ME 356: Hypersonic Aerothermodynamics, Spring 2019
Stanford University
Homework 1: Hypersonic Inviscid Flows (I)
Due Tuesday, April 23, in class.

Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. In the calculations, assume a calorically perfect gas unless stated otherwise.

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Questions (40 pts)

1. Consider a normal shock in a Mach-10 wind tunnel operating with air at temperature $T_1 = 80$ K and pressure $P_1 = 1$ bar in the test section. Calculate the stagnation enthalpy $h_0$, the ratio of the post-shock temperature $T_2$ to the stagnation temperature $T_0$, and the ratio of the post-shock pressure $P_2$ to twice the upstream dynamic pressure $\rho_1 U_1^2$. Compare the value of $h_0$ to the value of typical stagnation enthalpies involved in the re-entry of the Shuttle Orbiter from LEO.

2. Sketch the hypersonic flow over a slender two-dimensional wedge of semi-angle $\delta = 35^\circ$ at a free-stream Mach number $M_a = 8$ in the stratosphere. Briefly describe the main characteristics of the flow and outline possible thermochemical effects that might develop in this case as a result of the high temperatures prevailing in the shock layer.
Problem 1 (60 pts)

Reusable first-stage boosters have recently become relevant assets for decreasing the cost and increasing the operational lifetime of space launch systems. Panel (a) in the figure below provides an approximate velocity-altitude trajectory of the reusable first stage of the Falcon-9 rocket that can be described as follows. The trajectory begins with an ascent of the first and second stages up to the separation point at an approximate altitude of 90 km, where the first stage is jettisoned and the second stage continues in an ascending trajectory to orbit. The first stage is then reoriented for re-entry using cold-gas thrusters, and its main rocket engines perform a boostback burn to decrease the speed. The boostback burn is the first of the three retrofirings performed during the landing sequence. A supplementary reorientation maneuver is then commanded to the cold-gas thrusters while grid fins are deployed for aerodynamic guidance [see panel (b)]. Grid fins are control surfaces used for steering and stabilizing the first stage, and they have been traditionally used in high-speed aerodynamics as stabilizers for control of thermobaric weapons such as the GBU-43/B [see panel (c)]. At 4,500 km/h and 55 km of altitude, an entry burn decreases the speed by half during 20 km, whereas the landing burn at approximately 10 km altitude brings down the first stage safely to the ground.
The remainder of this exercise will be focused on point “N” of the descent trajectory in panel 1(a), which corresponds approximately to an altitude of \( z = 60 \) km above sea level while descending at a velocity \( U_{\infty} = 4,400 \) km/h with a negligible angle of attack. Assume that the length and width of the first stage are \( L = 45 \) m and \( D = 3.7 \) m, respectively, and that the grid fins are located at \( L_f = 3 \) m from the top of the first stage.

a) Calculate the Mach number \( M_a_{\infty} \) corresponding to the relative motion of the air around the first stage at the aforementioned conditions of velocity and altitude.

b) For simplicity, assume that the bottom side of the first stage resembles a planar wedge of semiangle \( \alpha = 30^\circ \) and depth \( D \), as shown in panel (d). Calculate the shock-incidence angle \( \beta \), along with the Mach number \( M_a_2 \) and the pressure \( P_2 \) downstream of the oblique shock emanating from the geometry.

c) Calculate the turning angle through the expansion fans emanating from the junction between the wedge and the fairing, along with the Mach number \( M_a_3 \) and pressure \( P_3 \) downstream.

d) The high-speed flow through the grid fins is complex and beyond the scope of this exercise. For further details, the reader is referred to the NATO report RTO-MP-AVT-135 (2006) available for download at this repository. For simplicity, assume that the first stage deploys two grid fins of side length \( L_g = 1.5 \) m normal to the flow, and their leading edges resemble planar wedges of width \( w = 5 \) cm and semiangle \( \theta/2 = 20^\circ \), which create oblique shocks as sketched in panel (e). Calculate the post-shock pressure \( P_4 \) acting on the leading edges of the grid fins.

*extra credit:*

e) Using the results obtained from parts (a), (b), (c), and (d), determine the maximum height \( L_{cg} \) of the center of gravity such that the first stage remains aerodynamically stable to yaw rotation during flight at point “N” of the velocity-altitude trajectory (to solve this question you will have to assume a small angle of attack).

f) Redo part (e) using the standard Newtonian theory and compare the results. In doing so, assume that the incident Mach number upstream of the grid fins corresponds to the value of \( P_3 \) calculated in part (e).
**Solutions to Questions 1 and 2**

1. **Normal Shock**

   - $T_1 = 800K$
   - $M_a = 10$
   - $P_1 = 1 \text{ bar}$

   **Stagnation Enthalpy**
   \[ h_0 = C_p T_0 = C_p T \left(1 + \left(\frac{\gamma-1}{2}\right) M_a^2 \right) = 1680 \text{ K} \]

   **Post-Shock Temperature**
   \[ \frac{T_2}{T_1} = \frac{\left[2 \gamma M_a^2 - (\gamma-1)\right] \left[2 + (\gamma-1) M_a^2\right]}{(\gamma+1)^2 M_a^2} = 20.4 \]
   \[ \Rightarrow T_2 = 1632 \text{ K} \]

   Therefore
   \[ \frac{T_2}{T_1} = \frac{h_0}{C_p} = 0.97 \]

   **Post-Shock Pressure**
   \[ \frac{P_2}{P_1} = \frac{P_2}{P_1} = \frac{P_2}{P_1} \frac{1}{\gamma M_a^2} = \frac{1}{\gamma M_a^2} \left(1 + \frac{2 \gamma}{\gamma+1} (M_a^2 - 1)\right) = 0.83 \]

   **For the Space Shuttle**: $h_{0,85} \approx \frac{U_{28}}{2}$, where $U_{28} = \sqrt{\frac{9}{8} R_0}$

   \[ \approx 7.9 \text{ km/s} \Rightarrow h_{0,85} \approx 31.2 \text{ MJ/kg} \]

   Therefore
   \[ \frac{h_0}{h_{0,85}} \approx 50\% \], thereby indicating that the present wind tunnel provides stagnation enthalpies that are much smaller than those encountered during STS re-entry.

2. **Hypersonic Flow with High Post-Shock Temperatures**

   - $M_a = 8$ and $\alpha = 25^\circ$ => Oblique-Shock Chart
   \[ \beta \approx 46^\circ \]

   Therefore
   \[ M_{a1} = M_a \sin \beta = 7.2 \text{ and } \frac{T_2}{T_1} = 14 \]

   Since $T_1 \approx 220 \text{ K} - 260 \text{ K} in the stratosphere$, then
   \[ T_2 \approx 2420 \text{ K} - 2860 \text{ K} \Rightarrow \text{ post-shock temperature} \]

   $h_{a1} \approx a$ => Hypersonic flow with high post-shock temperatures involving vibrational excitation and $O_2$ dissociation.
SOLUTION PROBLEM 1

\( \beta = 20^\circ \)

\( V_0 = 44100 \text{ km/h} \) at \( h = 60 \text{ km} \)

\[ \Rightarrow \text{US STANDARD ATMOSPHERE: } P_0 = 0.2 \text{ mbar} \]

\[ a_0 = 315 \text{ m/s} \]

\[ \Rightarrow \text{M}_\infty = \frac{V_0}{a_0} = 3.87 \]

b) With \( \text{M}_\infty = 3.87 \) and \( \alpha = 30^\circ \) = OBlique Shock Chart, \( \beta = 46^\circ \)

\[ \text{M}_{\infty,sh} = \text{M}_\infty \sin \beta = 2.78 \]

\[ \frac{P_2}{P_0} = 1 + \frac{2\gamma}{\gamma - 1} (\text{M}_{\infty,sh}^2 - 1) = 8.65 \Rightarrow P_2 = 1.73 \text{ mbar} \]

AND \( \text{M}_{2,sh} = \left(\frac{2 + (\gamma - 1)\text{M}_{\infty,sh}^2}{2\text{M}_{\infty,sh}^2 + \gamma - 1}\right)^{\frac{1}{\gamma - 1}} \)

\[ = 0.49 \Rightarrow \text{M}_2 = \frac{\text{M}_{2,sh}}{\sin(\beta - \alpha)} = 1.77 \]

\[ \text{TURNING ANGLE: } \theta = \alpha = 30^\circ \]

ISENropic FLOW: \[ \frac{P_3}{P_2} = \left(\frac{1 + (\gamma - 1)\text{M}_2^2}{1 + (\gamma - 1)\text{M}_3^2}\right)^{\frac{\gamma}{\gamma - 1}} = 0.14 \]

WITH \( \text{M}_3 \) BEING CALCULATED BY NOTICING

\[ \theta(\text{M}_3) = 20^\circ \Rightarrow \theta(\text{M}_2) = \alpha + \theta(\text{M}_3) = 50^\circ \]

\[ \text{PRANDTL-MEYER} \]

\[ \text{EXPANSION FAN TABLES} \]

\[ \text{THEN } \text{M}_3 = 3.0 \text{ AND } P_3 = 0.26 \text{ mbar} \]

c) AT ZERO ANGLE OF ATTACK, THE CENTER OF PRESSURE IS UNDEFINED ALONG THE ROCKET AXIS (IT IS LOCATED AT AN INFINITE DISTANCE DOWNSHIFT). THIS CAN BE SEEN BY NOTICING THAT THE ENTIRE SYSTEM OF AERODYNAMIC FORCES ACTING ON THE ROCKET YIELDS ZERO MOMENT EVERYWHERE ALONG THE AXIS (SEE FIG. A(a) BELOW). AS THE ANGLE OF ATTACK INCREASES, FORCES NORMAL TO THE ROCKET
AXIS DEVELOP THAT YIELD NON-ZERO MOMENT
ABOUT THE CENTER OF GRAVITY (SEE FIG. 1 (b)).
THE CENTER OF PRESSURE IS DEFINED AS THE
POINT OF APPLICATION OF THE RESULTING TOTAL
NORMAL FORCE \( F_{\text{Ntot}} \), IN SUCH A WAY THAT THE
MOMENT OF \( F_{\text{Ntot}} \) ABOUT THE CENTER OF PRESSURE
IS ZERO. IF THE CENTER OF PRESSURE IS
LOCATED ABOVE THE CENTRE OF GRAVITY, THEN
THE ROCKET IS STABLE (SEE FIG. 1(c)), IN THAT
\( F_{\text{Ntot}} \) TENDS TO ROTATE THE ROCKET AROUND
THE CENTER OF GRAVITY SO AS TO DECREASE
THE ANGLE OF ATTACK. TO SOLVE THIS QUESTION,

ASSUME A SMALL ANGLE OF ATTACK \( \theta = 5^\circ \). IN THIS WAY, THE NORMAL FORCES

ACTING ON THE ROCKET ARE:

**Nose**

- UPPER SIDE: \( \mathcal{M}_{n} = 3.89 \)  
  \( \alpha - \Gamma = 25^\circ \)  
  OBLIQUE SHOCK CHART
  \( \beta = 39^\circ \)
  \( \mathcal{M}_{n} = \mathcal{M}_{\infty} \sin \beta = 2.48 \)
  \( \mathcal{P}_{n} = 6.72 \), \( \mathcal{M}_{2} = 2.14 \)
  \( \mathcal{P}_{2} = 55^\circ \)
  \( \mathcal{M}_{n} = \mathcal{M}_{\infty} \sin \beta = 2.17 \)
  \( \mathcal{P}_{2} = 55^\circ \), \( \mathcal{M}_{2} = 2.14 \)

- LOWER SIDE: \( \mathcal{M}_{n} = 3.87 \)  
  \( \alpha + \Gamma = 35^\circ \)  
  OBLIQUE SHOCK CHART
  \( \beta = 39^\circ \)
  \( \mathcal{M}_{n} = \mathcal{M}_{\infty} \sin \beta = 2.48 \)
  \( \mathcal{P}_{n} = 6.72 \), \( \mathcal{M}_{2} = 2.14 \)
  \( \mathcal{P}_{2} = 55^\circ \)
  \( \mathcal{M}_{n} = \mathcal{M}_{\infty} \sin \beta = 2.17 \)
  \( \mathcal{P}_{2} = 55^\circ \), \( \mathcal{M}_{2} = 2.14 \)

**Total Normal Force**

\[
F_{\text{Ntot}} = \mathcal{P}_{n} \left( \frac{D}{2 \tan \gamma} \right) \left( \frac{P_{2L}}{P_{2}} - \frac{P_{2U}}{P_{2U}} \right) \text{ APPLIED AT } x = \frac{D}{2 \tan \gamma} \text{ FROM THE NOSE EDGE}
\]

**Body**

- UPPER SIDE: \( \Theta = \alpha \), \( \beta = 39^\circ \)
  \( \mathcal{M}_{2} = 2.14 \)
  \( \mathcal{P}_{2L} = 0.122 \)
  \( \mathcal{P}_{2U} = 0.21 \)

- LOWER SIDE: \( \Theta = \alpha \), \( \beta = 39^\circ \)
  \( \mathcal{M}_{2} = 2.14 \)
  \( \mathcal{P}_{2L} = 0.122 \)
  \( \mathcal{P}_{2U} = 0.21 \)

**Total Normal Force**

\[
F_{\text{Nbody}} = \mathcal{P}_{n} \left( L-L_{c} - \frac{D}{2 \tan \gamma} \right) D \left( \frac{P_{2L}}{P_{2}} \mathcal{P}_{n} - \frac{P_{2U}}{P_{2U}} \mathcal{P}_{n} \right) \text{ APPLIED AT } x = \frac{D}{2 \tan \gamma} \text{ FROM THE NOSE EDGE}
\]

**Fins**

(WILL ONLY CONSIDER FORCES ON THE WEDGE EDGES)

\[
F_{\text{Nfins}} = 0
\]

THE WEDGES DO NOT PRODUCE ANY NORMAL FORCE BECAUSE THE FLOW ALONG THE BODY (i.e., STAGE "S") IS ALIGNED WITH THE BODY AND THE FINS ARE OPENED AT 90° DEGREES. \( F_{\text{Nfins}} = 0 \)
The center of pressure is therefore located at $X_{cp}$ given by

$$(F_{nose} + F_{body})X_{cp} = F_{nose}X_{nose} + F_{body}X_{body}$$

$\Rightarrow X_{cp} = \frac{P_0 D^3}{2 \tan \alpha} \left( \frac{P_2L - P_2U}{P_2o} \right) + P_0 \left( L - L_f - \frac{D}{2 \tan \alpha} \right) D \left( \frac{P_3L}{P_2L} \frac{P_2L - P_2U}{P_2o} \frac{P_2U}{P_2o} \right) \left( \frac{D}{2} + \frac{1}{2} \left( L - L_f - \frac{D}{2 \tan \alpha} \right) \right)$

$= 16.8 \text{ m}$

Therefore the center of gravity must be, at most, at $X_{cg} = 16.8 \text{ m}$ or below for the first stage to remain stable under yaw motion. Note that this answer depends on the angle of attack, in that the maximum allowable value of $X_{cg}$ increases with decreasing $\beta$.

f) **nose**

- Upper side: $\frac{P_{2U}}{P_2o} = 1 + \frac{\rho o U_e^2 \sin^2 (\kappa - \Pi)}{P_2o} = 1 + \gamma M_a^2 \sin^2 (\kappa - \Pi) = 4.74$

- Lower side: $\frac{P_{2L}}{P_2o} = 1 + \frac{\rho o U_e^2 \sin^2 (\kappa + \Pi)}{P_2o} = 1 + \gamma M_a^2 \sin^2 (\kappa + \Pi) = 7.89$

**body**

The forces on the body remain the same as in e) (Note that the Newtonian theory cannot predict what occurs through an expansion fan).

$\Rightarrow$ Then the center of pressure is at $X_{cp} = 16.4 \text{ m}$ and therefore the center of gravity must be located at $X_{cg} = 16.4 \text{ m}$ or below.