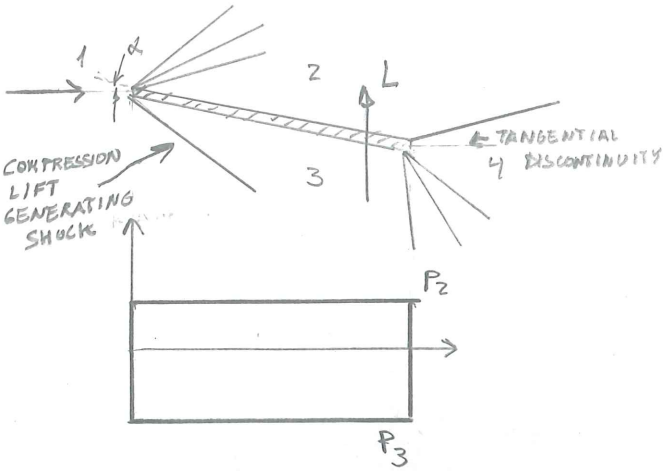


FIG 17. LIFTING FLAT PLATE IN A SUPERSONIC STREAM

A DIFFERENT BUT IMPORTANT ASPECT OF HIGH SPEED FLIGHT IS THE GENERATION OF COMPRESSION LIFT THROUGH THE UTILIZATION OF SHOCK WAVES CREATED BY THE FUSELAGE. AN EXAMPLE OF THIS PHENOMENON IS SHOWN IN FIG. 17 FOR A FLAT PLATE AT A SMALL ANGLE OF ATTACK. THE LIFT GENERATED ON THE PLATE IS

$$L = (P_3 - P_2)\, c \cos\alpha$$

WHERE C IS THE CHORD.

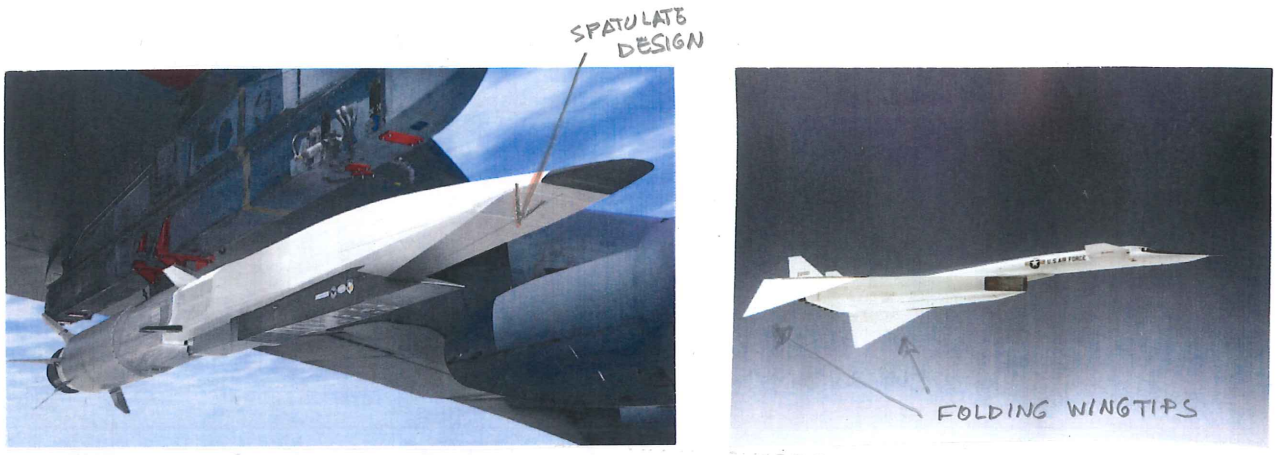


FIG 18. COMPRESSION LIFT IN AIRCRAFT DESIGN, HYPERSONIC WAVERIDERS (LEFT, BOEING X-51) AND SUPERSONIC BOMBERS (RIGHT, XB-70).

COMPRESSION LIFT HAS BEEN USED IN HIGH-SPEED AIRCRAFT DESIGNS. EXAMPLES OF THESE ARE SHOWN IN FIG. 18, SUCH AS "WAVERIDER" TECHNOLOGY TO INCREASE LIFT-TO-DRAG RATIOS AT HYPERSONIC SPEED THROUGH THE USE OF SPATULATE-LIKE DESIGNS, OR IN SUPERSONIC AIRCRAFTS THROUGH THE UTILIZATION OF FOLDING WINGTIPS FOR A BETTER CONTAINMENT OF THE OVERPRESSURE ON THE UNDERBELLY. THESE DESIGNS HAVE CONTRIBUTED TO BREAKING THE "LD BARRIER" THAT ARISES IN HYPERSONIC VEHICLES AS A RESULT OF INCREASED VISCOUS AND WAVE DRAGS AS M_∞ INCREASES

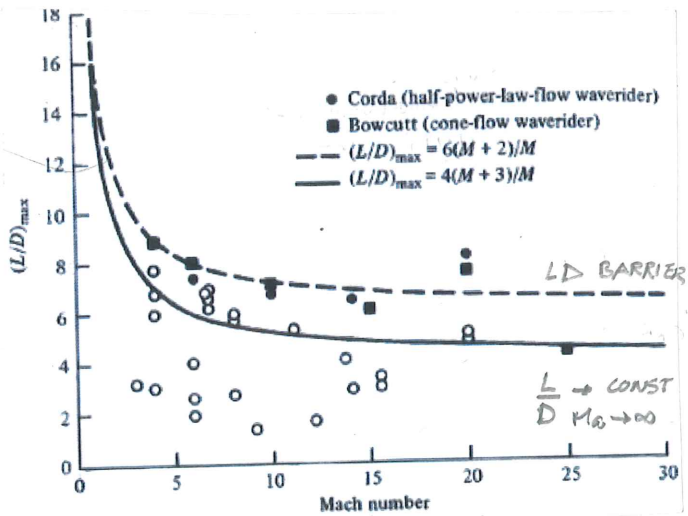


FIG. 19: LIFT-TO-DRAG RATIOS AS A FUNCTION OF MACH NUMBER FOR TYPICAL HYPERSONIC VEHICLES.

REACTING NORMAL DISCONTINUITIES

IN THE DISCUSSION OF THE RANKINE-HUGONIOT THEORY OF NORMAL DISCONTINUITIES (PAGE 21) IT WAS ARGUED THAT GASDYNAMICALLY THIN WAVES OF EXPANSION ($\mathcal{V}>1$, $\mathcal{P}<1$) WERE FORBIDDEN BY THE SECOND PRINCIPLE. SUCH CONCLUSIONS NEED TO BE REVISED IN THE PRESENCE OF CHEMICAL HEAT RELEASE IN REACTING MIXTURES, AS ENCOUNTERED IN MANY COMBUSTION APPLICATIONS. THE EXTENSION OF THE RANKINE-HUGONIOT THEORY TO COMBUSTION WAVES BEGINS BY REDEFINING THE ENTHALPY DIFFERENCE AS

$$h_2 - h_1 = \underbrace{c_p (T_2 - T_1)}_{\text{THERMAL ENTHALPY}} + \underbrace{\sum_{i=1}^{M} Y_{2i}\, h_i^0 - \sum_{i=1}^{M} Y_{1i}\, h_i^0}_{-q}$$

Y_i = MASS FRACTION OF COMPONENT "i"

h_i^0 = ENTHALPY OF FORMATION OF COMPONENT "i"

$-q$ = CHEMICAL HEAT RELEASED PER UNIT MASS OF REACTANT

TO INCLUDE THE HEAT RELEASED BY COMBUSTION. IN THIS WAY, THE HUGONIOT EQUATION (45) BECOMES

$$-q + \frac{\gamma}{\gamma-1}\left(\frac{P_2}{\rho_2} - \frac{P_1}{\rho_1}\right) = \frac{1}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right)(P_2 - P_1), \qquad (64)$$

WHICH, TOGETHER WITH THE RAYLEIGH LINE (44) FORM THE RANKINE-HUGONIOT EQUATIONS FOR REACTING WAVES. THESE EQUATIONS ARE ANALYZED BELOW.



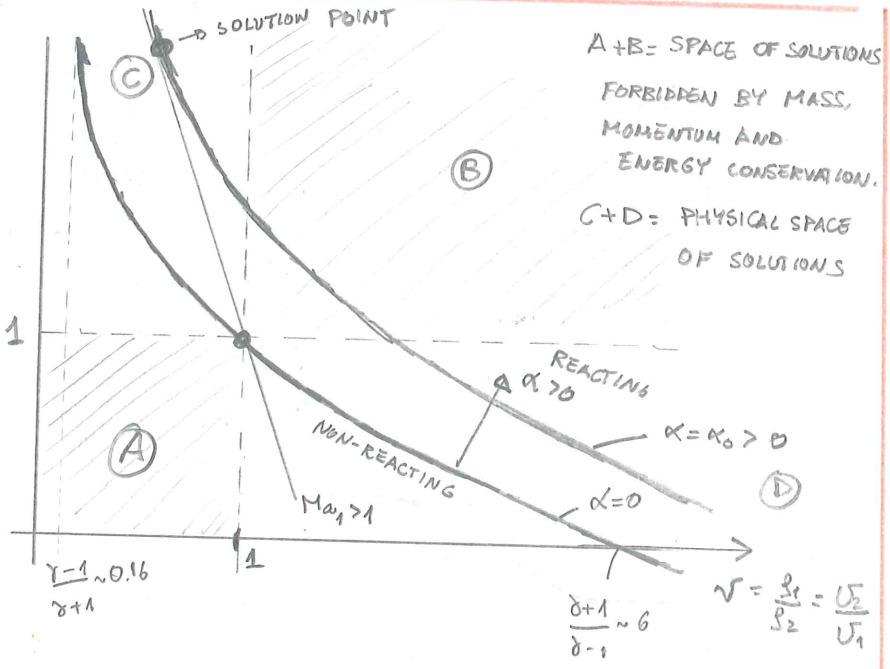
RANKINE-HUGONIOT EQUATIONS FOR REACTING WAVES:

$$\mathcal{P} = \frac{P_2}{P_1}, \quad \mathcal{V} = \frac{\rho_1}{\rho_2} = \frac{v_2}{v_1}$$

$$\mu = \frac{\left(\dot{m}''\right)^2}{P_1 \rho_1}\,\frac{v_1^2}{P_1/\rho_1} = \gamma M_{a_1}^2, \quad \alpha = \frac{q}{P_1/\rho_1} \;(>0)$$

$$\Rightarrow \boxed{\frac{\mathcal{P}-1}{\mathcal{V}-1} = -\mu} \qquad (65)$$

$$\boxed{\mathcal{P} = \frac{\left[2\alpha + (\gamma+1)/(\gamma-1)\right] - \mathcal{V}}{\frac{\gamma+1}{\gamma-1}\mathcal{V} - 1}} \qquad (66)$$

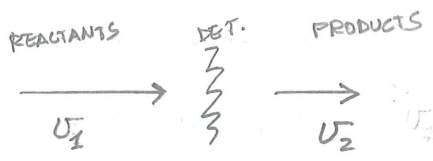


- THE SOLUTIONS IN THE QUADRANT (C) ($\mathcal{P}>1, \mathcal{V}<1$) ARE CALLED **DETONATIONS**.
- THE SOLUTIONS IN THE QUADRANT (D) ($\mathcal{P}<1, \mathcal{V}>1$) ARE CALLED **DEFLAGRATIONS**.

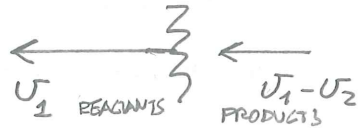
✱ DETONATION: WAVE THAT COMPRESSES THE GAS, DECELERATES IT, INCREASES ITS DENSITY AND MOVES IT IN THE DIRECTION OF WAVE PROPAGATION.

$$\Rightarrow \quad \frac{p_2}{p_1} > 1, \quad \frac{\rho_2}{\rho_1} > 1, \quad \frac{v_2}{v_1} < 1, \quad Ma_1 > 1, \quad \frac{T_2}{T_1} > 1, \quad Ma_2 \lesssim 1$$

SUPERSONIC WAVE!



WAVE FRAME



LABORATORY FRAME

↳ THE PRODUCTS CAN BE SUPER- SUB- OR SONIC.

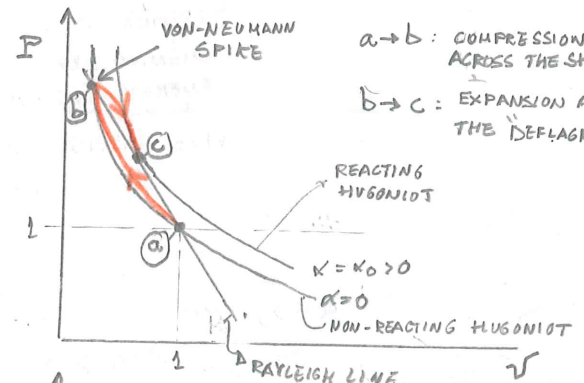
✱ $Ma_2 < 1$ ⇒ STRONG DETONATION

✱ $Ma_2 > 1$ ⇒ "WEAK DETONATION"

IT IS IMPORTANT TO NOTICE THAT, WHILE A SHOCK WAVE CANNOT EXIST WITHOUT AN EXTERNAL SOURCE OF ENERGY (FOR INSTANCE A CONTINUOUS FLOW), A DETONATION IS A SELF-SUSTAINED REACTING SHOCK WAVE WHOSE PROPAGATION SPEED IS DETERMINED BY THE AUTOIGNITIVE PROCESSES INVOLVED IN THE CORRESPONDING COMBUSTION REACTIONS AND BY THE THERMOCHEMICAL COMPOSITION OF THE GASEOUS MIXTURE. UNLIKE A SHOCK WAVE, IN EACH GIVEN MIXTURE THERE IS A DETONATION VELOCITY WHICH IS TYPICALLY OF THE ORDER OF 1–10 km/s ✱. THE DETONATION SUSTAINS ITSELF BY LIBERATING THE CHEMICAL ENERGY CONTAINED IN THE MIXTURE.

✱ → NOTE: TYPICAL DEFLAGRATION VELOCITIES ARE 0.1–1 m/s

THE DETAILED DESCRIPTION OF DETONATIONS IS BEYOND THE SCOPE OF THIS COURSE (✱). HOWEVER, A QUALITATIVE DISCUSSION IS PROVIDED HERE.



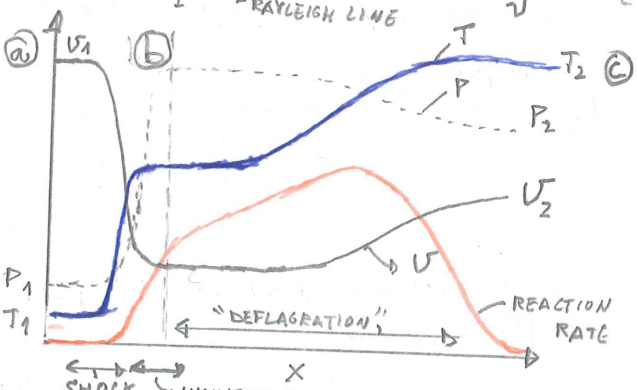
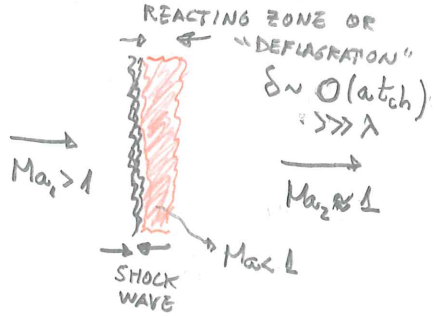


FIG 19. THERMODYNAMIC TRAJECTORY (RED, UPPER PANEL) AND "ZND" DETONATION STRUCTURE (LOWER AND RIGHT PANELS)



"ZELDOVICH – VON NEUMANN – DÖRING (ZND)" STRUCTURE OF A PLANAR DETONATION

($t_{ch} \approx$ AUTOIGNITION TIME)

(✱) SEE Y.B. ZEL'DOVICH AND A.S. KOMPANEETS, "THEORY OF DETONATION" (1960).

THE DETONATION STRUCTURE SHOWN IN FIG. 19 WAS FIRST PROPOSED BY ZEL'DOVICH (1940), VON NEUMANN (1942) AND DÖRING (1943), AND IS REFERRED TO AS "ZND". ALTHOUGH IT REPRESENTS A CANONICAL STRUCTURE OF PLANAR DETONATIONS, MORE RECENT WORKS HAVE SUGGESTED THAT THIS STRUCTURE IS UNSTABLE IN REALITY, IN THAT DETONATIONS TEND TO HAVE A COMPLEX THREE-DIMENSIONAL STRUCTURE THAT INVOLVES TRANSVERSE SHOCK WAVES AND CELLULAR PATTERNS (SEE FIG. 20). KNOWLEDGE OF ZND STRUCTURES IS HOWEVER ESSENTIAL TO THE UNDERSTANDING OF THOSE OTHER MORE COMPLEX STRUCTURES. THE ZND STRUCTURE IS SUBDIVIDED INTO A LEADING SHOCK, THAT INCREASES THE PRESSURE AND TEMPERATURE OF THE MIXTURE WITHOUT ANY CHEMICAL REACTION, AND A TRAILING "DEFLAGRATION" (*) IN WHICH EXOTHERMIC REACTIONS OCCUR THAT FURTHER ELEVATE THE TEMPERATURE. THE FLOW IMMEDIATELY DOWNSTREAM THE SHOCK IS HOT (TYPICALLY AT $T \sim 1800-2000\,K$) AND $P \sim 20\,P_1$ SUBSONIC. BECAUSE OF THIS, THE ENERGY RELEASE BEHIND THE SHOCK IS ACOUSTICALLY COUPLED WITH THE SHOCK WAVE AND CONTINUOUSLY FEEDS IT WITH ENERGY FOR THE SELF-PROPAGATION OF THE ENTIRE STRUCTURE. FOR TYPICALLY OBSERVED DETONATIONS, THE VELOCITY OF THE BURNT GASES RELAXES TO A LOCALLY SONIC SPEED WITH RESPECT TO THE MOVING FRONT, THUS GENERATING WHAT IS CALLED A CHAPMAN-JOUGUET DETONATION WHOSE PROPAGATION SPEED CAN BE COMPUTED FROM EQUATIONS (65)-(66) (SEE BELOW). THE SONIC CONDITIONS IN THE PRODUCTS WARRANTS A MAXIMUM ENTROPY INCREASE IN THE GAS WITH RESPECT TO ITS ORIGINAL STATE UPSTREAM OF THE DETONATION. LASTLY, IT IS WORTH EMPHASIZING THAT THE THICKNESS OF THE LEADING SHOCK (OF THE ORDER OF JUST A FEW MEAN FREE PATHS λ) IS MUCH SMALLER THAN THE THICKNESS OF THE REACTING ZONE, WHICH IS OF THE SAME ORDER AS THE SPEED OF SOUND MULTIPLIED BY THE CHEMICAL (AUTOIGNITION) TIME,

$\sim$ 400-500 m/s $\sim$ 1-10 μs

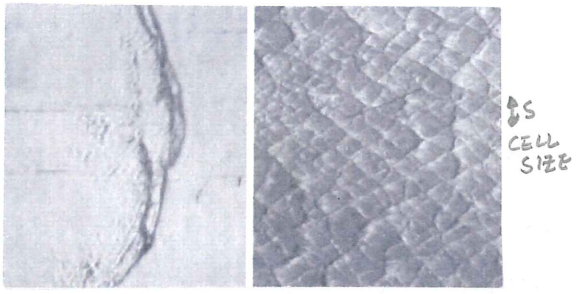


FIG. 20. EXPERIMENTAL OBSERVATIONS OF DETONATIONS (LEFT), AND A SMOKED FOIL RECORD OF A PROPAGATING DETONATION. INSCRIPTION ON A WALL (s = DETONATION CELL SIZE)

(*) QUOTATION MARKS ("") ARE USED SINCE THAT STRUCTURE IS NOT THAT OF A TYPICAL DEFLAGRATION, IN THAT HERE THERE IS NO TIME FOR HEAT CONDUCTION AND MOLECULAR DIFFUSION TO HAVE ANY NOTICEABLE EFFECTS ON THE TRANSPORT OF ENERGY AND MASS. HOWEVER, THE DEFLAGRATION IN THE ZND STRUCTURE ALSO EXPANDS AND ACCELERATES THE FLOW AS IN A NORMAL DEFLAGRATION.

* **DEFLAGRATIONS** = CONTRARY TO DETONATIONS, DEFLAGRATIONS EXPAND THE GAS, ACCELERATES IT, DECREASES ITS DENSITY AND LEAVE IT MOVING IN THE OPPOSITE DIRECTION OF WAVE PROPAGATION.

$$\Rightarrow \quad \frac{p_2}{p_1} < 1 \,, \quad \frac{\rho_2}{\rho_1} < 1 \,, \quad \frac{u_2}{u_1} > 1 \,, \quad \underline{\underline{Ma_1 < 1}} \,, \quad \frac{T_2}{T_1} > 1$$

SUBSONIC WAVE !



THE STUDY OF DEFLAGRATIONS (OR TYPICALLY CALLED PREMIXED FLAMES) IS BEYOND THE SCOPE OF THIS CLASS. IT SHOULD ONLY BE MENTIONED HERE THAT NORMAL DEFLAGRATIONS ARE NOT FLOW DISCONTINUITIES IN THE STRICT SENSE OF THE WORD, ALTHOUGH IN GASDYNAMIC SCALES THEY MAY APPEAR TO BE THIN COMPARED WITH OTHER LARGE-SCALE FLOW STRUCTURES. IN FACT, HEAT CONDUCTION AND MOLECULAR DIFFUSION ARE OF UTMOST IMPORTANCE WITHIN THE INTERIOR OF A DEFLAGRATION (AS OPPOSED TO DETONATIONS), THE THICKNESS OF A DEFLAGRATION, $\delta \sim \frac{D_T}{u_1} \sim \frac{\lambda}{Ma_1} >> \lambda$ IS MUCH LARGER THAN THE MEAN FREE PATH, IN ACCORDANCE WITH THE MULTIPLE COLLISIONS REQUIRED TO COMPLETE CHEMICAL REACTIONS. NOTE THAT A DEFLAGRATION IS AN EXPANSION WAVE.

* **CHAPMAN-JOUGUET CONDITIONS**

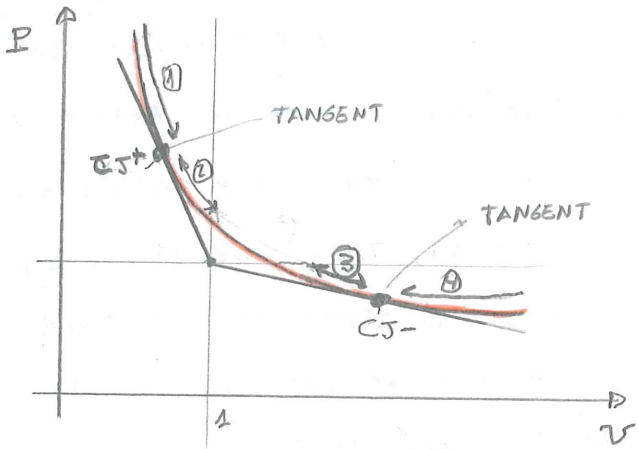


FIG. 20

THE CHAPMAN-JOUGUET (CJ) CONDITIONS ARE THOSE IN WHICH THE PRODUCTS MOVE AT LOCAL SONIC VELOCITIES WITH RESPECT TO THE WAVE.

- AT THE LOWER CJ POINT (CJ⁻ DEFLAGRATION):

$$Ma_1 = Ma_1^- < 1 \,, \quad Ma_2 = 1 \,, \quad P = P^- < 1 \,, \quad v = v^- > 1$$

- AT THE UPPER CJ POINT (CJ⁺ DETONATION)

$$Ma_1 = Ma_1^+ > 1 \,, \quad Ma_2 = 1 \,, \quad P = P^+ > 1 \,, \quad v = v^+ > 1$$

WHERE

$$\left.\begin{aligned} P^\pm &= 1 + \alpha(\gamma-1)\left\{1 \pm \left[1 + \frac{2\gamma}{\alpha(\gamma^2-1)}\right]^{1/2}\right\} \\ v^\pm &= 1 + \alpha\left(\frac{\gamma-1}{\gamma}\right)\left\{1 \mp \left[1 + \frac{2\gamma}{\alpha(\gamma^2-1)}\right]^{1/2}\right\} \\ Ma_1^\pm &= \left(\frac{P^\pm}{v^\pm}\right)^{1/2} = \left[1 + \frac{\alpha(\gamma^2-1)}{2\gamma}\right]^{1/2} \pm \left[\frac{\alpha(\gamma^2-1)}{2\gamma}\right]^{1/2} \end{aligned}\right\} \quad \text{CJ VALUES} \quad (67)$$

THESE EXPRESSIONS ALLOW FOR CALCULATIONS OF CJ SPEEDS. FOR INSTANCE, FOR A CJ DETONATION:

$$Ma_1^+ \approx \left(2\alpha \frac{(\gamma^2-1)}{\gamma} \right)^{1/2} \Rightarrow \boxed{U_1^+ = D_{CJ}^+ = \sqrt{2q(\gamma^2-1)}} \qquad (68) \qquad (\sim 1\text{-}10 \text{ km/s})$$

AND FOR A CJ DEFLAGRATION:

$$Ma_1^- \approx \left(\frac{\gamma}{2\alpha(\gamma^2-1)} \right)^{1/2} \Rightarrow \boxed{U_1^- = D_{CJ}^- = \frac{a_1^2}{\sqrt{2q(\gamma^2-1)}}} \qquad (69) \qquad (\sim 10\text{-}100 \text{ m/s})$$

WHERE (68)-(69) HAVE BEEN OBTAINED ASSUMING THAT $\alpha(\gamma^2-1)/2\gamma \gg 1$ IN (67).

NOTE THAT, EXCEPT FOR CJ CONDITIONS, THE PROPAGATION VELOCITIES OF DETONATIONS AND DEFLAGRATIONS CANNOT BE CALCULATED FROM JUST THE RANKINE-HUGONIOT EQUATIONS, BUT CONSIDERATIONS OF THE INTERNAL CHEMICAL PROCESSES HAVE TO BE ADDITIONALLY MADE.

IN FIG. 20, SOLUTIONS THAT ARE NOT CJ POINTS ARE DENOTED AS FOLLOWS:

(1) (ABOVE CJ^+): STRONG DETONATIONS

$\Rightarrow Ma_1^+ < Ma_1 < \infty$, $Ma_2 < 1$, RARELY OBSERVED, REQUIRES SPECIAL EXPERIMENTAL ARRANGEMENT

(2) (BELOW CJ^+): WEAK DETONATIONS

$\Rightarrow 1 < Ma_1 < Ma_1^+$, $Ma_2 > 1$, RARELY OBSERVED, REQUIRES SPECIAL GAS MIXTURES

(3) (ABOVE CJ^-): WEAK DEFLAGRATIONS

$\Rightarrow 0 < Ma_1 < Ma_1^-$, $Ma_2 < 1$, OFTEN OBSERVED, WAVE AT MOSTLY CONSTANT PRESSURE.

(4) (BELOW CJ^-): STRONG DEFLAGRATIONS

$\Rightarrow Ma_1 > Ma_1^-$, $Ma_2 > 1$, NOT OBSERVED, FORBIDDEN BY CONSIDERATIONS OF WAVE STRUCTURE

* COMPARISONS BETWEEN DETONATIONS AND DEFLAGRATIONS.

	DETONATION	DEFLAGRATION
Ma_1	5-10	10^{-4} - 10^{-2}
ρ_2/ρ_1	1.5-3	0.05-0.3
T_2/T_1	8-20	4-16
P_2/P_1	10-20	0.95-0.99.
U_2/U_1	0.4-0.7	4-6.

INTERACTIONS OF SHOCK WAVES WITH WALLS

* REFLECTION OF OBLIQUE SHOCKS

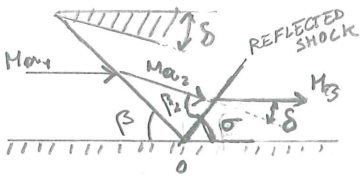


FIG. 21: REFLECTION OF AN OBLIQUE SHOCK

WHEN AN OBLIQUE SHOCK IS INTERCEPTED BY A WALL, IT IS REFLECTED AS SHOWN IN FIG. 21. THE INCIDENT SHOCK DEFLECTS THE FLOW THROUGH AN ANGLE δ TOWARD THE WALL, WHILE THE REFLECTED SHOCK TURNS IT BACK AGAIN AN AMOUNT δ TO ALIGN IT WITH THE WALL. THE INCLINATION OF THE INCIDENT SHOCK σ, OR ITS INCIDENCE ANGLE $\beta_2 = \sigma + \delta$, ARE EASILY CALCULATED FROM THE OBLIQUE SHOCK JUMP CONDITIONS (58) WHEN β AND Ma_1 ARE KNOWN. ALTHOUGH THE DEFLECTIONS PRODUCED BY THE TWO SHOCKS ARE EQUAL TO δ, THE PRESSURE RATIOS ARE NOT, AND THEREFORE THE REFLECTION IS NOT SPECULAR ($\beta \neq \sigma$ IN GENERAL).

* MACH REFLECTION

IF, IN FIG. 21, THE DEFLECTION ANGLE IS LARGER THAN THE MAXIMUM DEFLECTION ANGLE δ_{MAX} FOR Ma_2 (SEE FIG 13 AND COMMENTS IN PAGE 24), THEN THERE IS NO SOLUTION WITH A REFLECTED SHOCK. INSTEAD THE SOLUTION IS CALLED A MACH REFLECTION AND IS REPRESENTED IN FIG. 22.

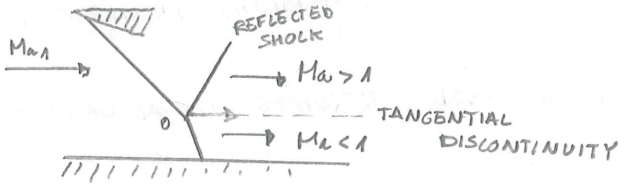
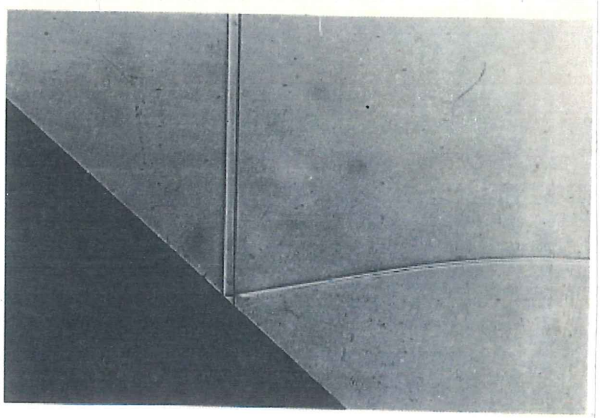
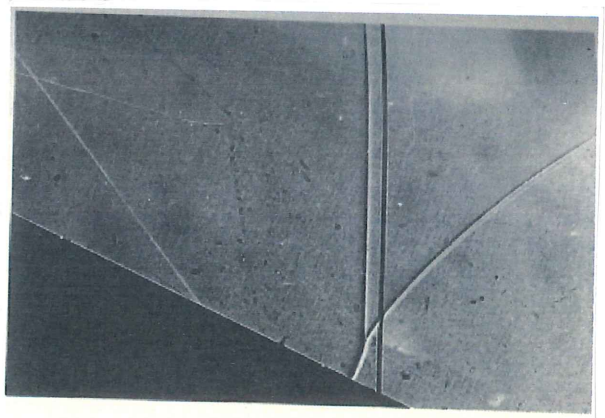


FIG. 22: MACH REFLECTION

A NEAR NORMAL SHOCK APPEARS NEAR THE WALL THAT MERGES WITH THE INCIDENT AND REFLECTED SHOCKS AT THE TRIPLE POINT 0. THE STREAMLINE THAT PASSES THROUGH 0 IS A TANGENTIAL DISCONTINUITY THAT IS A SOURCE OF VORTICITY AND TRANSFORMS INTO A SHEAR LAYER WHEN VISCOUS EFFECTS ARE CONSIDERED.



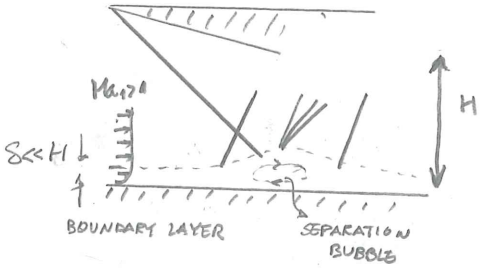
REGULAR REFLECTION



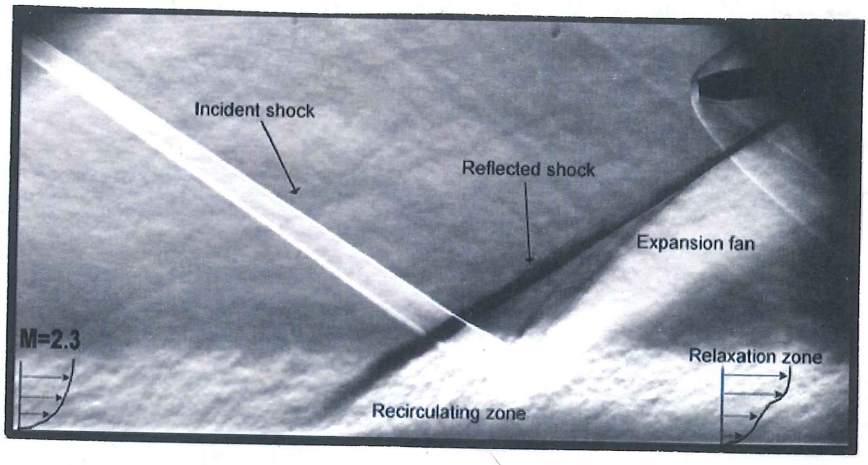
MACH REFLECTION.

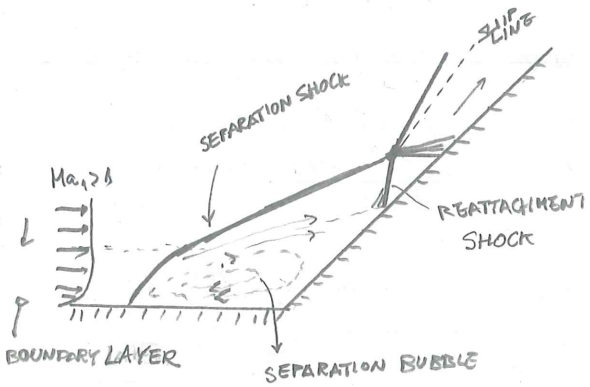
* VISCOUS EFFECTS

THE INTERACTIONS OF SHOCK WAVES WITH WALLS ARE OFTENTIMES COMPLICATED BY THE OCCURRENCE OF VISCOUS EFFECTS NEAR THE SURFACE, IN THE FORM OF BOUNDARY LAYERS. IN PARTICULAR, BOUNDARY LAYERS ARE THIN REGIONS OF THE FLOW ADJACENT TO WALL SURFACES ACROSS WHICH THE FLUID VELOCITY VARIES FROM THE INVISCID, FREE-STREAM VALUES TO ZERO AT THE WALL. THE INTERACTION OF SHOCK WAVES WITH BOUNDARY LAYERS LEADS TO A NUMBER OF IMPORTANT PHENOMENA SUCH AS SEPARATION, UNSTEADINESS, PRESSURE WAVES, TURBULENCE, ETC. (SEE FIG. 22). SOME OF THESE PHENOMENA HAVE IMPORTANT CONSEQUENCES IN TRANSONIC AIRFOILS, IN THAT THE BOUNDARY LAYER SEPARATION INDUCED BY THE SHOCK FORMING IN THE SUPERSONIC PORTION OF THE AIRFOIL LEADS TO DRAG INCREASE AND LIFT DECREASE.

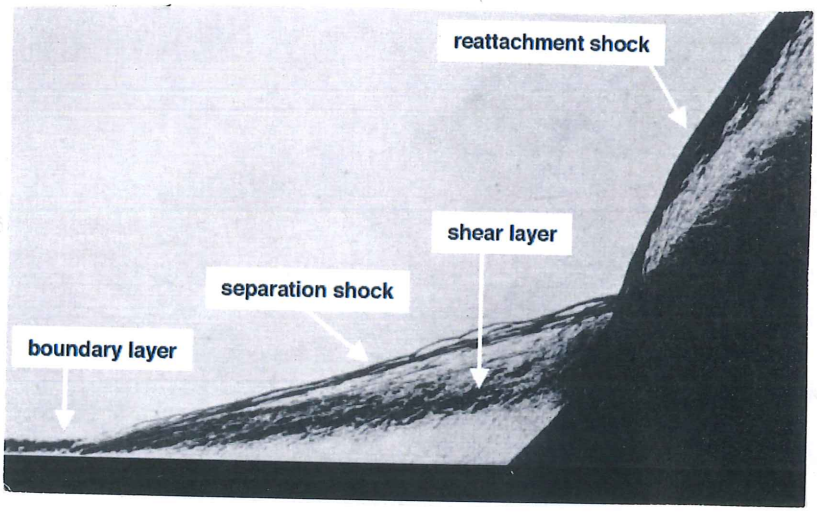


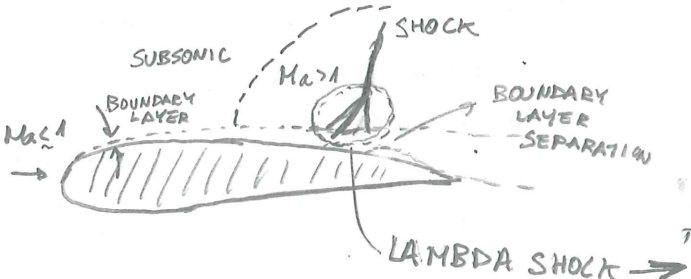
(a) SHOCK INTERACTING WITH A FLAT-PLATE BOUNDARY LAYER.

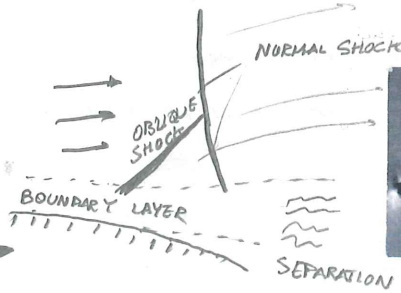


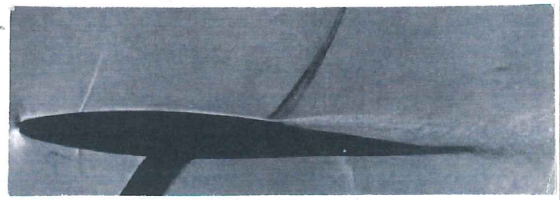


(b) RAMP FLOW WITH BOUNDARY-LAYER SEPARATION AT HIGH Ma.





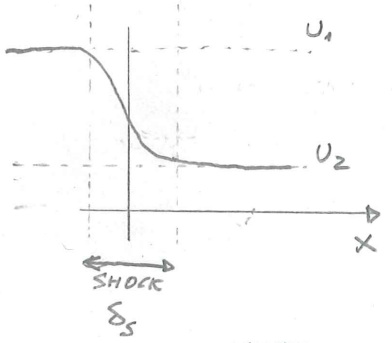




(c) TRANSONIC FLOW OVER AN AIRFOIL WITH BOUNDARY-LAYER SEPARATION

* INTERNAL STRUCTURE OF SHOCK WAVES AND NON-EQUILIBRIUM PROCESSES

IN THE THEORY DESCRIBED ABOVE, INCLUDING THE JUMP CONDITIONS DERIVED IN EQUATIONS (50)-(55), NO MENTION OF THE INTERNAL STRUCTURE OF THE SHOCK WAVE HAS BEEN MADE. IN FACT, IT IS REMARKABLE THAT THE DYNAMICS OF THE FLOW OUTSIDE THE SHOCK IS INDEPENDENT OF THE SHOCK INTERIOR. THE INTERIOR OF THE SHOCK IS DOMINATED BY NON-EQUILIBRIUM PROCESSES, WHICH IN THE SIMPLEST CASE OF MONO-ATOMIC GASES PERTAIN TO THE TRANSLATIONAL DEGREES OF FREEDOM AND ARE MANIFESTED THROUGH VISCOUS AND HEAT-CONDUCTION EFFECTS. THESE EFFECTS, HOWEVER, OPERATE ONLY INSIDE THE SHOCK IN A WAY THAT THE CORRECT ENTROPY INCREASE THROUGH THE SHOCK IS ESTABLISHED. THE QUANTIFICATION OF THESE PROCESSES WILL NOT BE PURSUED IN THIS COURSE, WHICH PRIMARILY DEALS WITH INVISCID FLOWS. THE READER IS REFERRED TO VINCENTI AND KRUGER'S "INTRODUCTION TO PHYSICAL GASDYNAMICS" (→ PAGE 412) FOR A DEEPER TREATMENT OF THIS SUBJECT. TO TREAT THE INTERNAL STRUCTURE OF SHOCK WAVES, EQS. (43) NEED TO BE AUGMENTED WITH THE DIFFUSIVE TRANSPORT OF MOMENTUM AND HEAT:



$$\left.\begin{aligned} &\rho u = \rho_1 u_1 \quad , \quad p = \rho R_g T \\ &p + \rho u^2 - \tau_{xx} = p_1 + \rho_1 u_1^2 \\ &\rho u\left(h + \frac{u^2}{2}\right) + q_x - u\tau_{xx} = \rho_1 u_1\left(h_1 + \frac{u_1^2}{2}\right) \end{aligned}\right\} \quad \text{SUBJECT TO } x \to +\infty,\ p \to p_2,\ u \to u_2,\ \rho \to \rho_2,\ h \to h_2$$

WITH $\tau_{xx} = \frac{4}{3}\mu\frac{du}{dx}$ THE SHEAR STRESS AND $q_x = -\kappa\frac{dT}{dx}$ THE HEAT FLUX. NOTE THAT THE EQUATIONS ABOVE REDUCE TO EQS. (43) FOR $x/\delta_s \gg 1$, WHERE GRADIENTS VANISH, WITH δ_s THE THICKNESS OF THE SHOCK.

IGNORE VARIATIONS IN μ AND κ

FOR WEAK SHOCKS $Ma_1^2 - 1 \ll 1$, THESE EQUATIONS CAN BE INTEGRATED TO GIVE THE VELOCITY

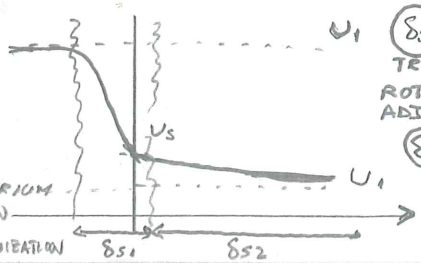
$$\frac{u - u_2}{u_1 - u_2} = \left\{1 + \exp\left[G\left(\frac{u_2 - u_2}{c_1}\right)\frac{x}{\lambda_1}\right]\right\}^{-1}$$

WHERE $c_1 = \left(\frac{8 R_g T_1}{\pi}\right)^{1/2}$ IS THE MEAN SPEED OF THE MOLECULES, $G = (\gamma+1)/\left[4/3 + (\gamma-1)/Pr\right]$ IS A CONSTANT, AND $\lambda_1 = \frac{\mu_1}{\frac{1}{2}\rho_1 c_1}$ IS THE MEAN FREE PATH. THE THICKNESS OF THE WEAK SHOCK IS

$$\delta_s \sim \frac{c_1\lambda_1}{G(u_1 - u_2)} \sim \frac{\lambda_1 u_1}{Ma_1 \Delta u} \sim \frac{\nu_1}{\Delta u}$$

WITH $\nu_1 = \mu_1/\rho_1$ THE KINEMATIC VISCOSITY AND $\Delta u = u_1 - u_2$. NOTE THAT δ_s IS A MULTIPLE OF λ_1. IN NORMAL SHOCKS, $\delta_s \sim 10-20\,\lambda$, AND ANALYSES SHOW THAT THE NAVIER-STOKES EQUATIONS STILL GIVE A GOOD APPROXIMATION OF THE INTERIOR OF SHOCKS DESPITE THE NON-CONTINUUM CHARACTER THAT THE FLOW APPROACHES IN THOSE REGIONS.

IN STRONG SHOCKS IN DIATOMIC AND POLYATOMIC GASES, A TAIL OF VIBRATIONAL AND CHEMICAL NON-EQUILIBRIUM ADJUSTMENT THAT CAN BE MUCH LONGER THAN δ_s, AND WHOSE SPATIAL EXTENT INCREASES WITH THE MACH NUMBER AND THE NON-EQUILIBRIUM TIMESCALES.



δ_{s1}: REGION OF TRANSLATIONAL + ROTATIONAL ADJUSTMENT.

δ_{s2}: REGION OF VIBRATIONAL AND CHEMICAL ADJUSTMENT

$T > 800\,K \Rightarrow$ VIBRATIONAL NONEQ
$T > 2000\,K \rightarrow O_2$ DISSOCIATION
$T > 4000\,K \Rightarrow N_2$ DISSOCIATION
$T > 9000\,K \Rightarrow N \rightarrow N^+ + e^-$ IONIZATION

3. COMPRESSIBLE FLOWS IN DUCTS AND NOZZLES

WHEN DESCRIBING THE FLOW OF GASES IN VARIABLE-AREA DUCTS IN PAGE 17, IT WAS OBSERVED THAT THE EFFECTS OF AREA VARIATIONS ARE VASTLY DIFFERENT IN SUBSONIC AND SUPERSONIC FLOWS. FOR INSTANCE, WHILE AN AREA INCREASE SLOWS DOWN AND INCREASES THE PRESSURE AND DENSITY IN SUBSONIC FLOWS, THE SAME AREA VARIATION CAUSES SUPERSONIC FLOWS TO ACCELERATE AND EXPAND. THIS EFFECT HAS IMPORTANT APPLICATIONS IN PROPULSION, AS DESCRIBED BELOW. OTHER NON-INTUITIVE PHENOMENA TO BE STUDIED IN THIS SECTION PERTAINS TO THE HEAT ADDITION IN GASEOUS FLOWS, WHICH LEADS TO DIFFERENT DYNAMICS DEPENDING ON WHETHER THE FLOW IS SUBSONIC OR SUPERSONIC.

DYNAMICS OF CONVERGENT-DIVERGENT NOZZLES.

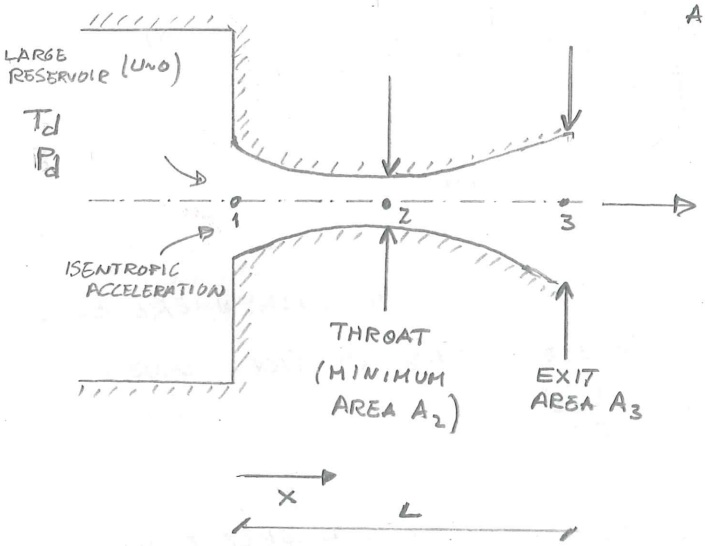


FIG. 23: CONVERGENT-DIVERGENT NOZZLE SCHEMATICS

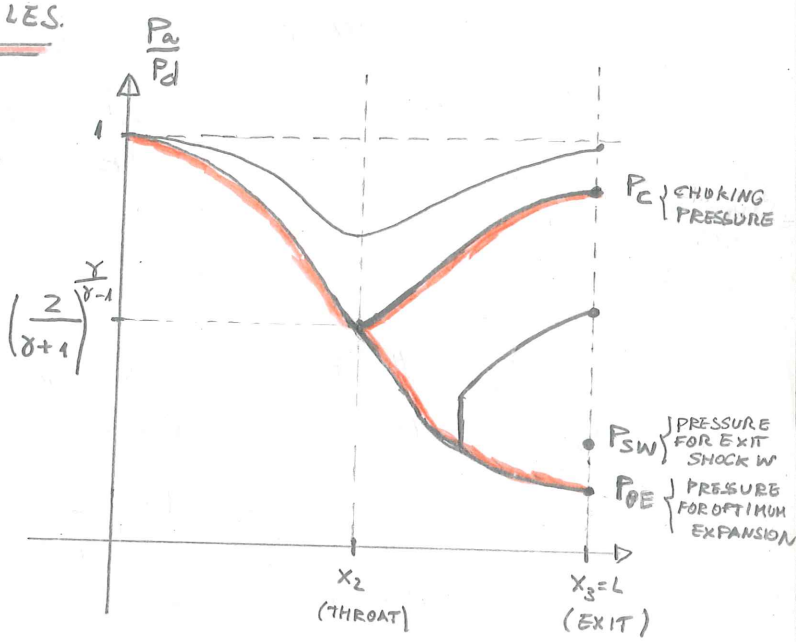


FIG. 24: CHARACTERISTIC OPERATING REGIMES AS A FUNCTION OF THE AMBIENT-TO-RESERVOIR PRESSURE RATIO P_a/P_d

- $\boxed{P_d > P_a > P_c}$ ⇒ SUBSONIC FLOW EVERYWHERE.

  HERE P_c IS THE AMBIENT PRESSURE FOR WHICH THE GAS BECOMES SONIC AT THE THROAT, WHICH IS CALCULATED AS FOLLOWS.

  THE MASS FLOW RATE IN THE NOZZLE IS

$$\dot{m} = \rho U A = \rho_d a_d A M_a \left(1 + \frac{(\gamma-1)}{2} M_a^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}, \quad \text{WHERE} \quad \rho_d a_d = P_d \sqrt{\frac{\gamma}{R_g T_d}}.$$

  FOR SONIC FLOW AT THE THROAT, $(P_2 = P^*, M_{a_2} = 1)$

$$\left(\frac{P_d}{P_2}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{P_d}{P^*}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{(\gamma-1)}{2} M_{a_2}^2 \overset{=1}{} = \frac{1+\gamma}{2}$$

  AND $\dot{m}^* = P_d \sqrt{\frac{\gamma}{R_g T_d}} A^* \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$, WHERE $A^* = A_2$.

BECAUSE OF CONTINUITY: $\dot{m}^* = P_d \sqrt{\frac{\gamma}{R_g T_d}} A_3 Ma_3 \left(1 + \frac{(\gamma-1)}{2} Ma_3^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} =$

$= P_d \sqrt{\frac{\gamma}{R_g T_d}} A_2 \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$,

WHICH GIVES $\boxed{\frac{A^*}{A_3} = \frac{A_2}{A_3} = Ma_3 \left(1 + \frac{(\gamma-1)}{2} Ma_3^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{1+\gamma}{2}\right)^{\frac{\gamma+1}{2(\gamma-1)}}}$ (70)

AS POINTED OUT IN PAGE 16, THIS FUNCTION GIVES 2 VALUES OF Ma_3, ONE SUBSONIC AND ANOTHER ONE SUPERSONIC.

FOR A GIVEN THROAT-TO-EXIT AREA RATIO $\frac{A_2}{A_3} = \frac{A^*}{A_3}$, IN CHOKED CONDITIONS TABLE I GIVES THE SOLUTION TO EQ. (70) IN TERMS OF TWO MACH NUMBERS ($Ma_3 > 1$, $Ma_3 < 1$) AND A CORRESPONDING PRESSURE RATIO $\frac{P_d}{P_e} = \left(1 + \frac{(\gamma-1)}{2} Ma_3^2\right)^{\frac{\gamma}{\gamma-1}}$ (71), WITH $P_e = P_a$ AT THE EXIT.

THIS IS THE $Ma_3 < 1$ SOLUTION.

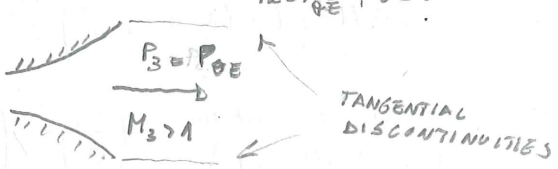
NOTE THAT THE STAGNATION PRESSURE IS CONSTANT EVERYWHERE SINCE THERE ARE NO SHOCKS $P_0 = P_d$

- $\boxed{P_a = P_c}$ THE FLOW IS SONIC AT THE THROAT AND SUBSONIC EVERYWHERE ELSE. THE MASS FLOW RATE THROUGH THE NOZZLE IS THE CRITICAL ONE,

$$\dot{m}^* = P_d \sqrt{\frac{\gamma}{R_g T_d}} A_2 \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}},$$

WHICH IS THE MAXIMUM ATTAINABLE VALUE. FURTHER DECREASE IN PRESSURE CAUSE NO VARIATIONS IN THE MASS FLOW RATE.

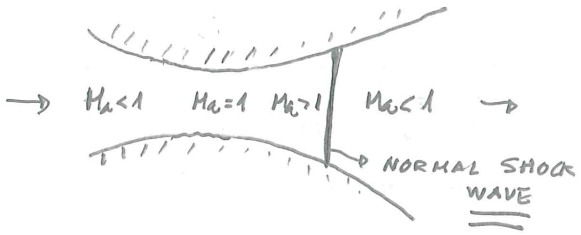
- $\boxed{P_a = P_{0E}}$ THE FLOW IS SONIC AT THE THROAT, SUBSONIC IN THE CONVERGING PART AND SUPERSONIC IN THE DIVERGING ONE. IN THESE CONDITIONS, THE FLOW IS OPTIMALLY EXPANDED AND THE EXHAUST CORRESPONDS TO A SUPERSONIC JET AT PRESSURE $P_3 = P_{0E} = P_a$ WITH A TANGENTIAL DISCONTINUITY BUT IN ABSENCE OF SHOCK WAVES

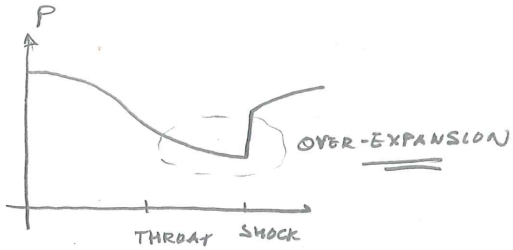


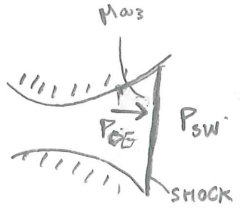
HERE P_{0E} IS GIVEN BY THE SOLUTION OF (70) CORRESPONDING TO THE SUPERSONIC M_3: $\frac{P_d}{P_{0E}} = \left(1 + \frac{\gamma-1}{2} Ma_3^2\right)^{\frac{\gamma}{\gamma-1}}$

THIS IS THE $Ma_3 > 1$ SOLUTION OF (70)

• $\boxed{P_c > P_a > P_{SW}}$ IN THIS RANGE OF PRESSURES, THE FLOW IS SUBSONIC IN THE CONVERGENT PART, SONIC AT THE THROAT, ACCELERATES SUPERSONICALLY IN THE DIVERGENT PART, BUT IT BECOMES SUBSONIC AGAIN SOMEWHERE IN THE DIVERGENT PART BECAUSE OF THE PRESENCE OF SHOCK WAVES THERE.







HERE P_{SW} IS THE AMBIENT PRESSURE SUCH THAT THE NORMAL SHOCK IS LOCATED AT THE NOZZLE EXIT, AND IT CAN BE COMPUTED AS FOLLOWS:

$$\frac{P_{SW}}{P_d} = \underbrace{\frac{P_{SW}}{P_{0E}}}_{②} \cdot \underbrace{\frac{P_{0E}}{P_d}}_{①}$$

① IS KNOWN FROM THE ISENTROPIC FLOW DESCRIBED ABOVE

② IS COMPUTED FROM THE NORMAL SHOCK JUMP CONDITION (EQ. (51), PAGE 22)

$$\Rightarrow \frac{P_{SW}}{P_d} = \underbrace{\left[1 + \frac{2\gamma}{\gamma+1}\left(Ma_3^2 - 1\right)\right]}_{①} \underbrace{\left(1 + \frac{(\gamma-1)}{2} Ma_3^2\right)^{-\frac{\gamma}{\gamma-1}}}_{②}$$

WITH $Ma_3 > 1$ THE SUPERSONIC SOLUTION TO EQ. (70).

IN THIS REGIME THE NOZZLE IS SAID TO BE OVER-EXPANDED, IN THAT THE PRESSURES IN THE SUPERSONIC REGIONS MAY BE BELOW THE AMBIENT EXIT PRESSURE. IT IS IMPORTANT TO NOTICE THAT THE ADVERSE PRESSURE GRADIENTS CREATED BY THE SHOCK IN THE INTERIOR OF THE NOZZLE CAN LEAD TO BOUNDARY-LAYER SEPARATION AND UNDERPERFORMANCE (SEE FIG. 25).

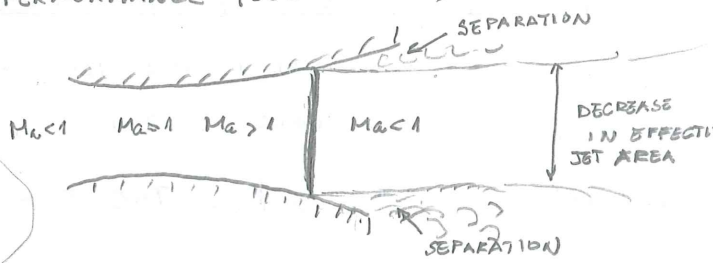
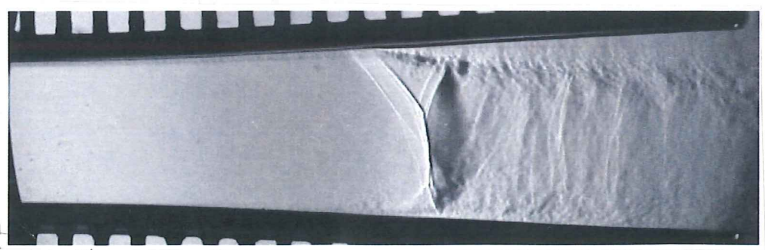
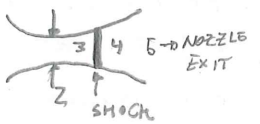


FIG. 25



NOTE: TO COMPUTE POSITION OF SHOCK WHEN $P_c > P_a > P_{SW}$, DO



1) $\frac{\rho_5}{\rho_d} = \frac{\rho_5}{\rho_4}\frac{\rho_4}{\rho_3}\frac{\rho_3}{\rho_d} = \left(\frac{P_5}{P_4}\right)^{1/\gamma}\left(\frac{P_3}{P_d}\right)^{1/\gamma}\frac{P_4}{P_3}\frac{T_3}{T_4} =$

$= \left(\frac{P_5}{P_4}\right)^{1/\gamma}\left(\frac{P_3}{P_d}\right)^{1/\gamma}\frac{P_4}{P_3}\left(\frac{P_3}{P_{03}}\right)^{\frac{\gamma-1}{\gamma}}\left(\frac{P_{04}}{P_4}\right)^{\frac{\gamma-1}{\gamma}}$

$= \left(\frac{P_5}{P_{04}}\right)^{1/\gamma}\left(\frac{P_{04}}{P_{03}}\right)^{\frac{\gamma-1}{\gamma}}$

$\underbrace{\left(\frac{P_a}{P_d}\right)^{1/\gamma}}_{\text{GIVEN}}$ $P_5 = P_a$!

2) $h_d = \frac{\gamma}{\gamma-1}\frac{P_d}{\rho_d} = \frac{\gamma}{(\gamma-1)}\frac{P_5}{\rho_5} + \frac{U_5^2}{2}$

$\Rightarrow U_5^2 = \frac{2\gamma}{\gamma-1}\frac{P_d}{\rho_d}\left(1 - \frac{\rho_d}{\rho_5}\frac{P_5}{P_d}\right)$

3) $\dot{m}^2 = \rho^2 U^2 A^2 = \dot{m}^{*2} = A_2^2\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}}\rho_d^2 a_d^2$

$= \rho_5^2 A_5^2 U_5^2$

SUBSTITUTE U_5 FROM 2)

$\Rightarrow \frac{A_2^2}{A_5^2}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{\gamma-1}}\left(\frac{\rho_d}{\rho_5}\right)^2\left(\frac{\gamma-1}{2}\right) +$

$+ \left(\frac{\rho_d}{\rho_5}\right)\left(\frac{P_5}{P_d}\right) - 1 = 0 \quad (P_5 = P_a)$

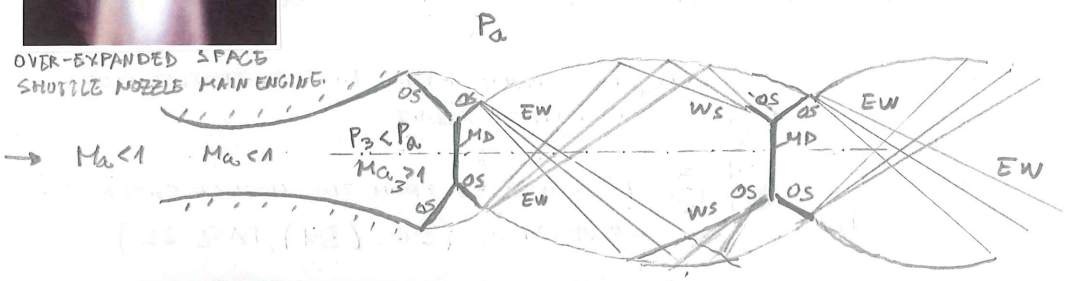
SOLVE FOR ρ_d/ρ_5, SUBSTITUTE IN 1), OBTAIN $\frac{P_{04}}{P_{03}} = \left(\frac{\rho_5}{\rho_d}\right)^{\frac{\gamma}{\gamma-1}}\left(\frac{P_d}{P_5}\right)^{\frac{1}{\gamma-1}}$ → READ Ma_3 FROM TABLE II NORMAL SHOCKS, COMPUTE A_3/A_2 FROM CONTINUITY.

- $P_a = P_{SW}$: THE NORMAL SHOCK WAVE IS LOCATED AT THE EXIT, AS DESCRIBED ABOVE.

- $P_{SW} > P_a > P_{0E}$: THE FLOW IS SUBSONIC IN THE CONVERGENT PART, SONIC AT THE THROAT AND SUPERSONIC IN THE DIVERGENT PART, INCLUDING THE EXIT SECTION. HOWEVER THE SUPERSONIC FLOW AT THE NOZZLE EXIT IS STILL OVER-EXPANDED, $P_3 < P_a$ (P_{0E} = P_3). THE FLOW RECOVERS THE AMBIENT PRESSURE THROUGH A COMPLEX SET OF OBLIQUE SHOCK WAVES AND INTERACTIONS AMONG THEM, WHICH LEAD TO A "DIAMOND-LIKE" JET PATTERN IN THE EXHAUST (SEE FIG. 26)



OVER-EXPANDED SPACE SHUTTLE NOZZLE MAIN ENGINE.



OS = OBLIQUE SHOCK
MD = MACH DISK (NORMAL SHOCK)
EW = EXPANSION WAVE
WS = WEAK SHOCK

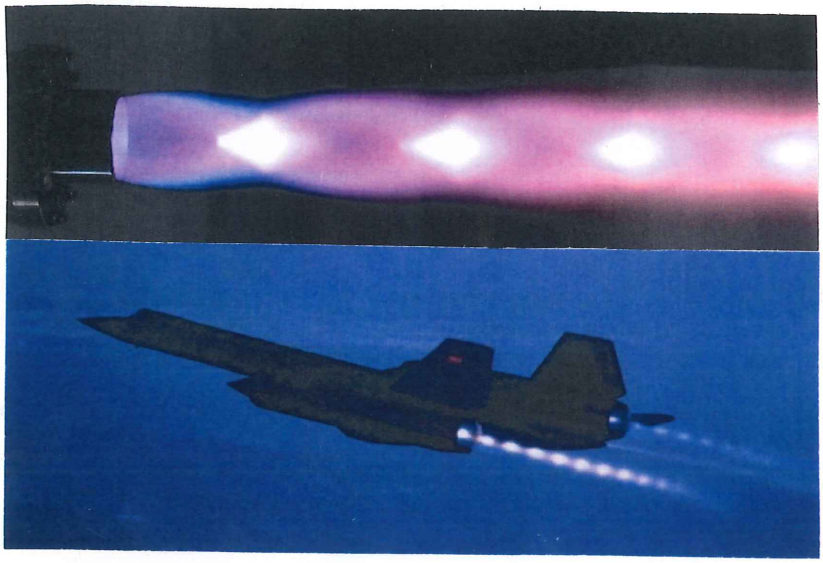
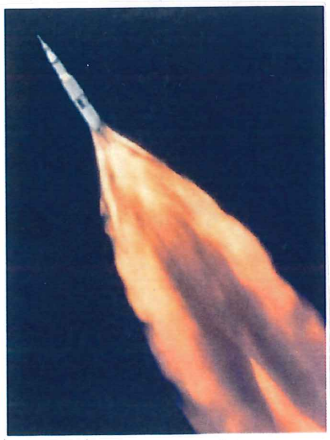
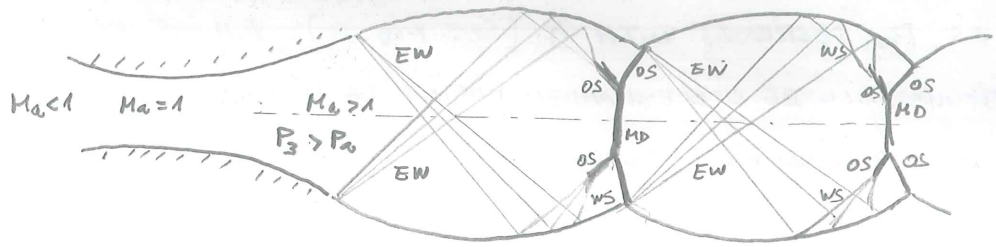


FIG. 26.
SCHEMATICS OF AN OVER-EXPANDED EXHAUST JET (UPPER PANEL), ALONG WITH VISUALIZATIONS. NOTE THAT THE SHOCK-SHOCK INTERACTIONS, AS IN THE MACH DISKS, LEAD TO A TEMPERATURE INCREASE AND CHANGE IN RADIATION EMISSIONS IN THE GAS JET.

- $P_a < P_{0E}$: THE FLOW IS SUBSONIC IN THE CONVERGENT PART, SONIC AT THE THROAT, AND SUPERSONIC IN THE DIVERGENT PART, INCLUDING THE NOZZLE EXIT. HOWEVER, THE FLOW IS UNDER-EXPANDED, IN THAT $P_3 = P_{0E} > P_a$. THE EXHAUST JET RECOVERS THE AMBIENT PRESSURE THROUGH A COMPLEX SET OF PRANDTL-MEYER EXPANSION FANS EMERGING FROM THE EXIT ORIFICE, AS DEPICTED IN FIG. 27, WHICH ALSO PRODUCE A "DIAMOND-LIKE" PATTERN.



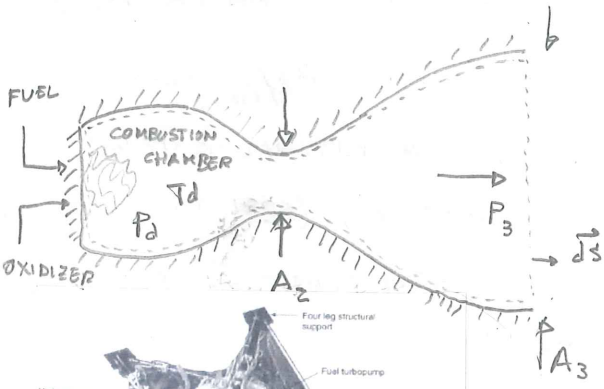
UNDER-EXPANDED PLUME OF SATURN-V AT HIGH ALTITUDE.

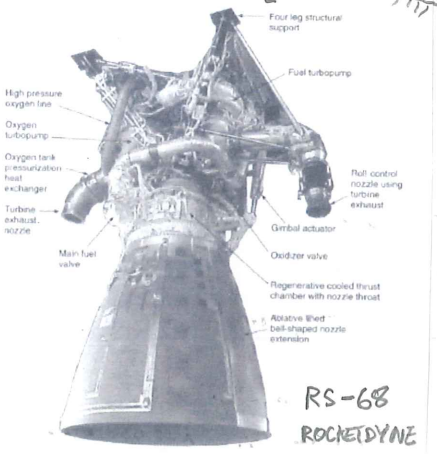


OS: OBLIQUE SHOCK
MD: MACH DISK
EW: EXPANSION WAVE
WS: WEAK SHOCK

FIG 27 - SCHEMATICS OF AN UNDER-EXPANDED JET.

* ROCKET-ENGINE NOZZLES





RS-68
ROCKETDYNE

THE THRUST PRODUCED BY THE ENGINE CAN BE CALCULATED USING AN INTEGRAL BALANCE OF MOMENTUM IN THE CONTROL VOLUME V_c;

$$\frac{d}{dt}\int_{V_c} \rho \vec{v}\, dV + \int_{S_c} \rho \vec{v}\vec{v}\cdot d\vec{S} = -\int_{S_c} P\, d\vec{S}$$

$$\dot{m} U_3 \vec{e}_x = \underbrace{-\int_{S_c - A_3} P\, d\vec{S}}_{-\vec{F}} \underbrace{-\int_{A_3} P\, d\vec{S}}_{(P_a - P_3) A_3 \vec{e}_x}$$

$$\rightarrow \boxed{\vec{F} = -\dot{m} U_3 \vec{e}_x - (P_3 - P_a) A_3 \vec{e}_x}$$

THRUST ON THE NOZZLE STRUCTURE

IN OPTIMAL EXPANSION, OR ABOVE THE SONIC PRESSURE.

THE EXHAUST VELOCITY U_3 CAN BE CALCULATED IN EACH OF THE REGIMES DESCRIBED IN THE PRECEDING SECTION. FOR INSTANCE, IN AN OPTIMALLY EXPANDED NOZZLE,

$$\dot{m} = \rho_3 U_3 A_3 = \dot{m}^* = P_d \sqrt{\frac{\gamma}{R_g T_d}} A_2 \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}},$$

SO THAT

$$U_3 = \frac{\rho_d}{\rho_3}\frac{P_d}{\rho_d}\sqrt{\frac{\gamma}{R_g T_d}}\frac{A_2}{A_3}\left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} = \left(\frac{P_d}{P_a}\right)^{1/\gamma}\sqrt{\gamma R_g T_d}\,\frac{A_2}{A_3}\left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

THE SPECIFIC IMPULSE I_s IS THE TOTAL IMPULSE PER UNIT WEIGHT OF PROPELLANT,

$$\left[I_s = \frac{\int_0^t F\, dt}{g\int_0^t \dot{m}\, dt} \sim \frac{F}{g\dot{m}}\right] \quad (72)$$

(FOR F AND $\dot{m}$ CONSTANT WITH t).

IN TYPICAL ROCKET ENGINES, $U_3 \sim 2000 - 3000\ m/s$, $I_s \sim 200 - 300\ s$, $\dot{m} \sim 0.02 - 0.03\ \frac{kg}{s}$

IT IS IMPORTANT TO NOTICE THAT THE PERFORMANCE OF A ROCKET ENGINE VARIES WITH THE FLIGHT ALTITUDE h AS P_a DECREASES WITH h (SEE FIG. 8). AN OPTIMALLY EXPANDED NOZZLE AT MID ALTITUDE WILL BE OVEREXPANDED DURING LAUNCH THUS CAUSING INTERNAL FLOW SEPARATION, AND IT WILL BE UNDEREXPANDED IN THE UPPER ATMOSPHERE. FOR THIS REASON, THE NOZZLES OF EACH ROCKET STAGE ARE DESIGNED DIFFERENTLY. IN PARTICULAR, UPPER STAGES TEND TO HAVE LARGER EXPANSION RATIOS A_3/A_2, SINCE LARGER A_3/A_2 LEADS TO SMALLER OPTIMAL-EXPANSION PRESSURES P_e (SEE EQS. 70-71). ALTITUDE-COMPENSATION TECHNIQUES EXIST TO WIDEN THE RANGE OF ENGINE PERFORMANCE THAT UTILIZE EXTENDIBLE NOZZLES, DROPPABLE INTERNAL INSERTS, OR DUAL BELL NOZZLES(*).

(*) SEE SUTTON (2001) ROCKET PROP. ELEMENTS

Stage	A_3/A_2 (exhaust to throat area ratio)	During flight: h(km)	During flight: I_s (sec)	During sealevel static tests: h(km)	During sealevel static tests: I_s (sec)
Booster or first stage	6	0	267	0	267
Second stage	10	24	312	0	254
Third stage	40	100	334	0	245

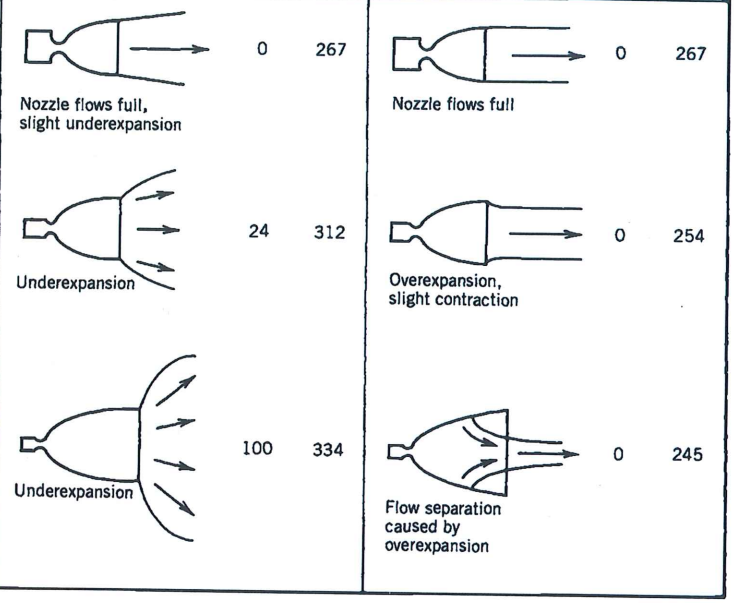
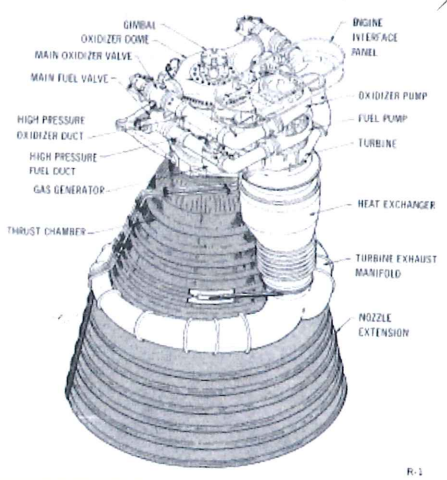


FIG. 28 Simplified sketches of exhaust gas behavior of three typical rocket nozzles for a three-stage launch vehicle. The first vehicle stage has the biggest chamber and the highest thrust but the lowest nozzle area ratio, and the top or third stage usually has the lower thrust but the highest nozzle area ratio. (**)

F-1 ENGINE FACT SHEET (1ST STAGE)

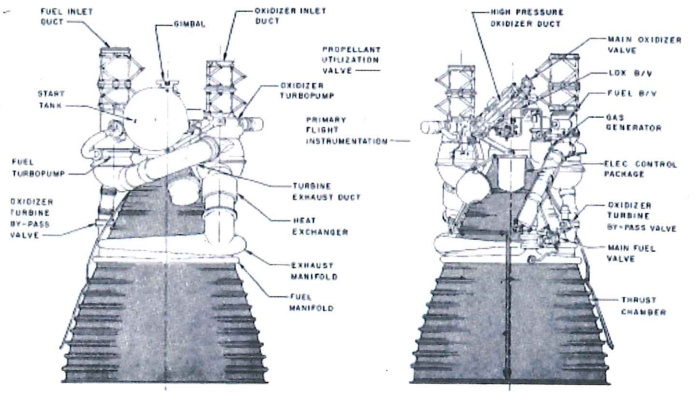


LENGTH	19 ft.
WIDTH	12 ft. 4 in.
THRUST (sea level)	1,500,000 lb.
SPECIFIC IMPULSE (minimum)	260 sec.
RATED RUN DURATION	150 sec.
FLOWRATE: Oxidizer	3,945 lb./sec. (24,811 gpm)
Fuel	1,738 lb./sec. (15,471 gpm)
MIXTURE RATIO	2.27:1 oxidizer to fuel
CHAMBER PRESSURE	965 psia
WEIGHT FLIGHT CONFIGURATION	18,500 lb. maximum
EXPANSION AREA RATIO (A_3/A_2)	16:1 with nozzle extension ←
	10:1 without nozzle extension ←
COMBUSTION TEMPERATURE: Thrust Chamber	5,970°F
Gas Generator	1,465°F
MAXIMUM NOZZLE EXIT DIAMETER	11 ft. 7 in.

NOTE: F-1 engine will be uprated to 1,522,000 lb. thrust for Vehicle 504 and all subsequent operational vehicles.

SATURN V NEWS REFERENCE — CHANGED DECEMBER 1968

J-2 ENGINE FACT SHEET (2ND AND 3rd STAGES)



LENGTH	11 ft. 1 in.
WIDTH	6 ft. 8½ in.
NOZZLE EXIT DIAMETER	6 ft. 5 in.
THRUST (altitude)	230,000 lb. (vehicle 503 2nd stage 225,000 lb.)
SPECIFIC IMPULSE	424 sec. (427 at 5:1 mixture ratio)
RATED RUN DURATION	500 sec.
FLOWRATE: Oxidizer	449 lb/sec (2,847 gpm)
Fuel	81.7 lb/sec (8,365 gpm)
MIXTURE RATIO	5.5:1 oxidizer to fuel
CHAMBER PRESSURE (Pc)	763 psia
WEIGHT, DRY, FLIGHT CONFIGURATION	3,480 lb.
EXPANSION AREA RATIO (A_3/A_2)	27.5:1 ←
COMBUSTION TEMPERATURE	5,750°F

FIG. 29. F-1 (1ST STAGE) AND J-2 (2ND AND 3RD STAGES) ENGINES OF THE SATURN-V ROCKET (44)

FLOWS IN DUCTS WITH HEAT ADDITION

CONSIDER THE CONSTANT-AREA DUCT IN FIG. 30 SUBJECT TO AN AMOUNT Q OF HEAT ADDED PER UNIT VOLUME AND TIME. THE HEAT IS ASSUMED TO BE ADDED VOLUMETRICALLY BECAUSE OF EXOTHERMICAL CHEMICAL REACTIONS OCCURRING IN THE GAS, OR THROUGH THE WALLS BY HEAT CONDUCTION. THE DETAILS OF THESE MECHANISMS WILL HOWEVER NOT BE STUDIED HERE.

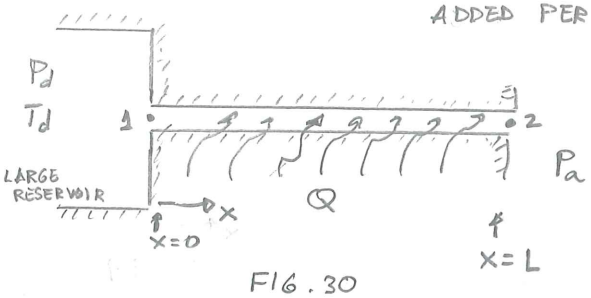


FIG. 30

THE EQUATIONS OF FLUID MOTION ARE GIVEN BY EQS. (11), (12) AND (19) IN PAGE 6 PARTICULARIZED FOR STEADY 1-DIMENSIONAL FLOWS:

$$\frac{\partial}{\partial x}(\rho u) = 0 \qquad (73)$$

$$\frac{\partial}{\partial x}(\rho u^2) = -\frac{\partial p}{\partial x} \qquad (74)$$

$$\frac{\partial}{\partial x}\left(h + \frac{u^2}{2}\right) = \underbrace{\frac{Q}{\rho u}}_{\text{HEAT ADDED PU. TIME AND VOLUME}} \qquad (75)$$

NOTE THAT THE FLOW IS NOT ISENTROPIC!

INTEGRATION OF (73) LEADS TO $\dot{m}'' = \rho u = \rho_1 u_1 = \text{CONST.}$ WHILE INTEGRATION OF (74) GIVES

$$p + \rho u^2 = \text{CONST.} = p_1 + \rho_1 u_1^2 \qquad (76)$$

SIMILARLY (75) YIELDS

$$h + \frac{u^2}{2} = \int_0^x \frac{Q(x)}{\dot{m}''}\,dx + h_1 + \frac{u_1^2}{2} \qquad (77)$$

$q(x) \equiv$ HEAT ADDED PER UNIT MASS

DIVIDING EQ. (76) BY p_1 AND USING THE MACH # DEFINITION:

$$\frac{p}{p_1} = 1 + \frac{\rho_1 u_1^2}{p_1} - \frac{\rho u^2}{p_1} = 1 + \gamma M_{a_1}^2 - \gamma M_a^2 \frac{p}{p_1}$$

$$\Rightarrow \boxed{\frac{p}{p_1} = \frac{1 + \gamma M_{a_1}^2}{1 + \gamma M_a^2}} \qquad (78)$$

SIMILARLY, USING CONTINUITY AND THE EQ. OF STATE:

$$\frac{\rho}{\rho_1} = \frac{u_1}{u} = \frac{u_1}{a_1}\frac{a_1}{a}\frac{a}{u} = \frac{M_{a_1}}{M_a}\left(\frac{T_1}{T}\right)^{1/2} = \frac{M_{a_1}}{M_a}\left(\frac{\rho}{\rho_1}\right)^{1/2}\left(\frac{p_1}{p}\right)^{1/2}$$

AND SUBSTITUTING (78):

$$\boxed{\frac{\rho}{\rho_1} = \left(\frac{M_{a_1}}{M_a}\right)^2 \frac{p_1}{p} = \left(\frac{M_{a_1}}{M_a}\right)^2 \left(\frac{1 + \gamma M_a^2}{1 + \gamma M_{a_1}^2}\right)} \qquad (79)$$

ADDITIONALLY, DIVIDING (77) BY h_1 GIVES

$$\frac{h}{h_1} + \frac{U^2}{2h_1} = \frac{q}{h_1} + 1 + \frac{U_1^2}{2h_1} \Rightarrow \frac{T}{T_1} + Ma^2 \left(\frac{T}{T_1}\right) \frac{(\gamma-1)}{2} = \frac{q}{h_1} + 1 + \frac{(\gamma-1)}{2} Ma_1^2$$

AND THEREFORE

$$\boxed{\frac{T}{T_1} = \frac{1 + \frac{(\gamma-1)}{2} Ma_1^2 + \frac{q}{h_1}}{1 + \frac{(\gamma-1)}{2} Ma^2}} \quad (80)$$

SUBSTITUTING (78)-(80) INTO THE EQUATION OF STATE:

$$\frac{P}{P_1} = \frac{\rho}{\rho_1} \frac{T}{T_1} \Rightarrow \frac{P_1}{P} \frac{\rho}{\rho_1} \frac{T}{T_1} = 1$$

YIELDS

$$\left(\frac{Ma_1}{Ma}\right)^2 \left(\frac{1+\gamma Ma^2}{1+\gamma Ma_1^2}\right)^2 \left\{ \frac{1 + \frac{(\gamma-1)}{2} Ma_1^2 + \frac{q}{h_1}}{1 + \frac{(\gamma-1)}{2} Ma^2} \right\} = 1$$

WHICH CAN BE EASILY REWRITTEN AS

$$\boxed{\frac{\mathcal{F}^2(Ma)}{\mathcal{F}^2(Ma_1)} = 1 + \frac{q}{h_{o_1}}} \quad (81)$$

* NOTE THAT

$$1 + \frac{q}{h_{o_1}} = 1 + \frac{h_o - h_{o_1}}{h_{o_1}} = \frac{h_o}{h_{o_1}} = \frac{T_o}{T_{o_1}}$$

WHERE $h_{o_1} = h_1 \left[1 + \frac{(\gamma-1)}{2} Ma_1^2\right]$ IS THE STAGNATION ENTHALPY AT THE ENTRANCE OF THE DUCT, AND

$$\boxed{\mathcal{F}(Ma) = \frac{Ma}{1+\gamma Ma^2} \left[1 + \frac{(\gamma-1)}{2} Ma^2\right]^{1/2}} \quad (82)$$

IS A FUNCTION OF THE LOCAL MACH NUMBER (SEE FIG. 30).

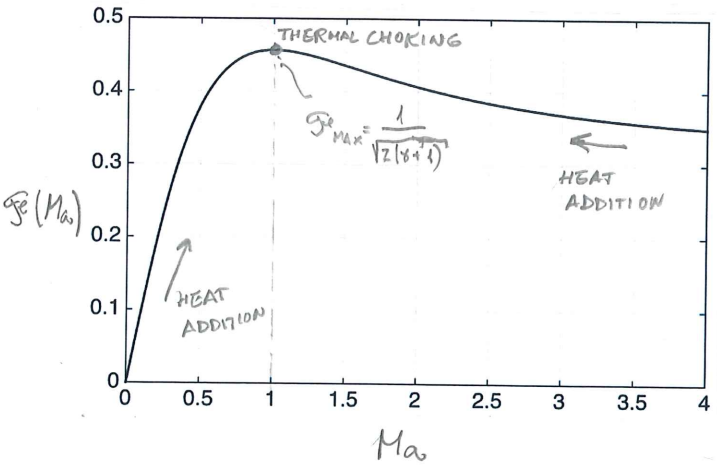


FIG. 30: THE FUNCTION $\mathcal{F}(Ma)$ IN EQ. (82) PARTICULARIZED FOR $\gamma = 1.4$

- FOR SUBSONIC FLOWS ($Ma_1 < 1$), THE ADDITION OF HEAT INCREASES THE MACH NUMBER (AS $\mathcal{F}^2(Ma)$ INCREASES WITH q AS IN EQ. (81)), UNTIL THE MAXIMUM $\mathcal{F}_{MAX} = \frac{1}{\sqrt{2(\gamma+1)}}$ AND $q_{MAX} = \left(\frac{\mathcal{F}^2_{MAX}}{\mathcal{F}^2(Ma_1)} - 1\right) h_{o_1}$ ARE REACHED, AND THE FLOW BECOMES SONIC OR THERMALLY CHOKED.

- FOR SUPERSONIC FLOWS ($Ma_1 > 1$), THE ADDITION OF HEAT DECREASES THE MACH NUMBER UNTIL SONIC CONDITIONS ARE ACHIEVED, WHERE THE FLOW BECOMES THERMALLY CHOKED.

* SCRAMJET UNSTART

PERHAPS ONE OF THE MOST IMPORTANT TECHNOLOGICAL PROBLEMS RELATED TO THE ADDITION OF HEAT IN SUPERSONIC FLOWS IS THAT OF THE UNSTART IN SUPERSONIC COMBUSTION RAMJETS (SCRAMJET). A SCRAMJET IS A CHEMICAL PROPULSION SYSTEM FOR HYPERSONIC VEHICLES IN WHICH THE ATMOSPHERIC AIR IS INGESTED SUPERSONICALLY AND BURNS WITH THE FUEL INJECTED IN THE COMBUSTION CHAMBER. THE BURNT GASES ARE ACCELERATED IN A SUPERSONIC NOZZLE PRODUCING THE THRUST OF THE VEHICLE (SEE FIG. 31)

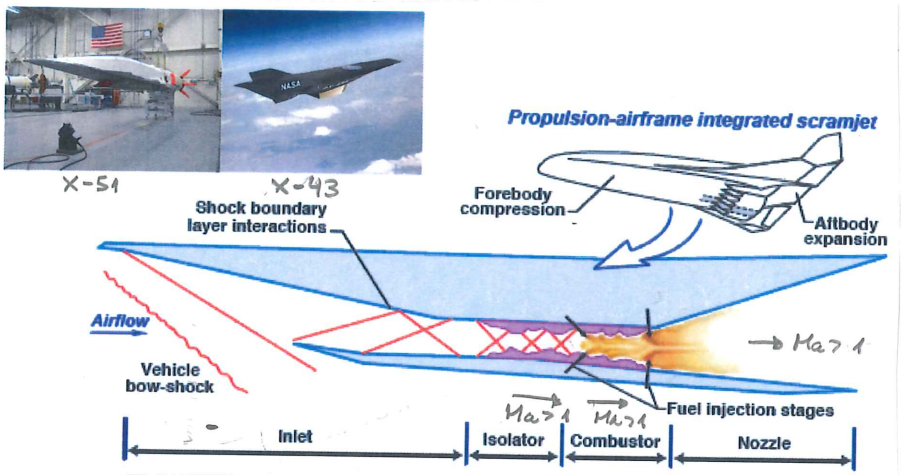


FIG. 31: SCRAMJET ENGINE SCHEMATICS.

IT IS IMPORTANT TO NOTICE THAT, FOR THE MOST PART, COMBUSTION IN SCRAMJETS OCCURS SUPERSONICALLY. THEREFORE, IN A FIRST APPROXIMATION, THE PROBLEM RESEMBLES THE ONE STUDIED IN THE PREVIOUS SECTION, ALTHOUGH HERE TURBULENT MIXING, BOUNDARY LAYERS, AND AEROTHERMOCHEMICAL COUPLING MAKE THE ANALYSIS MUCH MORE INVOLVED.

AN IMPORTANT TECHNOLOGICAL PROBLEM ASSOCIATED WITH SCRAMJET ENGINES IS THE OCURRENCE OF UNSTART DUE TO EXCESSIVE HEAT RELEASE. AS OBSERVED IN THE PREVIOUS SECTION, THE HEATING OF SUBSONIC OR SUPERSONIC FLOWS DRIVES THE SOLUTION TOWARD SONIC CONDITIONS, EXCESSIVE HEAT RELEASE ($q \geq q_{MAX}$) IS INCOMPATIBLE WITH THE STEADY FORMULATION. FOR INSTANCE, WHEN $q > q_{MAX}$ AND THE FLOW IS SUPERSONIC (AS IN A SCRAMJET COMBUSTOR) THE SOLUTION MUST INVOLVE A SHOCK PROPAGATING UPSTREAM AND LEAVING SUBSONIC FLOW DOWNSTREAM, WITH THE SHOCK EVENTUALLY REACHING THE INLET AND CHANGING THE BOUNDARY CONDITIONS THERE → TO SUBSONIC FLOW IN ORDER TO ACCOMODATE THE EXCESS HEAT. THIS IS NECESSARILY ASSOCIATED WITH A DECREASE IN THE MASS FLOW RATE THROUGH THE ENGINE, THEREBY LEADING TO UNSTART AND IN SOME CASES CATASTROPHIC FAILURE (SEE FIG. 32)

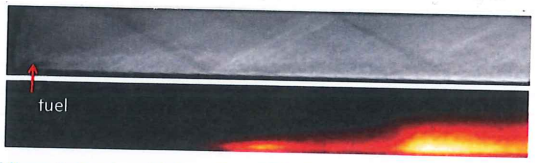


FIGURE 4. Schlieren (above) and false-colour OH* chemiluminescence (below) visualizations of the flow downstream of the injector (x=56-136 mm) for an equivalence ratio of 0.35.

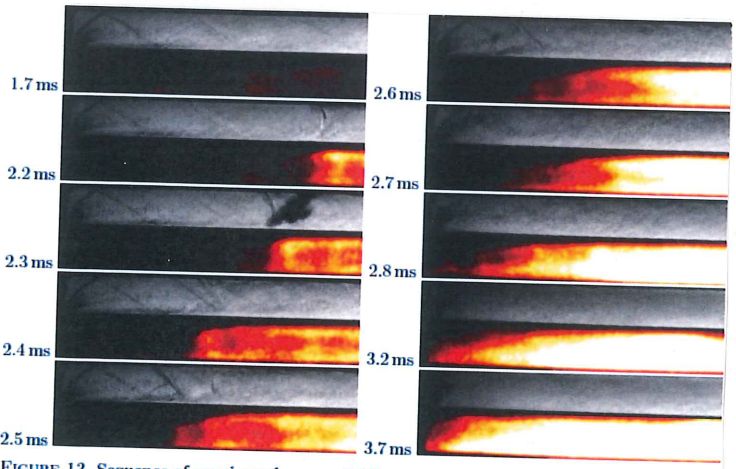


FIGURE 13. Sequence of quasi-synchronous Schlieren and OH* chemiluminescence images of the flow near the injector (x=56-136 mm) for an equivalence ratio of 1.1.

FIG 32. EXPERIMENTAL OBSERVATIONS OF UNSTART (LAURENCE ET AL, J. FLUID MECH. 2012)

4. WAVE MOTION IN COMPRESSIBLE FLOWS

THE SPEED OF SOUND IS RESPONSIBLE FOR THE PROPAGATION OF DISTURBANCES IN THE GASDYNAMICS EQUATIONS. IN THIS PART OF THE COURSE, THE FOCUS IS ON THE PROPAGATION OF THOSE DISTURBANCES IN THE FLOW FIELD. IT WILL BE DESCRIBED BELOW THAT THE ANALYSIS TURNS THE GASDYNAMICS EQUATIONS INTO WAVE EQUATIONS THAT REPRESENT THE SOLUTIONS ALONG CHARACTERISTIC LINES. THE ANALYSIS IS FIRST PERFORMED IN THE LINEAR LIMIT, FOR WHICH THE DISTURBANCES ARE SMALL, FOLLOWED BY CONSIDERATIONS ARISING FROM NON-LINEARITIES INDUCED BY ORDER-UNITY DISTURBANCES.

LINEAR SOUND WAVES

CONSIDER THE LIMITING CASE IN WHICH CHANGES IN DENSITY AND PRESSURE ACCOMPANYING THE FLUID MOTION ARE VERY SMALL IN COMPARISON WITH THE AVERAGE VALUES ρ_0 AND P_0,

$$\frac{\rho'}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} \ll 1 \quad \text{AND} \quad \frac{P'}{P_0} = \frac{P - P_0}{P_0} \ll 1 . \qquad (83)$$

THE ANALYSIS IS BETTER ILLUSTRATED BY CONSIDERING, FOR INSTANCE, THE PROPAGATION OF THESE DISTURBANCES IN A GAS FLOWING IN A PIPE AT SPEED U_0, WITH $U_0/a_0 = Ma \ll 1$.

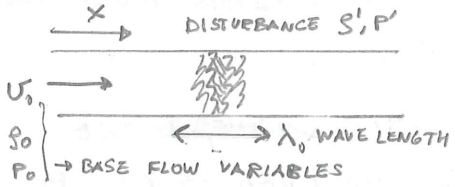


FIG. 7

IN ADDITION TO (83), THE RELEVANT LIMIT IS ONE IN WHICH THE PRESSURE AND DENSITY DISTURBANCES CREATE A VELOCITY PERTURBATION U' OF THE SAME ORDER AS THE VELOCITY OF THE BASE FLOW,

$$\frac{U'}{U_0} = \frac{U - U_0}{U_0} = O(1). \qquad (84)$$

THE PROPAGATION OF THE DISTURBANCES, IN THE ABSENCE OF VISCOUS OR HEAT-CONDUCTION EFFECTS, IS DESCRIBED BY THE ISENTROPIC EQUATIONS OF MOTION

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho U) = 0 \\ &\frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \frac{a^2}{\rho}\frac{\partial \rho}{\partial x} = 0 \end{aligned}\right\} \quad (85)$$

WHERE USE OF $a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$ HAS BEEN MADE IN ORDER TO ELIMINATE THE PRESSURE.

EQUATIONS (85) CAN BE FURTHER SIMPLIFIED BY USING THE LIMITS (83) AND (84). FOR INSTANCE, IN THE CONTINUITY EQUATION:

$$\underbrace{\frac{\partial \rho}{\partial t}}_{O\left(\frac{\rho'}{t_0}\right)} + \underbrace{U\frac{\partial \rho}{\partial x}}_{O\left(\frac{U_0 \rho'}{\lambda_0}\right)} + \underbrace{\rho\frac{\partial U}{\partial x}}_{O\left(\frac{\rho U_0}{\lambda_0}\right)} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial U'}{\partial x} = 0} \qquad (86)$$

SINCE $\frac{U_0 \rho'}{U' \rho_0} \sim \frac{\rho'}{\rho_0} \ll 1$.

NOTE ALSO THAT

$$\frac{\rho'}{t_0} \sim \frac{\rho_0 U_0}{\lambda_0} \Rightarrow \boxed{t_0 \sim \frac{\rho'}{\rho_0}\frac{\lambda_0}{U_0} \sim \frac{\lambda_0}{a_0}}$$

CHARACTERISTIC WAVE PERIOD

ALSO, IN THE MOMENTUM EQUATION:

$$\underbrace{\frac{\partial v}{\partial t}}_{O\left(\frac{U_0}{t_0}\right)} + \underbrace{v\frac{\partial v}{\partial x}}_{O\left(\frac{U_0^2}{\lambda}\right)} + \underbrace{\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}}_{O\left(\frac{a_0^2 \rho'}{\rho_0 \lambda}\right)} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial v'}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} = 0} \quad (87), \quad \text{WHERE } a_0^2 = \frac{\gamma P_0}{\rho_0}$$

SINCE $\dfrac{U_0^2/\lambda_0}{U_0/t_0} \sim \dfrac{t_0 U_0}{\lambda_0} \ll 1$

EQUATIONS (86) AND (87) RENDER THE ORDERS OF MAGNITUDE FOR ρ' AND P':

$$\frac{\partial v'}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} = 0 \Rightarrow \rho' \sim \frac{\rho_0}{a_0^2}\frac{U_0}{t_0}\lambda_0 \sim \frac{\rho_0 U_0}{a_0^2}\frac{\lambda_0}{t_0} \Rightarrow \boxed{\frac{\rho'}{\rho_0} \sim \frac{U_0}{a_0} = Ma \ll 1}$$

AND SINCE $dP = a_0^2 d\rho \Rightarrow (P' = a_0^2 \rho')$ $\boxed{\dfrac{P'}{P_0} \sim a_0^2 \dfrac{\rho'}{\rho_0}\dfrac{\rho_0}{P_0} = O(Ma) \ll 1}$ $(P' \sim \rho_0 a_0 v')$ ACOUSTIC PRESSURE VARIATIONS

IN SUMMARY, THE LINEAR APPROXIMATION (86) - (87) HOLDS WHEN

$$\boxed{\frac{\rho'}{\rho_0} \; O(Ma) \ll 1, \quad \frac{P'}{P_0} = O(Ma) \ll 1, \quad \frac{v'}{U_0} = O(1)} \quad (88)$$

THE WAVE-PROPAGATION REGIME DESCRIBED BY THE SCALING (88) IS CALLED "LINEAR ACOUSTICS". NOTE THAT $\dfrac{I'}{T_0} \sim \dfrac{P'}{P_0} + \dfrac{\rho'}{\rho_0}$ BECAUSE OF THE EQUATION OF STATE $P = \rho R_g T$, THEREFORE $\dfrac{I'}{T_0} = O(Ma) \ll 1$ IN LINEAR ACOUSTICS. FURTHERMORE, DIFFERENTIATING (86) WITH RESPECT TO TIME

$$\frac{\partial^2}{\partial t^2}\rho' + \rho_0 \frac{\partial}{\partial x}\left(\frac{\partial v'}{\partial t}\right) = 0$$

AND USING (87) YIELDS THE WAVE EQUATION FOR THE DENSITY CHANGE

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{\rho'}{\rho_0}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{\rho'}{\rho_0}\right).} \quad (89)$$

A SIMILAR MANIPULATION WITH EQ. (87) LEADS TO THE WAVE EQUATION FOR THE VELOCITY CHANGE

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{v'}{a_0}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{v'}{a_0}\right).} \quad (90)$$

ADDITIONALLY, THE RELATION $P' = a_0^2 \rho'$ MAKES POSSIBLE TO EXPRESS (89) IN TERMS OF THE PRESSURE CHANGE:

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{P'}{P_0}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{P'}{P_0}\right)} \quad (91)$$

EQUATIONS (89) - (91) DESCRIBE THE PROPAGATION OF LINEAR SOUND WAVES

THE GENERAL SOLUTION OF (89)-(91) IS MADE UP FROM TWO PARTICULAR SOLUTIONS, CORRESPONDING TO WAVES $F_{(\rightarrow)}$ AND $G_{(\leftarrow)}$ PROPAGATING IN THE $+x$ AND $-x$ DIRECTIONS;

$$\frac{\rho'}{\rho_0} = F_{(\rightarrow)}(x-a_0t) + G_{(\leftarrow)}(x+a_0t)$$

$$\frac{p'}{\rho_0 a_0^2} = F_{(\rightarrow)}(x-a_0t) + G_{(\leftarrow)}(x+a_0t)$$

$$\frac{u'}{a_0} = F_{(\leftarrow)}(x-a_0t) - G_{(\leftarrow)}(x+a_0t) \qquad (92)$$

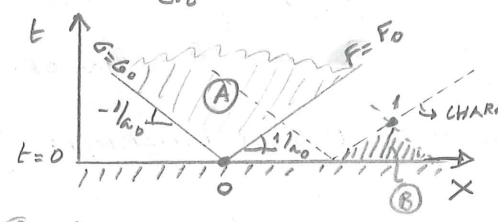
$F_{(\rightarrow)}$ = RIGHT-RUNNING WAVE

$G_{(\leftarrow)}$ = LEFT-RUNNING WAVE

WHERE THE PARTICULAR FORM OF F AND G DEPENDS ON THE INITIAL CONDITIONS OF THE DISTURBANCE. IT IS IMPORTANT TO REALIZE THAT EQS. (89)-(91) CAN BE REWRITTEN IN CHARACTERISTIC FORM AS

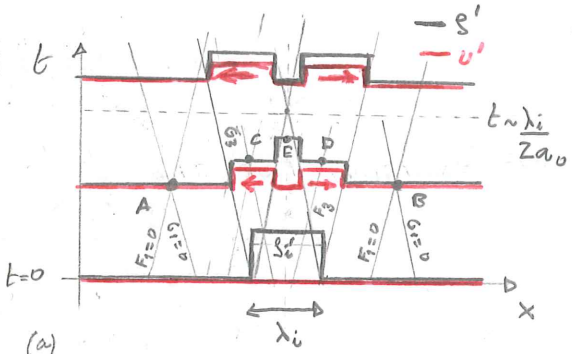
$$\left(\frac{\partial}{\partial t} \pm a_0 \frac{\partial}{\partial x}\right)\left(\frac{\partial \phi}{\partial t} \mp a_0 \frac{\partial \phi}{\partial x}\right) = 0 ,$$

WHERE $\frac{dx}{dt} = \pm a_0$ ARE THE "CHARACTERISTIC LINES". IN PARTICULAR, THE FLOW INFORMATION PROPAGATES ALONG THE CHARACTERISTICS, WITH THE HATCHED REGION IN THE SIDE FIGURE REPRESENTING THE DOMAIN OF INFLUENCE WHERE THE INFORMATION AT POINT 0 AT $t=0$ IS FELT AT A LATER TIME. IT WILL BE SHOWN LATER THAT IN NON-LINEAR ACOUSTICS, WHEN $U_0/a_0 \sim O(1)$, THE CHARACTERISTICS ARE INFLUENCED BY THE GAS VELOCITY IN ADDITION TO a_0.

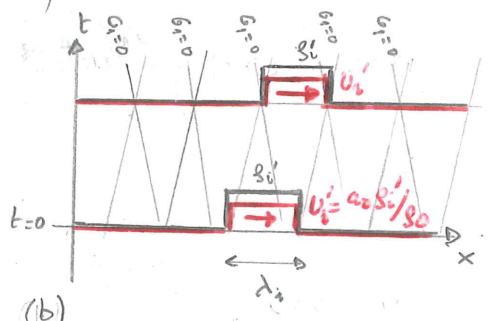


(A) : DOMAIN OF INFLUENCE OF 0

(B) : DOMAIN OF DEPENDENCE OF 1



(a)



(b)

FIG. 32 EXAMPLES OF WAVE PROPAGATION.

THE LINEAR COMBINATION OF EQUATIONS (92) EASILY LEADS TO THE RELATIONS

$$F = \frac{1}{2}\left(\frac{u'}{a_0} + \frac{\rho'}{\rho_0}\right) = \frac{1}{2}\left(\frac{u'}{a_0} + \frac{p'}{\rho_0 a_0^2}\right)$$

$$G = \frac{1}{2}\left(\frac{\rho'}{\rho_0} - \frac{u'}{a_0}\right) = \frac{1}{2}\left(\frac{p'}{\rho_0 a_0^2} - \frac{u'}{a_0}\right) \qquad (93)$$

CONSIDER THE EXAMPLE IN FIG. 32 (a), IN WHICH A DENSITY PULSE OF AMPLITUDE ρ_i'/ρ_0 ACROSS A DISTANCE λ_i IS LEFT TO EVOLVE IN GAS INITIALLY AT REST. AT POINTS A AND B THE GAS REMAINS AT REST SINCE

$$F_1 = 0 = \frac{1}{2}\left(\frac{u'}{a_0} + \frac{\rho'}{\rho_0}\right) \text{ AND } G_1 = 0 = \frac{1}{2}\left(\frac{u'}{a_0} - \frac{\rho'}{\rho_0}\right) \Rightarrow \begin{matrix} u_A' = u_B' = 0 \\ \rho_A' = \rho_B' = 0 \end{matrix}$$

AT POINT C:

$$F_1 = 0 = \frac{1}{2}\left(\frac{u_c'}{a_0} + \frac{\rho_c'}{\rho_0}\right) \text{ AND } G_3 = \frac{1}{2}\frac{\rho_i'}{\rho_0} = \frac{1}{2}\left(\frac{\rho_c'}{\rho_0} - \frac{u_c'}{a_0}\right)$$

AND THEREFORE $\frac{\rho_C'}{\rho_0} = \frac{1}{2}\frac{\rho_i'}{\rho_0}$ AND $\frac{U_C'}{a_0} = -\frac{\rho_C'}{\rho_0} \Rightarrow$ THE DENSITY WAVE AT C HAS HALF OF THE INITIAL AMPLITUDE, WHILE MASS CONSERVATION (EQ. (86)) MAKES THE GAS TO MOVE IN THE $-X$ DIRECTION. THE SAME OCCURS IN POINT D, BUT THE GAS MOVES THERE IN THE $+X$ DIRECTION:

$$G_1 = 0 = \frac{1}{2}\left(\frac{\rho_D'}{\rho_0} - \frac{U_D'}{a_0}\right) \text{ AND } F_3 = \frac{1}{2}\frac{\rho_i'}{\rho_0} = \frac{1}{2}\left(\frac{\rho_D'}{\rho_0} + \frac{U_D'}{a_0}\right) \Rightarrow \begin{cases} \frac{\rho_D'}{\rho_0} = \frac{1}{2}\frac{\rho_i'}{\rho_0} \\ \frac{U_D'}{a_0} = \frac{\rho_D'}{\rho_0} \end{cases}$$

NOTE THAT AT POINT E THE DENSITY PULSE REMAINS UNDISTURBED WITH RESPECT TO ITS INITIAL STATE,

$$G_3 = \frac{1}{2}\frac{\rho_i'}{\rho_0} = \frac{1}{2}\left(\frac{\rho_E'}{\rho_0} - \frac{U_E'}{a_0}\right) \text{ AND } F_3 = \frac{1}{2}\frac{\rho_i'}{\rho_0} = \frac{1}{2}\left(\frac{\rho_E'}{\rho_0} + \frac{U_E'}{a_0}\right) \Rightarrow \begin{cases} \frac{\rho_E'}{\rho_0} = \frac{\rho_b'}{\rho_0} \\ U_E' = 0 \end{cases}$$

AFTER A TIME $t = \frac{\lambda_i/2}{a_0}$ HAS PASSED THE RIGHT-RUNNING CHARACTERISTIC FROM THE LEFT BOUNDARY OF THE INITIAL PULSE CROSSES THE LEFT-RUNNING CHARACTERISTIC FROM THE RIGHT BOUNDARY OF THE INITIAL PULSE. THE DENSITY PULSE THEN BREAKS INTO A RIGHT-RUNNING AND A LEFT-RUNNING DENSITY AND VELOCITY PULSES OF DIMENSIONLESS AMPLITUDES $\rho'/\rho_0 = \frac{1}{2}\rho_i'/\rho_0$ AND $U'/a_0 = \pm\frac{1}{2}\rho_i'/\rho_0$.

CERTAIN INITIAL CONDITIONS LEAD TO CANCELLATION OF ONE OF THE WAVES, AS IN FIG. 32(b). IN PARTICULAR, IF THE DENSITY AND VELOCITY DISTRIBUTIONS ARE INITIALLY OF THE FORM SHOWN IN FIG. 32(b), WITH $\rho'/\rho_0 = \rho_i'/\rho_0$ AND $U'/a_0 = U_i'/a_0 = \rho_i'/\rho_0$, THEN $G = 0$ EVERYWHERE AS EASILY SEEN FROM EQS. (93). THEN THE RECTANGULAR PULSE PROPAGATES AT CONSTANT AMPLITUDE IN THE RIGHT DIRECTION ONLY.

A SIMILAR EXAMPLE IS OBSERVED WHEN A PISTON IS SET IN MOTION IN A DUCT OF LENGTH L FILLED WITH A GAS (SEE FIG. 33). ASSUME THAT THE PISTON MOVES (→ INITIALLY AT REST) WITH A CONSTANT SPEED U_P TO THE RIGHT, COMPRESSING THE GAS, WITH $U_P/a_0 \ll 1$ SUCH THAT THE PROBLEM CAN BE ANALYZED USING THE LINEAR FORMULATION GIVEN ABOVE. NOTE ALSO THAT IF $U_P/a_0 \ll 1$ THE DISTANCE TRAVELED BY THE PISTON AFTER AN ACOUSTIC PERIOD $\left(\sim \frac{L}{a_0}\right)$ IS IS OF ORDER $LU_P/a_0 \ll L$ AND THEREFORE CAN BE NEGLECTED IN THE FIRST APPROXIMATION.

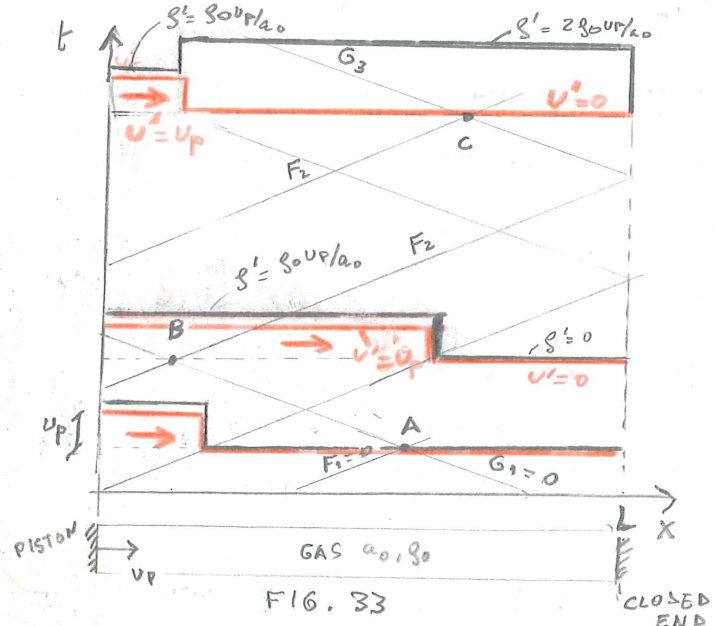


FIG. 33

$\Rightarrow$ AT POINT A: $F_1 = G_1 = 0 \Rightarrow U_A' = \rho_A' = P_A' = 0$

⇒ AT THE PISTON SURFACE: $u' = u_p$ AND $G_1 = 0$ (SEE THE DISCUSSION IN PREVIOUS EXAMPLE OF THE TRANSLATING PULSE), THEREFORE $0 = \frac{1}{2}\left(\frac{u_p}{a_0} - \frac{\rho_p'}{\rho_0}\right)$

SO THAT $\rho_p' = \rho_0\left(\frac{u_p}{a_0}\right)$ AND $p_p' = a_0^2 \rho_p' = \rho_0 u_p a_0$.

⇒ AT POINT B: $F_2 = \frac{1}{2}\left(\frac{u_B'}{a_0} + \frac{\rho_B'}{\rho_0}\right) = \frac{1}{2}\left(\frac{u_p}{a_0} + \frac{\rho_p'}{\rho_0}\right) = \frac{u_p}{a_0}$

$G_1 = 0$

THE GAS MOVES TO THE RIGHT AT A VELOCITY u_p WITH AN OVERPRESSURE $\rho_0 u_p a_0$.

NOTE THAT THE GAS IS AT REST FOR $x > a_0 t$ SINCE THE DISTURBANCE HAS NOT ARRIVED THERE YET, WHILE FOR $x < a_0 t$ IT MOVES TO THE RIGHT AT SPEED u_p CREATING AN OVERPRESSURE PULSE $p' = \rho_0 u_p a_0$. WHEN THIS ACOUSTIC WAVE REACHES THE END OF THE DUCT AT $x = L$, THE REBOUND DYNAMICS DEPENDS ON THE BOUNDARY CONDITIONS THERE. FOR INSTANCE, IF THE DUCT END IS CLOSED, THEN $u' = 0$ AT $x = L$, AND THE LEFT-RUNNING CHARACTERISTIC G_3 IS $G_3 = +F_2 = +\frac{u_p}{a_0}$ (SEE EQ. (92) AND USE $u' = 0$).

⇒ AT POINT C: $F_2 = \frac{u_p}{a_0} = \frac{1}{2}\left(\frac{u_c'}{a_0} + \frac{\rho_c'}{\rho_0}\right)$

$G_3 = \frac{u_p}{a_0} = \frac{1}{2}\left(\frac{\rho_c'}{\rho_0} - \frac{u_c'}{a_0}\right)$

$u_c' = 0$

$\rho_c' = 2\rho_0 \frac{u_p}{a_0} \Rightarrow p_c' = 2\rho_0 u_p a_0$

ON THE OTHER HAND, IF THE END OF THE DUCT IS OPEN TO AN ATMOSPHERE, THEN $p' = 0$ IS THE BOUNDARY CONDITION TO BE APPLIED AT $x = 0$. ($p = p_0$ AT EXIT SINCE AMBIENT IS ALSO AT p_0) THEN $G_3 = -F_2 = -u_p/a_0$ (SEE EQ (92) AND USE $p' = 0$), SO THAT $p_c' = 0 = \rho_c'$ AND $u_c' = 2u_p$.

NON-LINEAR ACOUSTIC WAVES

WHEN THE CONDITIONS GIVEN IN (88) ARE NOT SATISFIED, IN THAT THE MAGNITUDE OF THE PRESSURE AND DENSITY DISTURBANCES ARE NOT SMALL, THEN THE LINEAR THEORY GIVEN ABOVE IS NOT APPROPRIATE. IN USUAL SCENARIOS, HOWEVER, THE APPROXIMATION (88) IS SUFFICIENT. FOR INSTANCE, FOR A SOUND WAVE OF 100 dB PRESSURE LEVEL (LOUD HUMAN VOICE AT ~100 Hz), THE CHARACTERISTIC PRESSURE VARIATIONS ARE OF ORDER 10^{-5} TIMES THE ATMOSPHERIC PRESSURE, THE INDUCED VELOCITIES ARE OF ORDER 10^{-5} TIMES THE SPEED OF SOUND, AND THE AMPLITUDE OF THE DISPLACEMENT OF THE AIR PARTICLES IS OF ORDER 10^{-6} m. NON-LINEAR ACOUSTIC EFFECTS START BECOMING IMPORTANT ABOVE 190 dB, AND ARE ALSO FUNDAMENTAL TO THE FORMATION OF SHOCK WAVES. IN ORDER TO TACKLE THESE EFFECTS, CONSIDER AGAIN EQUATIONS (85), WHICH CORRESPOND TO ONE-DIMENSIONAL ISENTROPIC FLOW. SINCE

$$\frac{p}{\rho^\gamma} = \text{CONST.} \Rightarrow \frac{a^2}{\rho^{\gamma-1}} = \text{CONST.} \Rightarrow 2\frac{da}{a} = (\gamma-1)\frac{d\rho}{\rho}$$

SUBSTITUTING THE LAST RELATION INTO EQS.(85), THEY BECOME

$$\text{(CONTINUITY)}\quad \frac{\partial a}{\partial t} + u\frac{\partial a}{\partial x} + \frac{(\gamma-1)}{2} a\frac{\partial u}{\partial x} = 0 \qquad (94)$$

$$\text{(MOMENTUM)}\quad \frac{(\gamma-1)}{2}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}\right) + a\frac{\partial a}{\partial x} = 0 \qquad (95)$$

SUMMING (94) AND (95), ONE OBTAINS THE EQUATION

$$\left[\frac{\partial}{\partial t} + (u+a)\frac{\partial}{\partial x}\right]\left[a + \left(\frac{\gamma-1}{2}\right)u\right] = 0\,, \qquad (96)$$

WHILE SUBSTRACTING (95) FROM (94) LEADS TO

$$\left[\frac{\partial}{\partial t} + (u-a)\frac{\partial}{\partial x}\right]\left[a - \left(\frac{\gamma-1}{2}\right)u\right] = 0\,. \qquad (97)$$

EQUATIONS (96) AND (97) ARE THE DESCRIPTORS OF ONE-DIMENSIONAL NON-LINEAR ACOUSTIC WAVES. THEY CAN BE REWRITTEN IN A CONDENSED FORM AS

$$\left.\begin{aligned} \frac{\partial J^+}{\partial t} + (u+a)\frac{\partial J^+}{\partial x} &= 0 \\ \frac{\partial J^-}{\partial t} + (u-a)\frac{\partial J^-}{\partial x} &= 0 \end{aligned}\right\} \qquad (98)$$

WHERE

$$J^{\pm} = a \pm \left(\frac{\gamma-1}{2}\right)u$$

ARE THE "RIEMANN INVARIANTS"

IN PARTICULAR, THE INVARIANTS $J^{\pm}$ ARE THE NON-LINEAR ANALOGY TO THE FUNCTIONS F AND G IN LINEAR ACOUSTICS. IT IS EASY TO SEE THAT EQS. (98) ARE CHARACTERISTIC EQUATIONS WITH

$$C_+ : \frac{dx}{dt} = u + a\,, \qquad C_- : \frac{dx}{dt} = u - a \qquad (99)$$

THE RIGHT- AND LEFT-RUNNING CHARACTERISTIC CURVES, RESPECTIVELY. CONTRARY TO THEIR LINEAR COUNTERPARTS $\left(\frac{dx}{dt} = \pm a_0\right)$, THE CHARACTERISTIC LINES (99) ARE CURVED.

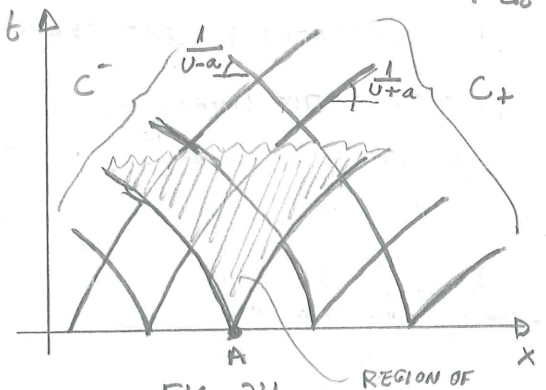


FIG. 34

EQS. (99) SHOW THAT J^+ REMAINS CONSTANT ALONG THE CURVES C_+, WHILE J^- REMAINS CONSTANT ALONG THE CURVES C_-. ALL SIGNALS, HOWEVER, MOVE AT SPEED a WITH RESPECT TO THE GAS.

IF THE INVARIANTS J^+ AND J^- WERE KNOWN IN A POINT OF SPACE AND TIME, THE GAS VELOCITY AND THE SPEED OF SOUND COULD BE COMPUTED AS

$$u = \frac{J^+ - J^-}{\gamma - 1} \quad \text{AND} \quad a = \frac{J^+ + J^-}{2} \qquad (100)$$

THE DIFFICULTY IN THE NON-LINEAR THEORY ARISES FROM THE FACT THAT GIVEN A POINT $\{x_0, t_0\}$ IT IS NOT POSSIBLE TO KNOW THE ORIGIN OF THE CURVES C_+ AND C_- INTERSECTING AT IT, UNLESS THE SOLUTION OF THE PROBLEM IS KNOWN. THIS IS BECAUSE THE CURVATURE OF C_+ IS INFLUENCED BY ALL THE PREVIOUS INTERSECTIONS WITH OTHER C_- CURVES BEFORE ARRIVING TO $\{x_0, t_0\}$, AND SIMILARLY THE CURVATURE OF C_- IS INFLUENCED BY ALL THE PREVIOUS INTERSECTIONS WITH C_+ CURVES. (SEE FIG. 35a)

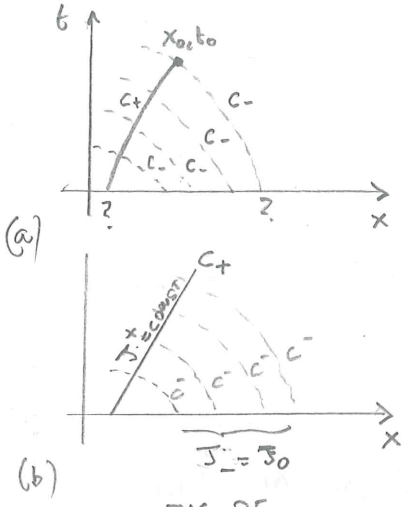


FIG. 35.

A PARTICULAR EXAMPLE IN WHICH THE PROBLEM CAN BE SOLVED ANALYTICALLY IS THE ONE WHERE ONE OF THE INVARIANTS $J_\pm$ IS THE SAME EVERYWHERE. FOR INSTANCE, IF $J_- = J_0 = \text{CONST}$ DOES NOT DEPEND ON THE POSITION, THEN THE GAS VELOCITY AND SPEED OF SOUND ARE GIVEN BY

$$u = \frac{J^+ - J_0}{\gamma - 1} \quad \text{AND} \quad a = \frac{J^+ + J_0}{2}$$

THESE ARE CALLED "NON-LINEAR SIMPLE WAVES"

SINCE $J^+ = \text{CONST}$. ALONG A GIVEN C_+, IT IS THEN FOUND THAT IN THIS PARTICULAR EXAMPLE THE C_+ CHARACTERISTICS ARE STRAIGHT LINES, IN THAT THE SLOPE $1/(u+a_0)$ IS CONSTANT (SEE FIG. 35b). A TYPICAL EXAMPLE OF THIS SITUATION IS THE ONSET OF MOTION OF A PISTON IN A DUCT FILLED WITH GAS INITIALLY AT REST. THE DUCT HAS A LENGHT L, AND THE PISTON IS SET IN MOTION AT $t>0$ WITH A SPEED $-U_p$ (SAY A LINEAR TIME RAMP) TO THE LEFT SUCH THAT THE GAS IS EXPANDED. THE RESULTING WAVES ARE THEREFORE EXPANSION WAVES. THE PISTON VELOCITY IS COMPARABLE TO THE SPEED OF SOUND, $U_p = O(a_0)$, SO THAT THE LINEAR THEORY IS NOT APPLICABLE. ALONG THE FIRST CHARACTERISTIC J_1^+, THE PISTON HAS NOT STARTED MOVING YET, $U_p = 0$, SO THAT $J_1^+ = a_0 = J_0 = J_1^-$. ALL POINTS BELOW THAT CHARACTERISTIC (FOR INSTANCE POINT "A") ARE AT REST SINCE THE INFORMATION OF THE PISTON'S MOTION HAS NOT ARRIVED THERE YET. ALONG THE SECOND CHARACTERISTIC, J_2^+ CAN BE EVALUATED FROM THE PISTON SURFACE ($U = -U_p$) TO BE

$$J_2^+ = a_p + u_p\left(\frac{\gamma-1}{2}\right),$$

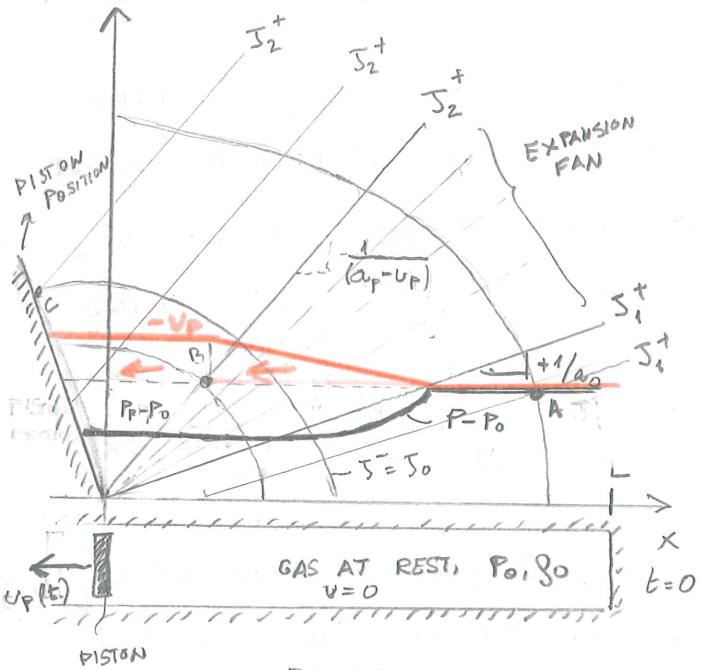


FIG. 36.

WHICH TOGETHER WITH $J_0 = a_0$, YIELDS

POINT C (PISTON SURFACE): $a_C = a_p = \frac{J_2^+ + J_0}{2} = \frac{a_p - U_p\left(\frac{\gamma-1}{2}\right) + a_0}{2} \Rightarrow \boxed{a_p = a_0 - U_p\left(\frac{\gamma-1}{2}\right)}$

($a_p < a_0$ DUE TO THE EXPANSION)

POINT B: $a_B = \frac{J_2^- + J_0}{2} = a_C = a_p$, $U_B = U_C = -U_p = \frac{J_2^+ - J_0}{\gamma-1}$.

SINCE THE SLOPE OF THE J^+ CHARACTERISTICS VARIES FROM $1/a_0$ TO $1/a_p - U_p > 1/a_0$, THERE IS AN EXPANSION FAN BETWEEN J_2^+ AND J_1^+, AS SHOWN IN FIG. 36. THE PRESSURE BEHIND THE PISTON DECREASES DUE TO THE EXPANSION AND ATTAINS A VALUE GIVEN BY THE ISENTROPIC RELATION $\frac{P_p}{P_0} = \left(\frac{a_p}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} = \left(1 - \left(\frac{\gamma-1}{2}\right)\frac{U_p}{a_0}\right)^{\frac{2\gamma}{\gamma-1}}$, WHICH IS NOT EQUAL TO THE PRESSURE THAT WOULD BE OBTAINED BY ACCELERATING THE FLOW QUASI-STEADILY ($\frac{P_p}{P_0} = \left[1 + \left(\frac{\gamma-1}{2}\right)\left(\frac{U_p}{a_0}\right)^2\right]^{\frac{\gamma}{\gamma-1}}$).

FOR $U_p = 2a_0\gamma/\gamma-1$, THE VALUE OF SPEED OF SOUND ON THE PISTON SURFACE a_p BECOMES NEGATIVE AND THE SOLUTION GIVEN ABOVE FAILS. THIS REPRESENTS THE FACT THAT FOR TOO LARGE PISTON VELOCITIES THE GAS CANNOT FOLLOW THE PISTON AND THERE IS A ZONE OF VACUUM BETWEEN THE LAST CHARACTERISTIC J_2^+ AND THE PISTON SURFACE (SEE FIG. 37)

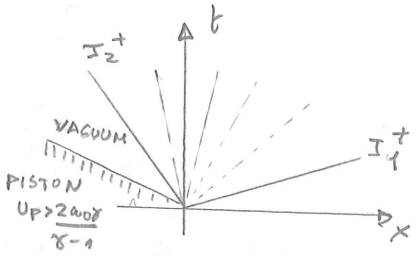
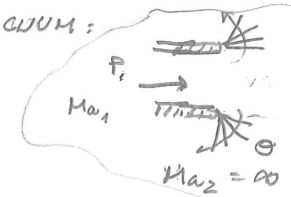


FIG. 37.

THE SITUATION IS SIMILAR TO OVERPASSING THE MAXIMUM DEFLECTION ANGLE OF A SUPERSONIC JET EXPANDING IN VACUUM:



$\left.\begin{array}{l} P_\infty = 0 \text{ (VACUUM)} \\ \nu(Ma_2) = 124.7° \end{array}\right\} \theta_{MAX} = \nu(Ma_2) - \nu(Ma_1)$

MAXIMUM TURNING ANGLE.

* FORMATION OF SHOCK WAVES: COALESCENCE OF CHARACTERISTICS

CONSIDER THE CASE IN WHICH THE PISTON MOVES TO THE RIGHT INSTEAD OF THE LEFT, THEREBY COMPRESSING THE GAS INSTEAD OF EXPANDING IT. THE VELOCITY OF THE PISTON U_p IS OF THE SAME ORDER AS THE SPEED OF SOUND, a_0, SO THAT THE LINEAR THEORY IS NOT APPLICABLE. SIMILAR TO THE EXAMPLE OF GAS EXPANSION SOLVED ABOVE, THE FIRST CHARACTERISTIC C_1^+ THAT LEAVES THE PISTON SURFACE UPON ITS IMMINENT ACCELERATION TO U_p HAS AN INVARIANT $J_1^+ = a_0$. ONCE THE PISTON REACHES ITS CRUISE SPEED U_p, THE CHARACTERISTICS C_2^+ LEAVING THE PISTON HAVE A SMALLER SLOPE $1/a_p + U_p$, WITH $a_p = a_0 + U_p\left(\frac{\gamma-1}{2}\right)$ ($a_p > a_0$ DUE TO THE COMPRESSION) AND $J_2^+ = a_p + U_p\left(\frac{\gamma-1}{2}\right)$. THEREFORE THE CHARACTERISTICS C_2^+ TEND TO RUN INTO THE CHARACTERISTICS C_1^+. WHEN THESE TWO CHARACTERISTICS INTERSECT, THE SOLUTION OF THE PROBLEM IS MULTIVALUED AT THE INTERSECTION POINT, WHICH LEADS TO THE FORMATION OF A SHOCK WAVE. THE ISENTROPIC FORMULATION (98) IS NO LONGER VALID AND THE SOLUTION BREAKS DOWN. NOTICE THAT THE FORMATION OF THE SHOCK IS A CONSEQUENCE OF THE NON-LINEAR PROPAGATION VELOCITY OF THE WAVES. IN THE LINEAR THEORY THE CHARACTERISTICS DON'T INTERSECT EACH OTHER.

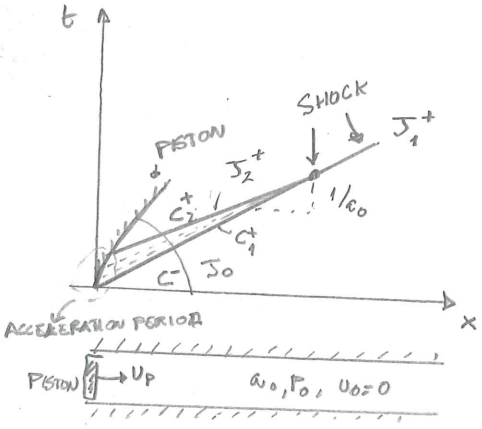


FIG 38

SPHERICAL SOUND WAVES

IN THE ABSENCE OF VISCOSITY AND HEAT CONDUCTION, THE PLANAR PULSES DESCRIBED ABOVE CONTINUE PERPETUALLY WITHOUT CHANGES IN THEIR FORM OR AMPLITUDE. THIS IS NOT THE CASE FOR SPHERICAL WAVES. THE EQUATIONS ANALOGOUS TO (86)-(87) IN SPHERICAL COORDINATES ARE

$$\frac{\partial \rho'}{\partial t} + \frac{\rho_0}{r^2}\frac{\partial}{\partial r}\left(u' r^2\right) = 0 \qquad (101)$$

$$\frac{\partial u'}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial r} = 0 \qquad (102)$$

FOR $M_{a_0} = \frac{U_0}{a_0} \ll 1$ (LINEAR ACOUSTICS)

SO THAT THE CORRESPONDING PROPAGATION EQUATIONS FOR THE DENSITY AND VELOCITY DISTURBANCES ARE

$$\frac{\partial^2}{\partial t^2}\left(\frac{\rho'}{\rho_0}\right) = \frac{a_0^2}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{\rho'}{\rho_0}\right)\right) \qquad (103)$$

→ SAME EQUATION FOR $p' = a_0^2 \rho'$

$$\frac{\partial^2}{\partial t^2}\left(\frac{u'}{U_0}\right) = a_0^2 \frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u'\right)\right) \qquad (104)$$

THE SOLUTION FOR A SPHERICAL DENSITY WAVE SPREADING OUT FROM THE CENTER IS

$$\frac{\rho'}{\rho_0} = \frac{p'}{\rho a_0^2} = \frac{F(r - a_0 t)}{r} \qquad (105)$$

⇒ SHAPE DOES NOT CHANGE BUT THE AMPLITUDE DECREASES WITH r

ASSUME AN ACOUSTIC PULSE OF WIDTH Δr EMANATING FROM THE CENTER. THE ENERGY PER UNIT VOLUME OF THE ACOUSTIC WAVE IS PROPORTIONAL TO ρ'^2. THE TOTAL ENERGY OF THE WAVE, $\rho'^2 \cdot 4\pi r^2 \Delta r$, MUST REMAIN CONSTANT DURING ITS PROPAGATION, SO $\rho'^2 \sim 1/r^2$ AS IMPLIED BY EQ. (105). SUBSTITUTING (105) IN (102) THE RELATION

$$\frac{\partial u}{\partial t} = -a_0^2\left[\frac{F'(r - a_0 t)}{r} - \frac{F(r - a_0 t)}{r^2}\right] \qquad (106)$$

$$\left(f'(r - a_0 t) = \frac{df(\xi)}{d\xi} \text{ WITH } \xi = r - a_0 t\right)$$

SO THAT

$$u = a_0\left[\frac{1}{r}\int^t \frac{dF(\xi)}{d\xi}dt - \frac{1}{r^2}\int^t F(\xi)\,dt\right] = \qquad \left\{ dt = -\frac{d\xi}{a_0} \right.$$

$$= a_0\left[\frac{F(r - a_0 t)}{r} - \frac{1}{r^2}\int^{r - a_0 t} F(\xi)\,d\xi\right] = a_0\left[\frac{\rho'}{\rho_0} - \frac{\varphi(r - a_0 t)}{r^2}\right] \qquad (107)$$

NOTICE THAT (107) BECOMES THE PLANE SOLUTION (92) FOR $r \to \infty$. IN THE SPHERICAL SOLUTION THERE IS AN ADDITIONAL TERM $\varphi(r - a_0 t)/r^2$. IN PLANE WAVES, DUE TO THE PROPORTIONALITY $u'/a_0 \sim \rho'/\rho_0$, THE FLUID CAN BE COMPRESSED ONLY IN THE REGION OF THE DISTURBANCE. THIS DOES NOT NECESSARILY OCCUR IN SPHERICAL WAVES. IN THESE, BECAUSE OF MASS CONSERVATION, A COMPRESSION MUST BE FOLLOWED BY AN EXPANSION. THE VELOCITY IS NOT PROPORTIONAL TO THE ρ' OR p' AND CHANGES SIGN WITHIN THE DISTURBANCE REGION (SEE FIG. 39)

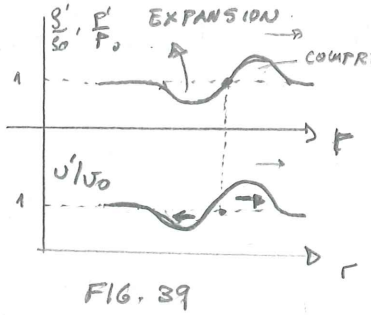
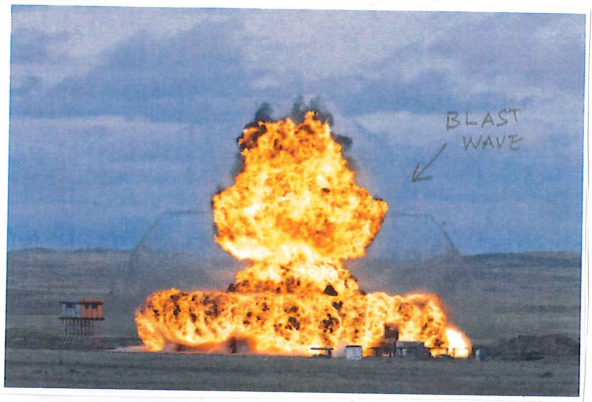


FIG. 39

BLAST WAVES

A BLAST WAVE IS A SPHERICAL PRESSURE WAVE THAT MOVES SUPERSONICALLY OUTWARDS FROM AN EXPLOSIVE CORE. BLAST WAVES ARE THE RESULT OF RAPID DEPOSITION OF ENERGY, WHICH IS OFTENTIMES PROVIDED BY THE FAST EXPANSION OF EXPLOSION PRODUCTS. BLAST WAVES ARE PRODUCED BY HIGH EXPLOSIVES (I.E. EXPLOSIVE MATERIALS THAT CAN DETONATE, SUCH AS TRINITROTOLUENE WHICH HAS A EXPLOSION VELOCITY OF ORDER 10 km/s) AND BY NUCLEAR EXPLOSIONS (SEE FIG. 40). THE PROBLEM OF BLAST-WAVE PROPAGATION IS A COMPLEX ONE THAT IS OFTEN TREATED IN A SIMPLIFIED MANNER, AS DESCRIBED BELOW.





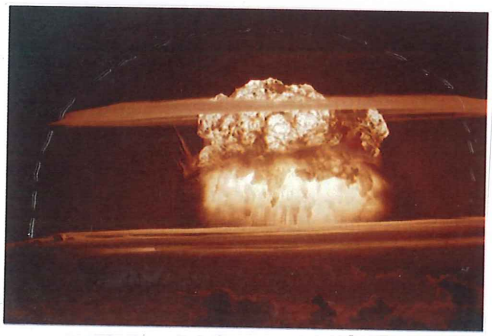
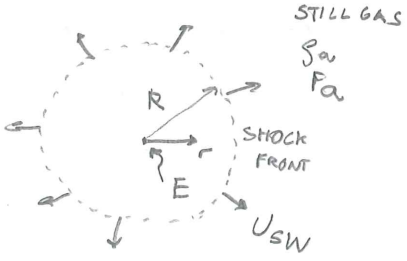


FIG. 40: EXAMPLES OF BLAST WAVES.

CONSIDER A GAS OF DENSITY ρ_a INITIALLY AT REST, IN WHICH A LARGE AMOUNT OF ENERGY E IS LIBERATED IN A SMALL VOLUME DURING A SHORT TIME INTERVAL, RESEMBLING, FOR INSTANCE, THE SITUATION FOUND DURING AN INTENSE EXPLOSION. A SHOCK WAVE WILL PROPAGATE THROUGH THE GAS STARTING FROM THE POINT WHERE THE ENERGY WAS ORIGINALLY RELEASED. THE EXPANSION CAUSED BY THE EXPLOSION IS TO FORCE THE GAS BOUNDED BY THE SHOCK INTO A THIN SHELL INSIDE THE FRONT. CONSIDER THE LIMIT IN WHICH THE RADIUS OF THE FRONT R IS MUCH LARGER THAN THE CHARACTERISTIC SIZE OF THE ZONE WHERE THE ENERGY WAS RELEASED. IN THIS LIMIT, THE ENERGY E IS RELEASED INSTANTANEOUSLY AND AT A POINT $r = 0$ IN SPACE, WHERE r IS THE RADIAL DISTANCE FROM THE DEPOSITION POINT. IF THE PRESSURE OF THE STILL GAS P_a IS NEGLIGIBLE COMPARED TO THE PRESSURE BEHIND THE SHOCK (A GOOD APPROXIMATION FOR NOT TOO LARGE R SINCE THE MAXIMUM PRESSURE DECAYS AS THE WAVE GROWS), THEN THE INITIAL INTERNAL ENERGY OF THE GAS THAT IS SET IN MOTION IS MUCH SMALLER THAN E, AND THE SPEED OF SOUND a_a IS MUCH SMALLER THAN THE VELOCITY OF BOTH GAS AND WAVE FRONT.



THEN THE GAS MOTION IS DETERMINED BY E AND ρ_a. THESE PARAMETERS HAVE TO BE COMBINED WITH r AND t, IN THE ABSENCE OF SPATIAL AND TEMPORAL SCALES, THEREBY GIVING RISE TO A SELF-SIMILAR FLOW IN WHICH THE VARIABLES CHANGE WITH TIME IN SUCH A MANNER THAT THEIR DISTRIBUTIONS WITH RESPECT TO r ARE SELF-SIMILAR IN TIME. NOTE THAT THE PROBLEM, AS STATED, DOES NOT HAVE A CHARACTERISTIC VELOCITY. IF THE COUNTER-PRESSURE p_a IS NEGLECTED, THEN $a_a = 0$ AND THE SPEED OF SOUND IN THE STILL GAS DOES NOT APPEAR IN THE FORMULATION.

THE DIMENSIONLESS SIMILARITY VARIABLE IS OBTAINED THROUGH COMBINATION OF E/ρ_a (WITH UNITS m^5/s^2) WITH r AND t, NAMELY:

$$\xi = r \left(\frac{\rho_a}{E t^2} \right)^{1/5}. \qquad (108)$$

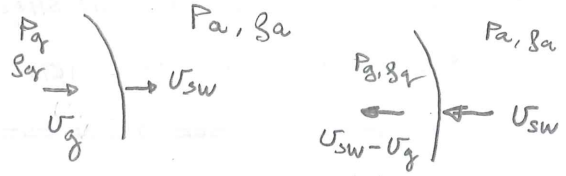
THE SHOCK RADIUS IS DEFINED BY A $\xi = \xi_0$, WITH $\xi_0 \sim \mathcal{O}(1)$, AND

$$R = \xi_0 \left(\frac{E}{\rho_a} \right)^{1/5} t^{2/5}. \qquad (109)$$

THE PROPAGATION VELOCITY OF THE SHOCK FRONT IS

$$U_{sw} = \frac{dR}{dt} = \frac{2}{5} \frac{R}{t} = \frac{2}{5} \xi_0 \left(\frac{E}{\rho_a} \right)^{1/5} t^{-3/5} = \frac{2}{5} \xi_0^{5/2} \left(\frac{E}{\rho_a} \right)^{1/2} R^{-3/2} \qquad (110),$$

WHICH INDICATES THAT THE SHOCK SPEED DECAYS WITH R. THE VELOCITY U_g, PRESSURE p_g AND DENSITY ρ_g BEHIND THE FRONT CAN BE COMPUTED FROM THE JUMP CONDITIONS (50), (51) AND (53) PARTICULARIZED FOR $Ma_1 \to \infty$,



$$\frac{U_{sw} - U_g}{U_{sw}} = \frac{2 + (\gamma-1) Ma_1^2}{(\gamma+1) Ma_1^2} \approx \frac{\gamma-1}{\gamma+1} \Rightarrow \frac{U_g}{U_{sw}} = \frac{2}{\gamma+1} \qquad (111)$$

$$\frac{\rho_g}{\rho_a} = \frac{(\gamma+1) Ma_1^2}{2 + (\gamma-1) Ma_1^2} \approx \frac{\gamma+1}{\gamma-1} \qquad (112)$$

$$\frac{p_g}{p_a} = 1 + \frac{2\gamma}{\gamma+1} (Ma_1^2 - 1) \approx \frac{2\gamma}{\gamma+1} Ma_1^2 = \frac{2\gamma}{\gamma+1} \frac{U_{sw}^2}{\gamma p_a / \rho_a}$$

$$\Rightarrow p_g = \frac{2}{\gamma+1} \rho_a U_{sw}^2 \qquad (113)$$

NOTE THAT WHILE ρ_g REMAINS CONSTANT AND EQUAL TO ITS INFINITE-MACH LIMIT $\frac{\gamma+1}{\gamma-1}\rho_a$, THE PRESSURE OF THE GAS BEHIND THE SHOCK DECREASES WITH TIME ACCORDING TO

$$p_g = \frac{2}{\gamma+1} \rho_a U_{sw}^2 \sim \rho_a U_{sw}^2 \sim \rho_a^{3/5} E^{2/5} t^{-6/5} \sim \frac{E}{R^3} \qquad (114)$$

WHEN R IS LARGE ENOUGH, SUCH THAT $p_g \sim p_a \left(\frac{\gamma+1}{\gamma-1} \right)$, THEN ρ_g STARTS DEPARTING

FROM THE LIMITING VALUE $\frac{\gamma+1}{\gamma-1}\rho_a$ (SEE FIG IN PAGE 21), AND THE ANALYSIS IS NO LONGER VALID, IN THAT THE EFFECT OF THE COUNTERPRESSURE p_a NEEDS TO BE INCORPORATED.

THE SOLUTION PROFILES $u(r,t)$, $\rho(r,t)$ AND $p(r,t)$ INSIDE A BLAST WAVE WERE FIRST OBTAINED BY SEDOV (1946), TAYLOR (1950), AND VON NEUMANN (1963) IN THE LIMIT ABOVE. THESE INVESTIGATIONS WERE AMBIENTED IN A TIME WHEN ATOMIC WEAPONS WERE FIRST DEVELOPED, THE PROBLEM OF THE BLAST WAVE BEING FUNDAMENTAL TO UNDERSTANDING THE DAMAGING POWER OF THE SUPERSONIC OVERPRESSURE PULSE IN DESTROYING ENEMY ASSETS. WHILE TAYLOR ADDRESED THE NUMERICAL SOLUTION (TAYLOR, PROC. ROYAL SOC. LONDON 22, 1950), THE OTHER TWO AUTHORS USED AN ENERGY INTEGRAL TO OBTAIN THE ANALYTICAL SOLUTION (SEDOV, DOKLADY AKAD NAUK, USSR, 1946; VON NEUMANN, 1963). THE ANALYSIS IS REFERRED TO THE POINT-SOURCE SOLUTION, IN THAT THE ENERGY E IS ASSUMED TO BE RELEASED AT A POINT.

CONSIDER THE SLIGHTLY MODIFIED SIMILARITY VARIABLE $\xi = r/R(t)$, WHERE $R(t)$ IS GIVEN BY (109). BASED ON EQS (110)-(113), IT IS CONVENIENT TO DEFINE THE SELF-SIMILAR VELOCITY, DENSITY AND PRESSURE

$$V = \frac{u}{2r/5t} = \frac{u}{\xi \dot{R}}, \quad D = \frac{\rho}{\rho_a}, \quad P = \frac{p}{\frac{4}{25}\rho_a \frac{r^2}{t^2}} = \frac{p}{\xi^2 \dot{R}^2 \rho_a} \tag{115}$$

WHERE u, ρ AND p ARE FUNCTIONS OF r AND t. IN ACCORDANCE WITH EQS. (111)-(113), THE BOUNDARY VALUES OF V, D AND P AT $\xi = 1$ ARE

$$V = \frac{2}{\gamma+1}, \quad D = \frac{\gamma+1}{\gamma-1}, \quad P = \frac{2}{\gamma+1}. \tag{116}$$

THE EQUATIONS FOR THE EXPANDING GAS ARE:

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho u r^2\right) = 0 \tag{117}$$ (CONTINUITY)

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \tag{118}$$ (MOMENTUM)

$$\frac{\partial}{\partial t}\left(\ln \frac{p}{\rho^\gamma}\right) + u\frac{\partial}{\partial r}\left(\ln \frac{p}{\rho^\gamma}\right) = 0 \tag{119}$$ (ENTROPY CONSERVATION)

NOTE THAT THE PROBLEM IS UNSTEADY ($\partial s \neq 0$) AND THE SHOCK INCREASES THE ENTROPY OF THE GAS AS IT PASSES THROUGH IT

SUBSTITUTION OF THE VARIABLES (115) INTO (117)-(119) GIVES THE EQUATIONS

$$-\xi\frac{dD}{d\xi} + 3VD + \xi\frac{d}{d\xi}(VD) = 0 \tag{120}$$ (CONTINUITY)

$$V\left(1-\frac{2}{5}V\right)+\frac{2}{5}\left(\xi-3V\right)\frac{dV}{d\xi}=-\frac{2}{3D}\left(2P+5\,\frac{dP}{d\xi}\right) \qquad (121)\ \text{(MOMENTUM)}$$

$$\frac{\xi}{a}\frac{d}{d\xi}(\ln P)-\frac{\gamma\xi}{a}\frac{d}{d\xi}(\ln D)=\frac{5-2V}{V-1} \qquad (122)\ \text{(ENTROPY)}$$

EXAMPLES:

$$\frac{\partial V}{\partial t}=\frac{\partial}{\partial t}\left(\frac{2}{5}\frac{r}{t}V\right)=-\frac{2r}{5t^2}V$$

$$-\frac{2}{5}\frac{r}{t}\frac{2}{5}\frac{5}{t}V'=-\frac{2r}{5t^2}\left(V+\frac{2}{5}\xi V'\right)$$

$$\frac{\partial V}{\partial r}=\frac{2}{5t}\left(\frac{2r}{5t}V\right)=\frac{4r}{25t^2}V\left(V+\xi V'\right)$$

WHERE USE OF $\frac{\partial}{\partial r}=\frac{\partial\xi}{\partial r}\frac{\partial}{\partial\xi}=\frac{1}{R}\frac{d}{d\xi}$ AND $\frac{\partial}{\partial t}=\frac{\partial\xi}{\partial t}\frac{\partial}{\partial\xi}=-\xi\frac{\dot R}{R}\frac{d}{d\xi}$ HAS BEEN MADE TO CHANGE THE DIFFERENTIAL OPERATORS. EQUATIONS (120)–(122) ARE SUBJECTED TO THE BOUNDARY CONDITIONS (116) AT THE SHOCK FRONT $\xi=1$. AN EXACT ANALYTICAL SOLUTION OF THIS SYSTEM EXISTS DUE TO SEDOV (CALCULATION DETAILS IN "SIMILARITY AND DIMENSIONAL METHODS IN MECHANICS", L.I. SEDOV (1959)):

$$\frac{\gamma P}{D}=\frac{\gamma(\gamma-1)(1-V)V^2}{2(\gamma V-1)} \rightarrow \text{"ENERGY INTEGRAL"}$$

(OBTAINED FROM $\frac{\partial}{\partial t}\left(\rho\left(e+\frac{v^2}{2}\right)\right)+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho v r^2\left(h+\frac{v^2}{2}\right)\right)=0$)

$$\xi^5=\frac{\left\{\frac{\gamma+1}{7-\gamma}\left[5-(3\gamma-1)V\right]\right\}^a\left[\frac{(\gamma+1)(\gamma V-1)}{\gamma-1}\right]^b}{\left[\frac{1}{2}(\gamma+1)V\right]^2}$$

→ OBTAINED BY SUBSTITUTING $\xi\frac{dD}{d\xi}$ FROM (120) INTO (122), AND INTEGRATING THE RESULTING EQUATION.

$$D=\frac{\gamma+1}{\gamma-1}\left[\frac{\gamma+1}{\gamma-1}(\gamma V-1)\right]^c\left\{\frac{(\gamma+1)}{7-\gamma}\left[5-(3\gamma-1)V\right]\right\}^d\left[\frac{\gamma+1}{\gamma-1}(1-V)\right]^f$$

→ FROM (120)

WITH $a=-\frac{13\gamma^2-7\gamma+12}{(3\gamma-1)(2\gamma+1)}$, $b=\frac{5(\gamma-1)}{2\gamma+1}$, $c=\frac{3}{2\gamma+1}$, $d=-\frac{a}{2-\gamma}$, $f=-\frac{2}{2-\gamma}$,

THE CONSTANT ξ_0 IN THE DEFINITION OF THE SIMILARITY VARIABLE ξ (THROUGH THE FRONT RADIUS (109)) IS OBTAINED BY IMPOSING THE ENERGY CONSTRAINT,

$$E=\int_0^R 4\pi r^2\rho\left[e+\frac{v^2}{2}\right]dr \qquad (123)$$

WHERE, AS MENTIONED BEFORE, THE INITIAL THERMAL ENERGY OF THE GAS HAS BEEN NEGLECTED IN COMPARISON WITH E. INTRODUCING THE SELF-SIMILAR VARIABLES (115) IN (123) AND NOTICING THAT $e=\frac{P}{(\gamma-1)\rho}$, ONE OBTAINS

$$E=\int_0^1 4\pi R^3\xi^4\dot R^2\rho_a D\left[\frac{P}{(\gamma-1)D}+\frac{V^2}{2}\right]d\xi \Rightarrow \frac{\rho}{\xi_0^5}\frac{16\pi}{25}\int_0^1 D\left[\frac{P}{(\gamma-1)D}+\frac{V^2}{2}\right]\xi^4 d\xi=1$$

$$\dot R^2R^3=\xi_0^3\left(\frac{E}{\rho_a}\right)^{3/5}t^{6/5}\cdot\frac{4}{25}\xi_0^2\left(\frac{E}{\rho_a}\right)^{2/5}t^{-6/5}=\frac{4}{25}\xi_0^5E/\rho_a$$

FROM WHERE $\xi_0=1.033$ IS OBTAINED FOR $\gamma=1.4$.

THE SOLUTION PROFILES OBTAINED FROM THIS ANALYSIS ARE GIVEN BELOW IN FIG. 41. THE GAS DENSITY DECREASES VERY FAST BEHIND THE SHOCK WAVE, INDICATING THAT MOST OF THE GAS THAT WAS INITIALLY UNIFORMLY DISTRIBUTED INSIDE THE SPHERE OF RADIUS R, IS NOW CONTAINED WITHIN A THIN LAYER NEAR THE FRONT

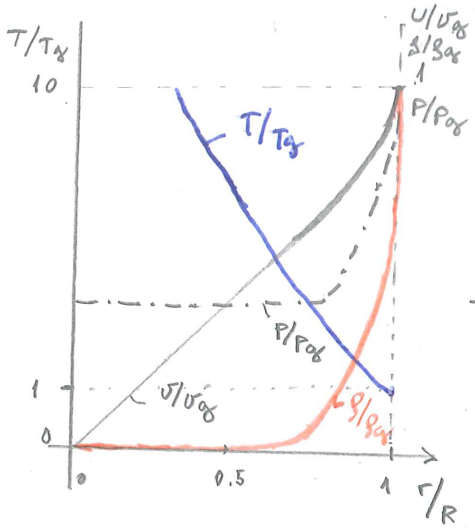


FIG. 41. POINT-SOURCE SOLUTION FOR $\gamma = 1.4$

ADDITIONALLY, THE PRESSURE DECREASES AWAY FROM THE FRONT TO THE CENTER BY A FACTOR OF 2 TO 3, AND REMAINS APROXIMATELY CONSTANT INSIDE THE SPHERE. REMARKABLY, THE TEMPERATURE INCREASES AWAY FROM THE FRONT TO THE CENTER BY A FACTOR OF ORDER 10, AND THEORETICALLY TO INFINITY AT THE CENTER. THIS IS DUE TO THE FACT THAT THE GAS NEAR THE CENTER WAS HEATED BY A VERY STRONG SHOCK WAVE.

AT A LATER STAGE OF THE EXPLOSION, WHEN $P_a \sim \frac{\gamma+1}{\gamma-1} P_\infty$, THE SELF-SIMILAR SOLUTION OF THE BLAST WAVE NO LONGER HOLDS AND THE COUNTER PRESSURE P_∞ NEEDS TO BE TAKEN INTO ACCOUNT. THIS OCCURS AT A RADUS $R \sim \left(E/P_\infty\right)^{1/3}$. FOR $E = 10^{14}$ J (CORRESPONDING TO THE ENERGY LIBERATED IN EXPLODING 20.000 TONS OF TNT), THE RADIUS R IS OF ORDER 1 km, WHICH IS REACHED IN CHARACTERISTIC TIMES $t_0 \sim R/a_a \sim 3$ SECS. AFTER THIS, THE STRONG SHOCK BECOMES A SPHERICAL SOUND WAVE OF THE TYPE ANALYZED IN PAGE 56, WITH $R \sim t^{2/5}$ GRADUALLY TRANSITIONING INTO THE CHARACTERISTIC $R = a_a t$. THE COMPRESSION ZONE ACCROSS SUCH A WAVE IS FOLLOWED BY AN EXPANSION REGION, WHICH BRINGS ALONG A WIND DIRECTED TOWARDS THE CENTER. (SEE FIG. 42)

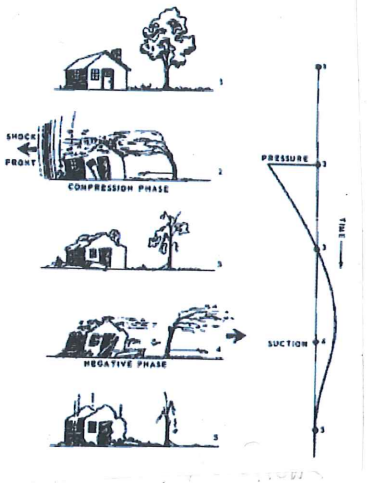


FIG. 42. VARIATIONS OF BLAST EFFECTS ASSOCIATED WITH COMPRESSION AND EXPANSION

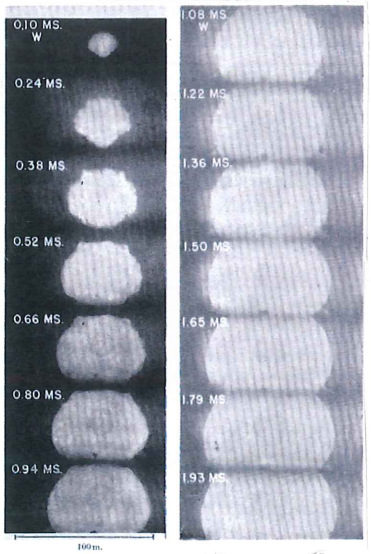


FIG. 43. TRINITY EXPLOSION (1945)

THE NUMERICAL SOLUTIONS OF EQS. (120)-(122) WAS PROVIDED BY TAYLOR IN 1950. IN A SECOND PAPER, TAYLOR VALIDATED HIS CALCULATIONS AGAINST THE DATA OBTAINED FROM THE FIRST NUCLEAR EXPLOSION TEST PERFORMED IN NEW MEXICO IN THE MID FORTIES (TRINITY BOMB). THE DATA HAD BECOME DECLASSIFIED BY THEN. HIS RESULTS, WHICH RESEMBLED THOSE IN FIG 41, WERE SUPPLEMENTED WITH A SERIES OF PHOTOGRAPHS OF THE EXPLOSION (SEE FIGS. 43-44). ANALYZING THESE IMAGES, TAYLOR PLOTTED THE TIME EVOLUTION OF THE BLAST RADIUS AND

COMPARED IT AGAINST THE THEORETICAL PREDICTION IN (109), NAMELY

$$R^{5/2} = \xi_0^{5/2}\left(\frac{E}{\rho_a}\right)^{1/2} t \Rightarrow \frac{5}{2}\log_{10} R = \log_{10}\left(\xi_0^{5/2}\left(\frac{E}{\rho_a}\right)^{1/2}\right) + \log_{10} t$$

THE COMPARISON IS SHOWN IN FIG. 45 FOR $\log_{10}\left(\xi_0^{5/2}\left(\frac{E}{\rho_a}\right)^{1/2}\right) = 11.915$. BY USING THE VALUE OF $\xi_0 = 1.033$ OBTAINED FROM INTEGRATION OF (120) - (122) WITH $\gamma = 1.40$, HE WAS ABLE TO CALCULATE THE DEPOSITED ENERGY $E = 7.14 \cdot 10^{13}$ J, EQUIVALENT TO 16800 TONS OF TNT. THE CHARACTERISTIC BLAST PRESSURE P_g COMPUTED FROM (114) IS OF ORDER 10000 bar AT $R = 30$ m. THE PREDICTIONS OF TAYLOR PROVED TO BE ACCURATE IN THE RANGE $R = 20$ TO 185 m, THAT IS FOR NOT TOO SMALL R (WHERE THE DETAILS OF THE ENERGY DEPOSITION ARE IMPORTANT AND THE POINT-SOURCE APPROXIMATION IS NO LONGER VALID) AND FOR NOT TOO LARGE R (WHERE THE POST-SHOCK PRESSURE BECOMES COMPARABLE TO THE STILL AIR PRESSURE). IN THAT INTERMEDIATE RANGE OF R, THE PROBLEM IS SELF-SIMILAR, IN THAT IT DOES NOT CONTAIN ANY LENGTH OR TIME SCALES OTHER THAN r AND t, WITH ξ IN (108) BEING THE SIMILARITY VARIABLE. THIS, IN FACT, IS AN ARCHETYPAL EXAMPLE OF "INTERMEDIATE ASYMPTOTICS" (SEE "SIMILARITY, SELF-SIMILARITY AND INTERMEDIATE ASYMPTOTICS", G.I. BARENBLATT, 1979).

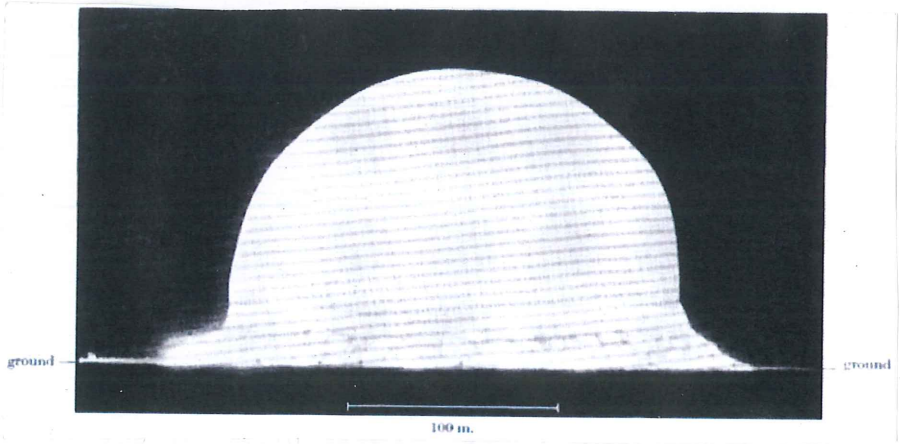


FIG. 44. TRINITY EXPLOSION AT $t = 15$ ms.

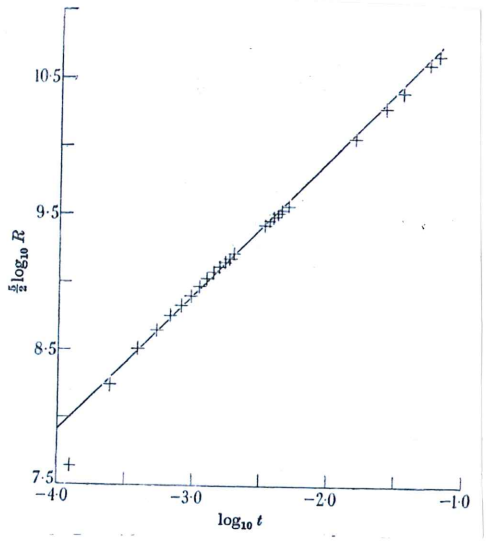


FIG. 45. LOGARITHMIC PLOT COMPARING THEORY (LINES) WITH DATA (POINTS).

THREE IMPORTANT EFFECTS HAVE NOT BEEN INCLUDED IN THE DESCRIPTION ABOVE: THE VARIATIONS OF γ WITH TEMPERATURE, THE RADIATION COOLING OF THE HOT CORE, AND THE FORMATION OF MACH REFLECTIONS NEAR THE GROUND.



V. TWO-DIMENSIONAL COMPRESSIBLE FLOWS

THE FLOWS STUDIED IN THIS SECTION PERTAIN TO STEADY TWO-DIMENSIONAL SCENARIOS. IT WILL BE SHOWN BELOW THAT THE CONSERVATION EQUATIONS REDUCE TO AN EQUATION FOR THE VELOCITY POTENTIAL, AND THAT DEPENDING ON THE GEOMETRY AND FLOW REGIME THE PROBLEM CAN BE CONVENIENTLY LINEARIZED TO CALCULATE, FOR INSTANCE, AERODYNAMIC FORCES.

NATURAL COORDINATES

IN PART I OF THIS COURSE IT WAS OBSERVED THAT SOME QUANTITIES, SUCH AS THE ENTROPY AND STAGNATION OR TOTAL ENTHALPY, ARE CONSERVED ALONG STREAMLINES IN STEADY, INVISCID ADIABATIC FLOWS. IT IS THEREFORE CONVENIENT TO REFORMULATE THE CONSERVATION EQUATIONS IN TERMS OF "INTRINSIC" OR "NATURAL" ORTHOGONAL COORDINATES, WHICH CONSIST OF A COORDINATE s TANGENT TO THE STREAMLINES AND A COORDINATE n NORMAL TO THEM (SEE FIG. 46). IN THESE COORDINATES, THE CONSERVATION EQUATIONS BECOME

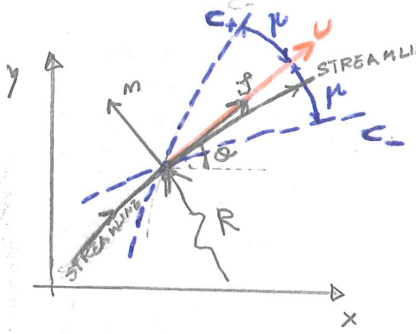


FIG. 46: NATURAL COORDINATES n AND s

$$\frac{\partial}{\partial s}(\rho u) + \rho u \frac{\partial \theta}{\partial n} = 0 \qquad \text{(CONTINUITY)} \qquad (124)$$

EQUIVALENTLY: $\rho u \Delta n = \text{CONST}$ WITH $\frac{1}{\Delta n}\frac{\partial \Delta n}{\partial s} = \frac{\partial \theta}{\partial s}$

$$\rho u \frac{\partial u}{\partial s} + \frac{\partial P}{\partial s} = 0 \qquad (s\text{-DIRECTION}) \qquad (125)$$

$$\rho u^2 \frac{\partial \theta}{\partial s} + \frac{\partial P}{\partial n} = 0 \qquad (n\text{-DIRECTION}) \qquad (126)$$

EQUIVALENTLY $\frac{\rho U^2}{R}$, WITH $\frac{1}{R} \equiv \frac{\partial \theta}{\partial s}$

EQUATIONS (124)-(126) ARE SUPPLEMENTED WITH THE EQUATION OF STATE. IN EQ. (126), $\frac{\partial \theta}{\partial s} = \frac{1}{R}$ IS THE INVERSE OF THE LOCAL RADIUS OF CURVATURE OF THE STREAMLINE. USING THE FIRST PRINCIPLE, $Tds = dh - \frac{1}{\rho}dP$, WITH $dh = dh_0 - udu$, THE ENTROPY EQUATIONS FOR AN ADIABATIC FLOW (I.E. NO THERMAL ENERGY INPUT/OUTPUT ALONG STREAMLINES, $\partial h_0/\partial s = 0$) BECOME

$$\left.\begin{aligned} T\frac{\partial s}{\partial s} &= -\left(u\frac{\partial u}{\partial s} + \frac{1}{\rho}\frac{\partial P}{\partial s}\right) = 0 \\ T\frac{\partial s}{\partial n} &= -\left(u\frac{\partial u}{\partial n} + \frac{1}{\rho}\frac{\partial P}{\partial n}\right) + \frac{dh_0}{dn} \end{aligned}\right\} \qquad (127)$$

THE FIRST OF THESE EQUATIONS STATES THAT THE ENTROPY REMAINS CONSTANT ALONG s, ALTHOUGH IN PRINCIPLE IT CAN CHANGE ACROSS STREAMLINES. HOWEVER, IF $h_0 = \text{CONST.} \neq f(n)$ AND $u\frac{\partial u}{\partial n} + \frac{1}{\rho}\frac{\partial P}{\partial n} = u\left(\frac{\partial u}{\partial n} - u\frac{\partial \theta}{\partial s}\right) = 0$ EVERYWHERE, THE FLOW IS ISENTROPIC THROUGHOUT (ALSO CALLED HOMENTROPIC, SEE PAGE 12). THE FIRST CONDITION REQUIRES UNIFORM STAGNATION ENTHALPY THROUGHOUT, WHILE THE SECOND IS EQUIVALENT TO IRROTATIONAL FLOW, WITH $\omega = \frac{\partial u}{\partial n} - u\frac{\partial \theta}{\partial s} = 0$ THE VORTICITY. THESE CONSIDERATIONS INDICATE THAT ZERO VORTICITY IMPLIES UNIFORM ENTROPY IF THE STAGNATION ENTHALPY IS UNIFORM. IF ONLY h_0 IS UNIFORM, THEN THE VORTICITY CORRESPONDS TO VARIATIONS OF ENTROPY ACROSS

STREAMLINES, AND ALSO TO VARIATIONS OF STAGNATION PRESSURE ACROSS STREAMLINES,

$$\omega = \frac{I}{U}\frac{\partial s}{\partial n} = -\frac{RT}{U}\frac{\partial}{\partial n}\left(\ln P_0\right). \qquad (128)$$

THIS ABOVE IS ACTUALLY AN EQUIVALENT FORM OF THE "CROCCO'S THEOREM", WHICH CAN BE MORE GENERALLY STATED USING THE MOMENTUM EQUATION

$$\nabla\left(\frac{|\vec{U}|^2}{2}\right) - \vec{U}\wedge\nabla\wedge\vec{U} = -\frac{\nabla P}{\rho}$$

WITH $T\nabla s = \nabla h - \frac{\nabla P}{\rho}$, WHICH LEADS TO

$$\nabla\left(\frac{|\vec{U}|^2}{2} + h\right) - \vec{U}\wedge\vec{\omega} = T\nabla s$$

WITH $\vec{\omega} = \nabla\wedge\vec{U}$ THE VORTICITY. WHEN $\nabla\left(h + \frac{|\vec{U}|^2}{2}\right) = \nabla h_0 = 0$, SO THAT h_0 IS CONSTANT THROUGHOUT, THE ENTROPY VARIATIONS ARE RELATED TO THE VORTICITY AS

$$\boxed{T\nabla s = -\vec{U}\wedge\vec{\omega}} \qquad (129)$$

OR EQUIVALENTLY, TO THE STAGNATION PRESSURE VARIATIONS

$$c_v T\,\nabla\left(\ln P_0\right) = -\vec{U}\wedge\vec{\omega} \qquad (130).$$

NOTE: $s - s_0 = c_v \ln\left(\frac{P}{P_0}\left(\frac{\rho_0}{\rho}\right)^{\gamma}\right) =$

$= c_v \ln\left(\frac{P}{P_0}\left(\frac{T}{T_0}\right)^{-\frac{\gamma}{\gamma-1}}\right) =$

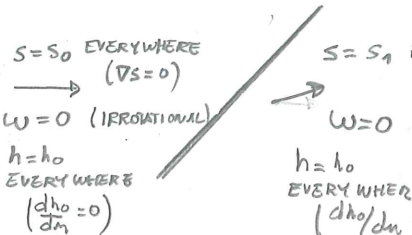
$= c_v \ln\left(\frac{P_0'}{P_0}\left(1+\frac{\gamma-1}{2}M_a^2\right)^{\frac{\gamma}{\gamma-1}} \times\right.$

$\left.\times\left(\frac{T_0'}{T_0}\right)^{-\frac{\gamma}{\gamma-1}}\left(1+\frac{\gamma-1}{2}M_a^2\right)^{-\frac{\gamma}{\gamma-1}}\right) =$

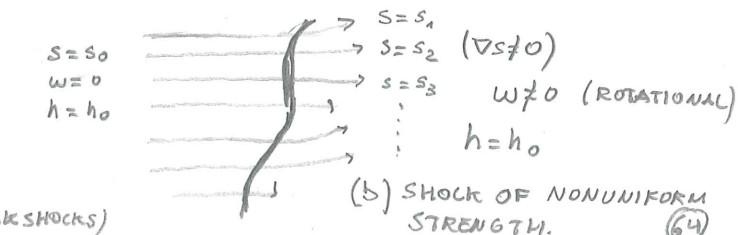
$h_0 = \text{const}$ ($T_0 = \text{const}$) $= c_v \ln\left(\frac{P_0'}{P_0}\right)$

⇒ IN ADIABATIC FLOWS, THE VARIATIONS OF ENTROPY ARE DIRECTLY RELATED TO VARIATIONS IN STAGNATION PRESSURE.

TO ILLUSTRATE THESE CONCEPTS, CONSIDER AN INVISCID FLOW PASSING THROUGH A SHOCK WAVE. IT WAS SEEN IN PART III THAT h_0 = CONST. THROUGH THE SHOCK. IF THE SHOCK STRENGTH IS UNIFORM ALONG ITS LENGTH, THAT IS, IF THE CONDITIONS UPSTREAM ARE UNIFORM ($\frac{d h_0}{dn} = 0$) AND THE ANGLE OF INCIDENCE IS CONSTANT, THEN THERE IS A JUMP IN ENTROPY ACROSS THE SHOCK BUT THIS JUMP IS CONSTANT ALONG THE SHOCK ($\frac{\partial s}{\partial n} = 0$), SO THAT $\nabla s = 0$ BEHIND THE SHOCK. THEREFORE THE DOWNSTREAM FLOW HAS $\vec{U}\wedge\vec{\omega} = 0$ ACCORDING TO EQ. (129). THIS IMPLIES THAT EITHER $\vec{U}$ AND $\vec{\omega}$ ARE EVERYWHERE PARALLEL, WHICH IS NOT THE CASE (EVEN LESS IN 2D SINCE $\vec{U}\perp\vec{\omega}$!) SINCE THE COMPONENT OF $\vec{\omega}$ NORMAL TO THE SHOCK IS ZERO (AS IT INVOLVES TANGENTIAL DERIVATIVES OF THE TANGENTIAL VELOCITY, WHICH IS CONTINUOUS), OR $\vec{\omega} = 0$, WHICH LEADS TO THE CONCLUSION THAT THE INITIALLY IRROTATIONAL FLOW REMAINS IRROTATIONAL DOWNSTREAM OF THE SHOCK. ON THE OTHER HAND, THE FLOW DOWNSTREAM FROM A CURVED SHOCK IS GENERALLY ROTATIONAL AND THEREFORE NON HOMENTROPIC (ALTHOUGH IT CAN BE ISENTROPIC IN THAT S IS CONSTANT ALONG THE POST-SHOCK STREAMLINES, SEE FIG. 46).



(a) SHOCK OF UNIFORM STRENGTH (ALSO WEAK SHOCKS)



(b) SHOCK OF NONUNIFORM STRENGTH.

THE VELOCITY - POTENTIAL EQUATION.

CONSIDER AN IRROTATIONAL FLOW, $\nabla \wedge \vec{U} = 0$, THEN THE VELOCITY IS RELATED TO THE POTENTIAL ϕ AS

$$\vec{U} = \nabla \phi \qquad (131)$$

SO THAT $\nabla \wedge (\nabla \phi) = 0$. NOTE THAT CROCCO'S THEOREM (129) IMPLIES THAT AN ADIABATIC $\left(\nabla \cdot \left(h + \frac{|\vec{U}|^2}{2}\right) = 0\right)$ AND IRROTATIONAL $(\nabla \wedge \vec{U} = 0)$ FLOW HAS CONSTANT ENTROPY $(\nabla S = 0)$ THROUGHOUT (I.E. IT IS HOMENTROPIC). A GENERAL EQUATION FOR THE VELOCITY POTENTIAL (IN STEADY FLOWS) CAN BE DERIVED FROM EULER'S MOMENTUM EQUATION

$$\vec{U} \cdot \nabla \vec{U} = -\frac{1}{\rho} \nabla P = -\frac{a^2}{\rho} \nabla \rho \qquad (dP = a^2 d\rho)$$

AND ELIMINATING THE DENSITY GRADIENT FROM CONTINUITY, $\vec{U} \cdot \nabla \rho = -\rho \nabla \cdot \vec{U}$,

$$\vec{U} \cdot (\vec{U} \cdot \nabla \vec{U}) = -\frac{a^2}{\rho} \vec{U} \cdot \nabla \rho = a^2 \nabla \cdot \vec{U} .$$

INTRODUCING THE VELOCITY POTENTIAL (131) IN THE ABOVE EQUATION,

$$\boxed{a^2 \nabla^2 \phi - \nabla \phi \cdot (\nabla \phi \cdot \nabla) \nabla \phi = 0} \quad , \qquad (132)$$

IN CARTESIAN COORDINATES THIS EQUATION CAN BE EXPRESSED AS

$$a^2 \frac{\partial^2 \phi}{\partial x_i \partial x_i} - \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0$$

OR EQUIVALENTLY, AS

$$(a^2 - \phi_x^2)\phi_{xx} + (a^2 - \phi_y^2)\phi_{yy} + (a^2 - \phi_z^2)\phi_{zz} = 2(\phi_x \phi_y \phi_{xy} + \phi_y \phi_z \phi_{yz} + \phi_z \phi_x \phi_{zx}) \qquad (133)$$

WHERE THE SPEED OF SOUND a CAN BE EXPRESSED AS A FUNCTION OF THE VELOCITY (OR $\nabla \phi$) BY MAKING USE OF THE ISENTROPIC RELATION (22), NAMELY

$$a^2 = a_0^2 - \left(\frac{\gamma - 1}{2}\right) |\nabla \phi|^2 . \qquad (134)$$

EQUATION (133) IS OF ELLIPTIC TYPE WHEN $Ma = |\vec{U}|/a < 1$ AND HYPERBOLIC WHEN $Ma = \frac{|\vec{U}|}{a} > 1$.

AN IMPORTANT PROPERTY OF HYPERBOLIC EQUATIONS IS THE EXISTENCE OF CHARACTERISTIC DIRECTIONS. IN TWO DIMENSIONS, THESE CAN BE OBTAINED IN THE NATURAL FRAME BY REWRITING (125) USING ISENTROPICITY,

$$U \frac{\partial U}{\partial s} = -\frac{1}{\rho} \frac{\partial P}{\partial s} = -\frac{a^2}{\rho} \frac{\partial \rho}{\partial s}$$

AND USING $\frac{\partial \rho}{\partial s} = -\rho \frac{\partial \theta}{\partial n} - \frac{\rho}{U} \frac{\partial U}{\partial s}$ FROM (124), WHICH GIVES

$$\frac{\cot\tan^2\mu}{U}\frac{\partial U}{\partial s}-\frac{\partial\Theta}{\partial n}=0 \qquad (135)$$

WHICH IS EQUIVALENT TO THE 2D VERSION OF (133) WHEN SUPPLIED WITH THE IRROTATIONALITY CONDITION

$$\frac{1}{U}\frac{\partial U}{\partial n}-\frac{\partial\Theta}{\partial s}=0 \qquad (136).$$

IN EQUATION (135), $\cot\tan^2\mu = Ma^2-1$, WITH μ THE MACH ANGLE IN TERMS OF $Ma = U/a$. EQS. (135)-(136) CAN BE PUT IN CHARACTERISTIC FORM BY USING THE PRANDTL-MEYER FUNCTION $\nu = \int \frac{\cot\tan\mu}{U}\,dU \Rightarrow d\nu = \cot\tan\mu\,\frac{dU}{U}$, SO THAT (135)-(136) BECOME

$$\frac{\partial\nu}{\partial s}-\tan\mu\frac{\partial\Theta}{\partial n}=0 \quad \text{AND} \quad \tan\mu\frac{\partial\nu}{\partial n}-\frac{\partial\Theta}{\partial s}=0.$$

SUBSTRACTING AND ADDING THESE EQUATIONS GIVES

$$\left.\begin{aligned}\frac{\partial R}{\partial s}+\tan\mu\frac{\partial R}{\partial n}&=0\\ \frac{\partial Q}{\partial s}-\tan\mu\frac{\partial Q}{\partial n}&=0\end{aligned}\right\} \quad \text{WITH} \quad \left.\begin{aligned}R&=\nu-\Theta\\ Q&=\nu+\Theta\end{aligned}\right\} \quad \text{RIEMANN INVARIANTS}$$

NOTE THAT THE CHARACTERISTIC LINES ARE GIVEN BY (137)

C_+ : $\frac{dn}{ds}=\tan\mu$ (LEFT-RUNNING), EQUIVALENTLY $\frac{dy}{dx}=\tan(\mu+\Theta)$

C_- : $\frac{dn}{ds}=-\tan\mu$ (RIGHT-RUNNING), EQUIVALENTLY $\frac{dy}{dx}=\tan(\Theta-\mu)$

WITH $\tan\mu=\frac{1}{\sqrt{Ma^2-1}}$. NOTE THAT THE CHARACTERISTICS ARE MACH LINES, WHICH ARE GENERALLY CURVED AS THE MACH NUMBER VARIES IN SPACE (SEE FIGS. 46-47). THE INFORMATION AT POINT A INFLUENCES ONLY THE REGION BETWEEN THE TWO CHARACTERISTICS (SEE MACH LINES IN FIG. 4b IN PAGE 8). SIMILARLY, THE SOLUTION AT POINT A DEPENDS ONLY ON THE PORTION OF THE BOUNDARY INTERCEPTED BY THE CHARACTERISTICS.

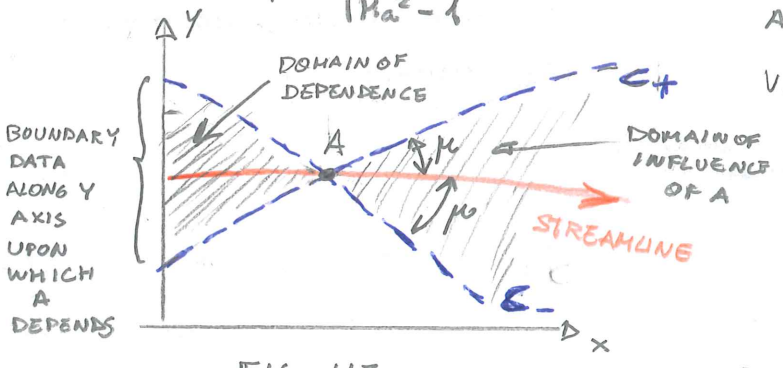


FIG. 47

IT SHOULD BE EMPHASIZED THAT THE VELOCITY POTENTIAL EQ. (133) IS NON-LINEAR, AND THERE IS NO GENERAL SOLUTION TO IT. IT IS HOWEVER CUSTOMARY IN AERODYNAMIC THEORY TO LINEARIZE IT AS DESCRIBED BELOW, ALTHOUGH SOME PROBLEMS SUCH AS TRANSONIC AND HYPERSONIC FLOWS ARE INHERENTLY NON-LINEAR. FOR THESE PROBLEMS IT IS CHALLENGING

TO OBTAIN GENERAL SOLUTIONS, AND OTHER METHODS BASED ON SIMILARITY MUST BE USED. IN WHAT FOLLOWS, THE FOCUS IS ON LINEAR ANALYSIS OF (133) TO TACKLE PROBLEMS IN SUBSONIC AND SUPERSONIC AERODYNAMICS.

SMALL-PERTURBATION THEORY

CONSIDER THE PROBLEM OF A UNIFORM STREAM U_∞ FLOWING AROUND A THIN AIRFOIL OR OTHER GEOMETRICAL OBSTACLE (SEE FIGS 48-49.) THAT CREATES ONLY A SMALL DISTURBANCE IN THE FLOW.

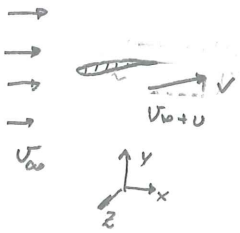


FIG. 48

THE MACH NUMBER OF THE FREE STREAM IS $Ma = U_\infty / a_\infty$, WITH $a_\infty = \left(\frac{\gamma P_\infty}{\rho_\infty}\right)^{1/2}$.

THE VELOCITY FIELD MAY BE WRITTEN AS

$$\vec{V} = (U_\infty + u, v, w), \quad \text{WITH } \frac{u}{U_\infty}, \frac{v}{U_\infty}, \frac{w}{U_\infty} \ll 1 \qquad (138)$$

THE EQUATIONS OF MOTION FOR A STEADY INVISCID FLOW ARE

$$\left.\begin{aligned} u_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} &= 0 \\ u_i \frac{\partial u_i}{\partial x_i} &= -\frac{1}{\rho}\frac{\partial P}{\partial x_i} \end{aligned}\right\}$$



FIG. 49

CONTINUITY

FOR CONSTANT ENTROPY ($\nabla S = 0$), $u_i \partial P/\partial x_i = a^2 u_i \partial \rho/\partial x_i = -a^2 \rho u_i \partial u_i/\partial x_i$, THEREFORE THE EQUATIONS ABOVE SIMPLIFY TO

$$u_i u_j \frac{\partial u_i}{\partial x_j} = a^2 \frac{\partial u_k}{\partial x_k}.$$

EXPANDING THIS EQUATION USING THE VELOCITY FIELD $\vec{V}$ (138) YIELDS

$$a^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right) = (U_\infty+u)^2\frac{\partial u}{\partial x} + v^2\frac{\partial v}{\partial y} + w^2\frac{\partial w}{\partial z} + \qquad (139)$$
$$+ (U_\infty+u)v\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) + vw\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right) + w(U_\infty+u)\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)$$

WHERE a^2 IS OBTAINED FROM THE CONSERVATION OF THE STAGNATION ENTHALPY AS

$$\frac{(U_\infty+u)^2+v^2+w^2}{2} + \frac{a^2}{\gamma-1} = \frac{U_\infty^2}{2} + \frac{a_\infty^2}{\gamma-1} \Rightarrow a^2 = a_\infty^2 - \frac{\gamma-1}{2}\left(2uU_\infty + u^2+v^2+w^2\right). \qquad (140)$$

SUBSTITUTING (140) INTO (139) GIVES

$$\begin{aligned}(1-Ma_\infty^2)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = {} & Ma_\infty^2\left[(\gamma+1)\frac{u}{U_\infty}+\frac{\gamma+1}{2}\frac{u^2}{U_\infty^2}+\frac{\gamma-1}{2}\frac{v^2+w^2}{U_\infty^2}\right]\frac{\partial u}{\partial x} \\ & + Ma_\infty^2\left[(\gamma-1)\frac{u}{U_\infty}+\frac{\gamma+1}{2}\frac{v^2}{U_\infty^2}+\frac{\gamma-1}{2}\frac{u^2+w^2}{U_\infty^2}\right]\frac{\partial v}{\partial y} \\ & + Ma_\infty^2\left[(\gamma-1)\frac{u}{U_\infty}+\frac{\gamma+1}{2}\frac{w^2}{U_\infty^2}+\frac{\gamma-1}{2}\frac{u^2+v^2}{U_\infty^2}\right]\frac{\partial w}{\partial z} \\ & + Ma_\infty^2\left[\frac{v}{U_\infty}\left(1+\frac{u}{U_\infty}\right)\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\frac{w}{U_\infty}\left(1+\frac{u}{U_\infty}\right)\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)+\frac{vw}{U_\infty^2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right]\end{aligned} \qquad (140)$$

ADDITIONAL SUBSTITUTION OF THE VELOCITY POTENTIAL, $v_i = \partial\phi/\partial x_i$, TRANSFORMS (140) INTO A FULL NON-LINEAR EQUATION FOR ϕ WHERE NO SIMPLIFICATION HAS BEEN MADE SO FAR. SOME SIMPLIFICATIONS CAN BE MADE IN ORDER TO SOLVE (140). FOR INSTANCE, IF THE PERTURBATION VELOCITIES ARE SMALL, u/v_∞, v/v_∞, $w/v_\infty \ll 1$, THEN ONE COULD NEGLECT THEIR SQUARES, SO THAT (140) BECOMES

$$\left(1-Ma_\infty^2\right)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=Ma_\infty^2(\gamma+1)\frac{u}{v_\infty}\frac{\partial u}{\partial x}+Ma_\infty^2(\gamma-1)\frac{u}{v_\infty}\left(\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

$$+Ma_\infty^2\frac{v}{v_\infty}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+Ma_\infty^2\frac{w}{v_\infty}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right) \qquad (141)$$

FURTHER SIMPLIFICATION CAN BE MADE BY NEGLECTING CROSS-PRODUCTS OF THE PERTURBATION VELOCITIES WHICH LEADS TO THE LINEAR EQUATION

$$\left(1-Ma_\infty^2\right)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \qquad (142)$$

IN SOME PROBLEMS, HOWEVER, SUCH LINEARIZATION IS NOT POSSIBLE. FOR INSTANCE, IN TRANSONIC FLOW $Ma_\infty \to 1$, AND IT IS NOT POSSIBLE TO NEGLECT THE FIRST TERM ON THE RIGHT-HAND SIDE OF (141) WHEN COMPARED TO THE FIRST TERM ON THE LEFT HAND SIDE. IN GENERAL, THAT TERM CAN BE RETAINED, WHICH GIVES

$$\left(1-Ma_\infty^2\right)\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=Ma_\infty^2(\gamma+1)\frac{u}{v_\infty}\frac{\partial u}{\partial x} \qquad (143)$$

CORRESPONDING TO AN EQUATION VALID IN SUBSONIC, TRANSONIC, AND SUPERSONIC FLOW. SUBSTITUTING THE VELOCITY POTENTIAL INTO (142) AND (143) GIVES

$$\boxed{\left(1-Ma_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0} \qquad (144)$$

AND

$$\boxed{\left(1-Ma_\infty^2\right)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\frac{Ma_\infty^2(\gamma+1)}{v_\infty}\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2}} \qquad (145)$$

THE LINEAR POTENTIAL EQUATION (144) CAN BE USED FOR SUBSONIC AND SUPERSONIC FLOWS. THE NON-LINEAR POTENTIAL EQUATION (145) HAS TO BE USED IN TRANSONIC AERODYNAMICS, ALTHOUGH IT CAN ALSO BE USED IN SUBSONIC AND SUPERSONIC FLOWS TO A HIGHER DEGREE OF ACCURACY. NOTE THAT THE CHARACTERISTICS OF EQ. (144) CAN BE EASILY COMPUTED BY NOTICING THAT, BY DEFINITION, THE FIRST DERIVATIVES OF ϕ (I.E. THE VELOCITY)

ARE CONTINUOUS ACROSS CHARACTERISTICS BUT THE SECOND DERIVATIVES (I.E. THE VELOCITY GRADIENTS) ARE NOT. THUS, USING

$$(1-M_{a_\infty}^2)\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$$

IN CONJUNCTION WITH

$$d\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial^2\phi}{\partial x^2}dx+\frac{\partial^2\phi}{\partial x\partial y}dy$$

$$d\left(\frac{\partial\phi}{\partial y}\right)=\frac{\partial^2\phi}{\partial x\partial y}dx+\frac{\partial^2\phi}{\partial y^2}dy$$

ONE CAN SOLVE FOR THE SECOND DERIVATIVES BY CRAMER'S ELIMINATION RULE

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\begin{vmatrix}0 & 0 & 1\\ d\left(\frac{\partial\phi}{\partial x}\right) & dy & 0\\ d\left(\frac{\partial\phi}{\partial y}\right) & dx & dy\end{vmatrix}}{\begin{vmatrix}(1-M_{a_\infty}^2) & 0 & 1\\ dx & dy & 0\\ 0 & dx & dy\end{vmatrix}},\qquad \frac{\partial^2\phi}{\partial y^2}=\frac{\begin{vmatrix}(1-M_{a_\infty}^2) & 0 & 0\\ dx & dy & d\left(\frac{\partial\phi}{\partial x}\right)\\ 0 & dx & d\left(\frac{\partial\phi}{\partial y}\right)\end{vmatrix}}{\begin{vmatrix}(1-M_{a_\infty}^2) & 0 & 1\\ dx & dy & 0\\ 0 & dx & dy\end{vmatrix}}.$$

SETTING THE DETERMINATE IN THE DENOMINATOR TO ZERO, WHICH MAKES THE SECOND DERIVATIVES INDETERMINATE, GIVES THE EQUATION OF THE CHARACTERISTICS.

$$\frac{dy}{dx}=\pm\sqrt{\frac{1}{M_{a_\infty}^2-1}}=\pm\tan\mu$$

FOR $M_{a_\infty}<1$ THE CHARACTERISTICS ARE COMPLEX AND EQ. (144) IS ELLIPTIC. ON THE OTHER HAND, FOR $M_{a_\infty}>1$ THE CHARACTERISTICS ARE REAL AND EQ. (144) IS HYPERBOLIC.

* PRESSURE COEFFICIENT

AN EXPRESSION FOR THE PRESSURE COEFFICIENT CAN BE EASILY OBTAINED IN THE CONTEXT OF SMALL PERTURBATION THEORY. CONSIDER THE DEFINITION

$$C_p=\frac{p-p_\infty}{\frac{1}{2}\rho_\infty U_\infty^2}=\frac{2}{\gamma M_{a_\infty}^2}\left(\frac{p}{p_\infty}-1\right)\qquad(146)$$

AND USE THE ISENTROPIC RELATION $\frac{p}{p_\infty}=\left[\frac{1+\left(\frac{\gamma-1}{2}\right)M_{a_\infty}^2}{1+\left(\frac{\gamma-1}{2}\right)M_a^2}\right]^{\frac{\gamma}{\gamma-1}}$ ALONG WITH THE CONSTANT STAGNATION ENTHALPY RELATION (140). SUCH APPROACH TRANSFORMS (146) INTO

$$C_p=\frac{2}{\gamma M_{a_\infty}^2}\left\{\left[1-\underbrace{\frac{\gamma-1}{2}M_{a_\infty}^2\left(\frac{2u}{U_\infty}+\frac{u^2+v^2+w^2}{U_\infty^2}\right)}_{\varepsilon}\right]^{\frac{\gamma}{\gamma-1}}-1\right\}$$

WHICH CAN BE FURTHER SIMPLIFIED FOR $\varepsilon\ll 1$ TO GIVE

$$C_p=-\left[\frac{2u}{U_\infty}+\left(1-M_{a_\infty}^2\right)\frac{u^2}{U_\infty^2}+\frac{v^2+w^2}{U_\infty^2}\right].\qquad(147)$$

FOR SUBSONIC OR SUPERSONIC PLANAR FLOWS THE SECOND TERM IN (147) CAN BE NEGLECTED, THEREBY YIELDING THE SIMPLIFIED EXPRESSION

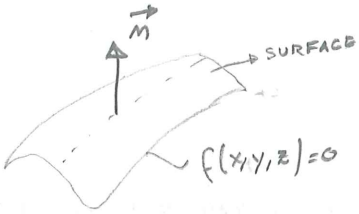
$$C_p = -\frac{2u}{U_\infty} \qquad (148)$$

WHICH ESTABLISHES THAT ANY POSITIVE PERTURBATION OF THE STREAMWISE VELOCITY CAUSES A DECREASE IN THE LOCAL STATIC PRESSURE AND THEREFORE A LOCALLY NEGATIVE PRESSURE COEFFICIENT.

* BOUNDARY CONDITIONS

THE BOUNDARY CONDITIONS TO BE APPLIED FOR INTEGRATING EQUATIONS (144) OR (145) ARE THE FAR-FIELD ONE $u, v, w \to 0$, AT $|\vec{x}| \to \infty$, SO THAT $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial z} = 0$

AND THE NO-PENETRATION CONDITION, IN THAT THE DIRECTION OF THE FLOW IS TANGENTIAL TO THE SOLID SURFACE. DENOTING THE LATTER BY $f(x, y, z) = 0$, THE NO-PENETRATION CONDITION BECOMES



$$\vec{V} \cdot \nabla f = 0 \qquad (149)$$

SINCE $\vec{m} = \nabla f / |\nabla f|$ IS THE NORMAL VECTOR TO THE SURFACE.

INTRODUCING THE PERTURBATION VELOCITIES (138) INTO (149) GIVES

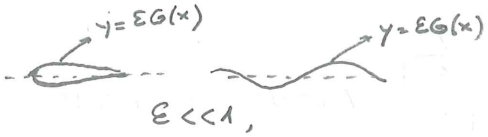
$$U_\infty \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + w \frac{\partial f}{\partial z} = 0 \qquad (150)$$

WHERE IT HAS BEEN USED THAT $u/U_\infty \ll 1$. IN TWO DIMENSIONS, (150) CAN BE REWRITTEN AS

$$\frac{v}{U_\infty} = \frac{\partial f/\partial x}{\partial f/\partial y} = \frac{dy_w}{dx} \qquad (151)$$

(y_w: WALL $y_w(x)$)

SO THAT THE VERTICAL VELOCITY ON THE SURFACE EQUALS THE SLOPE MULTIPLIED BY U_∞. IN SMALL-PERTURBATION THEORY, THE INCLINATION OF THE SURFACE OF THE BODY VARIES SLIGHTLY FROM ZERO, AS IN $y = \varepsilon G(x)$ WITH $\varepsilon \ll 1$ AND $G(x)$ AN ARBITRARY SHAPE FUNCTION. IT IS THEREFORE CONVENIENT TO TRANSFER THE BOUNDARY CONDITION (151) TO $y = 0$ IN THE FIRST APPROXIMATION IN ε, NAMELY,



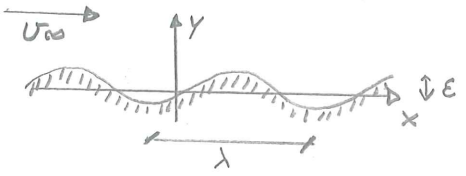
$$v(x,y) = v(x, \varepsilon G(x)) \sim v(x,0) + \left.\frac{\partial v}{\partial y}\right|_{y=0} \varepsilon G(x) + \mathcal{O}(\varepsilon^2) \sim v(x,0)$$

SO THAT (151) BECOMES

$$\frac{v(x,0)}{U_\infty} \approx \frac{dy_w}{dx} = \frac{1}{U_\infty} \frac{\partial \phi}{\partial y}(x,0) \qquad (152)$$

(TO LEADING ORDER)

FLOW PAST A WAVY WALL



CONSIDER THE FLOW PAST A SINUSOIDAL SOLID BOUNDARY GIVEN BY

$$y = \varepsilon \lambda \sin\left(\frac{2\pi x}{\lambda}\right)$$

WHERE λ IS THE WAVELENGTH AND $\varepsilon \ll 1$.

THE SUPERSONIC AND SUBSONIC FLOWS ARE DESCRIBED BY EQUATION (144). SUBJECT TO $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$ IN THE FAR FIELD $y \to \infty$, AND TO $\frac{\partial \phi}{\partial y}(x, y_w) = U_\infty \frac{dy_w}{dx} = U_\infty \varepsilon \cos\left(\frac{2\pi x}{\lambda}\right)$.

USING THE NON-DIMENSIONAL VARIABLES $x^* = \frac{2\pi x}{\lambda}$, $y^* = \frac{2\pi y}{\lambda}$, $u^* = u/\varepsilon U_\infty$, $v^* = v/\varepsilon U_\infty$, THE PROBLEM TO SOLVE IS

$$(1 - Ma_\infty^2)\frac{\partial^2 \phi^*}{\partial x^{*2}} + \frac{\partial^2 \phi^*}{\partial y^{*2}} = 0 \quad , \quad \text{SUBJECT TO} \quad \begin{cases} y^* \to +\infty : & \frac{\partial \phi^*}{\partial x^*} = \frac{\partial \phi^*}{\partial y^*} = 0 \\ y^* = 0 : & \frac{\partial \phi^*}{\partial y^*} = \sin(x^*) \quad \text{(TRANSFERRED TO } y=0\text{)} \end{cases}$$

NOTE THAT THE INDUCED VELOCITY DISTURBANCES ARE OF ORDER ε. ↑ OR FINITE FOR $Ma_\infty \neq 1$

ASSUMING FIRST SUBSONIC FLOW, $Ma_\infty < 1$, THEN THE VELOCITY-POTENTIAL EQUATION BECOMES

$$\frac{\partial^2 \phi^*}{\partial x^{*2}} + \frac{1}{m^2}\frac{\partial^2 \phi^*}{\partial y^{*2}} = 0 \quad , \quad \text{WITH } m^2 = 1 - Ma_\infty^2 > 0$$

SEPARATING VARIABLES AS $\phi = F(x)G(y)$, THE ABOVE EQUATION BECOMES $\frac{F''}{F} + \frac{1}{m^2}\frac{G''}{G} = 0$ AND THEREFORE $\frac{F''}{F} = -\frac{1}{m^2}\frac{G''}{G} = -k^2$, WITH k A CONSTANT TO BE DETERMINED FROM THE BOUNDARY CONDITIONS. THE SOLUTION TO THESE EQUATIONS IS $F(x^*) = A_1 \sin(kx^*) + A_2 \cos(kx^*)$ AND $G(y^*) = B_1 e^{-mky^*} + B_2 e^{mky^*}$. THE FAR-FIELD BOUNDARY CONDITIONS REQUIRE $B_2 = 0$, WHILE THE BOUNDARY CONDITION AT $y = 0$ REQUIRES $A_1 = 0$, $k = 1$, AND $A_2 B_1 = -1/m$. THE RESULTING VELOCITY POTENTIAL IS

$$\phi^* = -\frac{1}{\sqrt{1 - Ma_\infty^2}}\, e^{-\sqrt{1-Ma_\infty^2}\, y^*} \cos x^*$$

WHICH GIVES THE VELOCITY PERTURBATION FIELD

$$\begin{cases} u^* = \frac{\partial \phi^*}{\partial x^*} = \frac{1}{\sqrt{1 - Ma_\infty^2}}\, e^{-\sqrt{1-Ma_\infty^2}\, y^*} \sin x^* \\ v^* = \frac{\partial \phi^*}{\partial y^*} = -e^{-\sqrt{1-Ma_\infty^2}\, y^*} \cos x^* \end{cases}$$

THE CORRESPONDING PRESSURE COEFFICIENT IS GIVEN BY EQ. (148), WITH u THERE REFERRING TO THE DIMENSIONAL VELOCITY (I.E. THE ABOVE EXPRESSION OF u MULTIPLIED BY εU_∞),

⇒ AT THE WALL ($y = 0$):

$$\boxed{C_{p_w} = -2u/U_\infty \Big|_{y=0} = \frac{-2\varepsilon}{\sqrt{1 - Ma_\infty^2}} \sin\left(\frac{2\pi x}{\lambda}\right)} \qquad (153)$$

NOTE THAT THE SOLUTION PREDICTS ZERO DRAG FORCE, AS C_p IS IN PHASE WITH THE WALL SURFACE. SIMILARLY, AS $Ma_\infty \to 1$, THE VERTICAL EXTENSION OF THE DISTURBANCES GROWS LARGER, WHICH IS ACCOMPANIED WITH A DIVERGENCE OF C_p. THE FACTOR $1/\sqrt{1 - Ma_\infty^2}$ IN (153) IS NAMED THE PRANDTL-GLAUERT FACTOR, AND APPEARS UBIQUITOUSLY IN THE

THEORY OF SUBSONIC AERODYNAMICS AND CONSTITUTES THE BASIS OF THE PRANDTL-GLAUERT TRANSFORMATION. IN THIS TRANSFORMATION, y IS RE-SCALED AS $y^* = y\sqrt{1-Ma_\infty^2}$, WHILE ϕ BECOMES $\phi^* = (1-Ma_\infty^2)\phi$ AND x REMAINS UNCHANGED. IN THESE VARIABLES, (144) BECOMES

$$\frac{\partial^2\phi^*}{\partial x^{*2}} + \frac{\partial^2\phi^*}{\partial y^{*2}} = 0 \quad \Rightarrow \text{INCOMPRESSIBLE VELOCITY-POTENTIAL EQUATION.}$$

THE PRANDTL-GLAUERT FACTOR PREDICTS AN UNBOUNDED INCREASE OF THE PRESSURE COEFFICIENT AS TRANSONIC CONDITIONS ARE APPROACHED. IT SHOULD HOWEVER BE NOTICED THAT NON-LINEAR EFFECTS BECOME IMPORTANT IN TRANSONIC FLOW, WITH EQ. (145) BEING REQUIRED INSTEAD OF EQ. (144).

THE SOLUTION OF THE SUBSONIC PROBLEM MAY BE STATED IN THE FORM OF SIMILARITY LAW

$$C_p\frac{\sqrt{1-Ma_\infty^2}}{\varepsilon} = f\left(\frac{2\pi x}{\lambda}, \frac{2\pi y\sqrt{1-Ma_\infty^2}}{\lambda}\right) \qquad (154)$$

WHERE f IS EASILY OBTAINED FROM THE SOLUTIONS GIVEN ABOVE. IF THE ARGUMENT OF f IS KEPT CONSTANT, THEN C_p IS A SOLE FUNCTION OF ε AND THE PRANDTL-GLAUERT FACTOR. NOTE ALSO FROM THE SOLUTION THAT THE SMALL-PERTURBATION ASSUMPTION $u/U_\infty \ll 1$, REQUIRES A SMALL AMPLITUDE OF THE WALL DISTURBANCES, SUCH THAT

$$\varepsilon \ll \sqrt{1-Ma_\infty^2}.$$

FOR SUPERSONIC FLOW, $Ma_\infty > 1$, THE VELOCITY-POTENTIAL EQUATION CAN BE WRITTEN AS

$$\frac{\partial^2\phi^*}{\partial x^{*2}} - \frac{1}{m^2}\frac{\partial^2\phi^*}{\partial y^{*2}} = 0, \quad \text{WITH } m^2 = Ma_\infty^2 - 1 > 0$$

WHICH RESEMBLES A WAVE EQUATION WHOSE GENERAL SOLUTION IS

$$\phi^* = F(x^* - my^*) + G(x^* + my^*).$$

NOTE THAT THERE ARE TWO FAMILIES OF CHARACTERISTIC LINES, $\frac{dy^*}{dx^*} = \pm\frac{1}{m} = \pm\frac{1}{\sqrt{1-Ma_\infty^2}}$, ALONG WHICH THE

$$\begin{cases} C_+ : \dfrac{dy^*}{dx^*} = \dfrac{1}{\sqrt{1-Ma_\infty^2}} & (F = \text{CONST}) \\ C_- : \dfrac{dy^*}{dx^*} = \dfrac{-1}{\sqrt{1-Ma_\infty^2}} & (G = \text{CONST}) \end{cases}$$



WAVES F AND G PROPAGATE. WHEN THE STATE OF THE GAS AT $y^* \to \infty$ IS THAT OF A UNIFORM FREE STREAM, AS IN THE PROBLEM ADDRESSED HERE, THEN $G = 0$, IN THAT THE RIGHT RUNNING WAVES ARE BORN IN UNDISTURBED REGIONS. THEREFORE THE SOLUTION IS $\phi^* = F(x^* - my^*)$, WITH F OBTAINED FROM THE BOUNDARY CONDITION AT THE WALL AS

$$\frac{\partial\phi^*}{\partial y^*} = -m\left.\frac{dF}{d\xi}\right|_{y=0} = \cos x^*, \quad \text{WITH } \xi = x^* - my^*, \Rightarrow F(\xi) = -\frac{\sin\xi}{m} =$$

$$F = -\frac{\sin(x^* - my^*)}{m}$$

THEREFORE, THE VELOCITY POTENTIAL IS GIVEN BY

$$\phi^* = -\frac{1}{\sqrt{Ma_\infty^2 - 1}} \sin\left[x^* - y^*\sqrt{Ma_\infty^2 - 1}\right]$$

NOTE THAT $\frac{\partial \phi}{\partial x} \neq 0$ AND $\frac{\partial \phi}{\partial y} \neq 0$ AT $y \to +\infty$ BUT THEY REMAIN FINITE

AND THE VELOCITY PERTURBATION FIELD IS

$$u^* = \frac{\partial \phi^*}{\partial x^*} = -\frac{1}{\sqrt{Ma_\infty^2 - 1}} \cos\left[x^* - y^*\sqrt{Ma_\infty^2 - 1}\right]$$

$$v^* = \frac{\partial \phi^*}{\partial y^*} = \cos\left[x^* - y^*\sqrt{Ma_\infty^2 - 1}\right].$$

SIMILARLY, THE PRESSURE COEFFICENT BECOMES

$$C_p = \frac{2\varepsilon}{\sqrt{Ma_\infty^2 - 1}} \cos\left[\frac{2\pi x}{\lambda} - \frac{2\pi y}{\lambda}\sqrt{Ma_\infty^2 - 1}\right]$$ ← (WITH x AND y BEING EXPRESSED BACK IN DIMENSIONAL FORM).

WHICH ATTAINS THE WALL VALUE ($y = 0$)

$$\boxed{C_{p_w} = \frac{2\varepsilon}{\sqrt{Ma_\infty^2 - 1}} \cos\left(\frac{2\pi x}{\lambda}\right)} \qquad (155)$$

NOTE THAT C_{p_w} CAN BE REWRITTEN AS

$$C_{p_w} = \frac{2}{\sqrt{Ma_\infty^2 - 1}} \frac{dy_w}{dx}, \quad \frac{dy_w}{dx} \to \varepsilon \cos\left(x/\lambda\right)$$

THE CORRESPONDING DRAG COEFFICENT C_D PER UNIT WAVELENGTH IS

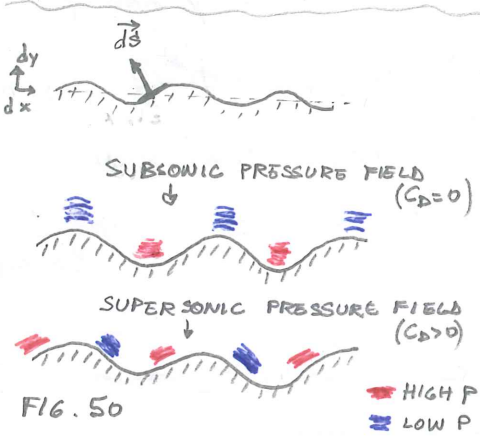


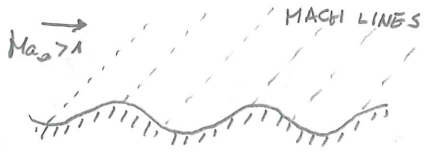
FIG. 50

$$C_D = \frac{1}{\lambda}\int_0^\lambda C_{p_w}\, dy_w \quad (dy_w: \text{VERTICAL PROJECTION OF THE SURFACE ELEMENT}) = \frac{1}{\lambda}\int_0^\lambda C_{p_w}\frac{dy_w}{dx}\,dx =$$

$$= \frac{1}{\lambda}\int_0^\lambda \frac{2}{\sqrt{Ma_\infty^2 - 1}}\left(\frac{dy_w}{dx}\right)^2 dx = \frac{2}{\sqrt{Ma_\infty^2 - 1}}\,\frac{1}{\lambda}\int_0^\lambda \left(\frac{dy_w}{dx}\right)^2 dx$$

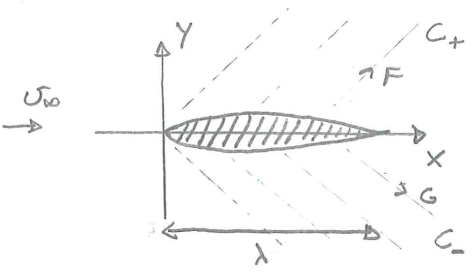
$$= \frac{2}{\sqrt{Ma_\infty^2 - 1}}\,\frac{1}{2\pi}\int_0^{2\pi} \cos^2\xi\, d\xi = \boxed{\frac{1}{\sqrt{Ma_\infty^2 - 1}}}. \qquad (156)$$

NOTE THAT THE SUPERSONIC VELOCITY DISTURBANCES u AND v DO NOT DECAY AWAY FROM THE WALL. RATHER, THE DISTURBANCES REMAIN CONSTANT ALONG THE CHARACTERISTICS $y = (\tan\mu)x + \text{const}$, WHICH CORRESPOND TO MACH LINES. IT IS ALSO REMARKABLE THAT THE PRESSURE FIELD (C_p) NEAR THE WALL TRANSITS FROM AN ANTISYMMETRIC DISTRIBUTION AT $Ma_\infty < 1$, WHERE THE MAXIMUM PRESSURE OCCURS IN THE TROUGHS AND THE MINIMUM IN THE CRESTS (THEREBY YIELDING ZERO DRAG), TO A SHIFTED PROFILE AT $Ma_\infty > 1$ WHERE THE MAXIMUM PRESSURE OCCURS ON THE WINDWARD SIDE OF THE CRESTS AND THE LOWEST PRESSURE IS ON THE LEEWARD SIDE (THEREBY YIELDING A POSITIVE DRAG FORCE OF NON-VISCOUS ORIGIN) ⟶ SUPERSONIC WAVE DRAG



FLOW OVER A SUPERSONIC THIN AIRFOIL

SOME OF THE RESULTS OBTAINED FOR THE WAVY WALL PROBLEM ARE ACTUALLY MORE GENERAL AND APPEAR AS SOLUTIONS OF A NUMBER OF PROBLEMS. TO SEE THIS, CONSIDER A THIN AIRFOIL



OF ARBITRARY SHAPE, AS DEPICTED ON THE SIDE FIGURE, WITH

$$\left.\begin{aligned} y_U &= \varepsilon\lambda Z(x) \quad \text{ON THE UPPER SURFACE} \\ y_L &= -\varepsilon\lambda H(x) \quad \text{ON THE LOWER SURFACE} \end{aligned}\right\}$$

Z AND H BEING TWO ARBITRARY (DIMENSIONLESS) FUNCTIONS AND ε A DIMENSIONLESS PARAMETER THAT IS PROPORTIONAL TO THE RATIO OF THE AIRFOIL WIDTH TO ITS CHORD LENGTH, WITH $\varepsilon \ll 1$. THE GENERAL SOLUTION TO THE SUPERSONIC FLOW PROBLEM

$$(1-Ma_\infty^2)\frac{\partial^2\phi^*}{\partial x^{*2}} + \frac{\partial^2\phi}{\partial y^{*2}} = 0$$

SUBJECT TO

$$\left.\begin{aligned} &y^* \to \pm\infty, \quad \frac{\partial\phi^*}{\partial x^*} = \frac{\partial\phi^*}{\partial y^*} = \text{FINITE}. \\ &y^* \to y_U^*; \quad \frac{\partial\phi^*}{\partial y^*} = \frac{dy_U^*}{dx^*} \\ &y^* \to y_L^*; \quad \frac{\partial\phi^*}{\partial y^*} = \frac{dy_L^*}{dx^*} \end{aligned}\right.$$

NOTE THIS IS < 0 AND $\frac{dy_L^*}{dx^*}$

WISE $\phi^* = F(x^* - my^*) + G(x^* + my^*)$, WHERE $m^2 = Ma_\infty^2 - 1$ AND x, y AND ϕ HAVE BEEN NORMALIZED AS $x^* = x/\lambda$, $y^* = y/\lambda$, $y_L^* = \frac{y_L}{\varepsilon\lambda}$, $y_U^* = \frac{y_U}{\varepsilon\lambda}$, AND $\phi^* = \frac{\phi}{\varepsilon U_\infty \lambda}$, AS IN THE PREVIOUS SECTIONS. NOTE THAT THE SOLUTION CONTAINS TWO WAVES THAT RUN ALONG THE CHARACTERISTIC LINES $C_+ : \frac{dy^*}{dx^*} = \tan\mu = \frac{1}{\sqrt{Ma_\infty^2 - 1}}$ AND $C_- : \frac{dy^*}{dx^*} = -\tan\mu = \frac{-1}{\sqrt{Ma_\infty^2 - 1}}$, WITH F CONSTANT ALONG C_+, AND G CONSTANT ALONG C_-.

ON THE UPPER SURFACE, THE RELEVANT CHARACTERISTIC IS C_+, SINCE C_- JUST BRINGS INFORMATION OF ZERO DISTURBANCE FROM INFINITY ($G = 0$). THEN

$$\phi = F(x^* - my^*) \quad \text{FOR } y^* > 0$$

AND SINCE $\frac{\partial\phi^*}{\partial y^*} = -m\frac{dF}{d\xi}\Big|_{y=0} = \frac{dy_U^*}{dx^*}$, WITH $\xi = x - my$, THEN $F'(x) = -\frac{1}{m}\frac{dy_U}{dx}$. (157)

SIMILARLY, ON THE LOWER SURFACE, THE RELEVANT CHARACTERISTIC IS C_-, SINCE C_+ JUST BRINGS INFORMATION OF ZERO DISTURBANCE FROM INFINITY ($F = 0$). THEN

$$\phi^* = G(x^* + my^*) \quad \text{FOR } y^* < 0$$

AND SINCE $\frac{\partial\phi^*}{\partial y^*} = +m\frac{dG}{d\eta}\Big|_{y=0} = \frac{dy_L^*}{dx^*}$, WITH $\eta = x + my$, THEN $G'(x) = \frac{1}{m}\frac{dy_L^*}{dx^*}$. (158)

IN SUMMARY, THE DIMENSIONAL DISTURBANCE VELOCITIES ARE (IN DIMENSIONAL FORM)

$$u = \frac{\partial\phi}{\partial x} = -\frac{U_\infty}{\sqrt{Ma_\infty^2 - 1}}\frac{dy_U}{dx} \quad \text{(ON UPPER SURFACE)}, \qquad u = \frac{\partial\phi}{\partial x} = \frac{U_\infty}{\sqrt{Ma_\infty^2 - 1}}\frac{dy_L}{dx} \quad \text{(ON LOWER SURFACE)}$$

THEN THE PRESSURE COEFFICIENT (148) BECOMES

$$C_{P_U} = -\frac{2u}{U_\infty} = \frac{2}{\sqrt{Ma_\infty^2 - 1}}\left(\frac{dy_U}{dx}\right) \qquad \text{(ON UPPER SURFACE)}$$

$$C_{P_L} = -\frac{2u}{U_\infty} = \frac{2}{\sqrt{Ma_\infty^2 - 1}}\left(-\frac{dy_L}{dx}\right) \qquad \text{(ON LOWER SURFACE)} \qquad (159)$$

WITH $\frac{dy_U}{dx} = \varepsilon\lambda\frac{dZ}{dx}$ AND $\frac{dy_L}{dx} = -\varepsilon\lambda\frac{dH}{dx}$ DEPENDING ON THE PARTICULAR GEOMETRY OF THE AIRFOIL. NOTE THAT THE CORRESPONDING DRAG COEFFICIENT IS

$$C_D = C_{D_U} + C_{D_L} = \frac{1}{\lambda}\int_0^\lambda C_{P_U}\,dy_U - \frac{1}{\lambda}\int_0^\lambda C_{P_L}\,dy_L =$$

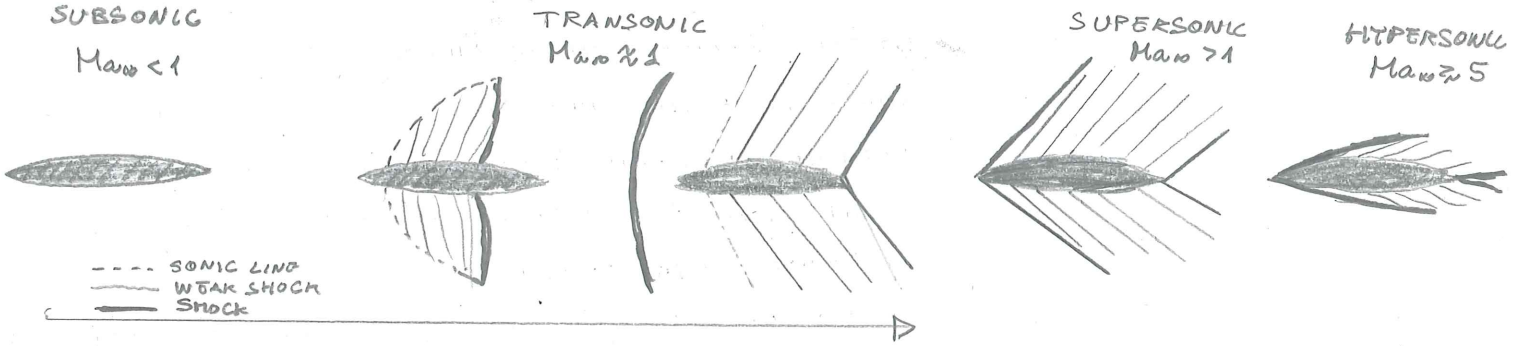
$$= \frac{2}{\sqrt{Ma_\infty^2 - 1}}\left\{\frac{1}{\lambda}\int_0^\lambda\left[\left(\frac{dy_U}{dx}\right)^2 + \left(\frac{dy_L}{dx}\right)^2\right]dx\right\} \qquad (160)$$

RESULTS (159) AND (160) EXPRESS AN IMPORTANT CHARACTERISTIC OF SUPERSONIC WAVE DRAG, NAMELY, THAT SUPERSONIC WAVE DRAG IS HIGHLY DEPENDENT ON THE LOCAL SLOPE OF THE PROFILE, OR EQUIVALENTLY, ON THE RATE OF CROSS-SECTIONAL AREA VARIATION OF THE AIRFOIL THICKNESS. THIS RESULT IS CENTRAL TO SUPERSONIC WAVE-DRAG MITIGATION, AS IN WHITCOMB'S AREA RULE REFERRED TO IN PAGE 28.

THEREFORE, IN GENERAL

$$C_P = \frac{2\Theta}{\sqrt{Ma_\infty^2 - 1}} \qquad (161)$$

ACCORDING TO THE TWO-DIMENSIONAL LINEAR THEORY OF SUPERSONIC AERODYNAMICS, WITH Θ THE LOCAL INCLINATION ANGLE OF THE AIRFOIL SURFACE.



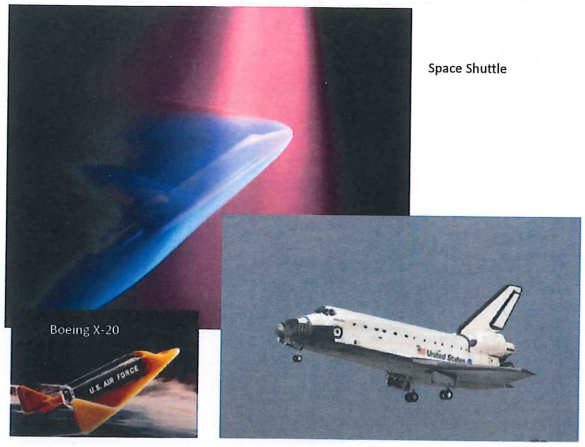
TRANSITION OF COMPRESSIBLE FLOW STRUCTURE WITH INCREASING MACH NUMBERS.

VI. HYPERSONIC AERODYNAMICS

THE FLOWS ANALYZED IN THE PREVIOUS PARTS OF THIS COURSE PERTAIN TO MODERATE MACH NUMBERS. FOR $Ma \gtrsim 5$-6, ADDITIONAL EFFECTS MUST BE ACCOUNTED FOR IN THE DESCRIPTION OF COMPRESSIBLE FLOWS. THESE EFFECTS INVOLVE HIGH-TEMPERATURE CHEMISTRY, SUCH AS IONIZATION AND DISSOCIATION OF THE GAS, NON-EQUILIBRIUM EFFECTS, SUCH AS VIBRATIONAL EXCITATION OF THE GAS MOLECULES, VISCOUS INTERACTIONS OF SHOCK WAVES WITH BOUNDARY LAYERS, RADIATION, AND RAREFACTION OF THE ENVIRONMENT AT HIGH ALTITUDES. THESE EFFECTS SET THE HYPERSONIC RANGE OF MACH NUMBERS ($Ma \gtrsim 5$-6) APART FROM LOWER-MACH # CONVENTIONAL COMPRESSIBLE FLOWS.

THE GAS ENVIRONMENT IN HYPERSONIC FLOWS

THE MAIN FIELD OF APPLICATION FOR HYPERSONIC FLOWS IS THAT OF HIGH-SPEED FLIGHT. HYPERSONIC VEHICLES SUCH AS X-15 ($Ma \sim 7$) OR THE SPACE SHUTTLE ($Ma \sim 25$ DURING REENTRY) EXPERIENCE AERODYNAMIC EFFECTS THAT CANNOT BE SOLELY EXPLAINED BY CONVENTIONAL SUPERSONIC AERODYNAMICS. REENTRY VEHICLES, SUCH AS SPACE CAPSULES AND LIFTING BODIES, ARE TYPICALLY DESIGNED TO HAVE BLUNT NOSES AND ROUND EDGES. THIS IS TO MAXIMIZE DRAG AND SLOW DOWN SPACE CAPSULES, TO MAXIMIZE THE LIFT-TO-DRAG RATIO IN LIFTING BODIES, AND TO SEPARATE THE FRONT SHOCK FROM THE FUSELAGE THEREBY REDUCING AERODYNAMIC HEATING LOADS ON THE VEHICLE'S AIRFRAME. IN HYPERSONIC AIRCRAFTS, THE AIRFRAMES ARE TYPICALLY DESIGNED TO HAVE SHARP NOSE TO REDUCE DRAG. ONE IMPORTANT DIFFERENCE BETWEEN HYPERSONIC AND CONVENTIONAL SUBSONIC AND SUPERSONIC ATMOSPHERIC AIRCRAFTS, IS THAT THE LIFTING SURFACES, PROPULSION ELEMENTS AND AIRFRAME ARE STRONGLY COUPLED IN HYPERSONIC AIRPLANES, WHILE THERE IS ONLY WEAK COUPLING BETWEEN COMPONENTS IN SUBSONIC AND SUPERSONIC ONES. AN EXAMPLE OF THIS STRONG COUPLING IS THE REQUIRED INTEGRATION OF THE PROPULSION SYSTEM SCRAMJET IN THE AIRFRAME OF HYPERSONIC VEHICLES, WHERE THE ENTIRE UNDERSURFACE OF THE AIRFRAME IS ACTUALLY PART OF THE SCRAMJET ENGINE AND CONTRIBUTES TO THE AIR COMPRESSION. FOR INSTANCE, THE INITIAL COMPRESSION IS PERFORMED BY THE LEADING BOW SHOCK FROM THE NOSE OF THE AIRCRAFT AND IT IS CONTINUED THROUGH A SERIES OF RAMPS. THE HOT GASES ARE THEN SUPERSONICALLY EXPANDED THROUGH A NOZZLE FORMED BY THE PROPER STREAMLINING OF THE REAR OF THE AIRCRAFT.



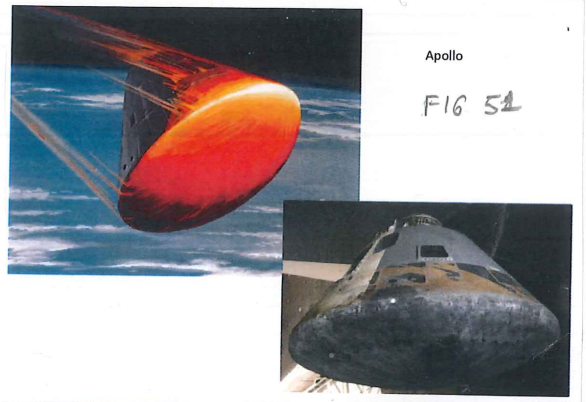
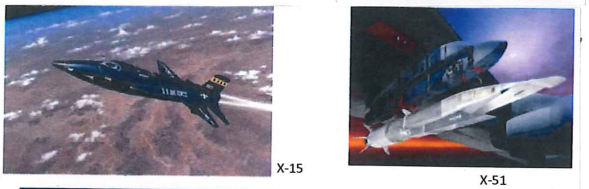


FIG 51



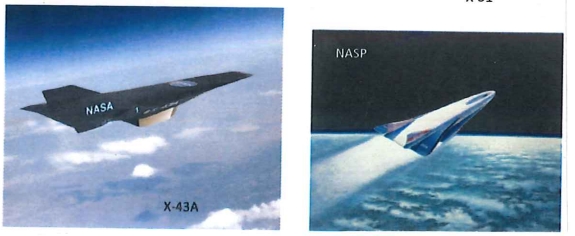


FIG. 52

* SHOCK LAYERS

AN IMPORTANT CHARACTERISTIC OF HYPERSONIC FLOWS IS THAT THE SHOCK WAVES CAUSED BY THE FUSELAGE SHAPE TEND TO BE VERY CLOSE TO THE FUSELAGE SURFACE AT HYPERSONIC MACH NUMBERS, THEREBY FORMING AN ENVELOPE AROUND THE SURFACE THAT MAY EVEN PENETRATE IN THE BOUNDARY LAYER. THE THIN GAP BETWEEN THE SHOCK WAVE AND THE SURFACE IS CALLED THE SHOCK LAYER (SEE FIG. 53), THIS LAYER IS THIN AND IS FORMED BY HOT GASES THAT EXERT HEATING LOADS ON THE FUSELAGE. IN THE LIMIT OF INFINITE MACH NUMBERS, THE SURFACE OF THE FUSELAGE IS UNDISTINGUISHABLE FROM THE SHOCK. THIS IS CALLED THE NEWTONIAN LIMIT AND GIVES RISE TO LOCAL SURFACE-INCLINATION METHODS TO COMPUTE DRAG FORCES AT HYPERSONIC MACH NUMBERS.

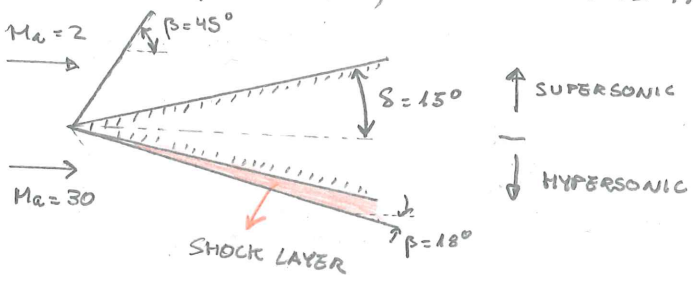


FIG. 53

* ENTROPY LAYERS

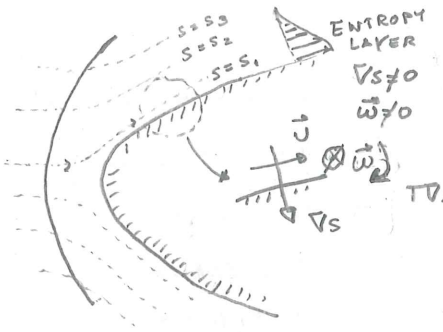


FIG. 54.

AS DISCUSSED IN THE CONTEXT OF CROCCO'S THEOREM (PAGE 64), A CURVED SHOCK WAVE LEADS TO TRANSVERSE ENTROPY GRADIENTS. IN HYPERSONIC BLUNT-NOSED VEHICLES (SEE FIG. 54), THE BOW SHOCK IS NEARLY NORMAL CLOSE TO THE NOSE AND GETS INCREASINGLY SHALLOWER AWAY FROM IT. THEREFORE, A FLUID PARTICLE CROSSING THE BOW SHOCK NEAR THE NOSE WILL EXPERIENCE A LARGER ENTROPY INCREASE THAN ANOTHER ONE CROSSING THE SHOCK THROUGH A PERIPHERAL REGION. THIS CAUSES A GRADIENT OF ENTROPY NORMAL TO THE SURFACE THAT INDUCES VORTICITY IN THE BOUNDARY LAYER AS INDICATED IN FIG. 54. THE REGION WHERE THE ENTROPY IS NON-UNIFORM IS CALLED THE ENTROPY LAYER.

* VISCOUS DISSIPATION

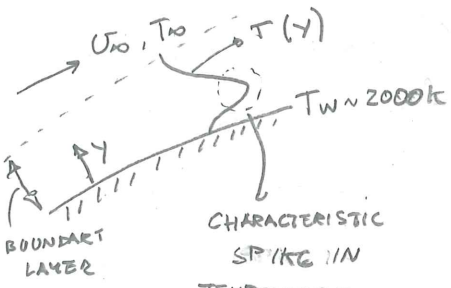


FIG. 55.

AT LOW MACH NUMBERS, THE VISCOUS DISSIPATION PLAYS A NEGLIGIBLE ROLE IN THE TRANSPORT OF THERMAL ENERGY. THIS CAN BE SEEN BY ANALYZING THE CORRESPONDING CONSERVATION EQUATION:

$$\rho \frac{\partial e}{\partial t} + \rho \vec{v} \cdot \nabla e = -P \nabla \cdot \vec{v} + \nabla \cdot (\kappa \nabla T) - \underbrace{\tau : \nabla \vec{v}}_{\text{VISCOUS DISSIPATION}}$$

WHILE THE CONVECTIVE TERMS ARE OF ORDER $\rho_\infty U_\infty c_v \frac{\Delta T}{L}$, THE VISCOUS DISSIPATION IS OF ORDER $\mu_\infty \frac{U_\infty^2}{\delta^2}$, THE RATIO OF THE LATTER TO THE FORMER BEING OF ORDER

$$\frac{\mu_\infty U_\infty^2/\delta^2}{\rho_\infty U_\infty c_v \Delta T/L} \sim \frac{L^2}{\delta^2} \frac{\mu}{\rho U_\infty L} \frac{U_\infty^2}{a_\infty^2} \sim M_{a_\infty}^2 ,$$

FOR $M_{a_\infty} \gg 1$, THE EFFECT OF THE VISCOUS DISSIPATION IS TO TRANSFORM THE KINETIC ENERGY OF THE OUTER STREAM INTO ORDER-UNITY INCREASES OF THERMAL ENERGY IN THE BOUNDARY LAYER.

* HIGH-TEMPERATURE GASDYNAMIC EFFECTS

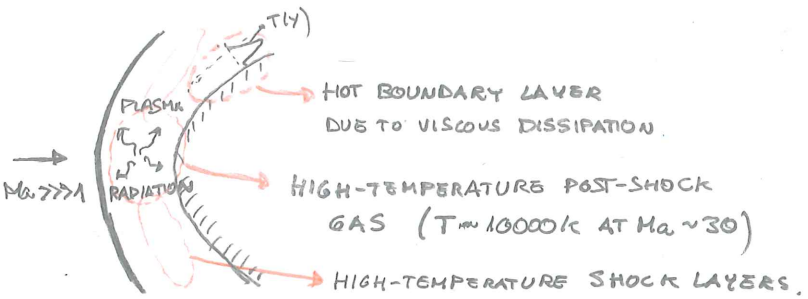


FIG. 56

TYPICAL REENTRY VELOCITIES

APOLLO: $U_\infty \sim 11$ km/s, $Ma_\infty \sim 36$

HIGH-TEMPERATURE EFFECTS ARE MANIFESTED IN HYPERSONIC FLOWS IN SEVERAL WAYS:

- HYPERSONIC BOUNDARY LAYERS ARE HOTTER DUE TO THE CONVERSION OF ORDER-UNITY AMOUNTS OF KINETIC ENERGY INTO ORDER-UNITY AMOUNTS OF THERMAL ENERGY. THIS TYPICALLY LEADS TO CHEMICALLY-REACTING BOUNDARY LAYERS, AS EXPLAINED BELOW.
- THE GAS SHOCKED BY THE BOW SHOCK GETS EXTREMELY HOT AT HYPERSONIC MACH NUMBERS, WHICH GIVES RISE TO RADIATIVE HEAT TRANSFER FROM THE GAS TO THE SURROUNDING SURFACES. ADDITIONALLY, THE GAS TEMPERATURE IS SO HIGH THAT IT CAN BECOME A PARTIALLY IONIZED PLASMA. SUCH IONIZATION CAN CAUSE COMMUNICATIONS BLACKOUT, IN THAT THE FREE ELECTRONS ABSORB THE RADIO WAVES FROM COMMUNICATION BETWEEN THE VEHICLE AND GROUND CREWS.

THE INCREASE IN TEMPERATURES AT HYPERSONIC SPEEDS PRODUCES A NUMBER OF PHYSICOCHEMICAL PHENOMENA SUCH AS

- EXCITATION OF VIBRATIONAL ENERGY OF GAS MOLECULES ($T > 1000$ K). THE PASSAGE OF GAS THROUGH SHOCK WAVES PRODUCES VIBRATIONAL NON-EQUILIBRIUM (SEE DISCUSSION IN PAGE 38), IN SUCH A WAY THAT MOLECULES STORE PART OF ITS THERMAL ENERGY INTO VIBRATIONAL MODES, WHICH CAN BE LATER EXCHANGED WITH TRANSLATIONAL THERMAL ENERGY AND KINETIC ENERGY. THE TYPICAL VIBRATIONAL RELAXATION TIMES ARE OF ORDER 1-10 μs, WHICH, WHEN COMBINED WITH TRANSLATIONAL HYPERSONIC VELOCITIES OF ORDER 10 km/s, LEAD TO PROLONGED REGIONS OF ORDER 10 cm - 1 m IN WHICH THE NON-EQUILIBRIUM EXCITATIONS RELAX TO EQUILIBRIUM.
- MOLECULAR DISSOCIATION ($O_2 \rightarrow 2O$, $T \gtrsim 2000$ K; $N_2 \rightarrow 2N$, $T \gtrsim 4000$ K)

  AT ELEVATED TEMPERATURES, OXYGEN AND NITROGEN MOLECULES DISSOCIATE, AND THEIR DISSOCIATION PRODUCTS CAN REACT AMONG THEMSELVES OR WITH THE ORIGINAL MOLECULES TO PRODUCE FOR INSTANCE "NO" MOLECULES. ONCE THE DISSOCIATION PRODUCTS COLLIDE WITH THE SURFACE OF THE THERMAL PROTECTION SYSTEMS AND RECOMBINE THERE (THROUGH CATALYSIS) IN EXOTHERMIC REACTIONS, THEREBY CAUSING EXTRA HEATING LOADS ON THE AIRFRAME AND DEPLETION OF THE THERMAL PROTECTION SYSTEM.
- IONIZATION ($N \rightarrow N^+ + e^-$, $O \rightarrow O^+ + e^-$; $T \gtrsim 9000$ K). AT SUFFICIENTLY HIGH TEMPERATURES, THE GAS BECOMES A PARTIALLY-IONIZED PLASMA WHERE IONS AND ELECTRONS MOVE FREELY.

IT IS IMPORTANT TO NOTICE THAT THE EFFECTS DESCRIBED ABOVE CANNOT BE TYPICALLY HANDLED BY THE CALORICALLY PERFECT GAS MODEL, BUT THAT THE CHEMICAL CONVERSION AND NONEQUILIBRIUM EFFECTS COMPLICATE THE CHARACTERIZATION OF THE GAS PROPERTIES, AND THE DESCRIPTION OF THE ENERGY TRANSFER.

* RARIFIED FLOWS

AS DESCRIBED IN PAGE 2, HIGH ALTITUDE ENVIRONMENTS ARE CHARACTERIZED BY LOW DENSITIES AND CORRESPONDINGLY LARGE MEAN FREE PATHS ($\lambda \sim 1\,m$ AT ALTITUDE 100 km). THE RAREFACTION OF THE FLOW PRODUCES MOLECULAR SLIP ON THE SURFACES AND PROLONGED NON-EQUILIBRIUM EFFECTS.

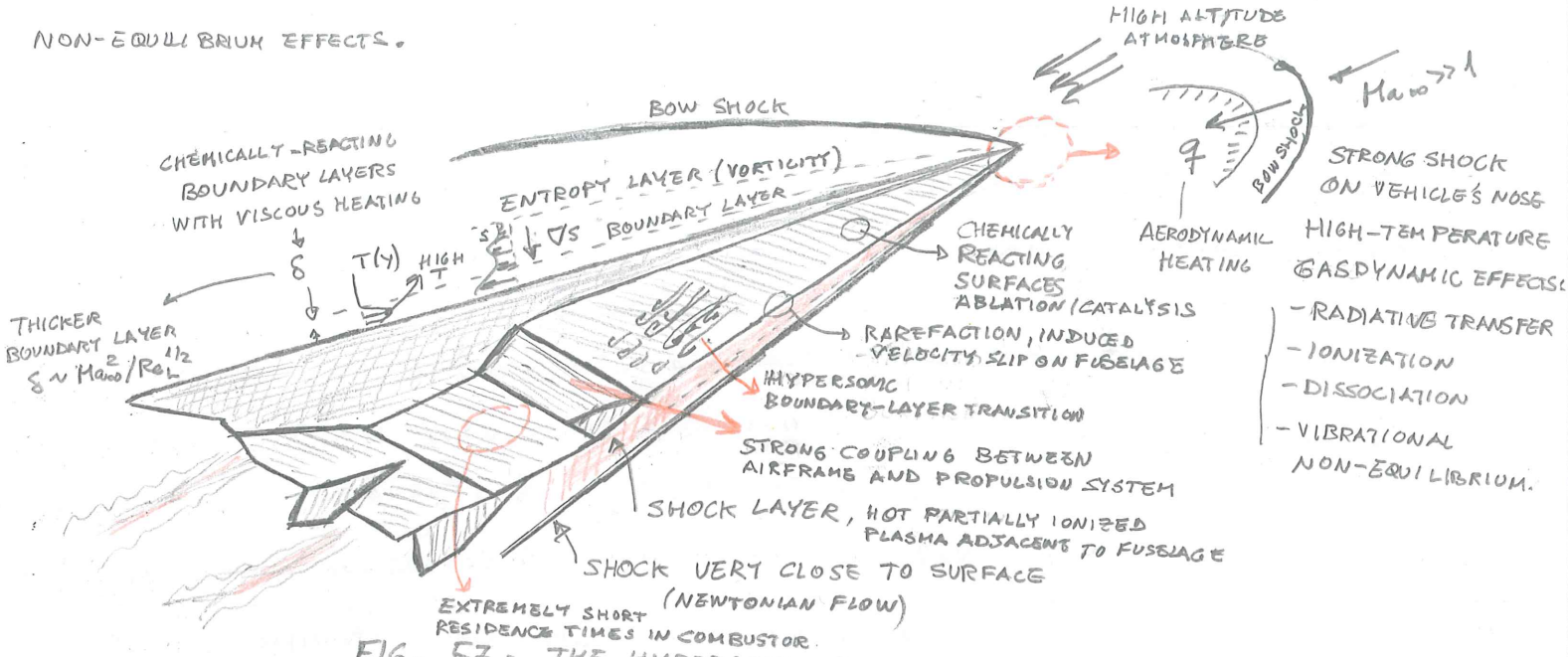


FIG. 57 : THE HYPERSONIC FLIGHT ENVIRONMENT (SUMMARY)

INVISCID HYPERSONIC FLOWS

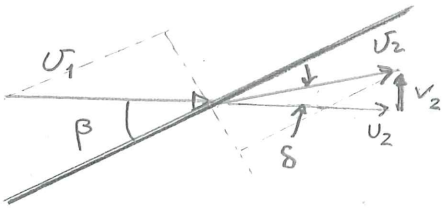
DESPITE THE COMPLEX VISCOUS INTERACTIONS AND HIGH-TEMPERATURE EFFECTS DESCRIBED ABOVE, IT IS INSTRUCTIVE TO STUDY THE ELEMENTARY THEORY OF INVISCID HYPERSONIC FLOWS WHERE THOSE EFFECTS ARE ACTUALLY NOT ACCOUNTED FOR. HOWEVER, THE SIMPLIFICATIONS LEAD TO RESULTS THAT OFTENTIMES ARE USEFUL AS FIRST APPROXIMATIONS TO MORE COMPLEX CALCULATIONS.

CONSIDER FIRST THE HYPERSONIC LIMIT OF THE OBLIQUE-SHOCK JUMP RELATIONS (58). THIS LIMIT IS OBTAINED BY MAKING $Ma_1^2 \sin^2\beta \gg 1$, WHICH YIELDS

$$\frac{p_2}{p_1} = \frac{2\gamma}{\gamma+1} Ma_1^2 \sin^2\beta \qquad (162)$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \qquad (163)$$

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1) Ma_1^2 \sin^2\beta}{(\gamma+1)^2} \qquad (164)$$



U_2, V_2 = HORIZONTAL AND VERTICAL COMPONENTS OF U_2

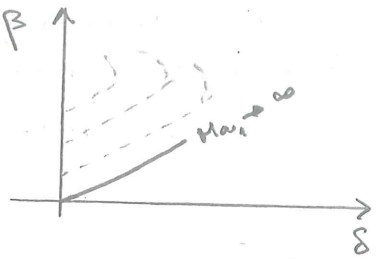
$$\frac{U_2}{U_1} = 1 - \frac{2\sin^2\beta}{\gamma+1}, \quad \frac{V_2}{U_1} = \frac{\sin(2\beta)}{\gamma+1} \qquad (165)$$

IN THE HYPERSONIC LIMIT, THE PRESSURE COEFFICIENT BECOMES

$$C_p = \frac{p_2 - p_1}{\frac{1}{2}\rho_1 U_1^2} = \frac{2}{\gamma Ma_1^2}\left(\frac{p_2}{p_1} - 1\right) = \frac{4}{\gamma+1}\sin^2\beta \qquad (166),$$

SIMILARLY, THE RELATION BETWEEN THE INCIDENCE ANGLE β AND THE STREAMWISE DEFLECTION ANGLE δ (Eq. (59)),

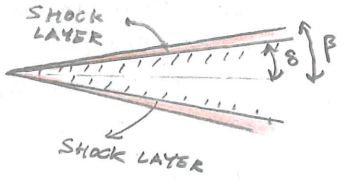
$$\tan\delta = \frac{2\cot\beta\left(Ma_1^2\sin^2\beta - 1\right)}{2 + Ma_1^2\left[\gamma + \cos(2\beta)\right]},$$



CAN BE SIMPLIFIED FOR SMALL δ (SLENDER BODIES) AND $Ma_1^2\sin^2\beta \gg 1$. TO DO SO, USE $\tan\delta \sim \delta$, $\sin\beta \sim \beta$, $\cos(2\beta) \sim 1$, BUT $Ma_1^2\sin^2\beta \sim Ma_1^2\beta^2 \gg 1$, SO THAT THE ABOVE RELATION BECOMES

$$\boxed{\beta = \left(\frac{\gamma+1}{2}\right)\delta} \qquad (167).$$

EQ. (167) PREDICTS THAT THE HYPERSONIC FLOW AROUND A SLENDER WEDGE OF SMALL SEMIANGLE δ, INVOLVES A SHOCK WAVE WHOSE INCLINATION ANGLE β IS JUST 1.2 TIMES δ (FOR $\gamma = 1.4$), WHICH ILLUSTRATES THE RELEVANCE OF THIN SHOCK LAYERS IN HYPERSONIC FLOWS



IN OBTAINING EXPRESSION (167), IT WAS ASSUMED THAT $\beta Ma_1 \gg 1$. OFTENTIMES, HOWEVER THIS PARAMETER IS OFTENTIMES OF ORDER UNITY AND ACTUALLY REPRESENTS ONE OF THE MOST IMPORTANT PARAMETERS IN HYPERSONIC FLOW THEORY:

$$\boxed{K = \beta Ma_1} \qquad \underline{\text{HYPERSONIC SIMILARITY PARAMETER}} \qquad (168)$$
(FOR δ SMALL ONLY)

THE HYPERSONIC SIMILARITY PARAMETER APPEARS PROMINENTLY IN THE FORMULATIONS OF HYPERSONIC FLOWS, ITS USEFULNESS RESIDING IN THE FACT THAT, TOGETHER WITH γ, IT IS THE ONLY PARAMETER ON WHICH THE SOLUTION TO MANY HYPERSONIC INVISCID PROBLEMS DEPEND. FOR INSTANCE, TWO BODIES OF DIFFERENT δ (DIFFERENT β) HAVE THE SAME DRAG COEFFICIENT WHEN THE MACH NUMBER Ma_1 IS CHANGED TO MAKE THEIR SIMILARITY PARAMETERS EQUAL. CORRECTIONS TO EQUATIONS (162) - (167) CAN BE DERIVED FROM (58) FOR FINITE K (I.E. FINITE MACH NUMBER), WHICH GIVES

$$\frac{\beta}{\delta} = \frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} \qquad (169)$$

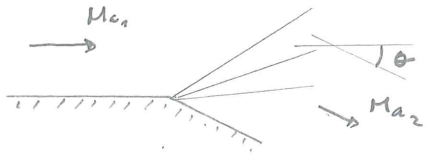
$$\frac{p_2}{p_1} = 1 + \frac{\gamma(\gamma+1)}{4}K^2 + \gamma K^2\sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^2}} \qquad (170)$$

$$C_p = 2\delta^2\left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{K^4}}\right] \qquad (171)$$

(SEE ANDERSON'S "HYPERSONIC AND HIGH-TEMPERATURE GASDYNAMICS", 2006, FOR FULL DERIVATION)

NOTE THAT $C_p = \delta^2 F(K, \gamma)$ EMERGES FROM (171) AS AN IMPORTANT SIMILARITY RELATION.

SIMILAR SIMPLIFICATIONS CAN BE MADE FOR EXPANSION WAVES IN THE HYPERSONIC LIMIT. RECALL FROM PAGE 27 THAT THE TURNING ANGLE Θ AND THE MACH NUMBERS Ma_1 AND Ma_2 ARE RELATED AS



$$\Theta = \nu(Ma_2) - \nu(Ma_1) , \qquad (172)$$

WHERE $\nu(Ma)$ IS THE PRANDTL-MEYER FUNCTION

$$\nu(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\sqrt{\frac{\gamma-1}{\gamma+1}(Ma^2-1)} - \arctan\sqrt{Ma^2-1} .$$

FOR LARGE Ma, THE PRANDTL-MEYER FUNCTION BECOMES

$$\nu(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan\left(Ma\sqrt{\frac{\gamma-1}{\gamma+1}}\right) - \arctan Ma$$

WHICH CAN BE FURTHER SIMPLIFIED BY NOTING THAT

$$\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \frac{1}{x} - \mathcal{O}\left(\frac{1}{x^3}\right) \quad \text{FOR} \quad x \to \infty$$

SO THAT

$$\nu(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}}\left(\frac{\pi}{2} - \sqrt{\frac{\gamma+1}{\gamma-1}}\frac{1}{Ma} + \mathcal{O}\left(\frac{1}{Ma^3}\right)\right) - \left(\frac{\pi}{2} - \frac{1}{Ma} + \mathcal{O}\left(\frac{1}{Ma^3}\right)\right)$$

WHICH YIELDS THE APPROXIMATE EXPRESSION

$$\boxed{\nu(Ma) \approx \left(\sqrt{\frac{\gamma+1}{\gamma-1}} - 1\right)\frac{\pi}{2} - \frac{2}{\gamma-1}\frac{1}{Ma}} \qquad (173)$$

SUBSTITUTION OF (173) INTO (172) YIELDS THE EXPRESSION

$$\boxed{\Theta = \frac{2}{\gamma-1}\left(\frac{1}{Ma_1} - \frac{1}{Ma_2}\right)} \qquad (174)$$

FOR LARGE Ma, THE CHANGE IN PRESSURE THROUGH THE EXPANSION FAN IS GIVEN BY

$$\frac{p_2}{p_1} = \left(\frac{1 + \left(\frac{\gamma-1}{2}\right)Ma_1^2}{1 + \left(\frac{\gamma-1}{2}\right)Ma_2^2}\right)^{\frac{\gamma}{\gamma-1}} \approx \left(\frac{Ma_1}{Ma_2}\right)^{\frac{2\gamma}{\gamma-1}}$$

AND SINCE $\frac{Ma_1}{Ma_2} = 1 - \left(\frac{\gamma-1}{2}\right)Ma_1\Theta$ FROM (174), $\frac{p_2}{p_1}$ BECOMES

$$\frac{p_2}{p_1} = \left(1 - \frac{\gamma-1}{2}K\right)^{\frac{2\gamma}{\gamma-1}} , \quad \text{WITH} \quad \boxed{K = Ma_1\Theta} \quad \text{THE SIMILARITY PARAMETER} \qquad (175)$$

USING THIS IN THE DEFINITION OF C_p LEADS TO

$$C_p = \frac{2}{\gamma Ma_1^2}\left(\frac{p_2}{p_1} - 1\right) \approx \frac{2}{\gamma Ma_1^2}\left[\left(1 - \left(\frac{\gamma-1}{2}\right)K\right)^{\frac{2\gamma}{\gamma-1}} - 1\right] \qquad (176)$$

WHICH ALSO INDICATES A SIMILARITY LAW OF THE TYPE

$$C_p = \Theta^2 F(K, \gamma) .$$

* NEWTONIAN THEORY OF HYPERSONIC FLOWS

IN HIS EARLY THEORY OF FLUID MOTION (PRINCIPIA, 1687), NEWTON ENVISIONED THE FORCE EXERTED ON A BODY SUBMERGED IN A FLUID AS THE EFFECT OF A COLLECTION OF FLUID PARTICLES

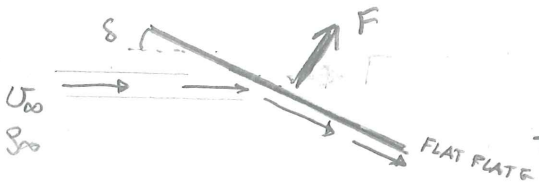


FIG. 58

INITIALLY MOVING RECTILINEARLY AND THEN COLLIDING WITH THE BODY SURFACE. AT COLLISION, THE PARTICLES WOULD TRANSFER THEIR NORMAL MOMENTUM ENTIRELY TO THE BODY AND THEY WOULD SUBSEQUENTLY SLIDE ALONG THE BODY SURFACE, AS DEPICTED IN FIG. 58. THE RESULTING FORCE ON THE BODY IS JUST THE NORMAL COMPONENT OF THE VELOCITY $U_\infty \sin\delta$ MULTIPLIED BY THE MASS FLOW RATE INCIDENT ON THE BODY SURFACE, $U_\infty \rho_\infty A \sin\delta$, WHICH GIVES $F = A\rho_\infty U_\infty^2 \sin^2\delta$. NEWTON HAD NOT THE CURRENT KNOWLEDGE THE COMMUNITY HAS ON THE IMPORTANCE OF RANDOM MOTION OF MOLECULES (PRESSURE FORCES), BOUNDARY LAYERS, SHOCK WAVES OR COMPRESSIBILITY EFFECTS, THAT ARE IMPORTANT IN COMPUTING THE AERODYNAMIC FORCES. YEARS LATER D'ALEMBERT PROVED THAT IN AN INVISCID FLOW THE DRAG FORCE IS ZERO, AND PRANDTL DISCOVERED THE IMPORTANCE OF BOUNDARY LAYERS IN DETERMINING THE ACTUAL FORCE ON BODIES. ALTHOUGH NEWTON'S THEORY WAS ERRONEOUS, IT HAS RECENTLY (SINCE 1950's) FOUND APPLICATION IN THE FIELD OF HYPERSONICS, IN THAT THE TIGHTLY ENVELOPING SHOCK

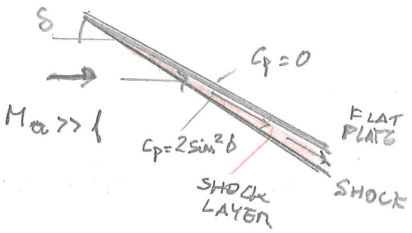


FIG. 59.

THAT DEVELOPS AROUND THE BODY MAKES THE FLOW STREAMLINES TO DEFLECT PARALLEL TO THE SURFACE, PRECISELY AS IN THE WAY NEWTON ENVISIONED. THE CHARACTERISTIC PRESSURE COEFFICIENT FROM NEWTON'S THEORY IS

$$C_p = \frac{P - P_\infty}{\frac{1}{2}\rho_\infty U_\infty^2} = \frac{\rho_\infty U_\infty^2 \sin^2\delta}{\frac{1}{2}\rho_\infty U_\infty^2} = 2\sin^2\delta \qquad (177)$$

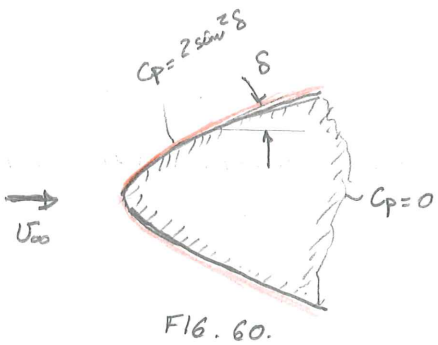


FIG. 60.

IN THE NEWTONIAN THEORY APPLIED TO COMPLEX BODIES, THE ANGLE δ IN (177) IS <u>THE LOCAL DEFLECTION ANGLE OF THE SURFACE</u>. IN THIS MODEL, THE FLUID PARTICLES IMPACT ONLY THE FRONTAL AREA OF THE BODY EXPOSED TO THE FREE STREAM, WITH THE AFT PART BEING IN THE SHADOW WITH $C_p = 0$. FOR INSTANCE, FOR THE FLAT PLATE IN FIG. 59, $C_p = 2\sin^2\delta$ ON THE LOWER SURFACE, AND $C_p = 0$ ON THE UPPER SURFACE (δ = ANGLE OF ATTACK), WITH $C_D = C_p \sin\delta = 2\sin^3\delta$ THE DRAG COEFFICIENT AND $C_L = C_p \cos\delta = 2\sin^2\delta\cos\delta$ THE LIFT COEFFICIENT. (MAXIMUM C_L AT $\delta \sim 55°$)

AN IMPORTANT CHARACTERISTIC OF THE NEWTONIAN THEORY (177) IS THAT IT DOES NOT DEPEND ON THE MACH NUMBER. THIS IS A FUNDAMENTAL RESULT OF THE HYPERSONIC INVISCID THEORY, THE MACH-NUMBER INDEPENDENCE PRINCIPLE, IN THAT C_L, C_D, C_p AND THE FLOW STRUCTURE BECOME INDEPENDENT OF THE MACH NUMBER AT HIGH MACH NUMBERS. (SEE FIG. 61)

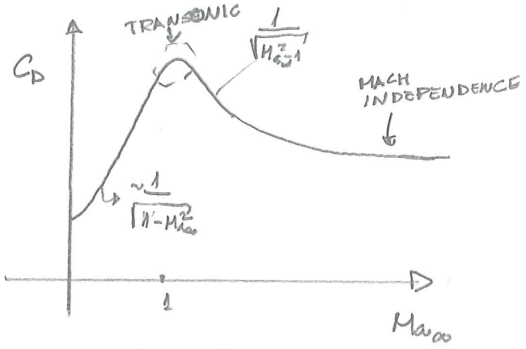


FIG. 61

IT IS INTERESTING TO COMPARE (177) WITH THE ASYMPTOTIC APPROXIMATION (166) FOR C_p FOR AN OBLIQUE SHOCK AT $M_{a_1}^2 \sin^2\beta \gg 1$:

$$C_p = \frac{4}{\gamma+1}\sin^2\beta \qquad C_p = 2\sin^2\beta$$

(OBLIQUE SHOCK) (NEWTONIAN)

FOR $\gamma = 1$ BOTH EXPRESSIONS ARE THE SAME AND YIELD

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1} \to \infty \quad , \quad \beta = \delta\left(\frac{\gamma+1}{2}\right) \to \delta$$

SO THAT THE DENSITY IS INFINITE BEHIND THE SHOCK, AND THE SHOCK IS COINCIDENT WITH THE BODY SURFACE. THEN ONE CAN RECOVER NEWTON'S THEORY BY TAKING THE SIMULTANEOUS LIMIT $\gamma \to 1$, $M_a \to \infty$.

A NUMBER OF MODIFICATIONS EXIST TO CORRECT THE NEWTONIAN THEORY, PERHAPS THE MOST FAMOUS ONE BEING THE NEWTON-BUSEMANN CORRECTION FOR THE CENTRIFUGAL PRESSURE GRADIENT THAT ARISES FROM THE FUSELAGE CURVATURE AND WHICH IS NOT INCLUDED IN THE NEWTONIAN FORMULA (177).

Homeworks and Exams

ME 355: Compressible Flows
Stanford University

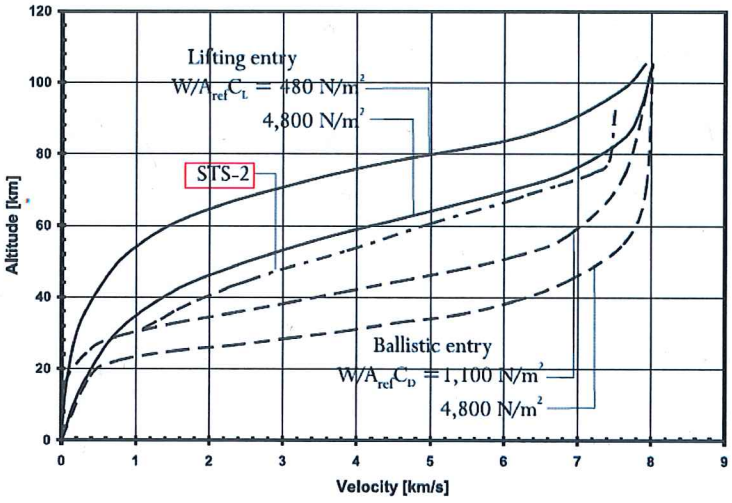
Warmup Problem Set

Spring 2016

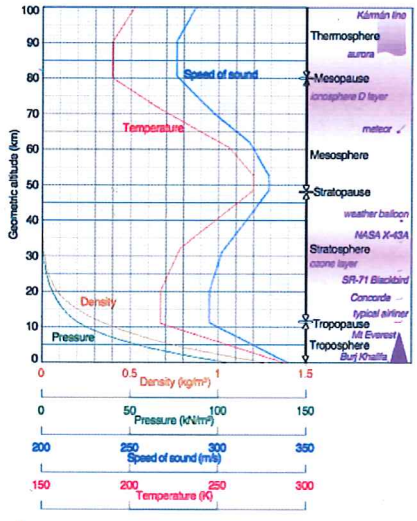
Javier Urzay
jurzay@stanford.edu

Problem 1

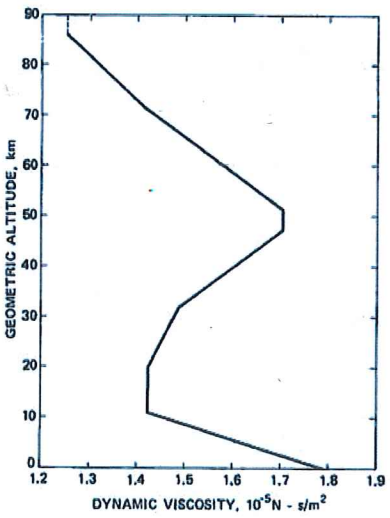
The altitude-velocity flightpath of the Space Shuttle Columbia during descent is shown in Fig. 1(a), where it is denoted by the label "STS-2". Variations of the properties and thermodynamic variables in the atmosphere are provided in Fig. 1(b,c). Assume that the density decreases with altitude as $\rho = \rho_A \exp(-h/\alpha)$ where $\alpha = 7.2$ km and $\rho_A = 1.2$ kg/m^3.



(a)



(b)



(c)

Assuming that the characteristic length of the Space Shuttle's thermal shield is $L = 32.82$ m, calculate at points A (h =80 km altitude), B (60 km) and C (20 km) of the trajectory the following parameters:

a) Knudsen number
b) Reynolds number
c) Mach number

KNUDSEN #: $Kn = \frac{\lambda}{L}$, $\lambda = \frac{m}{\sqrt{2}\pi d^2 \rho} = \frac{m}{\sqrt{2}\pi d^2 \rho_A} e^{+h/\alpha}$ WITH $\begin{cases} m = 4.79 \cdot 10^{-26} \frac{kg}{molecule} \\ d = 390 \cdot 10^{-12} m \end{cases}$ AIR

REYNOLDS #: $Re_L = \frac{\rho U L}{\mu}$, WITH $\rho = \rho_A e^{-h/\alpha}$

MACH #: $Ma = \frac{U}{a}$.

FROM THE FIGURES AND THE DENSITY VS ALTITUDE EQUATION :

	$h(km)$	$U(km/s)$	$\rho\ (kg/m^3)$	$\lambda(m)$	$\mu\ (\frac{Ns}{m^2})$	$a(m/s)$	Kn	Re_L	Ma
A	80	7.4	$1.79 \cdot 10^{-5}$	$3.9 \cdot 10^{-3}$	$1.28 \cdot 10^{-5}$	~275	$1.2 \cdot 10^{-4}$	$3.4 \cdot 10^{5}$	26.9
B	60	5.0	$2.88 \cdot 10^{-4}$	$0.7 \cdot 10^{-3}$	$1.6 \cdot 10^{-5}$	~325	$6.1 \cdot 10^{-6}$	$2.9 \cdot 10^{6}$	15.3
C	20	~0.9	0.074	$0.95 \cdot 10^{-6}$	$1.42 \cdot 10^{-5}$	~290	$2.8 \cdot 10^{-8}$	$1.5 \cdot 10^{8}$	3.1

NOTE THAT AN ESTIMATION OF THESE NUMBERS AT $h \sim 130$ km (NOT SHOWN IN FIGURES) GIVES

$\rho \sim 1.7 \cdot 10^{-8}\ kg/m^3$

$\mu \sim 10^{-5}\ \frac{Ns}{m^2}$

$\lambda \sim 4.1\ m$

$a \sim 310\ m/s$

$U \sim 8\ km/s$

$Re_L \sim 446$ ← LOW REYNOLDS

$Kn \sim 0.12$ ← SLIP / NON-CONTINUUM FLOW

$Ma \sim 25.8$

Problem 2

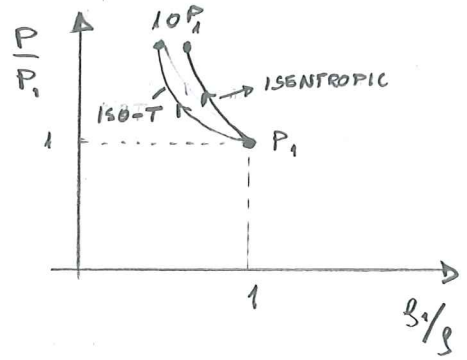
An ideal gas at pressure $P_1 = 1$ atm and density $\rho_1 = 1$ kg/m^3 is compressed isothermally to $P_2 = 10P_1$. The adiabatic constant of the gas is $\gamma = 1.4$ and its molecular weight is $W = 10$ g/mol.

a) What is the density of the compressed gas?

b) Calculate the increments of thermal energy, entropy and enthalpy in the gas

c) Compute the ratio of the mean distances between molecules in the gas before and after compression

d) Answer to parts a-c if the compression process is isentropic instead of isothermal.



GAS CONSTANT: $R_g = \frac{R^\circ}{W} = \frac{8.31}{10^{-2}} = 831\ J/kg\,K$

SPECIFIC HEATS: $c_p = \frac{\gamma R_g}{\gamma - 1} = 2.9\ kJ/kg\,K$

$c_v = c_p/\gamma = 2.1\ kJ/kg\,K$

* ISOTHERMAL TRAJECTORY: $T = \frac{P_1}{\rho_1 R_g} = \frac{P_2}{\rho_2 R_g} \Rightarrow \frac{\rho_2}{\rho_1} = \frac{P_2}{P_1} = 10 \Rightarrow \boxed{\rho_2 = 10\ \frac{kg}{m^3}}$

* INCREMENT OF THERMAL ENERGY: $e_2 - e_1 = c_v (T_2 - T_1) = 0$

" ENTHALPY: $h_2 - h_1 = c_p (T_2 - T_1) = 0$

" ENTROPY: $\left[s_2 - s_1 = c_v \ln \left[\frac{P_2}{P_1} \left(\frac{\rho_1}{\rho_2} \right)^\gamma \right] = c_v \ln \left[\frac{10}{10^{1.4}} \right] = -1.9\ \frac{kJ}{K} \right]$

* MEAN INTERMOLECULAR DISTANCE: $\delta \sim n^{-1/3}$, WHERE $n = \frac{\rho N_A}{W}$ IS THE NUMBER DENSITY AND N_A IS THE AVOGADROS' NUMBER.

$$\Rightarrow \frac{\delta_2}{\delta_1} = \left(\frac{n_2}{n_1} \right)^{-1/3} = \left(\frac{\rho_2}{\rho_1} \right)^{-1/3} = 10^{-1/3} = \boxed{0.46}$$

IF THE PROCESS IS ISENTROPIC INSTEAD:

$$\frac{P_2}{P_1} = \left(\frac{\rho_2}{\rho_1} \right)^\gamma = 10 \Rightarrow \frac{\rho_2}{\rho_1} = 10^{1/\gamma} = 5.2, \quad \text{AND} \quad \frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{\rho_1}{\rho_2} = \left(\frac{\rho_2}{\rho_1} \right)^{\gamma - 1} = 1.9$$

$$\Rightarrow e_2 - e_1 = c_v (T_2 - T_1) = c_v T_1 \left(T_2/T_1 - 1 \right) = \frac{P_1}{(\gamma - 1)\rho_1} \left[\left(\frac{\rho_2}{\rho_1} \right)^{\gamma - 1} - 1 \right] = \boxed{230\ kJ/kg}$$

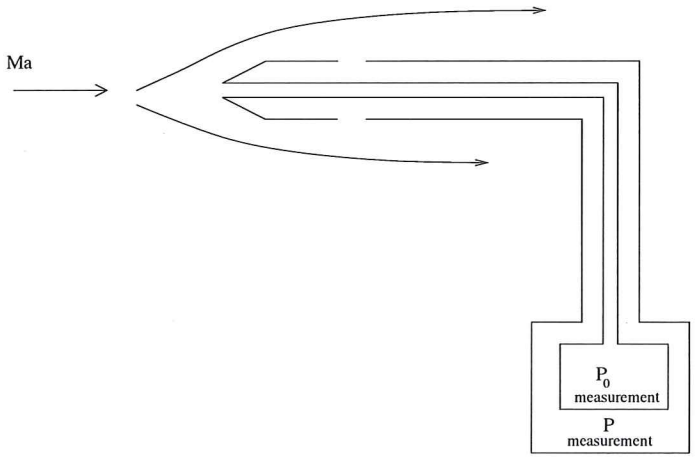
$$h_2 - h_1 = \gamma (e_2 - e_1) = \boxed{322\ kJ/kg}$$

$$\boxed{s_2 - s_1 = 0}$$

AND $\frac{\rho_2}{\rho_1} = \left(\frac{P_2}{P_1} \right)^{-1/\gamma} = \boxed{0.57}$

Problem 3

Consider the subsonic Pitot tube in the figure below. Knowing that the static and stagnation pressures measured are, respectively, $P_0 = 2.45$ bar and $P = 2$ bar, compute the velocity of the free stream assuming that the gas is air at a stagnation temperature $T_0 = 550$K.



RECALL FROM BERNOULLI'S EQUATION THAT $h + \frac{1}{2}v^2 = \text{CONST.}$

$$\Rightarrow \quad C_p T + \frac{v^2}{2} = C_p T_0 \quad \Rightarrow \quad \frac{T_0}{T} = 1 + \frac{(\gamma-1)}{2} M_a^2$$

SINCE $$\frac{P_0}{P} = \left(\frac{\rho_0}{\rho}\right)^{\gamma} = \left(\frac{T_0}{T}\right)^{\frac{\gamma}{\gamma-1}} = \left[1 + \frac{(\gamma-1)}{2} M_a^2\right]^{\frac{\gamma}{\gamma-1}} = \frac{2.45}{2} = 1.22.$$

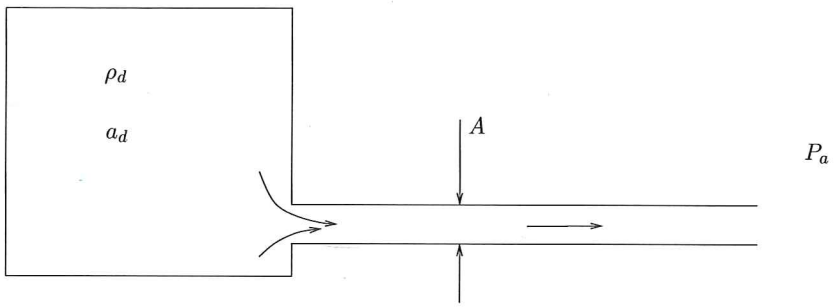
THEN $\boxed{M_a = 0.55}$ (SEE ALSO TABLE I)

THE STATIC TEMPERATURE IS $$T = \frac{T_0}{1 + \frac{(\gamma-1)}{2} M_a^2} = 517\ \text{K}$$

AND $$v = M_a \sqrt{\gamma R_g T} = \boxed{250\ \text{m/s}}$$

Problem 4

A straight duct of cross sectional area A is attached to a large gas reservoir as shown in the figure. The atmospheric pressure is P_a. Assuming that the values of the density ρ_d and speed of sound a_d in the reservoir are known, provide the expressions to compute the pressure, velocity, speed of sound, Mach number, mass flow rate and density in the duct. Consider the cases $P_d/P_a < [(\gamma+1)/2]^{\gamma/(\gamma-1)}$ and $P_d/P_a > [(\gamma+1)/2]^{\gamma/(\gamma-1)}$ separately in your calculations.



MASS FLOW RATE: $\dot{m} = \rho_c U_c A = \rho_d a_d \frac{\rho_c}{\rho_d} \frac{a_c}{a_d} \frac{U_c}{a_c} A =$

($\rho_c, U_c, P_c, a_c, Ma_c$ REFER TO THE DUCT)

$$= \rho_d a_d A \left(\frac{T_c}{T_d}\right)^{1/2} \frac{\rho_c}{\rho_d} Ma_c A$$

$$= \rho_d a_d A \left[1 + \frac{(\gamma-1)}{2} Ma_c^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} Ma_c \quad (1)$$

FOR SUBSONIC FLOW IN THE DUCT, Ma_c IS CALCULATED BY IMPOSING $\boxed{P_c = P_a}$ (2)

$$\Rightarrow \frac{P_d}{P_a} = \left[1 + \frac{\gamma-1}{2} Ma_c^2\right]^{\frac{\gamma}{\gamma-1}} \Rightarrow \boxed{Ma_c = \left\{\frac{2}{\gamma-1}\left[\left(\frac{P_d}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]\right\}^{1/2}} \quad (3)$$

WITH $P_d = \frac{\rho_d a_d^2}{\gamma}$ THE PRESSURE IN THE RESERVOIR. SUBSTITUTING (3) INTO (1):

$$\boxed{\dot{m} = \rho_d a_d A \left(\frac{P_d}{P_a}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \left[\left(\frac{P_d}{P_a}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\frac{2}{\gamma-1}\right)^{1/2}} \quad (4)$$

WHILE THE DENSITY AND SPEED OF SOUND IN THE DUCT ARE $\boxed{\frac{\rho_c}{\rho_d} = \left(\frac{P_a}{P_d}\right)^{1/\gamma}}$ AND $\boxed{\frac{a_c}{a_d} = \left(\frac{P_a}{P_d}\right)^{\frac{\gamma-1}{2\gamma}}}$ (5)

THAT $P_c = P_a$ ONLY HOLDS FOR $Ma_c < 1$, OR EQUIVALENTLY, $\frac{P_d}{P_a} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$ ACCORDING TO EQ. (3)

FOR $\frac{P_d}{P_a} > \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$, THEN $\boxed{Ma_c = 1}$ (6) AND $P_c \neq P_a$, WITH P_c GIVEN BY

$$\boxed{\frac{P_d}{P_c} = \left[1 + \frac{(\gamma-1)}{2} Ma_c^2\right]^{\frac{\gamma}{\gamma-1}} = \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}} \quad (7).$$

SIMILARLY, $\dot{m}$ IS GIVEN BY

$$\dot{m} = \rho_d a_d A \left[1 + \frac{(\gamma-1)}{2} M_{a_c}^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}} M_{a_c}^1 = \rho_d a_d A \left(\frac{(\gamma+1)}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \quad (8)$$

WHILE $$\frac{\rho_c}{\rho_d} = \left(\frac{P_c}{P_d}\right)^{1/\gamma} \quad \text{AND} \quad \frac{a_c}{a_d} = \left(\frac{P_c}{P_d}\right)^{\frac{\gamma-1}{2\gamma}} \quad (9)$$

IN SUMMARY, (1) IS ALWAYS VALID,

(2), (3), (4) AND (5) ARE VALID IF $\frac{P_d}{P_a} < \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$

(6), (7), (8) AND (9) ARE VALID IF $\frac{P_d}{P_a} > \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$ (I.E. THE DUCT IS CHOKED)

ME 355: Compressible Flows, Spring 2016

Stanford University

Homework 1

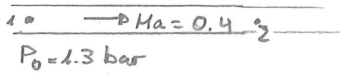
Due Tuesday, April 19, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student's Name: JAVIER URZAY **Student's ID:**.....

Questions (40 pts)

1. Air flows isentropically through a constant-area duct at Ma = 0.4. The stagnation pressure at point 1 is $P_0 = 1.3$ bar. What is the stagnation pressure at a downstream point 2?



SINCE THE FLOW IS ISENTROPIC, THE STAGNATION PRESSURE REMAINS CONSTANT. $P_{0_1} = P_{0_2} = 1.3$ bar

✓ 10/10

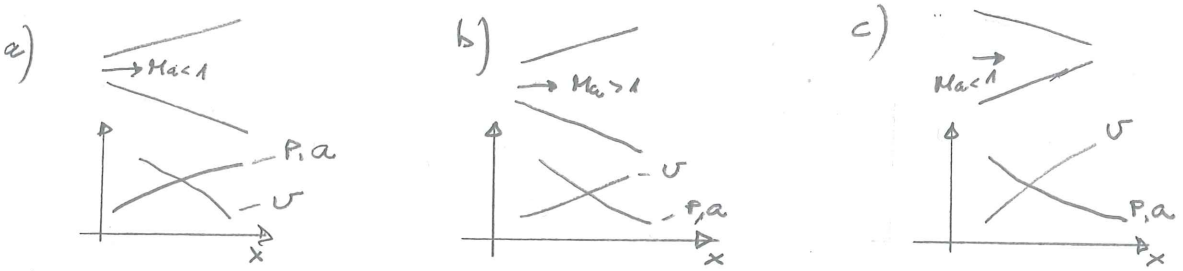
2. In addition to the conditions described in the previous question, it is also known that the stagnation density at point 1 is $\rho_0 = 1.0$ kg/m^3. What is the static enthalpy at the downstream point 2?

$$\frac{P_1}{P_0} = \left(1 + \frac{\gamma-1}{2} M_a^2\right)^{\frac{\gamma}{1-\gamma}} = 0.89 = \left(\frac{\rho_1}{\rho_0}\right)^{\gamma} = \frac{P_2}{P_0} = \left(\frac{\rho_2}{\rho_0}\right)^{\gamma} \Rightarrow h_2 = \frac{h_2}{h_0} \cdot h_0 = \frac{T_2}{T_0} h_0 = \left(\frac{P_2}{P_0}\right)^{\frac{\gamma-1}{\gamma}} \frac{\gamma}{\gamma-1} \frac{P_0}{\rho_0} = 3.38 \frac{P_0}{\rho_0}$$

$$= 440 \text{ kJ/kg}$$

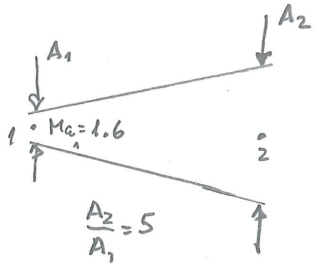
✓ 10/10

3. Sketch the streamwise variations of pressure, speed of sound and flow velocity for a) a subsonic flow through a divergent nozzle, b) a supersonic flow through a divergent nozzle, and c) a subsonic flow through a convergent nozzle.



✓ 10/10

4. A gas with $\gamma = 1.4$ flows supersonically through a diverging duct. At section 1, the cross-sectional area is A_1 and the Mach number is $Ma_1 = 1.6$. At section 2, the cross-sectional area is $A_2 = 5A_1$. Calculate the Mach number Ma_2 at section 2.



CONTINUITY: $\dot{m} = \rho U A = \text{CONST.}$

$$\Rightarrow \dot{m} = \rho_0 a_0 \frac{U}{a} \frac{a}{a_0} \frac{\rho}{\rho_0} A = \rho_0 a_0 A Ma \left[1 + \frac{(\gamma-1)}{2} M_a^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}$$

THEREFORE $$\frac{A_1}{A_2} = \frac{Ma_2 \left[1 + \frac{(\gamma-1)}{2} Ma_2^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}}{Ma_1 \left[1 + \frac{(\gamma-1)}{2} Ma_1^2\right]^{-\frac{\gamma+1}{2(\gamma-1)}}} = \frac{1}{5} \Rightarrow \text{OBTAIN } Ma_2 \quad (Ma_2 > 1)$$

$\frac{1}{0.79}$ FROM TABLE I

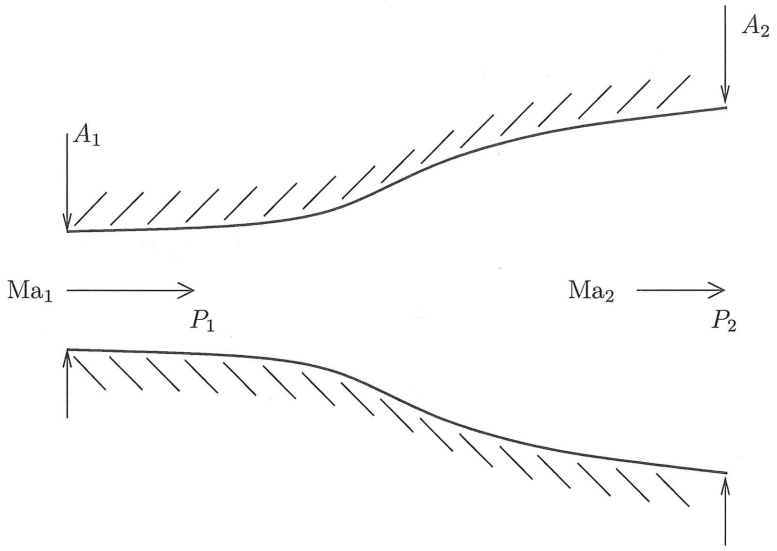
THIS EQ. CAN BE SOLVED BY DOING $\frac{A_2}{A^*} = \frac{A_2}{A_1} \cdot \frac{A_1}{A^*} = 5 \frac{A_1}{A^*} = 6.28 \Rightarrow Ma_2 = 3.4$

FROM TABLE I

✓ 10/10

Problem 1 (30 pts)

Consider the subsonic diffuser shown in the figure where the inlet Mach number $\mathrm{Ma}_1 < 1$ and the area ratio $A_2/A_1 > 1$ are known parameters.



a) Derive an expression to compute the static-pressure ratio P_2/P_1 as a function of Ma_1, A_2/A_1 and γ.

b) Calculate the static-pressure ratio P_2/P_1 for $\mathrm{Ma}_1 = 0.4$ and $A_2/A_1 = 4$.

c) Prove that the expression

$$\frac{P_0}{P_1} = \left[1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2\right]^{\gamma/(\gamma-1)}$$

leads to the incompressible Bernoulli equation

$$P_1 + \rho U_1^2/2 = P_0$$

in the incompressible limit $\mathrm{Ma}^2 \ll 1$, where P_0 is the stagnation pressure, U_1 is the flow velocity, and ρ is the density.

---

a) FROM MASS CONSERVATION: $\dot{m} = \rho U A = \frac{\rho}{\rho_0}\frac{U}{a}\frac{a}{a_0}\rho_0 a_0 A = \text{CONST} \Rightarrow \frac{\rho_1}{\rho_0}\mathrm{Ma}_1\left(\frac{T_1}{T_0}\right)^{1/2}A_1 = \frac{\rho_2}{\rho_0}\mathrm{Ma}_2\left(\frac{T_2}{T_0}\right)^{1/2}A_2$

AND SINCE $\rho_1 = \frac{P_1}{R_g T_1}$ AND $\rho_2 = \frac{P_2}{R_g T_2}$, THEN $\frac{P_2}{P_1}\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{T_1}{T_2}\right)^{1/2}\frac{A_2}{A_1} = 1$. (1)

SUBSTITUTING THE RELATION $\frac{T_2}{T_1} = \frac{T_2}{T_0}\frac{T_0}{T_1} = \left(1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2\right)\Big/\left(1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2\right)$ INTO EQ (1)

YIELDS: $\frac{P_2}{P_1}\frac{\mathrm{Ma}_2}{\mathrm{Ma}_1}\left(\frac{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2}{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2}\right)^{1/2}\frac{A_2}{A_1} = 1$, WHICH GIVES THE QUADRATIC EQUATION

$$\mathrm{Ma}_2^2\left(1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2\right) = \mathrm{Ma}_1^2\left(1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2\right)\left(\frac{A_1}{A_2}\right)^2\left(\frac{P_1}{P_2}\right)^2$$

FOR Ma_2.

THE SOLUTION IS

$$(\gamma-1)\,Ma_2^2 = -1 + \left[1 + \frac{2(\gamma-1)Ma_1^2\left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)}{\left(\frac{P_2}{P_1}\right)^2\left(\frac{A_2}{A_1}\right)}\right]^{1/2}$$

SUBSTITUTING THIS EXPRESSION INTO THE ISENTROPIC RELATIONS

$$\frac{P_2}{P_1} = \frac{P_2}{P_0}\frac{P_0}{P_1} = \left(\frac{1+\frac{(\gamma-1)}{2}Ma_2^2}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right)^{\frac{\gamma}{1-\gamma}}$$

GIVES

$$\boxed{\frac{P_2}{P_1} = \left\{\frac{\frac{1}{2}+\frac{1}{2}\left[1+\frac{2(\gamma-1)Ma_1^2\left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)}{\left(\frac{P_2}{P_1}\right)^2\left(\frac{A_2}{A_1}\right)^2}\right]^{1/2}}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right\}^{\frac{\gamma}{1-\gamma}}} \quad (2)$$

✓ 10/10

WHICH REPRESENTS THE IMPLICIT RELATION BETWEEN $\frac{P_2}{P_1}$, $\frac{A_2}{A_1}$ AND Ma_1.

b) FOR $Ma_1 = 0.4$ AND $\frac{A_2}{A_1} = 4$, EQ. (2) CAN BE SOLVED ITERATIVELY TO GIVE $\boxed{\frac{P_2}{P_1} \approx 1.11}$ ✓ 10/10

c) $\frac{P_0}{P_1} = \left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}}$ IS OF THE FORM $(1+\varepsilon)^b$, WHICH CAN BE EXPANDED IN TAYLOR SERIES AS $(1+\varepsilon)^b \sim 1+b\varepsilon$ FOR $\varepsilon \ll 1$. IN THIS WAY,

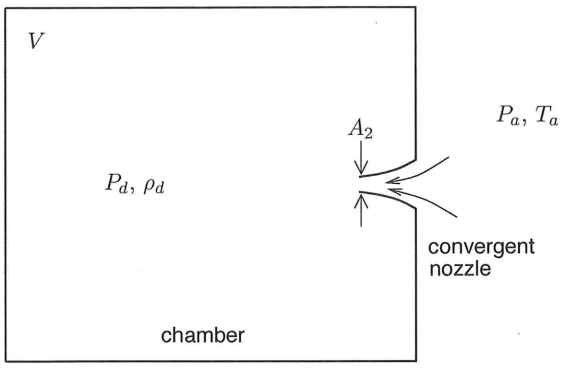
$$\frac{P_0}{P_1} = \left(1+\frac{(\gamma-1)}{2}Ma_1^2\right)^{\frac{\gamma}{1-\gamma}} \sim 1+\frac{\gamma}{2}Ma_1^2 = 1+\frac{\gamma U_1^2}{2\gamma P_1/\rho} = 1+\frac{\rho U_1^2}{2P_1}$$

$$\Rightarrow \boxed{P_0 = P_1 + \frac{\rho U_1^2}{2}}$$

✓ 10/10

Problem 2 (30 pts)

A small hole is perforated on the wall of a chamber of volume V that is initially void of gas. The hole resembles a convergent nozzle of minimum cross-sectional area $A_2 \ll V^{2/3}$. Air from the atmosphere flows through the hole filling up the chamber, as shown in the figure below. The atmosphere is at pressure P_a and temperature T_a.



a) Show that mass and total internal-energy conservation constraints require that the time evolution of the density ρ_d and pressure P_d of the gas in the chamber are given by

$$\frac{d\rho_d}{dt} = \frac{\dot{m}}{V} \tag{1}$$

and

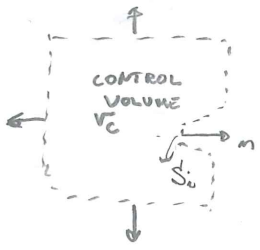
$$\frac{dP_d}{dt} = \frac{\dot{m}}{V} a_a^2 \tag{2}$$

where $\dot{m}$ is the mass flow rate and a_a is the speed of sound in the atmospheric air.

b) Obtain an expression for the time $t^\star$ during which the hole remains choked as a function of γ, V, A_2, T_a and the gas constant R_g.

c) Plot the time evolution of the density ρ_d/ρ_a and pressure P_d/P_a of the gas in the chamber during the interval $0 < t/t^\star < 5$ for $\gamma = 1.4$.

a)



* CONTINUITY: $\frac{d}{dt}\int_{V_c} \rho\, dV + \int_{S_c} \rho \vec{v}\cdot d\vec{S} = 0 \Rightarrow \boxed{V\frac{d\rho_d}{dt} = \dot{m}}$ (1)

$V\frac{d\rho_d}{dt}$; $-\rho_2 U_2 A_2 = -\dot{m}$

✓ 10/10

* TOTAL INTERNAL ENERGY: $\frac{d}{dt}\int_{V_c} \rho\left(e + \frac{|\vec{v}|^2}{2}\right) dV + \int_{S_c} \rho\left(e + \frac{|\vec{v}|^2}{2} + \frac{P}{\rho}\right)\vec{v}\cdot d\vec{S} = 0$ (NEGLIGIBLE)

$\frac{d}{dt}(\rho_d e_d) =$

$= c_v V \frac{d}{dt}(\rho_d T_d) =$

$= \frac{V}{\gamma-1}\frac{dP_d}{dt}$

$-\dot{m}\left(h_2 + \frac{U_2^2}{2}\right) = -\dot{m} h_a =$

$= -\frac{\dot{m} a_a^2}{\gamma - 1}$

$\Rightarrow \boxed{V\frac{dP_d}{dt} = \dot{m} a_a^2}$ (2)

✓ 10/10

b) INITIALLY THERE IS NO GAS IN THE CHAMBER $\Rightarrow \rho_d = P_d = 0$ AT $t = 0$. AS GAS FLOWS THROUGH THE HOLE, THE PRESSURE P_d AND DENSITY ρ_d INCREASE ACCORDING TO EQS (1) + (2). IT IS HOWEVER NECESSARY TO OBTAIN AN EXPRESSION FOR THE MASS FLOW RATE $\dot{m}$. AT THE BEGINNING, THE PRESSURE IN THE CHAMBER IS SMALL AND THE NOZZLE IS CHOKED, WITH $Ma_2 = 1$ AND

$$\dot{m} = \dot{m}^* = P_a \sqrt{\frac{\gamma}{R_g T_a}} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \qquad (3)$$

USING (3) IN (2) LEADS TO

$$\left.\begin{aligned} \frac{dP_d}{dt} &= \frac{P_a \sqrt{\gamma^3 R_g T_a}}{V} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \\ P_d(0) &= 0 \end{aligned}\right\} \qquad (4)$$

SO THAT $\frac{P_d(t)}{P_a} = \frac{\sqrt{\gamma^3 R_g T_a}}{V} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} t$. THE NOZZLE REMAINS CHOKED AS LONG AS $\frac{P_a}{P_d} \geq \left(\frac{\gamma+1}{2}\right)^{\frac{\gamma}{\gamma-1}}$, SO THAT $\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} = \frac{\sqrt{\gamma^3 R_g T_a}}{V} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} t^*$

$$\Rightarrow \boxed{t^* = \left(\frac{2}{\gamma+1}\right)^{1/2} \frac{V}{\sqrt{\gamma^3 R_g T_a} A_2}} \qquad (5)$$

✓ 10/10

c) FOR $0 < t < t^*$ THE NOZZLE IS CHOKED ($\dot{m} = \dot{m}^*$) AND

$$\left.\begin{aligned} \frac{d\rho_d}{dt} &= \frac{P_a}{V}\sqrt{\frac{\gamma}{R_g T_a}} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}, & \rho_d(0) &= 0 \\ \frac{dP_d}{dt} &= \frac{P_a}{V}\sqrt{\gamma^3 R_g T_a} A_2 \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}, & P_d(0) &= 0 \end{aligned}\right\} \qquad (6)$$

FOR $t > t^*$ THE NOZZLE CEASES TO BE CHOKED AND $\dot{m} = P_a \sqrt{\frac{\gamma}{R_g T_a}} A_2 \left(\frac{2}{\gamma-1}\right)^{1/2} \left[\left(\frac{P_a}{P_d}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\frac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2\gamma}}$

SO THAT

$$\left.\begin{aligned} \frac{d\rho_d}{dt} &= \frac{P_a}{V}\sqrt{\frac{\gamma}{R_g T_a}} A_2 \left(\frac{2}{\gamma-1}\right)^{1/2} \left[\left(\frac{P_a}{P_d}\right)^{\frac{\gamma-1}{\gamma}} - 1\right]^{1/2} \left(\frac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2\gamma}}, & \rho_d(t^*) &= \rho_d^* \\ \frac{dP_d}{dt} &= \frac{P_a}{V}\sqrt{\gamma^3 R_g T_a} A_2 \left(\frac{2}{\gamma-1}\right)^{1/2} \left[\left(\frac{P_a}{P_d}\right)^{\frac{(\gamma-1)}{\gamma}} - 1\right]^{1/2} \left(\frac{P_a}{P_d}\right)^{-\frac{\gamma+1}{2\gamma}}, & P_d(t^*) &= P_d^* \end{aligned}\right\} \qquad (7)$$

WHERE

$$\left.\begin{aligned} \rho_d^* &= \frac{P_a}{\cancel{V}}\sqrt{\frac{\gamma}{R_g T_a}} \cancel{A_2} \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} \left(\frac{2}{\gamma+1}\right)^{1/2} \frac{\cancel{V}}{\sqrt{\gamma^3 R_g T_a}\,\cancel{A_2}} = \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} \frac{P_a}{\gamma R_g T_a} \\ P_d^* &= \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} P_a \end{aligned}\right\}$$

ARE THE CRITICAL VALUES OF DENSITY AND TEMPERATURE. NORMALIZING $\tilde{\rho} = \rho_d/\rho_a = \frac{\rho_d R_g T_a}{P_a}$, $\tilde{P} = P_d/P_a$ AND $\tau = t/t^*$, EQUATIONS (6) AND (7) BECOME

$$\left.\begin{aligned} \frac{d\tilde{\rho}}{d\tau} &= \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} \\ \frac{d\tilde{P}}{d\tau} &= \left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} \end{aligned}\right\} \quad (0<\tau<1),\ \text{SUBJECT TO } \tilde{P}(0) = \tilde{\rho}(0) = 0 \qquad \text{CHOKED FLOW } (\dot{m} = \text{CONST.}) \qquad (8)$$

✓

→

AND

$$\frac{d\tilde{\rho}}{d\tau} = \frac{1}{\gamma}\left(\frac{2}{\gamma+1}\right)^{1/2}\left(\frac{2}{\gamma-1}\right)^{1/2}\left(\tilde{P}^{\frac{1-\gamma}{\gamma}}-1\right)^{1/2}\tilde{P}^{\frac{\gamma+1}{2\gamma}}$$

$$\frac{d\tilde{P}}{d\tau} = \left(\frac{2}{\gamma+1}\right)^{1/2}\left(\frac{2}{\gamma-1}\right)^{1/2}\left(\tilde{P}^{\frac{1-\gamma}{\gamma}}-1\right)^{1/2}\tilde{P}^{\frac{\gamma+1}{2\gamma}}$$

$(1 < \tau < \infty)$ (9)

SUBJECT TO

$$\tilde{\rho} = \tilde{\rho}^{**} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}$$

AND $\tilde{P} = \tilde{P}^{**} = \gamma\rho^{**}$ AT $\tau = 1$

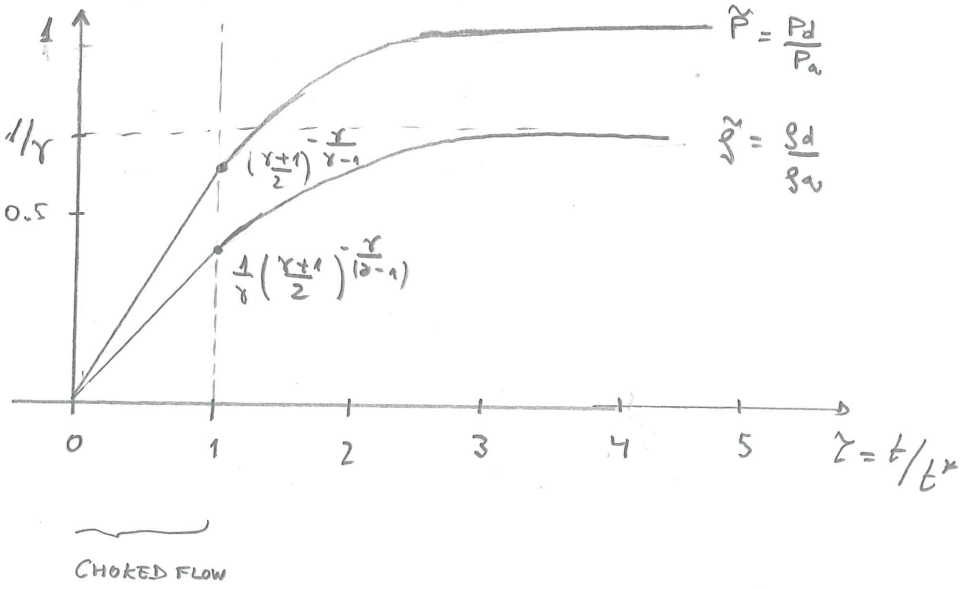
NON-CHOKED FLOW ($\dot{m} \neq$ CONST)

✓ 10/10

THE INTEGRATION OF EQS. (8) IS TRIVIAL AND YIELDS THE LINEAR VARIATIONS

$$\tilde{\rho} = \frac{1}{\gamma}\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}}\tau \quad , \qquad \tilde{P} = \gamma\tilde{\rho} \qquad (10).$$

THE INTEGRATION OF EQS. (9) DETERMINES THE TIME EVOLUTIONS $\tilde{P}(\tau)$ AND $\tilde{\rho}(\tau)$ WHEN THE CHAMBER IS FILLED ENOUGH SUCH THAT THE FLOW IS NOT CHOKED ANYMORE. THE INTEGRATION IS NOT STRAIGHTFORWARD, BUT A SKETCH OF THE SOLUTION IS GIVEN BELOW



ME 355: Compressible Flows, Spring 2016

Stanford University

Homework 2: Shocks and expansion waves

Due Thursday, May 5, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ for all problems.

Student's Name: JAVIER URZAY **Student's ID:**

Questions (40 pts)

1. A supersonic air flow at $\mathrm{Ma}_1 = 2.4$ and $P_1 = 1$ bar encounters a normal shock wave. Compute the Mach number in the post-shock flow Ma_2 as well as the relative variation in the stagnation pressure $(P_{01} - P_{02})/P_{01}$ and specific entropy $(s_1 - s_2)/c_v$.

TABLE II: $\frac{s_2}{s_1} = 3.2$

$\mathrm{Ma}_1 = 2.4$, $P_1 = 1$ bar $\rightarrow$ Ma_2

$$\mathrm{Ma}_2 = \left(\frac{2 + (\gamma-1)\mathrm{Ma}_1^2}{2\gamma\mathrm{Ma}_1^2 + 1 - \gamma}\right)^{1/2} = 0.52 \quad \checkmark$$

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}(\mathrm{Ma}_1^2 - 1) = 6.5$$

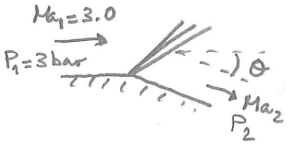
$$\frac{P_{02}}{P_{01}} = \left(\frac{1 + \frac{\gamma-1}{2}\mathrm{Ma}_1^2}{1 + \frac{\gamma-1}{2}\mathrm{Ma}_2^2}\right)^{\frac{\gamma}{\gamma-1}} \frac{P_2}{P_1} = 0.54$$

$$\Rightarrow \frac{P_{01} - P_{02}}{P_{01}} = 0.46 \quad \checkmark \quad 10/10$$

$$\frac{s_1 - s_2}{c_v} = -\ln\left(\frac{P_2}{P_1}\frac{s_2^\gamma}{s_1^\gamma}\right) = -\ln\left(\frac{6.5}{3.2^\gamma}\right) = -0.24 \quad \checkmark$$

$(s_2 > s_1)$

2. A supersonic flow parallel to a wall at $\mathrm{Ma}_1 = 3.0$ and $P_1 = 3$ bar encounters a corner that deflects the stream outwards at an angle θ creating an expansion fan. Compute the angle θ required to decrease the static pressure of the gas to $P_2 = 0.1$ bar. What is the associated Mach number Ma_2?



ISENTROPIC FLOW:

$$\frac{P_2}{P_1} = \frac{0.1}{3} = \left(\frac{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2}{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow \mathrm{Ma}_2 = 5.65 \quad \checkmark$$

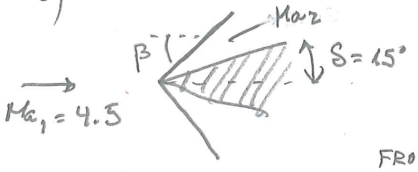
$$\theta = \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1)$$

TABLE III: $82.5 \quad 49.7$

$$\Rightarrow \theta = 32.7^\circ \quad \checkmark \quad 10/10$$

3. Sketch i) the supersonic flow at $\mathrm{Ma}_1 = 4.5$ around a symmetric wedge of semi-angle $\delta = 15^\circ$ at pressure $P_1 = 1$ bar, ii) the same flow when the semi-angle is increased to $\delta = 70^\circ$.

i) ATTACHED SHOCK



OBLIQUE-SHOCK CHART:

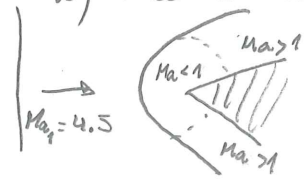
$\mathrm{Ma}_1 = 4.5$, $\delta \Rightarrow \beta = 26^\circ$ ✓

$\Rightarrow \mathrm{Ma}_{1n} = \mathrm{Ma}_1 \sin\beta = 1.97$

FROM TABLE II: $\mathrm{Ma}_{2n} = 0.58 = \mathrm{Ma}_2/\sin(\beta - \delta)$

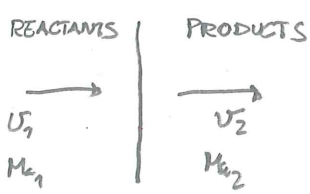
$\Rightarrow \mathrm{Ma}_2 = 3.03$

ii) SINCE $\delta > \delta_{MAX} = 40^\circ \Rightarrow$ DETACHED SHOCK



✓ 10/10

4. Compute the Chapman-Jouget deflagration and detonation velocities, as well as the Mach number of the reactants and products in the wave frame, for a stoichiometric H_2-air mixture at temperature $T_1 = 298$ K and pressure $P_1 = 1$ bar. Assume that the heat released by the combustion reaction is $q = 3423$ KJ/Kg and that the gas constant is $R_g = 383$ J/KgK.



REACTANTS DENSITY: $\rho_1 = \frac{P_1}{R_g T_1} = 0.88\ \mathrm{kg/m^3} \Rightarrow \alpha = \frac{q}{P_1/\rho_1} = 29.7$

$\Rightarrow$ CJ$^+$ DETONATION: $U_1 = D_{CJ}^+ = \left(2q(\gamma^2-1)\right)^{1/2} = 2563\ \mathrm{m/s}$ ✓ 10/10

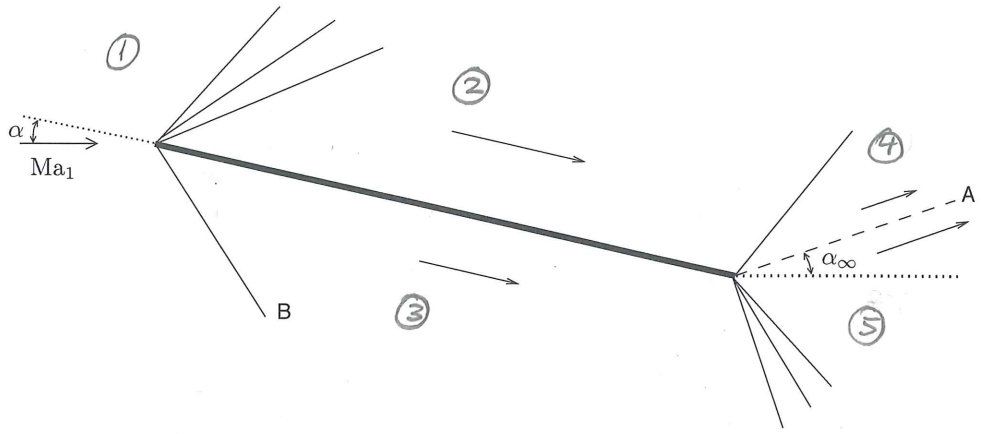
$\mathrm{Ma}_1 = U_1/\sqrt{\gamma R_g T_1} = 6.41$ ✓, $\mathrm{Ma}_2 = 1$ ✓ BY DEFINITION OF CJ POINT

$\Rightarrow$ CJ$^-$ DEFLAGRATION: $U_1 = D_{CJ}^- = \frac{\gamma R_g T_1}{D_{CJ}^+} = 62.3\ \mathrm{m/s}$ ✓

$\mathrm{Ma}_1 = U_1/\sqrt{\gamma R_g T_1} = 0.15$ ✓, $\mathrm{Ma}_2 = 1$ ✓ BY DEFINITION OF CJ POINT

Problem 1 (30 pts)

Consider the flat plate depicted below as a model of a supersonic interceptor aircraft flying in the stratosphere at an altitude $h = 20$ km. The aircraft flies at $\mathrm{Ma}_1 = 3.0$ with an angle of attack $\alpha = 30°$.

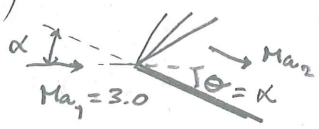


a) Determine the wave-drag and compression-lift coefficients given by $C_D = (1/2)D/(\rho_1 U_1^2 S)$ and $C_L = (1/2)L/(\rho_1 U_1^2 S)$, respectively. In this formulation, S is the length of the plate, while D and L are the drag and lift forces per unit spanwise length.

b) Because of the non-zero angle of attack, the solution in the wake of the aircraft involves a tangential discontinuity A that in reality leads to a shear layer with constant static pressure across. Compute the streamline angle of deflection α_∞ with respect to the incident horizontal free stream.

c) Far from the aircraft, the leading shock B becomes weaker and resembles a weak shock or compression Mach wave whose streamline deflection angle is just 0.1% of the aircraft angle of attack α. This wave creates a sonic boom that is perceived in populated areas on the ground. Compute the angle of inclination of the Mach wave on the ground and the associated sound pressure level (in decibels) as it passes through the populated areas.

a) • 1 → 2 EXPANSION WAVE

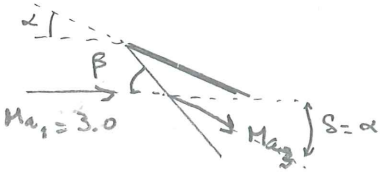


$\alpha = \nu(\mathrm{Ma}_2) - \nu(\mathrm{Ma}_1) = 30° \Rightarrow \nu(\mathrm{Ma}_2) = 30° + 49.7° = 79.7°$

TABLE II: 49.7 → TABLE III: $\mathrm{Ma}_2 = 5.3$

ISENTROPIC EXPANSION: $\frac{P_2}{P_1} = \left(\frac{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_1^2}{1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.049$

• 1 → 3 OBLIQUE SHOCK



$\mathrm{Ma}_1 = 3.0$, $\delta = \alpha = 30°$ } USE CHART: $\beta = 52°$ $\Rightarrow$ $\mathrm{Ma}_{1n} = \mathrm{Ma}_1 \sin\beta = 2.36$

↳ FROM TABLE II NORMAL SHOCKS → $\frac{P_3}{P_1} = 6.27$, $\mathrm{Ma}_{3n} = 0.53$

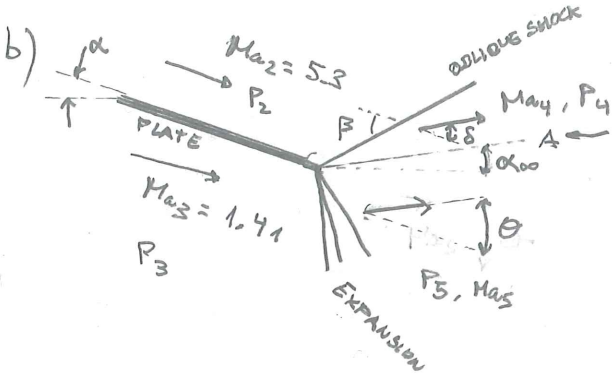
SO THAT $\mathrm{Ma}_3 = \frac{\mathrm{Ma}_{3n}}{\sin(\beta - \delta)} = 1.41$

* DRAG COEFFICIENT: $C_D = \frac{D}{\frac{1}{2}\rho_1 U_1^2 S} = \frac{(P_3 - P_2)\,S \sin\alpha}{\frac{1}{2}\rho_1 U_1^2 S} = \left(\frac{P_3}{P_1} - \frac{P_2}{P_1}\right)\frac{\sin\alpha}{\frac{1}{2}\gamma Ma_1^2} = \boxed{0.493}$ ✓

10/10

* LIFT COEFFICIENT: $C_L = \frac{L}{\frac{1}{2}\rho_1 U_1^2 S} = \frac{(P_3 - P_2)\,S\cos\alpha}{\frac{1}{2}\rho_1 U_1^2 S} = \left(\frac{P_3}{P_1} - \frac{P_2}{P_1}\right)\frac{\cos\alpha}{\frac{1}{2}\gamma Ma_1^2} = \boxed{0.855}$ ✓

b)



THE EQUILIBRIUM CONDITION ACROSS THE TANGENTIAL DISCONTINUITY IS $\boxed{P_4 = P_5}$

NOTE THAT $\theta = \alpha + \alpha_\infty = \delta$

↳ NEED TO ITERATE IN α_∞ UNTIL $P_4 = P_5$

STREAMS 4 AND 5 MUST BE PARALLEL

• 2 → 4 : OBLIQUE SHOCK

$Ma_2 = 5.3$, $\delta = \alpha + \alpha_\infty$ } FOR A GIVEN α_∞, β CAN BE OBTAINED FROM CHART

↳ $Ma_{3n} = Ma_3 \sin\beta \Rightarrow \frac{P_4}{P_2}$ FROM TABLE II

• 3 → 5 : EXPANSION WAVE

$Ma_3 = 1.41$, $\theta = \alpha + \alpha_\infty$ } FOR A GIVEN α_∞; θ IS KNOWN:

$\theta = \mathcal{D}(Ma_5) - \mathcal{D}(Ma_3) = \alpha + \alpha_\infty$

$\mathcal{D}(Ma_3) = 9.2°$ (TABLE II)

THEN $\mathcal{D}(Ma_5) = \alpha + \alpha_\infty + \mathcal{D}(Ma_3)$ GIVES Ma_5 FROM TABLE II

$\frac{P_5}{P_3} = \left(\frac{1 + \frac{\gamma-1}{2} Ma_3^2}{1 + \frac{\gamma-1}{2} Ma_5^2}\right)^{\frac{\gamma}{\gamma-1}}$

⇒ CONTINUE ITERATION IN α_∞ UNTIL $P_4 = P_5$

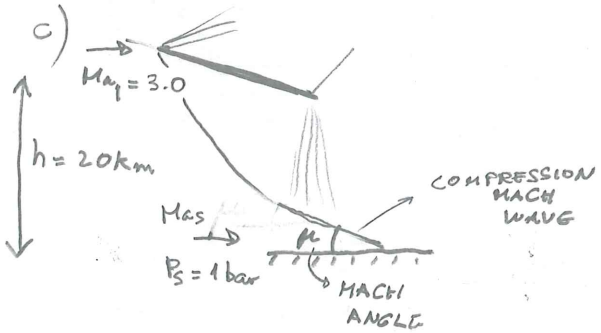
ANSWER IS $\boxed{\alpha_\infty \approx 5°}$ ✓

10/10

c)

⇒ $\begin{cases} \delta = 35°,\ \beta \approx 49°,\ Ma_{3n} = 4.0 \Rightarrow \frac{P_4}{P_2} = 18.5 \Rightarrow \frac{P_4}{P_1} = \frac{P_4}{P_2}\frac{P_2}{P_1} \approx 0.9 \\ \theta = 35°,\ \mathcal{D}(Ma_5) = 44.2°,\ Ma_5 = 2.7,\ \frac{P_5}{P_3} = 0.14 \Rightarrow \frac{P_5}{P_1} = \frac{P_5}{P_3}\frac{P_3}{P_1} \approx 0.9 \end{cases}$

c)



THE ANGLE FORMED BY THE WEAK SHOCK AND THE GROUND SURFACE IS THE MACH ANGLE:

$$\mu = \arcsin\left(\frac{1}{\sqrt{Ma_5^2 - 1}}\right)$$

WHERE Ma_5 IS THE MACH NUMBER ON THE GROUND. NOTE THAT THE SPEED OF SOUND IS $a \sim 290$ m/s AT 20 km AND 346 m/s ON THE GROUND (SEE PAGE 11 KLASS NOTES). THEREFORE $Ma_5 \sim \frac{290}{346} Ma_1 = 2.51$

⇒ $\boxed{\mu = 23.4°}$ ✓

$P_5 = 1\,bar = 10^5\,Pa$

IF THE STREAM DEFLECTION ANGLE CAUSED BY THE WEAK SHOCK IS $\delta = 0.001\alpha$ ⇒ $\frac{\Delta P}{P_5} = \frac{\gamma Ma_5^2 \delta}{\sqrt{Ma_5^2 - 1}} = 2 \cdot 10^{-3}$ ($\delta = 0.02$ rad) (PAGE 26 CLASS NOTES)

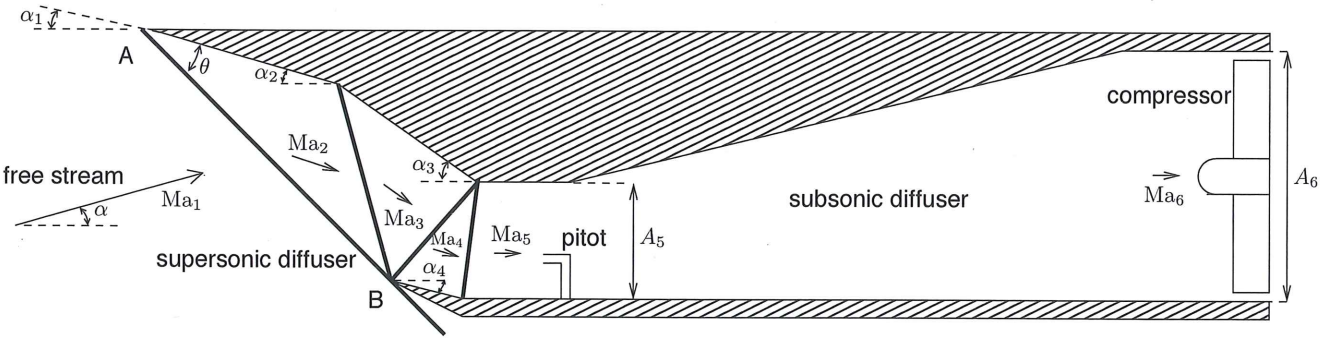
⇒ $SPL = 20\log_{10}\left(\frac{2 \cdot 10^{-3} P_5}{P_{ref}}\right) = 20\log_{10}\left(\frac{2 \cdot 10^{-3} \cdot 10^5\,Pa}{20 \cdot 10^{-6}\,Pa}\right) = \boxed{140\,dB}$ ✓

10/10

Problem 3 (30 pts)

The Concorde was powered by four Rolls-Royce/Snecma Olympus-593 turbojet engines capable of producing 38,050 lbs of thrust. Each one included compressor, combustor, turbine and afterburner stages. At supersonic flight speeds, the air had to be decelerated to subsonic velocities before reaching the compressor. However, slowing down a supersonic stream necessarily involves shock waves and loss of stagnation pressure. For that reason, the intake design for Concorde's engines was especially critical and involved complex ramp and nozzle assemblies, including a variable-geometry intake control system that shifted from a straight duct at subsonic speeds to increasing the ramp angles at supersonic speeds, thus transforming the intake front into a supersonic diffuser. Downstream from the shock train, the flow became subsonic and it was further decelerated through a subsonic diffuser for increasing as much as possible the static pressure upstream from the compressor. A model description of the intake at design conditions is depicted in the figure below, where $\mathrm{Ma}_1 = 2.2$, $\alpha_1 = 10°$, $P_1 = 0.3$ bar and $A_6/A_5 = 1.4$. A pitot-tube measurement right downstream from the shock train gives $\mathrm{Ma}_5 = 0.7$ and $P_5 = 1.4$ bar. Consider only the weak solution for oblique shocks.



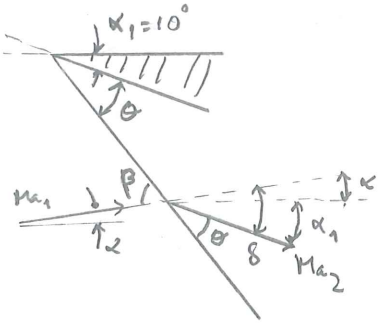


a) Calculate the angle of attack α such that the angle formed by the leading shock and the upper cowl lip is $\theta = 30°$, as shown in the figure.

b) Calculate the Mach number Ma_6 upstream from the compressor.

c) Determine the pressure-recovery ratio P_6/P_{01}.

a)



$$\delta = \kappa + \kappa_1$$
$$\beta = \theta + \delta = \theta + \kappa + \kappa_1$$

FOR $Ma_1 = 2.2$, $\theta = 30°$ AND $\kappa_1 = 10°$, ONE NEEDS TO FIND κ_1 SUCH THAT THE RELATION

$$\tan\delta = \frac{\cot\alpha\beta \left(Ma_1^2 \sin^2\beta - 1 \right)}{2 + Ma_1^2 \left(\gamma + \cos(2\beta) \right)}$$

IS SATISFIED

AFTER ITERATION $\Rightarrow$ $\boxed{\kappa = 12.8°}$ ✓ 10/10

b) FROM PITOT : $Ma_5 = 0.7$

MASS CONSERVATION: $\dot{m} = \rho U A = \rho_{05} a_{05} \frac{U_5}{a_5} \frac{a_5}{a_{05}} \frac{\rho_5}{\rho_{05}} A_5 =$

$$= \rho_{05} a_{05} A_5 Ma_5 \left(1 + \frac{(\gamma-1)}{2} Ma_5^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}} =$$
$$= \rho_{06} a_{06} A_6 Ma_6 \left(1 + \frac{(\gamma-1)}{2} Ma_6^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

SINCE $5 \to 6$ IS ISENTROPIC, THE STAGNATION QUANTITIES ARE CONSERVED AND

$$\frac{A_6}{A_5} = \frac{Ma_5 \left(1 + \frac{(\gamma-1)}{2} Ma_5^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}}}{Ma_6 \left(1 + \frac{(\gamma-1)}{2} Ma_6^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}}} = 1.4 = \frac{A_6}{A^*} \cdot \frac{A^*}{A_5}$$

$= 0.91$ FROM TABLE I AT $Ma_5 = 0.7$

THEN $\frac{A^*}{A_6} = \frac{1}{1.4} \frac{A^*}{A_5} = 0.65 \Rightarrow \boxed{Ma_6 = 0.42}$ FROM TABLE I ✓ 10/10

c) $\frac{P_6}{P_{01}} = \frac{P_6}{P_5} \frac{P_5}{P_1} \frac{P_1}{P_{01}}$ (① ② ③)

① : $\frac{P_6}{P_5} = \left(\frac{1 + \frac{(\gamma-1)}{2} Ma_5^2}{1 + \frac{(\gamma-1)}{2} Ma_6^2} \right)^{\frac{\gamma}{\gamma-1}} = 1.22$

② $\frac{P_5}{P_1} = \frac{1.4}{0.3} = 4.6$

③ $\frac{P_1}{P_{01}} = \frac{1}{\left(1 + \frac{\gamma-1}{2} Ma_1^2 \right)^{\frac{\gamma}{\gamma-1}}} = 0.09$

$\Rightarrow$ $\boxed{\frac{P_6}{P_{01}} = 0.50}$ ✓ 10/10

FULL PRESSURE RECOVERY

NOTE THAT IN AN IDEAL INLET $P_6 \sim P_{01}$, IN REALITY, LOSSES FROM SHOCK WAVES, BOUNDARY-LAYER SEPARATION AND INCOMPLETE DECELERATION LEAD TO JUST PARTIAL RECOVERY OF THE FREE-STREAM STAGNATION PRESSURE.

ME 355: Compressible Flows, Spring 2016

Stanford University

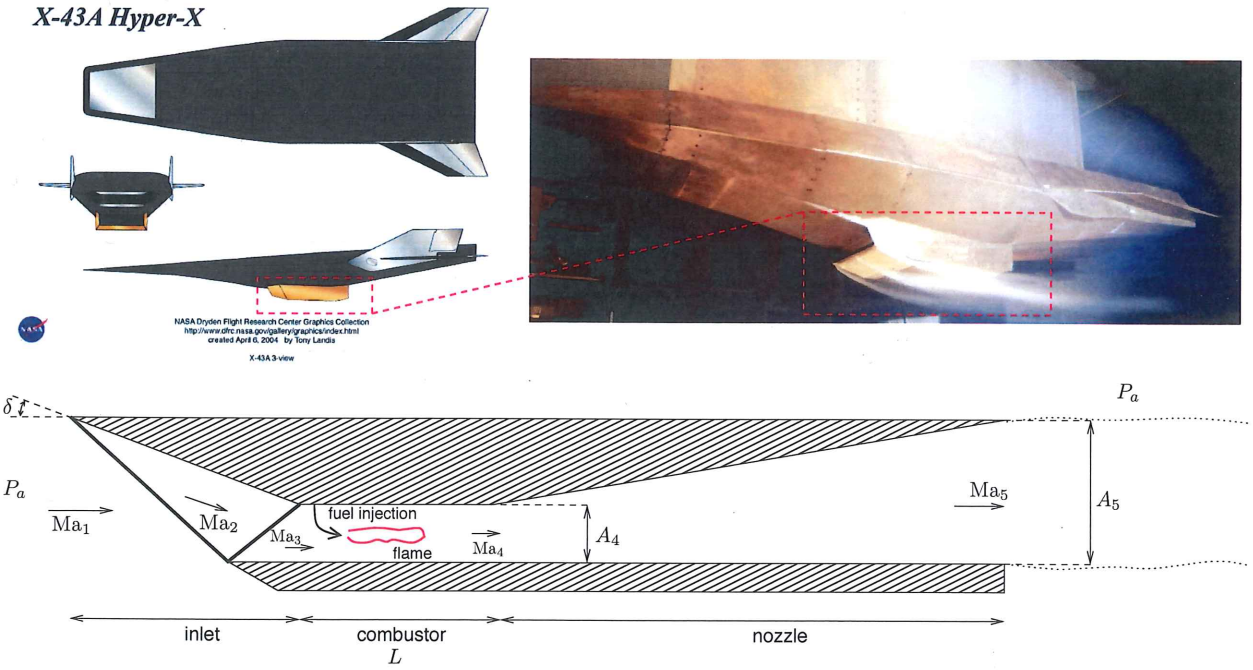
Homework 3: Convergent-divergent nozzles, and flows with heat addition

Due Tuesday, May 17, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1$ KJ/KgK for all problems.

Student's Name: JAVIER URZAY **Student's ID:**.....

SCRAMJETS (supersonic combustion ramjets) are utilized as propulsion systems for suborbital hypersonic vehicles. SCRAMJET-powered vehicles carry the fuel on board, and obtain the oxidizer by ingestion of atmospheric oxygen. SCRAMJET engines are composed of three basic components: an inlet supersonic diffuser, where incoming air is compressed through a shock train; a combustor, where gaseous fuel burns supersonically with atmospheric oxygen to produce heat; and a diverging nozzle, where the hot combustion products are accelerated to produce thrust. A model description of the SCRAMJET power plant of the X-43A hypersonic aircraft is provided in the figure below, where $\mathrm{Ma}_1 = 5$, $\delta = 15°$, $P_a = 0.1$ bar, $T_a = 270$ K and $A_4/A_5 = 0.5$.



a) Compute the Mach number Ma_3, static pressure P_3 and static temperature T_3 at the combustor inlet.

b) Calculate the amount of heat release per unit mass that is required in the combustor to attain sonic conditions $\mathrm{Ma}_4 = 1.0$ at the nozzle inlet section (combustor exit). What are the associated static temperature T_4 and pressure P_4?

c) Determine the Mach number Ma_5 and static pressure P_5 at the nozzle exit.

d) Show that the thrust F produced by the engine can be written as

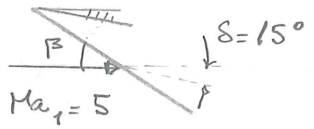
$$\frac{F}{P_1 A_5} = \frac{P_5}{P_1}(1+\gamma \mathrm{Ma}_5^2) - (1+\gamma \mathrm{Ma}_1^2), \tag{1}$$

and compute its numerical value for $A_5 = 0.03\ \mathrm{m}^2$.

e) Characterize the jet flow at the exhaust. Compute the streamline deflection angle and Mach number downstream from the first set of expansion fans.

f) The length of the combustor is $L = 10$ cm, and the autoignition time of the fuel+air mixture at temperature T_3 and pressure P_3 is $t_{ig} = 10\ \mu s$. Estimate the ratio of the characteristic residence time of a fluid particle in the combustor to the autoignition time. The ratio represents the Damköhler number Da. Based on the estimated value of Da, is the residence time in the combustor sufficiently large to warrant combustion of the reactants?

a) 1→2: OBLIQUE SHOCK



FROM CHART:

$\beta = 24^\circ \Rightarrow \mathrm{Ma}_{1n} = \mathrm{Ma}_1 \sin\beta = 2.03$

TABLE II NORMAL SHOCK

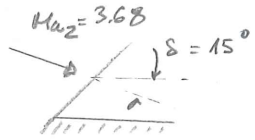
$P_2/P_a = 4.5$

$T_2/T_a = 1.68$

$\mathrm{Ma}_{2n} = 0.57$

THEN $\mathrm{Ma}_2 = \mathrm{Ma}_{2n}/\sin(\beta-\delta) = 3.68$

2→3 OBLIQUE SHOCK



FROM CHART

$\beta = 28^\circ \Rightarrow \mathrm{Ma}_{2n} = \mathrm{Ma}_2 \sin\beta = 1.73$

TABLE II NORMAL SHOCK

$P_3/P_2 = 3.30$

$T_3/T_2 = 1.48$

$\mathrm{Ma}_{3n} = 0.63$

THEN $\mathrm{Ma}_3 = \mathrm{Ma}_{3n}/\sin(\beta-\delta) = 2.8$

$\Rightarrow P_3 = \frac{P_3}{P_2}\frac{P_2}{P_a} P_a = 1.48$ bar, $T_3 = \frac{T_3}{T_2}\frac{T_2}{T_a} T_a = 671.3$ K

b) FOR SONIC CONDITIONS $\mathrm{Ma}_4 = 1.0$, THE HEAT ADDED MUST BE

$$\frac{q_{MAX}}{h_{o_3}} = \frac{1}{2(\gamma+1)\mathcal{F}^2(\mathrm{Ma}_3)} - 1, \quad \text{WITH } \mathcal{F}(\mathrm{Ma}_3) = \frac{\mathrm{Ma}_3}{1+\gamma \mathrm{Ma}_3^2}\left[1+\frac{(\gamma-1)}{2}\mathrm{Ma}_3^2\right]^{1/2} = 0.37$$

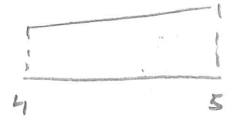
$\Rightarrow \frac{q_{MAX}}{h_{o_3}} = 0.52 \Rightarrow q_{MAX} = 0.52 c_p T_3 \left(1+\frac{(\gamma-1)}{2}\mathrm{Ma}_3^2\right) = 896.4$ KJ/kg

SINCE $q_{MAX} = c_p(T_{o_4} - T_{o_3}) \Rightarrow T_{o_4} = \frac{q_{MAX}}{c_p} + T_3\left(1+\frac{(\gamma-1)}{2}\mathrm{Ma}_3^2\right) = 2620.2\,\mathrm{K} = T_4\left(1+\frac{(\gamma-1)}{2}\mathrm{Ma}_4^2\right)$

$\Rightarrow T_4 = 2183.5$ K

AND $\frac{P_4}{P_3} = \frac{1+\gamma Ma_3^2}{1+\gamma \cancel{Ma_4^2}^{1}} = 4.99$ (CONS. OF MOMENTUM) $\Rightarrow P_4 = 4.99\, P_3 = \boxed{7.38 \text{ bar}}$

c) ISENTROPIC FLOW:



$Ma_4 = 1$ $\quad Ma_5 = ?$

$A_4/A_5 = 0.5$

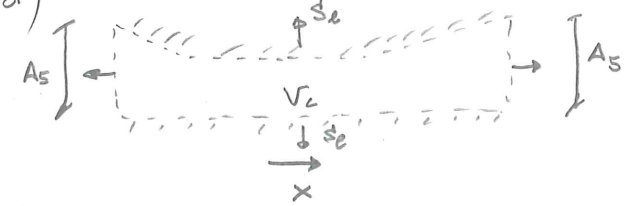
$$\dot{m} = \rho U A = \text{const} \Rightarrow A_5 Ma_5 \left(1+\frac{\gamma-1}{2}Ma_5^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} = \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}} A_4$$

$$\frac{Ma_5\left(1+\frac{(\gamma-1)}{2}Ma_5^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}}{\left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}} = \frac{A_4}{A_5} = 0.5 = \frac{A^*}{A_5}$$

↳ FROM TABLE I (ISENTROPIC FLOW): $\boxed{Ma_5 = 2.18}$

$$\frac{P_5}{P_4} = \left(\frac{1+\frac{\gamma-1}{2}\cancel{Ma_4^2}^{1}}{1+\frac{(\gamma-1)}{2}Ma_5^2}\right)^{\frac{\gamma}{\gamma-1}} = 0.18 \Rightarrow \boxed{P_5 = 1.34 \text{ bar}}$$

d)



$$\int_{S_c} \rho \vec{v}\vec{v}\cdot d\vec{S} = -\int_{S_c} P\, d\vec{S}$$

$$\left(-\rho_1 U_1^2 A_5 + \rho_5 U_5^2 A_5\right)\vec{e}_x = -\int_{S_c} P\, d\vec{S} + (P_1 A_5 - P_5 A_5)\vec{e}_x$$

$-\vec{F}$ (over S_c)

$$\Rightarrow \vec{F} = (P_1 - P_5)A_5 \vec{e}_x + (\rho_1 U_1^2 - \rho_5 U_5^2)A_5 \vec{e}_x$$

$$\frac{\vec{F}}{P_1 A_5} = \left(1 - \frac{P_5}{P_1} + \frac{\rho_1 U_1^2}{P_1} - \frac{\rho_5 U_5^2}{P_1}\right)\vec{e}_x = \left(1 - \frac{P_5}{P_1} + \gamma Ma_1^2 - \gamma Ma_5^2 \frac{P_5}{P_1}\right)\vec{e}_x$$

$$\Rightarrow \frac{\vec{F}}{P_1 A_5} = \left\{-\frac{P_5}{P_1}\left(1+\gamma Ma_5^2\right) + \left(1+\gamma Ma_1^2\right)\right\}\vec{e}_x$$

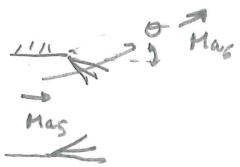
THEN THE THRUST IS ($-x$ DIR) $\quad \frac{F}{P_1 A_5} = -\frac{P_5}{P_1}\left(1+\gamma Ma_5^2\right) - \left(1+\gamma Ma_1^2\right)$ //

SUBSTITUTING VALUES: $P_1 = 0.1$ bar, $Ma_5 = 2.18$, $P_5 = 1.34$ bar, $Ma_1 = 5$, $A_5 = 0.03\ m^2$

$$\left.\begin{array}{l}\frac{F}{P_1 A} = 66.5 \Rightarrow F = 19.9 \text{ kN} \\ \end{array}\right\}$$

e) AT THE EXHAUST, THIS FLOW IS SUPERSONIC ($Ma_5 = 2.18$) AND UNDEREXPANDED ($P_5 > P_a$)

$\Rightarrow$ EXPANSION FANS (SEE FIG 27 IN PAGE 43 NOTES)



THE TURNING ANGLE IS $\theta = \nu(Ma_6) - \nu(Ma_5)$ (MACH OF DEFLECTED STREAM)

$\nu(Ma_5) = 31.73°$ TABLE III

SINCE $\frac{P_6}{P_5} = \frac{P_a}{P_5} = 0.074 = \left(\frac{1+\frac{(\gamma-1)}{2}Ma_5^2}{1+\frac{(\gamma-1)}{2}Ma_6^2}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow Ma_6 = 3.94$

TABLE II: $\nu(Ma_6) = 65.1°$

$\Rightarrow \boxed{\theta = 33.3°}$

f) THE CHARACTERISTIC RESIDENCE TIME OF A FLUID PARTICLE IN THE COMBUSTOR IS $t_R \approx \frac{L}{U_3}$, WHERE $U_3 = Ma_3 \sqrt{\gamma R_g T_3} = 1451.6\ m/s$ IS THE FLOW VELOCITY AT THE COMBUSTOR ENTRANCE. THEN $t_R \sim 68.9\ \mu s$.

→ DAMKÖHLER: $Da = \frac{t_R}{t_{ig}} = 6.89 > 1$, THE RESIDENCE TIME IS LONG ENOUGH TO WARRANT AUTOIGNITION.

ME 355: Compressible Flows, Spring 2016

Stanford University

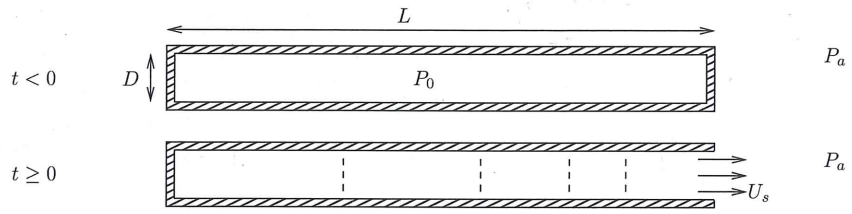
Homework 4: Wave motion

Due Tuesday, May 31, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1$ KJ/KgK for all problems.

Student's Name: JAVIER URZAY ..... **Student's ID:** .....

A cylinder of diameter D and length L initially closed at both ends is filled with an stagnant gas at pressure P_0 and density ρ_0. The cylinder is surrounded by air at pressure $P_a < P_0$. At $t = 0$, a pyrotechnic system breaks the right-end wall of the duct, so that the gas starts flowing out of the cylinder. This problem addresses the dynamics shortly after rupture of the right-end wall during times of order L/a_0, where a_0 is the speed of sound in the gas. Depending on the value of the overpressure $\epsilon = (P_0 - P_a)/P_0$, the disturbances produced in the gas may be small ($\epsilon \ll 1$), so that the linear-acoustics theory is applicable, or large ($\epsilon = O(1)$), in which case the linear approximation is no longer valid and the non-linear acoustics formulation needs to be employed.



Consider first the case in which the air pressure differs slightly from the pressure of the gas in the cylinder, $\epsilon = (P_0 - P_a)/P_0 \ll 1$.

20 **a)** Compute the velocity of the gas at the cylinder exit U_s.

10 **b)** Calculate the force on the cylinder.

20 **c)** Characterize the spatial profiles of velocity and pressure inside the cylinder during the time interval $0 < t < 2L/a_0$.

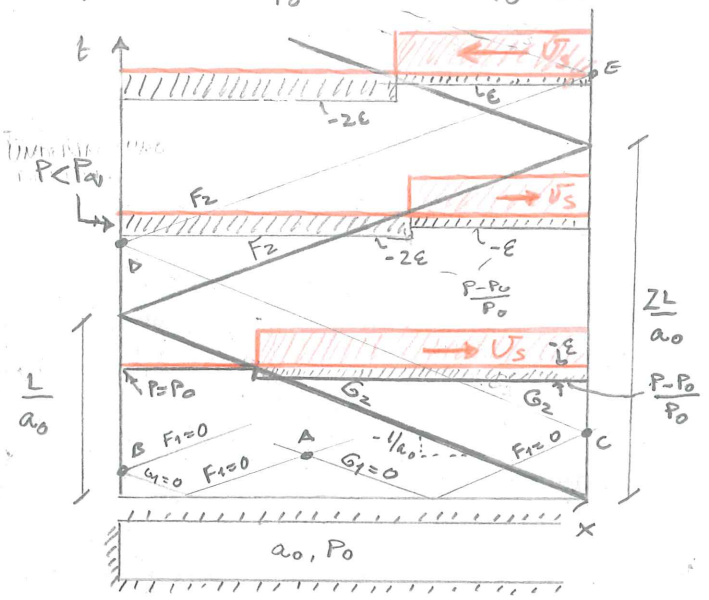
Secondly, consider the case in which the air pressure differs from the pressure of the gas by an amount of order unity, $\epsilon = (P_0 - P_a)/P_0 = O(1)$.

30 **d)** Compute the velocity of the gas at the cylinder exit U_s. Check that U_s becomes the one obtained in part a) in the limit $\epsilon \ll 1$.

10 **e)** Calculate the force on the cylinder.

10 **f)** Characterize the spatial profiles of velocity and pressure inside the cylinder during the time interval $0 < t < L/a_0$.

a) SINCE $\frac{P_0-P_a}{P_0}=\varepsilon\ll 1 \Rightarrow \frac{\Delta P}{P_0}\sim \gamma M_a^2 \ll 1 \Rightarrow$ USE LINEAR THEORY



THE CONDITIONS AT $t=0$ ARE $U'=P'=0$

FROM LINEAR THEORY: $\frac{U'}{a_0}=F+G$, $\frac{P'}{\rho_0 a_0^2}=F-G$

- AT POINT A: $F_1=0$, $G_1=0$ $\Rightarrow U'_A=P'_A=0$ $\Rightarrow$ THE FLOW IS AT ITS INITIAL STATE
- AT POINT B: $G_1=0$, $U'_B=0$ (BC) $\Rightarrow$ THE REBOUND WAVE IS $F=F_1=0$ $\Rightarrow P'_B=0$
- AT POINT C: $F_1=0$, $P=P_a$ (BC) $\Rightarrow \frac{P_a-P_0}{\rho_0 a_0^2}=\cancel{F_1}^{\,0}-G_2$

$$\frac{U'_C}{a_0}=G_2=\frac{P_0-P_a}{\rho_0 a_0^2}$$

$$\Rightarrow \boxed{U_S=U'_C=\frac{P_0-P_a}{\rho_0 a_0}}=\frac{\varepsilon P_0}{\rho_0 a_0}=\frac{\varepsilon a_0}{\gamma}\ll a_0$$

b) SEE PROFILES ABOVE.

VELOCITY U'

PRESSURE $(P-P_0)/P_0$

- AT POINT D: $G_2=\frac{P_0-P_a}{\rho_0 a_0^2}$, $U'_D=0$ (BC) $\Rightarrow F_2=-G_2$, $\frac{P'_D}{\rho_0 a_0^2}=\frac{2(P_a-P_0)}{\rho_0 a_0^2}=-\frac{2\varepsilon P_0}{\rho_0 a_0^2}$
- AT POINT E: $F_2=\frac{P_a-P_0}{\rho_0 a_0^2}$, $P=P_a$ (BC) $\Rightarrow \frac{P_a-P_0}{\rho_0 a_0^2}=\frac{P_a-P_0}{\rho_0 a_0^2}-G_3 \Rightarrow G_3=0$

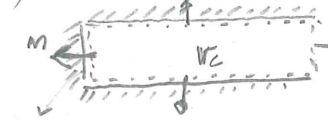
$$\frac{U'_E}{a_0}=F_2=-\frac{U_S}{a_0}$$

NOTE: YOU COULD ALSO COMPUTE $\vec{F}$ IN A MORE DIFFICULT WAY:

$$\vec{F}=-\frac{d}{dt}\int_{V_0}\rho\vec{u}\,dV-\int_{S_C}\rho\vec{u}\,\vec{u}\cdot d\vec{S}=\left\{-\frac{d}{dt}\left(\frac{\pi D^4}{4}\rho_0\left(1-\frac{\varepsilon}{\gamma}\right)U_S a_0 t\right)-\rho_0\left(1-\frac{\varepsilon}{\gamma}\right)U_S^2\frac{\pi D^4}{4}\vec{e}_x\right\}$$

$$=-\frac{\pi D^4}{4}\rho_0\left(1-\frac{\varepsilon}{\gamma}\right)\varepsilon a_0^2\left[\frac{\varepsilon}{\gamma}+1\right]\vec{e}_x\approx_{\varepsilon\ll 1}-\frac{\pi D^4}{4}\rho_0 a_0^2\frac{\varepsilon}{\gamma}\vec{e}_x=-\frac{\pi D^4}{4}\varepsilon P_0\vec{e}_x$$

c) MOMENTUM CONSERVATION EQUATION:



$$\frac{d}{dt}\int_{V_0}\rho\vec{u}\,dV+\int_{S_C}\rho\vec{u}\,\vec{u}\cdot d\vec{S}=-\int_{S_C}P\,d\vec{S}$$

FORCE IS ONLY IN X DIRECTION

$$-\int_{S_C}P\,d\vec{S}=\underbrace{-\int_{\text{CLOSED END}}P\,d\vec{S}}_{-F_{\text{FLUID}\to\text{CYLINDER}}}\underbrace{-\int_{\text{LATERAL SURFACES}}P\,d\vec{S}}_{0\ \text{(SYMMETRY)}}\underbrace{-\int_{\text{EXIT}}P\,d\vec{S}}_{-P_a\frac{\pi D^2}{4}}=$$

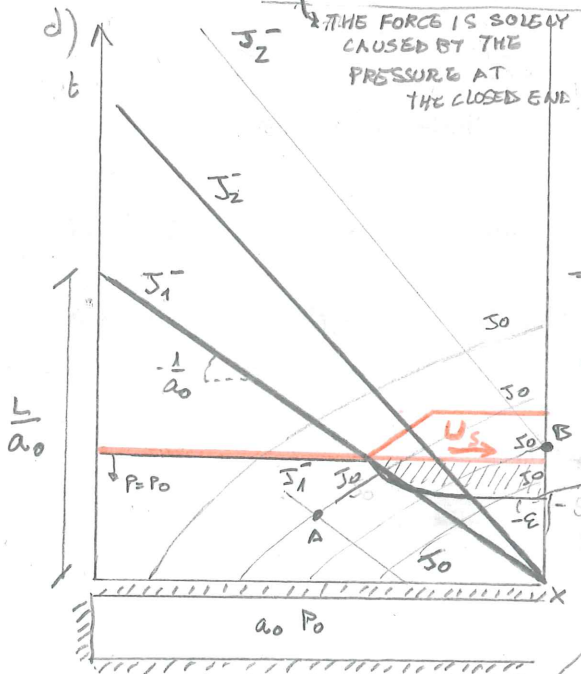
* EASY WAY:

$$\vec{F}=\int_{\text{CLOSED END}}(P-P_a)\,d\vec{S}=\begin{cases}-\varepsilon P_0\frac{\pi D^4}{4}\vec{e}_x & \left(0<t<\frac{1}{a_0}\right)\\ +\varepsilon P_0\frac{\pi D^4}{4}\vec{e}_x & \left(\frac{L}{a_0}<t<\frac{2L}{a_0}\right)\end{cases}$$

CONTRIBUTION OF THE ATMOSPHERE

$\Rightarrow$ THE FORCE IS SOLELY CAUSED BY THE PRESSURE AT THE CLOSED END

d)



SINCE $\frac{P_0-P_a}{P_0}\sim 1 \Rightarrow \frac{\Delta P}{P_0}\sim\gamma M_a^2\sim O(1) \Rightarrow$ USE NON-LINEAR THEORY

AGAIN, CONDITIONS AT $t=0$ ARE $U=0$, $P=P_0$.

FROM THE THEORY: $J^+=a+U\left(\frac{\gamma-1}{2}\right)$, $J^-=a-U\left(\frac{\gamma-1}{2}\right)$ $\Rightarrow U=\frac{J^+-J^-}{\gamma-1}$, $a=\frac{J^++J^-}{2}$ AND $\left(\frac{a_0}{a}\right)^2=\left(\frac{P_0}{P}\right)^{\frac{\gamma-1}{\gamma}}=\left(\frac{T_0}{T}\right)\approx 1+\frac{\gamma-1}{2}M_a^2$

$\Rightarrow$ FROM THE INITIAL CONDITIONS: $J^+=J_0=a_0 \Rightarrow$ NON-LINEAR SIMPLE WAVES (C^- CHARACTERISTICS ARE STRAIGHT LINES)

- AT POINT A: $J^+=J_0=a_0$, $J_1^-=a_0$ $\Rightarrow U_A=0 \Rightarrow$ THE GAS IS AT REST AND HAS NOT NOTICED THE EFFECT OF THE OPEN END BELOW THE FIRST CRITICAL CHARACTERISTIC EMERGING FROM $x=L$ AT $t=0$
- AT POINT B (EXIT): THE PRESSURE CONDITION DEPENDS ON WHETHER THE DUCT IS CHOKED OR NOT. FOR $P_a\geq P_S^*$ (P^* CALCULATED BELOW)

$\Rightarrow$ THE DUCT CANNOT BE CHOKED AND $P_B=P_a$ (SUBSONIC FLOW)

$$\Rightarrow P_B=P_a,\ J^+=J_0=a_0 \Rightarrow \left(\frac{P_0}{P_a}\right)^{\frac{\gamma-1}{\gamma}}=\left(\frac{a_0}{a_B}\right)^2 \Rightarrow a_B=a_0\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}=a_S$$

RECALL A STRAIGHT DUCT CAN ONLY ACHIEVE SONIC CONDITIONS AT EXIT (AT MOST)!

AS A RESULT: $u_s = a_B = a_0 \left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}}$; $J^+ = J_0 = a_0$

$\Big\} a_s = a_B = \frac{J_0 + J_2^-}{2} \Rightarrow \quad J_2^- = a_0 \left[2\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1 \right]$

THE INVARIANT J_2^- RUNS ALONG A CHARACTERISTIC OF SLOPE $\frac{-1}{u_s + a_B}$, WITH $J_2^- = a_B - \left(\frac{\gamma-1}{2}\right) u_s$.

THEN $\left[u_s = \frac{J_0 - J_2^-}{\gamma-1} = \frac{a_0}{\gamma-1} - \frac{a_0}{\gamma-1}\left[2\left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] = \frac{2}{\gamma-1} a_0 \left[1 - \left(\frac{P_a}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} \right] \right]$

SINCE $\frac{P_a}{P_0} = \frac{P_a}{P_0} + 1 - 1 = -\varepsilon + 1 \Rightarrow u_s = \frac{2}{\gamma-1} a_0 \left[1 - (1-\varepsilon)^{\frac{\gamma-1}{2\gamma}} \right] \overset{\varepsilon \ll 1}{\sim} \frac{2}{\gamma-1} a_0 \left(\frac{\gamma-1}{2\gamma}\right)\varepsilon = a_0 \frac{\varepsilon}{\gamma} = \boxed{\frac{P_0 - P_a}{\rho_0 a_0}}$

* NOTE THE SOLUTION ABOVE IS VALID AS LONG AS $P_a > P_s^*$, WITH P^* GIVEN BY THE CONDITION

$$Ma_s = \frac{u_s}{a_s} = \frac{2}{\gamma-1} \frac{a_0}{a_s} \left[1 - \left(\frac{P_s^*}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} \right] = \frac{2}{\gamma-1} \left[\left(\frac{P_0}{P_s^*}\right)^{\frac{\gamma-1}{2\gamma}} - 1 \right] = 1$$

$\left(\frac{P_0}{P_s^*}\right)^{\frac{\gamma-1}{2\gamma}}$

$$\Rightarrow \boxed{P_s^* = \left(\frac{\gamma+1}{2}\right)^{-\frac{2\gamma}{\gamma-1}} P_0 \simeq 0.28 P_0} \quad \text{FOR } \gamma = 1.4$$

THE PRESSURE P_s^* IS THE CRITICAL AMBIENT PRESSURE FOR WHICH $Ma_s = 1$. FOR $P_a > P_s^*$ THE FLOW REMAINS SUBSONIC EVERYWHERE, $Ma_s < 1$, AND THE SOLUTION IS THE ONE GIVEN ABOVE. THE VELOCITY AND PRESSURE PROFILES IN THE EXPANSION FAN ARE OBTAINED BY MAKING

$\gamma - 1 \quad 1 + 1 = 1$

CONSTANCY OF $J^+ = J_0$; CONSTANCY OF C_- SLOPES

$$J^+ = J_0 = a_0 = a + \left(\frac{\gamma-1}{2}\right) u \quad \text{AND} \quad \frac{dx}{dt} = u + a = \text{CONST} = \frac{x}{t}$$

COMBINING THESE EQUATIONS: $u = \frac{x}{t} + \left(\frac{\gamma-1}{2}\right) u = a_0 \Rightarrow$

$$\left\{ \begin{array}{l} \boxed{u = \frac{2}{\gamma+1}\left(a_0 + \frac{x}{t}\right)} \\ \boxed{a = \frac{2a_0}{\gamma+1} + \frac{1-\gamma}{\gamma+1}\frac{x}{t}} \end{array} \right. \quad \left\{ \begin{array}{l} (x=0 \text{ IS TAKEN AT THE OPEN END}) \\ \rightarrow \text{IN THE EXPANSION FAN.} \end{array} \right.$$

$\Rightarrow$ NOTE THAT THE VELOCITY IS LINEAR IN x ACROSS THE FAN.

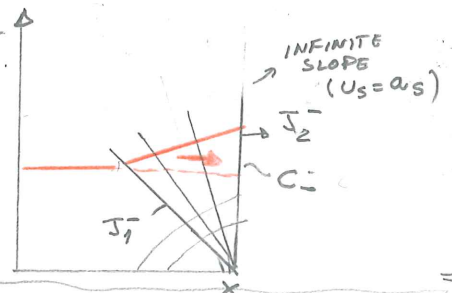
$\hookrightarrow u = 0,\ a = a_0$ AT $\frac{x}{t} = -a_0$

THE CORRESPONDING PRESSURE VARIATIONS ACROSS THE FAN ARE $\left(\frac{P}{P_0}\right)^{\frac{\gamma-1}{\gamma}} = \left(\frac{a}{a_0}\right)^2 \Rightarrow \frac{P}{P_0} = \left(\frac{a}{a_0}\right)^{\frac{2\gamma}{\gamma-1}} = \left[\frac{2}{\gamma+1} + \frac{1-\gamma}{\gamma+1}\frac{x}{t a_0} \right]^{\frac{2\gamma}{\gamma-1}} = \left(\frac{2}{\gamma+1}\right)^{\frac{2\gamma}{\gamma-1}} \left[1 + \frac{1-\gamma}{2}\frac{x}{t a_0} \right]^{\frac{2\gamma}{\gamma-1}}$

WITH $P = P_0$ AT $\frac{x}{t} = -a_0$. SIMILARLY, $\frac{\rho}{\rho_0} = \left(\frac{P}{P_0}\right)^{1/\gamma} = \left(\frac{2}{\gamma+1}\right)^{\frac{2}{\gamma-1}} \left[1 + \left(\frac{1-\gamma}{2}\right)\frac{x}{t a_0} \right]^{\frac{2}{\gamma-1}}$

FOR $P_a < P_s^*$, THE DUCT IS CHOKED ($Ma_s = 1$), AND $P_s \neq P_a$, BUT $P_s = P_s^* = \left(\frac{\gamma+1}{2}\right)^{-\frac{2\gamma}{\gamma-1}} P_0$.

NOTE THIS VALUE IS LOWER THAN THE ONE PREDICTED FROM STEADY THEORY! $\left(\left(\frac{\gamma+1}{2}\right)^{-\frac{\gamma}{\gamma-1}} P_0 \right)$ PAGE 18 NOTES.



THIS CASE IS SOLVED IN THE SAME WAY AS BEFORE, WITH THE DIFFERENCE THAT THE BOUNDARY CONDITION AT THE OPEN END IS $P_s = P_s^*$ INSTEAD OF $P_s = P_a$. THEN $a_s = a_0 \left(\frac{P_s^*}{P_0}\right)^{\frac{\gamma-1}{2\gamma}} = \left(\frac{\gamma+1}{2}\right) a_0 = u_s$

$\Rightarrow$ NOTE THAT THE INVARIANT $J_2^- = a_s - \left(\frac{\gamma-1}{2}\right) u_s = \frac{3-\gamma}{2}$ RUNS ALONG A VERTICAL CHARACTERISTIC C^- SINCE THE FLOW IS SONIC AT THE EXIT.

e) FORCE ON CYLINDER: (SEE c)

$$F = \int (P - P_a)\, d\vec{S} = -\varepsilon P_0 \frac{\pi D^2}{4} \vec{e}_x \quad \left(0 < t < \frac{L}{a_0}\right)$$

ME 355: Compressible Flows, Spring 2016

Stanford University

Midterm Exam

Tuesday, May 10

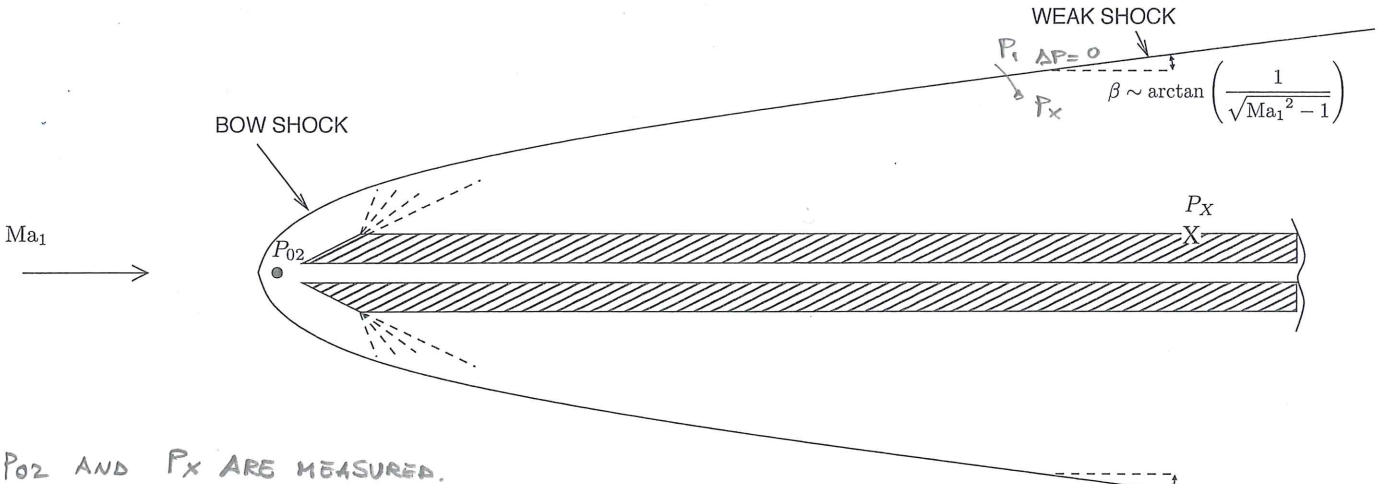
Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name: JAVIER URZAY **Student's ID:**.....

PART I: Closed books, closed notes, calculators allowed
Time: 20 mins

Questions (30 pts)

1. Explain what is supersonic wave drag and how it differs from viscous drag. ⇒ SEE PAGE 28 OF MY NOTES
2. The stagnation temperature upstream from a normal shock wave at $\mathrm{Ma}_1 = 1.5$ is $T_{01} = 335$ K. The static pressure downstream is $P_2 = 3$ bar. What is the stagnation enthalpy downstream h_{02}? ⇒ $h_0 = \text{CONST.}$ $h_{02} = c_p T_{02} = c_p T_{01} = 335$ KJ/Kg
3. A Machmeter for a supersonic aircraft consists of a very long central duct connected to a manometer that measures the post-shock stagnation pressure P_{02}, along with a lateral probe far away from the tube nose that measures the static pressure P_X at that position X (see fig. below). Based on those two quantities, how would you compute the Mach number of the free stream Ma_1?



P_{02} AND P_X ARE MEASURED.

$$\frac{P_{02}}{P_X} = \frac{P_{02}}{P_2}\frac{P_2}{P_X} \quad (1), \quad \text{WITH} \quad \frac{P_{02}}{P_2} = \left(1 + \frac{(\gamma-1)}{2}\mathrm{Ma}_2^2\right)^{\frac{\gamma}{\gamma-1}} \quad (2), \quad \mathrm{Ma}_2^2 = \frac{2 + (\gamma-1)\mathrm{Ma}_1^2}{2\gamma\mathrm{Ma}_1^2 + 1 - \gamma} \quad (3)$$

SINCE THE SHOCK VERY FAR FROM THE TUBE NOSE IS WEAK: ΔP IS VERY SMALL $\Rightarrow P_X \sim P_1$

$$\Rightarrow \frac{P_2}{P_X} \sim \frac{P_2}{P_1} = 1 + \frac{2}{\gamma+1}\left(\mathrm{Ma}_1^2 - 1\right) \quad (4)$$

WITH (1), (2), (3) AND (4) $\Rightarrow \mathrm{Ma}_1 = f\left(\frac{P_{02}}{P_X}\right)$ //

$$\left\{ \frac{P_{02}}{P_X} = \frac{\left(\frac{\gamma+1}{2}\mathrm{Ma}_1^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(\frac{2\gamma}{\gamma+1}\mathrm{Ma}_1^2 - \frac{\gamma-1}{\gamma+1}\right)^{\frac{1}{\gamma-1}}} \right\}$$

ME 355: Compressible Flows, Spring 2016

Stanford University

Midterm Exam

Tuesday, May 10

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name: JAVIER URZAY ..... **Student's ID:** .....

PART II: Open books, open notes, calculators allowed

Time: 60 mins

Problem 1 (60 pts)

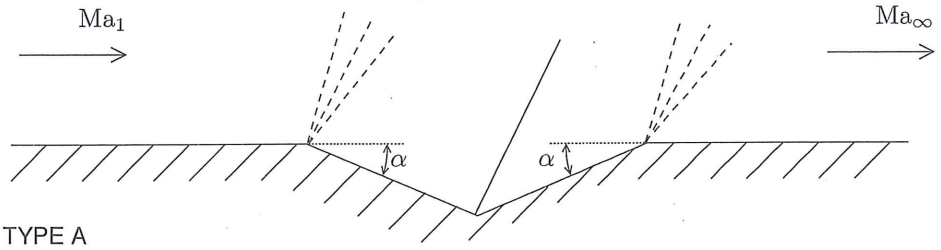
A supersonic stream at $\mathrm{Ma}_1 = 4.0$ flows parallel to a wall and encounters a geometrical disturbance of type A or B, as depicted below. Type A is an indentation and type B is a protrusion, both having the same deflection angle $\alpha = 20°$. Determine which configuration (type A or type B) yields

a) the maximum decrease in static pressure, $(P_1 - P_\infty)/P_1$

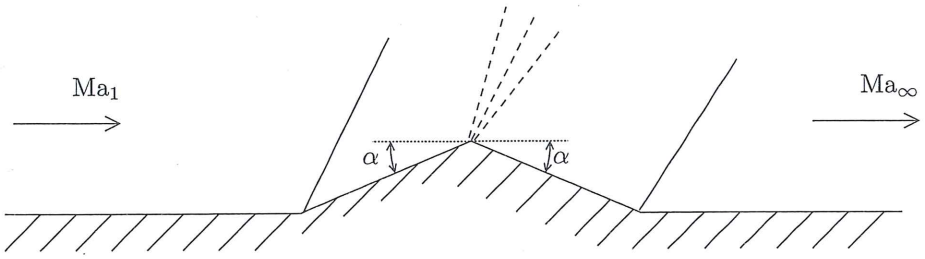
b) the maximum decrease in stagnation pressure, $(P_{01} - P_{0\infty})/P_{01}$

c) the maximum exit Mach number, Ma_∞

d) the minimum entropy increase, $(s_\infty - s_1)/c_v$

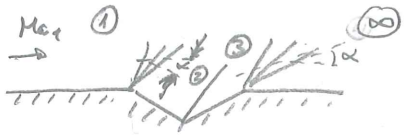


TYPE A



TYPE B

TYPE A :



• 1 → 2 : PRANDTL-MEYER EXPANSION. (TABLE III)

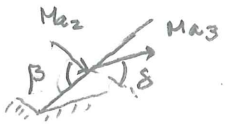
$$\Theta = \nu(Ma_2) - \underbrace{\nu(Ma_1)}_{65.78} = \alpha = 20^\circ$$

$$\Rightarrow \nu(Ma_2) = 85.78^\circ \Rightarrow Ma_2 = 6.1$$

ISENTROPIC FLOW: $(S_2 = S_1)$, $(P_{02} = P_{01})$

$$\frac{P_2}{P_1} = \left(\frac{1 + \frac{(\delta-1)}{2} Ma_1^2}{1 + \frac{(\delta-1)}{2} Ma_2^2} \right)^{\frac{\gamma}{\gamma-1}} = 0.087$$

• 2 → 3



$\delta = 2\alpha = 40^\circ$, $Ma_2 = 6.1$ } OBLIQUE-SHOCK CHART $\beta \sim 57^\circ$

$$\Rightarrow Ma_{2n} = Ma_2 \sin\beta = 5.11$$

TABLE II : $Ma_{3n} = 0.41 = Ma_3 \sin(\beta - \delta) \Rightarrow Ma_3 = 1.4$

TABLE II : $P_3/P_2 = 30.1$; $\rho_3/\rho_2 = 5.03$; $P_{03}/P_{02} = 0.057$

ENTROPY VARIATION: $\frac{S_3 - S_2}{C_v} = \ln\left(\frac{P_3}{P_2} \left(\frac{\rho_2}{\rho_3} \right)^\gamma \right) = 1.15$

• 3 → ∞, PRANDTL-MEYER EXPANSION WAVE (TABLE III)

$$\Theta = \nu(Ma_\infty) - \underbrace{\nu(Ma_3)}_{8.9^\circ} = 20^\circ \Rightarrow \nu(Ma_\infty) = 28.9^\circ$$

$$\Rightarrow Ma_\infty = 2.1$$

ISENTROPIC FLOW $(S_3 = S_\infty)$, $(P_{03} = P_{0\infty})$

$$\frac{P_\infty}{P_3} = \left(\frac{1 + \frac{(\gamma-1)}{2} Ma_3^2}{1 + \frac{(\gamma-1)}{2} Ma_\infty^2} \right)^{\frac{\gamma}{\gamma-1}} = 0.35$$

$$\Rightarrow \frac{P_1 - P_\infty}{P_1} = 1 - \frac{P_\infty}{P_1} = 1 - \frac{P_\infty}{P_3} \frac{P_3}{P_2} \frac{P_2}{P_1} = 0.083$$

$$\frac{P_{01} - P_{0\infty}}{P_{01}} = 1 - \frac{P_{0\infty}}{P_{01}} = 1 - \frac{P_{0\infty}}{P_{03}} \frac{P_{03}}{P_{02}} \frac{P_{02}}{P_{01}} = 0.94$$

TYPE B :



• 1 → 2 OBLIQUE SHOCK CHART



$\delta = \alpha = 20^\circ$, $Ma_1 = 4.0$ } $\beta = 32^\circ$

$$Ma_{1n} = Ma_1 \sin\beta = 2.11$$

TABLE II : $Ma_{2n} = 0.56$; $\frac{P_2}{P_1} = 4.97$; $\frac{\rho_2}{\rho_1} = 2.81$

$$\Rightarrow \frac{S_2 - S_1}{C_v} = \ln\left(\frac{P_2}{P_1} \left(\frac{\rho_1}{\rho_2} \right)^\gamma \right) = 0.15$$

$$Ma_2 = \frac{Ma_{2n}}{\sin(\beta - \delta)} = 2.69 \qquad \frac{P_{02}}{P_{01}} = 0.67$$

• 2 → 3 : PRANDTL-MEYER EXPANSION WAVE (TABLE III)

$$\Theta = \nu(Ma_3) - \underbrace{\nu(Ma_2)}_{43.6^\circ} = 2\alpha = 40^\circ \Rightarrow \nu(Ma_3) = 83.6^\circ \Rightarrow Ma_3 = 5.8$$

ISENTROPIC FLOW ($P_{O_2} = P_{O_3}$, $S_2 = S_3$):

$$\frac{P_3}{P_2} = \left(\frac{1 + \frac{(\gamma-1)}{2} Ma_2^2}{1 + \frac{(\gamma-1)}{2} Ma_3^2} \right)^{\frac{\gamma}{\gamma-1}} = \boxed{0.017}$$

- $3 \to \infty$ OBLIQUE SHOCK



$Ma_3 = 5.8$, $\delta = \alpha = 20°$ } CHART: $\beta \sim 28°$

$Ma_{3n} = Ma_3 \sin\beta = 2.72$

FROM TABLE II:

$\frac{P_{\infty}}{P_3} = 8.33$, $\frac{\rho_{\infty}}{\rho_3} = 3.60$,

$Ma_{\infty n} = 0.49$, $\frac{P_{0\infty}}{P_0} = 0.42$

$$\Rightarrow Ma_{\infty} = \frac{Ma_{\infty n}}{\sin(\beta - \delta)} = \boxed{3.52}$$

$$\frac{S_{\infty} - S_3}{c_v} = \ln\left(\frac{P_{\infty}}{P_3} \frac{\rho_3^{\gamma}}{\rho_{\infty}^{\gamma}} \right) = \boxed{0.32}$$

THEN:

$$\frac{P_1 - P_{\infty}}{P_1} = 1 - \frac{P_{\infty}}{P_3} \frac{P_3}{P_2} \frac{P_2}{P_1} = \boxed{0.29}$$

$$\frac{P_{01} - P_{0\infty}}{P_{01}} = 1 - \frac{P_{0\infty}}{P_{03}} \frac{P_{03}}{P_{02}} \frac{P_{02}}{P_{01}} = \boxed{0.71}$$

($\frac{P_{03}}{P_{02}} \to 1$)

$$\frac{S_{\infty} - S_1}{c_v} = \frac{S_{\infty} - S_3}{c_v} + \frac{S_2 - S_1}{c_v} = \boxed{0.47}$$

ANSWERS:

a) MAX. DECREASE IN STATIC PRESSURE: TYPE B

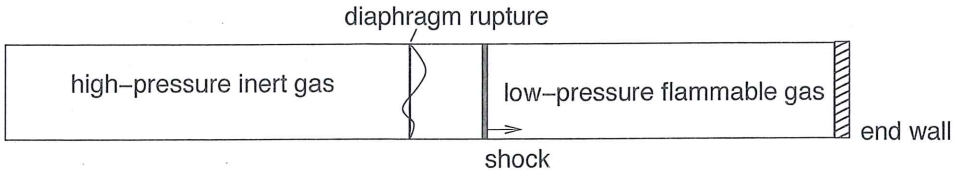
b) MAX. DECREASE IN STAGNATION PRESSURE: TYPE A

c) MAX. MACH NUMBER: TYPE B

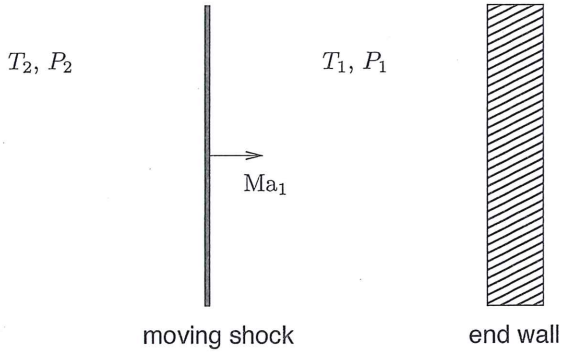
d) MIN ENTROPY INCREASE: TYPE B.

Problem 2 (10 pts)

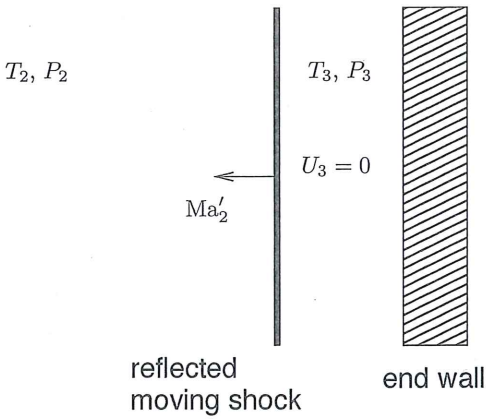
Shock tubes are typically used for studies of combustion chemical kinetics. In a shock tube, a diaphragm separates high pressure inert gas from a lower-pressure, flammable gas mixture at temperature $T_1 = 300\text{K}$ and pressure $P_1 = 0.1$ bar. At $t = 0$, the diaphragm is ruptured and the over-pressure created by the inert gas produces a shock wave that propagates at $\text{Ma}_1 = 2$ into the reacting gaseous mixture at rest. The reflection of the shock wave plays an important role in the ignition of the reactants. In the first approximation, however, neglect the combustion chemical processes in the shock tube.



a) Compute the pressure P_2 and temperature T_2 of the flammable gas after the shock has passed (see fig. below).



b) When the moving shock reaches the wall, it is reflected back towards the left leaving the gas at rest behind, $U_3 = 0$, in accordance with the non-penetration boundary condition at the wall. Calculate the Mach number of the reflected shock Ma_2', along with the pressure P_3 and temperature T_3 of the gas between the reflected shock and the end wall.



a) LAB FRAME WAVE FRAME

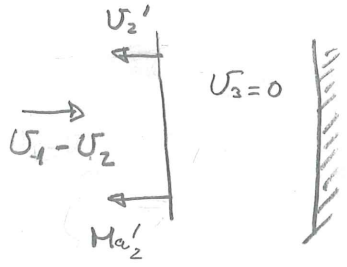


SINCE $Ma_1 = 2$, FROM TABLE II NORMAL SHOCK RELATIONS:

$$\frac{P_2}{P_1} = 4.5 \quad , \quad \frac{T_2}{T_1} = 1.7 \quad , \quad Ma_2 = 0.57.$$

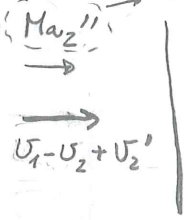
$$\Rightarrow \boxed{P_2 = 0.45 \text{ bar}} \quad \boxed{T_2 = 510 \text{ K}}$$

b) LAB FRAME



WAVE FRAME

MACH NORMAL SHOCK FOR JUMP RELATIONS



WITH: $Ma_2'' = \dfrac{U_1 - U_2 + U_2'}{a_2} =$

$$= Ma_1 \left(\frac{T_1}{T_2}\right)^{1/2} - Ma_2 + Ma_2'$$

THEREFORE: $$Ma_2'' = Ma_1 \left(\frac{T_1}{T_2}\right)^{1/2} - Ma_2 + Ma_2' \quad (1)$$ (REFLECTION + $U_3 = 0$)

AND FROM THE NORMAL-SHOCK RELATIONS:

$$Ma_3^2 = \frac{2 + (\gamma - 1) Ma_2''^2}{2\gamma Ma_2''^2 + 1 - \gamma} \quad (2)$$ (POST-SHOCK MACH) $Ma_3 = U_2'/a_3$

WITH $$Ma_3 = \frac{U_2'}{a_3} = \left(\frac{T_2}{T_3}\right)^{1/2} Ma_2' \quad (3)$$

AND $$\frac{T_3}{T_2} = \frac{[2\gamma Ma_2''^2 - (\gamma - 1)][2 + (\gamma - 1) Ma_2''^2]}{(\gamma + 1)^2 Ma_2''^2} \quad (4)$$

THE PROBLEM IS CLOSED BY SOLVING (1)-(4) WITH $Ma_1 = 2$, $\frac{T_2}{T_1} = 1.7$, AND $Ma_2 = 0.57$ FROM PART a)

ITERATION: (TABLE II OR EQS (2) AND (4); EQ (3))

$Ma_2' = 1.8$, $Ma_2'' = 2.76$, $Ma_3 = 0.49$, $\frac{T_3}{T_2} = 2.4 \Rightarrow Ma_2' = 0.76$ ⊗

$Ma_2' = 0.8$ $Ma_2'' = 1.80$ $Ma_3 = 0.61$, $\frac{T_3}{T_2} = 1.23 \Rightarrow Ma_2' = 0.67$ ⊗

$Ma_2' = 0.74$ $Ma_2'' = \underline{1.70}$ $Ma_3 = \underline{0.64}$ $\frac{T_3}{T_2} = \underline{1.46} \Rightarrow Ma_2' = 0.77$ ✓

$\Rightarrow T_3 = 1.46 \cdot 510 = \underline{744.6 \text{ K}}$, $P_3 = 3.2 \cdot 0.45 = \underline{1.44 \text{ bar}}$

ME 355: Compressible Flows, Spring 2016

Stanford University

Final Exam

Friday, June 3

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name: JAVIER VIEZAY **Student's ID:**.....

PART 1: Closed books, closed notes, calculators allowed
Time: 40 mins

Questions (30 pts)

1. Using the conservation equations in a control volume that includes an oblique shock front, state the continuity, momentum and energy jump constraints across the shock wave, indicating which variables (or group of variables) remain invariant and which ones are discontinuous. SEE PAGE 19 NOTES
2. Explain why a curved shock wave generates vorticity in an initially irrotational flow. SEE PAGE 64 NOTES
3. Explain the structure of a ZND detonation indicating its main regions in terms of pressure, temperature and velocity variations across the front, and sketch the approximate thermodynamic trajectory of a fluid particle in a pressure / specific-volume diagram. SEE PAGE 82 NOTES

ME 355: Compressible Flows, Spring 2016

Stanford University

Final Exam

Friday, June 3

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name: JAVIER UREAY **Student's ID:**.....

PART II: Open books, open notes, calculators allowed
Time: 120 mins

Problem 1 (40 pts)

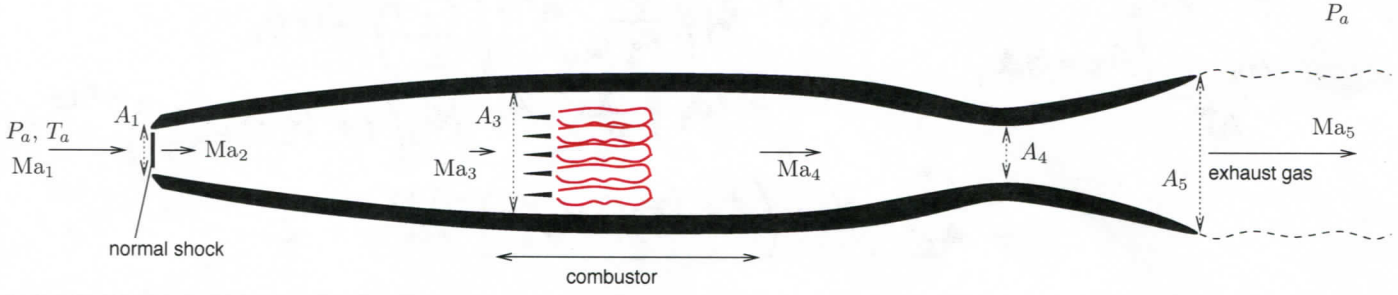
A simplified model of a RAMJET engine is depicted in the figure below. RAMJETS differ from SCRAMJETS in that combustion in the former occurs subsonically rather than supersonically. Supersonic air at $U_1 = 2$ km/s, $T_a = 270$ K and $P_a = 0.1$ bar is ingested through a nozzle of cross section $A_1 = 20$ cm^2 that creates a normal shock. The flow passes through a diffuser of area ratio $A_3/A_1 = 2$ before entering the combustion chamber. The heat release from combustion leads to a 20% increase in the stagnation temperature. The combustion products flow through a converging-diverging nozzle of area ratio $A_4/A_5 = 0.2$, with $A_4/A_3 = 0.5$.

a) The Mach number, pressure and temperature of the air entering the combustor.

b) The Mach number, pressure and temperature of the gas leaving the combustor.

c) The Mach number, pressure and temperature of the exhaust gases.

d) The specific impulse of the engine, $I_s = F/(\dot{m}g)$, where F is the engine thrust, $\dot{m}$ is the mass flow rate, and g is the gravitational acceleration.



a) NORMAL SHOCK (TABLE II)

$6.1 = Ma_1 = \frac{U_1}{\sqrt{\gamma R T_a}}$; P_a, T_a | Ma_2 ; $P_2\ T_2$

$\Rightarrow Ma_2 = 0.4$

$P_2/P_a = 43.2$, $T_2/T_a = 8.2 \rightarrow$ $P_2 = 4.3$ bar, $T_2 = 221.4$ K

TO ISENTROPIC RELATION TO OBTAIN: $T_{0_2} = T_2\left(1 + \left(\frac{\gamma-1}{2}\right)Ma_2^2\right) = 2284.5\,K = T_{03}$

2 → 3 ISENTROPIC DIFFUSER: $A_2 Ma_2 \left(1 + \frac{(\gamma-1)}{2}Ma_2^2\right)^{\frac{-\gamma+1}{2(\gamma-1)}} = A_3 Ma_3 \left(1 + \frac{(\gamma-1)}{2}Ma_3^2\right)^{\frac{-\gamma+1}{2(\gamma-1)}}$

FLOW WITH HEAT ADDITION: $A_2 = A_1$

$$\Rightarrow \frac{A_3}{A_2} = \frac{Ma_2\left(1+\left(\frac{\gamma-1}{2}\right)Ma_2^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}}{Ma_3\left(1+\left(\frac{\gamma-1}{2}\right)Ma_3^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}} = 2 \Rightarrow \text{OBTAIN SOLUTION } Ma_3 \ (Ma_3<1)$$

$$\frac{1}{0.628} = 1.592$$

$$\frac{A_3}{A^*} = \frac{A_3}{A_2}\left(\frac{A_2}{A^*}\right) = 2 \cdot 1.592 = 3.18 \Rightarrow \text{DOES NOT COME IN MY TABLE!}$$

ITERATE TO SOLVE FOR $Ma_3 \Rightarrow$ **$Ma_3 = 0.19$**

$$\Rightarrow \frac{P_3}{P_2} = \frac{\left(1+\left(\frac{\gamma-1}{2}\right)Ma_2^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\left(\frac{\gamma-1}{2}\right)Ma_3^2\right)^{\frac{\gamma}{\gamma-1}}} = 1.09 \Rightarrow \boxed{P_3 = 4.67 \text{ bar}}$$

$$\frac{T_3}{T_2} = \left(\frac{P_3}{P_2}\right)^{\frac{\gamma-1}{\gamma}} = 1.02 \Rightarrow \boxed{T_3 = 2268.5}$$

b) $T_{04} = 1.2\, T_{03} = 1.2\, T_{02} =$

$$\Rightarrow q = c_p(T_{04}-T_{03}) = c_p T_{03}\left(\frac{T_{04}}{T_{03}}-1\right) = \frac{c_p T_{03}}{5}$$

$$\Rightarrow Fe^2(Ma_4) = Fe^2(Ma_3)\left(1+\frac{q}{c_p T_{03}}\right) = 1.2\, Fe^2(Ma_3) = 0.0395$$

$Fe^2(Ma_3) = 0.0329$ FROM $Fe(Ma) = \frac{Ma}{1+\gamma Ma^2}\left[1+\left(\frac{\gamma-1}{2}\right)Ma^2\right]^{1/2}$

$\Downarrow$

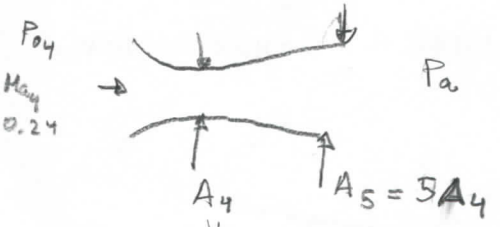
$Ma_4 \approx 0.24$

THEN $T_{04} = 2741\ K \Rightarrow$ **$T_4 = \frac{T_{04}}{1+\frac{\gamma-1}{2}Ma_4^2} = 2710\ K$**; $\frac{P_4}{P_3} = \frac{1+\gamma Ma_3^2}{1+\gamma Ma_4^2} = 0.972$

$\Downarrow$

$P_4 = 4.54$ bar

c) IN ORDER TO KNOW THE CONDITIONS AT THE EXIT WE NEED TO COMPUTE THE CHARACTERISTIC PRESSURES OF THE CD NOZZLE:



IF THE FLOW IS CHOKED,

$$\dot{m} = \dot{m}^* = P_{04}\sqrt{\frac{\gamma}{R_g T_{04}}}\,A^*\left/\left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}\right. =$$

$$= P_{04}\sqrt{\frac{\gamma}{R_g T_{04}}}\,A_5\, Ma_5\left(1+\frac{\gamma-1}{2}Ma_5^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

$$\Rightarrow \frac{A^*}{A_5} = Ma_5\left(1+\frac{\gamma-1}{2}Ma_5^2\right)^{-\frac{\gamma+1}{2(\gamma-1)}} = \frac{1}{5}$$

- $\Rightarrow Ma_5^{SUP} = 3.17$ TABLES
- $\Rightarrow Ma_5^{SUB} = 0.21$ ITERATION

$\Rightarrow$ PRESSURE FOR SONIC THROAT (STILL SUBSONIC FLOW AT EXIT): $\boxed{P_C = P_{04}\left(1+\frac{\gamma-1}{2}Ma_5^{2\,SUB}\right)^{-\frac{\gamma}{\gamma-1}} = \left(\frac{1+\frac{\gamma-1}{2}Ma_4^2}{1+\frac{\gamma-1}{2}Ma_5^2}\right)^{\frac{\gamma}{\gamma-1}} P_4 = 1.01\, P_4 = 4.5 \text{ bar}}$

$\Rightarrow$ PRESSURE FOR SONIC THROAT AND SUPERSONIC FLOW AT EXIT: $\boxed{P_{0E} = \frac{\left(1+\frac{\gamma-1}{2}Ma_4^2\right)^{\frac{\gamma}{\gamma-1}}}{\left(1+\frac{\gamma-1}{2}Ma_5^{2\,SUP}\right)^{\frac{\gamma}{\gamma-1}}} P_4 = 0.022\, P_4 = 0.1 \text{ bar}}$

SINCE $P_{0E} \approx P_a$, THE CD NOZZLE IS OPTIMALLY EXPANDED,

WITH $Ma_5 = \boxed{3.17}$

$$P_5 = P_{0E} = P_a = \boxed{0.1 \text{ bar}}$$

$$T_5 = \frac{T_{04}}{1 + \frac{\gamma-1}{2} Ma_5^2} = \boxed{910.7 \text{ K}}$$

$$\Big\} \quad U_5 = Ma_5 \sqrt{\gamma R_g T_5} = 1.91 \text{ km/s}$$

d) MASS FLOW RATE:

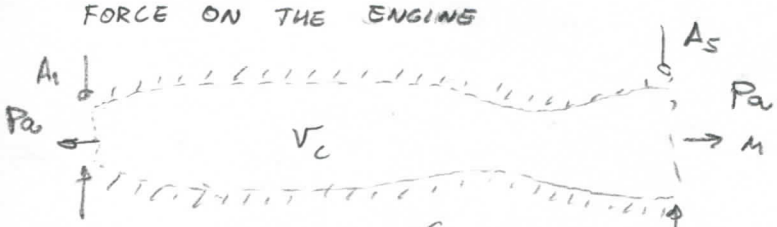
$$\dot{m} = \dot{m}^* = P_{04} \sqrt{\frac{\gamma}{R_g T_{04}}} A^* \left(\frac{1+\gamma}{2}\right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

WITH $P_{04} = P_4 \left(1 + \frac{\gamma-1}{2} Ma_4^2\right)^{\frac{\gamma}{\gamma-1}} = 4.72 \text{ bar}$

$$A^* = \frac{A^*}{A_3} \cdot \frac{A_3}{A_2} A_1 = 0.5 \cdot 2 A_1 = A_1 = 20 \text{ cm}^2 \qquad (\overset{\text{NOTE}}{A^* = A_4})$$

SUBSTITUTING ABOVE: $\dot{m} = \underline{0.73 \text{ kg/s}}$

FORCE ON THE ENGINE



$$A_5 = \frac{A_5}{A_4} \cdot \frac{A_4}{A_3} \frac{A_3}{A_1} A_1 = 5 \cdot 0.5 \cdot 2 A_1 = 5 A_1$$

$$\frac{d}{dt} \int_{V_c} \rho \vec{v} \, dV + \int_{S_c} \rho \vec{v} \vec{v} \, d\vec{s} = - \int_{S_c} P \, d\vec{s}$$

$$-\rho_1 v_1^2 A_1 \vec{e}_x + \rho_5 v_5^2 A_5 \vec{e}_x = - \int_{S_{LATERAL}} P \, d\vec{s} = -\vec{F}$$

$$\vec{F} = -\dot{m} (v_5 - v_1) \vec{e}_x = \underline{65.7 \text{ N}} \Rightarrow \text{IN THE + x DIRECTION!}$$

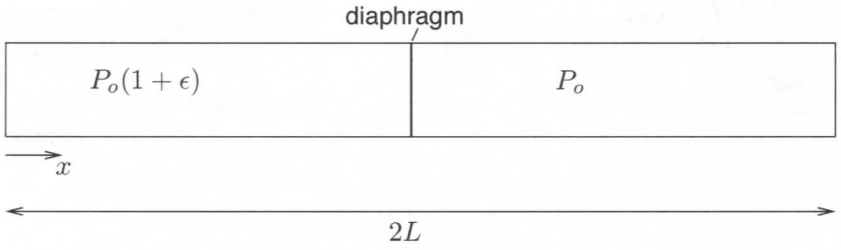
THE NORMAL SHOCK AT THE ENTRANCE PRODUCES MORE DRAG THAN THE THRUST THE ENGINE CAN PRODUCE

$$I_S = \frac{F_S}{\dot{m} g} = \underline{8.59 \text{ s}}$$

USELESS NUMBER SINCE THERE IS NO THRUST FORWARD!

Problem 2 (30 pts)

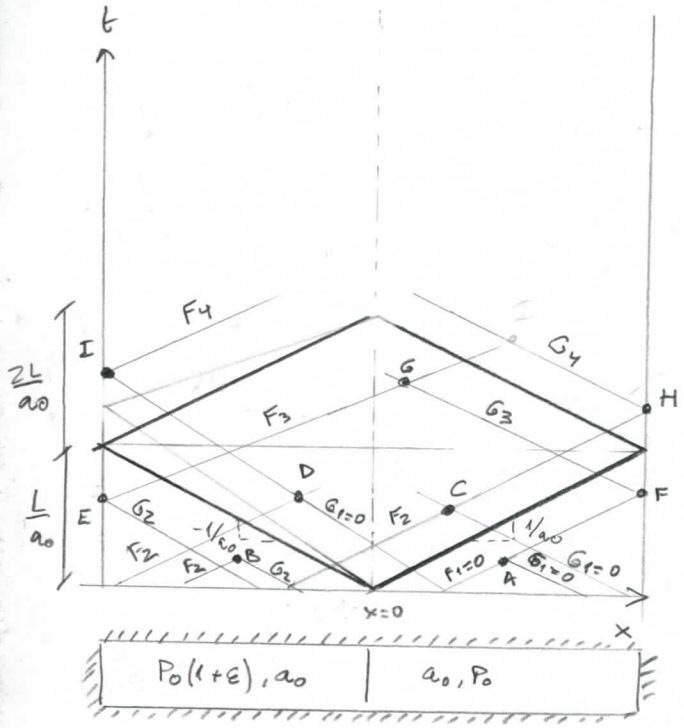
A closed duct of length $2L$ is initially divided into two equal cavities filled with the same gas at the same temperature T_o and separated by a diaphragm. Initially, the pressure of the gas in the right cavity is P_o, while the pressure of the gas in the left cavity is $P_o(1+\epsilon)$, with $\epsilon \ll 1$. At $t = 0$ the diaphragm is removed and acoustic waves propagate into both cavities. Draw the velocity and pressure profiles in the duct as a function of x for a) $t = L/(2a_0)$, b) $t = L/a_0$, and c) $t = 2L/a_0$, where a_0 is the speed of sound. Justify your sketches by appropriately computing the flow variables using the acoustics theory.



SINCE $\epsilon \ll 1$ ⇒ USE LINEAR THEORY.

$$F = \frac{1}{2}\left(\frac{u'}{a_0} + \frac{p'}{\rho_0 a_0^2}\right) \qquad G = \frac{1}{2}\left(\frac{p'}{\rho_0 a_0^2} - \frac{u'}{a_0}\right)$$

$$\frac{p'}{\rho_0 a_0^2} = F + G \qquad \frac{u'}{a_0} = F - G$$

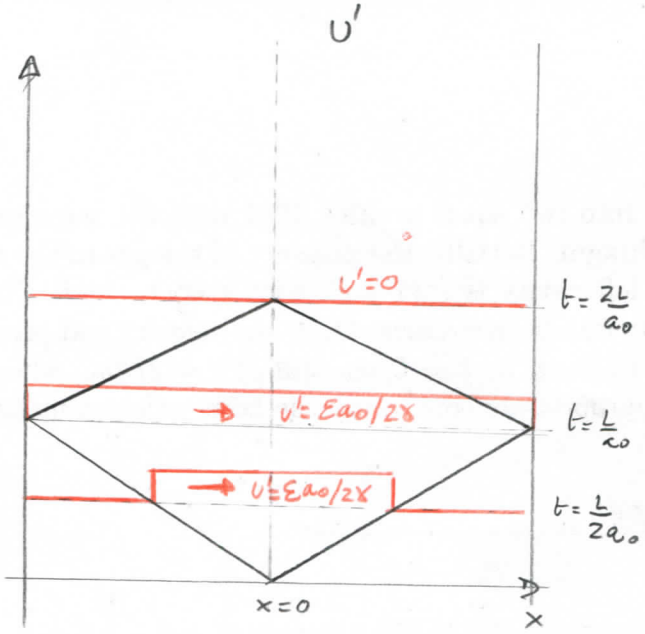


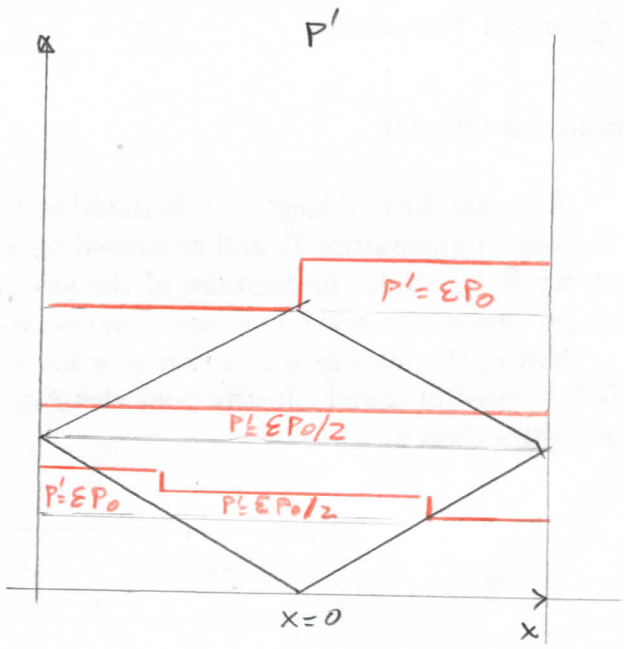
- AT POINT A: $F_1 = G_1 = 0 \Rightarrow u'_A = p'_A = 0$
- AT POINT B: $F_2 = \frac{1}{2}\frac{\epsilon P_o}{\rho_0 a_0^2} = \frac{\epsilon}{2\gamma}$, $G_2 = \frac{1}{2}\frac{\epsilon P_o}{\rho_0 a_0^2} = \frac{\epsilon}{2\gamma}$ ⇒ $u'_B = 0$, $p'_B = \epsilon P_o$

THE GAS BELOW THE TWO CHARACTERISTICS EMERGING FROM $x=0, t=0$ HAS NOT NOTICED THE RUPTURE OF THE DIAPHRAGM

- AT POINT C: $F_2 = \frac{\epsilon}{2\gamma}$, $G_1 = 0$ ⇒ $\frac{u'_C}{a_0} = F_2 - G_1 = \frac{\epsilon}{2\gamma}$; $\frac{p'_C}{\rho_0 a_0^2} = F_2 + G_1 = \frac{\epsilon}{2\gamma} \Rightarrow p'_C = \frac{\epsilon P_o}{2}$
- AT POINT D: SAME AS C.
- AT POINT E: $G_2 = \frac{\epsilon}{2\gamma}$, $u'_E = 0$ ⇒ $F_3 = G_2 = \frac{\epsilon}{2\gamma}$; $\frac{p'_E}{\rho_0 a_0^2} = F_3 + G_3 = \frac{\epsilon}{\gamma} \Rightarrow p'_E = \epsilon P_o$
- AT POINT F: $F_1 = 0$, $u'_F = 0$ ⇒ $G_3 = F_1 = 0$, $p'_F = 0$
- AT POINT G: $F_3 = \frac{\epsilon}{2\gamma}$, $G_3 = 0$ ⇒ $\frac{u'_G}{a_0} = \frac{\epsilon}{2\gamma}$; $\frac{p'_G}{\rho_0 a_0^2} = \frac{\epsilon}{2\gamma} \Rightarrow p'_G = \frac{\epsilon P_o}{2}$
- AT POINT H: $F_2 = \frac{\epsilon}{2\gamma}$, $u'_H = 0$ ⇒ $G_4 = F_2 = \epsilon/2\gamma$; $\frac{p'_H}{\rho_0 a_0^2} = F_2 + G_4 = \frac{\epsilon}{\gamma} \Rightarrow p'_H = \epsilon P_o$
- AT POINT I: $G_1 = 0$, $u'_I = 0$ ⇒ $F_4 = G_1 = 0$, $p'_I = 0$

DRAWINGS →


U'
U'=0
t = 2L/a₀
U'= Ea₀/2ε
t = L/a₀
U'=Ea₀/2ε
t = L/2a₀
x=0
x


P'
P'= εP₀
P'= εP₀/2
P'= εP₀
P'= εP₀/2
x=0
x

Supplementary slides

ME 355:
COMPRESSIBLE FLOWS

Stanford University
Spring 2016



Javier Urzay, PhD.



Captain Chuck Yeager



Bell X-1

Breaking of the sound barrier (1947)

Link to the Breaking of the Sound Barrier video

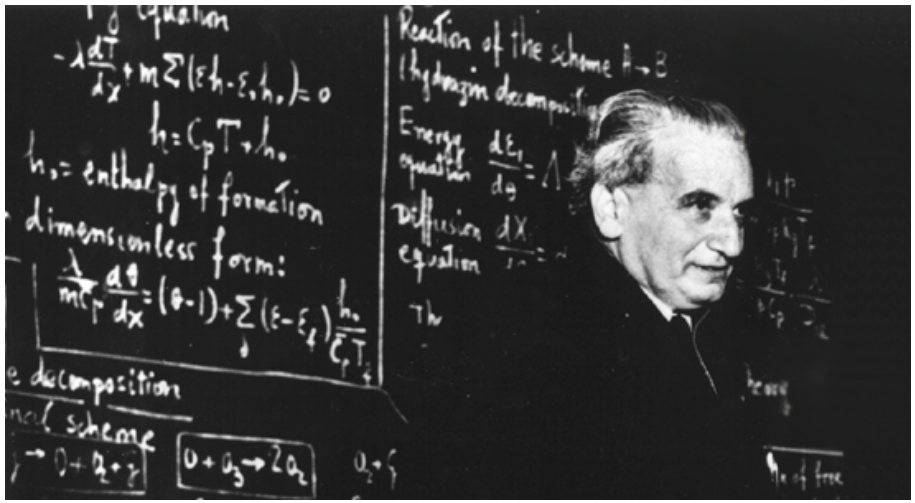
https://youtu.be/eQJha-aeP1o

HISTORY OF RESEARCH IN COMPRESSIBILITY EFFECTS IN AERODYNAMICS

"We (Dryden and I) called the speed range just below and just above the sonic speed - Mach number nearly equal to 1 - the transonic range.

I introduced the term in this form in a report to the Air Force. I am not sure whether the general who read it knew what it meant. . . . ☺

I will remember this period (about 1941) when designers were rather frantic because of the unexpected difficulties of transonic flight.
They thought the troubles indicated a failure in aerodynamic theory."



Theodore von Kármán,
Aerodynamics (Cornell Univ.Press, 1954).

I. SCIENCE

THE SPEED OF SOUND

Isaac Newton

Calculation of the speed of sound considering isothermal wave propagation (1687).



Pierre Laplace

Calculation of the speed of sound considering adiabatic wave propagation (1816).



SHOCKS

Bernhard Riemann

Calculation of isentropic shocks (1858) and sound waves.



William Rankine and Pierre Hugoniot

Flow across a normal shock (1870).

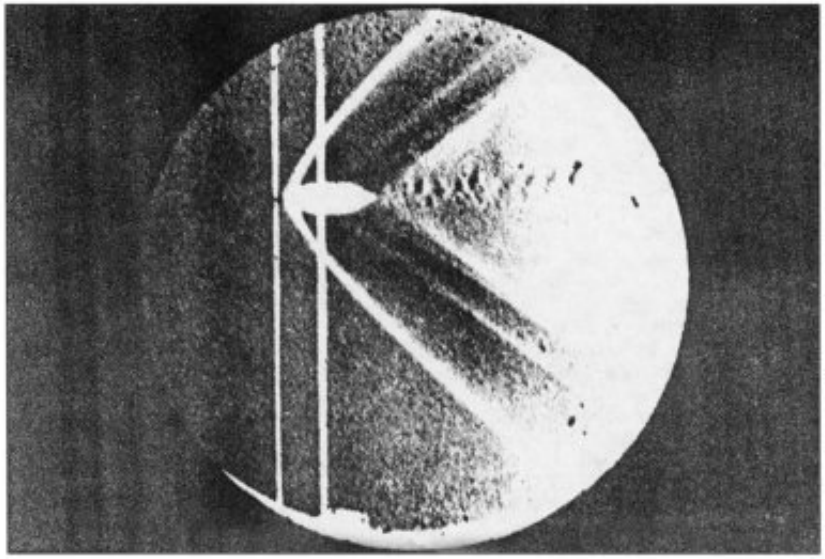




SHOCKS

Ernst Mach

First experimental observation of shock waves on flying bullets using shadowgraph (1887)



Ludwig Prandtl and Theodore Meyer

Flow across oblique shocks and expansion fans (1908).

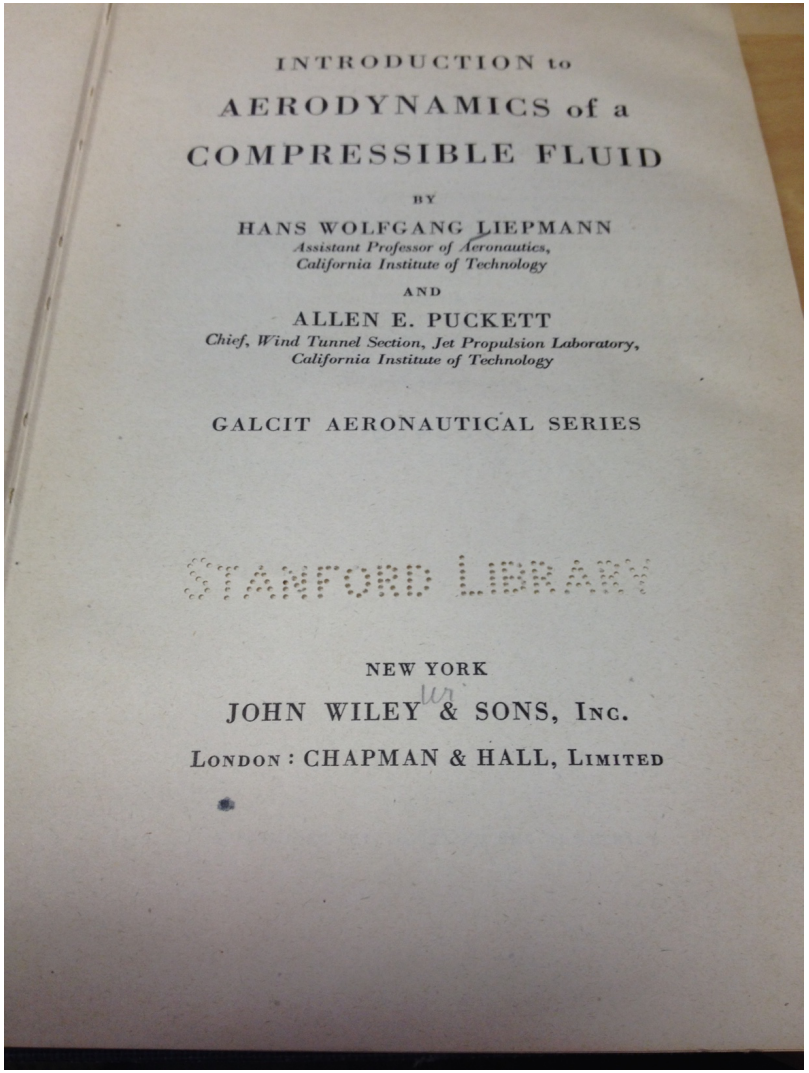


Hans Liepmann and Allen E. Puckett

First textbook on compressible flows (1947)

"The resulting treatment is believed to be as up to date as possible in a subject which is developing as rapidly as compressible fluid hydrodynamics, and in which military classification often remains in effect."

In Preface by von Kármán & Millikan


INTRODUCTION to
AERODYNAMICS of a
COMPRESSIBLE FLUID

BY

HANS WOLFGANG LIEPMANN
Assistant Professor of Aeronautics,
California Institute of Technology

AND

ALLEN E. PUCKETT
Chief, Wind Tunnel Section, Jet Propulsion Laboratory,
California Institute of Technology

GALCIT AERONAUTICAL SERIES

STANFORD LIBRARY

NEW YORK
JOHN WILEY & SONS, INC.
LONDON: CHAPMAN & HALL, LIMITED

+Many others

Landau, Zel'dovich, Taylor, Shapiro, Lighthill, Witham, Crocco, Ferri…

II. ENGINEERING SCIENCE

FLOW AROUND PROPELLERS IN WWII AIRCRAFTS



Subsonic Flow

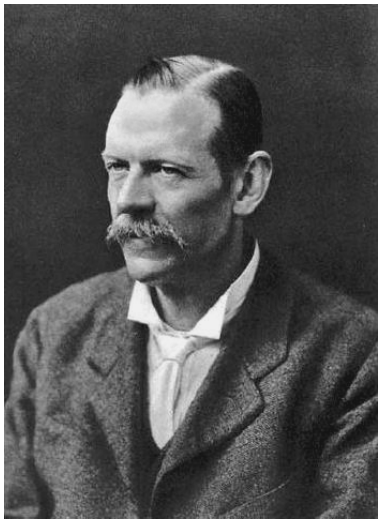
Much higher speed flow relative to the propeller tip

RAF Spitfire

HIGH SPEED FLOW AROUND PROPELLERS

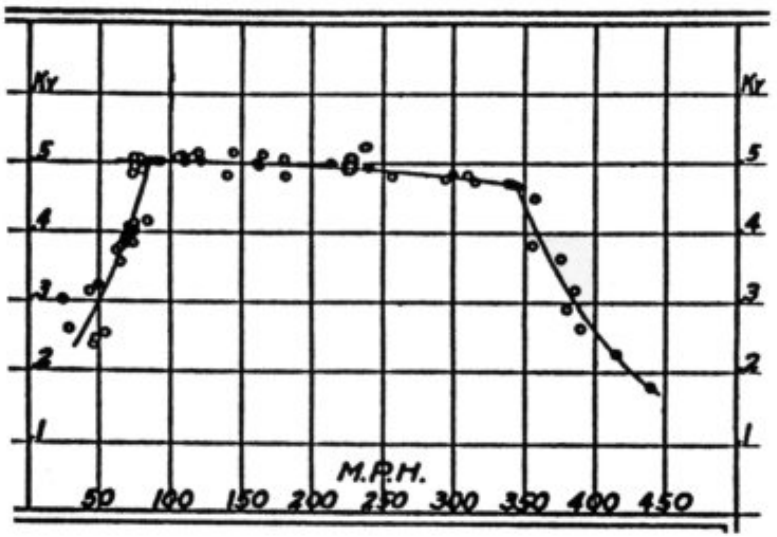
George Bryan

Theoretical analyses of subsonic and supersonic flows over cylinders (1919)



Frank Caldwell and Elisha Fates

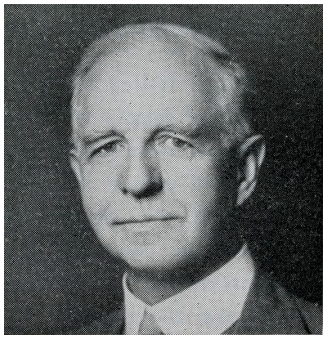
Experiments of high-speed flow effects on propeller blades (1920).



Hugh Dryden and Lyman Briggs

Pressure measurements on high-speed airfoils (1926)





Lift decrease

AIRFOIL 3R20

LIFT COEFFICIENTS c_L

V/c	Angle of attack											
	−4	−2	0	2	4	6	8	10	12	14	16	20
0. 50	0. 275	0. 309	0. 363	0. 399	0. 442	0. 489	0. 534	0. 564	0. 624	0. 687	0. 749	0. 635
. 65	. 309	. 313	. 343	. 388	. 444	. 481	. 523	. 417	. 443	. 503	. 566	. 811 . 612
. 80	. 147	. 103	. 168	. 230	. 292	. 327	. 373	. 536 . 411	. 462	. 518	. 552	. 627
. 95	−. 026	. 010	. 076	. 160	. 220	. 269	. 297	. 344	. 386	. 428	. 464	. 547
1. 08	--------	−. 057	. 000	. 059	. 121	. 179	. 233	. 260	. 297	. 452 . 332	. 382	. 464

Drag increase

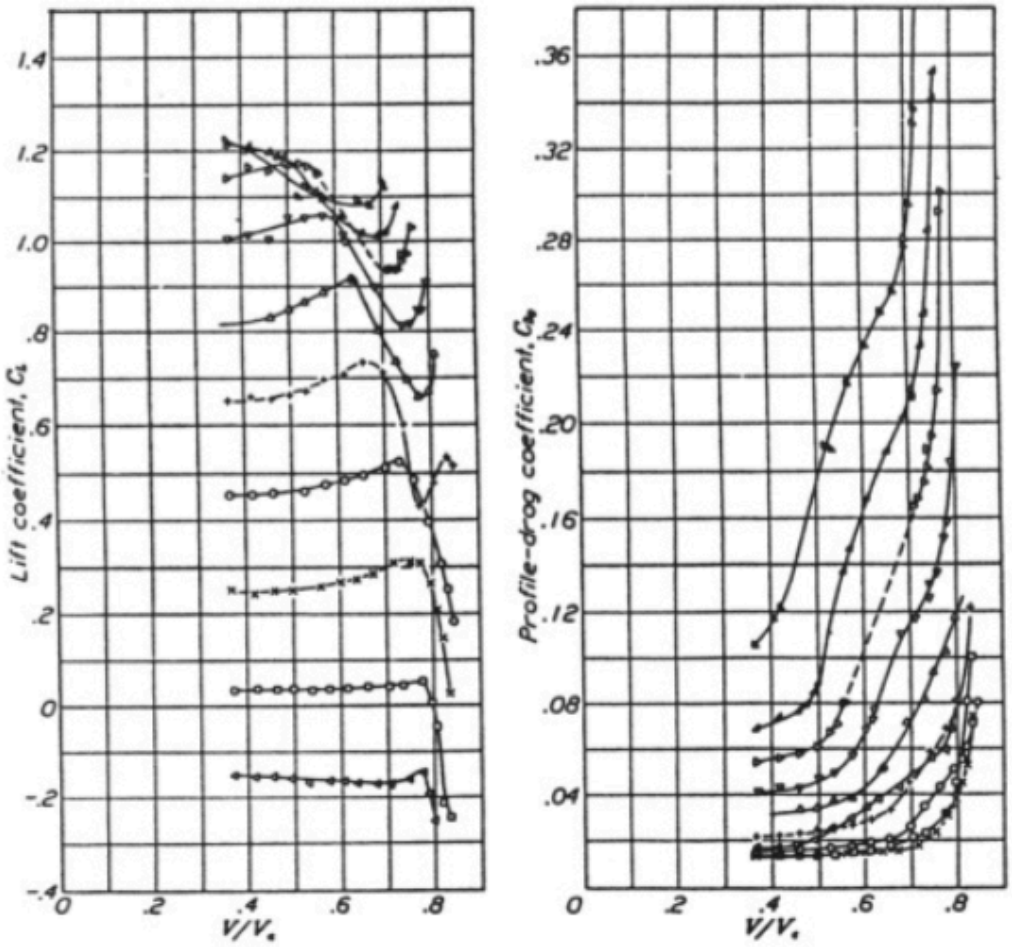
DRAG COEFFICIENTS c_D

0. 50	0. 097	0. 083	0. 078	0. 076	0. 080	0. 088	0. 099	0. 112	0. 129	0. 145	0. 164	0. 271
. 65	. 109	. 092	. 080	. 078	. 081	. 089	. 099	. 148	. 173	. 200	. 223	. 279
. 80	. 118	. 115	. 115	. 120	. 131	. 142	. 157	. 174	. 192	. 216	. 243	. 297
. 95	. 172	. 167	. 163	. 165	. 167	. 177	. 188	. 199	. 219	. 240	. 265	. 316
1. 08	--------	. 183	. 177	. 175	. 177	. 181	. 190	. 205	. 219	. 236	. 257	. 300

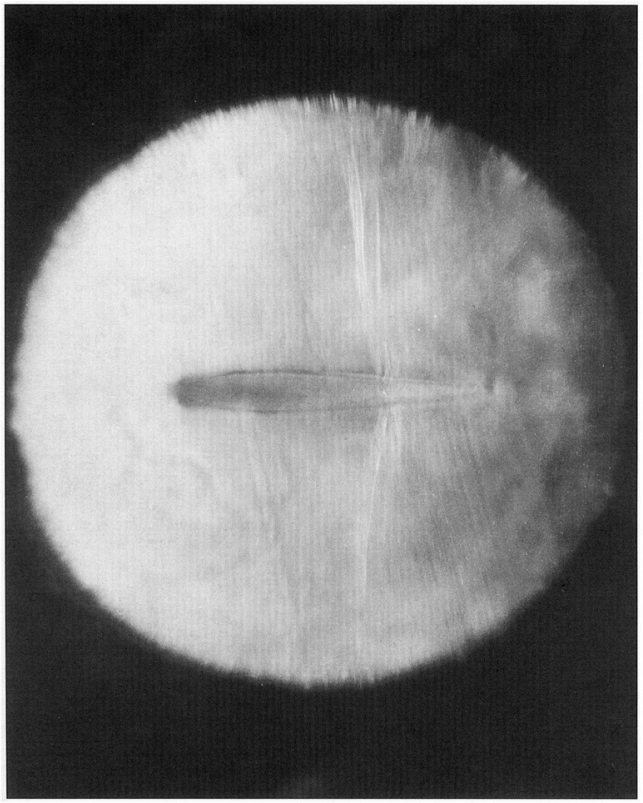
TRANSONIC FLOWS ON AIRFOILS

John Stack

Pressure measurements on high-speed propeller sections and airfoils (1934)







TRANSONIC FLOWS ON AIRFOILS

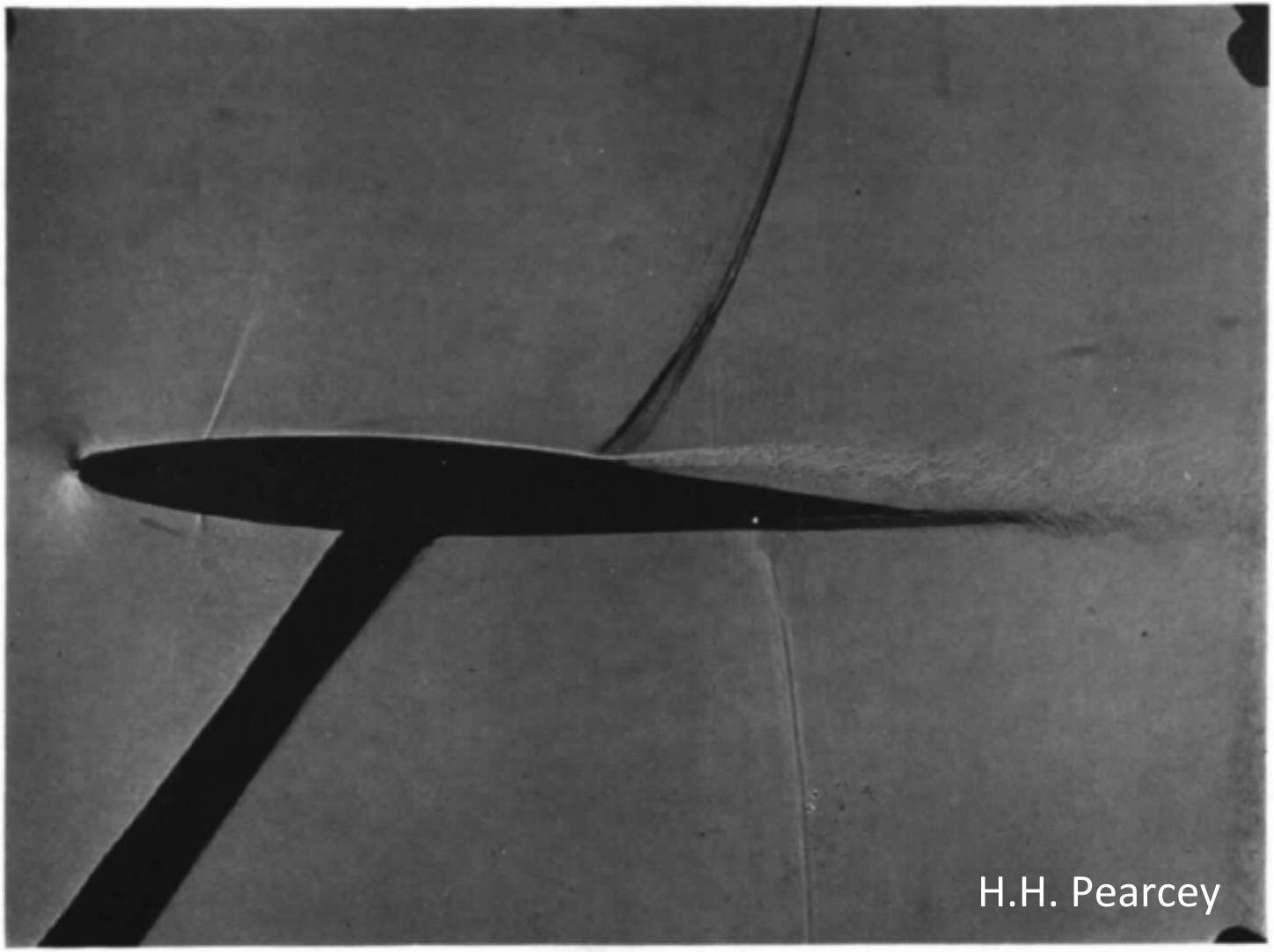
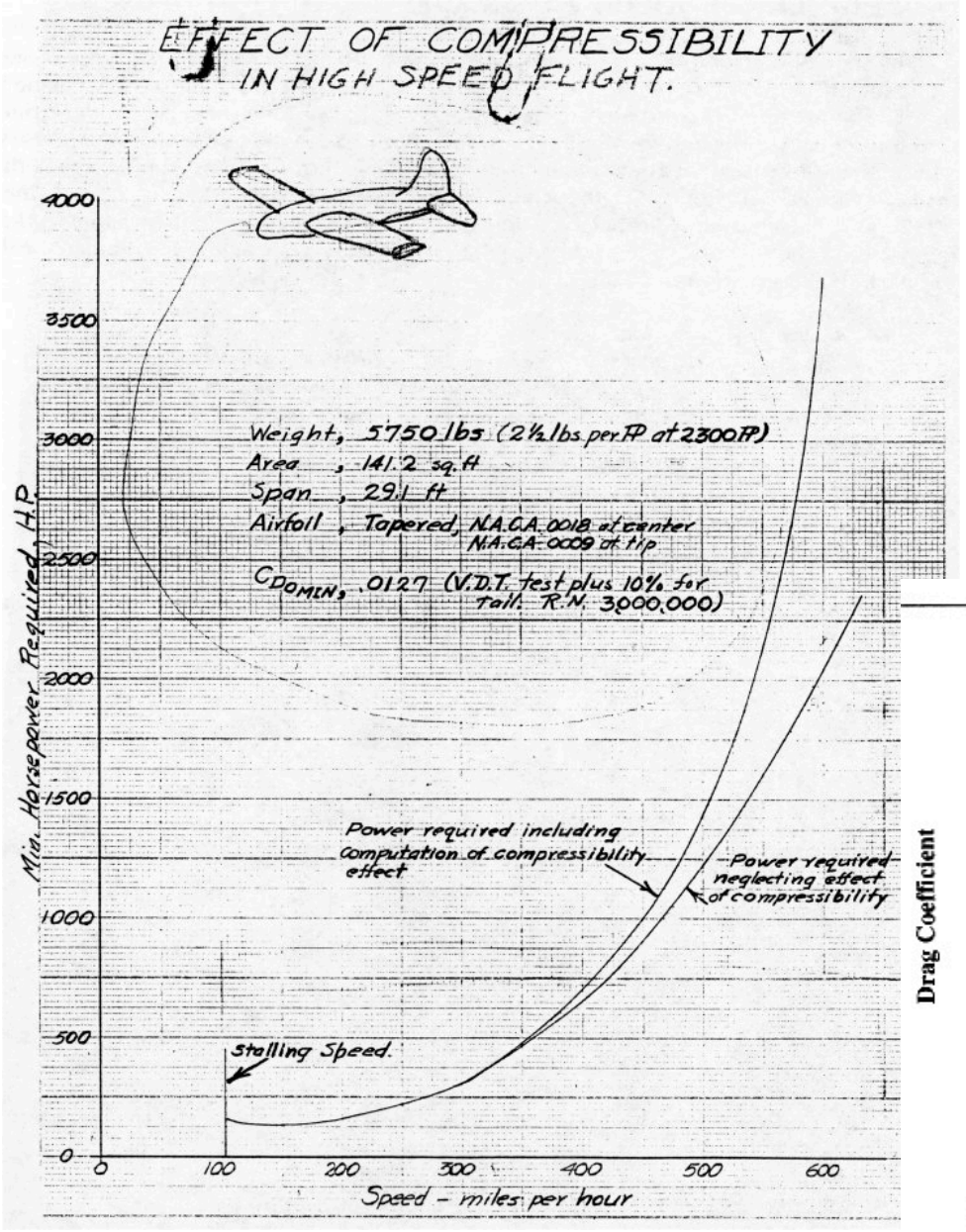


FIG. 2. Schlieren photograph of the flow past an aerofoil with severe separation at the foot of the upper-surface shock; 10 per cent RAE 102 section at 2-deg incidence ($M_0 = 0{\cdot}88$).

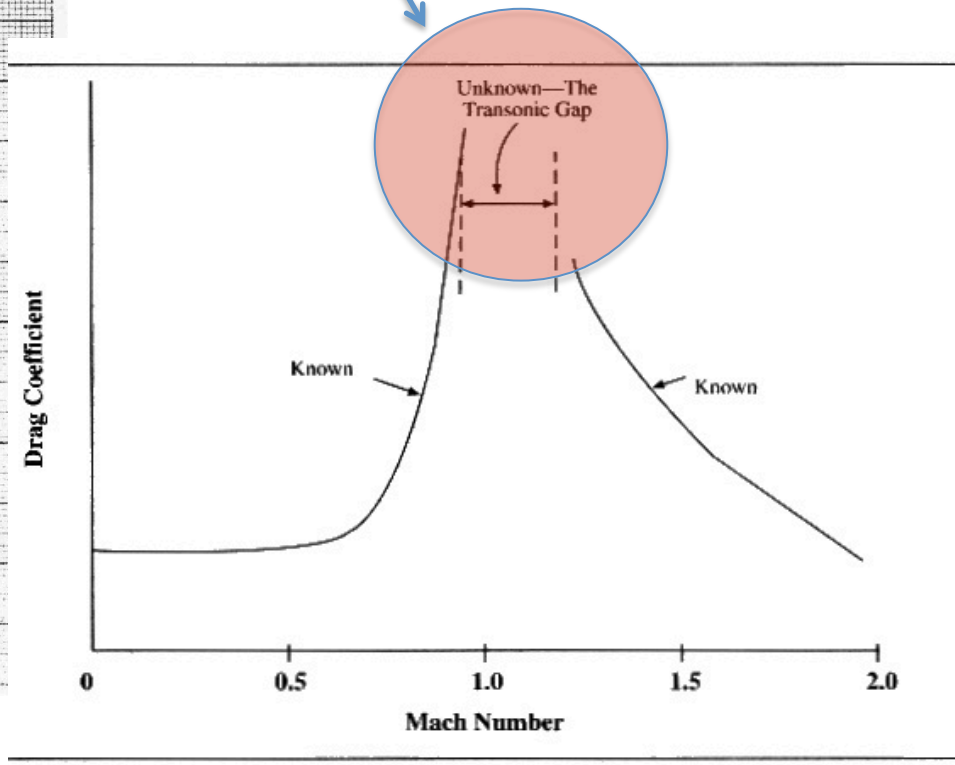
THE TRANSONIC PROBLEM



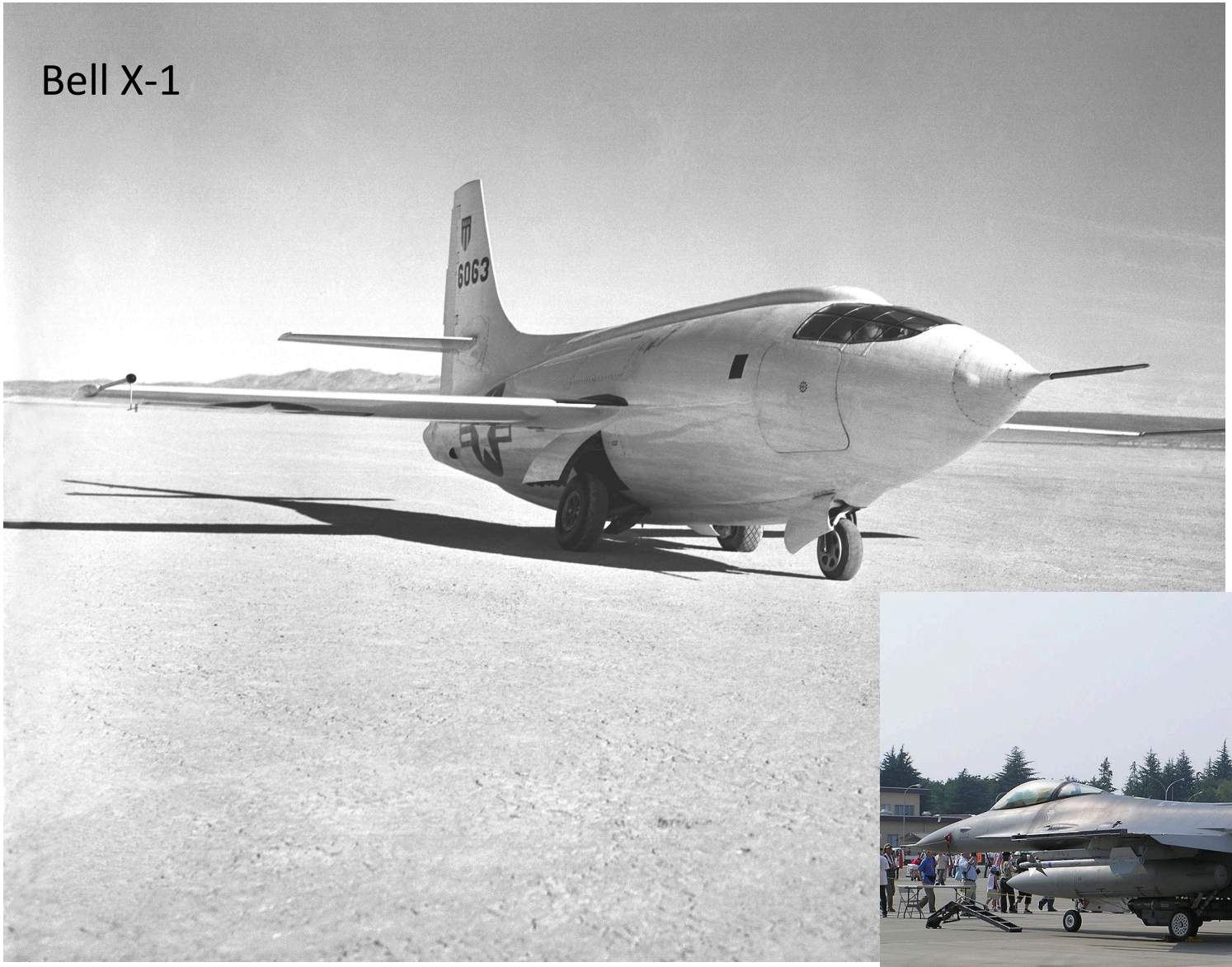
Skecth by J. Stack. NACA (1933)

"Mach tuck"/ "Compressibility burble":

- Lift decrease,
- Drag increase,
- Center of pressure moves back toward the trailing edge
- Hinged control surfaces become locked



Bell X-1



Stabilator or flying tail



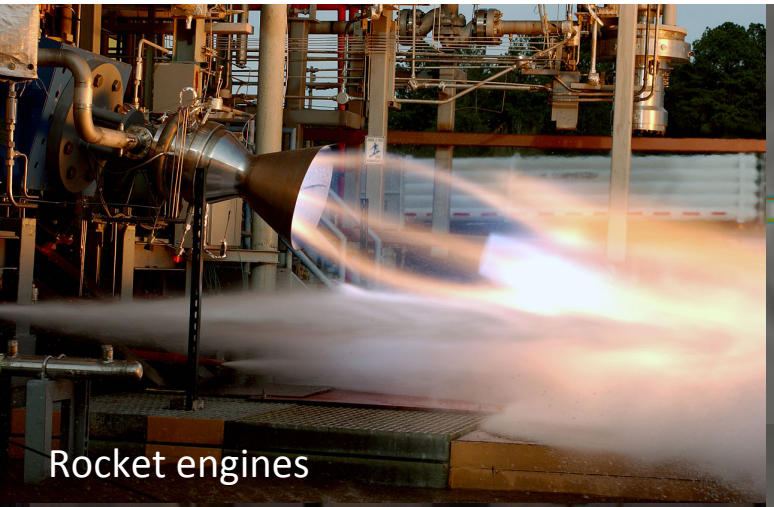
F-16

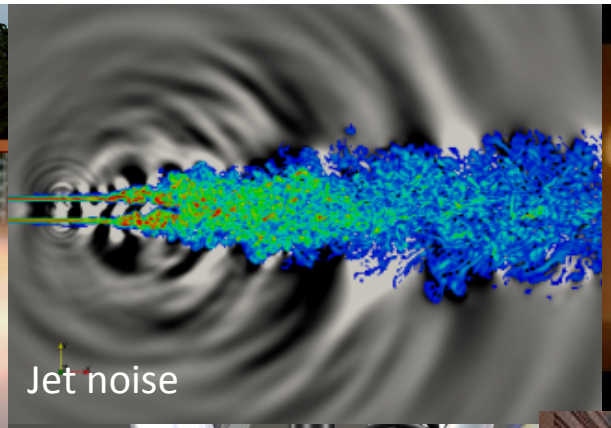

AMERICAN
X-PLANES
TOP SECRET

Link to X-planes video

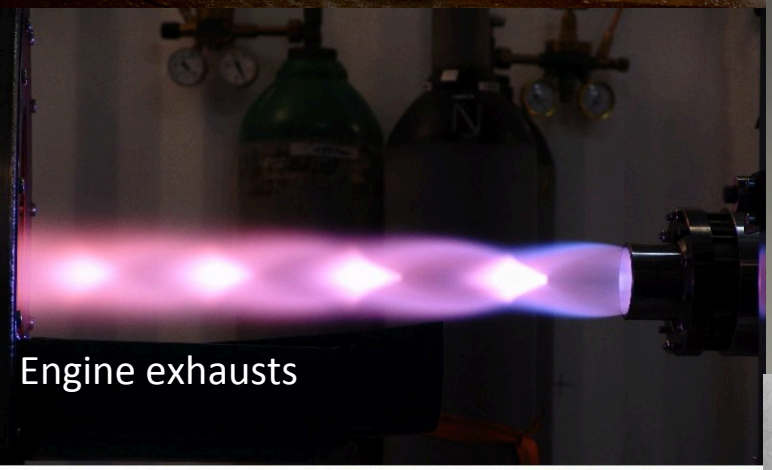
https://youtu.be/jCz1U1o5L5U

OTHER APPLICATIONS OF COMPRESSIBLE FLOWS


Rocket engines

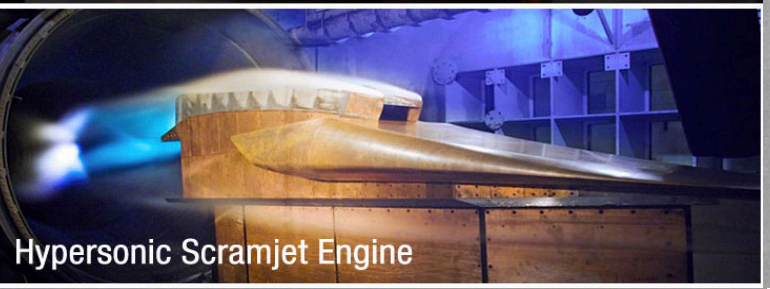

Jet noise

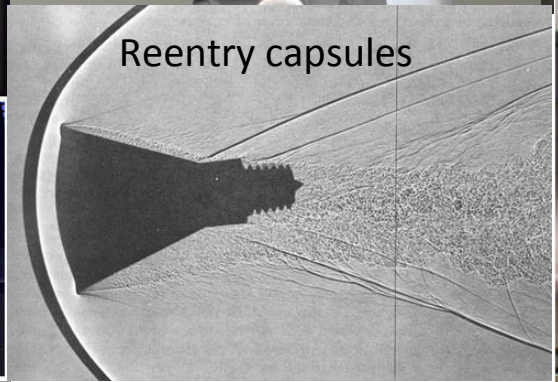

Weaponry


Engine exhausts


Internal combustion engines




Hypersonic Scramjet Engine


Reentry capsules



Engine Intakes

ME 355:
COMPRESSIBLE FLOWS

1. Conservation Equations
2. Shocks and Expansion Waves
3. Compressible Flows in Ducts and Nozzles
4. Wave Motion in Compressible Flows
5. Two-Dimensional Theory of Compressible Flows
6. Hypersonic Aerodynamics

ME 355:
COMPRESSIBLE FLOWS

Instructor: Javier Urzay, Ph.D.
E-mail: jurzay@stanford.edu
Office: 206 CTR building,
Office Hours: Mondays 4:00 PM-6:00 PM.

Lectures: Tuesdays and Thursdays, 1:30 PM-2:50 PM at room 380-381T.

Homeworks: There will be 4 homework assignments.

Exams: Midterm Exam: Tuesday, May 10 (TBA), in class.
Final Exam: (TBA).

Grading Scheme: 30% Homeworks + 30% Midterm Exam + 40% Final Exam.

Website: http://www.stanford.edu/~jurzay/ME_355

ME 355:
COMPRESSIBLE FLOWS

Reference Textbooks (not required):

- M.A. Saad, *"Compressible Flow"*, Prentice Hall, 1993.

- A.H. Shapiro, *"The Dynamics and Thermodynamics of Compressible Fluid Flow"*, Wiley, 1953.

- H.W. Liepmann & A. Roshko, *"Elements of Gas Dynamics"*, Dover, 1957.

- J.D. Anderson, *"Hypersonic and High-Temperature Gas Dynamics"*, AIAA, 2006.

- Y.B. Zel'dovich & Y.P. Raizer, "*Physics of Shock Waves and High-Temperature Hydrodynamic Phenomena"*, Dover, 2002.

- L.D. Landau & E.M. Lifshitz, *"Fluid Mechanics"*, BH, 1959.

ME 355:
COMPRESSIBLE FLOWS

Other Textbooks of relevance for this topic:

- E.H. Hirschel & C.W. Weiland, *"Selected Aerothermodynamic Design Problems of Hypersonic Vehicles"*, Springer, 2009. (applied hypersonics)

- W.G. Vincenti & C.H. Kruger, *"Introduction to Physical Gas Dynamics"*, Krieger 1965. (kinetic theory of gases)

- Y.B. Zel'dovich & A.S. Kompaneets, *"Theory of Detonation"*, Academic Press, 1960. (detonations)

- G.S. Sutton & O. Bliblarz, "Rocket Propulsion Elements", Wiley, 2010. (rocket nozzles)

- M. van Dyke, "Perturbation Methods in Fluid Mechanics", Parabolic, 1975. (two-dimensional aerodynamic theory)

- J.D. Anderson, *"Computational Fluid Dynamics"*, McGraw-Hill, 1995. (numerical methods for compressible flows)

- R.J. le Veque, "Numerical Methods for Conservation Laws", Birkhauser, 1992. (numerical methods for hyperbolic equations)