I. THE HYPersonic GAS ENVIRONMENT

Hypersonic flows are a special class of flows that arise around aerodynamic shapes moving in gaseous environments at exceedingly high velocities compared to the speed of the sound waves in the gas. In most practical applications related to hypersonics, the velocities associated with aircrafts and spacecrafts piercing through the atmosphere are in the range \(U_{\infty} \approx 1700 \text{ m/s} - 4000 \text{ m/s}\) (i.e., \(3800 \text{ mph} - 34000 \text{ mph}\)), although these can be much higher in the context of re-entry flows in other planets of the solar system (i.e., \(40000 \text{ m/s}\) in Jupiter). The aforementioned range of velocities translates into Mach numbers in the range \(5 \leq M_a \leq 40\). The lower end of this interval corresponds to applications of low-altitude high-speed flight and missile impact, while the upper end represents conditions approached by spacecrafts reentering in the terrestrial atmosphere during return from lunar or inter-planetary missions.

Note that an aircraft flying at Mach 40 can turn around the world along the equator in less than 4 hours.

**Fig. 1**: The gas environment near a notional hypersonic vehicle.
The magnitude of the hypersonics problem is perhaps best illustrated by analyzing the associated kinetic energies. For instance, the specific kinetic energy of hypersonic flows around high-speed vehicles are of order \( \frac{U_0^2}{2} \approx 2 \text{ MJ/kg} \). These energies are comparable or larger than the specific heats of vaporization of water (2.3 MJ/kg), carbon (60 MJ/kg), silica (5.8 MJ/kg), and iron (7.0 MJ/kg), and they are much higher than the specific thermal energy of air at normal conditions (300 MJ/kg). At hypersonic Mach numbers, in the presence of shock waves, these kinetic energies are approximately transformed into specific thermal energy behind the shock front, which causes exceedingly large temperatures in the post-shock gases of order

\[
\begin{cases}
  T_2 \approx 3000 \text{ K} \quad \text{for } M_{\infty} \approx 10 \quad \text{hypersonic cruise aircraft (X-43)} \\
  T_2 \approx 7000 \text{ K} \quad \text{for } M_{\infty} \approx 25 \quad \text{re-entry spacecraft from low Earth orbit (Space Shuttle orbiter)} \\
  T_2 \approx 11000 \text{ K} \quad \text{for } M_{\infty} \approx 36 \quad \text{re-entry spacecraft at escape velocity from lunar or inter-planetary return trajectory (Apollo capsule or Mars-return capsule)}
\end{cases}
\]

The high post-shock temperatures do not translate into similar wall temperatures, \( T_w \), since radiative equilibrium tends to lower the latter, and additionally, techniques for moderating wall temperatures have been discovered over the years to reject (or conveniently absorb in isolated parts of the structure) the intense aerodynamic heating that the airframe is subjected to at hypersonic speeds.

The thermal barrier, the realm of hypersonics is therefore not one associated with the onset of compressibility effects as passing through the transonic regime into the supersonic regime. That is the focus of transonic and low-supersonic aerodynamics and their challenges were greatly overcome by the breaking of the sound barrier in 1947. Instead, the problem of

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**Subsonic (\( M_{\infty} < 1 \))**

**Transonic (\( M_{\infty} \approx 1 \))**

**Supersonic (\( M_{\infty} > 1 \))**

**Highly Supersonic (\( M_{\infty} > 5 \))**

**Hypersonic (\( M_{\infty} > 5 \))**

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**Fig. 2**

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*Increasing Mach number*
Hypersonics is inexorably related to the everlasting and an ever-increasing thermal barrier that is encountered as the Mach number is increased (Fig. 2), and which poses severe constraints in airframe materials and aerodynamic design. As an example, consider the Space Shuttle orbiter, $m = 100.10^2$ kg, orbiting around the Earth at an altitude $285,000$ km. Therefore, with a circular orbital velocity $v_{cs} \approx \sqrt{\frac{GM}{R_e}} \approx 7.8$ km/s, with $R_e = 6371$ km, and $g = 9.8$ m/s$^2$, the Earth's radius and gravitational acceleration, the difference between the mechanical energies of the shuttle in orbit and at rest on the Earth's surface is

$$\Delta E = \frac{mv_{cs}^2}{2} + \frac{mgR_e}{2} = 3.3 \times 10^7 \text{ J}.$$ 

This energy difference must be transformed into heat during atmospheric re-entry.

The average gas + electricity consumption for a single-person home in Northern California is of order $2.5 \text{kWh/day} \times \frac{94 \text{ MJ}}{1 \text{kWh}} \times 365 \text{ days/year}$. As a result, all the heat generated during the Space Shuttle re-entry at hypersonically through the atmosphere could be used for heating and cooking of a single-person home for $360,000$ days $\approx 1000$ years. However, note that only a small fraction ($< 1\%$) of $\Delta E$ is transferred as heat to the entry system, the rest being dissipated into heat in the surrounding air. This exact fraction is a solution to the fluid mechanical problem of atmospheric re-entry. An order of magnitude is obtained in Chapter V that indicates

$$\frac{Q}{\Delta E} \approx \frac{1}{2} \frac{C_f S}{\rho A},$$

where $Q$ is the total amount of heat (J) absorbed by the spacecraft, $C_f$ is the mean friction coefficient, $S$ is the wetted surface, $A$ is the frontal area, and $\rho$ is the drag coefficient. This estimate (originally obtained by Allen and Essers in 1958), is perhaps one of the most crucial results of the theory of hypersonics, since it establishes that blunt bodies (large $C_d$ and $A$) with comparatively low skin friction and small wetted areas are the most practical for re-entry applications, because such aerodynamic shapes enable maximum dissipation of energy in the form of heat into the gas environment (rather than into the spacecraft structure).

The high temperatures associated with hypersonic flight pose severe challenges for the airframe surface materials. Many of the most important refractory materials
AND ADVANCED ALLOYS USED IN HIGH-TEMPERATURE APPLICATIONS HAVE MELTING POINTS SIMILAR
TO SOME OF THE WALL TEMPERATURES CITED ABOVE (2760 K FOR HEBYDIUM, 2200 K FOR
ZIRCONIUM, 1700 K FOR INCONEL-X, OR 1560 K FOR BERYLLIUM). THESE HIGH WALL TEMPERATURES
IN HYPersonicS ARE TO BE CONTRASTED WITH THOSE FOUND IN SUPERSONIC AIRCRAFT SUCH
AS THE CONCORDE (M∞ ~ 2, T∞ ~ 380 K) OR THE SR-71 BLACKBIRD (M∞ ~ 3.1, T∞ ~ 600 K).

THIS MAKES NECESSARY THE UTILIZATION OF THERMAL PROTECTION SYSTEMS (TPS) IN HYPersonicS,
INCLUDING REFRACTORY SHIELDS, AND ALSO HEAT SINKS AND ABLATIVE SURFACES. THE STUDY
OF TPS IS NOT THE SUBJECT OF THIS COURSE.

HOWEVER IT IS WORTH MENTIONING THAT THE TPS
REQUIREMENTS ARE FUNDAMENTALLY DIFFERENT DEPENDING
ON THE APPLICATION. THIS CAN BE SEEN IN THE
HEATING PROFILES IN FIG. 3, WHICH ILLUSTRATE
THE TRENDS OF HEAT FLUXES ENTERING IN TWO
TYPES OF VEHICLES AS A FUNCTION OF ENTRY TIME.

IT IS OBSERVED THAT INTER-CONTINENTAL BAllISTIC MISSILES, WHICH HAVE A SLENDER SHAPE,
TEND TO BE SUBJECTED TO VERY HIGH HEAT FLUXES BUT FOR A SHORT AMOUNT OF TIME (~30 s).
IN CONTRAST, MANNED ENTRY CAPSULES ARE MUCH LESS SLENDER AND CONSEQUENTLY ARE
SUBJECTED TO MUCH SMALLER HEAT FLUXES (~ 1/100 SMALLER THAN ICBMs) BUT FOR A MUCH
LONGER DURATION (~ 30 mins) IN CHOOSING A TPS, THE MAXIMUM HEAT FLUX IMPOSES
THE TYPE OF MATERIAL, WHILE THE TOTAL HEAT LOAD IMPOSES THE VOLUME OF TPS MATERIAL.

THE DIFFERENCES IN MISSION TIMES IN FIG. 3 AS WELL AS THE LARGE DISPARITIES IN
THE HEAT FLUXES, WILL BE FURTHER EXPLAINED LATER IN THE COURSE.

HIGH-TEMPERATURE EFFECTS IN HYPersonicS

IN EXPLAINING THE DISTINGUISHED CONDITIONS, IT IS IMPORTANT TO NOTE THAT THE

FLIGHT MACH NUMBER
\[ M_{\infty} = \frac{U_{\infty}}{a_{\infty}} \]

IS PROPORTIONAL TO THE SQUARE ROOT OF THE RATIO OF THE KINETIC ENERGY AND
SPECIFIC THERMAL ENTHALPY OF THE SURROUNDING AIR,

\[ M_{\infty}^2 = \frac{U_{\infty}^2}{a_{\infty}^2} = \frac{U_{\infty}^2}{\sqrt{RgT}} = \frac{2}{(y-1)} \frac{U_{\infty}^2}{C_p T_{\infty}} \]
As a result, hypersonic mach numbers \( (Ma > 5) \) involve kinetic energies larger than the thermal enthalpy of the surrounding gas. At high mach numbers into the hypersonic range \( (Ma > 5) \), this ratio becomes exceedingly large and it has profound consequences in the structure of the flow field that constitute the core of the problem of hypersonics. Such complex effects, unique to the hypersonic environment, are driven by the high temperatures attained in the gas as a result of transformation of kinetic energy into thermal energy. This transformation occurs ubiquitously in the flow field downstream from shocks, where the flow is decelerated because of the much higher pressures of the post-shock gases in comparison with the free-stream pressure, and near solid surfaces where the kinetic energy is dissipated into heat (Fig. 4). This transformation is, to a good approximation, described by the relation

\[
h_{\text{local}} = h_{\infty} + \frac{U_{\infty}^2}{2} \approx h + \frac{U_{\infty}^2}{2},
\]

which establishes that, at hypersonic mach numbers, the stagnation enthalpy of the free stream is mostly of kinetic origin, and as a result, becomes transformed into local enthalpy in regions where the gas is strongly decelerated (Fig. 4), thus creating enormous increments in the local temperature. The problem is, remarkably, more pronounced in the terrestrial atmosphere where most hypersonic vehicles are designed for, since the free-stream temperatures are relatively high \( (220k \sim 300 k) \), and therefore become premultiplied across shocks and boundary layers by multiples of the mach number. The high temperatures generated this way are sufficient to activate complex fluid-mechanical and thermo-chemical effects as follows (Fig. 4):

- Hypersonic boundary layers tend to be thicker and hotter than their low speed counterparts, and they tend to develop chemical conversion in the gas and near surfaces as explained below.
- At high Mach numbers ($M_{\infty} \gtrsim 20$), the gas downstream of the bow shock emerging from the nose of the spacecraft can get very hot and give rise to radiative heat transfer to the fuselage.

- Non-calorically-perfect effects are rapidly attained at relatively low flight speeds. At temperatures $T \gtrsim 800$ K, corresponding to post-shock conditions in the stratosphere at $M_{\infty} \gtrsim 4-5$ ($U_{\infty} \approx 4200-1500$ m/s), the vibrational degrees of freedom are excited (and) give rise to temperature dependences in specific heats. Additionally, air dissociation of oxygen and nitrogen molecules arise at higher speeds whose values depend on the flight altitude. At normal pressure, oxygen begins to dissociate ($O_2 \rightarrow 2O$) at $T \gtrsim 2000$ K, while nitrogen begins to dissociate ($N_2 \rightarrow 2N$) at much higher temperatures $T \gtrsim 4000$ K. At $T \approx 9000$ K all the air is virtually dissociated into $N$ and $O$ molecules. These effects have profound effects in the post-shock density that are crucial for the computation of temperatures in hypersonic flows. In the stratosphere, the onset of dissociation begins at $M_{\infty} \sim 7$ (for $O_2$) and $M_{\infty} \sim 15$ (for $N_2$), as shown in Fig. 5. Air ionization begins at $T \approx 9000$ K at normal pressure ($N \rightarrow N^+ + e^−$, $O \rightarrow O^+ + e^−$), translating into $M_{\infty} \sim 30$ in the stratosphere. This produces a plasma sheath of weakly ionized gas around the aircraft that can cause a communications blackout during a portion of the re-entry.

**Diagram:**

- In addition to the aforementioned effects, non-equilibrium thermodynamic and chemical phenomena can arise at high speeds when the characteristic flow time around the aircraft is of the same order as the time required by molecular collisions to equilibrate the dynamics.

*The flight speed expressed in kts is equivalent to the Mach number at 35 km altitude (Fig. 5)*
II. INVIScid HYPERSONIC FLOWS

Despite the wealth of complexities encountered in hypersonic flows, it is instructive to first study the simpler limit in which the flow is inviscid. In this limit, the effects introduced by viscosity and heat conduction are small. This condition is satisfied when the Reynolds number

$$Re_{\infty} = \frac{U_\infty l}{\nu} \gg 1$$

(1)

is moderately large. Inviscid conditions are preferentially attained in free-stream regions away from surfaces, wakes and flow discontinuities (Fig. 6), provided that the free-stream flow does not contain disturbances or these are negligibly small compared to the mean flow. In this way, the flow is slender (i.e., streamlined) and is not subject to energy losses by friction or other dissipation mechanisms except for energy losses across shock waves. Viscous effects are studied in Chapter III.

The hypersonic limit for shock-wave jump conditions.

\[
\begin{align*}
\beta &= \text{streamline deflection angle} \\
\theta &= \text{angle of incidence} \\
U_1 &= \text{pre-shock flow velocity} \\
U_2 &= \text{post-shock flow velocity}
\end{align*}
\]

In previous courses it was studied that the integral form of the conservation equations across an oblique shock becomes:

\[
\begin{align*}
\text{Continuity of mass:} & \quad \frac{\partial}{\partial t} U_1 + \frac{\partial}{\partial x} U_{1m} = \frac{\partial}{\partial x} U_{2m} = 0 \\
\text{Conservation of normal momentum:} & \quad \frac{\partial}{\partial t} U_{1n}^2 + P_1 = \frac{\partial}{\partial x} U_{2n}^2 + P_2 \\
\text{Conservation of tangential momentum:} & \quad U_{1t} = U_{2t} \\
\text{Conservation of total enthalpy:} & \quad h_0 = h_1 + \frac{U_{1m}^2}{2} = h_2 + \frac{U_{2m}^2}{2} = h_2
\end{align*}
\]

In Eq. (5), the symbol \( h_0 \) denotes a total or stagnation enthalpy. The same superscript is used throughout these notes to denote other stagnation quantities.
It can be shown that equations (2) - (5) lead to the Rankine-Hugoniot jump conditions

\[ \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma + 1} \left( \frac{M_{A2}^2}{M_{A1}^2} - 1 \right), \quad (7) \]

\[ \frac{T_2}{T_1} = \frac{P_2}{P_1} \frac{S_2}{S_1} = \frac{2\gamma \frac{M_{A2}^2}{M_{A1}^2} - (\gamma - 1)}{(\gamma + 1)^2} \left[ 1 + (\gamma - 1) \frac{M_{A2}^2}{M_{A1}^2} \right], \quad (8) \]

\[ \frac{U_{2m}}{U_{1m}} = \frac{S_2}{S_1} \left( \frac{M_{A2}^2}{M_{A1}^2} \right) \quad (9), \quad \frac{M_{A2}^2}{M_{A1}^2} = 2 + (\gamma - 1) \frac{M_{A2}^2}{M_{A1}^2} \quad (10) \]

\[ \frac{P_2}{P_1} = \frac{S_2}{S_1} = \left( \frac{1 + (\gamma - 1) \frac{M_{A2}^2}{M_{A1}^2}}{1 + (\gamma - 1) \frac{M_{A2}^2}{M_{A1}^2}} \right) \frac{1}{P_1} \quad (11) \]

In this formulation, \( \gamma = \frac{C_p}{C_v} \) is the adiabatic coefficient, with \( C_p \) and \( C_v \)
the specific heats at constant pressure and volume, respectively, which for now are assumed to be constant and related through the expression \( C_p - C_v = R_o \)
for an ideal gas, where \( R_o = \frac{R}{M} \) is the gas constant, \( R = 8.31 \frac{J}{molK} \)
is the universal gas constant, and \( M \) is the molecular weight.

Correspondingly, the equation of state is

\[ P = \frac{\gamma}{\gamma - 1} R_o \frac{T}{\rho}, \quad (12) \]

where \( P, \rho \) and \( T \) are static values of pressure, density, and temperature.

Static and stagnation quantities are related by the Bernoulli equation

\[ h + \frac{1}{2} \frac{U^2}{\gamma} = h_0 \Rightarrow \frac{C_p}{\gamma} + \frac{1}{2} \frac{U^2}{\gamma} = C_p T_0, \quad (13) \]

or, equivalently,

\[ \frac{T_0}{T} = 1 + \frac{\frac{U^2}{\gamma}}{2C_p T} = 1 + \frac{(\gamma - 1) \frac{M_a^2}{2}}{1}, \quad (14) \]

where \( M_a = \frac{\frac{U}{\rho}}{\gamma} \) is the Mach number and
\[
\alpha_o = \sqrt{\frac{\gamma R_o T}{\rho}} = \sqrt{\frac{\gamma P}{\rho}} \quad (15)
\]
is the speed of the soundwaves in the gas. The stagnation quantities correspond to the state of a gas isentropically brought to rest from its initial velocity \( U \). As a result, the isentropic relations
\[ \frac{T_0}{T} = \left( \frac{\rho_0}{\rho} \right)^{\gamma-1} = \left( \frac{P_0}{P} \right)^{\gamma-1} \]  \hspace{1cm} (16)

Between stagnation and static quantities are satisfied. Upon combining Eqs. (14) and (16), the expressions

\[ \frac{h_0}{h} = \frac{T_0}{T} = \frac{\rho_0}{\rho} = \left( \frac{a_0}{a} \right)^2 = \left( \frac{P_0}{P} \right)^{\gamma-1} = 1 + \left( \frac{\gamma-1}{2} \right) M_{a,\text{in}}^2 \]  \hspace{1cm} (17)

As a function of the local Mach number. Note that Eq. (17) is applicable only along each streamline and on each side of the shock wave but never across it, since (17) requires the flow be isentropic. This condition is not satisfied across the shock, which creates a positive entropy jump in the flow because of viscous and heat conduction within the shock.

In the jump conditions (6) - (11), \( M_{a,\text{in}} \) refers to the normal Mach number,

\[ M_{a,\text{in}} = \frac{U_{a,\text{in}}}{a_1} \]  \hspace{1cm} (18)

where \( U_{a,\text{in}} = U_1 \sin \beta \) is the flow velocity normal to the shock (see Fig. 2).

Similarly, \( U_{a,\text{t}} = U_1 \cos \beta \) is the tangential flow velocity, and

\[ U_{2a,\text{t}} = U_2 \cos (\beta - \delta) \]

\[ U_{2a,\text{e}} = U_2 \cos (\beta - \delta) \]

are the corresponding velocity components of the post-shock gas. Combining the two expressions for \( U_{a,\text{in}} \) and \( U_{a,\text{e}} \), one obtains

\[ M_{a,\text{in}} = \frac{U_{a,\text{in}}}{a_1} = M_{a,1} \sin \beta \]

\[ M_{a,\text{e}} = \frac{U_{a,\text{e}}}{a_2} = M_{a,2} \sin (\beta - \delta) \]  \hspace{1cm} (19)

For the Mach numbers in the normal direction. Additionally, a relation between the angle of incidence \( \beta \) and the angle of streamline deflection \( \delta \) can be obtained by using

\[ \frac{U_{a,\text{e}}}{U_{a,\text{in}}} = \frac{U_{2a,\text{e}}/U_{2a,\text{t}}}{U_{2a,\text{t}}/U_{2a,\text{e}}} = \frac{\tan (\beta - \delta)}{\tan \beta} \]

Along with the jump condition (4) for \( U_{a,\text{e}}/U_{a,\text{in}} \), which gives
\[
\frac{\tan (\beta - \delta)}{\tan \beta} = \frac{Z + (Y-1) \frac{M_0^2 \sin^2 \beta}{(\gamma+1)} \frac{M_0^2 \sin^2 \beta}{(\gamma+1)}}{Z + M_0^2} \approx \frac{2 \cotan \beta \left( \frac{M_0^2 \sin^2 \beta - 1}{2 + M_0^2} \right)}{2 + M_0^2 \left( \frac{M_0^2 \sin^2 \beta}{2 + \cos (2\beta)} \right)}
\]

The contours of Eq. (20) are shown schematically in Fig. 8. Note that, for a given deflection angle \( \delta = \delta_0 \), and a given Mach number, \( M_0 \), Eq. (20) has 2 solutions \( \beta_W \) and \( \beta_S \), with \( \beta_W < \beta_S \). The solution \( \beta_S \) corresponds to strong shocks, for which the post-shock flow is subsonic \( (M_0 < 1) \). Conversely, the solution \( \beta_W \) corresponds to weak shocks, for which the post-shock flow is supersonic \( (M_0 > 1) \). Zero deflection \( (\delta = 0) \) of the streamlines occurs only in the following situations:

- For normal shock waves \( (\beta = 90^\circ) \) \( \Rightarrow \cotan \beta = 0 \), which implies \( M_0^2 < 1 \).
- For compression Mach wave \( (\text{also called weak shocks}) \) in a particularized sense with respect to that used above \( \) such that \( M_0^2 \sin^2 \beta = 1 \), or equivalently \( \sin \beta = \frac{1}{M_0} \), where \( \mu_m \) is the Mach angle.

Note that \( M_0^2 \sin^2 \beta = 1 \) implies \( \mu_m = 1 \) because of (20), so that the pre-shock flow in weak shocks is asymptotically sonic.

It is worth mentioning that Eqs. (6) - (11) imply the following inequalities across the shock:

\[
\frac{P_2}{P_1} > 1, \quad \frac{S_2}{S_1} > 1, \quad \frac{T_2}{T_1} > 1, \quad \frac{U_2}{U_1} < 1, \quad \frac{P_02}{P_01} < 1, \quad \frac{S_02}{S_01} < 1, \quad \frac{T_02}{T_01} = 1
\]

\[
\frac{S_2}{S_1} > 1, \quad \text{and} \quad M_0 < 1 \quad (21)
\]

- The flow behind a shock has higher pressure, temperature, entropy, and density, the same stagnation temperature as the flow upstream, and lower values of stagnation pressure and density. In addition, the normal flow behind a shock is always subsonic.
The hypersonic limit of the Rankine-Hugoniot jump conditions \((6)-(11)\) is obtained by making \(M_{a1}^2 \sin^2 \beta \gg 1\), or equivalently, \(M_{a1}^2 \gg 1\), which yields

\[
\frac{P_2}{P_1} = \frac{2\gamma}{\gamma+1} M_{a1}^2 \sin^2 \beta, \quad (22) \quad \frac{T_2}{T_1} = \frac{2\gamma}{(\gamma+1)^2} M_{a1}^2 \sin^2 \beta, \quad (23)
\]

\[
\frac{s_2}{s_1} = \frac{\gamma+1}{\gamma-1}, \quad (24) \quad \frac{U_{2n}}{U_{1n}} = \frac{S_2}{s_1} = \frac{\gamma-1}{\gamma+1}. \quad (25)
\]

For the pressure, temperature, density and normal-velocity jumps, similarly, the relation between the incidence angle \(\beta\) and the streamwise deflection angle \(\delta\) [Eq. (20)] can be simplified for slender bodies \((\delta \ll \beta)\) and high Mach numbers \((M_{a1}^2 \sin^2 \beta \gg 1)\) by making \(M_{a1} \sin \delta \approx s, \quad \sin \beta \approx \beta, \quad \cos (2\beta) \approx 1\) and \(M_{a1}^2 \sin^2 \beta \approx M_{a1}^2 \beta^2 \gg 1\), which gives

\[
\delta \approx \frac{2}{\beta} \frac{\beta^2}{\gamma+1} \Rightarrow \delta = \frac{2\beta}{\gamma+1}. \quad (26)
\]

Equation (26) predicts that the hypersonic flow around a slender wedge of small semi-angle \(\delta\) (see Fig. 9), involves a shock wave whose inclination angle \(\beta\) is just \(\beta \approx 1.2\delta\) for \(\delta = 1.4\), in this way, in hypersonic regimes the shock wave envelopes the wedge at a very short distance \(s_{sc} = \left(\frac{\gamma-1}{2}\right) s_x\), where \(s_x\) is the horizontal distance from the leading edge. Note that \(s_{sc}\) becomes smaller than the characteristic boundary-layer thickness \(s_B \approx (x/R_{c1/2}) M_{a1}^2\) when \(\frac{2}{\beta-1} \frac{M_{a1}^2}{R_{c1/2}} \beta \gg 1\), or equivalently, for wedge angles \(\delta \approx \frac{2}{(\beta-1) R_{c1/2}}\). In this regime, a complex viscous interaction occurs in which the hot shock layer merges with the boundary layer.

However, viscous effects will be excluded from the formulation for now.

An additional simplification of Eqs \((6)-(11)\) in the hypersonic limit, that is useful for slender-body theory, concerns the post-shock flow-velocity components parallel and perpendicular to the upstream flow, which are denoted here by the lowercase symbols \(u_2\) and \(v_2\), respectively (Fig. 5).
In particular, $u_2$ and $v_2$ represent flow disturbances associated with the streamline deflection induced by the oblique shock. Their ratios to the free stream velocity, $u_2/u_1$ and $v_2/u_1$, can be calculated as follows for any Mach numbers:

$$\frac{u_2}{u_1} = \frac{U_2}{U_1} \frac{U_{2m}}{U_{1m}} \frac{U_{2n}}{U_{1n}} = \cos \delta \frac{1}{\sin (\beta - \delta)} \left[ \frac{Z + (7-1)M_{1m}^2}{(7+1)M_{1m}^2} \right] \sin \beta \tag{27}$$

$$\frac{v_2}{u_1} = \frac{V_2}{U_1} \frac{U_{2m}}{U_{1m}} \frac{U_{2n}}{U_{1n}} = \sin \delta \frac{1}{\sin (\beta - \delta)} \left[ \frac{Z + (7-1)M_{1m}^2}{(7+1)M_{1m}^2} \right] \sin \beta \tag{28}$$

In simplifying the above expressions, it is useful to note that

$$\frac{\sin \beta \cos \delta}{\sin (\beta - \delta)} = \frac{\sin \beta \cos \delta}{\sin \beta \cos \delta - \cos \beta \sin \delta} = \frac{\tan \beta}{\tan \beta - \tan \delta} = \frac{\tan \beta}{1 + \tan \delta \tan \beta}$$

$$\frac{\sin \beta \sin \delta}{\sin (\beta - \delta)} = \frac{\sin \beta \cos \delta}{\sin \beta \cos \delta - \cos \beta \sin \delta} = \frac{\tan \beta \tan \delta}{\tan (\beta - \delta) (1 + \tan \delta \tan \beta)} \tag{29}$$

Substituting (29) into (27) and making use of (20):

$$\frac{u_2}{u_1} = \frac{1}{1 + \tan \delta \tan \beta} = \frac{1 - 2 \left( M_{1m}^2 \sin^2 \beta - 1 \right)}{M_{1m}^2 (7+1)} \tag{31}$$

Similarly, substituting (30) into (28) and making use of (20):

$$\frac{v_2}{u_1} = \frac{\tan \delta \tan \beta}{1 + \tan \delta \tan \beta} \frac{1}{\tan \beta} = \frac{2 (M_{1m}^2 \sin^2 \beta - 1) \cot \beta}{M_{1m}^2 (7+1)} \tag{32}$$

In the hypersonic limit, (31) and (32) become

$$\frac{u_2}{u_1} = 1 - \frac{2 \sin^2 \beta}{(7+1)} \quad \frac{v_2}{u_1} = \frac{\sin \left( \frac{2 \beta}{\gamma + 1} \right)}{\gamma + 1} \tag{33}$$

Note that $\frac{v_2}{u_1} = O(\delta)$ for $\delta$ small, while $\frac{u_2}{u_1} = 1 = O(\delta^2)$. First order!

An important quantity in high-speed aerodynamics is the pressure coefficient,

$$C_p = \frac{P_2 - P_1}{\frac{1}{2} q_1 U_1^2}$$

where $q_1 = \frac{1}{2} \frac{1}{\gamma} U_1^2$ is the dynamic pressure upstream from the shock.

Note that at high Mach the sum of static and dynamic pressures does not equal the stagnation pressure, i.e., $p_1 + q_1 \neq P_0$. Second order!
IN PARTICULAR, $q_1$ CAN BE REWRITTEN AS

$$q_1 = \frac{1}{2} \frac{2}{\gamma P_1} \gamma P_1 U_1^2 = \frac{1}{\gamma P_1} \gamma P_1 H_{a_1}^2,$$

AND THE PRESSURE COEFFICIENT BECOMES

$$C_p = \frac{\frac{P_2 - P_1}{q_1}}{q_1} = \frac{2}{\gamma} \left( \frac{P_2}{P_1} - 1 \right) = \frac{4}{(\gamma+1)H_{a_1}^2} \left( M_{a_1}^2 - 1 \right).$$  \hfill (34)

IN THE HYPERSONIC LIMIT, (34) SIMPLIFIES TO

$$C_p = \frac{4}{\gamma+1} \sin^2 \beta$$  \hfill (35)

COMBINATION OF (33) AND (25) RENDERS THE USEFUL RELATION

$$C_p = 2 \left( 1 - \frac{U_2}{U_1} \right)$$  \hfill \hfill (36)

WHICH INDICATES THAT THE DECREASE IN THE PARALLEL COMPONENT OF THE VELOCITY ACROSS THE SHOCK PRODUCES AN INCREASE IN THE STATIC PRESSURE AND A CORRESPONDING POSITIVE VALUE OF THE PRESSURE COEFFICIENT. EQUATION (36) IS EQUIVALENT TO THE CLASSIC RESULT THAT CAN BE DERIVED FROM THE AERODYNAMICS OF SLENDER BODIES (I.E., EQ. (148) IN THE MESSER NOTES), WHERE $U = U_2 - U_1$ PLAYS THE ROLE OF THE HORIZONTAL DISTURBANCE OF THE FREE STREAM VELOCITY. NOTE THAT (35) (OR (36)) INDICATES THAT $q_1$ IS A SECOND-ORDER QUANTITY IN THE STREAMLINE DEFLECTION ANGLE, I.E., $C_p \sim O(\delta^2) \sim O(\delta^2)$, EQUATION (35) IS ALSO REMARKABLY SIMILAR TO THE RESULTS OBTAINED BY A MUCH SIMPLER FORMULATION KNOWN AS NEWTONIAN THEORY, IN WHICH THE AERODYNAMIC FORCES ON A BODY AT HYPERSONIC SPEEDS ARE A SIMPLE FUNCTION OF THE INCLINATION ANGLE $\delta$ OF ITS SURFACES WITH RESPECT TO THE FREE STREAM, AS DISCUSSED FURTHER BELOW.

AS A SUMMARY, THE JUMP CONDITIONS (6) - (14) ARE PLOTTED IN FIG. 11 ALONG WITH THEIR HYPERSONIC LIMIT. IT IS WORTH HIGHLIGHTING A COUPLE OF ASPECTS OF THE LIMITING EXPRESSIONS (22) - (25). FIRST, THE PRESSURE AND TEMPERATURE RATIOS APPEAR TO INCREASE UNBOUNDEDLY WITH THE MACH NUMBER SQUARED. THIS BEHAVIOR, FOR INSTANCE, ENABLES THE SIMPLIFICATION OF THE
FORMULATION OF BLAST WAVES \textit{(Sedov, 1946; Taylor, 1950; von Neumann, 1953)}.

A blast wave is a spherical pressure wave that moves supersonically outwards from an explosive core. The actual motion is hypersonic as a result of the exceedingly high speeds involved in the detonation of high explosives (\( \approx 10 \text{ km/s} \) for TNT). That \( \frac{P_2}{P_1} \) is proportional to \( \frac{M_0^2}{M_1^2} \) in \( \text{(22)} \) is used in blast-wave theory to neglect the effect of the ambient pressure, which is \( \frac{1}{M_0^2} \left( \frac{\gamma}{\gamma-1} \right)_1 \) times smaller than the pressure behind the shock front. This enables the construction of a famous self-similar theory for blast waves (see page 57 in the notes). Note however that the density reaches a maximum compression ratio of order \( \frac{\gamma+1}{\gamma-1} \).

An additional consequence of the proportionality of \( \frac{T_2}{T_1} \) with \( M_0^2 \) at large \( M_0 \) is the prediction of unrealistically high temperatures in the shock layer. For instance, at Mach 25, which is representative of reentry from a decaying low-Earth orbit, the temperature jump increase across the shock is of order 100, which would render temperatures in the shock layer of order 30,000 K, corresponding to 6 times the surface temperature of the sun. Similarly, in reentry from deflected interplanetary orbits, the Mach number is about 35, as in the Apollo reentry from the moon, which would lead to even higher shock-layer temperatures of order 100,000 K. According to Fig. 6, these values are unrealistically high because the analysis that leads to \( \text{(6)-(9)} \) assumes a calorically perfect non-reacting gas. The realm of hypersonics is that, as \( M_0 \) increases, the high temperatures in the post-shock gas...
Along with the comparable timescales of the flow and of the relaxation of gasdynamic effects such as vibrational non-equilibrium or electronic excitation lead to a chemically-reacting non-equilibrium gas in which, to begin with, $CP$ and $C_V$ are no longer constants, the correct calculation of the shock layer that takes into account these complex effects renders shock-layer temperature of order 7,000 K for Mach 25 and 11,000 K for Mach 36. These complex effects will be discussed in Chapter IV of these class notes.

**The Hypersonic Limit for Expansion Waves**

![Diagram](image)

The expansion of a supersonic flow can be achieved through a convex corner, as in Fig. 42. This generates an expansion fan through which the flow is accelerated and expanded isentropically. In contrast to shock waves, the expansion fan occupies a finite-thickness region of space, since sharp discontinuities are forbidden in expansion processes because of the 2nd principle of thermodynamics (see page 24 in the Mises notes). A classic analysis of the stream deflection and associated isentropic expansion leads to the relation

$$
\Theta = \mathcal{J}(M_{a2}) - \mathcal{J}(M_{a1})
$$

(37)

between the deflection angle $\Theta$ and the upstream and downstream Mach numbers $M_{a1}$ and $M_{a2}$ (see page 27 in the Mises notes). In Eq. (38), $\mathcal{J}(M_a)$ is the Prandtl-Meyer function

$$
\mathcal{J}(M_a) = \sqrt{\frac{y+1}{y-1}} \arctan \left[ \frac{y-1}{y+1} \left( M_a^{y-1} - 1 \right) \right] - \arctan \sqrt{M_a^{2} - 1}
$$

(38)

For $M_a >> 1$, the Prandtl-Meyer function becomes

$$
\mathcal{J}(M_a) = \sqrt{\frac{y+1}{y-1}} \arctan \left( M_a \frac{y-1}{y+1} \right) - \arctan M_a
$$

which can be further simplified by noting that
\[
\arctan(x) = \frac{\pi}{2} - \arctan \left( \frac{4}{x} \right) = \frac{\pi}{2} - \frac{4}{x} - O\left( \frac{1}{x^2} \right) \quad \text{for } x \to \infty.
\]

So that
\[
0(M_a) = \sqrt{\frac{y+1}{y-1}} \left[ \frac{\pi}{2} - \frac{4}{y-1} \frac{1}{M_a} + O\left( \frac{1}{M_a^2} \right) \right] = \left[ \frac{\pi}{2} - \frac{4}{y-1} + O\left( \frac{1}{M_a^2} \right) \right],
\]

which yields the approximate expression

\[
0(M_a) = \left[ \sqrt{\frac{y+1}{y-1}} - 1 \right] \frac{\pi}{2} - \frac{4}{y-1} \frac{1}{M_a},
\]

(39)

To order \( \frac{1}{M_a^2} \), substitution of (39) into (37) gives

\[
0 = \frac{2}{y-1} \left( \frac{1}{M_a} - \frac{1}{M_{a_2}} \right) \quad (40)
\]

For large \( M_a \), while the isentropic change in pressure through the expansion fan is

\[
P_2 = \frac{P_2}{P_1} = \frac{P_2}{P_1} = \frac{\left( 1 + \frac{y-1}{2} \right) M_{a_1}^2}{\left( 1 + \frac{y-1}{2} \right) M_{a_2}^2} \times \left( \frac{M_{a_1}}{M_{a_2}} \right)^{\frac{2}{y-1}}
\]

Where use of equation (47) has been made along with the fact that the stagnation pressure is constant along streamlines.

**NEWTONIAN THEORY OF HYPERSONIC FLOWS**

In his early theory of fluid motion (Principia, 1687), Newton envisioned the force exerted on a body submerged in a fluid as a collective effect of fluid particles initially moving rectilinearly and then colliding with the body surface, as in Fig. 13. At collision, the fluid particles would entirely transfer their normal momentum to the body and they would subsequently move parallel to the body surface. Newton associated this description to a rarefied medium in which he postulates no interaction between individual particles and in which he only accounts for the impact force \( F \) caused by the normal transfer of momentum of the particles impinging on the body surface. The resulting force is the normal component of the velocity,

\[
U_m = U_{m_0} \sin \theta
\]

(41)

Multiplied by the mass flow rate incident on the body surface

\[
\dot{m}_w = \dot{m}_w \sin \theta \quad (42)
\]
COMBINING (41) AND (42) GIVES
\[ F = \rho_0 U_a^2, \delta U_a = \rho_0 U_o^2, \delta \sin^2 \delta \]  
(43)

THE CORRESPONDING PRESSURE ON THE WINDWARD SIDE OF THE BODY IN FIG. 8 IS
\[ P - P_o = \rho_0 U_o^2 \cos^2 \delta \]  
(44)

AND THE PRESSURE COEFFICIENT IS
\[ C_p = \frac{P - P_o}{\frac{1}{2} \rho_0 U_o^2} = 2 \sin^2 \delta \]  
(45)

EQUATIONS (43) - (45) WERE NOT ORIGINALLY OBTAINED BY NEWTON BUT BY EPSTEIN IN
"ON THE AIR RESISTANCE OF PROJECTILES," PNAS 1931. IT SHOULD BE NOTED THAT THE THEORY
PRESENTED ABOVE IS A ROUGH ONE THAT DOES NOT ACCOUNT FOR MULTIPLE EFFECTS THAT TO DAY
WE KNOW EXISTS AND INFLUENCE THE AERODYNAMICS OF BODIES IMMERSED IN FLUIDS. FOR INSTANCE,
D'ALEMBERT CORRECTLY FORMULATED THE ZERO FORCE THAT A BODY MUST EXPERIENCE IN AN INVISCOUS
FLUID AS THAT TREATED ABOVE. ADDITIONALLY, BOUNDARY-LAYER EFFECTS WERE INTRODUCED LATER
AND WERE FOUND IMPORTANT IN DETERMINING THE FORCE, PARTICULARLY AT MODERATE SPEEDS.

![Diagram](image)

FIG. 44: NEWTONIAN THEORY

HOWEVER, DESPITE ITS SHORTCOMINGS, THE NEWTONIAN THEORY HAS PROVEN REMARKABLY USEFUL IN HYPERSOUND
FOR THE FOLLOWING REASONS. FIRST IS THAT THE
FLOW IMMEDIATELY UPSTREAM FROM THE BODY REMAINS
UNDISTURBED SINCE \( M_o \ll 1 \). AS A RESULT, THE
STREAMLINES ARE STRAIGHT AND THE FLUID PARTICLES
DO APPROACH THE BODY SURFACE IN THE SAME
BALLISTIC WAY AS ENVISIONED BY NEWTON. SECOND IS THAT AT \( M_o \ll 1 \) THE SHOCK WAVE
ENVELOPS THE BODY VERY CLOSE TO ITS SURFACE, AS SUGGESTED BY EQUATION (26),
SO THAT THE BALLISTIC APPROACH PERDURES UNTIL VERY CLOSE TO THE SURFACE, WHERE
THE DIRECTION OF THE FLUID PARTICLES MUST NECESSARILY CHANGE TO A NEAR TANGENTIAL
TRAJECTORY. THIRD IS THAT THE WAVE DRAG EMERGING FROM THE INTERACTION OF THE
FLOW AND THE BODY (\( \xi \)), THE PRESSURE DRAG \( C_p = \frac{P - P_o}{\frac{1}{2} \rho_0 U_o^2} = 2 \sin^2 \delta \) EMERGING FROM THE PRESSURE
ACROSS THE NOSE SHOCK), TENDS TO BE A VERY IMPORTANT COMPONENT OF THE TOTAL DRAG FORCE
THAT IN MANY SITUATIONS SUPERSEDES THE VISCOUS FRICTION DRAG BECAUSE OF THE
HIGH REYNOLDS NUMBERS INVOLVED.

IN EQ. (45), THE ANGLE \( \delta \) IS GENERALLY THE LOCAL DEFLECTION OR INCLINATION ANGLE
OF THE SURFACE, THEREBY GIVING RISE TO A GENERAL SET OF METHODS FOR COMPUTING AERODYNAMIC
...
Forces in hypersonics that are called "local surface inclination methods" include the tangent-cone, tangent-wedge and shock-expansion methods, along with modifications to the Newtonian theory, some of which are discussed further below.

The Newtonian theory is attractive by its simplicity, however, some important aspects should be taken into consideration when employing this theory:

i) The pressure on any leeward surface is equal to the ambient pressure $P_0$, thereby yielding $q_p = 0$ locally there. This approximation clearly breaks down as $M_\infty$ decreases.

ii) The lift coefficient $q_L = q_p \cos \delta = 2 \sin^2 \delta \cos \delta$ on a flat plate becomes zero at zero angle of attack. However, the drag coefficient $q_D = q_p \sin \delta = 2 \sin^3 \delta$ approaches zero faster as $\delta \to 0$, such that the lift-to-drag ratio $L/D$ becomes infinite $L/D \to \infty$ at $\delta \to 0$ (see Fig. 15). This result is clearly counter-intuitive and arises because of neglecting the viscous forces generated by friction, which would lead to finite drag at $\delta \to 0$ and zero lift due to flow symmetry ($L/D = 0$).

iii) The asymptotic behavior of the drag coefficient $q_D \sim \delta^2$ for $\delta$ small is in contrast with the $q_D \sim \delta^2$ behavior obtained from the thin-airfoil theory for supersonic flows (e.g., see Eq. 160 in page 75 of the MESH notes). This highlights the non-linearities typically associated with hypersonic flows that complicate the analysis of the small-disturbance equations as shown further below.

![Fig. 15: Lift and Drag Coefficients and Lift-to-Drag Ratio on a Flat Plate at $M_\infty = 1$.](image)

![Fig. 16: Surface Pressure Coefficient (left) and Incidence Angle (right) for a 15° cone.](image)
The results from the Newtonian Theory do not depend on the Mach number. This is a manifestation of the "Mach-Number Independence Principle", which will be studied further below within the context of the small-disturbance equation. And states that the non-dimensional variables (and the drag and lift coefficients) become mostly independent of $Ma_{1o}$ at $Ma_{1o} > 0$. This independency has been observed in experimental drag coefficients (Figs. 16-19), which appear to plateau at sufficiently large values of the Mach number.

At finite Mach numbers, the Newtonian Theory requires modifications (see Fig. 16 and 17). One relevant modification useful for blunt bodies is the one proposed by Lees (1955),

$$ C_D = C_{D_{\text{max}}} \sin^2 \theta $$  \hspace{1cm} (46)

Where $C_{D_{\text{max}}}$ is the maximum surface pressure coefficient evaluated behind the normal shock wave created by a blunt body (Fig. 18), namely

$$ C_{D_{\text{max}}} = \frac{P_2 - P_\infty}{\frac{1}{2} \rho_\infty U_\infty^2} = \frac{2}{\gamma M_{\text{sh}}^2} \left( \frac{P_2}{P_\infty} - 1 \right) $$  \hspace{1cm} (47)

Where $P_2$ is the stagnation pressure behind the shock (Note that a stagnation-point-like flow is formed in between the body and the quasi-normal portion of the nose shock so as to make the flow almost stagnant there). Since

$$ \frac{P_2}{P_\infty} = \frac{P_2}{P_0} \frac{P_{0o}}{P_0} $$  \hspace{1cm} (here $P_0$ is equivalent to the preshock state $P_{0o}$ in the previous sections)

And substituting Eq. (11) for $P_2/P_0$, Eq (17) for $P_{0o}/P_0$ and and Eq. (10) for $Ma_2^2$, one obtains

$$ \frac{P_2}{P_0} = \left( \frac{1}{2} \frac{\gamma+1}{\gamma-1} Ma_{1o}^2 \right)^{1/(\gamma+1)} \left( \frac{1 - \gamma + 2 \gamma Ma_{1o}^2}{\gamma + 1} \right) $$  \hspace{1cm} (47)

For a spherical body,

$$ C_D = \frac{16 \pi}{3} \frac{L}{D} \frac{U_\infty^2}{\frac{1}{2} \rho_\infty U_\infty^2} $$  \hspace{1cm} (48)
In this way, \( q_{\text{Pmax}} \) in Eq. (47) becomes

\[
q_{\text{Pmax}} = \frac{Z}{\gamma HA_{20}^2} \left[ \frac{(8+1) HA_{20}^2}{48 HA_{20}^2 - 2(8+1)} \right]^{\frac{1}{8+1}} \left[ 1 - \frac{y + 2 HA_{20}^2}{y+1} \right]^{-1}
\]

The modified Newtonian theory becomes particularly useful in flows past blunt bodies, since the correction (48) makes the local pressure behind the normal section of the nose shock to become the approximately right one. For instance, in the straight Newtonian theory, the pressure on the nose of the body is

\[
P = P_{\infty} + \frac{q_{\text{Pmax}}}{2} U_{\infty}^2 \sin \frac{8}{y} \sin \frac{8}{y+1}
\]

As implied by Eq. (44) evaluated for \( \xi = \pi/2 \). As a result, the surface pressure coefficient there is

\[
q_p = \frac{1}{2} \frac{P-P_{\infty}}{P_{\infty}} = 2
\]

The ratio of (44) to the true stagnation pressure is

\[
\frac{P}{P_{\infty}} = \frac{P_{\infty}}{P_{\infty}} \left( 1 + \frac{q_{\text{Pmax}}}{2} U_{\infty}^2 \frac{1}{P_{\infty}} \right) = \frac{P_{\infty}}{P_{\infty}} \left( 1 + \gamma HA_{20}^2 \right)
\]

With \( P_{\infty}/P_{\infty} \) given by the inverse of (47). As a result, for \( HA_{20} >> 1 \), (54) becomes

\[
\frac{P}{P_{\infty}} \sim \frac{2}{2} \left( 2 HA_{20}^2 \right) - \frac{8+1}{8+1} \left( 2 HA_{20}^2 \right) - \frac{8+1}{8+1} \left( 2 HA_{20}^2 \right)
\]

And it can be trivially seen that \( \frac{P}{P_{\infty}} = \frac{P_{\infty}}{P_{\infty}} \left( 1 + \gamma HA_{20}^2 q_{\text{Pmax}}^2 \right) = 1 \) in this case. The improved performance of the modified Newtonian theory is illustrated in Fig. 19, where the flow over a paraboloid \( y = A x^2 + B \) is analyzed. Note that the entire pressure distribution along the body \( P(y)/P_{\infty} \) can be derived by making \( \sin^2 \xi = (1 + 4 A y^2)^{-1} \) in Eqs. (49) and (53).
Neither the straight Newtonian theory (42) nor the modified Newtonian theory (46) take into account the pressure variations induced by the curvature of the body surface. In particular, centrifugal forces tend to induce pressure gradients in the radial direction. As a result, the pressure on the body surface tends to be smaller than the pressure on the edge of the shock layer in flows over blunt bodies (see Fig. 20). The latter pressure is the nominal one predicted by the straight Newtonian theory (44).

Centrifugal corrections to (45) were first formulated by Adolf Busemann (1932) and are known as the Newton-Busemann theory, which is summarized below.

Consider the calculation of the surface-pressure coefficient at point ", where the local deflection angle is $\delta_s$ and the local pressure is $P_s$. The local normal direction is denoted by $n$, whereas the local radius of curvature of the surface is $R$ (see Fig. 21). In accordance with the Newtonian theory, the free stream penetrates in the thin shock layer and turns suddenly there along a trajectory following very closely the surface of the body. At every point of entry into the shock layer, the incoming streamtubes form thin laminae along paths enveloping the body, as shown schematically in Fig. 16. Mass continuity through this process requires

$$\int_{\omega} u_s \delta s \, dy = \int_{\omega} u \, dm, \quad (53)$$

where $\omega$ and $u$ are densities and velocities along the laminae. In addition, the conservation of momentum requires a balance between the centrifugal acceleration and the pressure gradient, namely

$$\frac{dP}{dm} = \frac{\rho \delta_s^2}{R}, \quad (54)$$

Integrating (54) from a point on the body surface to another one immediately above on the edge of the shock layer, one obtains

$$\int_{P_s}^{P} \rho \delta_s \, dy = \int_{0}^{\delta_s} \rho \delta_s \cos \delta_s \, dy \quad (55)$$
Where use of Eq. (53) has been made in the formulation, \( \delta_{sl} \) is the thickness of the shock layer. In the Newtonian limit, \( \delta_{sl} \to 0 \), (55) becomes

\[
P - P_s = \int_0^{y_s} \frac{\rho_0 U_0^2}{R} U dy \quad (56)
\]

where \( y_s \) is the vertical location of the shock-layer edge, since \( R \), which is given by \( R = (\sin \delta)^{-1} \frac{\partial y}{\partial \phi} \), and represents the radius of curvature of the streamlines, is constant along \( \delta \) (i.e., all streamlines are deflected by an angle \( \delta \) upon entering the shock layer), then (56) becomes

\[
P_s = P + \frac{\rho_0 U_0^2}{2} \left( \frac{\partial \delta}{\partial y} \right)_s \sin \delta \int_0^{y_s} \cos \delta dy \quad (57)
\]

where the Newtonian relation \( U = U_\infty \cos \phi \) has been used indicating that the velocity within the shock layer is parallel to the local surface. Equation (57) can be easily rewritten as

\[
\frac{q_{ps}}{2} = 2 \sin^2 \delta + \frac{1}{2} \frac{\rho_0 U_0^2}{2} \left( \frac{\partial \delta}{\partial y} \right)_s \sin \delta \int_0^{y_s} \cos \delta dy \quad (58)
\]

which corresponds to the Newton-Busemann formula. In particular, the second term in (58) corresponds to a centrifugal correction to the Newton Eq. (45) that depends on the accumulated history of the streamlines upstream of that point. For slender bodies, \( \sin \delta \approx \delta \) and \( \cos \delta \approx 1 \), so that (58) becomes

\[
\frac{q_{ps}}{2} \approx 2 \delta^2 + 2 \kappa y \quad (59)
\]

where \( \kappa = 1/R \) is the local curvature. It is worth remarking, however, that the utilization of (58) generally leads to worse results than the straight and modified Newtonian theories, as shown in Fig. 22(a).

**Fig. 22. Hypersonic flow around a bi-convex airfoil (from Anderson 2004, and Eggers et al., NASA TM-1983, 1983), including numerical, straight Newtonian, and Newton-Busemann solutions for (a) \( \gamma = 1.4 \) and (b) \( \gamma = 1.05 \).**
The Newtonian theory tends to be more accurate for three-dimensional bodies of revolution than two-dimensional ones. This aspect is elaborated further below within the context of the Taylor-Maccoll theory for the hypersonic flow over cones.

To summarize, the straight Newtonian theory is preferable for straight bodies such as wedges or cones, and its predictions become increasingly more accurate as the Mach number increases (see Fig. 16 and Fig. 17c). On the other hand, the modified Newtonian theory is preferable for blunt bodies (Figs. 19a and 19b), since it takes into account the overpressure behind the near-normal portion of the nose shock. The centrifugal corrections in the Newton-Busemann theory tend to degrade the predictions in all cases. Remarkably, the latter become more accurate than the straight and modified Newtonian theories in the limit \( \gamma \to 1 \) (see Fig. 22a). The relevance of the \( \gamma \to 1 \) is described below.

The combined limit \( M_\infty \to \infty \) and \( \gamma \to 1 \).

It is interesting to note that the Newton-Busemann theory generally worsens the predictions except for the case in which \( \gamma \to 1 \), as revealed in Fig. 22a, where it clearly matches the numerical simulations (also at \( \gamma \to 1 \)) better than the straight Newtonian theory, as expected. As a consequence, two important questions arise from these observations: 1) Is the Newtonian theory a simplified representation of the solution to the conservation equations in the limit \( \gamma \to 1 \), \( M_\infty \to \infty \), and \( \gamma \to 1 \)? 2) Why is the modified Newtonian theory accurate for some problems where \( \gamma \to 1 \)? The answer to the first question is yes, as shown below, and consequently, the answer to the second question is that the agreement appears to be fortuitous.

Consider the oblique-shock jump conditions (6)-(44) along with relation (20) between \( \beta \) and \( \delta \) in the limit \( M_\infty \sin^2 \beta \gg 1 \), corresponding to very large Mach numbers (particularly, large enough to make the normal Mach number \( M_{\infty, n} = M_\infty \sin \beta \) much larger than unity), then equations (6)-(9).
BECOME (22) - (25). NOTE THAT THOSE EXPRESSIONS ARE VALID FOR ANY ANGLE OF INCIDENCE IN SO FAR AS \( M_0, \sin^2 \beta \gg 1 \), OR EQUIVALENTLY, \( \sin \beta \gg \mu H \), so that \( \mu H \ll \beta \ll \frac{\pi}{2} \), with \( \mu H = \tan \left( \frac{L}{H} \right) \), THE MACH ANGLE.

IN THIS LIMIT, THE RELATION (20) BECOMES

\[
\tan \beta = \frac{2 \cos \beta \sin \beta}{\gamma + 1 - 2 \sin^2 \beta} \Rightarrow \tan (\beta - \delta) \approx \left( \frac{\gamma-1}{\gamma+1} \right) \tan \beta. \tag{60}
\]

IN INTERPRETING (22)-(25) AND (60), IT IS IMPORTANT TO NOTE THAT THE STREAMLINE DEFLECTION ANGLE \( \delta \) AND THE JUMPS OF DENSITY \( \rho_2/\rho_1 \) AND NORMAL VELOCITY \( U_{2n}/U_{1n} \) DO NOT DEPEND ON THE MACH NUMBER, WHEREAS THE PRESSURE AND TEMPERATURE JUMPS, \( P_2/P_1 \) AND \( T_2/T_1 \), ARE UNBOUNDED BY THE MACH NUMBER. A SIMILAR QUANTITY THAT BECOMES INDEPENDENT OF THE MACH NUMBER IS THE PRESSURE COEFFICIENT (35). THE FLOW STRUCTURE AT \( M_0, \sin^2 \beta \gg 1 \) THEREFORE RESEMBLES THE ONE IN FIG. 23A.

\[ M_0 \approx \sqrt{\frac{(\gamma-1)\epsilon}{\gamma+1}} \left( \frac{1 - \beta^2}{2} \right) \]

\[ U_{2n} = \left( \frac{1}{1-\beta^2} \right) U_1 \sin \beta \]

\[ \theta \sim O(1) \]

\[ \beta \sim O(1) \]

\[ \delta \ll O(1) \]

\[ \text{(b) } \delta \ll \lambda \quad \beta \ll \mu H \quad \text{SLENDER BODY} \]

IF, ADDITIONALLY, THE BODY IS SLENDER \( \delta \ll \lambda \), THEN (60) BECOMES

\[
\beta - \delta \sim \left( \frac{\gamma - 1}{\gamma + 1} \right) \beta \Rightarrow \beta \sim \left( \frac{\gamma + 1}{2} \right) \delta. \tag{61}
\]

WHICH IS THE SAME AS THE RESULT OBTAINED IN EQ. (21). AS STRESSED ABOVE, HOWEVER, \( \delta \) AND \( \beta \) CANNOT BE ARBITRARILY SMALL, BUT \( \beta \gg \mu H \) FOR THE HYPERSONIC LIMIT (22)-(25) TO HOLD (i.e. \( M_0, \sin^2 \beta \gg 1 \)), ONLY IN THE LIMIT \( M_0 \to \infty \) CAN \( \beta \) AND \( \delta \) TEND TO ZERO WHILE SATISFYING \( M_0, \sin^2 \beta \gg 1 \). \tag{29}
The structure of the flow for the slender case is sketched in Fig. 23b.

In deriving the expressions for the post-shock velocity and Mach number, it is convenient to define the parameter

\[ \varepsilon = \frac{\gamma - 1}{\gamma + 1} \]  

(62)

Using the definition (62), equations (22) - (25) can be expressed as

\[ \frac{P_2}{P_1} = (1 + \varepsilon) \frac{\gamma}{\gamma + 1} \frac{Ma_2 \sin^2 \beta}{\varepsilon} \], \[ \frac{S_2}{S_1} = \varepsilon \], \[ \frac{T_2}{T_1} = (1 + \varepsilon) \frac{\gamma}{\gamma + 1} \frac{Ma_2 \sin^2 \beta}{\varepsilon} \], \[ \frac{U_{2m}}{U_{1m}} = \varepsilon \]  

(62)

While Eq. (60) becomes

\[ \tan(\beta - \delta) = \varepsilon \frac{\tan \beta}{\sin(\beta - \delta)} \Rightarrow \sin(\beta - \delta) = \frac{\varepsilon \tan \beta}{\sqrt{1 + \varepsilon^2 \tan^2 \beta}} \]  

(64)

In addition, the hypersonic limit \( Ma_{2m} \gg 1 \) of Eq. (40) is

\[ Ma_{2m} \approx \sqrt{\frac{(\gamma - 1)}{2\gamma \varepsilon}} \]  

(65)

In these variables, the post-shock velocity in Fig. 18(a) is

\[ U_2 = \frac{U_{2m}}{\sin(\beta - \delta)} = \frac{\varepsilon}{\sqrt{\beta}} U_{1m} \left( \frac{\tan \beta + \varepsilon^2}{\tan^2 \beta} \right)^{1/2} = \left( \frac{\tan \beta + \varepsilon^2}{\tan^2 \beta} \right)^{1/2} \frac{U_1}{\sin \beta} \sin \beta \]  

(66)

While the Mach number of the post-shock flow is

\[ Ma_2 = \frac{Ma_{2m}}{\sin(\beta - \delta)} \approx \frac{\varepsilon}{\sqrt{1 + \varepsilon}} \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2} \frac{1}{\varepsilon} = \sqrt{\frac{\varepsilon}{1 + \varepsilon}} \frac{1}{\varepsilon} \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2} \]  

(67)

For the slender case \( \beta \ll 1 \), but \( \beta \gg \theta \), then Eq. (60) becomes

\[ \beta - \delta \approx \varepsilon \beta \Rightarrow \beta \approx \frac{\delta}{1 - \varepsilon} \]  

(68)

Which, after substitution of (62), is exactly the same expression as (26).

In this case, the post-shock velocity is simply

\[ U_2 = \frac{U_{2m}}{\beta - \delta} = U_2 \]  

(69)

While the corresponding Mach number is

\[ Ma_2 = \frac{Ma_{2m}}{\beta - \delta} \approx \frac{\varepsilon}{\sqrt{1 + \varepsilon}} \frac{1}{\varepsilon} \beta \]  

The surface pressure coefficients are

\[ \gamma \approx 2 \left( 1 - \varepsilon \right) \sin^2 \beta \]  

(71)
For the case in Fig. 23a, and
\[
G_1 = 2(1 - \varepsilon) \beta^2
\] (71)

For the case in Fig. 23b. It is illustrative to inquire about the limit
\( \varepsilon \to 0 \) (i.e., \( r \to 1 \)) in the above equations, which are solutions of the
Euler equations in the hypersonic limit, and ask whether the resulting
expressions have any resemblance with the Newtonian theory.

In the limit \( \varepsilon \to 0 \), the jump conditions (63) become
\[
\frac{P_2}{P_1} \to M_2^2 \sin^2 \beta, \quad \frac{S_2}{S_1} \to 1 \to \infty, \quad \frac{T_2}{T_1} \to \varepsilon M_2^2 \sin^2 \beta \quad \text{(indeterminate)}
\]
\[
\frac{U_{2m}}{U_{1m}} = \varepsilon \to 0,
\] (72)

Thereby expressing that the post-shock flow becomes infinitely dense, it
has zero post-shock normal velocity, and its temperature is indeterminate.

In addition, the normal Mach number of the post-shock flow is
\[
M_{2m} = \varepsilon^{1/2} \to 0
\] (73)

indicating that the flow normal to the shock wave is infinitely slow
compared to the local speed of sound.

For the case in Fig. 4b(a), the relation (64) becomes
\[
\beta = 8
\] (74)

so that the shock wave and the surface of the body are coincident,
thereby yielding a shock layer of zero thickness through which the
fluid moves parallel to the surface at a velocity
\[
U_2 \approx U_1 \cos \delta
\]
(75) \( \rightarrow \) see resemblance to Fig. 16

which corresponds to an infinite Mach number
\[
M_{2m} \approx \frac{1}{\varepsilon^{1/2} \tan \delta} \to \infty
\] (76).

The surface pressure coefficient is, however, finite and equals
\[
G_1 = 2 \sin^2 \delta
\] (77)

which is exactly the same as the one predicted by the Newtonian theory,
[Eq. (45)]. The \( \varepsilon \to 0 \) limit for the slender case in Fig. 23b, yields
Similar results as those presented above, with the shock wave and the body surface becoming coincident in space, as schematically shown in Fig. 24.

\[
\begin{align*}
\frac{\rho_2}{\rho_1} & \approx 2 \sin \beta \\
\frac{\rho_2}{\rho_1} & \approx \frac{1}{\varepsilon} \sin \beta \\
\frac{U_2}{U_1} & \approx \frac{1}{\varepsilon} \sin \beta \\
\end{align*}
\]

(a) \( \delta = O(1) \)  
(b) \( \delta < 1 \)  

\( \beta = O(1) \)  
\( \beta < \delta \)  

(But \( \delta < \delta_{max} \), i.e., attached shock)

\( \varepsilon \rightarrow 0 \)

Gl...
The shock, which becomes infinite in the limit \( \varepsilon \to 0 \). Before elaborating more on that limit, it is illustrative to obtain a general expression for \( \varepsilon \) that does not involve the assumption of a perfect gas. It is important to note that the equivalence between \( \varepsilon = 0 \) and \( \gamma = 1 \) emerges because of the assumption \( \gamma = \frac{c_p}{c_v} \) with \( c_p \) and \( c_v \) being constants in the formulation, an assumption corresponding to a (calorically and thermally) perfect gas that is required to derive the Rankine-Hugoniot jump conditions \((6)-(11)\). These relations are obtained by combining Eqs. \((2)-(5)\) to obtain

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{P_2 - P_1}{\frac{A_2 - A_1}{A_2}} = -\frac{\Delta h}{\Delta v}
\end{array} \right. \\
\text{(Rayleigh line, with } \Delta v = \frac{S_4}{S_2} U_{2m} = \frac{S_2}{S_4} U_{2m})
\end{align*}
\]

\[
\begin{align*}
\frac{h_2 - h_1}{\frac{S_4}{S_2}} = \frac{1}{\frac{S_4}{S_2}} \left( \frac{P_2 - P_1}{\frac{S_4}{S_2}} \right) \quad \text{(Hugoniot curve)}
\end{align*}
\]

and then applying the perfect-gas equations \( P = \rho R_T \) and \( h = c_p T \), with \( c_p \) a constant specific heat at constant pressure. Note however that Eqs. \((38)-(52)\) are general conservation laws valid also for non-perfect gases. Insofar as chemical and thermodynamic equilibrium conditions prevail in the flow (equilibrium conditions are discussed later in the text), in these general conditions, an expression for the density ratio can be obtained directly from \((79)\) as

\[
\varepsilon = \frac{\frac{S_4}{S_2}}{\frac{P_2 - P_1}{(h_2 - h_1)S_2 - (P_2 - P_1)}}
\]

In the high-Mach number limit, \( P_2/P_1 \gg 1 \) and \( h_2/h_1 \gg 1 \), and the above expression simplifies to

\[
\varepsilon = \frac{\frac{\frac{S_4}{S_2}}{h_2}}{\frac{P_2}{h_2} + \frac{P_2}{S_2}} = \frac{\frac{P_2}{S_2}}{h_2} + \varepsilon_2 (80)
\]

where use of the relation \( h_2 = e_2 + \frac{P_2}{S_2} \) has been made. In this way, \( \varepsilon \) is a sole function of the post-shock thermodynamic state. Note that for a perfect gas, \( P_2/S_2 = R_T T_2 \), \( h_2 = c_p T_2 \), \( e_2 = c_v T_2 \), \( \gamma = c_p/c_v \), and \((80)\) becomes

\[
\varepsilon = \frac{\frac{\frac{S_4}{S_2}}{h_2}}{\frac{P_2}{S_2} + \varepsilon_2} = \frac{\gamma - 1}{\gamma + 1} (86)
\]

*Note: In diatomic gases with vibrational excitation, \( \varepsilon \) increases, which increases the denominator of \((80)\) and decreases \( P_{2h} \) and \( P_{2v} \). In dissociation cases (endothermic process), \( h_2 = e_2 + \frac{P_2}{S_2} \), which increases \( h_2 \), \( \varepsilon_2 \to \infty \), and decreases \( \varepsilon \).*
The Newtonian limit therefore corresponds to the limit in which the post-shock gases are infinitely dense, which necessarily requires zero cross section of the shock layer by mass conservation. In perfect monoatomic gases, \( \gamma = 5/3 \), and \( \gamma = 4/3 \). In perfect diatomic gases, \( \gamma = 7/5 \) and \( \gamma = 1/6 \). In both of these cases, \( \gamma > 1 \) and the Newtonian approximation does not necessarily apply. In more realistic conditions typically encountered in the post-shock gases at hypersonic Mach numbers, the gases tend to undergo chemical dissociation and vibrational excitation, both of which tend to decrease \( \gamma \) in (80) below its perfect value (81), reaching density ratios of order \( \gamma \sim 1/5 \). This is to highlight that, despite the fundamental complexities associated with real hypersonic flow effects, these complexities tend to drive \( \gamma \) to smaller values in the realm of the Newtonian theory. It will be shown later in the course that the perfect-gas-based theory (6) - (11) leads to unrealistically high post-shock temperatures due to the finite and not-so-small density ratio (21) that is obtained by evaluating it with \( \gamma = 1.4 \).

For now, this can be understood by using the equation of state and expressions (6) - (7) for a calorically perfect gas,

\[
\frac{P_2}{P_1} = \frac{\gamma}{\gamma - 1} \left( \frac{T_2}{T_1} \right),
\]

which indicates that, for a fixed \( \gamma \), \( T_2 \) increases quadratically with the Mach number, thereby yielding unrealistically high post-shock temperatures of order \( T_2 \sim 60,000 \text{ K} \) at \( M_0 = 30 \) for a pre-shock gas at temperature \( T_1 = 300 \text{ K} \).

It will be shown later in the course that non-perfect effects, including chemical and vibrational excitation phenomena, come into play at high Mach numbers and moderate the temperature of the post-shock gases by influencing the denominator in equation (80).
The supersonic flow field around a cone differs from that around a wedge. In the latter, the velocity and thermodynamic state of the post-shock flow are uniform and equal to the solution prescribed by the shock jump conditions \((6)-\(11)\). On the other hand, the flow over a cone is an axisymmetric one where the streamlines diverge as the fluid particles move downstream. As a result, the velocity and thermodynamic state of the post-shock gas cannot be uniform. Note however that the shock jump conditions \((6)-\(11)\) are always locally valid, and they have to be matched by the post-shock gas flow at the shock wave.

The solution for the inviscid supersonic flow over a cone was first proposed by Busemann (1929) and later by Taylor and Maccoll (Proc. Roy. Soc. A 139, 1933). Assuming a semi-infinite cone, as in Fig. 26, a solution exists in which the flow variables only depend on the angle \(\Theta\). Equivalently, the flow variables are constant along every ray \(\Theta = \text{const.}\) emanating from the vertex, thereby rendering a conical flow field. It is important to note that such conical flow field is compatible with the boundary conditions at the body surface and at infinity if the flow is supersonic and inviscid. In contrast, such conicity is incompatible with subsonic flows or flows at finite Reynolds numbers subject to viscous effects near the body surface.

Consider a spherical coordinate system \((r, \Theta, \phi)\), where \(r\) is the distance from the cone vertex, \(\Theta\) is the latitude angle measured from the cone axis, and \(\phi\) is the longitude angle, which does not play any role in the present configuration at zero angle of attack. The flow velocity components in the radial and latitudinal directions are denoted as \(u\) and \(v\), respectively.
Since the flow is irrotational, the vorticity in the \( \phi \) direction must vanish,

\[
\omega_{\phi} = \frac{1}{\gamma} \left[ \frac{2}{\partial \tau} (rv) - \frac{\partial u}{\partial \phi} \right] = 0 \Rightarrow \frac{d v}{d \phi} - v = 0 \quad (82)
\]

Similarly, the equation of continuity requires

\[
\frac{1}{\gamma} \frac{\partial}{\partial \tau} (\rho v^2) + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} (\rho u \sin \phi) = 0 \Rightarrow \frac{\partial}{\partial \phi} (\rho u \sin \phi) = 0 \quad (83)
\]

Combining (82) and (83) leads to the equation

\[
\frac{d^2 u}{d \phi^2} + \frac{1}{\sin \phi} \frac{d u}{d \phi} + \cot \phi \frac{d u}{d \phi} + Z u = 0 \quad (84)
\]

which can be further simplified by making use of the Bernoulli equation (13)

written in the form

\[
\frac{\gamma}{\gamma - 1} \left( \frac{v^2}{2} + \frac{p}{\rho} \right) = \frac{\gamma}{\gamma - 1} \frac{p_{02}}{\rho_{02}} \quad (85)
\]

where the enthalpy \( h = c_p T \) has been rewritten as \( h = \frac{\gamma}{\gamma - 1} \left( \frac{p}{\rho} \right) \) by making use of the relation \( c_p - c_v = R_0 T \), the definition of the adiabatic coefficient \( \gamma = c_p/c_v \), and the ideal gas equation of state \( p = \rho R_0 T \). Note that this analysis assumes a calorically perfect gas. In eq. (85), \( p_{02} \) and \( \rho_{02} \) refer to the stagnation pressure and density of the post-shock gases. Note that these are uniform in the post-shock gases since the flow is homentropic. As a result, the stagnation pressure \( \rho_{02} \) and temperature \( T \) are uniform downstream of the shock since the shock is isentropic.

Pressure and density are also related to their corresponding static values through the isentropic trajectory

\[
\frac{p}{\rho_{02}} = \frac{p_{02}}{\rho_{02}} \quad (86)
\]

as observed from eq. (17). In this way, (85) can be rewritten as

\[
\left( \frac{\rho}{\rho_{02}} \right)^{\gamma - 1} = 1 - \left( \frac{\gamma - 1}{2} \right) \left( \frac{v^2 + u^2}{\rho_{02}^2} \right) \quad (87)
\]

where \( \alpha_{02} \) is the stagnation speed of sound of the post-shock gas (also defined in eq. 17). Note that, according to (82), the velocity \( \left[ \frac{2}{\gamma (\gamma - 1)} \right]^{\frac{1}{\gamma - 1}} \alpha_{02} \) corresponds to the velocity of the gas if it was flowing in near-vacuum conditions (§20).

Differentiating the density in (87) and substituting (82) gives

\[
\left( \frac{\rho}{\rho_{02}} \right)^{\gamma - 1} \frac{d \rho}{d \phi} + \frac{d u}{\rho_{02}^2} \left( \frac{d u}{d \phi} + \frac{d v}{d \phi} \left( \frac{d^2 u}{d \phi^2} \right) \right) = 0 \quad (88)
\]
However, an expression for \( \frac{d^2 u}{d \theta^2} \) can be also obtained from (84), namely

\[
\frac{d}{d \theta} \frac{d^3 u}{d \theta^2} = \frac{2u + \cosec \theta \frac{du}{d \theta} + \frac{d^2 u}{d \theta^2}}{du/d \theta} . \tag{84}
\]

Substituting this expression into Eq. (88) yields

\[
- \left( \frac{3}{8 \alpha_0^2} \right)^{\gamma - 1} \left[ \frac{2u + \cosec \theta \frac{du}{d \theta} + \frac{d^2 u}{d \theta^2}}{du/d \theta} \right] + \frac{1}{a_0^2} \left\{ \frac{du}{d \theta} \left( \frac{d^2 u}{d \theta^2} + \frac{du}{d \theta} \right) \right\} = 0 . \tag{90}
\]

Lastly, substituting (87) and (82) into (90) gives

\[
\left\{ \frac{1}{1 - \left( \frac{3}{2 \alpha_0^2} \right) \left[ \frac{u^2 + \left( \frac{du}{d \theta} \right)^2}{a_0^2 \frac{du}{d \theta}} \right]} \right\} \left[ \frac{2u + \cosec \theta \frac{du}{d \theta} + \frac{d^2 u}{d \theta^2}}{du/d \theta} \right] = \frac{1}{a_0^2 \left( \frac{du}{d \theta} \right)} \left[ \frac{d^2 u}{d \theta^2} \right] \frac{du}{d \theta} + \frac{d^2 u}{d \theta^2} \right\} . \tag{91}
\]

which represents a non-linear ordinary differential equation for \( u(\theta) \) given

a post-shock stagnation speed of sound \( \alpha_0^2 \). Note that \( \alpha_0 \) is related to

\( \alpha_0 \) by the continuity of the stagnation enthalpy across the shock, \( \alpha_0 = \alpha_0^2 \).

As a result, the solution to (91) is specified once the stagnation temperature

of the incoming flow is provided to the equation. In addition, the non-penetration

boundary condition

\[
v = \frac{du}{d \theta} = 0 \quad \text{AT} \quad \theta = \delta \quad \tag{92}
\]

has to be satisfied on the surface of the cone, while the post-shock

velocities

\[
U = U_{2*} = U_{4*} = U_4 \cos \beta \quad \text{AT} \quad \theta = \beta \quad \tag{93}
\]

\[
v = \frac{dU_2}{d \theta} = \frac{U_4}{\delta} = - \sin \beta \quad \tag{93}
\]

have to be recovered immediately downstream of the oblique shock. It is

important to note that the incidence angle \( \beta \) in (93) is part of the solution.

Note that \( \beta \) and \( \delta \) are not actually related through Eq. (20) in this problem

since the local deflection angle of the streamlines

is not necessarily \( \delta \) (see Fig. 27) because of the

3D flow displacement effect created by the conical geometry.

As a result, (91) - (93) has to be solved iteratively

by first assuming an angle \( \beta \), computing the values...
\[ \frac{u_0}{\theta} \text{ at } \theta = \theta \text{ from (9.3), integrating (9.4), and checking whether the condition (9.2) is satisfied.} \]

The stagnation speed of sound \( a_{02} \) is related to the speed of sound in the undisturbed gas \( a_{04} \) by

\[
\frac{a_{02}^2}{a_{04}^2} = \frac{a_{04}^2}{a_{04}^2} = 1 + \left( \frac{y-1}{2} \right) \frac{M_{04}^2}{2} \Rightarrow \alpha^2 = \frac{a_{02}^2}{1 + \left( \frac{y-1}{2} \right) M_{04}^2} \quad (9.4)
\]

Similarly, the distribution of \( P \) and \( \rho \) anywhere behind the shock is given by combining Eqs. (85) and (86) to eliminate the density, which gives

\[
\left( \frac{P}{P_{02}} \right)^{y-1} \left( \frac{\rho}{\rho_{02}} \right)^{y-1} = 1 - \left( \frac{y-1}{2} \right) \frac{(u^2 + v^2)}{a_{02}^2} \quad (9.5)
\]

with \( P_{02} \) and \( \rho_{02} \) being related to the stagnation quantities in the undisturbed stream, \( P_{04} \) and \( \rho_{04} \) through the shock jump condition (41).

There are a number of important differences between the flow over a 2D wedge and the flow over a cone. Perhaps the most important ones pertain to the relieving effect caused by the conical geometry, which, for the same Mach number, leads to a weaker shock in the cone case (i.e., \( P_{cone} < P_{wedge} \)), and correspondingly creates lower values of \( P, T \) and \( \rho \) on the cone surface, as sketched in Fig. 28(a). For the same reason, the nose shock tends to become detached earlier in the wedge than in the cone as the surface inclination angle increases at a given Mach number of the free stream, as depicted in Fig. 28(b).

Figures 29 and 30 show the shock angle \( \beta \) and the pressure coefficient \( C_p \) as a function of the Mach number of the free stream and for a number of surface inclination angles \( \theta \). Similar to the wedge case in Fig. 8, there exist strong and weak solutions of the shock, as well as a critical value of \( M_{04} \) below which there is no solution for an attached shock.
AND THE SHOCK MUST BECOME DETACHED AS IN FIG. 28b. AN EXAMPLE OF A THREAD OF EXPERIMENTAL PHOTOGRAPHS OF THE SUPersonic FLOW AROUND CONICAL PROJECTILES IN FLIGHT IS PROVIDED IN FIG. 31 THAT ILLUSTRATES THE TRAJECTORY A→B IN FIG. 29 BY WHICH THE DECREASE IN \( \text{Ma}_{2} \) CAUSES THE SHOCK TO BE DETACHED FROM THE PROJECTILE. WHILE THE MAXIMUM SURFACE INCLINATION ANGLE FOR WHICH AN ATTACHED SHOCK SOLUTION EXISTS IN A WEDGE IS \( \theta_{\text{max}} \approx 45.6^\circ \) (SEE FIG. 8), THE CORRESPONDING VALUE FOR THE CONE CAN BE OBTAINED NUMERICALLY AND IS EQUAL TO \( \theta_{\text{max}} \approx 54.6^\circ \).

IN Figs. 24−25, THE HYPERSONIC LIMIT CORRESPONDS TO \( \text{Ma}_{1}^2 \sin^2 \beta \gg 1 \) \( (\text{Ma}_{1}^2 \gg 1) \), WHICH IS OBSERVED TO LEAD TO A PLATEAU IN ALL CURVES WHERE \( \beta \approx \delta \), WITH DIFFERENCES \( \beta - \delta = 0 \text{(8.6)} \), WHERE \( \delta = \delta_{1}/\delta_{2} \).

THIS POINT CAN BE FORMALLY PROVEN BY EXPANDING THE VELOCITIES \( U \) AND \( V \) NEAR THE CONE SURFACE
\[
(U \approx U_{S}(1-(\beta-\delta)^2), \quad V \approx U_{S}(\beta-\delta)),
\]
with \( U_{S} \approx U_{S} \cos \beta \) THE TANGENTIAL VELOCITY ON THE CONE SURFACE), AND THEN SUBSTITUTING THESE APPROXIMATIONS INTO (93). NOTE THAT THE STRAIGHT NEWTONIAN THEORY \( q_{1} = 2 \sin^{2} \delta \) PROVIDES A REASONABLE RESULT IN THE HYPERSONIC LIMIT WHEN COMPARED TO THE NUMERICAL SOLUTION (FIG. 30).
MACH-NUMBER INDEPENDENCE PRINCIPLE

Most engineering applications in hypersonics involve cruising or re-entering vehicles at Mach numbers in the range \( 5 \leq M_{\infty} \leq 30 \). However, most hypersonic test facilities are capable of achieving Mach numbers up to approximately \( M_{\infty} \approx 10 \) (see page 22 for a discussion on experimental facilities). The results presented in Figs. 16, 17 and 30 suggest that some of the results may become independent as the Mach number increases to large values. This independence tends to be invoked to overcome the deficiencies of experimental facilities in reproducing real flight conditions.

The Mach-number independence principle was derived by Oswatitsch (ZAMP, 249-264 (1951)) for inviscid flow. For blunt bodies, such as the one sketched in Fig. 32, the Mach-number independence is achieved for \( M_{\infty} \gg 1 \) (\( M_{\infty} > 5 \), see Fig. 17(a, b)) for slender bodies, the Mach-number independence is achieved for \( M_{\infty} \ll 1 \) so as to make the normal Mach number \( M_{\infty,n} \gg 1 \). As a result, the Mach number independence is achieved at much larger free-stream Mach numbers \((M_{\infty} \approx 10, \text{ see Figs. 16 and 17c})\) for slender bodies.

In Oswatitsch's description of this principle, the following variables become independent of the Mach number: the force and moment coefficients, the lift-to-drag ratio, the bow-shock shape, the bow-shock stand-off distance, the flow patterns, the velocity field \( \vec{U}/U_\infty \), the density field \( \rho/\rho_\infty \), the pressure field \( p/p_\infty U_\infty^2 \), and the temperature field \( T/T_\infty U_\infty^2/c_p \) (note the appropriate factors used for non-dimensionalization). It is important to note that there is currently no proof that the Mach-number independence principle holds also in the presence of viscous and complex effects inherent to hypersonic flows (i.e., chemical effects, vibrational excitation, radiative processes, etc.). The discussion below is therefore

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(33)
CONCERNED TO CALORICALLY-PERFECT INVISID FLOWS.

CONSIDER THE CONSERVATION EQUATIONS FOR AN INVISID, ADIABATIC FLOW

\[ \nabla \cdot (\mathbf{u}^T \mathbf{u}) = 0 \quad (96) \]

\[ \nabla \cdot (\mathbf{u}^T \mathbf{u}^T) = -\nabla P \quad (97) \]

\[ \nabla \cdot (\mathbf{u} \mathbf{u} \mathbf{u}) = 0 \quad (98) \]

EQUATIONS (96) - (98) REPRESENT, RESPECTIVELY, THE CONSERVATION OF MASS, MOMENTUM AND ENTRPY, AND ARE TO BE APPLIED TO SOLVE THE FLOW IN FIG. 27 IN THE REGION BETWEEN THE BODY SURFACE AND THE NOSE SHOCK. NOTE THAT THE FLOW IN THIS REGION IS ISENTROPIC SINCE THERE ARE NO SOURCES OF IRREVERSIBILITY OTHER THAN THE SHOCK.

THE BOUNDARY CONDITIONS ASSOCIATED WITH THE INTEGRATION OF (96) - (98) ARE THE NO-PENETRATION CONDITION ON THE BODY SURFACE

\[ \mathbf{u} \cdot \mathbf{n} = 0 \quad (99) \]

ALONG WITH THE POST-SHOCK JUMP CONDITIONS IMPOSED BY Eqs. (67) - (71) AND (72), NAMELY

\[ \frac{P}{P_\infty} = 1 + \frac{2\gamma}{\gamma+1} \left( \frac{M_{\infty}^2 \sin^2 \beta - 1}{\gamma+1} \right), \quad \frac{\mathbf{u}}{U_\infty} = \frac{M_{\infty}^2 \sin^2 \beta - 1}{\gamma+1} \left( M_{\infty}^2 - 1 \right) \cos \beta \quad (100) \]

AT THE SHOCK POSITION \( x = x_0 (y) \). EQUATIONS (96) - (100) ARE SUPPLEMENTED WITH THE DEFINITION OF ENTROPY FOR A CALORICALLY-PERFECT GAS

\[ S - S_0 = C_v \ln \left( \frac{P/\gamma}{P_0/\gamma_0} \right) \quad (101) \]

WHERE \( S_0, P_0, \) AND \( \gamma_0 \) CORRESPOND TO AN REFERENCE STATE. NORMALIZING THE SPATIAL VARIABLES WITH \( R_0, \) THE VELOCITIES WITH \( U_\infty, \) THE PRESSURE WITH \( P_0 U_\infty^2, \) THE DENSITY WITH \( \rho_0, \), AND THE ENTROPY WITH \( C_v, \) THE EQUATIONS (96) - (98) BECOME

\[ \nabla \cdot (\mathbf{u}^T \mathbf{u}^T) = 0 \quad (102) \]

\[ \nabla \cdot (\mathbf{u}^T \mathbf{u}^T \mathbf{u}^T) = -\nabla P \quad (103) \]
\( \nabla \cdot ( \rho^* U^* S^* ) = 0 \), \quad (104) \\

While (94) is simply 
\[ U^* \cdot \mathbf{M}^* = 0 \] \quad (105) \\

Similarly, in the limit \( M_{\infty}^2 \rightarrow 10 \), the boundary conditions (100) become 
\[ P^* = \frac{Z \gamma \sin^2 \beta}{Y + 1}, \quad S^* = \left( \frac{Y + 1}{Y - 1} \right) \] \quad (106) \\
\[ U^* = \nu = \frac{Z \sin^2 \beta}{(Y + 1)}, \quad V^* = \frac{\sin^2 \beta}{Y + 1} \]

With (104) being rewritten as 
\[ S^* = \nu \left( \frac{P^*}{S^* V^*} \right) \] \quad (107) \\

In equations (106), the shock angle is a solution of this free-boundary problem that becomes increasingly independent of the Mach number as observed from the limiting relation (60). As a result, the dimensionless problem (102)-(107) becomes independent of the Mach number as illustrated by the Oswatitsch's principle.

Lastly, it is interesting to note that, as altitude increases, the density decreases and the mean free path increases. For a given Mach number, this effect causes a decrease in the Reynolds number (recall that the Knudsen, Reynolds and Mach numbers are related as \( K_n \sim M_{\infty} / R_n \)). The decrease in the Reynolds number makes viscous effects to become increasingly important, thereby hindering the type of straightforward discussion presented in this section. Viscous effects are discussed in Chapter III of these notes.

**Small Disturbance Theory of Hypersonic Flows**

Before introducing the conservation equations for small disturbances in hypersonic flows, which are useful to examine high-speed flows over slender bodies, it is illustrative to summarize the characteristics of the flow fields around oblique shocks at supersonic and hypersonic velocities.
SUPERSONIC FLOW
OVER NON-SLENDER BODIES

Ma₁ > 1 BUT NOT LARGE
Ma₁ sin β = 0(1), Ma₁ sin δ = 0(1),

sin β ≤ 1 BUT NOT SMALL
(β = 0(1))

sin δ ≤ 1 BUT NOT SMALL
(δ = 0(1))

β = 8 + O(1)

U₁ = U₁, P₁ = P₁

U₂ = U₂, P₂ = P₂

THE SUPERSONIC FLOW OVER NON-SLENDER BODIES IS CHARACTERIZED
BY ORDER-UNITY DISTURBANCES IN ALL VARIABLES WITH
RESPECT TO THEIR FREE-STREAK VALUES.

SUPERSONIC FLOW
OVER SLENDER BODIES

Ma₁ > 1 BUT NOT NECESSARILY LARGE
Ma₁ sin β = 1, Ma₁ sin δ << 1,

sin β = 1

Ma₁ sin δ = d << 1

δ = 8 + O(1)

β = 8 + O(1)

U₁ = U₁, P₁ = P₁

U₂ = U₂, P₂ = P₂

THE SUPERSONIC FLOW OVER SLENDER BODIES RESULTS IN WEAK SHOCKS
THAT ARE CHARACTERIZED BY SMALL FLOW DEFORMATIONS, NEAR-UNITY
NORMAL MACH NUMBERS, AND SMALL DISTURBANCES IN ALL VARIABLES.

POST-SHOCK GAS:

\[ \frac{U_{2m}}{U_{1m}} = \frac{\gamma}{\gamma-1} \]

\[ \frac{P_{2m}}{P_{1m}} = 1 + O(M_{a}^{2} \sin^{2} \beta - 1) \]

\[ \frac{M_{2m}^{2}}{M_{1m}^{2}} = \frac{\gamma M_{a}^{2}}{\gamma-1} \frac{P_{1m}}{P_{2m}} \frac{1}{\sin^{2} \beta} \]

\[ \sin (\beta - 8) = \sin \beta - O(M_{a}^{2} \sin^{2} \beta - 1) \approx 8 + O(1) \]

\[ U_{2} = U_{1} - O(M_{a}^{2} \sin^{2} \beta - 1) \ll U_{1} \]

\[ V_{2} = 0(M_{a}^{2} \sin^{2} \beta - 1) \cdot U_{1} = 0(8) \ll U_{1} \]

WEAK SHOCK
(MACH WAVE)

\[ \frac{P_{2}}{P_{1}} \approx 1 + O(M_{a}^{2} \sin^{2} \beta - 1) \approx 1 \]

\[ \frac{M_{a}^{2}}{M_{1m}^{2}} = \frac{\gamma M_{a}^{2}}{\gamma-1} \frac{P_{1m}}{P_{2m}} \frac{1}{\sin^{2} \beta} \]

\[ \beta = 8 + O(1) \]

\[ \frac{U_{2}}{U_{1}} = O(8) \ll U_{1} \]
**Hypersonic Flow Over Non-Slender Bodies**

\[
E = \frac{U_{2m}}{U_{1m}} = \frac{\gamma + 1}{\gamma - 1} \quad \text{at } Ma_1 \to 0
\]

\[
P_2 \approx Ma_1^2 \sin^2 \beta \quad \text{at } Ma_1 \to 0
\]

\[
Ma_{2m} = E \ll 1
\]

\[
U_2 \approx U_1 \cos \beta
\]

\[
Ma_2 \approx \frac{1}{\epsilon^{1/2}} \gg 1
\]

\[
\sin(\beta - \delta) \approx E \tan \beta \to \beta \approx \delta + \mathcal{O}(E)
\]

\[
U_2 \approx [1 - \mathcal{O}(\sin^2 \beta)] U_1 < U_1
\]

\[
V_2 \approx 0 \quad \text{(small disturbances)}
\]

The hypersonic flow over non-slender bodies results in large normal Mach number of the incident stream, small normal Mach number of the post-shock gases (almost zero post-shock normal velocity), large pressure and density increments, and hypersonic post-shock gases with a shock enveloping the surface of the body, and becoming coincident with it in the limit \( \gamma \to 1 \).

---

**Hypersonic Flow Over Slender Bodies**

\[
E = \frac{U_{2m}}{U_{1m}} = \frac{\gamma + 1}{\gamma - 1} \quad \text{at } Ma_1 \to 0
\]

\[
P_2 \approx Ma_1^2 \sin^2 \beta \quad \text{at } Ma_1 \to 0
\]

\[
Ma_{2m} = E \ll 1
\]

\[
U_2 \approx U_1
\]

\[
Ma_2 \approx \frac{1}{\epsilon^{1/2}} \gg 1
\]

\[
\sin(\beta - \delta) \approx \beta - \delta \approx E \beta \to \beta \approx \delta/1 - E \ll 1
\]

\[
U_2 \approx \left(1 - \frac{2 \sin^2 \beta}{\delta + 1}\right) U_1 \quad \text{(small disturbances)}
\]

\[
V_2 \approx \frac{\sin(2\beta)}{\epsilon^{1/2}} U_1 = \mathcal{O}(E) U_2 < U_1
\]

The hypersonic flow over slender bodies is similar to the case over non-slender bodies, but here the disturbances of the streamwise and spanwise velocity components are small.
The results of these four subdivisions are summarized in Fig. 33 in the form of a $\beta$ vs $\delta$ chart analogous to that sketched in Fig. 8 and resulting from solving equation (20). For a slender body of semiangle $\delta \ll 1$, sufficiently large Mach numbers $M_a \gg \frac{1}{\sin \delta}$ are required to enter in the hypersonic range, such that the normal Mach $M_{\text{sh}} = M_1 \sin \delta \approx 1$ and the post-shock temperature is large. All other smaller Mach numbers will tend to yield supersonic flows with $M_{\text{sh}} \sin \beta \ll 1$ and weak shocks.

It is worth mentioning that, whereas the flows over non-slender bodies typically do not admit solutions in terms of small perturbations, both supersonic and hypersonic flows admit solutions in the form of small perturbation theory, the former yielding linear equations for the disturbance and the latter (hypersonic flow) requiring the retention of non-linear terms, as shown below.
Consider the hypersonic flow over the slender body sketched in Fig. 34. Because of the flow deflection caused by the enveloping shock, streamwise \((u')\) and spanwise \((v')\) velocity disturbances are created with respect to the streamwise pre-shock velocity \(U_{\infty}\). As shown in the analysis performed earlier in this section, the post-shock velocities \(u_2\) and \(v_2\) are:
\[
\begin{cases}
  u_2 = U_{\infty} + u' \\
  v_2 = v' = O(\delta^2 u_{\infty}) \text{ and positive}
\end{cases}
\]

Note that both \(u'\) and \(v'\) are much smaller than \(U_{\infty}\), but not necessarily smaller than the local speed of sound \(a_2\):
\[
\frac{u'}{a_2} = \frac{\delta^2 U_{\infty}}{a_\infty} = \delta^2 \frac{U_{\infty}}{a_\infty} \frac{a_{\infty}}{a_2} = \delta^2 \frac{U_{\infty}}{a_\infty} \left( \frac{P_{\infty}}{P_2} \frac{\alpha_2}{\alpha_{\infty}} \right)^{1/2} \approx \delta^2 \frac{U_{\infty}}{a_\infty} \left( \frac{1}{\rho_{\infty}} \frac{\delta^2}{\epsilon} \right)^{1/2} = \frac{\delta}{\epsilon^{1/2}}
\]

\[
\frac{v'}{a_2} = \frac{\delta U_{\infty}}{a_\infty} \frac{a_{\infty}}{a_2} = \frac{\delta}{\epsilon^{1/2}}
\]

Where \(\delta \ll \lambda\) and \(\epsilon \ll \lambda\).

As mentioned earlier, the post-shock gas is deflected at an angle comparable to the local inclination angle of the surface \(\delta\). As a result, \(v' = \frac{dy}{dx} = O\left(\frac{\epsilon}{\lambda}\right)\) \((108)\)

With \(y = y_s(x)\) the equation of the body surface, \(c\) the width of the body and \(\lambda\) its length. Since the body is slender:
\[
\frac{c}{\lambda} \approx \tan 5 = \delta = \gamma \ll \lambda
\]

Where \(\gamma\) is called the aspect or slenderness ratio and is assumed here to be a small parameter. In this limit, and since \(u'/U_{\infty} \ll 1\), equation \((108)\) becomes:
\[
\frac{v'}{U_{\infty}} = O(\gamma) \ll 1 \Rightarrow \frac{v'}{a_{\infty}} = O\left(\frac{\delta}{\rho_{\infty}} \frac{\epsilon}{\gamma}\right) \quad (110)
\]

Whereas
\[
\frac{u'}{a_{\infty}} = O\left(\delta^2 \rho_{\infty}\right) = O\left[2 \left(\frac{\rho_{\infty}}{\rho_{\infty}}\right)^{1/2}\right] \quad (111)
\]

It will be shown below that the parameter
\[
\gamma = M_{\infty} \gamma (112) \Rightarrow \text{Tsiens Hypersonic Similarity Parameter (Tsiens, J. Math. Phys. 1946)}
\]

is an important similarity parameter to describe hypersonic flows over slender bodies.
In many applications it is desired to have aerodynamic shapes that cause small flow disturbances, as in Fig. 35.1. In supersonic flows, the assumption of slenderness (109) leads to a useful linearized theory (see pages 67-75 of the ME 355 class notes). The linearized theory, however, breaks down in the transonic range and at hypersonic flow speeds. In both of those regimes, non-linearities play an essential role despite the fact that the velocity disturbances are small compared with \( U_0 \) (note that viscous effects must be necessarily neglected in this context since the non-slip condition leads to no longer small velocity disturbances of order \( U_0 \)).

While the linearized supersonic theory requires \( \frac{c}{U_0} \ll 1 \) (or more precisely, \( \sqrt{\frac{c^2}{M_{in}^2} - 1} \ll 1 \)), such that the maximum slope of the body is much smaller than the slope of the free-stream Mach cone (i.e., \( \tan \theta M = \frac{1}{\sqrt{\frac{c^2}{M_{in}^2} - 1}} \)), the small-disturbances theory of hypersonics requires \( c \ll 1 \), so that \( \frac{c}{U_0} = 0(1) \) or much larger than unity.

The notation below follows Anderson's book notation, which follows the notation in Van Dyke's "A Study of Hypersonic Small-Disturbance Theory." NACA Tech. Note 3473, but the formulation here is illustrated only for 2D flows. Consider the slender body in Fig. 34. A non-linear formulation based on small velocity disturbances exists that provides the solution to the rotational flow behind the nose shock in the following way. Particularizing Eqs. (96) - (98) for a Cartesian system gives

\[
\frac{\partial}{\partial x} (u^2) + \frac{\partial}{\partial y} (uv) = 0 \quad (114)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -\frac{1}{S} \frac{\partial p}{\partial y} \quad (115)
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (117)
\]
WITH \( S \) GIVEN BY (104). SIMILARLY, THE ASSOCIATED BOUNDARY CONDITIONS ARE GIVEN BY (99) (NON-Penetration ON THE BODY SURFACE) AND (100) (POST-SHOCK GAS CONDITIONS). IN THE RESCALED (BARRED) VARIABLES

\[
\begin{align*}
\vec{x} &= \bar{x} \lambda, \quad \vec{y} &= \bar{y} \lambda \bar{c}, \quad \vec{u} &= U_0 \left( 1 + \bar{c}^2 \bar{u} \right), \quad \vec{v} &= U_0 \bar{c} \bar{v}, \\
\vec{P} &= \bar{P} \bar{M}_w^2 \bar{c}^2 \bar{P}_0 \bar{P}, \quad \bar{s} &= \bar{S} \bar{S}_0, \quad (118)
\end{align*}
\]

THE CONSERVATION EQUATIONS BECOME

\[
\begin{align*}
\frac{\partial \vec{u}}{\partial \bar{x}} + \frac{\partial \vec{v}}{\partial \bar{y}} &= 0 \quad (119) \quad \frac{\partial \bar{P}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{P}}{\partial \bar{y}} &= -\frac{1}{\bar{S}} \frac{\partial \bar{S}}{\partial \bar{y}} \quad (121) \\
\frac{\partial \vec{v}}{\partial \bar{x}} + \bar{u} \frac{\partial \vec{v}}{\partial \bar{y}} &= -\frac{1}{\bar{S}} \frac{\partial \bar{P}}{\partial \bar{x}} \quad (120) \quad \frac{\partial}{\partial \bar{x}} \left( \bar{P} \frac{\partial \bar{y}}{\partial \bar{x}} \right) + \bar{v} \frac{\partial}{\partial \bar{y}} \left( \bar{P} \frac{\partial \bar{y}}{\partial \bar{x}} \right) &= 0 \quad (122)
\end{align*}
\]

WHERE TERMS OF ORDER \( \bar{c}^2 \) HAVE BEEN NEGLECTED. NOTE THAT \( \bar{u} \) AND \( \bar{v} \) PLAY THE ROLE OF NON-DIMENSIONAL VELOCITY DISTURBANCES SCALED WITH \( \bar{c} \) ACCORDING TO THE REASONING EXPLAINED AT THE BEGINNING OF THIS SECTION. IN CONTRAST, \( \bar{P} \) AND \( \bar{S} \) ARE NOT DISTURBANCES SINCE IT MAKES LITTLE SENSE TO EXPAND THESE VARIABLES BECAUSE THE PRESSURE AND DENSITY VARIATIONS ARE TYPICALLY LARGE COMPARED TO \( P_0 \) AND \( S_0 \) IN HYPERSONIC FLOWS.

IN THESE VARIABLES, THE BOUNDARY CONDITION (99) BECOMES

\[
\bar{M}_x + \bar{m}_y \bar{v} = 0
\]

WHERE TERMS OF ORDER \( \bar{c}^2 \) HAVE BEEN NEGLECTED. SIMILARLY, THE CONDITIONS (100) BEHIND THE SHOCK BECOME

\[
\begin{align*}
\bar{P} &= \frac{2}{\gamma + 1} \left[ \left( \frac{d \bar{y}_s}{d \bar{x}} \right)^2 + \frac{1 - \gamma}{2 \gamma} \frac{1}{\bar{M}_w^2 \bar{c}^2} \right] \quad (123) \\
\bar{s} &= \left( \frac{\gamma + 1}{\gamma - 1} \right) \left[ \frac{d \bar{y}_s}{d \bar{x}} \right]^2 \frac{1}{\left( \frac{d \bar{y}_s}{d \bar{x}} \right)^2 + \frac{2}{(\gamma - 1)} \frac{1}{\bar{M}_w^2 \bar{c}^2} \right] \quad (124) \\
\bar{u} &= \frac{2}{\gamma + 1} \left[ \left( \frac{d \bar{y}_s}{d \bar{x}} \right)^2 - \frac{1}{\bar{M}_w^2 \bar{c}^2} \right] \quad (125) \\
\bar{v} &= \frac{2}{\gamma + 1} \left[ \left( \frac{d \bar{y}_s}{d \bar{x}} \right)^2 - \frac{1}{\bar{M}_w^2 \bar{c}^2} \right] \frac{1}{\left( \frac{d \bar{y}_s}{d \bar{x}} \right)^2} \quad (126)
\end{align*}
\]

WHERE \( \bar{y} = \bar{y}_s(\bar{r}) \) IS THE RESCALED BODY SURFACE SUCH THAT, FOR HYPERSONIC SPEEDS AND SLENDER BODIES,

\[
\sin \bar{\beta} = \bar{S} \approx \frac{d \bar{y}_s}{d \bar{x}} = \frac{d \bar{y}_s}{d \bar{x}} \quad (127)
\]
with \( \frac{d^2 \Phi}{d x^2} \) a quantity of order unity. From the analysis above it is observed that the problem depends only on 2 parameters: \( \gamma \) and \( M_{\infty} \) \( \sqrt{\gamma} \). As a result the solutions are expected to be of the form

\[
\begin{align*}
\Phi &= \Phi \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right), \\
\Psi &= \Psi \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right), \\
\Omega &= \Omega \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right), \\
\Theta &= \Theta \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right).
\end{align*}
\]

The corresponding surface pressure coefficient is

\[
\frac{d \Phi}{c^2} = F \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right)
\]

\( F \) is a functional obtained by solving the entire problem.

\[
\frac{d \Phi}{c^2} = F \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right)
\]

Equation (129) represents an important similarity rule, in that it correlates the surface pressure coefficient of hypersonic flows past slender bodies that differ from one another only by a uniform expansion or contraction of their thickness.

For instance, in Fig. 36, as the thickness is reduced from \( c_2 \) to \( c_1 \), in hypersonic flow, the similarity rule (129) states that the ratio of the local pressure coefficient at \( Q \) and the square of the thickness \( c^2 \) is the same for both bodies as long as Tsien's similarity parameter \( M_{\infty} \gamma \) is the same in both flows (provided that \( \gamma \) is also the same). This is illustrated in Fig. 36, which shows that, provided one

\[
\frac{d \Phi}{c^2} = F \left( \vec{x}, \vec{y}, \gamma, M_{\infty} \right)
\]

The utilization of the similarity rule (129) to transform that surface pressure into the one for a \( 5^\circ \) semivertex cone (i.e., \( \frac{d \Phi}{c^2} = \text{const} \)) yields a result.
That is close to the numerical result as long as $M_{\infty} \gg 1$. As the Mach number decreases, however, the similarity parameter becomes small, $M_{\infty} \ll 1$, and the small-disturbances theory described above is no longer applicable (note that $p$, $\bar{u}$, and $\bar{v}$ cease to be of order unity when $M_{\infty} \ll 1$ as observed from (123)-(126)). The limit $M_{\infty} \ll 1$ corresponds to the linearized theory of supersonic flows over slender bodies, in which the maximum slope of the body is smaller than the Mach angle.

The non-linear theory of small disturbances in hypersonics cannot however be bridged continuously with the linear theory of small disturbances of supersonic flows. Nonetheless, Van Dyke (J. Aeronaut. Sci. 1954) proposed a combined similarity rule for supersonic and hypersonic flows up to just above transonic range,

$$\frac{\varphi}{c'} = F \left( \frac{x}{y}, \gamma, \zeta \sqrt{\frac{M_{\infty}^2 - 1}{2}} \right) \quad \text{(130)}$$

where $y$ must remain fixed in the transformation. As shown in Fig. 37, the combined rule improves substantially the predictions over the supersonic range. Note that the linearized theory of supersonic flows over slender bodies yields a pressure coefficient of the form $\varphi = \frac{2}{\sqrt{M_{\infty}^2 - 1}}$, with $\gamma$ the local surface inclination ($\gamma \sim \chi$ for slender bodies), such that $\varphi/c' = \frac{2}{\sqrt{2(M_{\infty}^2 - 1)}}$ (see pages 74-75 of the M355 class notes), which constitutes the lower supersonic limit of the relation (130) proposed by van Dyke. As $M_{\infty} \to \infty$ in the hypersonic range, however, the linearized supersonic theory would render $\varphi \to 0$, which is wrong. As $M_{\infty} \to \infty$, the similarity parameter becomes large, $M_{\infty} \gg 1$, the formulation (119)-(126) ceases to depend on the Mach number in a manner similar to that predicted by Oswatitsch in the Mach-number independence principle, and the similarity relations (129) or (130) become independent of the Mach number, thereby yielding a scaling $\varphi \sim \zeta^2$ that resembles the one predicted by the Newtonian
THEORY (45).

Similarity relations can be obtained for the drag and lift coefficients on a two-dimensional body,

\[
\frac{C_D}{\lambda} = \frac{1}{\lambda} \int_{0}^{\lambda} \left( \frac{C_{lpu} + C_{lpl}}{C^2} \right) \frac{dy}{C} = \int_{0}^{\lambda} \left( \frac{C_{lpu} - C_{lpl}}{C^2} \right) \frac{dy}{C} = G \left( \gamma, M_\infty, C \right) \tag{131}
\]

\[
\frac{C_L}{\lambda^2} = \frac{1}{\lambda^2} \int_{0}^{\lambda} \left( \frac{C_{lpu} - C_{lpl}}{C^2} \right) \frac{dx}{C} = \int_{0}^{\lambda} \left( \frac{C_{lpu} - C_{lpl}}{C^2} \right) \frac{dx}{C} = H \left( \gamma, M_\infty, C \right) \tag{132}
\]

where \( C_{lpu} \) and \( C_{lpl} \) are pressure coefficients on the upper and lower surfaces that follow the similarity rule (129). Similarly to the discussion above, in the limit \( M_\infty \rightarrow \infty \), the drag and lift coefficients scale as \( C_D \sim \lambda^2 \) and \( C_L \sim \lambda^2 \), while their ratio (i.e., the lift to drag ratio \( L/D \)) scales as \( L/D \sim 1/\lambda \) and becomes a sole function of the geometry, which is typically termed as the \( L/D \) barrier in the literature and represents the fact that there is only so much lift that can be extracted from an aerodynamic shape with a given geometry by just increasing the Mach number (Fig. 38). Note that these results were already anticipated in the analysis of the Newtonian flow on page 42, where it was highlighted the increased sensitivity of the wave drag to the aspect ratio of the body in hypersonic flow (i.e., \( C_D \sim \lambda^3 \)), in comparison with the milder dependency found in supersonic flows (i.e., \( C_D \sim \lambda^2 \), see Eq. (160) in the Mereu class notes).

Lastly, it is worth mentioning that in using the similarity relations (129)-132, the results tend to be more accurate when \( \lambda \) is taken to be the maximum slope of the surface in bodies with non-uniform surface-inclination angles.
Blunt bodies are widely employed in hypersonic applications. It will be shown in the next chapters that a blunt aerodynamic shape of the nose of an aerospace vehicle largely decreases the local rate of heat transfer created by viscous friction. However, the aerodynamics of blunt bodies at high speeds are fundamentally different from the case of sharp pointed bodies such as the projectiles shown in Fig. 39. Two important aspects are treated in this section: The first is that aerodynamic blunt shapes lead to detached or bow shocks. An analytical solution to the stand-off distance of the shock is described for a calorically perfect gas. The second aspect is related to the shock curvature, which introduces vorticity into the initially irrotational flow along a relatively thick layer that sits on top of the boundary layer around the body.

While the first aspect illustrates the usefulness of the Newtonian theory, the second one connects with the next chapter on viscous hypersonic flows.

**The Shock Stand-Off Distance in the Newtonian Limit**

Consider the problem sketched in Fig. 40. A hypersonic stream at Mach $M_a$, density $\rho_0$, pressure $P_0$, and velocity $U_{10}$. The gas is assumed to be calorically perfect with a constant adiabatic coefficient $\gamma$. An axisymmetric coordinate system \( \Gamma, \beta \) is used, where $X = X_b(\beta)$ denotes the surface of the body and $X = 0$ is centered at the shock tip, similarly the axis $\Gamma = 0$ runs through the axis of symmetry of the body, which is impinged by the free stream with a zero angle of attack.

It is known from the theory of oblique shocks that no solution exists when the deflection angle exceeds a critical value $\alpha_{max}$ that
INCREASES WITH THE MACH NUMBER TO A MAXIMUM OF 45.6° IN PLANE WEDGES AND 57.6° IN CONES AT INFINITE MACH NUMBER. INSTEAD, THE SOLUTION TO THE PROBLEM MUST INVOLVE A DETACHED SHOCK STANDING OFF AT A DISTANCE $x_0$ FROM THE BLUNT NOSE (SEE FIG. 40) THAT NEEDS TO BE CALCULATED AS PART OF THE SOLUTION.

THE FLOW VARIABLES ARE NORMALIZED AS

$$U = \frac{U}{U_\infty}, \quad V = \frac{V}{U_\infty}, \quad P = \frac{P}{P_\infty U_\infty^2}, \quad \rho = \frac{\rho}{\rho_\infty}, \quad y = \frac{x}{x_0}, \quad \gamma = \frac{\gamma}{\gamma_0},$$

WHERE $x_0$ IS THE CHARACTERISTIC RADIUS OF CURVATURE OF THE BLUNT NOSE, AND $\epsilon$ IS THE FUNDAMENTAL DIMENSIONLESS PARAMETER (62) EMERGING FROM THE NEWTONIAN THEORY, NAMELY

$$\epsilon = \frac{y - 1}{y + 1},$$

IN SUCH A WAY THAT THE DENSITY IN (133) IS NORMALIZED WITH THE POST-SHOCK DENSITY.

IN THESE VARIABLES, THE CONSERVATION EQUATIONS FOR THE INVISCID ISENTROPIC FLOW BEHIND THE BOWSHOCK ARE

$$\frac{\partial}{\partial x} (\rho u u) + \frac{\partial}{\partial y} (\rho v u) = 0 \quad (135)$$

$$\frac{\partial}{\partial x} (\rho u v) + \frac{\partial}{\partial y} (\rho v v) = -\frac{\epsilon}{\gamma} \frac{\partial \rho}{\partial y} \quad (137)$$

$$\frac{\partial}{\partial x} (\rho u u) + \frac{\partial}{\partial y} (\rho v v) = 0 \quad (138)$$

WHERE THE ASTERISKS INDICATING DIMENSIONLESS VARIABLES ARE OMITTED.

EQUATION (135) INDICATES THAT THERE EXISTS A STREAM FUNCTION $\psi$ SUCH THAT

$$\frac{\partial \psi}{\partial y} = \rho u \quad \text{AND} \quad \frac{\partial \psi}{\partial x} = -\rho v,$$

WHICH AUTOMATICALLY SATISFIES EQ. (135). UPSTREAM FROM THE BOW SHOCK, IN THE UNIFORM FLOW, $\psi = \frac{1}{2} \frac{y^2}{x}$. ADDITIONALLY, ON THE SURFACE OF THE BODY THE STREAM FUNCTION IS CHOSEN TO BE $\psi = 0$.

THE ANALYSIS IS FACILITATED WHEN EQUATIONS (135) - (138) ARE TRANSFORMED TO AN EQUIVALENT BUT MORE INSIGHTFUL FORM USING THE VON MISES TRANSFORMATION, WHERE

$$\xi = \psi (x, y) \quad \text{AND} \quad \zeta = \gamma$$

ARE CHOSEN AS INDEPENDENT VARIABLES.

NOTE THAT

$$\frac{\partial}{\partial x} \frac{\partial \xi}{\partial \xi} = -\frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y}$$

AND

$$\frac{\partial}{\partial y} \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \gamma} \frac{\partial \xi}{\partial \gamma} = \frac{\partial y}{\partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \gamma}$$
So that the relation
\[ \frac{\partial}{\partial \xi} \left( \begin{array}{c} u^2 + v^2 \\ u \partial_x + v \partial_y \end{array} \right) = \frac{\partial}{\partial \xi} \left( \begin{array}{c} \partial_x \\ \partial_y \end{array} \right) \]
(141)
is satisfied. Physically, the von-Mises transformation rewrites the material derivative, which is an operator acting along a streamline, in the form (141) by taking advantage of changing one of the independent variables (x) by one that is constant along the streamlines. Using these new variables, it is not difficult to show that Eqs. (135) - (139) become
\[ \frac{2}{\xi} \left( \frac{\partial}{\partial \xi} \left( \begin{array}{c} u \\ u \partial_x + v \partial_y \end{array} \right) \right) = 0, \quad (142) \]
\[ u^2 + v^2 + (1 + E) \frac{\partial}{\partial \xi} P = l + \frac{(1 - 6)}{2 \xi} \]
(144)
\[- \frac{\partial}{\partial \xi} \frac{1}{\gamma - 1} \frac{\partial}{\partial \xi} = 0, \quad (145) \]
where (142) is continuity, (143) is the conservation of streamwise momentum, (144) is the conservation of radial momentum along with the conservation of energy (i.e., Bernoulli equation along a streamline); and (145) is the conservation of entropy. While (142), (143) and (145) are straightforward to derive, Eq. (144) requires some treatment of the pressure gradient in (139). Note that, upon transforming (137), the equation
\[ \frac{\partial}{\partial \xi} \left( \begin{array}{c} u \\ u \partial_x + v \partial_y \end{array} \right) \right) = - \frac{\partial}{\partial \xi} \left( \begin{array}{c} \partial_x \\ \partial_y \end{array} \right) = - \frac{\partial}{\partial \xi} \left( \begin{array}{c} \frac{\partial P}{\partial \xi} \\ \frac{\partial P}{\partial \xi} \end{array} \right) \]
(146)
is obtained, which represents the momentum equation across streamlines. While the left-hand side of (146) can be written as an exact differential of the kinetic energy, the right-hand side requires introduction of the density in the pressure gradient, which can be made by noticing that the flow is isentropic (i.e., Eq. (138))

And as a result, 1st principle of thermodynamics
\[ h = enthalpy = \frac{u}{\gamma - 1} \]
(146)
and therefore (146) becomes
\[ \frac{\partial}{\partial \xi} \left( \begin{array}{c} u^2 + v^2 + \frac{\gamma - 1}{\gamma} P \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \end{array} \right) = 0 \Rightarrow \frac{u^2 + v^2 + \frac{\gamma - 1}{\gamma} P}{\gamma - 1} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \left( \begin{array}{c} \frac{u}{\gamma - 1} \end{array} \right), \]
(147)

With \( \zeta \) a stagnation enthalpy evaluated in the freestream \( \zeta = 1 + \frac{\gamma - 1}{2} \left( \frac{\frac{1}{\gamma - 1} \frac{A}{\gamma M^2 \zeta} \right) = \left( \begin{array}{c} \frac{u}{\gamma - 1} \end{array} \right), \) where \( u = V = \frac{A}{\gamma M^2 \zeta} \) and \( P = \frac{A}{\gamma M^2 \zeta}. \)
(147) INTO (144) IS MADE EASILY BY USING THE RELATIONS \[
\frac{1}{\gamma - 1} = \frac{1 - \varepsilon}{2 \varepsilon} \quad \text{AND} \quad \frac{\gamma}{\gamma - 1} = \frac{1 + \varepsilon}{2 \varepsilon}
\]
EMERGING FROM (134). LASTLY, IN EQ. (145), \(f(\xi)\) IS AN ARBITRARY FUNCTION THAT
RESULTS FROM INTEGRATING (138) IN THE TRANSFORMED VARIABLES, NAMELY \(\frac{d}{d\xi} = \frac{d}{d\xi}\).

PHYSICALLY, IT REPRESENTS THE LOGARITHM OF THE ENTROPY BEHIND THE SHOCK, WHICH
IN PRINCIPLE IS A FUNCTION OF THE STREAMLINE BECAUSE OF THE SHOCK CURVATURE (SEE
NEXT SECTION).

EQUATIONS (142) - (145) ARE SUBJECT TO THE NON-PENETRATION CONDITION AT \(\xi = \xi = 0\),
\[
0 \cdot \frac{d}{d\xi} = u \cdot x + v \cdot m = 0 \Rightarrow u + v \cdot x_b \cdot r = 0 \quad (146)
\]
WHERE \(x_b = \frac{\partial x_b}{\partial \xi}\), AND TO THE DENSITY \((\rho)\), PRESSURE \((P)\), STREAMWISE VELOCITY \((u)\)
AND SPANWISE VELOCITY \((v)\) JUMP CONDITIONS AT THE BOW SHOCK, \(x = x_b \cdot r\) (WITH \(x_b(0) = 0\)),
WHICH, IN DIMENSIONLESS FORM, CAN BE WRITTEN AS
\[
P = (1 - \varepsilon) \left( \frac{1}{1 + x_{sr}^2} \right) \frac{\varepsilon}{(1 + \varepsilon) \rho_{m}^2} \quad (149)
\]
\[
\rho = \frac{1}{1 + (1 - \varepsilon) \varepsilon^{-4} \rho_{m}^{-2} (1 + x_{sr}^2)} \quad (150)
\]
\[
u = u - (1 - \varepsilon) \left\{ \frac{1}{1 + x_{sr}^2} - \frac{1}{\rho_{m}^2} \right\} \quad (151)
\]
\[
v = (1 - \varepsilon) x_{sr} \left\{ \frac{1}{1 + x_{sr}^2} - \frac{1}{\rho_{m}^2} \right\} \quad (152)
\]
WHERE \(x_{sr} = \frac{\partial x_s}{\partial \xi} = \frac{1}{\cos \beta}\) IS THE INVERSE OF THE LOCAL SLOPE OF THE SHOCK, SO THAT
\(\sin \beta = \frac{1}{1 + x_{sr}^2}\) IN THE SHOCK JUMP CONDITIONS. IN THE TRANSFORMED VARIABLES, EQUATIONS
(149) - (152) ARE APPLIED AT \(\xi = r / \xi_{sr} \).

INTEGRATION OF THE SECOND EQUATION IN (139) GIVES A RELATION BETWEEN THE EQUATION
OF THE BODY AND THE EQUATION OF THE SHOCK, NAMELY
\[
\int_{x_b}^{x_s} dx = - \int_{0}^{1/2} \frac{1}{s_{vr}} \frac{1}{E d \psi} \Rightarrow x_b = x_s + \varepsilon \int_{0}^{1/2} \left( \frac{1}{s_{vr}} \right) d \psi \quad (153)
\]
IN SUCH A WAY THAT THE STAND-OFF DISTANCE IS GIVEN BY
\[
x_0 = x_b(0) - x_s(0) = \varepsilon \lim_{r \to 0} \int_{0}^{1/2} \left( \frac{1}{s_{vr}} \right) d \psi \quad (154).
\]
AT THE SHOCK, NOTE THAT $\psi = r^{1/2}$, AND CONSEQUENTLY

$$X_{SF} = \frac{\partial \psi}{\partial \theta} \frac{\partial X}{\partial \psi} = \frac{r}{X_{SF}} = (2\psi)^{1/2}X_{SF}, \quad (155)$$

IN EQ. (145), $\int (\psi)$ CAN BE OBTAINED BY USING (149) - (155) AND (155), NAMELY

$$f(\psi) = (1 - \epsilon) \left[ \frac{1}{1 + 2\psi X_{SF}^2} - \epsilon \frac{E M_{a_{oo}}^{-2}}{1 + \epsilon} \right] \left[ 1 + (1 - \epsilon) E^{-1} M_{a_{oo}}^{-2} \left( 1 + 2\psi X_{SF}^2 \right) \right] \frac{1 + \epsilon}{1 - \epsilon}, \quad (156)$$

WHICH IS VALID EVERYWHERE BEHIND THE SHOCK SINCE $f(\psi)$, THE EXPONENTIAL OF THE ENTROPY, IS CONSTANT ALONG STREAMLINES.

OF PARTICULAR RELEVANCE IN THE ABOVE EQUATIONS IS THE DIMENSIONLESS GROUP

$$\frac{1}{E M_{a_{oo}}^{-2}}. \quad (157)$$

NOTE THAT THE NEWTONIAN LIMIT IS $\epsilon = 0$ (i.e., $\psi = 1$), WHILE HYPERSONIC SPEEDS REQUIRE $M_{a_{oo}} \to \infty$. AS A RESULT, THE NEWTONIAN LIMIT CONSISTS OF THE SIMULTANEOUS LIMIT

$$\epsilon \to 0 \quad \text{AND} \quad M_{a_{oo}}^{-2} \to 0 \quad (158)$$

SUCH THAT

$$\epsilon^{-1} M_{a_{oo}}^{-2} = 0 \quad (1) \quad (159)$$

TAKING THE LIMIT (158) - (159), EQUATIONS (142) - (145) REDUCE TO

$$2 \frac{\partial \psi}{\partial \theta} \frac{1 + 2\psi X_{SF}^2}{1 + 2\psi X_{SF}^2} = 0 \quad (160)$$

$$U + V = \frac{2\psi X_{SF}^2}{1 + 2\psi X_{SF}^2} \quad (162)$$

$$- \frac{\partial U}{\partial \theta} + \frac{\partial P}{\partial \psi} = 0 \quad (161)$$

$$\frac{p}{\rho} = \frac{1}{1 + 2\psi X_{SF}^2} + \epsilon^{-1} M_{a_{oo}}^{-2} \quad (163)$$

WHILE EQ. (158) ESTABLISHES $X_0 = X_{SF}$, NAMELY THAT THE SHOCK AND THE BODY SURFACE ARE COINCIDENT AND THEREFORE THE SHOCK STAND-OFF DISTANCE IS ZERO,

$$X_0 = 0 \quad (164)$$

SIMILARLY, THE BOUNDARY CONDITIONS (149) - (152) BECOME

$$P = \frac{A}{(1 + X_{SF}^2)} \quad (165)$$

$$U = A \frac{1}{1 + X_{SF}^2} \quad (167)$$

$$S = \frac{A}{1 + \epsilon^{-1} M_{a_{oo}}^{-2} (1 + X_{SF}^2)} \quad (166)$$

$$V = \frac{X_{SF}}{1 + X_{SF}^2} \quad (168)$$

NOTE THAT (164), (165), (167) AND (168) COINCIDE WITH THE NEWTONIAN LIMIT

OF THE OBlique SHock RELATIONS ANTICIPATED IN PREVIOUS SECTIONS, AND THESE EQUATIONS...
The incidence angle of the bow shock is related to \( \chi_{sr} \), as \( \tan \beta = \frac{1}{X_{sr}} \), so that \( (1 + X_{sr}^2)^{-1} = \sin^2 \beta \) and \( X_{sr} (1 + X_{sr}^2)^{-1} = \frac{1}{2} \sin (2 \beta) \).

Equations (160) and (162) provide the leading-order approximations for the velocities

\[
U = X_{sr} \quad V = \frac{X_{sr} \left( \frac{1}{2} \right) \left( 2 \psi \right)^{3/2} X_{s\psi} (\psi)}{\left( X_{sr}^2 (\frac{1}{2} + 1) \right)^{1/2} \left( 1 + 2 \psi X_{s\psi}^2 (\psi) \right)^{1/2}}, \quad (169)
\]

and substitution of this result into (161) yields

\[
P = \frac{A}{1 + X_{sr}^2} + \int \frac{A}{1 + X_{sr}^2} \frac{\partial U}{\partial \psi} \, d\psi = \frac{A}{1 + X_{sr}^2} + \frac{A}{B} \frac{d}{d \psi} \left( \frac{X_{sr}}{1 + X_{sr}^2 (\frac{3}{2})^{3/2}} \right) \int \frac{A}{1 + X_{sr}^2} \left( 2 \psi \right)^{3/2} X_{s\psi} (\psi) \left( 1 + 2 \psi X_{s\psi}^2 (\psi) \right)^{1/2} \, d\psi = \frac{P_2}{1 + X_{sr}^2} + \frac{A}{B} \sin \beta \frac{d \beta}{d \psi} \int R \cos \beta \, dR \quad (170)
\]

which represents the Busemann centrifugal correction for axisymmetric bodies, with \( P_2 \) the post-shock pressure, despite the heuristics involved in the derivation of the Busemann corrections earlier in this chapter; it is remarkable that it emerges from the conservation equations in the limit (458) - (459).

The rest of the analysis is quite extensive and will be omitted here. It consists of successive approximations to Eqs. (142) - (145) in the expansion parameter

\[
\mathcal{d} = E + M_0^2, \quad (171)
\]

The analysis is greatly facilitated by assuming a parabolic bow shock \( X_s = \frac{R}{2} \) and then calculating the equation of the body surface \( X = X_b (R) \) that is consistent with \( X_s \) and with the conservation equations. Such a method is typically referred to as "inverse method." It is therefore important to notice that the shock stand-off distance at hypersonic speeds ceases to be zero in the second approximation to the Newtonian limit, i.e., when \( E > 0 \) (302 > 3) and correspondingly the post-shock density becomes finite. After lengthy calculations that can be found in Chester, J. Fluid Mech. 1, 490 - 496 (1956), the asymptotic expansion

\[
\frac{A}{R_0} = E - \left( \frac{8}{3} \right)^{1/2} E^{3/2} + \frac{13}{5} E^2 - \frac{463}{168} \left( \frac{8}{3} \right)^{1/2} E^{5/2} + \ldots \quad (172)
\]

THE GENERATION OF VORTICITY IN CURVED SHOCKS

CONSIDER THE MOMENTUM EQUATION

$$\frac{\partial \mathbf{u}}{\partial t} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla p,$$  \hspace{1cm} (173)

WHERE THE CONVECTIVE TERM CAN BE WRITTEN IN TERMS OF THE VORTICITY \( \mathbf{\omega} = \nabla \times \mathbf{u} \) BY MAKING USE OF THE VECTOR IDENTITY

$$\mathbf{u} \cdot \nabla \mathbf{u} = \nabla \left( \frac{|\mathbf{u}|^2}{2} \right) - \mathbf{\omega} \cdot \mathbf{\omega},$$  \hspace{1cm} (174)

SO THAT (173) BECOMES

$$\frac{\partial \mathbf{u}}{\partial t} - \rho \mathbf{\omega} \cdot \mathbf{\omega} = -\nabla p - \rho \nabla \left( \frac{|\mathbf{u}|^2}{2} \right).$$  \hspace{1cm} (175)

THE FIRST PRINCIPLE OF THERMODYNAMICS PROVIDES THE ENERGY-CONSERVATION RELATION

$$TdS = de + pd \left( \frac{1}{\gamma} \right),$$  \hspace{1cm} (176)

WHERE \( e = h - \frac{p}{\gamma} \) IS THE INTERNAL ENERGY. IN TERMS OF THE ENTHALPY \( h \), EQ. (176) CAN BE REWRITTEN AS

$$TdS = dh - \frac{dp}{\gamma},$$  \hspace{1cm} (177)

WHICH CAN BE TRANSFORMED INTO

$$T \nabla S = \nabla h - \nabla p / \gamma.$$  \hspace{1cm} (178)

SUBSTITUTING (178) INTO (175), THE MODIFIED MOMENTUM EQUATION

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{\omega} \cdot \mathbf{\omega} = T \nabla S - \nabla \left( h + \frac{|\mathbf{u}|^2}{2} \right).$$  \hspace{1cm} (179)

IS OBTAINED, WHICH, IN THE STEADY LIMIT BECOMES

$$\mathbf{\omega} \cdot \mathbf{\omega} = T \nabla S - \nabla \left( h + \frac{|\mathbf{u}|^2}{2} \right)$$ \hspace{1cm} (CROCCO'S EQUATION) \hspace{1cm} (180)

WHERE \( h_0 = h + \frac{|\mathbf{u}|^2}{2} \) IS THE STAGNATION ENTHALPY AS DEFINED IN EQ. (18). IF \( h_0 \) IS UNIFORM IN THE FLOW, OR EQUIVALENTLY, WHEN THE FLOW IS ADIABATIC (I.E., THERE IS NO EXTERNAL HEAT ADDITION OR HEAT LOSSES), THEN (180) BECOMES

$$\mathbf{\omega} \cdot \mathbf{\omega} = T \nabla S$$ \hspace{1cm} (181)

WHICH ESTABLISHES A DIRECT RELATION BETWEEN VORTICITY AND ENTROPY GRADIENTS.
At this point, recall that flows with uniform entropy, $\nabla S = 0$, are referred to as *homentropic*. In homentropic flows, $S = S_0$ everywhere, but $S_0$ can vary with time (Fig. 41a). In contrast, inviscid compressible flows satisfying the relation

$$\frac{D S}{D t} = \nabla \cdot S = 0 \quad (182)$$

are referred to as *isentropic*, in that the initial entropy of a fluid particle, $S = S_p$, is conserved along pathlines, although $S_p$ can vary in space and time. In steady flows, (182) simplifies to

$$\nabla \cdot S = 0, \quad (183)$$

which corresponds to an *isentropic flow* in which the entropy is constant along streamlines (Fig 41b).

The implications of Eq. (183) in hypersonics are important. Consider, for instance a flow over a two-dimensional wedge or an axisymmetric cone, in which the nose shock is attached as in Fig. 42a. The flow upstream is inviscid, irrotational, and adiabatic, and therefore by Eq. (183) it is also homentropic. As it passes through the oblique shock, an entropy jump is generated that increases the entropy behind the shock. However, since the shock is straight, its strength is uniform, and consequent

In Fig. 42b, we see the flow pattern for a blunt body. The flow is non-isentropic at the shock, but downstream of the shock, it becomes isentropic. The entropy increases near the nose, but then decreases downstream.


IN PRACTICAL APPLICATIONS, THERE IS ALWAYS A THIN BOUNDARY LAYER CLOSE TO THE WALL WHERE VISCOS EFFECTS ARE IMPORTANT (FIG. 44). THE BOUNDARY LAYER DISPLACES THE INVISCID STREAM AROUND THE BODY, WHICH CAUSES A CHANGE IN THE SHAPE OF THE BOW SHOCK PREDICTED BY AN INVISCID ANALYSIS. THE GROWTH OF THE BOUNDARY LAYER AROUND THE BODY IS INFLUENCED BY THE PRESSURE GRADIENT ABOVE AND BY THE VELOCITY AND TEMPERATURE AT THE EDGE OF THE BOUNDARY LAYER. THE DISPLACEMENT CREATED BY THE BOUNDARY LAYER CAN CAUSE AN ORDER-UNITY INCREASE OF THE POST-SHOCK GAS PRESSURE NEAR THE NOSE, WHERE THE PERCENTAGE OF THE SHOCK-LAYER THICKNESS OCCUPIED BY THE BOUNDARY LAYER IS LARGEST. IN ADDITION, AS THE BOUNDARY LAYER GROWS,
It entrains the entropy layer above it and its corresponding vorticity. These complex effects caused by the finite Reynolds numbers involved, are typically referred to as viscous/inviscid interactions and become important at high altitudes in hypersonic flows, where the effects of viscosity must enter in the description. Viscous effects are discussed in the next chapter.

III. VISCOUS HYPERSONIC FLOWS

In the previous chapter, the focus has been on flows where the effects of viscosity are negligible, or more precisely, on the calculation of quantities that do not fundamentally depend on viscosity. Examples of such quantities are the shock angles in the free stream around an aerodynamic shape, which do not depend on viscosity up to the viscous/inviscid interactions briefly mentioned above, or the pressure, drag and lift coefficients, which, at high Mach numbers, are mostly determined by the pattern of shock waves instead of friction (i.e., recall the concept of supersonic wave drag in page 28 of the ME 355 class notes).

It should however be mentioned that viscous effects are inexorably present in all practical applications, even though they may not crucially influence some quantities, they are nonetheless essential for the calculation of a number of relevant engineering metrics of performance for an aerospace vehicle. These include, for instance, the heat flux into the vehicle structure at all flight altitudes, the friction drag and lift forces at high altitudes, or the ablation rate of thermal protection systems shielding the vehicle from the high thermal loads upon reentry. They also play a role in advanced aerodynamic aspects related to boundary-layer transition to turbulence and the associated thermal loads caused by this effect.
THE ROLE OF FLIGHT ALTITUDE

The emergence of viscous effects is perhaps more critical at high altitudes for the following reasons. Figure 45 provides the Earth's atmosphere strata definitions. A change in strata coincides with approximate altitudes where the temperature remains constant with altitude (i.e., zero thermal lapse, $dT/dz = 0$). The region of the atmosphere between sea level and $z = 14$ km is the troposphere. The density is observed to decay by 63% (i.e., by a factor $e^{-1}$) within this layer (see Fig. 4). In the atmosphere model where it is assumed that the temperature, composition, and gravitational acceleration are constant with height, the density can be expressed as

$$\rho = \rho_0 e^{-\beta z}$$

where $\rho_0$ and $\beta$ are adjustable parameters and $z$ is the altitude, while $\rho_0 \approx 1.22 \text{ kg/m}^3$ is the sea-level density, $\beta \approx 4.7 \times 10^{-4} \text{ m}^2/\text{ s}$ is a scale height that measures the e-folding of the density. Above the troposphere are the stratosphere and mesosphere. The latter ends at an altitude of 86 km and is considered approximately the limit height above which aerodynamic loads become negligible fractions of the vehicle weight. Correspondingly, above the mesosphere the motion of the vehicle is almost always suborbital and is to a good approximation solely described by gravitational mechanics. Note that the atmosphere above 86 km is highly dynamic and highly rarefied, and its composition and physical properties are very influenced by solar radiation. Figure 47 provides temperature variations in earth's ATMOSPHERE CONCURRENT WITH THE DECREASE IN DENSITY IS AN INCREASE IN THE KINEMATIC VISCOSITY, $\nu = \rho u$, with altitude, as shown in Fig. 48. Although it is difficult to define a physically significant Reynolds number that can illustrate the character of the aerodynamic field over the entire vehicle, since there are large changes in temperature in the flow around the vehicle and near the surface because of compressibility effects, it is clear...
that a vehicle characteristic length $L$ flying at a speed $U_0$ will experience a Reynolds number $10^5$ times smaller in the mesosphere relative to that at sea level. In re-entry applications, however, the Reynolds number increases along the re-entry trajectory but not by a factor of order $10^5$. It increases instead by a smaller factor of order $10^3$ because at high altitudes the flight velocity is much larger than close to the ground, which somewhat compensates for the larger kinematic viscosity at high altitudes so as to not render too small Reynolds numbers.

**Boundary-layer transition**

The increase of the Reynolds number along re-entry is greatly illustrated by the in-flight measurements of the shuttle orbiter reported by Goodrich et al., AIAA-83-0485 (1983). There, a series of thermocouples along the spine of the orbiter’s underbelly were employed, along with onboard and telemetry data, to determine characteristic Reynolds and Mach numbers upon re-entry, and most importantly, to determine the location of boundary-layer transition to turbulence along the orbiter’s underbelly. (Fig. 49).

In standard missions, the space shuttle re-enters the atmosphere from low-earth orbits at altitudes of order 123 km, angles of attack $\alpha \approx 40^\circ$, and Mach numbers $M_a \approx 25$, which correspond to circular orbit velocities of order $V_0 \approx 7.4 \text{ km/s}$. The velocity/altitude curve of the shuttle STS-2 mission.
NOTE THAT THE REYNOLDS NUMBER OF THE ORBITER VARKS FROM \( R_{e, \text{L}} \sim 10^5 \) AT THE ENTRY INTERFACE \((\sim 123 \text{ km})\) TO \( R_{e, \text{L}} \sim 10^8 \) NEAR THE GROUND AFTER APPROXIMATELY \( 1800 \text{ s \approx 30 mi} \) OF TOTAL MANEUVERING FLIGHT. MOST REMARKABLY, WITHIN THAT RANGE OF REYNOLDS NUMBERS, THERE OCCURS THE CRITICAL PHENOMENON OF BOUNDARY-LAYER TRANSITION FROM LAMINAR TO TURBULENT FLOW NEAR THE SURFACE OF THE FUSELAGE. THIS BEGINS NEAR AN ALTITUDE OF \( 50 \text{ km} \) WHEN THE ORBITER IS DESCENDING AT \( M_{\infty} \sim 10 \) \((\sim 3800 \text{ km/s})\), AND IT IS SPATIALLY LOCALIZED NEAR THE AFTMOST PART OF THE UNDERBELLY, WHICH CREATES A SIGNIFICANT THERMAL LOADING (FIG. 52C). THE REGION OF TRANSITION THEN MOVES RAPIDLY UPSTREAM ALONG THE ORBITER'S UNDERBELLY AND REACHES THE FOREBODY AFTER APPROXIMATELY \( 300 \text{ s \approx 5 mi} \) AFTER TRANSITION ONSET (FIG. 52A AND TABLE 1) WHEN THE ORBITER IS DESCENDING AT \( M_{\infty} \sim 6 \) \((\sim 1900 \text{ km/s})\) AT \( 140 \text{ km} \) ALTITUDE. IN THE FOREBODY, TRANSITION TO TURBULENCE CAUSES A LESS SEvere THERMAL LOADING COMPARED TO PEAK.

### TABLE 1: FREE-STREAM FLOW CONDITIONS AT TIMES OF BOUNDARY-LAYER TRANSITION ALONG THE ORBITER WINDWARD PITCH PLANE (GOODRICH ET AL., 1983)

<table>
<thead>
<tr>
<th>( M_{\infty} )</th>
<th>( V_{\infty} ) (km/s)</th>
<th>( T_{\infty} ) (K)</th>
<th>( P_{\infty} ) (Pa)</th>
<th>( R_{e, \text{L}} \times 10^5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>1630</td>
<td>39.1</td>
<td>772</td>
<td>384</td>
</tr>
<tr>
<td>0.1</td>
<td>1352</td>
<td>131.4</td>
<td>6386</td>
<td>447</td>
</tr>
<tr>
<td>0.15</td>
<td>1383</td>
<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
<td>0.2</td>
<td>1383</td>
<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
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<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
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<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
<td>0.4</td>
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<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
<td>0.5</td>
<td>1383</td>
<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
<td>0.6</td>
<td>1383</td>
<td>149.2</td>
<td>8100</td>
<td>32.7</td>
</tr>
<tr>
<td>0.7</td>
<td>1170</td>
<td>159.6</td>
<td>9911</td>
<td>36.0</td>
</tr>
<tr>
<td>0.8</td>
<td>1077</td>
<td>167.6</td>
<td>10397</td>
<td>37.5</td>
</tr>
<tr>
<td>0.9</td>
<td>1106</td>
<td>167.9</td>
<td>10397</td>
<td>37.5</td>
</tr>
<tr>
<td>0.99</td>
<td>1182</td>
<td>168.7</td>
<td>11348</td>
<td>38.7</td>
</tr>
</tbody>
</table>

(Fig. 50) VELOCITY/ALTITUDE PARAMETERS FOR SEVERAL RE-ENTRY VEHICLES (BERTIN, 1994)
CONDITIONS, which are attained well before at higher altitudes where the flow is laminar (Fig. 5.5a). In general, transition in lifting re-entry vehicles such as the Shuttle Orbiter occurs well past peak heating and therefore contributes very little (but causes locally large heat fluxes) to the total heat load during re-entry (Goodrich et al., 1985). In the case of the Shuttle Orbiter, transition was mostly induced by a 1 mm roughness in the tiles of the TPS covering the underbelly.

Boundary-layer transition is a complex phenomenon that is subject of research in high-speed flows and is beyond the scope of this course. Note however that transition is linked to viscous effects and involves growth of perturbations in the laminar boundary layer, which can be caused by roughness or free-stream disturbances and destabilize the boundary layer downstream, creating first large-scale streamwise vortices that increase friction and are effective at heating the surface, and which then become a fully turbulent, mostly disorganized forest of vortices downstream (Figs. 53 and 54).

At hypersonic velocities, the problem described above is greatly complicated by several other factors, including the delay of transition caused by increasing Mach numbers (Fig. 55) and cold wall temperatures, both of which represent stabilizing effects, and the influences of thermochemical effects, recession of ablative surfaces of TPS, and free-stream disturbances, the latter becoming important for the transition of boundary layers along smooth surfaces. Additional effects, such as viscous/inviscid interactions caused by vorticity in the entropy layer overriding the boundary layer, are open research questions.
Depending on the particular targeted application, hypersonic flight trajectories can be subdivided into:

1. **Trans-atmospheric trajectories**, which involve ascent and crossing of the mesopause, into low-Earth orbit, as well as atmospheric re-entry from circular or parabolic orbits.

2. **Endo-atmospheric trajectories**, which involve ascent, cruise flight (most often in the stratosphere) and descent, all without leaving the confines of the atmosphere.

While vehicles undergoing the trans-atmospheric trajectories, such as the Apollo capsule or the shuttle orbiter, typically encounter laminar, transitional, and turbulent boundary layers, vehicles undergoing endo-atmospheric trajectories are most likely to encounter turbulent boundary layers for the most part of the flight time. In fact, each type of mission trajectory demands a specific type of hypersonic vehicle (Figs. 56-57). The remarkable adaptation of the fuselage to induce high-altitude breaking with minimum heat flux (blunt bodies; space capsules and lifting re-entry bodies) is in stark contrast with the sharp aerodynamic shapes prevailing in endo-atmospheric vehicles ( scramjets, rocket-powered aircrafts) and also unmanned trans-atmospheric vehicles (boost-glide weapons, intercontinental ballistic missiles). This remarkable adaptation is absent in most low-speed aircraft design.
Since the hypersonic flow environment imposes severe constraints on the aircraft structure, aerodynamic trim, thermal protection system, radio communications and propulsion system, in particular, endo-atmospheric vehicles are designed so as to minimize wave drag (which is inviscid in origin) and heating loads, the latter being closely connected with viscous friction. On the other hand, trans-atmospheric vehicles have blunt shapes to increase wave drag, decrease deceleration forces harmful to crews, and decrease local heat fluxes in the nose area (exceptions that are less blunt are missiles and boost glide weapons for reasons that will become more evident in the analysis of trajectories performed in Chapter IV). Table II and Figure 50 summarize some of these aspects.

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>ENDO-ATMOSPHERIC VEHICLES (X-15, X-51, etc.)</th>
<th>TRANS-ATMOSPHERIC VEHICLES</th>
<th>UNHEATED MISSILES (HYPERSONIC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>MACH NUMBER RANGE</td>
<td>0-10</td>
<td>0-25</td>
<td>0-25</td>
</tr>
<tr>
<td>CONFIGURATION</td>
<td>SLIGHTER</td>
<td>BLUNT NOSE, SLIGHTER LIFTING BODY</td>
<td>SOMEWHAT BLUNT NOSE, SLIGHTER NON-LIFTING BODY</td>
</tr>
<tr>
<td>FLIGHT TIME</td>
<td>LONG (&gt;1 hour)</td>
<td>SHORT (30 mins)</td>
<td>SHORT (30 mins)</td>
</tr>
<tr>
<td>FLIGHT ALTITUDE</td>
<td>0-30 km (up to within the stratosphere)</td>
<td>0-200 km (low-earth orbit)</td>
<td>0-1200 km (interplanetary orbit)</td>
</tr>
<tr>
<td>ANGLE OF ATTACK</td>
<td>SMALL (K=0-2°)</td>
<td>LARGE (K=40°)</td>
<td>(K=0°/HERCULES)</td>
</tr>
<tr>
<td>BOUNDARY LAYERS</td>
<td>MORE TURBULENT THAN LAMINAR; IN SCRAM Jets, transition is tripped purposely in the forebody</td>
<td>LAMINAR+ TRANSITIONAL+ TURBULENT; TRANSITION: 1,500 km</td>
<td>LAMINAR+ TRANSITIONAL+ TURBULENT; TRANSITION: 1,500 km</td>
</tr>
<tr>
<td>WAVE DRAG</td>
<td>IMPORTANT</td>
<td>VERY IMPORTANT</td>
<td>IMPORTANT</td>
</tr>
<tr>
<td>SKIN DRAG</td>
<td>VERY IMPORTANT</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
</tr>
<tr>
<td>LIFT-TO-DRAG RATIO</td>
<td>~10</td>
<td>~1</td>
<td>~0.3</td>
</tr>
<tr>
<td>ALTITUDE OF MAX. DECELERATION</td>
<td>—</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
</tr>
<tr>
<td>MAX. DECELERATION/ACCELERATION</td>
<td>2g (ACCEL. X-15)</td>
<td>3-6g</td>
<td>3-6g</td>
</tr>
<tr>
<td>ALTITUDE OF MAX. HEAT FLUX (W/4m^2)</td>
<td>—</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
</tr>
<tr>
<td>TOTAL HEAT LOAD (J)</td>
<td>HIGH</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
</tr>
<tr>
<td>RAREFACTION EFFECTS</td>
<td>NOT IMPORTANT</td>
<td>MILD</td>
<td>MILD/IMPORTANT</td>
</tr>
<tr>
<td>AIR DISSOCIATION EFFECTS</td>
<td>MILD/IMPORTANT (O2)</td>
<td>VERY IMPORTANT (O2+N2)</td>
<td>VERY IMPORTANT (O2+N2)</td>
</tr>
<tr>
<td>AIR IONIZATION EFFECTS</td>
<td>MILD/IMPORTANT</td>
<td>MILD/IMPORTANT</td>
<td>MILD/IMPORTANT</td>
</tr>
<tr>
<td>VIBRATIONAL NON-EQUILIBRIUM POST-SHOCK GAS RADIATION</td>
<td>MILD NOT IMPORTANT</td>
<td>MILD/IMPORTANT</td>
<td>MILD/IMPORTANT</td>
</tr>
</tbody>
</table>
**RAREFACTION EFFECTS**

In close connection with viscosity are rarefaction effects arising at high altitudes as a result of the exponential decrease in density (183). The Navier-Stokes equations are based on a "field formulation" of the fluid variables $\mathbf{v}(x,t)$, where $\mathbf{v}$ and its gradients are continuous in space. This is a good approximation when the mean free path between molecular collisions $\lambda$ is much smaller than the characteristic length of the flow $L$, or equivalently, when $Kn = \lambda/L \ll 1$, where $Kn$ is the Knudsen number. For instance, consider a gas of molecules of diameter $d$ with uniform number density $n_0$. A molecule labelled "A" in Fig. 58 will then collide with other molecules if their centers penetrate into a sphere of radius $d$ centered at "A", assuming that the molecule moves at a mean thermal speed $v_0 = (8 Kn k_B T/m_0)^{1/2}$, with $k_B$ the Boltzmann constant and $m_0$ the mass of a molecule, then the "A" molecule sweeps a volume $\pi d^2/2$ per unit time. The corresponding collision frequency is

$$\omega = \pi d^2 \frac{v_0}{m_0} \quad (184)$$

In collisions per unit time, whereas
\[ \lambda_{oo} = \frac{C_{oo}}{\Omega_{oo}} = \frac{1}{\frac{1}{2} \pi n_{oo} d^2} = \frac{m}{\frac{1}{2} \pi \rho_{oo} d^2} \]  

(185)

is the mean free path. In Eq. (185), the factor \( \frac{1}{2} \) arises from the fact that \( C_{oo} \) is not the relevant velocity to consider, but rather \( \frac{1}{2} \bar{C} \) which is the mean relative speed of the molecules (e.g., see rigorous derivation of (185) on pages 48-54 of Vincenti and Kruger (2002) "Introduction to Physical Gas Dynamics"). At sea level,

\[ T_{oo} = 298 \text{ K} \]
\[ P_{oo} = 101325 \text{ Pa} \]
\[ m = W_{\text{air}} = \frac{28.9 \times 10^{-3} \text{ kg/mol}}{N_{\Lambda}} \cdot \frac{6.022 \times 10^{23} \text{ molecules/mol}}{} = 4.79 \times 10^{-26} \text{ kg/mol} \]
\[ d = d_{N_2} = 390 \times 10^{-12} \text{ m} \]
\[ R = R_{\text{air}} = 8.31 \frac{\text{J}}{\text{mol} \cdot \text{K}} \]
\[ \frac{R_{oo}}{R_{\text{air}}} = 1.2 \frac{\text{J}}{\text{mol} \cdot \text{K}} \rightarrow m_{oo} = \frac{m}{m} = 2.4 \times 10^{25} \text{ molecules/m}^3 \]
\[ \bar{C}_{oo} = \left( \frac{8 k_{B} T_{oo}}{\pi m} \right)^{1/2} \approx 467 \text{ m/s} \]

\[ \lambda_{oo} \approx 60 \times 10^{-9} \text{ m} \]
\[ \Omega_{oo} \approx 10^{10} \text{ collisions/s} \]
\[ \delta_{oo} \approx 3.4 \times 10^{-9} \text{ m} \]

\( \delta_{oo} \) is the mean intermolecular distance.

At sea level, \( \lambda_{oo} \) is always small compared to most characteristic lengths of interest, so that all relevant high-speed flows around bodies tend to satisfy the continuum assumption that \( \kappa_{m} = \lambda_{oo}/L \ll L \). However, \( \lambda_{oo} \) is a strong function of altitude because of its dependence on \( \rho \), as shown in Fig. 51a. (Note that \( \Omega_{oo} \) is additionally dependent on \( C_{oo} \), whose variations are similar to those of the speed of sound \( C_{oo} \) and are not as strong as those of \( \rho \), as observed in Fig. 58b).

As a result, the mean free path increases with altitude as

\[ \lambda_{oo}/\lambda_{oo}^{(2=0)} \approx e^{Bz} \]  

(186)

at a critical altitude \( z = \frac{L}{B} \), the Knudsen number becomes unity and \( L = \lambda_{oo} \). In this limit, the flow is said to be strongly rarefied, in that every molecule on average collides once or less with other molecules in distances of order \( L \). Note that for the radius of the nose of the Shuttle orbiter \( L/2 \approx 0.79 \text{ m} \), the critical height is \( z = 120 \text{ km} \), which is similar to the altitude of the entry interface of its flight.

Fig. 58: US Standard Atmosphere 1976.
For this reason, rarefaction effects are not crucial in re-entry applications, except for inter-planetary space capsules or ballistic missiles, whose descent begins in the outermost skins of the atmosphere. Even in the latter cases, the bulk of the aerodynamic interactions, including aerobraking and heating, occurs below the mesopause where the density is sufficiently large to enable continuum effects of friction and wave drag. In addition, note that for aerodynamic bodies flying at velocities much higher than the speed of sound \( \lambda_0 \) is not the appropriate mean free path but something of order \( \sqrt{2 \left( \frac{2}{n} \right)} \lambda_0 \), since the volume swept by the molecules between collisions is \( 7T \). Instead (note that \( \lambda_0 \) is measured in the gas frame but the relevant mean free path in the flow around a hypersonic vehicle must be measured in the vehicle frame), and \( \lambda_0 \) the number density of the gas around the hypersonic vehicle varies largely in space and cannot be univocally characterized by a single number. For these reasons, and many others that are described thoroughly by Probst in "Shock Wave and Flow Field Development in Hypersonic Re-Entry" Aras Journal (1961), great caution should be used when using \( \lambda_0 \) or \( \left( \frac{2}{n} \right) \lambda_0 \) to characterize the altitude \( z_0 \) for the onset of free-molecule flow (\( \kappa_M = \lambda_0 \)). In fact, the correction factor \( \left( \frac{2}{n} \right) \lambda_0 \) to \( \lambda_0 \) described above, already suggests that the true altitude for free-molecule flow is actually higher than the altitude for which \( \lambda_0 = \lambda_0 \) in hypersonic flight (4-8, /20/0 \( \frac{LH_2}{\lambda_0 (re)} \)).

The flow field around a blunter body as it descends into the atmosphere is qualitatively sketched in Fig. 59. Note that viscous effects at high altitudes in the continuum limit first as a thick fully-viscous (and laminar) shock layer, which, as \( R_{\text{sh}} \lambda_0 \) increases.
Becomes a shock layer of the type in F16.43, with a thin viscous boundary layer near the surface (which may be laminar, transitional or turbulent depending on the value of $Re_{visc}$, $Ma_{visc}$ and other parameters as discussed in the previous section), along with a rotational entropy layer right above yet within the shock layer. (F16.60)

Note that the analysis of such boundary-layer flow is by no means straightforward even in the laminar regime, since "uniform freestream" values are not found for the velocity and other quantities at the boundary-layer edge as in traditional boundary-layer analysis (here only the stagnation enthalpy reaches the "uniform freestream" value - yet within the shock layer - if the gas is non-radiating). The reason for this is that the flow in the entropy layer is rotational, and its vorticity can as well compete with the vorticity in the boundary layer. Eventually, the entropy layer is swallowed by the boundary layer at many nose radii downstream. In summary, note that although the flow may be laminar around hypersonic vehicles at high altitudes, in that the Reynolds number may be relatively small, the characteristics of the flow may be enormously complicated by non-continuum effects, entropy layers, and a number of thermochemical effects some of which are listed in Table II and are the subject of Chapter IV.

Before finishing this section, it is worth mentioning that the viscosity and mean free path are approximately related by the kinetic theory as

$$\mu \sim \frac{\lambda}{2 \sqrt{\kappa \lambda}}, \quad (187)$$

As a result, rarefaction effects typically induce an increase in the skin drag relative to the pressure drag (see F16.61).

The rest of this chapter is dedicated to compressible laminar boundary layers including the near-stagnation region at the nose of blunt hypersonic vehicles.

Fig. 61: Skin, pressure and total drag on a hypersonic circular cylinder (Bertin, 1994)
Boundary layers are thin regions close to solid surfaces where viscous effects become important. At high Reynolds numbers, across the boundary layer, the aero-thermal variables transit from their values in the free stream (which henceforth will be referred to as edge values with a subscript “e”) to emphasize that these may be different from those in the free stream upstream from the aerospace vehicle because of the presence of shocks) to their values at the wall. At high Reynolds numbers, $Re_{e} = \frac{2eU_{e}L}{\mu_{e}} \gg 1$, the boundary layer is slender in the sense

$$\frac{\delta_{e}}{L} << 1$$

(188)

where $L$ is a characteristic size of the vehicle. This slenderness can be exploited in simplifying the Navier-Stokes equations, as discussed in classical texts.

High-speed boundary layers differ from their incompressible counterparts in three main aspects:

1. The momentum and thermal fields are strongly coupled through the density.
2. Such coupling is aggravated by the dependence of viscosity and thermal conductivity on temperature.
3. At order-unity Mach numbers, the kinetic energy and the thermal energy are comparable, and as a result, the viscous dissipation of kinetic energy plays a fundamental role in the structure of the temperature field.

* The recovery factor and adiabatic wall temperature.

An example of the coupling described above is illustrated schematically in Fig. 62 for a high-speed boundary layer over an adiabatic wall. While in the overriding inviscid stream the static temperature is $T_{0}$, the viscous dissipation in the boundary layer causes an increase of temperature there. In this example, steady-state conditions are achieved by balancing that heat with convection and conduction away from the wall. The resulting temperature at the wall is referred to as the adiabatic wall temperature $T_{aw}$, which generally satisfies

$$T_{aw} < T_{0e}$$

(189)
WHERE $T_{\infty}$ IS THE STAGNATION TEMPERATURE IN THE INVISCID STREAM. NOTE THAT, ALTHOUGH THE FLUID IS NECESSARILY BROUGHT TO REST AT THE WALL, THE DECCELERATION IS BY NO MEANS ISENTROPIC, AND THEREFORE THE STAGNATION TEMPERATURE $T_{\infty}$ IS NOT RECOVERED AT THE WALL. INSTEAD, A RECOVERY FACTOR

$$q = \frac{T_{aw} - T_{\infty}}{T_{aw} - T_{\infty}} = \frac{T_{aw} - T_{\infty}}{U_e^2 / 2C_p} = \frac{Z}{(y-1) M_a^2} \left( \frac{T_{aw} - T_{\infty}}{T_{aw} - T_{\infty}} \right)$$

(189)

CAN BE DEFINED THAT IS SMALLER THAN UNITY. A TYPICAL PROFILE OF STAGNATION TEMPERATURE $T_{\infty}$ ACROSS THE BOUNDARY LAYER IS ALSO PROVIDED IN FIG. 65 AND WILL BE DISCUSSED FURTHER BELOW.

WITH NON-ADIABATIC WALLS, THE TEMPERATURE PROFILE ARRIVES AT THE WALL WITH NON-ZERO SLOPE AND CAN ATTAIN THE SHAPES SKETCHED IN FIG. 68, DEPENDING ON WHETHER THE WALL TEMPERATURE IS LOWER OR HIGHER THAN THE ADIABATIC WALL TEMPERATURE. IN THE FORMER CASE, THE TEMPERATURE IN THE BOUNDARY LAYER NEEDS NOT BE BOUNDED BY $T_{\infty}$ OR $T_{aw}$, BUT IT CAN DEVELOP AN INTERNAL MAXIMUM DUE TO THE HEAT CREATED BY THE VISCOS DISSIPATION, WHICH IS COMMONLY REFERRED TO AS "AERODYNAMIC HEATING" (HERE THIS TERM SHOULD BE UNDERSTOOD TO BE LIMITED TO VISCOS EFFECTS, SINCE OTHER AERODYNAMICALLY-DERIVED HEATING PHENOMENA CAN OCCUR, FOR INSTANCE, BY SHOCK/SHOCK INTERACTIONS). IN GENERAL, THE ADIABATIC WALL TEMPERATURE IS A SOLUTION OF THE BOUNDARY-LAYER EQUATIONS, WHICH ARE DESCRIBED BELOW.

* THE BOUNDARY-LAYER EQUATIONS

THE SLENDERNESS CONDITION (188) ENABLES THE SIMPLIFICATION OF THE NAVIER-STOKES EQUATIONS BY NEGLIGING SECOND-ORDER FLOW PHENOMENA RELATED TO $\omega$) THE STREAMWISE VISCOS STRESS OF THE TWO VELOCITY COMPONENTS $\mu_{ii}$) STREAMWISE HEAT CONDUCTION, $\mu_{ii}$) VISCOS DISSIPATION GENERATED BY ALL GRADIENTS OF THE TRANSVERSE VELOCITY, $\mu_{ii}$) VISCOS DISSIPATION GENERATED BY THE STREAMWISE GRADIENTS OF THE STREAMWISE VELOCITY. IN ADDITION, IF THE MACH NUMBER IS NOT TOO LARGE COMPARED TO $L/e$, THE PRESSURE VARIATIONS ACROSS THE BOUNDARY LAYER CAN BE SAFELY NEGLECTED. WITH THESE SIMPLIFICATIONS, THE NAVIER-STOKES EQUATIONS BECOME

---

(68)
Continuity (190)

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = 0
\]

Streamwise Momentum Conservation (191)

\[
\frac{\partial}{\partial x} (\rho u) + \frac{\partial}{\partial y} (\rho v) = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right)
\]

Transverse Momentum Conservation (192)

\[
\frac{\partial}{\partial y} (\rho u) = P = P_0(x)
\]

Enthalpy Conservation (193)

\[
\frac{\partial}{\partial x} (\rho h) + \frac{\partial}{\partial y} (\rho v h) = \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) + u \frac{dP}{dx} + \mu \left( \frac{\partial v}{\partial y} \right)^2
\]

Non-Radiating Gas (194)

EQUATIONS (190)-(193) ARE THE BASIC BOUNDARY-LAYER EQUATIONS FOR COMPRESSIBLE FLOWS IN THERMODYNAMIC AND CHEMICAL EQUILIBRIUM. (THIS CONCEPT WILL BE DEFINED IN MORE DEPTH IN CHAPTER IV.) IN EQ. (193), \( h \) IS A STATIC ENTHALPY GENERALLY DEFINED AS FOR NON-REACTING GASES

\[
h = \int_{T_0}^{T} C_p(T) dT
\]

WHERE \( T_0 \) IS AN ARBITRARY REFERENCE TEMPERATURE AND \( C_p \) IS A CONSTANT.

PRESSURE SPECIFIC HEAT THAT MAY GENERALLY VARY WITH TEMPERATURE DEPENDING ON THE DEGREE OF VIBRATIONAL OR ELECTRONIC EXCITATION (SEE CHAPTER IV). EVEN IF \( C_p \) VARIES WITH TEMPERATURE, NOTE THAT \( \partial h/\partial x = C_p \partial T/\partial x \) AND \( \partial h/\partial y = C_p \partial T/\partial y \) BY THE LEIBNIZ INTEGRAL RULE, AND THEREFORE (193) CAN BE WRITTEN AS

\[
\frac{\partial}{\partial x} (\rho h) + \frac{\partial}{\partial y} (\rho v h) = \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} \right) + u \frac{dP}{dx} + \mu \left( \frac{\partial v}{\partial y} \right)^2
\]

WHERE \( k \) IS THE THERMAL CONDUCTIVITY, IN CALORICALLY PERFECT GASES (\( C_p = \text{Const.} \)) (194) REMAINS THE SAME, BUT A USEFUL COMBINATION OF (191) AND (194) CAN BE MADE BY NOTICING THAT

\[
T_0 = T + \frac{u^2 + v^2}{2C_p} = T + \frac{u^2}{2C_p}
\]

AND SUMMING THE KINETIC-ENERGY EQUATION

\[
\frac{\partial}{\partial x} \left( \frac{u^2}{2} \right) + \frac{\partial}{\partial y} \left( \frac{v^2}{2} \right) = -\frac{dP}{dx} + \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial v}{\partial y} \right)^2
\]

TO EQ. (194) (NOTE THAT (195) IS OBTAINED BY MULTIPLYING (191) BY \( u \)), WHICH GIVES

\[
\frac{\partial}{\partial x} (\rho h) + \frac{\partial}{\partial y} (\rho v h) = \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} + \mu \frac{\partial u}{\partial y} \left( \frac{u^2}{2} \right) \right) = \frac{\partial}{\partial y} \left( \kappa \frac{\partial T}{\partial y} + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right)^2 \right)
\]

AND SINCE \( C_p \) IS CONSTANT, THEN THE ABOVE EQUATION BECOMES

\[
\frac{\partial}{\partial x} (\rho h) + \frac{\partial}{\partial y} (\rho v h) = \frac{\partial}{\partial y} \left( \mu \frac{\partial T}{\partial y} \right) + \frac{\partial}{\partial y} \left[ \frac{1}{\Pr} - 1 \right] \mu \frac{\partial T}{\partial y}
\]

WHERE \( \Pr = \mu C_p / \kappa = \frac{D}{D_T} \) IS THE PRANDTL NUMBER AND \( D_T = k / \rho C_p \) IS THE THERMAL DIFFUSIVITY. THE PROBLEM OF A NON-REACTING, COMPRESSIBLE LAMINAR BOUNDARY LAYER
IN LOCAL THERMODYNAMIC EQUILIBRIUM IS DEFINED BY Eqs. (190) - (192) ALONG WITH
(194) OR (197), THE LATTER BEING ONLY VALID IF \( CP = \text{const.} \). THESE EQUATIONS NEED TO
BE SUPPLEMENTED WITH THE EQUATION OF STATE

\[
P_0 = \rho R_0 T \quad (198)
\]

ALONG WITH AN EXPRESSION FOR THE THERMODYNAMIC PRESSURE AT THE BOUNDARY-LAYER
EDGE AS A FUNCTION OF \( x \). NOTE THAT GRADIENTS \( dP_0/dx \) CAN BE EXTERNALLY IMPOSED
OR RESULT FROM THE INVISID SOLUTION OVER CURVED BODIES OBTAINED, FOR EXAMPLE,
BY USING THE NEWTONIAN HYPERSONIC THEORY (45).

THE BOUNDARY CONDITIONS AT THE WALL MAY INCLUDE

\[
\begin{align*}
U &= U_0 \quad (\text{NON-SLIP}) \\
V &= V_{\infty} \quad (\text{NON-SLIP} \quad V_{\infty} = 0 \quad \text{OR SUCTION/BLowing} \quad V_{\infty} \neq 0) \\
T &= T_{\infty} \quad (\text{THERMAL CONTACT WITH UNIFORM OR VARIABLE WALL TEMPERATURE}) \\
\frac{\partial T}{\partial y} &= 0 \quad (\text{ADIABATIC WALL})
\end{align*}
\]

AND OTHER MIXED THERMAL BOUNDARY CONDITIONS THAT CAN BE USED WITH RADIATIVE OR ABETIVE
WALLS.

IN ADDITION, CONSTITUTIVE LAWS FOR THE VARIATIONS OF \( CP, \mu, K \) AND \( Pr \) MUST BE
INCORPORATED IN THESE EQUATIONS IN THE MOST GENERAL CASE. IN WHAT FOLLOWS, THE
VARIATIONS OF \( CP \) AND \( Pr \) WITH TEMPERATURE WILL BE NEGLECTED, SO THAT \( \mu \)
AND \( K \) BECOME PROPORTIONAL TO EACH OTHER AND TO A PRESCRIBED POWER OF \( T \).

BEFORE SOLVING THE SET OF EQUATIONS OUTLINED ABOVE, IT IS WORTH ANALYZING FIRST
A SIMPLIFIED CASE THAT NONETHELESS IS CONCEPTUALLY USEFUL AND PROVIDES GROUNDS FOR
ANALOGIES BETWEEN THE VELOCITY AND TEMPERATURE FIELDS.

# THE CASE OF UNITY PRANDTL NUMBER

EQUATION (197) IS REMINISCENT OF THE MOMENTUM EQUATION (191), BUT NOT QUITE THE SAME.
CONSIDER FIRST MAKING \( Pr = 1 \) IN (197), WHICH GIVES

\[
\frac{\partial}{\partial x} (\rho U \frac{\partial T}{\partial x}) + \frac{\partial}{\partial y} \left[ \rho \frac{\partial T}{\partial y} + \mu \frac{\partial U}{\partial y} \right] = \frac{2}{\partial y} \left( \frac{\mu \frac{\partial T}{\partial y}}{\partial y} \right) \quad (199)
\]

OR EQUIVALENTLY,

\[
\frac{2}{\partial y} \left[ \frac{K}{\partial y} \frac{\partial T}{\partial y} + \mu \frac{\partial U}{\partial y} \right] = \frac{2}{\partial y} \left( \frac{\mu \frac{\partial T}{\partial y}}{\partial y} \right) \quad (200)
\]

AS OBSERVED FROM THE EQUATION IMMEDIATELY ABOVE EQUATION (197), AN IMPORTANT SOLUTION
TO (199) THAT SATISFIES THE BOUNDARY CONDITION \( T_y = T_{y_0} \) AT \( y \to \infty \) IS THE ONE IN
IN WHICH THE STAGNATION TEMPERATURE IS UNIFORM IN THE BOUNDARY LAYER,

\[ T_0 = T_{0a} \quad \text{everywhere,} \quad (201) \]

IN SUCH A WAY THAT (200) GIVES

\[ \kappa \frac{\partial T}{\partial y} + \mu \frac{\partial u}{\partial y} = \text{const.} = 0 \quad \text{at} \quad y = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} \quad \text{is assumed to be zero} \quad (202) \]

THIS CONSTRAINT INDICATES THAT THE HEAT CONDUCTION AND THE WORK DONE BY THE VISCOUS STRESS ARE IN BALANCE EVERYWHERE, IN PARTICULAR, AT THE WALL

\[ q_w = -\kappa \frac{\partial T}{\partial y} \bigg|_{y=0} - \mu \frac{\partial u}{\partial y} \bigg|_{y=0} = 0 \quad (203) \]

SINCE \( u = 0 \) AT THE WALL. AS A RESULT, A COMPRESSIBLE BOUNDARY LAYER OF A GAS WITH UNITY PRANDTL NUMBER OVER AN ADIABATIC WALL NECESSARILY REQUIRES THE STAGNATION TEMPERATURE (OR ENTHALPY) TO BE UNIFORM AND EQUAL TO \( T_{0a} \).

BECAUSE OF (145), THE ADIABATIC WALL TEMPERATURE EQUALS THE STAGNATION TEMPERATURE,

\[ \overline{T_{0a}} = T_{aw} + \overline{U^2} \overline{2c_p} = T_{aw} \quad (204) \]

THERBY INDICATING A UNITY RECOVERY FACTOR, \( \bar{a} = 1 \). NOTE THAT THE SAME CONCLUSION CAN BE INFERRED FROM (202),

\[ \frac{\partial}{\partial y} \left( T + \overline{P_r U^2} \overline{2c_p} \right) = 0 \quad \Rightarrow \quad T + \overline{U^2} \overline{2c_p} = T_{0a} \quad \text{everywhere} \quad (205). \]

EQUATIONS (204) - (205) SUGGEST THAT, FOR GASES WITH \( P_r < 1 \), IT IS EXPECTED THAT \( \gamma < 1 \), BUT SINCE \( P_r \approx 0.72 \) IN PRACTICAL APPLICATIONS, THE ADIABATIC WALL TEMPERATURE IS ALWAYS MUCH CLOSER TO THE STAGNATION TEMPERATURE OF THE OVERRIDING INVISID STREAM THAN TO ITS STATIC TEMPERATURE. AT HYPERSONIC MACH NUMBERS, \( \frac{T_{0a}}{T_0} \gg 1 \), AND AS A CONSEQUENCE, \( T_{aw} \) IS VERY HIGH, AND THE SURFACE OF THE ADIABATIC WALL GETS VERY HOT (NOTE THAT IN PRACTICE, RADIATION FROM THE SURFACE ESTABLISHES LOWER WALL TEMPERATURES ALTHOUGH THEY ARE SUFFICIENTLY HOT TO MELT THERMAL-RESISTANT ALLOYS). IT IS FOR THESE REASONS THAT SOME OF THE MOST IMPORTANT CHALLENGES OF HYPERSONIC FLIGHT ARE NOT ASSOCIATED WITH BREAKING THE SOUND BARRIER, BUT RATHER WITH BREAKING THIS "THERMAL BARRIER".

CONSIDER NOW AN ADDITIONAL SIMPLIFICATION OF THE PROBLEM BY MAKING A ZERO PRESSURE GRADIENT (\( \frac{dP_0}{dx} = 0 \)) IN (194), WHICH NOW BECOMES
\[
\frac{\partial}{\partial x} \left( \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) = \frac{2}{\gamma} \left( \frac{\partial u}{\partial y} \right). \tag{206}
\]

This equation is very similar to Eq. (199) for the stagnation temperature at \( Pr = 1 \). (Recall also that \( \mu = \kappa \) since \( c_p = \text{const} \) if the wall is isothermal.) It is straightforward to see that the similarity relation

\[
\frac{T_0 - T_W}{T_{oe} - T_W} = \frac{U}{U_0} \tag{207}
\]

must be necessarily satisfied, since (199) and (206) along with the boundary conditions \( U = U_0 \), \( T_0 = T_{oe} \) \((\gamma = \infty)\), and \( U = 0 \), \( T_0 = T_W \) \((\gamma = 0)\) are formally equivalent problems when formulated in the variables \( \frac{U}{U_0} \) and \( \frac{T_0 - T_W}{T_{oe} - T_W} \).

Equation (207) indicates that isolevels of velocity are also isolevels of stagnation temperature \((\text{Fig. 64})\), and as a result the thickness of the thermal and momentum boundary layers have equal thicknesses.

Upon substituting (195) into (205), the relation between the velocity and static temperature

\[
T - T_W = (T_{oe} - T_W) \left( \frac{U}{U_0} \right) - \frac{\gamma-4}{2} \frac{Ma^2}{\kappa} \frac{T_{oe}}{U_0} \left( \frac{U}{U_0} \right)^2 \tag{208}
\]

is obtained, which enables the calculation of the local heat flux

\[
q = -\kappa \frac{\partial T}{\partial y} = \kappa \frac{U}{U_0} \frac{\partial U}{\partial y} \left[ T_W - T_{oe} + (\gamma - 4) \frac{Ma^2}{\kappa} \frac{T_{oe}}{U_0} \frac{U}{U_0} \right]. \tag{209}
\]

It is interesting to note that, since \( \frac{\partial U}{\partial y} > 0 \) for attached boundary layers, then \( q > 0 \) \((\text{i.e., } \frac{\partial T}{\partial y} < 0)\) everywhere if \( T_W > T_{oe} \), thereby resembling the blue curve in Fig. 56 in which heat is always conducted away from the hot plate. On the other hand, if \( T_W < T_{oe} \), then \( q \) may be positive or negative depending on position, indicating a non-monotonic behavior in the temperature as in the red curve in Fig. 63. (Note that \( T_W = T_{oe} \) when \( Pr = 1 \) as elucidated above).

In all cases, the wall heat flux is given by

\[
q_w = -\kappa \frac{\partial T}{\partial y} \bigg|_{y=0} = C_p \frac{U_0}{U_0} (T_W - T_{oe}), \tag{210}
\]
WHERE \( \tau_w = \mu \partial v / \partial y \bigg|_{y=0} \) IS THE SHEAR STRESS AT THE WALL. EQUATION (240) CAN BE RECAST INTO THE FAMILIAR FORM

\[
St = C_f / 2
\]

REYNOLDS ANALOGY (211)

WHERE

\[
St = \frac{\tau_w}{\rho u_0 c_f (T_{w0} - T_w)}
\]

IS THE STANTON NUMBER, AND

\[
C_f = \frac{T_w}{\frac{1}{2} \rho u_0 u_*^2}
\]

IS THE SKIN FRICTION COEFFICIENT.

EQUATION (211), THE REYNOLDS ANALOGY, ESTABLISHES AN ANALOGY BETWEEN HEAT AND MOMENTUM TRANSFER, OR BETWEEN HEAT TRANSFER AND VISCOUS FRICTION. IN DERIVING SUCH ANALOGY RECALL THAT IT HAS BEEN ASSUMED THE FOLLOWING:

1) THE GAS IS CALORICALLY PERFECT AND NON-RADIATING, 66) THE PRANDTL NUMBER IS \( Pr = 1 \), 67) THE WALL IS ISO THERMAL AND ITS TEMPERATURE IS CONSTANT ALONG THE WALL, AND 68) THE PRESSURE GRADIENT IS ZERO, \( dP/dx = 0 \). NOTE THAT NO ASSUMPTIONS HAVE BEEN MADE REGARDING \( \mu \) OR \( \kappa \), WHICH CAN VARY ARBITRARILY WITH TEMPERATURE.

IT IS NOT DIFFICULT TO REFORMULATE THE REYNOLDS ANALOGY (211) IN TERMS OF THE NUSSELT NUMBER, \( Nu_x = h x / k_0 \), WITH \( h = \frac{\tau_w}{(T_w - T_{w0})} \) THE CONVEKTIVE COEFFICIENT, AND THE REYNOLDS NUMBER, \( Re_x = \rho u_0 x / \mu_0 \), WHICH GIVES

\[
Nu_x = \frac{C_f}{2} Re_x \quad (212)
\]

FOR PRACTICAL PURPOSES, THE ANALYSIS ABOVE SUGGESTS THAT THE DIRECTION OF THE HEAT FLUX AT THE WALL DEPENDS ON THE DIFFERENCE \( T_w - T_{w0} \), WITH \( T_{w0} \) BEING OF ORDER \( T_0 \), RATHER THAN ON THE DIFFERENCE \( T_w - T_0 \), WHICH INSTEAD BECOMES RELEVANT FOR HEAT TRANSFER ONLY IN LOW-SPEED BOUNDARY LAYERS.

SELF-SIMILAR BOUNDARY-LAYER EQUATIONS

THE CONSIDERATIONS GIVEN ABOVE ONLY PERTAIN TO INTEGRAL RELATIONS BETWEEN \( T \) AND \( u \). INSTEAD, THIS SECTION FOCUSES ON DERIVING A SELF-SIMILAR SYSTEM OF EQUATIONS EQUIVALENT TO THE GENERAL ONES GIVEN BY (190)-(193). FOR THIS PURPOSE, THE FOLLOWING CHANGE OF VARIABLES IS INTRODUCED, WHICH WAS PROPOSED BY LEES IN "LAMINAR HEAT TRANSFER OVER BLUNTED-BODIES AT HYPERSONIC SPEEDS", JET PROPULSION 26 (1956), AND WHICH
COMBINES THE LEVY AND HANGLER AND HOWARTH-DORODNITZYN TRANSFORMS. THE CHANGE OF INDEPENDENT VARIABLES IS GIVEN BY

$$
\xi = \xi(x) = \int_0^x \frac{\rho_0}{\rho_0} \frac{\rho \xi}{\rho} \, dx \quad , \quad \eta = \eta(x, y) = \frac{\rho_0}{\rho} \int_0^y \, dy
$$

(213)

IN SUCH A WAY THAT

$$
\frac{\partial \xi}{\partial x} = \frac{\rho_0}{\rho_0} \frac{\rho \xi}{\rho} \quad , \quad \frac{\partial \eta}{\partial y} = \frac{\rho_0}{\rho} - \frac{\rho \xi}{\rho}
$$

(214)

THE ANALYSIS IS ALSO FACILITATED BY INTRODUCING THE STREAM FUNCTION \( \psi \), WITH

$$
\psi_u = \frac{\partial \psi}{\partial y} \quad \text{AND} \quad \psi_v = -\frac{\partial \psi}{\partial x}
$$

(215)

AND BY TRANSFORMING THE DEPENDENT VARIABLES \( \psi, u \) AND \( h \) AS

$$
\psi = \sqrt{2 \xi} f(t) \quad , \quad u = U_0(x) f'(t) \quad \text{AND} \quad h = h_0(x) g(h)
$$

(216)

WHERE \( U_0(x) \) AND \( h_0(x) \) ARE DISTRIBUTIONS AT THE EDGE OF THE BOUNDARY LAYER BY THE OVERRIDING INVISCO STREAM, AND \( f \) AND \( g \) ARE SOLE FUNCTIONS OF \( t \) THAT ARE TO BE SOLVED FOR, INTRODUCING THIS CHANGE IN EQUATIONS (190)-(193) THE SELF-SIMILAR FORMS

$$
\left( \frac{\partial f''}{\partial \xi} \right) + \frac{f f''}{u} \frac{\partial f'}{\partial \xi} = \frac{2 \xi}{u} \frac{\partial f'}{\partial \xi} \quad , \quad \text{MOMENTUM CONSERVATION (218)}
$$

$$
\left( \frac{\partial h'}{\partial \xi} \right) + \frac{h f}{\rho_0} \frac{\partial f'}{\partial \xi} = \frac{2 \xi}{\rho_0} \frac{\partial h}{\partial \xi} \quad , \quad \text{MOMENTUM CONSERVATION (218)}
$$

(218)

ARE OBTAINED ALONG WITH THE UNIFORMITY OF THE PRESSURE \( \frac{\partial f'}{\partial \xi} = 0 \) ACROSS THE BOUNDARY LAYER. IN DERIVING THESE EQUATIONS, USE OF THE EULER EQUATION IN THE INVISCO STREAM TO REDUCE THE INITIAL BOUNDARY CONDITIONS AT THE BOUNDARY HAS BEEN MADE. IN ADDITION, THE PARAMETER \( \gamma \) IS GIVEN BY

$$
\gamma = \frac{3}{4} \frac{\psi}{\nu_0 \rho_0}
$$

(220)

THE TRANSVERSE VELOCITY COMPONENT \( v \) CAN BE OBTAINED FROM THE SECOND EQUATION IN (215),

$$
\psi_v = -\frac{\partial \psi}{\partial x} = -\frac{\partial \eta}{\partial x} \, \frac{\partial f'}{\partial \xi} = -\frac{\mu_0 \rho \xi}{\rho_0} \frac{\partial f'}{\partial \xi} - \frac{\eta_0}{\rho_0} \frac{\partial f'}{\partial \xi}
$$

(221)

WHERE \( \frac{\partial f'}{\partial \xi} \) CAN BE COMPUTED THE FOLLOWING WAY. FOR UNIFORM FREE-SKREAM CONDITIONS, THE INTEGRATION OF (214) YIELDS \( f = \sqrt{2 \xi} \left[ \frac{2}{R_{\infty}} \int_0^x g(t) \, dt \right] \), SO THAT \( \frac{\partial f}{\partial \xi} = -\frac{1}{\sqrt{2 \xi}} \left( \frac{2}{R_{\infty}} \right) \frac{\mu_0 \rho \xi}{\rho_0} f' \). SUBSTITUTING THIS INTO (221) YIELDS \( v \rho_0 = \left( \frac{1}{2 \xi} \right) f' - \frac{f}{R_{\infty}} \), WHERE \( R_{\infty} = \frac{\rho_0 \nu_0 \mu_0}{\rho_0} \) IS THE LOCAL REYNOLDS NUMBER. THE BOUNDARY CONDITIONS ASSOCIATED WITH (217) AND (218) ARE
\[ \xi = \xi' = 0 \text{ AT THE WALL } \xi = 0, \text{ CORRESPONDING TO NON-SLIP AND NON-PERMEABLE WALL,} \]
\[ \eta = \eta' = 0 \text{ AT THE WALL } \eta = 0, \text{ CORRESPONDING TO ADIABATIC WALL,} \]
\[ \text{OR } \eta = \eta_0 \text{ AT THE WALL } \eta = 0, \text{ CORRESPONDING TO ISOTHERMAL WALL,} \]
\[ \xi' = \eta' = 0 \text{ AT THE BOUNDARY LAYER EDGE CORRESPONDING TO THE INVIScid STREAM VELOCITY AND ENTHALPY.} \]

It is important to note that (214)–(218) are generally valid for a non-reacting, non-radiating gas at local thermodynamic equilibrium. In the formulation, \( \beta, \mu, \kappa, \rho, \) and \( \xi \) may vary with temperature, and the edge conditions \( \eta_0, \eta_0', \mu_0, \kappa_0 \) and \( \rho_0 \) as well as the wall temperature \( T_w \) may vary with \( x \) (or \( \xi \)). As a consequence, very rarely in hypersonic flows the equations (214)–(218) are fully self-similar, in that they only depend on \( \xi \). For such self-similar conditions to be satisfied, the following conditions must be simultaneously satisfied:

1) The right-hand side of (214) must be a function of \( \xi \) or a constant. This condition is trivially satisfied in flat-plate boundary layers with zero pressure gradient in the streamwise direction, which, by equation (21a), implies that \( \partial U_\xi / \partial x = 0 \).

2) The Chapman–Rubensin parameter must be a function of \( \xi \), i.e., \( \xi = \xi(\eta) \), or equal to a constant, it is anticipated that this occurs when

\[ \mu \text{ varies as } \xi / \eta, \Rightarrow \xi = \mu / \mu_0 \eta, \quad \xi = x \quad (\mu = \mu_0, \quad \xi = \xi_0) \]

The gas is calorically perfect and \( \mu \) varies with a power of the temperature,

\[ \xi = \mu / \mu_0 \eta = (T / T_\infty)^{\sigma - 1} \quad (\text{Note } \mu / \mu_0 = (T / T_\infty)^{\sigma - 1} \text{ here, } \sigma > 0) \]

The gas is calorically perfect, \( \mu \) varies with \( T \) according to the Sutherland's law, and \( T_\infty \) is uniform in \( x \):,

\[ \xi = \mu / \mu_0 \eta = (T / T_\infty)^{3/2} \left[ (T_\infty + A) / T_\infty \right]^{1/2} = \eta_0 \mu_0 \right)^{3/2} \left[ (1 + A / T_\infty) / \eta_0 \right]^{1/2} = F(\eta) \]

(Note that Sutherland's law for viscosity is

\[ \mu / \mu_0 = (T / T_\infty)^{3/2} \left[ (T_\infty + A) / T_\infty \right] \], with \( A = \text{const} \))

3) The Prandtl number must be constant or a function of \( \eta \) alone, as with the \( \xi \) parameter, this condition is satisfied when the gas is calorically perfect, and \( \mu \) varies with a power of the temperature, or with Sutherland's
LAW UNDER ZERO STREAMWISE VARIATIONS OF $T_o$.

4) THE WALL TEMPERATURE, IF IMPOSED BY BOUNDARY CONDITIONS, MUST BE A CONSTANT.

5) THE RIGHT-HAND SIDE OF EQUATION (218) MUST BE A CONSTANT OR A FUNCTION OF $\gamma$ ALONE. THIS CONDITION IS TRIVIALLY SATISFIED IN FLAT-PLATE BOUNDARY LAYERS UNDER ZERO PRESSURE GRADIENT, SINCE THIS IMPLIES ZERO ENTHALPY GRADIENT IN THE STREAMWISE DIRECTION ALONG THE EDGE

$$\rho_0 U_0 \frac{dh_0}{dx} = U_0 \frac{dP_e}{dx} = 0. \quad (223)$$

IN THESE CONDITIONS, THE AERODYNAMIC HEATING TERM IN (218) SCALES AS

$$\frac{U_0^2}{h_0} \frac{\nabla}{\nabla h_0} = \frac{\gamma}{\gamma-1} \frac{P_e}{S_e} = \frac{1}{M_0^2} \quad (224)$$

WITH $M_0$ THE MACH NUMBER AT THE BOUNDARY-LAYER EDGE.

THE CASE OF A BOUNDARY LAYER OVER A FLAT PLATE WITH $\frac{dP_e}{dx} = 0$

FOR A FLAT-PLATE BOUNDARY LAYER WITH ZERO PRESSURE GRADIENT, $\frac{dP_e}{dx} = \frac{dU_0^2}{dx} = \frac{dh_0}{dx} = 0$.

AND $\frac{dP_e}{dx} = dS_e/dx = 0$. CORRESPONDINGLY, Eqs. (219) - (218) BECOME

$$\left( \frac{Q_{m'}}{Pr} \right)' + f Q' = 0 \quad (225)$$

$$\left( \frac{Q_{m'}}{Pr} \right)' + f' \phi' = \left( \frac{1}{Pr} - 1 \right) \left( 1 + \frac{\gamma-1}{2} M_0^2 \right)^{-\frac{1}{2}} \frac{d}{dx} \left( \frac{P_e}{S_e} \right) = 0 \quad (226)$$

SUBJECT TO $f(0) = f'(0) = 0$, $\phi_0(0) = 0$ (OR $\phi_0(0) = \phi_0$), AND $f'(\infty) = \phi'(\infty) = \frac{1}{2}$.

WHERE A CALORICALLY PERFECT GAS HAS BEEN ASSUMED, SO AS TO EXPRESS THE LAST TERM OF (218) USING (224). NOTE THAT AN EQUATION FOR THE STAGNATION TEMPERATURE

$$\gamma = \frac{T_0}{T_{oe}} = \frac{\gamma}{\gamma-1} \left( 1 + \frac{\gamma-1}{2} M_0^2 \right)^{-\frac{1}{2}} = \gamma \left( \frac{1}{Pr} \right) \frac{P_e}{S_e} \quad (227)$$

CAN BE EASILY DERIVED BY MULTIPLYING (225) BY $\frac{U_0^2}{Pr T_{oe}}$ AND ADDING THE RESULTING EQUATION TO (226) MULTIPLIED BY $\frac{P_e}{Pr T_{oe}}$, WHICH GIVES

$$\left( \frac{Q_{m'}}{Pr} \right)' + f m' + \left[ \frac{Q_{m'}}{Pr} \left( \frac{1}{Pr} - 1 \right) \left( 1 + \frac{\gamma-1}{2} M_0^2 \right)^{-\frac{1}{2}} \right]' = 0 \quad (228)$$

OR EQUIVALENTLY

$$\left( \frac{Q_{m'}}{Pr} \right)' + f m' + \left( \frac{\gamma-1}{2} M_0^2 \left[ 1 + \frac{(\gamma-1)}{2} M_0^2 \right]^{-\frac{1}{2}} \right)' \left( \frac{1}{Pr} \right) \frac{P_e}{S_e} = 0 \quad (229)$$
IN PARTICULAR, (223) IS THE SELF-SIMILAR FORM EQUIVALENT TO (194), AND IT ALSO LEADS TO
THE CONSTANT STRATIFICATION-TEMPERATURE RESULT \( m = 1 \) FOR ADIABATIC WALLS AND \( \Gamma = 1 \),
in such a way that the adiabatic wall temperature in self-similar form is
\[
\frac{T_{aw}}{T_e} = 1 + \left( \frac{V_e - V}{E} \right) M_{aw}^2
\]
which is the same as the stagnation temperature of the inviscid stream
(i.e., unity recovery factor \( \Gamma = 1 \)).

Profiles of \( U/U_e \) and \( T/T_e \) are shown
in Fig. 65 for adiabatic and isothermal
walls and varying edge Mach numbers.
The vertical axis refers to \( \sqrt{Re_{st}} \) with
\[
\delta_{st}(x) = \frac{x}{\sqrt{Re_{st}x}} = \frac{x}{(Re_{st}x/\mu_e)^{1/2}}
\]
The classic boundary-layer thickness estimate.
Results are also provided for \( M_{aw} = 0 \). Note
that in this limit, (225) does not collapse
to the Blasius problem unless \( \partial \frac{\partial}{\partial y} \), \( \Gamma = 1 \),
in which case \( \Gamma = 1 \) (\( T = T_e \)), \( \Gamma = 1 \),
\( \delta_{st} = \sqrt{Re_{st}} \delta_{st}(x) \) is the classic self-similar variable
for incompressible boundary layers, and (225)
becomes completely decoupled from (226).
The skin friction coefficient distribution on the
flat plate is given by
\[
C_f = \frac{2 \mu w}{\rho_e U_e^2} \left[ \frac{\partial^2 u}{\partial y^2} \right]_{y=0} = \frac{2 \mu w}{\rho_e U_e^2} \left( \frac{\rho_e U_e^2}{\sqrt{Re_{st}}} \right) f''(0) = \frac{2 \mu w}{\rho_e U_e^2} \left( \frac{\rho_e U_e^2}{\sqrt{Re_{st}}} \right) f''(0)
\]
where \( q_w = \frac{\partial}{\partial y} w \) is the Chapman-Enskog parameter at the wall, and

\[
Re_{st,x} = \frac{\rho_e U_e x}{\mu_e}
\]

Fig. 65: (a) velocity and
(b) temperature profiles for adiabatic (left)
and isothermal \( (T_w/T_e = 1/4) \) walls (right).
In these results, \( \Gamma = 0.75 \), \( \gamma = 1.4 \),
\( \beta = \text{const.} \), and \( A/T_e = 0.505 \) (Van Driest,
NACA TN 2594, (1952)) data digitized
by Anderson (2006)
is the local Reynolds number based on the edge conditions. In general, note from (225) - (226), the associated boundary conditions, and (2.34), that \( \zeta' \) is a complex function of several parameters in compressible boundary layers, \( \zeta' = F(M_{\infty}, \text{Pr}, T_e, T_w/T_e) \).

This in contrast with the simpler incompressible case, \( \zeta' = 0.664/\sqrt{Re_{\infty}} \).

A similar derivation can be made for the Stanton number,

\[
St = \frac{q_w}{\rho_e U_e(h_m - h_w)} = \frac{1}{\rho_e U_e(h_m - h_w)} \left[ \frac{K T_l}{Q T} \right] = \frac{\frac{K w}{\rho_e U_e Cp(T_w - T_m)}}{\frac{1}{2} \frac{T_e}{T_\infty} \frac{\alpha'(0)}{Re_{\infty}}} \]

For \( \text{Pr} = 1 \), the adiabatic wall temperature is equal to the stagnation temperature of the inviscid stream, \( T_w = T_{m,\infty} \). Additionally, for \( \text{Pr} = 1 \) and isothermal wall, the equations (225) and (226) are similar to each other in terms of the variables

\[
\frac{m(h)}{q_w/q_w} = f(h) \quad (234)
\]

\( \text{Pr} \neq 1 \), the adiabatic wall temperature must be computed numerically. However, it is worth mentioning that, in all cases, the recovery factor \( M_{\infty} \) tends to scale very accurately as

\[
q_w \approx \sqrt{Pr} \quad (235)
\]

This relation tends to hold for wide ranges of \( \text{Pr} \) and \( M_{\infty} \), and for constant and

FIG. 68. (a) Skin friction coefficient and (b) Stanton number for flat-plate compressible boundary layers with \( \text{Pr} = 0.75 \), \( \gamma = 1.4 \), \( \zeta' = \text{const} \), and \( A/T_e = 0.505 \) (van Driest, NACA TN # 2597, 1952). Data digitized by Anderson (2005).

Isothermal walls, the Reynolds analogy (244) is recovered.

When \( \text{Pr} \neq 1 \), the adiabatic wall temperature must be computed numerically.
The boundary-layer profiles shown in Fig. 65 are along with the variations of $q_T$ and $S_b$ in Fig. 66 indicate a number of important results that can be summarized as follows:

1) The boundary-layer thickness increases rapidly, with $M_a$. This is clearly observed in Fig. 65, suggesting that $S_b$ is no longer an appropriate scale at high Mach numbers. In contrast, when

$$\delta_b = \frac{x}{\left(\frac{\rho_w u_w}{\mu_w}\right)^{1/2}}$$

is employed to scale the results, an improved collapse is observed (Fig. 68). Note that (236) can be written in terms of the Mach number by doing

$$\delta_b = \frac{x}{\left(\frac{\rho_w u_w}{\mu_w}\right)^{1/2}} = \left(\frac{\rho_w}{\rho_w}\right)^{1/2} \delta_{b_0} = \left(\frac{\rho_w}{\rho_w}\right)^{1/2} S_{b_0}$$

with $\delta_{b_0} = \frac{\rho_w \mu_w}{\rho_w \mu_w} \left[\left(\frac{T_w}{T_e}\right) \left(\frac{T_w}{T_e}\right)^{1/2} \left(\frac{T_w}{T_e}\right)^{1/2}\right]^{1/2} \approx \frac{T_w}{T_e}$

and $\frac{\rho_w}{\rho_w} = \frac{T_w}{T_e}$, so that

$$\delta_b = \left(\frac{T_w}{T_e}\right)^{1/2} \delta_{b_0}$$

(237)

with $\delta_{b_0}$ the incompressible scaling given by (236). Assuming $T_w \approx T_{in}$ and $\gamma = 1$, then $T_w/T_e \approx T_{in}/T_e = T_{in}/T_e = 1 + (\gamma - 1) \frac{T_{in}}{T_e}$. Introducing this approximation in (237) and taking the limit $M_a \to 1$ gives

$$\delta_b = M_a^2 \delta_{b_0} = M_a^2 \frac{x}{\rho w u_w^{1/2}}$$

(238)

where the temperature exponent $\gamma$ for the viscosity has been assumed to be close to unity as is usually the case.

2) The boundary layer becomes thinner as the wall temperature decreases.

This effect is clearly illustrated by comparing the adiabatic and isothermal cases in Fig. 58 and noting that the wall temperature in the former is much higher. A decrease in $T_w$ increases the density in the boundary layer, which therefore renders a thinner
Boundary layer that can carry the same amount of momentum deficit as a thicker boundary layer with a higher wall temperature. This scaling is also ratified by (237).

The peak temperature in the boundary layer increases rapidly with $M_{w0}$.

This is an important effect visualized in Fig. 65b that is caused by the increase of the viscous dissipation term in Eq. (226). As a result, as $M_{w0}$ increases, an increasing amount of heat is transferred to the surface that is usually referred to as hypersonic aerodynamic heating, and which plays an important role in the design of aerospace vehicles.

The aerodynamic heating rate is typically of the same order as the flux of stagnation enthalpy impinging on the vehicle. This can be seen by taking Eq. (238), i.e., assuming that $\dot{h}_{nw} \gg \dot{h}_w$ as is usually the case at $M_{w0} \gg 1$, and approximating $\dot{h}_{nw}$ by $\dot{h}_{w0}$ (i.e., $\gamma \approx 1$), which gives

$$\dot{q}_w \approx \frac{1}{2} \dot{e}_w \frac{C}{2} \approx \frac{1}{2} \dot{e}_w C_f$$  

(239)

Where use of the definition $\dot{h}_{w0} = \dot{h}_w + \frac{1}{2} \dot{e}_w$ has been made in the limit $M_{w0} \gg 1$. Note that the aerodynamic heating rate (239) is, in the first approximation proportional to $U_0^3$ (i.e., up to other dependencies in the Stanton number $St = \frac{F(M_{w0}, Pr, \gamma, T_w/T_0)}{1 + \frac{1}{2} C_f}$). This cubic dependence on $U_0$ is in contrast with the quadratic dependence of the friction drag on the velocity $D = \frac{1}{2} \rho_0 U_0^2 S C_f$,

(Where "to" indicates true free-stream conditions) and highlights the importance of aerodynamic heating at high speeds.

Note: Eq. (239) is one form of aerodynamic heating created by viscous friction. The term "aerodynamic heating" is also widely employed to refer to other aerodynamically-derived heating mechanisms that are not necessarily linked to viscous dissipation and which may involve shock/shock interactions, shock/boundary-layer interactions, shock-induced radiative heating, or low-speed heat transfer in stagnation flow regions. Remarkably, however (239) already elucidates the importance of minimizing friction (i.e., $St \approx C_f$) on vehicles flying at high speeds in denser (lower) parts of the atmosphere ($\approx 10^5$ km).
The skin friction coefficient and the Stanton number decrease as Mach increases. This is clearly observed in Fig. 66 but should not be misinterpreted as a decrease in the wall shear stress or wall heat flux, since these two increase quadratically and cubically with velocity.

The skin friction coefficient and the Stanton number increase as the wall is cooled. This is also clearly observed in Fig. 66 and is expected from the fact that the boundary layer becomes thinner as $T_w$ decreases.

The considerations above illustrate the peculiarities found in compressible laminar boundary layers on flat plates. Attention is now diverted to a different type of flow that can also be solved with Eqs. (217)-(218) and which is of technological relevance for hypersonic flows over blunt bodies.

The case of the stagnation-region flow field around a blunt body.

As mentioned previously, the hypersonic flow over a blunt body generates a bow shock that sits close to the body surface. Near the normal portion of this shock, a stagnation streamline appears that divides the flow into upwards and downwards turning portions (Fig. 69). It is important to note, however, that this dividing streamline, which terminates at a stagnation point, does not generally pass through the normal portion of the shock under non-zero angles of attack, but instead tends to drift towards the side of the body with the largest curvature (see Fig. 70). Focus here will nevertheless be limited to the simpler symmetric case depicted in Fig. 69.
The flow field around the stagnation region does not fulfill the conditions for self-similarity, since $U_e$, $T_e$, $S_e$ and $\mu_e$ generally vary in the streamwise direction along the blunt body surface as specified by the corresponding inviscid description of the post-shock flow (see Chapter III). However, one can always use "local self-similarity" to describe the flow locally in the stagnation point region assuming that such streamwise variations are slow. In this way, one can still use Eqs. (217) - (218) by letting $dU_e/d\xi$ and $dS_e/d\xi$ be functions of $\xi$ that nonetheless vary slowly, and in this particular problem, render again a formulation that only depends on $\beta$. To see this, consider the following approximations:

i) Near the stagnation point, the edge velocity $U_e$ is small, in particular, the flow in this region is subsonic, and therefore the viscous heating term does not play any role in the enthalpy equation (218). In addition, because of the associated small values of $M_0$, the enthalpy along the edge is assumed to be locally uniform and equal to $h_{2e} = h_{2oo}$, so that the term involving $dS_e/d\xi$ in (218) vanishes.

ii) Near the stagnation point, the velocity field in the inviscid stream can be approximated by a local strain rate $dU_e/dx \bigg|_{x=0}$ multiplied by the distance $x$ from the stagnation streamline along the curved body, namely

$$U_e \approx \left( \frac{dU_e}{d\xi} \right) \bigg|_{x=0} \times x \quad (240)$$

(This flow is typically referred to as Hieke's after Hieke [1914] and represents a particular case of the Falkner-Skan family of solutions). An important consequence of (240) is that

$$\bar{\xi} = \int_0^x s_e \frac{\mu_e}{U_e} d\xi = \int_0^x \frac{dU_e}{d\xi} \bigg|_{x=0} \times x \frac{\mu_e}{\mu_e} \bigg|_{x=0} \frac{x^2}{2} \quad (241)$$

And therefore

$$\frac{dU_e}{d\xi} \bigg|_{x=0} \frac{d\xi}{d\bar{\xi}} = \left( \frac{dU_e}{d\xi} \bigg|_{x=0} \right) \frac{1}{\frac{d\xi}{d\bar{\xi}}} \Rightarrow \frac{2\bar{\xi}}{U_e} \frac{dU_e}{d\bar{\xi}} = 4 \quad (242)$$
Equation (242) helps in getting rid of the $\delta$ dependence of the right-hand side of the momentum equation (241). Additionally, it is not difficult to see that the coefficient $2 \xi \left( \frac{U_0}{\delta h_0} \right) \frac{dU_0}{d\delta}$ on the right-hand side of the energy equation (242) can be rewritten as

$$2 \xi \frac{\delta U_0}{\delta h_0} \frac{dU_0}{d\delta} = 2 \left[ \frac{\delta U_0}{\delta h_0} \left( \frac{dU_0}{dx} \right) _{x=0} ^{x=2} \right] \left[ \left( \frac{dU_0}{dx} \right) _{x=0} ^{x=2} \right] \frac{1}{\delta h_0 \delta h_0} = \frac{\delta U_0}{\delta h_0} \left( \frac{dU_0}{dx} \right) _{x=0} ^{x=2} \frac{1}{\delta h_0}$$

which becomes vanishingly small near $x=0$ and can be neglected in the first approximation.

For calorically perfect gases, $C_p = \text{const.}$, and the term $\delta h_0 / \delta$ on the right-hand side of the momentum equation (241) can be rewritten as

$$\delta \frac{\delta h_0}{\delta} = \frac{T}{T_0} = \frac{h}{h_0} = \delta \left( \frac{h}{h_0} \right)$$

With these $1^\circ$-$4^\circ$ approximations, the self-similar conservation equations for the stagnation-point flow become

\begin{align*}
(\gamma f''(f))' + ff' &= f'^2 - \gamma (243) \\
(\gamma h'_{Fr})' + f'q' &= 0 (244)
\end{align*}

Subject to $f(0) = f'(0) = 0$, $q'(0) = 0$ (or $\delta (0) = \delta w$), and $f'(\infty) = q'(\infty) = 1$.

It is interesting to compare (243) - (244) with their counterparts for the flat-plate problem (225) - (226) and realize that here the streamwise convection of momentum and the streamwise pressure gradient do play a role, while viscous heating does not play any role, all the aerodynamic heating is due here to the thermal energy recovery by the gas deceleration behind the shock, where the temperature increases significantly ($\delta \xi$, $T_e / T_{\infty} \to T_e / T_{\infty}$), and the heat is transferred by conduction to the body surface (Fig. 7.1) across the boundary layer.

The wall heat flux near the stagnation region can be expressed as

$$q_w = \frac{S_t \delta h_0 U_0 C_p}{T_{aw} - T_w} (245)$$

where $T_{aw} = T_e + \frac{U_0^2}{2C_p} \sim T_e \sim T_{aw}$, while $S_t$ is

$$S_t < 1 \quad T_e = \text{const.} \quad \text{across shock} + \text{Mach 1}$$

Fig. 7.1
A STANTON NUMBER THAT NEEDS TO BE OBTAINED BY NUMERICAL INTEGRATION OF Eqs. (243)-(244). FOR CYLINDERS AT ZERO MACH NUMBERS

\[ S_t = 0.570 \sqrt{\frac{Pr}{Re}} (246) \]

WHILE FOR SPHERES IT IS BEST CORRELATED WITH

\[ S_t = 0.763 \sqrt{\frac{Pr}{Re}} (247) \]

IN BOTH CASES, (245) ATTAINS THE FORM

\[ \frac{q}{\mu} = \text{const.} \frac{Pr}{Re} \frac{dU_e}{dx} \frac{C_F}{(T_{oo}-T_w)} \]  

WALL HEAT FLUX IN THE STANZATION REGION

(248)

WHICH IS TYPICALLY REFERRED TO AS "FYF-RIDDELL" CORRELATION AFTER FYF AND RIDDLEL" THEOREY OF STANZATION POINT HEAT TRANSFER IN DISASSOCIATED AIR" J. AERONAUT. SCI. 25 (1958),

ALTHOUGH FYF AND RIDDLEL'S EXPRESSION IS A MORE GENERAL ONE THAT CONTAINS CHEMICAL EFFECTS RESULTING FROM AIR DISASSOCIATION.


\[ \frac{dP_e}{dx} = - \frac{\rho_0 U_e}{\rho_e} \frac{dU_e}{dx} + \frac{\rho_0 U_e}{\rho_e} \frac{dU_e^2}{dx} \]

ALONG WITH THE NEWTONIAN APPROXIMATION

\[ P_e = \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \sin \delta = \rho_0 + \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \cos^2 \theta \]

WHICH GIVES

\[ \frac{dP_e}{dx} \approx \frac{2}{\gamma_0} \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \cos \theta \sin \theta \frac{d\theta}{dx} \approx \frac{2}{\gamma_0} \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \frac{d\theta}{dx} + \frac{2}{\gamma_0} \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \frac{2}{R_0} \]

AND THEREFORE

\[ \frac{dU_e}{dx} \approx \frac{1}{R_0} \left( \frac{2}{\gamma_0} \frac{\rho_0}{\gamma_0} \frac{U_e}{\gamma_0} \right) \]

INTRODUCING THIS EXPRESSION INTO (248),

IT IS OBSERVED THAT THE WALL HEAT FLUX IN THE STANZATION REGION SCALES AS

\[ \frac{q}{\mu} \approx \frac{2}{\gamma_0} \text{const.} \frac{Pr}{Re} \frac{U_e}{\gamma_0} \frac{5}{2} \]

(249)

WHICH REVEALS A CUBIC DEPENDENCE ON \( U_e \) SIMILARLY TO (239), AND MOST IMPORTANTLY, THAT THE STANZATION POINT HEATING VARIES INVERSELY WITH THE SQUARE ROOT OF THE RADIUS OF CURVATURE OF THE NOSE. THE RESULT (249) IS OF PARAMOUNT IMPORTANCE FOR THE DESIGN OF HYPERSONIC VEHICLES AND ESTABLISHES THAT SHARP EDGES, ALBEIT
RELEVANT FOR REDUCING DRAG, ARE IMPRACTICAL IN HYPERSONICS SINCE THE MATERIAL WOULD NOT BE CAPABLE OF WITHSTANDING THE INDUCED HEATING AND WOULD QUICKLY MELT. NOTE THAT THE HEAT FLUX DECREASES RAPIDLY AWAY FROM THE STAGNATION REGION AS $\phi_0$ AND $\mu_0$ DECREASE, AND THEREFORE THE HEAT TRANSFER IS MAXIMUM AT THE STAGNATION POINT.

IT IS FOR THESE REASONS THAT HYPERSONIC VEHICLES FUNDAMENTALLY CONSIST OF BLUNT NOSES AND BLUNT LEADING EDGES.

\* AERODYNAMIC HEATING

THE ABOVE CONSIDERATIONS ILLUSTRATE THE IMPORTANCE OF VISCOS EFFECTS IN CAUSING LARGE THERMAL LOADS AT HYPERSONIC SPEEDS. AS MENTIONED IN THE FOOTNOTE BELOW EQ. (239), HOWEVER, NOT ALL THE GENERATION MECHANISMS OF AERODYNAMIC HEATING ARE VISCOSITY RELATED. THE MOST COMMON MECHANISMS OF AERODYNAMIC HEATING ARE ILLUSTRATED IN FIG. 72 BELOW, AND A SURFACE TEMPERATURE DISTRIBUTION IS GIVEN IN FIG. 73 FOR THE X-15 HYPERSONIC CRUISE AIRCRAFT.

\[ M_{\infty} \rightarrow 4 \]
\[ \rightarrow \]
\[ \rightarrow \]

A) STAGNATION POINT HEATING

\[ M_{\infty} > A \]

B) SKIN-FRICTION HEATING

C) BOUNDARY-LAYER TRANSITION HEATING

D) HOT PLUMS HEATING

E) COMPRESSION CORNER HEATING

F) SHOCK-SHOCK INTERFERENCE HEATING

FIG. 72: TYPICAL HOT SPOTS IN HYPERSONIC AIRCRAFTS AND ASSOCIATED AERODYNAMIC HEATING MODES.

A practical example of aerodynamic heating by shock-shock interactions, which have not been studied here, is provided in Fig. 74 for the X-15A2 last flight. Shock-shock interactions of type IV are particularly critical for aerodynamic heating (Figs. 74d and 72f) and occur at intersections between bow shocks from canopies, rudders, wings or engine pylons, with incident shocks from the aircraft nose or inlet spikes. The interaction produces a hot supersonic jet directed to the fuselage.

FIG. 74:
(a) X-15A2 during its last Mach 6.7 record-breaking flight in 1967. (b) Magnified view showing the dummy ramjet pylon and the approximate location of the shock-shock interaction (shock emanating from the inlet spike and bow shock from pylon). (c) During flight, the interaction melted the inconel-X airframe causing perforation and unintentional separation of the dummy ramjet. (Credit: Watts (1968), NASA Tech. Note #1669). (d) Type-IV simulations, NASA CR-138308.)
IV HIGH-SPEED THERMOCHEMICAL EFFECTS

As mentioned in Chapter I, the high temperatures involved in hypersonic phenomena resulting from partial conversion of kinetic into thermal energy lead to a number of thermochemical processes that increase the complexity of the hypersonic flow theory in a multi-fold manner. These processes are the subject of this chapter.

THE ROLE OF THERMOCHEMICAL EFFECTS IN HYPERSONICS

As discussed within the context of Fig. 7.5, a mostly sequential onset of complex processes occurs as the flow velocity is increased that are related to vibrational excitation of the air molecules, air dissociation and ionization. It is important to understand that the mapping of these different regions in Fig. 7.5 is obtained by solving a normal shock problem subject to the velocity \( U_0 = U_{10} \) of the pre-shock gases indicated along the abscissa, along with pre-shock gas conditions \( P_1 \) and \( T_1 \) at the corresponding altitude given by a standard model atmosphere analyses of the aforementioned normal shock problem supplemented with vibrational excitation, air dissociation and ionization, are provided further below and constitute a central part of this chapter in contrast with the perfect gas calculations that have been performed in Chapters II and III.

Before introducing the details of these complex thermochemical effects, it is worth emphasizing that they have a profound quantitative impact on the solution, as shown in Fig. 7.5 by comparing the post-shock temperature at an altitude of \( z = 52 \text{ km} \) obtained using the perfect gas model (i.e., \( C_p = \text{const.} \) with \( \gamma = 1.4 = \text{const.} \) and governed by the ideal gas equation of state) and an equilibrium chemically reacting (non-perfect, dissociating, and ionizing) gas.
The post-shock temperature predicted by the calorically-perfect gas model is systematically much higher than that predicted by including some of the thermochemical effects enabled at high temperatures, as observed in Fig. 75. This disparity can be qualitatively explained by the following considerations:

1) As mentioned in Chapter I and ratified in subsequent chapters, at hypersonic velocities the kinetic energy of the pre-shock gases is mostly transformed into thermal energy in the post-shock gas. In a calorically perfect gas, this excess of thermal energy is dedicated to increasing the energy of the translational and rotational motion of the molecules. In a non-calorically perfect gas, the excess of thermal energy is dedicated to increasing the energy of the translational, rotational, and vibrational motion of the molecules, to exciting the electrons, and, in chemically-reacting gases, to providing chemical energy for the chemical conversion of the air molecules. Since only the translational motion contributes to the temperature in an ideal gas, it is clear that a calorically-perfect gas would always have a much higher temperature behind the shock than a non-calorically perfect one for the same kinetic energy of the pre-shock gases. For instance, at \( \mathbf{U}_0 \approx 11 \text{ km/s} \) (corresponding to \( \mathbf{M}_0 \approx 33 \) at \( z = 52 \text{ km} \)), the shock temperature jumps are

\[
\begin{align*}
T_2/T_1 &\approx 210 \\
T_2/T_1 &\approx 44
\end{align*}
\]

using the shock jump condition (Eq. (18)) for calorically-perfect gases including vibrational excitation, and air dissociation and ionization in chemical equilibrium.

2) Dissociation and ionization reactions are endothermic, i.e., energy is always required to break the bonds of nitrogen and oxygen molecules and to strip electrons from their orbits. This energy cannot be but drained from the gas itself, thereby decreasing its temperature with respect to the case in which these effects are ignored.

3) The post-shock pressure - in contrast to the temperature - is mostly insensitive to the aforementioned thermochemical effects. This is due to conservation of momentum (Eq. (3)) across the shock, which, at hypersonic speeds, simplifies
APPROXIMATELY to \( P_2 \approx S_1 U_1^2 \) since \( P_1 \ll S_1 U_1^2 \) \((6-8, M_{Ad} \gg 1)\) and \( P_2 \gg S_2 U_2^2 \) \((i.e., M_{Ad} \ll 1)\). In this way, the post-shock static pressure is mostly imposed by the dynamic pressure of the free stream, which, at typical free-stream temperatures, \(220-300\) K of flight in the mesosphere and stratosphere, is mostly independent of the above-mentioned thermochemical effects since these do not play any role in the free-stream gases. For instance, at \( U_{\infty} = 14\) km/s in Fig. 7.5 (corresponding to \( M_{Ad} \approx 23\) at \( z = 52\) km), the shock pressure jumps are

\[
\begin{align*}
\frac{P_2}{P_1} & \approx 12.70 & \text{using the shock-jump condition (Eq. (7)) for calorically-perfect gases} \\
\frac{P_2}{P_1} & \approx 13.90 & \text{including vibrational excitation, and air dissociation and ionization in chemical equilibrium.}
\end{align*}
\]

Concurrent with Points 1) and 2) above is a large increase in the post-shock density \( S_2 \) when the thermochemical effects are included, as prescribed by the ideal gas equation of state

\[ P_2 = \frac{S_2 R_0 T_2}{\rho_2} \]

\( \Rightarrow \)

\( \rho_2 \)

Increases a lot when thermo-chemical effects are accounted for.

As a result, the density ratio \( \frac{\rho_2}{\rho_1} \) is much smaller than that predicted by the shock theory for calorically-perfect gases \( \text{(Note that } \frac{\rho_2}{\rho_1} \approx 1/6 \text{ for } M_{Ad} \rightarrow \infty \text{ in Eq. (6)} \text{.)}\). The maximum compression ratio allowed by the theory, for instance, at \( U_{\infty} = 14\) km/s in Fig. 7.5 (corresponding to \( M_{Ad} \approx 23\) at \( z = 52\) km), the shock density jumps are

\[
\begin{align*}
\frac{\rho_2}{\rho_1} & \approx 5.97 & (\varepsilon \approx 0.17) \text{ using the shock-jump condition (Eq. (6)) for calorically-perfect gases.} \\
\frac{\rho_2}{\rho_1} & \approx 15 & (\varepsilon \approx 0.07) \text{ including vibrational excitation, and air dissociation and ionization in chemical equilibrium.}
\end{align*}
\]

It is worth highlighting at this point the resemblance of the above results when thermo-chemical effects are accounted for, with the Newtonian theory in
IN CHAPTER II, WHICH REQUIRES VANISHINGLY SMALL DENSITY RATIOS $\varepsilon \to 0$ (i.e., $R \to 1$ in the realm of the theory for calorically-perfect gases) to hold, to some extent, the incorporation of thermochemical effects in the solution makes it more "Newtonian-like", which remarkably stresses the importance of the Newtonian theory despite its heuristic origins. In what follows, thermodynamic and chemical basic concepts are introduced to assist in quantifying these effects.

BASIC CONCEPTS IN THERMODYNAMICS AND PHYSICAL CHEMISTRY

* IDEAL GAS

Most applications in hypersonics are well described by the ideal-gas equation of state

$$p = \frac{n R^* T}{w} \quad (250)$$

which pertains to gases in which intermolecular forces are negligible (as opposed to real gases in which such interactions are important). The ideal-gas equation (250) can be derived from statistical mechanics under the assumptions that i) the different degrees of freedom of the molecules (i.e., translation, rotation, vibration, and electronic excitation) are energetically decoupled from each other, and ii) each degree of freedom is fully equilibrated so as to yield only one temperature in the system. These concepts will be explained further below. Equation (250) tends to be a good approximation for gases at not too large pressures ($p < 100$ bar) and not too low temperatures ($T > 100$ K), but also for not too low pressures where there are sufficient collisions in the system to render a meaningful temperature, and for not too high temperatures to not cause high levels of ionization and different temperatures for the charged particles (i.e., thermodynamic non-equilibrium).

* CALORICALLY-PERFECT SINGLE-COMPONENT GAS

A calorically-perfect gas is one in which (250) is satisfied and the thermal energy is linearly proportional to the temperature,

$$\varepsilon = C_V T \quad (251)$$

specific internal energy

with $C_V = \frac{\partial \varepsilon}{\partial T}$, = const. A specific heat at constant volume that remains constant, because of the first principle of thermodynamics,
\[ \frac{dq}{dT} = \frac{de}{dT} + P \frac{dl}{dT}, \quad (252) \]

With \( q \) the heat into the system \((\frac{dq}{dT} > 0)\) or into the surroundings \((\frac{dq}{dT} < 0)\), then

\[ \frac{dq}{dT}_p = \frac{dq}{dT}_p - \frac{P}{s^2} \frac{dl}{dT}_p = \frac{C_v}{C_p} - \frac{R_0}{C_p}, \quad \Rightarrow \quad \frac{C_p}{C_v} = \frac{R_0}{C_p} \quad (253) \]

Where \( Q = \frac{dq}{dt}_p \) is the specific heat at constant pressure, which remains constant with temperature. Note that the enthalpy \( h = e + p/s \) can be written as

\[ h = C_v T + R_0 \quad (254) \]

Similarly, the adiabatic coefficient

\[ \gamma = \frac{C_p}{C_v} = \text{constant} \quad (255) \]

Remains constant. It is also not difficult to show that an adiabatic trajectory \((\frac{dq}{dT} = 0)\) in thermodynamic space is represented by

\[ \frac{dq}{dT} = \frac{de}{dT} + P \frac{dl}{dT} = 0 \quad \Rightarrow \quad C_v \frac{dT}{dP} = \frac{P}{s^2} \frac{dl}{dT} \quad \Rightarrow \quad \frac{C_p}{C_v} = \gamma = \frac{P}{s} \frac{dl}{dT} \]

From \( h = e + p/s \)

\[ \text{and} \quad (251) \]

Which can be integrated to give

\[ \int_{s_0}^{s} \frac{dP}{P} = \int_{T_0}^{T} \gamma ds \quad \Rightarrow \quad \frac{P}{s} \gamma = P_0/\gamma \quad (256) \]

The adiabatic trajectory \((256)\) coincides with an isentropic when the process is reversible, namely

\[ dS_{\text{universe}} = dS_{\text{system}} + dS_{\text{surroundings}} \geq 0 \quad (= 0 \text{ for reversible processes}) \]

\[ \Rightarrow \quad ds = \frac{de}{T} + \frac{P}{T} \frac{dl}{T} \quad (257) \]

Integrating:

\[ S - S_0 = C_v \ln \left( \frac{T}{T_0} \right) - R_0 \frac{\ln \left( \frac{P}{P_0} \right)}{\gamma} \]

And eliminating \( \ln \left( \frac{T}{T_0} \right) \) from the two equations, one obtains the definition of the entropy

\[ S - S_0 = C_v \ln \left( \frac{P/s}{P_0/s_0} \right) \quad \text{specific entropy} \quad (258) \]

For a calorically perfect gas. Note that \( S = \text{const.} \) (isentropic) when \((256)\) and \((257)\) are satisfied simultaneously. In summary, a calorically perfect gas is one
IN WHICH \( P, \rho, \) AND \( T \) are related by the ideal equation of state (250) and \( C_v = \text{constant} \), which automatically leads to all other relations (253), (254), (255), (256), and (258). Note that the shock jump relations (7)-(11), as well as the definition of the stagnation temperature (14) and other stagnation variables (17), are all based on a calorically perfect gas. As a matter of fact, almost all mathematical developments in Chapters II and III (with exception of Eqs. (217)-(218) which are generally valid) are based on a calorically perfect gas, since the formulation becomes much more involved otherwise and requires the numerical solution of the original set of conservation equations.

**Thermally Perfect Single-Component Gas**

A thermally perfect gas is one in which (250) is satisfied and the internal energy is a sole — but non-linear — function of temperature, namely

\[
\frac{d\varepsilon}{dT} = C_v(T) \frac{dT}{T} \quad \Rightarrow \quad \varepsilon - \varepsilon_0 = \int_{T_0}^{T} C_v(T) \, dT \quad (259)
\]

with \( C_v = C_v(T) = \frac{\partial \varepsilon}{\partial T} \) varying with temperature. Note that Eq. (253) still holds, and as a consequence,

\[
\frac{dh}{h} = \frac{dv}{v} + \frac{dP}{P} \frac{d}{dP} = C_v dT + \frac{R_q}{R_g} dT = C_p(T) dT \Rightarrow h - h_0 = \int_{T_0}^{T} C_p(T) \, dT \quad (260)
\]

with \( C_p = C_p(T) = \frac{dP}{dT} |_P = \frac{\partial h}{\partial T} |_P \) varying with temperature so as to satisfy \( C_p = C_v = \text{const} \). Correspondingly, the adiabatic coefficient

\[
\gamma = \gamma(T) = \frac{C_p(T)}{C_v(T)} = 1 + \frac{R_q}{R_g} = \frac{1}{1 - \frac{R_q}{C_p(T)}} \quad (261)
\]

is not constant and tends to decrease with \( T \), since in many cases \( C_p \) increases with \( T \) (see Fig. 76). Note also that the integrations leading to the adiabatic trajectory (256) and to the entropy definition (258) are impeded now and can only be addressed numerically. For the same reasons, the jump relations (7)-(11), as well as the definition of the stagnation temperature (14) and other stagnation variables (17), cannot be used. However, the general conservation relations across the shock (2)-(5) can be used even if the gas is thermally perfect. In fact, it can be shown...
Fig. 76: Specific heat at constant pressure for selected gases (Pommeret-Vermaas 2008), which represent the Rankine-Hugoniot relations. In particular, the iterative resolution of (262)-(263) together with (250) and (260), provides the complete solution of the shock-wave problem for a given value of the mass flow rate per unit area $\dot{m}$ and for pre-shock values $h_1, P_1$ and $\rho_1$. A normalized version of the problem can be easily written by defining the dimensionless variables $\Theta = \frac{C_p(T_1)}{\gamma} H = \frac{h_1}{U^2_1}, \quad \Theta = \frac{C_p(T)}{C_v(T)}, \quad \gamma = \frac{C_p(T)}{C_v(T)} \quad \text{which gives} \quad P = \frac{P_2}{\rho_1^2 U^2_1}, \quad U = \frac{S_2}{S_1}, \quad \frac{P}{\rho_1^2 U^2_1} = -1 \quad \text{and} \quad H = \frac{2 h_2}{U^2_1} = (1 + \gamma) \left( P - \frac{1}{\gamma_1 M_1^2} \right)

As dimensionless versions of (262)-(263) and

$$P \gamma = \frac{(\gamma - 1)}{2 \gamma_1} \Theta \quad \text{and} \quad H = \int \Theta \frac{C_p(\Theta)}{\gamma_1 M_1^2} d\Theta$$

As dimensionless versions of (250) and (260), in these equations, $M_1 = U_1 / a_1$ is a Mach number of the pre-shock gases based on the speed of sound for a single-component thermally-perfect gas,

$$\left[ \Phi = \frac{\partial P}{\partial T} \right] \gamma = \frac{\partial P}{\partial T} \left( \frac{\partial T}{\partial \Phi} \right)_s = \left( R_0 T + \frac{R_0 T}{\gamma v} \left( \frac{P}{\gamma_1 C_v} \right) \right) = \left( 1 + \frac{R_0}{C_v} \right) R_0 T = \frac{C_p(T)}{C_v(T)} \frac{R_0 T}{\gamma_v(T)}$$

(250) with $R_0 \approx \text{const.}$

This is an inverse of an Eckert number.

Which resembles the one for calorically-perfect gases but with variable $\gamma$. Note that the parameter $\frac{2 h_1}{U_1^2}$ arises in the Hugoniot equation in addition to $M_1$ but for typical pre-shock temperatures, $T_1 \sim 220-300$K in air, $h_1 \sim C_p(T_1) T_1$, and therefore $\frac{2 h_1}{U_1^2} \sim \frac{2}{(\gamma - 1)} \gamma_1 M_1^2$. At high Mach numbers, the formulation above becomes
\[ P = 1 - \sqrt{1 - (1 + \sqrt{1 - (1 + \sqrt{1 - \ldots})})}, \quad H = (1 + \sqrt{1 - (1 + \sqrt{1 - \ldots})}) \cdot P, \quad PV = \frac{X_i}{2 \gamma_i}, \quad H = \int_0^\infty \phi(\theta) d\theta, \]

which remarkably does not depend on the Mach number as expected from the independence principle. For calorically-perfect gases, the above formulation collapses to

\[ P_i = \frac{2}{3} \left( \frac{\gamma_i}{\gamma_i + 1} \right) \left( \frac{E_i}{U_{i}^2} \right)^{\frac{\gamma_i}{\gamma_i + 1}}, \quad P_i \sim \frac{Y_i}{Y_i + 1}, \quad \text{consistent with the shock-tube relation} \quad (22)-(25), \]

it will be further explained below that the temperature variations of the specific heats in ideal gases is caused by the excitation of vibrational energy in the molecules and by the excitation of electrons in their orbits within the atoms of the molecules.

### CHEMICALLY-REACTING MULTI-COMPONENT IDEAL GAS

A multi-component ideal gas (i.e., N components) that is chemically reacting is characterized by satisfying (250), namely,

\[ P = \rho R_0 T, \quad \text{where} \quad R_0 = \frac{R_0}{\overline{w}}, \quad \overline{w} = \text{mean molecular weight}, \quad (264) \]

and it also satisfies that each of the components in the mixture is thermally perfect, namely

\[ e_i - e_i^0 = \int_0^T C_{v_i}(T) dT \quad \text{(265)} \quad \text{partial specific internal energy}, \quad e_i = e_i^0(T) \]

with \( C_{v_i} = \frac{\partial e_i}{\partial T} \) \( \frac{\gamma}{\gamma - 1} \) \( \gamma_i \), which generally varies with temperature.

For convenience, it is useful to define the molar fraction

\[ \chi_i = \frac{m_i}{\sum m_i}, \]

where \( m_i = \frac{M_i}{\overline{w}} \) is the number of moles of species \( i \), \( M_i \) is the corresponding mass, and consequently

\[ \chi_i = \frac{Y_i}{\sum Y_i / \overline{w}}, \quad \text{where} \quad Y_i = \frac{M_i}{\sum M_i} = \frac{m_i}{\sum m_i} = \frac{1}{\overline{w}} \]

is the mass fraction.

The mean molecular weight of the mixture is

\[ \overline{w} = \frac{\sum m_i \overline{w}}{\sum m_i} = \frac{N}{\sum \chi_i \overline{w}}, \quad \text{where use of the relations} \quad \sum_{i=1}^N \chi_i = \sum_{i=1}^N Y_i = 1 \quad \text{has been made,} \]

in eq. (265), \( e_i^0 \) is a formation energy. Note that, for every species, the ideal gas equation is

\[ P_i = \rho_i R_0 T / \overline{w}, \quad \text{with} \quad P_i = \chi_i \overline{P} \quad \text{the partial pressure} \quad \text{(266)} \]
IN SUCH A WAY THAT (264) IS RECOVERED UPON SUMMING OVER ALL SPECIES,

$$\sum_{i=1}^{N} P_{i} = \sum_{i=1}^{N} X_{i} = P = \sum_{i=1}^{N} \frac{s_{i}}{W_{i}} R_{i}^{0} T = \sum_{i=1}^{N} \frac{Y_{i}}{W_{i}} R_{i}^{0} T = \frac{R_{0} T}{W}.$$ 

SIMILARLY, THE FIRST PRINCIPLE FOR EVERY SPECIES IS

$$T ds_{i} = de_{i} + P_{i} d\left(\frac{V_{i}}{s_{i}}\right) = \frac{\partial e_{i}}{\partial T} \frac{dT}{s_{i}} + P_{i} d\left(\frac{1}{s_{i}}\right)$$

SO THAT

$$T \frac{\partial s_{i}}{\partial T} \left|_{P,Y_{i}} = \frac{\partial e_{i}}{\partial T} \left|_{P,Y_{i}} + P_{i} \frac{dV_{i}}{dT} \left|_{P,Y_{i}} = C_{V_{i}} - \frac{R_{0}}{W_{i}} = \frac{\partial h_{i}}{\partial T} \right|_{P,Y_{i}} = C_{p_{i}} \right.$$  

WHICH INDICATES THAT MEYER'S RELATION

$$C_{p_{i}} - C_{V_{i}} = \frac{R_{0}}{W_{i}} \tag{267}$$

IS SATISFIED PER COMPONENT (BUT NOT GLOBALY). USING (266) AND (267), EQ. (265) CAN BE EASILY REWRITTEN IN TERMS OF THE PARTIAL SPECIFIC ENTHALPY

$$h_{i} = e_{i} + P_{i} h_{i} \tag{268}$$

PARTIAL SPECIFIC ENTHALPY

WHERE $$C_{p_{i}} = \frac{\partial h_{i}}{\partial T} \left|_{P,Y_{i}} \right.$$ AND $$h_{i}^{0}$$ ARE, RESPECTIVELY, THE SPECIFIC HEAT AND FORMATION ENTHALPY OF SPECIES $$i$$.

IN TERMS OF MIXTURE QUANTITIES, THE FORMULATION ABOVE LEADS TO

$$h = \sum_{i=1}^{N} Y_{i} h_{i}^{0} + \sum_{i=1}^{N} Y_{i} \int_{T_{0}}^{T} C_{p_{i}} \, dT \tag{269} \Rightarrow h = h(Y_{i}, T)$$

AND

$$e = \sum_{i=1}^{N} Y_{i} h_{i}^{0} + \sum_{i=1}^{N} Y_{i} \int_{T_{0}}^{T} C_{V_{i}} \, dT \tag{270} \Rightarrow e = e(Y_{i}, T)$$

WHERE $$T_{0} = 298 \text{ K}$$ IS AN ARBITRARY REFERENCE TEMPERATURE AT WHICH THE FORMATION VALUES ARE TYPICALLY TABULATED. NOTE THAT THE FORMATION COMPONENT REPRESENTS ENTHALPY OR ENERGY VARIATIONS DUE TO CHEMICAL CONVERSION. SIMILARLY, THE FIRST PRINCIPLE FOR EACH SPECIES

$$T ds_{i} = dh_{i} - \frac{dP_{i}}{s_{i}} \tag{271}$$

CAN BE TRANSFORMED INTO THE FIRST PRINCIPLE FOR THE CHEMICALLY-REACTING MIXTURE
BY MULTIPLYING BY \( Y_0 \) AND SUMMING OVER ALL COMPONENTS, NAMELY
\[
T \sum_{i=1}^{N} Y_0 dS_i = \sum_{i=1}^{N} Y_0 dh_i - \sum_{i=1}^{N} Y_0 dP_i,
\]
and since \( dh = \sum_{i=1}^{N} Y_i dh_i + \sum_{i=1}^{N} h_i dY_i \) and \( ds = \sum_{i=1}^{N} Y_0 dS_i + \sum_{i=1}^{N} S_i dY_i \), then the above expression becomes
\[
TdS = dh - dP + \sum_{i=1}^{N} \left( \frac{1}{S_i} \right) dq_i dY_i \quad (272)
\]
Where \( q_i = h_i - Ts_i \) is the partial specific Gibbs free energy (also called the specific chemical potential). Equivalently, in terms of \( \Theta \), (271) can be written as
\[
Tds = d\Theta + P d\left( \frac{1}{S} \right) + \sum_{i=1}^{N} q_i dY_i \quad (273)
\]
In writing down the conservation equations for chemically-reacting flows, additional terms are sometimes required to account for molecular diffusion and chemical reactions.

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{\partial \Theta}{\partial t} + \nabla \cdot \mathbf{\Phi}_\Theta = -\nabla \cdot \left( \rho \phi \mathbf{U} \right) + \nabla \cdot \left( \frac{\Theta}{\rho} \mathbf{U} \right) - \nabla \cdot \mathbf{q}^\Theta \\
\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} = -\nabla p + \nabla \cdot \left( \frac{\rho \mathbf{U}}{S} \right) - \nabla \cdot \mathbf{q}^\mathbf{U}
\end{array} \right. \quad (274)
\end{align*}
\]
Steady-state (total)

\[
\begin{align*}
\Theta &= \Theta + \frac{1}{2} \sum_{i=1}^{N} Y_i (h_i + \int_{0}^{T} \sum_{j=1}^{N} S_j dY_j) + \frac{1}{2}
\end{align*}
\]
Enthalpy (total)

\[
\begin{align*}
h &= h + \frac{1}{2} \sum_{i=1}^{N} Y_i \left( h_i + \int_{0}^{T} \sum_{j=1}^{N} S_j dY_j \right) + \frac{1}{2}
\end{align*}
\]
Viscous stress tensor

\[
\mathbf{q}^\mathbf{U} = -k \nabla T + \sum_{i=1}^{N} \Phi_i Y_i \mathbf{V}_i \quad (275)
\]
Heat flux

\[
\mathbf{\Phi}_\Theta = \mu \left( \nabla \mathbf{U} + \nabla \mathbf{U}^T \right) + \left( \frac{\kappa - \frac{p}{3} }{\rho} \right) \nabla \cdot \mathbf{U} \mathbf{I}, \quad (276)
\]
Species conservation

\[
\begin{align*}
\frac{\partial Y_i}{\partial t} + \nabla \cdot \left( \rho Y_i \mathbf{V}_i \right) &= -\nabla \cdot \left( \frac{\rho \mathbf{V}_i Y_i}{S} \right) + \dot{w}_i \\
\dot{w}_i &= \dot{W}_i \sum_{j=1}^{M} \left( \frac{N_i - N_j}{W_j} \right) \dot{w}_j \\
(277)
\end{align*}
\]
Diffusion velocity

where \( \dot{w}_i = \dot{W}_i \sum_{j=1}^{M} \left( \frac{N_i - N_j}{W_j} \right) \dot{w}_j \)
(278) is the chemical mass production rate

\[
\begin{align*}
\text{in } \left[ \frac{\text{kmol} / \text{s m}^3}{\text{kmol} / \text{s m}^3} \right] \text{ for a mechanism of } M \text{ elementary reactions}
\end{align*}
\]

\[
\sum_{i=1}^{N} \mathbf{R}_i = \sum_{i=1}^{N} \mathbf{Q}_i \quad \text{CHEMICAL SPECIES}
\]

where \( Y_i \)’s are stoichiometric coefficients, the chemical rate of reaction \( \dot{w}_i \) has units \( [\text{kmol} / \text{s m}^3] \).

\[
\dot{w}_i = \frac{1}{(\dot{r}_i - \dot{r}_j)} \frac{d [\mathbf{R}_i]}{dt} = K_{bi} \left[ \sum_{i=1}^{N} \mathbf{Q}_i \right] \dot{w}_i - K_{bi} \left[ \sum_{i=1}^{N} \mathbf{Q}_i \right] \dot{w}_i \\
(280)
\]

where \([\cdot]\) = \( \sum Y_i / W_j \) indicates molar concentration, and \( K_{bi} \) and \( K_{bi} \) are forward and backward reaction constants of reaction step \( \dot{w}_i \) to the reaction constants.
$K_i$ or $K_i^*$ are usually written in the Arrhenius form

$$K_i = A_i T^{-m_i} e^{-E_i/R_o T} \quad (281)$$

where $A_i$ = pre-exponential factor, $m_i$ = temperature exponent, $E_i$ = activation energy.

Note that the chemical heat release rate is absent from (274) and (275) because $h$ and $e$ already incorporate the chemical energy through the formation terms. As a result, Eqs. (2) - (5) are for instance also valid for chemically-reacting shock waves (although an additional equation for conservation of species may be necessary in situations of chemical non-equilibrium as explained further below). Additional closures are required for the diffusion velocity $\overrightarrow{V}_D$ in (276), but these are beyond the scope of this course. The simplest form of $\overrightarrow{V}_D$, which neglects thermal diffusion and barodiffusion is

$$\overrightarrow{V}_{i, D} = -\frac{\nabla P_i}{\rho_i} \cdot (D_i W_i \nabla X_i - \sum_{k=1}^N D_{ik} W_k Y_k \nabla X_k) \quad (282)$$

### Molar Entropy and Gibbs Free Energy

Reformulating the 1st Principle (271) applied to species $i$ in terms of molar entropy $S_i = S_i^{\circ} W_i$, molar enthalpy $h_i = \tilde{h}_i W_i$, and specific volume $V_i = W_i / \rho_i$ yields $S_i - S_i^{\circ} = \int_{T_0}^T C_{pi} \frac{dT}{T} - R \ln \frac{P_i}{P_0}$ (283)

where $C_{pi} = C_p W_i$ is the molar specific heat at constant pressure. The molar entropy of the mixture is then given by

$$\tilde{S} = \sum_{i=1}^N X_i \tilde{S}_i^{\circ} \quad (284)$$

Note that (283) - (284) become (258) when the gas is calorically perfect.

The partial molar Gibbs free energy, $\tilde{\gamma}_i = \tilde{h}_i - T \tilde{S}_i$, can be expressed as

$$\tilde{\gamma}_i^{\circ} = \tilde{S}_i^{\circ} = \int_{T_0}^T C_{p_i} \frac{dT}{T} - \int_{T_0}^T C_{p_i} \frac{dT}{T} + R_0 T \ln \left( \frac{P_i}{P_0} \right) \quad (285)$$

More compactly, letting $T_0 = T$, then

$$\tilde{\gamma}_i^{\circ}(P, T) = \tilde{S}_i^{\circ}(T) + R_0 T \ln \left( \frac{P_i}{P_0} \right) \quad (286)$$

which provides an explicit dependence of $\tilde{\gamma}_i$ on the partial pressure of component $i$.

Similarly to (284), the molar Gibbs free energy of the mixture is

$$\tilde{\gamma}_i = \sum_{i=1}^N X_i \tilde{\gamma}_i^{\circ} = \tilde{\gamma}_i^{\circ}(P, T, X_i) \quad (287)$$
CHEMICAL EQUILIBRIUM

The exact differential of the molar Gibbs free energy \( \delta \bar{G} \) of the mixture can be expressed as

\[
d\bar{G} = \frac{\partial \bar{G}}{\partial T} \, dT + \frac{\partial \bar{G}}{\partial P} \, dP + \sum_{i=1}^{N} \frac{\partial \bar{G}}{\partial X_i} \, dX_i,
\]

and since \( \frac{\partial \bar{G}}{\partial T} \) (i.e., \( \frac{\partial \bar{G}}{\partial T} \)) \( p_i, x_i, i = 1, \ldots, N \)

\[= \bar{S}, \quad \frac{\partial \bar{G}}{\partial P} \]

\[= \bar{U}, \quad \text{and}
\]

\[\frac{\partial \bar{G}}{\partial X_i} \]

\[= \bar{a}_i, \quad \text{then the above expression becomes}
\]

\[
d\bar{G} = -\bar{S} \, dT + \bar{U} \, dP + \sum_{i=1}^{N} \bar{a}_i \, dX_i. \tag{287}
\]

AT CONSTANT PRESSURE AND TEMPERATURE, EQ. (287) REDUCES TO

\[
d\bar{G} = \sum_{i=1}^{N} \bar{a}_i \, dX_i. \tag{288}
\]

BUT SINCE\(\sum_{i=1}^{N} X_i \, d\bar{a}_i + \sum_{i=1}^{N} \bar{a}_i \, dX_i\), THEN\(\sum_{i=1}^{N} X_i \, d\bar{a}_i = 0\) IN THESE CONDITIONS. IN FACT, IT IS EASY TO SEE FROM (Z 287) THAT THE RELATION

\[
\frac{\sum_{i=1}^{N} X_i \, d\bar{a}_i}{\sum_{i=1}^{N} \bar{a}_i} = -\frac{\bar{S}}{\bar{U}} \, dT + \bar{U} \, dP \tag{289} \quad \text{(GIBBS-DUHEM RELATION)}
\]

IS SATISFIED. EXPRESSION (288) IS PARTICULARLY USEFUL IN SYSTEMS AT CONSTANT PRESSURE AND TEMPERATURE. IN PARTICULAR, EQUATION (272) WRITTEN ON A MOLAR BASIS YIELDS

\[
Td\bar{S} + \sum_{i=1}^{N} \bar{a}_i \, dX_i = d\bar{H} - \bar{U} \, dP
\]

\[
\Rightarrow \frac{d\bar{G}}{T} = \bar{S}
\]

AND SINCE\(d\bar{G} = d\bar{S}_{\text{SURROUNDINGS}} + d\bar{S}_{\text{UNIVERSE}}\), WITH\(d\bar{S}_{\text{SURROUNDINGS}} = \frac{d\bar{G}}{T}\), THEN

\[
Td\bar{S}_{\text{UNIVERSE}} = \sum_{i=1}^{N} \bar{a}_i \, dX_i = d\bar{\bar{a}} \tag{290}
\]

AT CONSTANT PRESSURE AND TEMPERATURE. EQUIVALENTLY, (290) STATES THAT IN REVERSIBLE EQUILIBRIUM, \(Td\bar{S}_{\text{UNIVERSE}} = 0\), THE GIBBS FREE ENERGY SATISFIES

\[
d\bar{G} = \sum_{i=1}^{N} \bar{a}_i \, dX_i = 0 \quad \text{AT CONSTANT P AND T.} \tag{291}
\]

NOTE THAT, FOR A REVERSIBLE REACTION \(\sum_{i=1}^{N} \bar{a}_i \, \Delta_i \equiv \sum_{i=1}^{N} \bar{a}_i \, \Delta_i\), THE PROGRESS RATE \(x_0\)

INDICATES \(d[\Delta_i]/(\Delta_i - \Delta_i')\) IS THE SAME FOR ALL REACTANTS, AND SINCE \(d[\Delta_i] = \frac{B}{R_pT} \, dX_0\)

AT CONSTANT P AND T, THEN (291) REQUIRES THAT \(\sum_{i=1}^{N} (\Delta_i - \Delta_i') \, \bar{a}_i = 0\). USING (287) IN THIS EXPRESSION LEADS TO

\[
\frac{1}{T} \, \left( \frac{P_b}{P_0} \right) \sum_{i=1}^{N} (\Delta_i - \Delta_i') \, g_i(T) = R_0 \, T \, \ln \left\{ \prod_{i=1}^{N} \left( \frac{P_b}{P_0} \right) \right\} (\Delta_i - \Delta_i')
\]

\[= \sum_{i=1}^{N} (\bar{a}_i - \bar{a}_i') \, g_i(T)\]
Equation (292) establishes a relation \( F (p, T, x_i, i = 1 \ldots N) = 0 \) that the mixture composition must satisfy in equilibrium. In addition to this constraint and to \( \sum_{i=1}^{N} x_i = 1 \), \( N-2 \) other equations pertaining to the conservation of atoms must be specified in order to determine the composition in chemical equilibrium (see Workup Problem Set for an example).

Two aspects are worth emphasizing in the above formulation:

1) Air dissociation and ionization reactions are typically molecule-producing reactions \( (N_2 \rightarrow N + N, \quad O_2 \rightarrow O + O, \quad N \rightarrow N^+ + e^-) \), in that \( \sum_{i=1}^{N} (\rho_i x_i \rightarrow \rho_{i+1} x_{i+1}) > 0 \), so that the exponent of the pressure ratio in (292) is positive. As a result, a decrease in pressure (for instance by increasing altitude) is accompanied by a displacement of equilibrium concentrations towards the products, which partly explains the negative slopes of the boundaries for dissociation and ionization in the velocity-altitude map in Fig. 5 (i.e., these thermochemical effects become more prominent at increasing altitudes).

2) Chemical equilibrium entails zero net rate of reaction, since the forward and backward rates become equal. This is a good approximation when there exist sufficient collisions to warrant equilibrium. At high altitudes, however, the collisions are not so numerous and it becomes necessary to retain finite-rate chemistry effects, which also enter in competition with fluid mechanical transport. The latter phenomenon requires the study of chemical non-equilibrium. The opposite to chemical equilibrium is frozen flow, in which the number of collisions is not sufficient to trigger any chemical processes and the mixture flows without undergoing any chemical conversion; such is the case at low temperatures (\( T < 1500-2000 \text{ K} \)) where air can be considered as non-reacting.

High-temperature air in chemical equilibrium

An important application of chemical equilibrium in hypersonics is the analysis of dissociation and ionization that occurs in air at high temperatures. However, as mentioned above, equilibrium conditions are only reached in situations when forward and backward reaction rates are very fast compared to any transport phenomenon. In these
Conditions, a typical chemical equilibrium configuration is described by the reaction set in Table II. At room temperature, the molar composition of air is approximately 78\% N\(_2\), 21\% O\(_2\), 1\% Ar, along with trace amounts of CO\(_2\). At higher temperatures, however, other chemical species emerge as a result of dissociation and ionization. In particular, the description in Table II exploits 30 species:

\[
\text{O}_2, \text{N}_2, \text{O}, \text{N}, \text{O}^-, \text{O}_2^-, \text{O}_2^+, \text{O}^{++}, \text{N}^+, \text{N}^{++}, \text{N}_2^+, \text{CO}_2, \text{CO}, \text{CO}^+, \text{C}, \text{C}^+, \text{C}^{++}, \text{NO}_2, \text{NO}, \text{N}_2\text{O}, \text{NO}^+, \text{Ar}, \text{Ar}^+, \text{Ar}^{++}, \text{Ne}, \text{Ne}^+, \text{and e}^-.
\]

The solution to the chemical equilibrium problem supplied with the many species listed above is cumbersome and needs to be obtained numerically. One such solution is provided in Fig. 76 for pressure conditions resembling 30 km altitude (stratosphere). There, it is worth noting that

1) For \(T < 2500\text{K}\), the air composition is mostly the same as at ambient temperature, with only minute amounts of \(\text{O}^- (\sim 1\%)\) and \(\text{NO} (\sim 1\%)\).

2) For \(2500\text{K} < T < 4000\text{K}\), oxygen dissociates significantly \((X_0 > 0.3)\) and NO reaches a peak concentration \((\sim 4\%, 3000\text{K})\). At 10000 K, the \(\text{O}_2\) has dissociated almost completely.

3) For \(4000\text{K} < T < 8000\text{K}\), \(\text{N}_2\) dissociates significantly \((X_n > 0.7)\) and \(\text{NO}^+\) ions and electrons are present in trace amounts \((\sim 0.1\%)\) at \(T = 8000\text{K}\). The gas is mainly composed of \(\text{O}^+\) and \(\text{N}_2\).

4) For \(T > 8000\text{K}\), ionization of atomic species is important. At \(T = 10000\text{K}\), the gas is mainly composed of \(\text{N}_0, \text{N}^+, \text{O}^+, \text{N}_2, \text{Ar}^+, \text{Ne}^+, \text{and e}^-\).
In order to analyze basic effects of the high temperatures encountered in hypersonic flows on the behavior of the air environment, it is necessary to introduce some fundamental concepts of statistical mechanics. These, for instance, describe departures of the specific heat $C_p$ from the calorically perfect value as the temperature increases, and are useful in discussing the corresponding decrease in postshock temperatures as compared to their counterparts in calorically perfect gases. The explanations here, however, cannot be but superficial and limited to summarizing the main results. Further discussions and more elaborated expositions can be found, for instance, in the book “Physical Gas Dynamics” by Vincenti and Kruger (2002). The discussion here will be focused on vibrational-excitation effects that arise at low to intermediate temperatures ($T \gtrsim 2000 \text{ K}$) and are responsible for thermochemical effects acting at $M_a \lesssim 4$ that tend to lower the post-shock temperature, yet without involving chemical reactions.

The characterization of the thermodynamic state of a gas requires the specification of the state of each one of the molecular constituents. This, in practice, is a futile endeavor given the large number of molecules (i.e., $10^{25}$ molecules in $1 \text{ m}^3$ at normal conditions) typically present. Only an “average” thermodynamic state can be often calculated using concepts from statistical mechanics, understanding the word average here as an state that exists and manifests itself with most probability, the calculation (and existence) of this average state is the realm of equilibrium thermodynamics and requires consideration of the statistics of the energetics of the constituent molecules (see schematics in Fig. 77).
THE MOLECULAR STATES CAN BE CALCULATED BY SOLVING THE SCHRÖDINGER EQUATION, WHICH PROVIDES THE ASSOCIATED WAVE FUNCTION. HOWEVER, THE CHARACTER OF THE SCHRÖDINGER EQUATION IS SUCH THAT SOLUTIONS (i.e., STATES) ONLY EXIST FOR DISCRETE VALUES (i.e., EIGENVALUES) OF THE TOTAL ENERGY OF THE MOLECULE \( E_i \), THE LATTER BEING COMPOSED OF TRANSLATIONAL, ROTATIONAL, VIBRATIONAL, AND ELECTRONIC COMPONENTS, Namely,

\[
E_i = E_m + E_{\text{trans}} + E_{\text{rot}} + E_{\text{vib}} + E_{\text{elec}}
\]

\[ (293) \]

IN EQ. (292), THE ENERGY COMPONENTS ALSO ONLY EXIST AT DISCRETE VALUES, WHICH ARE THE LEAST SPACED IN THE TRANSLATIONAL MOTION, AND THE MOST SPACED IN THE ELECTRONIC MOTION. IN THIS WAY, THE ENERGY LEVELS ARE QUANTIZED, THE QUANTUM NUMBERS \( (n, l, k, m) \) ARE EIGENVALUES OF THE SCHRÖDINGER EQUATION, AND THE TOTAL ENERGIES \( E_i \) CAN ONLY EXIST AT DISCRETE LEVELS \( E_i \). IT IS NONTHELESS FOUND THAT A NUMBER OF DIFFERENT STATES, OR EQUIVALENTLY, A NUMBER OF DIFFERENT COMBINATIONS OF MOLECULAR MOTION, CAN RENDER THE SAME ENERGY EIGENVALUE \( E_i \). THIS MULTIPOLICY IS QUANTIFIED BY THE DEGENERACY \( 0_N \) OF EACH ENERGY LEVEL \( E_i \).

IN A GAS OF \( N \) MOLECULES, IN WHICH \( N_i \) MOLECULES HAVE AN ENERGY \( E_i \), THE TOTAL ENERGY OF THE SYSTEM IS \( E = \sum_{i=1}^{N} N_i E_i \), WITH \( \sum_{i=1}^{N} N_i = N \). THE DISTRIBUTION \( N_i \) ACROSS THE ENERGY LEVELS IS CALLED THE MACROSTATE OF THE SYSTEM. THE MACROSTATE THAT OCCURS WITH MOST PROBABILITY, OR EQUIVALENTLY, THE MACROSTATE THAT OCCURS IN THERMODYNAMIC EQUILIBRIUM IS THE ONE THAT CONSISTS OF THE MAXIMUM NUMBER OF MICROSTATES \( W_{\text{max}} \). HERE, A MICROSTATE REFERS TO ONE OF THE MULTIPLE COMBINATIONS ENABLED BY THE DEGENERACIES \( 0_N \) THAT THE POPULATIONS OF MOLECULES \( N_i \) AT EACH ENERGY LEVEL HAVE FOR POPULATING THE DIFFERENT ENERGY MODES IN

\[ (293) \] WHILE CONSERVING THE SAME MACROSTATE CONFIGURATION, \( N_i (E_i) \).
It can be shown from statistical mechanics that in thermodynamic equilibrium the most probable macrostate satisfies the Boltzmann distribution:

\[ N_i = N \sum_{j=1}^{Q_i} \frac{g_j e^{-\varepsilon_j / kT}}{Q} \]  

(294)

with \( Q = \sum_{j=1}^{Q_i} g_j e^{-\varepsilon_j / kT} \) \( (295) \) the partition function. Note that (294) establishes that \( N_i \) decreases with \( \varepsilon_i \), in such a way that the population of molecules decreases as the energy level increases, and does so in a monotonic manner (Fig. 78).

Similarly, at a given temperature \( T \), most of the contribution to the sum in the partition function (295) is provided by energy levels such that \( \varepsilon_i < kT \), the population of molecules has decreased multi-fold at energy levels \( \varepsilon_i > kT \). In this way, increasing the temperature leads to a wider range of energy levels that can be activated (Fig. 78). Since the vibrational and electronic levels are much more spaced than the translational and rotational levels, such an increase in temperature typically enables the population of vibrational (\( T \approx 800K \)) and electronic (\( T \approx 9,000K \)) energy levels. It is important to note that the population distribution is monotonic in thermodynamic equilibrium, but that non-monotonicity may arise in situations concerning thermodynamic non-equilibrium, when, for instance, a vibrational energy level is suddenly populated by the passage of a gas through a shock. Provided that a sufficient number of collisions occurs, however, thermodynamic equilibrium is eventually achieved after a transient in which the population distribution is not given by (284) (note that this also means that during this transient, the internal energy and pressure of the gas are not given by the equilibrium expressions that are introduced right below).
THERMODYNAMIC NON-EQUILIBRIUM IS DEFERRED TO FURTHER BELOW IN THESE NOTES.

4. THE PARTITION FUNCTION


$$
e = \frac{T^2}{V} \frac{\partial \ln Q}{\partial T}, \quad P = T \frac{\partial \ln Q}{\partial V}, \quad (295)$$

WHERE $V$ IS THE SPECIFIC VOLUME. ONE IMPORTANT PROPERTY OF THE PARTITION FUNCTION (295) IS THAT IT IS EQUALS THE PRODUCT OF THE PARTITION FUNCTIONS OF EACH ENERGY MODE IN AS MUCH AS EACH ENERGY MODE IS SEPARABLE AS IN (293), NAMELY

$$
Q = \sum_i q_i e^{-\frac{E_i}{kT}} = \sum_i \sum_m q_m e^{\frac{E_m}{kT}} = \sum_i q_i e^{-\frac{E_i}{kT}} \sum_m q_m e^{\frac{E_m}{kT}} = Q_{\text{trans}} Q_{\text{rot}} Q_{\text{vib}} Q_{\text{elec}} \quad (297)
$$

THE SEPARATION ESTABLISHED BY (297) IS WELL JUSTIFIED FOR THE TRANSLATIONAL DEGREE OF FREEDOM IN IDEAL GASES. ELECTRONIC EXCITATIONS CAN ALSO BE SEPARATED FROM ROTATION AND VIBRATION AT CONVENTIONAL TEMPERATURES. HOWEVER, THE LATTER TWO CAN BE SEPARATED IN A LESS ACCURATE APPROXIMATION (ROTATION + CENTRIFUGAL, Etc., (297) IS A FUNDAMENTAL ASSUMPTION OF THE KINETIC THEORY OF IDEAL GASES THAT ENABLES THE CALCULATION OF MACROSCOPIC QUANTITIES AS IN (296). UNDER THE SEPARATION ASSUMED IN (297), THE INDIVIDUAL PARTITION FUNCTIONS CAN BE DETERMINED BY SOLVING THE ASSOCIATED SCHRÖDINGER EQUATION PER DEGREE OF FREEDOM, WHICH, FOR A DIATOMIC MOLECULE GIVES

$$
Q_{\text{trans}} = \left( \frac{2\pi m kT}{\hbar^2} \right)^{3/2} \quad (298) \quad Q_{\text{vib}} = \frac{1}{1 - e^{-\frac{\hbar\nu}{kT}}} \quad (300)
$$

$$
Q_{\text{rot}} = \frac{8\pi^2 m kT}{\hbar^2} \quad (299) \quad Q_{\text{elec}} = q_0 + q_1 e^{-\frac{E_1}{kT}} + q_2 e^{-\frac{E_2}{kT}} \quad (301)
$$

WHERE $\hbar$ IS THE PLANCK CONSTANT, $I$ IS THE MOMENT OF INERTIA OF THE MOLECULE, $\nu$ IS A VIBRATION FREQUENCY, AND $E_1$ AND $E_2$ ARE THE FIRST TWO ENERGY LEVELS OF THE ELECTRONIC MODE, WHICH ARE THE ONLY IMPORTANT ONES FOR $T < 15000 \, K$.

IT IS IMPORTANT TO NOTE THAT ONLY $Q_{\text{trans}}$ DEPENDS ON THE VOLUME, AND AS A CONSEQUENCE ONLY THE TRANSLATIONAL MODE CONTRIBUTES TO THE PRESSURE IN (296), NAMELY
\[ P = \frac{R_0 T}{\Theta_{\text{rot}}} \] (294)

Note that in real gases, the molecules interact through potentials and the partition functions need to be recalculated, which corresponds to the ideal gas equation of state. On the other hand, all modes contribute to the internal energy of the gas in (294). This can be seen by substituting (298) - (304) into (294), which gives

\[ e = \frac{3}{2} R_0 T + R_0 T + \frac{h v/k}{\Theta_{\text{vib}}} \frac{R_0 T}{\Theta_{\text{elec}}} + \Theta_{\text{elec}}(T) \] (302)

\[ \text{Internal Energy of a THERMALY-PERFECT NON-REACTING GAS} \]

\[ \Theta_v = \frac{h v}{k} \text{ TRANSLATIONAL TEMPERATURE} \]

\[ \Theta_{\text{rot}} = \frac{h^2}{8 \pi^2 k} \text{ ROTATIONAL TEMPERATURE} \]

\[ \Theta_{\text{elec}} = \frac{h^2}{8 \pi^2 k} \text{ ELECTRONIC TEMPERATURE} \]

\[ \Theta_{\text{elec}} = \frac{h^2}{8 \pi^2 k} \]

AND \( C_p = C_v + R_0 \). NOTE THAT, FOR ATOMS, THERE ARE NOT VIBRATIONAL OR ROTATIONAL ENERGY MODES, AND (302) - (303) SIMPLIFY TO

\[ e = \frac{3}{2} R_0 T + \Theta_{\text{elec}}(T), \quad C_v = \frac{3}{2} R_0 + \Theta_{\text{elec}}(T) \] \[ \text{ (304) } \]

IN EQ. (297) THE FACTOR \( h v/k \) CAN BE ASCRIBED TO A CHARACTERISTIC VIBRATIONAL TEMPERATURE

\[ \Theta_{\text{vb}} = \frac{h v}{k}, \] (305)

WHICH DELINESATES THE LIMIT WHERE VIBRATIONAL MOTION IS FULLY EXCITED. A SIMILAR ROTATIONAL TEMPERATURE \( \Theta_{\text{rot}} = \frac{h^2}{8 \pi^2 k} \) CAN BE DEFINED, AS WELL AS ELECTRONIC TEMPERATURES \( \Theta_{\text{elec}} = \Theta_{\text{elec}}' \) AND \( \Theta_{\text{elec}}'' = \Theta_{\text{elec}}''' \). TABLE IV PROVIDES SOME VALUES OF RELEVANCE FOR HIGH-TEMPERATURE AIR.

IT IS IMPORTANT TO NOTE THAT

3) AT CONVENTIONAL TEMPERATURES FOR HYPERSONIC FLIGHT, THE ROTATIONAL MODE IS ALWAYS FULLY EXCITED.

4) UP TO TEMPERATURES OF ORDER \( T \approx 600-800 \text{K} \), THE GAS BEHAVES CALORICALLY PERFECT WITH \( y = 1.4 \) AND \( C_p \approx 5 R_0 / 2 \).

5) FOR \( T > 600-800 \text{K} \), THE VIBRATIONAL MODE BECOMES PROGRESSIVELY MORE EXCITED, WITH \( C_v \) INCREASING WITH TEMPERATURE AND \( \Theta_v \) CORRESPONDINGLY DECREASING, IN SUCH A WAY.
That the gas ceases to be a calorically perfect one. The decreasing trend of γ with temperature makes the gas more amenable for Newtonian-type of analysis, as anticipated on Page 28. Note however that the dependence of Cu and γ on T is less trivial in monoatomic species or at higher temperatures T ≈ ν0 in diatomic species because of the subsequent excitation of electronic modes (Note that e → P0 and Cu → R0 for T ≈ ν0, indicating a clear separation of the vibrational degree of freedom, but by then the electronic excitations may have taken over).

High-Temperature Effects on Equilibrium Inviscid Flows

The general form of the conservation equations for high-temperature inviscid flows in chemical and thermodynamic equilibrium is given by:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad \text{(Continuity, 306)} \]
\[ \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P \quad \text{(Momentum, 307)} \]
\[ \frac{\partial h}{\partial t} + \mathbf{u} \cdot \nabla h = \frac{\partial P}{\partial t} \quad \text{(Total Enthalpy, 308)} \]

The latter being obtained by simplifying Eq. (275) neglecting molecular-diffusion terms. Note that h in (308) generally includes both thermal and chemical enthalpies as in (298),

\[ h = h(T, P, \rho) = \sum_{i=1}^{N} Y_i h_i^0 + \sum_{i=1}^{N} \int_{0}^{T} C_P(T) dT. \quad (309) \]

Additionally, the pressure, temperature and density are related by the equation of state

\[ P = \rho R_0 T, \quad \text{with} \quad R_0 = R_0^0/W = R_0^0 \left( \sum_{i=1}^{N} \frac{Y_i}{W_i} \right) \quad (310). \]

The composition in chemical equilibrium is obtained from the equilibrium constant (292)

\[ C = \prod_{i=1}^{N} \left( \frac{\rho_i}{\rho_0^0} \right)^{Y_i^0} = \left( \frac{P}{P_0^0} \right)^{\sum_{i=1}^{N} \left( \frac{\rho_i}{\rho_0^0} \right)^{Y_i^0}} \sum_{i=1}^{N} X_i^{\left( Y_i^0 - \rho_i^0 \right)} \quad (311) \]

which can be applied to the M chemical steps in equilibrium. The remaining N-M-1 equation needed to obtain the molar fractions \( X_i = Y_i W/W_i \) correspond to atom conservations. The N-th molar fraction is obtained from the constraint \( \sum_{i=1}^{N} X_i = 1 \). These relations, written symbolically, along with (311), provide the mixture composition in chemical equilibrium as a function of the local pressure and temperature,

\[ Y_i = Y_i(P, T) \quad (312) \]

The simultaneous resolution of (306)-(310) and (312), subject to appropriate
Boundary conditions and supplemented with relations for \( C_p^0(T) \) and \( \varphi^0(T) \) (obtained for instance from JANNAF tables or similar databases), provides the solution to the problem.

In the case of planar shocks, the above formulation simplifies to the integral conservation laws (2), (3) and (5), which in turn become (262) and (263), where \( n = \sqrt{\frac{1}{2} U_i \sin \beta} \). The solution is then direct, but in most cases requires numerical integration. Before describing qualitative aspects of the ensuing solutions, it is worth emphasizing the following:

\( \text{(i)} \) Equations (306)-(310) and (312) represent an appropriate formulation for systems in thermodynamic equilibrium, meaning that the distribution of molecules in the system across energy levels is approximately given by the Boltzmann distribution (294). This is achieved when a sufficiently large number of collisions occur to statistically relax the distribution of molecules to (294). In this limit, the different energy modes are equilibrated and their corresponding energies are given by the different components appearing in (302). Non-equilibrium phenomena, resulting from a deficitary number of collisions, are addressed further below.

\( \text{(ii)} \) Similarly, equations (306)-(310) and (312) represent an appropriate formulation for systems in chemical equilibrium, meaning that the chemical conversion timescales are infinitesimally small compared to the fluid-mechanical time scale. As a consequence, the distribution of chemical species has reached equilibrium values that only depend on the local pressure and temperature, in a way that the associated distribution of molecules of each species also follows a Boltzmann distribution. Note that chemical equilibrium also requires a sufficiently high number of collisions (in turn, it requires a much higher number of collisions than the one required for thermodynamic equilibrium).

The above considerations suggest that thermo-chemical equilibrium is an appropriate assumption when the fluid-mechanical time scales are much larger than the characteristic time scales of thermodynamic and chemical relaxation.
The aforementioned approximations break down as the flight speed increases (i.e., as the residence time of the air around the spacecraft decreases), as the pressure decreases (i.e., as the vibrational relaxation time increases and as the rate of recombination decrease), and as the temperature decreases (i.e., as the vibrational relaxation time increases). It is therefore clear that all processes that lead to a deficit of collisions will ultimately lead to non-equilibrium phenomena. As a result, increasing altitude and flight speeds degrade the thermo-chemical equilibrium built into Eqs. (306)-(310) and (312). (See Fig. 81).

*Thermo-Chemical Equilibrium Effects on Shock Waves*

As mentioned on pages 87-89, thermo-chemical effects arising at high temperatures in hypersonic regimes have profound consequences in the post-shock gases. Firstly, the temperature $T_2$ is much shallower than that predicted by the theory of calorically-perfect gases (Eq. (8)). This can be understood by noticing that the increment of static enthalpy across the shock, described by the Hugoniot equation (263), has to be dedicated to dissociate the air molecules and to excite molecular degrees of freedom of vibration and electronic motion (Fig. 82). This is illustrated in the numerical calculations shown in Fig. 83, where the departure from the calorically-perfect gas values is observed to increase as the flight speed (or Mach number) and altitude increase. In particular, the increase in altitude translates into a decrease in pressure, which favors air dissociation and ionization.


FIG. 85: VARIATION OF THE RATIO OF THE POST-SHOCK PRESSURE $P_2$ TO THE CALORICALLY-PERFECT VALUE $P_2^0$ FOR DIFFERENT FLIGHT SPEEDS AND ALTITUDES IN A NORMAL SHOCK (HUBER P.W., NASA TR R-163).
In contrast with the large effects on temperature, the post-shock pressure $p_2$ is only slightly affected by high-temperature thermo-chemical phenomena, as shown in the calculations in Fig. 85. As mentioned above, the post-shock pressure is mostly imposed by the pre-shock momentum, and therefore remains mostly insensitive up to departures of order 20% from the value predicted by the theory of calorically-perfect gases (Eq. (7)).

Concurrent with the decrease in $T_2$, the mostly invariant $p_2$, and a slight decrease in mean-molecular weight $W_2$, is a large increase in the post-shock density $\rho_2$ in such a way as to render density ratios $\varepsilon = \rho_1/\rho_2$ much smaller than the value $1/6$ predicted by the theory of calorically-perfect gases. As shown in Fig. 84, the density ratio may be as low as $\varepsilon \approx 1/20$ at high altitudes, in a way that resembles the limiting behavior $\varepsilon \to 0$ that is foundational to the Newtonian theory of inviscid hypersonic flows.

The increase in the post-shock density $\rho_2$ caused by thermo-chemical equilibrium effects leads to modifications in the shock patterns. For instance, since the shock standoff scales as $\Delta \approx \varepsilon R_0$, and $\varepsilon$ decreases in equilibrium air, then a decrease in $\Delta$ is observed when equilibrium effects are incorporated in the calculations. For the same reasons, since the angle of incidence $\beta$ and the streamline deflection angle are related through the kinematic constraint $\tan(\beta - \delta) \approx \varepsilon \tan\beta$ at high Mach numbers, the decrease in $\varepsilon$ by equilibrium effects translates into a decrease of $\beta$ in oblique shocks with respect to the value predicted by the theory of calorically-perfect gases and also into an increase of the maximum $\delta$ supported by an attached shock (see Mbeuk, NASA TN 3895, and Fig. 8).

As a general rule in hypersonic flow over cones, at $M_{\infty} \sin \delta \approx 10$, the surface density $\rho_\infty$ and temperature $T_\infty$ from the calorically-perfect gas model are off by
FACTORS OF ORDER Z \( (i.e., T_c \text{CO}_2 = 2 T_c \text{AIR} \text{ EQUILIBRIUM}, \ s \text{CO}_2 = \frac{4}{3} s \text{AIR} \text{ EQUILIBRIUM}) \)

AND THE DISCREPANCIES INCREASE WITH DECREASING Pressures \( (i.e., \text{INCREASING ALTITUDES}) \)

AS SHOWN IN FIG. 87. THE DISCREPANCIES HOWEVER VANISH AT \( \text{Ma}_w \sin \delta \leq 4 \) FOR ALL PRACTICAL FLIGHT ALTITUDES, WHERE THE POST-SHOCK TEMPERATURE \( T_2 \) CEASES TO BE SUFFICIENTLY LARGE TO ENABLE SIGNIFICANT VIBRATIONAL EXCITATIONS AND AIR DISSOCIATION.

IN THAT LIMIT, WHICH BELONGS TO THE SUPersonic RANGE, THE THEORY OF CALORICALLY PERFECT GASES PROVIDES AN ACCURATE RESULT.

FIG. 87: RATIO OF SURFACE TEMPERATURE AND SURFACE DENSITY TO FREE-STREAM VALUES AS A FUNCTION OF THE HYPERSONIC SIMILARITY PARAMETER \( \text{Ma}_w \sin \delta \) FOR A CONE OF SEMIANGLE \( \delta \). (VISHCH ET AL., J. SPACECRAFT 1967).

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THE EQUILIBRIUM SPECIES DISTRIBUTION BEHIND AN OBlique SHOCK AROUND A VEHICLE FOLLOWING A TYPICAL ENTRY TRAJECTORY IS SHOWN IN FIG. 88.

AT RELATIVELY LOW \( \text{Ma}_w \) (i.e., INCREASING ALTITUDES), THE POST-SHOCK GAS COMPOSITION DECRESSES \( \text{P}_g \) (i.e., INCREASING ALTITUDES)

GAS COMPOSITION RESEMBLES THAT OF NORMAL AIR, WITH minor traces of NO being formed as \( \text{Ma}_w \) NEARS 7.

ABOVE \( \text{Ma}_w = 7 \), THE FOLLOWING STAGES ARE DISCERNED:

1. \( 7 \leq \text{Ma}_w \leq 15 \): THE POST-SHOCK GAS IS FORMED BY (IN ORDER OF IMPORTANCE)
   \[ N_2, O_2, O, NO \text{ AND } Ar \text{ WITH TRACES OF } N. \]

2. \( 15 \leq \text{Ma}_w \leq 20 \): THE POST-SHOCK GAS IS FORMED BY (IN ORDER OF IMPORTANCE)
   \[ N_2, O, N, Ar, NO \text{, WITH TRACES OF } O_2^+ \text{ AND } NO^+ \]

3. \( 20 \leq \text{Ma}_w \leq 25 \): \[ N, N_2, O, Ar, NO, \text{ WITH TRACES OF } e^- \text{ AND } NO^+ \]
A classic effect arising in hypersonic flows is the production of electrons across shock waves because of the partial ionization of the air. This is illustrated in Fig. 89 (b), where trace amounts of electrons, $X_e \sim 10^{-4}$, are observed in equilibrium conditions in the post-shock gas. The maximum amount is attained in the portion of the flight envelope (Fig. 89a) related to high altitudes ($2 \times 170,000$ ft, i.e., above the stratopause) and high Mach numbers ($M_a > 20$).

Note however that $X_e \sim 10^{-4}$ translates into electron number densities

$$n_e = \frac{p_e}{W_e} N_A X_e \sim O \left(10^{-13} - 10^{-14}\right) \text{cm}^{-3}$$

The associated plasma frequencies are

$$f_p = \frac{1}{2\pi} \left(\frac{m_e - e^2}{e_0 n_e}\right)^{1/2} \sim O \left(10^{-10} - 10^{-12}\right) \text{GHz}$$

The resulting plasma sheath plays an important role in the radiocommunications of the hypersonic vehicle (Fig. 90) in the following way. Only waves at frequencies $f_e$ larger than the plasma frequency can penetrate the sheath. As a result, VLF ($1-10$ kHz) or VHF ($10-100$ kHz) radio waves will tend to be cut off.

A more general map of variations of $f_p$ with altitude and speed is provided in Fig. 89.
From the discussions in the previous sections it is evident that the establishment of thermodynamic and chemical equilibria requires a sufficiently large number of collisions as the fluid particle moves in the spatiotemporally varying flow field around a hypersonic vehicle. This implies the competition of two time scales:

\[ t_R = \text{Residence (Fluid-Mechanical) time} \]
\[ t_T = \text{Thermodynamic/Chemical relaxation time}. \]

The assumption of equilibrium requires that the readjustment time by collisions \( t_T \) be negligible compared to \( t_R \). In many cases of interest for hypersonics, however, this requirement is not met either locally or globally around the aircraft.

In the notation above,

\[ \frac{t_R}{t_T} \gg 1 \Rightarrow \text{Thermo-Chemical Equilibrium Flow} \]
\[ \frac{t_R}{t_T} < 1 \Rightarrow \text{Thermo-Chemically Frozen Flow} \]
\[ \frac{t_R}{t_T} = O(1) \Rightarrow \text{Thermo-Chemical Non-Equilibrium}. \]

The time scale \( t_T \) has different components, representative of molecular motion (translation \( t_{\text{trans}} \), rotation \( t_{\text{rot}} \), vibration \( t_{\text{vib}} \), or electronic \( t_{\text{el}} \)) and chemical conversion (\( t_{\text{chem}} \)).

The readjustment of translation and rotation requires only a few collisions, and the corresponding relaxation times are of the order of the collision time \( \Theta^{-1} \), with \( \Theta \) the collision frequency defined in Eq. (184). Non-equilibrium phenomena in these processes occur when the ratios \( \frac{t_R}{t_{\text{trans}}} \) and \( \frac{t_R}{t_{\text{rot}}} \) are finite, which typically occurs in flow regions with large gradients (e.g., shocks, and to a lesser extent boundary layers), and involve finite molecular transport effects (viscosity, thermal conductivity, and mass.
DIFFUSIVITY FOR TRANSLATIONAL NON-EQUILIBRIUM, AND BULK VISCOSITY FOR ROTATIONAL NON-EQUILIBRIUM. AS A RESULT, TRANSLATIONAL AND ROTATIONAL NON-EQUILIBRIUM ARE REPRESENTATIVE OF NON-INVIScid EFFECTS, WHICH DO NOT NECESSARILY REQUIRE HIGH TEMPERATURES TO EXIST.

IN MOST PRACTICAL SITUATIONS, FOR HYPERSONICS, WITH EXCEPTION OF HIGHLY RAREFIEd FLOWS FOR WHICH $\mathcal{C}_r = 0$, THE GAS IS IN TRANSLATIONAL AND ROTATIONAL EQUILIBRIUM, WITH SMALL DEPARTURES FROM EQUILIBRIUM BEING NOTICED IN BOUNDARY LAYERS AND MUCH MORE SEVERE DEPARTURES EXISTING IN SHOCK WAVES (NOTE THAT $\frac{\mathcal{C}_r}{U/L} \sim \frac{\mathcal{A}/L}{\mathcal{A}} \sim \mathcal{R}_k \mathcal{K}_m$).

IN CONTRAST, VIBRATIONAL AND CHEMICAL PROCESSES TAKE MUCH LONGER TO READJUST. AS A RESULT, EXTENDED REGIONS IN THE FLOW FIELD MAY OCCUR WHERE VIBRATIONAL AND CHEMICAL NON-EQUILIBRIUM EXIST, WITH THESE REGIONS NOT BEING NECESSARILY CONFINED TO REGIONS WITH LARGE GRADIENTS. IN GENERAL, VIBRATIONAL MOTION READJUSTS EARLIER THAN THE CHEMICAL COMPOSITION, ALTHOUGH BOTH ARE INTRICATELY COUPLED DURING THE NON-EQUILIBRIUM STAGE. IT IS IMPORTANT TO EMPHASIZE THAT BOTH VIBRATIONAL AND CHEMICAL PROCESSES REQUIRE HIGH TEMPERATURES TO OCCUR, AND ARE THEREFORE OF HIGH RELEVANCE IN THE POST-SHOCK CASES OF HYPERSONIC SHOCK WAVES (FIG. 92).

BASED ON THE ABOVE CONSIDERATION, THE FOLLOWING DEFINITIONS CAN BE MADE WITH REGARD TO THE TYPE OF GAS:

1) CALORICALLY-PERFECT GAS: TRANSLATIONALLY AND ROTATIONALLY EQUILIBRATED, VIBRATIONALLY FROZEN (UNEXCITED), AND CHEMICALLY FROZEN.

2) THERMALLY-PERFECT GAS: TRANSLATIONALLY, ROTATIONALLY, AND VIBRATIONALLY EQUILIBRATED, AND CHEMICALLY FROZEN.

3) CHEMICALLY-REACTING EQUILIBRIUM IDEAL GAS: TRANSLATIONALLY, ROTATIONALLY, VIBRATIONALLY, AND CHEMICALLY EQUILIBRATED (SEE FIG. 91).

Note that the degree of chemical equilibrium is measured by the time-scale ratio

$$D_\alpha = \frac{\mathcal{C}_r}{t_{chem}}$$

(314) DAMköHLER NUMBER

WITH $D_\alpha \rightarrow 0$ AND $D_\alpha \rightarrow \infty$ INDICATING, RESPECTIVELY, FROZEN AND EQUILIBRIUM CONDITIONS.
VIBRATIONAL AND CHEMICAL NON-EQUILIBRIUM

Non-equilibrium phenomena related to chemical and vibrational processes are important in hypersonics due to the following factors:

1) Chemical and vibrational processes tend to require many collisions to equilibrate compared to translational and rotational processes (the latter only require a few collisions to equilibrate). At hypersonic velocities, the collisions around the aircraft are hindered because of the small residence time of the fluid particles and because of the low ambient pressures involved in high altitude flight.

2) Hypersonic flows typically involve high temperatures that are necessary for the activation of vibrational and chemical processes. Note however that high temperatures tend to equilibrate the flow because of the corresponding decrease in the mean free path, thereby tending to counteract the non-equilibrating effect of the low pressure and high speed.

The subjects of chemical and vibrational non-equilibrium are disciplines on their own and cannot be treated in depth here. The discussion here will be limited to basic results. The reader is referred to Vincenti and Kruger’s “Introduction to Physical Gas Dynamics,” Chs. VII–VIII (1965) for further details.

The expression for the internal energy \( (302) \) contains different terms that pertain to different modes of molecular motion. Momentarily disregarding electronic excitations (which give raise to a plethora of radiative phenomena beyond the scope of this course), the internal energy in thermodynamic equilibrium can be written as

\[
E = E_{\text{trans}} + E_{\text{rot}} + E_{\text{vib}}, \quad \text{with} \quad \begin{cases} 
E_{\text{trans}} = \frac{3}{2} R_T T \\
E_{\text{rot}} = R_T T \\
E_{\text{vib}} = \frac{\Theta_V / T}{\Theta_V / T - 1} R_T T
\end{cases}
\]  

\( (315) \)

With \( \Theta_V = \frac{h \nu}{k} \) a characteristic vibrational temperature. It is important to note that the temperature \( T \) in \( (315) \) is the same for all components when the system is in thermodynamic equilibrium, and is equal to the translational kinetic temperature defined by

\[
E_{\text{trans}} = \frac{3}{2} R_T T = \frac{1}{2} \bar{c}^2
\]  

\( (316) \)

where \( \bar{c} = \frac{1}{\sqrt{3 R_T T}} = \left( \frac{3P}{\rho} \right)^{1/3} \) is the root-mean square speed of the gas molecules.
In situations where vibrational non-equilibrium prevails, $\epsilon_{\text{vib}}$ has not yet relaxed to the equilibrium value in (215). Instead, $\epsilon_{\text{vib}}$ is obtained from a supplementary equation that describes the relaxation process and which is given generally by

$$\frac{d\epsilon_{\text{vib}}}{dt} = \epsilon_{\text{vib}}^\infty - \epsilon_{\text{vib}} - F(K_{\text{vib}})$$  \hspace{1cm} (347)

where $\epsilon_{\text{vib}}^\infty = \epsilon_{\text{vib}}^0/T_0$, $T_0$ is the equilibrium value, $\tau_{\text{vib}}$ is a characteristic vibrational relaxation time, and $F(K_{\text{vib}})$ is a function that depends on reaction rates of dissociation (see Treado and Mittleman, Phys. Fluids (1962)). Note that there exists a coupling between the chemical reaction rates and the vibrational relaxation rate, in that molecules that are highly excited vibrationally are more readily dissociated by collisions, thus increasing the rate of dissociation (Hammerling et al., Phys. Fluids (1952)), while dissociation causes an additional drainage of mean vibrational energy given by the second term in (214).

The relaxation time $\tau_{\text{vib}}$ is usually obtained by the Landau-Teller (1936) theory

$$\tau_{\text{vib}} = \frac{\chi_{\text{vib}} e^{(\frac{K_{\text{vib}}}{T})^{1/2}}}{P}$$  \hspace{1cm} (348)

where $\chi_{\text{vib}}$ and $K_{\text{vib}}$ are constants that depend on the vibrating molecule and on the bath of translational molecules on which they collide (see Table V). The characteristic values of $\tau_{\text{vib}}$ are of order $10^{-5}$ to $10^{-7}$ at atmospheric pressures and high temperatures. For instance, consider pure $N_2$ at $T=3000 K$ and $P=1 atm$, so that $\tau_{\text{vib}} \approx 38 \mu s$ according to the values in Table V. Note that the collision frequency (184) at this temperature is $1.5 \times 10^{10}$ collisions/s, which indicates that during the vibrational relaxation period, approximately $\tau_{\text{vib}} \times 10^{10}$ collisions have taken place, at this pressure and temperature, which are representative of the gas behind a normal shock wave at Mach 10 in the stratosphere ($T_2 = 3200 K, \eta_2/\eta_1 \approx 8.5$ at $U_2 = 18600 ft/s$ and $2 \times 10^{12} ft^2$ in Fig. B2 (a,b)). The vibrational relaxation of $N_2$ lasts for $(5\epsilon_{1}/\eta_2)\tau_{\text{vib}} \approx 0.3 cm$ downstream of the normal shock. This distance over which vibrational non-equilibrium

**Table V:** Values of $\chi_{\text{vib}}$ and $K_{\text{vib}}$ in (348).

<table>
<thead>
<tr>
<th>Species</th>
<th>Heat-Bath Molecule</th>
<th>$C_{\text{vib}}$ atm-microsec</th>
<th>$K_{\text{vib}}$ K</th>
<th>Approximate Range of $T$, K</th>
</tr>
</thead>
<tbody>
<tr>
<td>O$_2$</td>
<td>O$_2$</td>
<td>$5.42 \times 10^{-4}$</td>
<td>2.95 $\times 10^4$</td>
<td>800-3200</td>
</tr>
<tr>
<td>O$_2$</td>
<td>Ar</td>
<td>$3.58 \times 10^{-4}$</td>
<td>2.95 $\times 10^4$</td>
<td>1300-4300</td>
</tr>
<tr>
<td>N$_2$</td>
<td>N$_2$</td>
<td>$7.12 \times 10^{-4}$</td>
<td>1.91 $\times 10^4$</td>
<td>800-6000</td>
</tr>
<tr>
<td>NO</td>
<td>NO</td>
<td>$4.86 \times 10^{-4}$</td>
<td>1.37 $\times 10^4$</td>
<td>1500-3000</td>
</tr>
<tr>
<td>NO</td>
<td>Ar</td>
<td>$6.16 \times 10^{-4}$</td>
<td>1.37 $\times 10^4$</td>
<td>1500-4600</td>
</tr>
</tbody>
</table>
PERSISTS IS ACTUALLY COMPARABLE TO CHARACTERISTIC SHOCK STANDOFF DISTANCES \( \Delta \approx (\frac{d}{v}) R_0 \) FROM A NOSE OF CURVATURE RADIUS \( R_0 \approx 10 \) cm (Fig. 93).

In the calculation of flows subject to vibrational non-equilibrium, equations (306) - (308) are integrated with \( h \) not including the vibrational enthalpy, namely

\[
h = \sum_{f=1}^{F} Y_f h_f^0 + \sum_{i=1}^{N} Y_i \left( h_{i,\text{trans}} + h_{i,\text{rot}} + h_{i,\text{vib}} \right)
\]

where \( h_{i,\text{vib}} = c_{i,\text{vib}} + P/S \) AND \( c_{i,\text{vib}} \) is obtained by integrating (317), while \( h_{i,\text{trans}} \) and \( h_{i,\text{rot}} \) are given by their equilibrium expressions.

Note that vibrational non-equilibrium in the post-shock gases typically coexists with chemical non-equilibrium whereby finite-rate chemistry becomes important in dissociating the air molecules (Fig. 93). As a result, Eqs. (306) - (308) need to be supplemented with the equation for species transport (278), which in the inviscid limit becomes

\[
\frac{\partial Y_i}{\partial t} + \mathbf{u} \cdot \nabla Y_i = \dot{W}_i \quad (319)
\]

where \( \dot{W}_i \) is a chemical production term that competes with the advection terms when the characteristic Damköhler number (314) is of order unity. As specified in (280) - (294), however, \( \dot{W}_i \) depends on temperature in situations where chemical and vibrational equilibrium coexist, as in the zone immediately behind a normal shock wave at hypersonic velocities (Fig. 93), the character of the temperature participating in the definition of \( \dot{W}_i \) is unclear because of the two-way coupling effect between dissociation and vibrational relaxation mentioned above. In fact, the successive relaxation of the different degrees of freedom motivates the definition of alternative temperatures,

\[
\begin{align*}
E_{\text{trans}} &= R_T T_{\text{trans}} \\
E_{\text{rot}} &= R_T T_{\text{rot}} \\
E_{\text{vib}} &= \frac{\mathcal{O}_V}{T_{\text{vib}}} R_T T_{\text{vib}} \left( e^{\mathcal{O}_V/T_{\text{vib}} - 1} \right)
\end{align*}
\]
where $T_{\text{trans}}$ is the kinetic translational temperature, $T_{\text{rot}}$ is the rotational temperature, and $T_{\text{vib}}$ is the vibrational temperature. As shown in Fig. 94, the translational and rotational temperatures equilibrate fast across the shock since only a few collisions are required to fully excite translational and rotational degrees of freedom. Note that the translational temperature is the one that participates in the equation of state, which can be considered still valid for small departures from thermodynamic equilibrium. On the other hand, the vibrational temperature takes much longer distances to equilibrate as implied by the large parameter $\frac{\nu}{T_{\text{vib}}} \gg 1$, which suggests that many collisions are needed for $T_{\text{vib}}$ to become the equilibrium value $T_{\text{eq}}$.

During the non-equilibrium period, $T \neq T_{\text{vib}}$ in the conservation equations, and it is customary to use a combined temperature $T_{\text{eq}} = (T_{\text{vib}}T)^{1/2}$ to evaluate the rates of the chemical reactions participating in the dissociation process (see the treatise "Non-equilibrium Hyperbolic Flows" by Park about this subject). Here, $T$ is the translational kinetic temperature (i.e., $T_{\text{vib}} \rightarrow T$ at distances much larger than $U_2^2 T_{\text{vib}}$ behind the shock as shown in Fig. 94).

*Non-equilibrium flow across a shock*

![Diagram showing non-equilibrium flow across a shock](image)

The characteristic temperature and density profiles across a normal shock are provided in Fig. 95 for $Ma = 10$ and $Z = 120 \, \text{kg} \, \text{m}^{-2} \, \text{s}^{-1}$. It is important to note that, because of the only few collisions across the shock, the values $T_{\text{eq}}$ and $Z_{\text{eq}}$ immediately downstream of the shock are those corresponding to frozen flow, or equivalently, those that would be obtained by applying the shock jump conditions (6) and (8). It is only after a downstream distance of order $L \sim U_2 T_{\text{eq}}$, (which incorporates both vibrational and chemical non-equilibrium since $T_{\text{eq}} > T_{\text{vib}}$ in conventional situations) that the flow variables relax.
To those predicted by the equilibrium theory (such as those in Figs. 83-85), the distance $\lambda_{eq}$ is typically of order $\lambda_{eq} = O(10)$ cm for normal shocks, and $\lambda_{eq} = O(100)$ cm for oblique shocks, the latter being longer because of the higher velocities and lower temperature associated with the post-shock gases.

It is interesting to note that the non-equilibrium processes also have important influences on the distribution of chemical species behind the shock. A control-volume analysis of Eq. (214) around the shock wave demands the continuity of the frozen composition across:

\[
\frac{V_1}{S_0} = \frac{V_2}{S_0} + \int_{V_c}^{V_c + V} \rho dV = \int_{V_c}^{V_c + V} \rho dV
\]

As a result, the composition immediately downstream of the shock is that of the pre-shock (non-dissociated) gas, and only after a downstream distance of order $\lambda_{eq}$ the equilibrium composition (i.e., the one shown in Fig. 96) is achieved (see Fig. 96).

Before closing this chapter, it is worth emphasizing that the considerations given above are in many cases oversimplified with respect to real non-equilibrium phenomena that may be encountered in flows around hypersonic vehicles. Real conditions may include flow unsteadiness, pre-shock chemically reacting gases passing through additional shocks, non-equilibrium effects in expansion waves, molecular recombination in walls, boundary-layer effects, and non-equilibrium radiative effects in the gas and in the wall surfaces. Most importantly, the non-equilibrium character of the flow is often set by the local flow conditions rather than by global flow indicators such as those depicted in Fig. 5, thereby making the prediction of thermo-chemical phenomena around hypersonic vehicles a challenging problem that remains subject of research.
V. RE-ENTRY AERO-MECHANICS

The previous chapters have been devoted to the study of hypersonic flow fields in several forms. The design of a hypersonic vehicle, however, heavily relies on the global characteristics of the flight trajectory. A problem that has received much attention over the years is the atmospheric re-entry from space, which involves hypersonic flows at exceedingly high temperatures that set severe constraints in the design of the spacecraft.

GENERAL CONSIDERATIONS

![Diagram of re-entry trajectory and atmosphere entry velocity](image)

**Fig. 9.1. Sketch of a re-entry trajectory in the terrestrial atmosphere, with \( R_0 = 6371 \text{ km} \) the Earth radius.**

The problem of re-entry involves the penetration of a body (meteor, space capsule, glider, warhead, satellite, asteroid, etc.) into the atmosphere at high velocities ranging from the circular orbital velocity (\( \approx 7.900 \text{ m/s} = 28500 \text{ km/h} \) for Earth) to the escape velocity (\( \approx 11200 \text{ m/s} = 40200 \text{ km/h} \) for Earth) and beyond. Outside the planetary atmosphere, the motion of the body is described by the Kepler's laws of celestial mechanics, as the body enters the atmosphere and more particularly after crossing the mesopause, the aerodynamic forces become increasingly important because of the increasingly larger value of the density as the ground is approached. The aerodynamic forces lead to a large deceleration and intense friction that more often than not cause disintegration of the body in the case of meteors, and risk the structural integrity in the case of spacecraft. In manned spaceflight, the re-entry stage becomes much more crucial, since the objective turns into slowing down the spacecraft from entry velocities of order of tens of thousands of miles per hour to zero velocity at landing, and doing it safely. An obvious but surely impractical solution to the problem is to employ retro-rockets to slow down the spacecraft through the atmosphere. However, it is characteristic of rocket propulsion that for every ton of payload lifted to space, many tons of fuel are required in the booster. As a result, the extra weight that retro-rocketing would require renders it impractical.
This alternative highly unattractive. To date, the alternative is to let the atmosphere to slow down the spacecraft by viscous friction. A typical re-entry trajectory is sketched in Fig. 97, where

\[
\begin{align*}
\gamma_e &= \text{entry flight-path angle (with respect to the local horizontal)} \\
U_e &= \text{entry velocity} \\
Z_e &= \text{entry-interface height (on Earth, } Z_e \approx 86 \text{ km)}
\end{align*}
\]

The passive deceleration caused by friction may look daunting at first, since most meteors are completely vaporized during re-entry due to the intense heating generated. However, throughout the years, solutions have been created that warrant decelerations within human tolerances and preserve the structural integrity of the spacecraft, although such solutions still lead to excessive mission costs because of the added weight necessary to fulfill these tolerances.

* Types of re-entry trajectories

The problem of atmospheric entry takes on different forms depending on the mission trajectory under investigation. Perhaps the three most common situations are re-entries produced as a result of (i) ballistic, or boost-glide trajectories of rocket payloads from one to another point on the Earth surface, (ii) deflected orbital trajectories of space crafts initially orbiting along circular or elliptical orbits around the Earth, and (iii) hyperbolic trajectories of spacecrafts returning from the moon or other planets. Examples of these spaceflight trajectories are provided in Fig. 98. In that figure, the following trajectories are highlighted:

- **Ballistic entry**: Non-lifting bodies \((L/D = 0)\) enter the atmosphere ballistically from space. The characteristic flight path angle \(\gamma_e\) is large, and as a result the deceleration and heat flux are large (see next section). The gravitational and centrifugal forces tend to be negligible once the spacecraft enters the atmosphere in ballistic mode.
Fig. 98: Main types of re-entry trajectories in a planetary atmosphere.
GLIDING ENTRY: LIFTING BODIES \( \dfrac{L}{D} = 0(1) \) CAN SMOOTHLY ENTER THE ATMOSPHERE AT ALMOST ZERO INITIAL FLIGHT-PATH ANGLE, \( \gamma = \Delta \). THIS TENDS TO REDUCE THE DECELERATION BUT INCREASES THE TOTAL HEAT LOAD BECAUSE OF AN INCREASED AMOUNT OF ENTRY TIME AND BECAUSE OF THE ADDITIONAL WEIGHT REQUIRED FOR A LIFT-SUPPORTING STRUCTURE WITH WINGS OR OTHER MEANS. IN THIS TYPE OF TRAJECTORY, THE VERTICAL COMPONENT OF THE ACCELERATION AND THE VERTICAL COMPONENT OF THE DRAG FORCE ARE NEGLIGIBLE.

SKIPPING ENTRY: VARIABLE VALUES OF THE LIFT-TO-DRA G RATIO CAN BE USED IN LIFTING BODIES TO LIFT-UP THE VEHICLE ONCE IT HAS ENTERED THE ATMOSPHERE, THE RESULTING UPWARDS MOTION MAY CAUSE THE VEHICLE TO HOSTLY EXIT THE ATMOSPHERE, AFTER WHICH A NEGATIVE \( \dfrac{L}{D} \) NEEDS TO BE APPLIED FOR IT TO PENE TRATE AGAIN INTO THE ATMOSPHERE. IN SKIPPING ENTRIES THE INITIAL FLIGHT-PATH ANGLE IS LARGE, AND THE GRAVITATIONAL AND CENTRIFUGAL FORCES ARE NEGLIGIBLE.


![Diagram of different types of entry](image)
**ENTRY VELOCITY:**

The characteristic entry velocity also depends on the trajectory under investigation. For satellites or manned spacecrafts orbiting in circular orbits around the Earth, the entry velocity \( U_e \) is of the order of the circular velocity,

\[
U_e = \left( \frac{\mu}{R_\oplus} \right)^{1/2} = 7.9 \text{ km/s} = 28,795 \text{ km/h} \space \text{"first cosmic velocity"}
\]

\[
= 17.7 \text{ mph} = 25,950 \text{ ft/s}
\]

with \( \mu = 3.986 \times 10^{14} \text{ m}^3/\text{s}^2 \) the standard gravitational parameter, and \( R_\oplus = 6371 \text{ km} \).

The radius of the Earth. Other trajectories involve supercircular entries at the parabolic velocity

\[
U_e = \left( \frac{2 \mu}{R_\oplus} \right)^{1/2} = 11.2 \text{ km/s} = 40,269 \text{ km/h} \space \text{"second cosmic velocity"}
\]

\[
= 25,022 \text{ mph} = 36,700 \text{ ft/s}
\]

As in the case of a spacecraft returning from the Moon, or at hyperbolic velocities

\[
U_e > \left( \frac{2 \mu}{R_\oplus} \right)^{1/2}
\]

For spacecrafts returning from other planets, in all cases, \( U_e \) is much larger than the peripheral velocity of the Earth as it rotates around its axis (i.e., the absolute velocity of the atmosphere), which is approximately 400 m/s \( \approx 1600 \text{ km/h} \) \( \approx 1000 \text{ mph} \) along the Earth's equator, and in this way direct and retrograde orbits tend to pose similar challenges for re-entry.

![Deboost Diagram](image)

The entry velocity discussed above is not generally the true velocity at the entry interface, since the spacecraft must typically execute trajectory control on approach to Earth.

One characteristic maneuver is the deboost from a circular orbit, which involves the firing of retro-rockets to deflect the spacecraft into a descent ellipse that intercepts the surface of the Earth (Fig. 100).
In general, however, the change in altitude along the deflected orbit is small compared to the distance of the deboost point to the center of the Earth, and the difference between the actual entry velocity $U_e$ and the circular orbital velocity is small. In most practical deboost maneuvers, the retrorockets need to provide a velocity increment $\Delta U$ that is of order $1/2$ to $1/3$ the circular orbital velocity $U_c$. A complete analysis of deboost maneuvers from circular and supercircular orbits is provided in the seminal report by Low G., "Nearly Circular Transfer Trajectories for Descending Satellites, NASA TR-R-3, (1959).

Entry Corridor

A problem in spaceflight is the safe re-entry of spacecrafts from supercircular orbits; for instance, in their return from the moon or other planets. In particular, an overshoot or an undershoot approach with respect to the established return trajectory can have fatal consequences, as a result of guidance errors. In an undershoot approach, the spacecraft enters the atmosphere at a large flight-path angle, which causes accelerations beyond human tolerances. In contrast, in an overshoot approach, the atmosphere is too little dense to provide the drag force needed to slow down the spacecraft, which will traverse the outer layer of the atmosphere and will continue on an orbit that will not allow it to re-enter until most likely the power and oxygen are exhausted. Although return passes would eventually slow down the vehicle, this would also involve repeated crossings of the Van Allen belts and therefore too much exposure to heavy radiation (see F in Fig. 98).

The entry-corridor depth is defined as the difference between the altitudes of the perigees of the undershoot and overshoot orbits resulting from vacuum trajectories. In a lunar return trajectory, for instance, the entry-corridor depth is of order 10 miles.
Which, compared to the 250,000 miles that separate the moon from Earth, requires an incredibly accurate guidance system. Entry corridor depths can be enlarged by the utilization of lift, since lift assists in correcting the entry angle. Through computations of entry corridors are provided by Chapman D.R., "An Analysis of the Corridor and Guidance Requirements for SuperCircular Entry into Planetary Atmospheres", NASA TR-R-55 (1960).

**Lift Force and Downrange Precision Landing**

The utilization of lift does not only increase the entry corridor depth with respect to a purely ballistic entry, but it also lengthens the path to the ground, decreases the maximum deceleration and heating rate, provides control capability for correcting too steep entry angles, and most interestingly, provides capabilities for landing within much wider areas on the surface of the Earth. Lift can be generated by changing the attitude of the spacecraft (Fig. 102). In non-lifting bodies, such as space capsules, lift is generated by shifting the center of gravity above the axis of symmetry. Negative lift can be produced by rotating the capsule around its axis of symmetry using small rockets pointed in the azimuth direction, in such a way as to shift the center of gravity downwards. This control authority was employed in the reentry of the Apollo Command Module (Fig. 103) for a "pull up" at about 180,000 ft and a "skip" to increase the maximum range of the spacecraft and to reduce the aerodynamic heating on the thermal shield. The control period during "pull-up" is employed to reduce the landing footprint to less than 2,000 miles length. The second control period fine tunes the

---

**Fig. 102:**

- Shuttle Orbiter
- L / D = 1.5
- Apollo Command Module
- L / D = 0.5

---

**Fig. 103:** Apollo 4 re-entry trajectory (adapted from Hills, NASA TN D53-99 (1969)).

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LANDING FOOTPRINT BY REDUCING ITS LENGTH TO 690 MILES. NOTE THAT AN ERROR IN THE ATTITUDE OF THE CAPSULE DURING THE FIRST CONTROL PERIOD, FOR INSTANCE INVOLVING TOO MUCH LIFT, COULD SEND THE SPACECRAFT UP AWAY FROM THE ATMOSPHERE LIKE A STONE SKIPPING ON A POND, AND INTO AN ORBIT THAT WOULD LAST BEYOND THE LIFETIME OF THE COMMAND MODULE SYSTEMS.

![Map Diagram](image)

**Fig. 104**: Different landing footprints that can be achieved by lifting bodies. Each curve represents possible landing points at the specified $L/D$. (Becker, 1961)

AN ILLUSTRATION OF A LANDING FOOTPRINT FOR A LIFTING VEHICLE IS PROVIDED IN FIG. 104. IN GENERAL, A VEHICLE WITH A LIFT-TO-Drag Ratio $L/D \approx 2$ CAN BE LANDED ANYWHERE WITHIN AN AREA EXTENDING APPROXIMATELY 5,000 MILES. THE CAPABILITY FOR A HYPERSONIC GLIDER TO CONTROL THE LANDING ZONE BY BANKING ITSELF AND VARYING ITS FLIGHT ATTITUDE, IS COMMONLY KNOWN AS "DYNAMIC SOARING" (HENCE THE PROJECT "DYNASOAR", 1957-1963 AIMED AT DEVELOPING A HYPERSONIC DYNAMIC SOARER FOR ACCESS AND RETURN FROM LOW-EARTH ORBIT).

**Ballistic Re-entry Aero-Mechanics**

Perhaps the simplest re-entry trajectory is the ballistic type. (Allen S.H, E66ers A.J., NASA TR 1381, 1958). Based on the schematics shown in Fig. 105, the acceleration of the spacecraft in the polar coordinate system $\dot{r}, \theta$ is

$$\ddot{r} = \rho - \left(\frac{dV}{dt} + \frac{V^2}{r}\right) \hat{e}_r + \left(\frac{dV}{dt} - 2V\dot{\theta}\right) \hat{e}_\theta,$$

WHERE $\hat{e}_r$ AND $\hat{e}_\theta$ ARE UNIT VECTORS IN THE RADIAL AND POLAR DIRECTIONS, RESPECTIVELY, AND $V$ AND $U$ ARE THE CORRESPONDING VELOCITY
Components, \( \mathbf{V} = -\mathbf{a} \| \mathbf{b} \| = U \sin \gamma \), and \( \mathbf{u} = \mathbf{c} \| \mathbf{d} \| = U \cos \gamma \), with \( U \) the velocity, magnitude.

The second Newton's law for this planar motion corresponds to

\[
- m \frac{dV}{dt} + m \frac{d^2r}{dt^2} = m \frac{d^2x}{dt^2} = m \frac{u^2}{r} + L \cos \gamma - m g \sin \gamma + D \sin \gamma, \quad (321)
\]

\[
m \frac{dv}{dt} = m r \frac{d^2\theta}{dt^2} = 2 uv + L \sin \gamma - D \cos \gamma. \quad (322)
\]

In the ballistic entry, the following approximations are made: 1) the lift is zero, 2) the drag coefficient is constant, 3) the centrifugal force is negligible, 4) the Coriolis force is negligible, 5) the weight of the vehicle is negligible, and 6) the flight-path angle is constant and equal to the entry value \( \gamma = \gamma_E \). In this way, \((321) - (322)\) simplify to

\[
m \frac{dv}{dt} = - \frac{1}{2} \rho A C_D U^2 \sin \gamma_E \quad (323), \text{ where } \rho = \rho_0 e^{-\beta z} \text{ (exponential atmosphere)}
\]

An equation for the acceleration magnitude \( m \frac{dU}{dt} \) is easily obtained by multiplying the first of the equations in \((323)\) by \( U \) and the second one by \( V \), and by making use of \( V^2 = U^2 + V^2 \). Upon summing both equations, which gives

\[
m \frac{dU}{dt} = - \frac{1}{2} \rho A C_D U^2, \text{ subject to } U = U_E \text{ (entry velocity) at } t = 0 \quad (324)
\]

Note that the approximations made above are accurate for not too small entry angles \( \gamma_E \), whereby the centrifugal, Coriolis and gravitational forces are all much smaller than the drag force \( D \), which is responsible for the large decelerations of order 100g's experienced by a vehicle in ballistic entry.

The solution to \((324)\) is direct and involves the replacement of time by altitude \( -V = \gamma_E U_E \) \( dV \) by \( dU \). By using the relation \( \frac{dU}{dt} = \frac{d^2V}{dt^2} = \frac{dV}{dt} \), which transforms \((324)\) into

\[
\int \frac{dV}{V} = \int \frac{1}{2} \rho_0 e^{-\beta z} A C_D \frac{dU}{dt} \quad \Rightarrow \quad \ln \left( \frac{V}{U} \right) = \frac{1}{2} \rho_0 A C_D \frac{e^{-\beta z}}{m \sin \gamma_E}
\]

Which simplifies to

\[
V = U_E e^{-\frac{1}{2} \rho_0 A C_D \frac{e^{-\beta z}}{m \sin \gamma_E}}, \quad (325)
\]
WHERE $\zeta$ IS A DIMENSIONLESS PARAMETER GIVEN BY

$$\zeta = \frac{3 \frac{m}{S_o A G_D}}{S_o A G_D}$$  \hspace{1cm} (326)

WHICH CAN BE REWRITTEN AS $\zeta = (\beta L) \left( \frac{S_S}{S_o} \right) \left( \frac{\sin Y_E}{G_D} \right)$, WHERE $S_S$ IS THE MEAN STRUCTURAL

DENSITY OF THE VEHICLE, AND $L = V / \rho$ IS A CHARACTERISTIC LENGTH OF THE VEHICLE BASED ON

ITS VOLUME $V = \frac{m}{\rho_s}$. ALSO PART OF $\zeta$ IS THE BALLISTIC COEFFICIENT $\Gamma = \frac{m}{G_D} \left[ \frac{k_{e_0}}{m^2} \right]$, IN

SUCH A WAY THAT $\zeta = (\beta H / \rho_s) \sin Y_E$.

NOTE THAT $U$ IN (325) IS A SUPER-EXponentially DEcreasing FUNCTION OF THE ALTITUDE,

SUCH RAPID DECREASE IS CAUSED BY THE ATMOSPHERIC DRAG, AND IT DECREASES INCREASINGLY

MORE RAPID AS $\zeta$ BECOMES SMALLER. THIS CAN BE SEEN BY CALCULATING THE DECELERATION

NORMALIZED BY THE GRAVITATIONAL ACCELERATION, NAMELY

$$-\frac{1}{q} \frac{dU}{dt} = \frac{U \sin Y_E}{q} \frac{dU}{dt} = \frac{\sin Y_E}{q} \frac{U E^2}{2} e^{-\frac{\beta \rho_s \rho_o}{G_D}} e^{-\frac{\beta \rho_s \rho_o}{G_D}} = \frac{U E^2}{2} e^{-\frac{\beta \rho_s \rho_o}{G_D}} e^{-\frac{\beta \rho_s \rho_o}{G_D}} = \frac{S_o}{\rho}$$ \hspace{1cm} (327)

THE EXPRESSION FOR THE ACCELERATION PROVIDED ABOVE HAS A MAXIMUM AT THE

CRITICAL ALTITUDE

$$z_A = \frac{1}{\beta} \ln \left( \frac{\frac{1}{\beta}}{5} \right) = \frac{1}{\beta} \ln \left( \frac{S_o A G_D}{\beta m \sin Y_E} \right)$$ \hspace{1cm} (328) ALTITUDE

FOR MAXIMUM

DECELERATION

WHERE THE MAXIMUM DECELERATION IS

$$-\frac{1}{q} \frac{dU}{dt} = \left( \frac{U E^2}{2} \right) \frac{\beta \sin Y_E}{2e^2}$$ \hspace{1cm} (329) MAXIMUM DECELERATION

FIG. 106.
A schematic of the solution represented by \((328)-(329)\) is provided in Fig. 105 for the case \(s < 1\) \((s < 0.5)\). A number of important aspects of the above formulation are worthy of discussion:

4) **The value of the maximum deceleration is independent of the mass of the vehicle.**

2) **The value of the maximum deceleration decreases with decreasing entry angles \(\gamma_E\). It is for this reason that shallow entries are preferable for they enable maximum decelerations within human tolerances.**

3) **For entries at the first cosmic velocity, \(U_e = \sqrt{\frac{4}{\beta R}}\), the maximum deceleration is proportional to \(\sqrt{\beta R_0}\). For earth, \(\beta R_0 \approx 950\), and as a result \(a_{\text{max}}/g_0 = \frac{\beta R_0 \sin \gamma_E}{2e^2} \approx 174 \sin \gamma_E\). For \(40^\circ\) of maximum acceleration, the entry angle must be \(\gamma_E \approx 3.3^\circ\). Note that in other planets \(\beta R_0\), \(4x/90\approx 0.32\) and \((\beta R_0)^* \approx 44.5\), so that \(a_{\text{max}}/g_0 \approx 3.1 \sin \gamma_E\). As a consequence, a much steeper entry angle \(\gamma_E \approx 18.7^\circ\) can be used to attain the same \(40^\circ\) of maximum acceleration.**

4) **The altitude for maximum deceleration \(z_1\) is independent of the entry velocity.**

5) **The altitude for maximum deceleration \(z_1\) increases as the entry angle \(\gamma_E\) becomes increasingly smaller. As a consequence, shallow entries lead to smaller maximum decelerations and to higher altitudes where the maximum deceleration occurs.**

6) **The velocity at the altitude of maximum deceleration can be easily obtained by substituting \((328)\) into \((323)\), which gives**

\[
U_1 = U_e e^{-\frac{4}{2}} \approx 0.64 U_e
\]

**Equivalently, maximum deceleration occurs when \(63\%\) of the initial kinetic energy has dissipated into heat, and this is independent of all parameters.**
7) The impact velocity of the vehicle with the ground (in the absence of retro-rockets or drogue parachutes to slow it down near the ground) is given by equation (326) particularized for \( z = 0 \), namely

\[
U_I = U_E e^{-\frac{1}{2z}}.
\]

Equation (334) Impact Velocity

Note that only a fraction \( e^{-\frac{1}{2z}} \) of the initial kinetic energy survives till impact. Equivalently, a fraction \( 1 - e^{-\frac{1}{2z}} \) of the initial kinetic energy is dissipated into heat during re-entry. In practical applications, \( 1 - e^{-\frac{1}{2}} \approx 99\% \), so that typically only \( 1\% \) of the initial kinetic energy \( \frac{U_E^2}{2} \) remains at impact.

8) The maximum deceleration represented by (329) occurs above the ground only when \( \frac{z}{L} < 1 \), so as to make \( z_L \) (328) defined. In re-entries with \( \frac{z}{L} > 1 \), the maximum acceleration occurs near impact, \( z = z_L = 0 \), where \( \frac{\alpha_{\text{max}}}{c} = \frac{1}{\frac{U_E^2}{2c}} \cdot \frac{\beta \sin \theta e}{L} e^{-\frac{1}{2z}} \).

Note that \( \alpha_L \) is not defined at impact. That this value is smaller than (329) by a factor \( e^{-\frac{1}{2z}} < 1 \), indicating that a projectile of high ballistic coefficient experiences lower decelerations prior to impact albeit the much deeper atmospheric penetration and the higher impact velocity.

Re-entry Heating

At the first cosmic speed, the kinetic energy per unit mass of a spacecraft is \( \frac{gR_\odot}{2} \approx 31 \text{ MJ/kg} \). As mentioned above, approximately 99% of this kinetic energy is dissipated into heat during re-entry. If 100% of this heat was to be absorbed by the spacecraft structure, the re-entry would be catastrophic. Note that a kg of steel (specific heat \( C = 0.48 \text{ KJ/kg} \)) would heat up by \( \Delta T = \frac{gR_\odot}{2c_p} \approx 65,000 \text{ K} \) if it absorbed that large amount of heat, and would vaporize upon re-entry. Fortunately,
Most of that heat is dissipated into the air flowing around the spacecraft. This heat diversion away from the spacecraft is mainly achieved by the action of strong shock waves, as shown below. For this reason, re-entry vehicles have blunt shapes that generate strong bow shocks. Despite this, a small portion of heat does always reach the spacecraft by heat conduction and radiation in shock layers and boundary layers, which can be however enough to destroy the spacecraft if it is not dealt with appropriately.

One way of absorbing the heat reaching the spacecraft is by using thick metal plates (heat sinks) that store the heat by increasing their temperature. As for instance, the beryllium panels used for the recovery canister of the Mercury space capsule (Fig. 107).

Other methods include heat rejection by materials with high emissivities that re-radiate the heat away from the spacecraft, or ablating materials that vaporize as they absorb the heat, such as the Avcoat 5026-39 (epoxy resin in a fiberglass matrix) that was used for the thermal shield of the Apollo command module.

Generally three figures of merit are important for characterizing the thermal load on a spacecraft: \( Q \) \( [\text{J}] \) absorbed during re-entry, \( \dot{Q} \) \( [\text{W/m}^2] \) the maximum time rate of spatially averaged heat input per unit area, \( \dot{Q}_{\text{max}} \) \( [\text{W/m}^2] \) the maximum time rate of the local heat input in a sensitive location (e.g., nose region), and \( Q_{\text{max}} \) \( [\text{W/m}^2] \). These quantities are estimated in what follows for ballistic re-entry (Allen and Eggers, NASA TR 1381, 1958) and neglecting radiative heating (a good approximation for not too high velocities \( M \approx 25-30 \)).
THE LOCAL HEAT FLUX INTO THE SPACECRAFT CAN BE WRITTEN BY USING THE DEFINITION OF THE STANTON NUMBER,

\[ \dot{q} = \frac{p_0 u_0 c_p}{T_{r0}} (T_r - T_w) \eta \]  

(3.32)

WHERE \( T_r \) IS A RECOVERY TEMPERATURE THAT FOR \( Pr = 1 \) IS GIVEN BY

\[ T_r = T_{ro} = T_0 \left( 1 + \left( \frac{y-1}{2} \right) M_{ao}^2 \right) \]  

(3.33)

(HERE THE SUBINDEX "0" REFERS TO EDGE QUANTITIES). SINCE THE STAGNATION TEMPERATURE IS CONSTANT ACROSS SHOCKS, IT IS INFERRED FROM (3.33) THAT \( T_r \) IS EQUAL TO THE FREE-STRAIN STAGNATION TEMPERATURE \( T_{ro} \) TO A GOOD APPROXIMATION,

\[ T_r = T_{ro} = T_0 \left( 1 + \left( \frac{y-1}{2} \right) M_{ao}^2 \right) \]  

(3.34)

(HERE THE SUBINDEX "oo" REFERS TO FREE-STRAIN CONDITIONS). SUBTRACTING THE LOCAL WALL TEMPERATURE \( T_w \) FROM (3.34) AND TAKING THE LIMIT \( M_{ao} \approx 1 \) YIELDS

\[ T_r - T_w = T_{ro} \left( \frac{y-1}{2} \right) M_{ao}^2 = \frac{U^2}{2 c_p} \]  

(3.35)

WHERE \( U = U_{oo} \) IS THE MAGNITUDE OF THE FLIGHT VELOCITY GIVEN BY (3.25). SUBSTITUTING (3.35) INTO (3.32) GIVES

\[ \dot{q} = \frac{p_0 u_0}{2} U^2 \eta \]  

(3.36)

AND USING THE REYNOLDS ANALOGY \( \eta = \frac{C_N}{2} \),

\[ \dot{q} = \frac{C_N}{4} \frac{p_0 u_0}{U^2} U^2 \]  

(3.37)

WHICH IS ANALOGOUS TO THE SCALING OBTAINED IN EQ. (2.39). THE SURFACE-AVERAGED HEAT FLUX OVER THE TOTAL WETTED SURFACE \( S \) OF THE SPACECRAFT IS

\[ \overline{\dot{q}} = \frac{1}{S} \int \dot{q} dS' = \frac{U^2}{2} \int \frac{p_0 u_0}{U} \dot{q} dS' = \frac{p_0}{2} \frac{U^2}{4} \overline{C_f} \]  

(3.38)

WHERE \( \overline{C_f} = \frac{1}{S} \int \frac{p_0 u_0}{U} \dot{q} dS' \) IS A SURFACE AVERAGED SKIN FRICTION.

SUBSTITUTION OF THE SPACECRAFT VELOCITY (3.25) INTO (3.38) GIVES THE EVOLUTION OF THE SURFACE-AVERAGED HEAT FLUX WITH ALTITUDE.
\( \frac{dQ}{dt} = \int q'' \, ds = \bar{q}'' \, s = \frac{80}{4} \, e^{-\beta z} \, \bar{q} \, U_e^3 \, e^{-\frac{3}{2}} \, e^{-\beta z} \) \[w\] (342)

The surface-averaged heat flux (339) is linked to the total heat input rate

\[ Q = \frac{80 \, q_e \, U_e^2}{4 \, \sin \gamma_e} \int_0^{+\infty} e^{-\frac{1}{5} \, e^{-\beta z}} \, e^{-\beta z} \, dz = \frac{4}{5} \left( \frac{q_e \, s}{C_D} \right) m U_e^2 (1 - e^{-\frac{1}{5}}) \] (342)

Total heat input into the spacecraft during ballistic re-entry.

Lastly, the heat flux on the blunt nose of the spacecraft is given by

\[(249)\] rewritten as

\[ q''_{\text{nose}} = \bar{q}'' \frac{2^{3/4} \cdot \text{const.}}{\epsilon^{1/4}} \left( \frac{M_{\text{in}} \, R_0}{P_{\text{in}}} \right)^{1/2} U_e^{3/2} \leq \text{const.} \left( \frac{M_{\text{in}} \, R_0}{P_{\text{in}}} \right)^{1/2} U_e^3 \leq \text{const.} \left( \frac{Y \cdot 1.14 \, M_{\text{in}} \, \rho_0 \, U_{\infty}^2}{R_{\text{in}}} \right)^{1/2} \] (343)

Where \( \mu_c \) has been approximated as \( \mu_c = \frac{1}{2} \left( \frac{T_e}{T_{\infty}} \right) \) (with \( T_{\infty} \) the approximate post-shock static temperature \( T_e \), and \( T_{\infty} \sim \frac{U_{\infty}^2}{2 \, c_p T_{\infty}} \) at high Mach numbers \( M_{\text{in}} \gg 1 \).
Substituting the spacecraft velocity \((32.5)\) and \(\beta_0\) into \((343)\) yields

\[
\frac{q_{\text{nose}}''}{2 \varepsilon'^3} = \text{const} \left( \frac{10}{(v_0)^2} \right) e^{-\frac{3}{2} \frac{\beta_0^2}{U_e^2}} e^{-\frac{2}{3} \frac{5}{3} e^{-\frac{\beta_0^2}{U_e^2}}} \left[ \frac{\text{W}}{\text{m}^2} \right] \tag{344}
\]

Heat flux on the blunt nose

which becomes maximum at the altitude (assuming that \(\varepsilon, k_0, \text{and } T_0\) are nearly constant)

\[
2_3 = \frac{1}{\beta} \ln \left( \frac{3}{5} \right) = \frac{1}{\beta} \ln \left( \frac{3 \beta_0 C_D A}{\beta m \sin Y_e} \right) = \frac{1}{\beta} \left( 2 + \frac{1}{\beta} \ln 3 \right)
\tag{345}

\sim 7.4 \text{ km}

on Earth

\[
\text{Note: } \varepsilon = \frac{3}{4} k_0 \text{ for calorically perfect gas}
\]

where \(q_{\text{nose}}''\) becomes maximum and equal to

\[
q_{\text{nose max}}'' = \text{const} \left( \frac{\beta m \beta_0 \sin Y_e}{3 A C_D (C_T)^{1/2} R_0} \right) \left( \frac{U_e^2}{4} \right)
\tag{346}

In examining the above results it is worth highlighting the following aspects:

1) The altitudes for maximum surface-averaged heat flux \((2_3)\) and maximum heat flux on the blunt nose \((2_3)\) do not depend on the entry velocity. However, they both increase as the entry angle becomes smaller.

2) The maximum surface-averaged heat flux \(q_{\text{max}}''\) and the maximum heat flux on the blunt nose \(q_{\text{nose max}}''\) decrease as the entry angle decreases. This, together with a similar observation made for the maximum deceleration \(a_{\text{max}}/g\), suggest that shallow entries are preferable for manned missions on ballistic trajectories, for which low deceleration and heating rates are desirable.

3) In contradistinction, shallow entries tend to increase \(s\), and as a result the total heat input \(Q\) increases due to the longer re-entry time.

4) While \(q_{\text{max}}''\) and \(q_{\text{nose max}}''\) provide information about the type of material necessary for the type of material of the thermal protection system, the total heat input \(Q\) provides information about the material volume required, thereby indicating that shallow entries require bulky thermal protection systems.
OF RELATIVELY HEAT UNRESISTANT MATERIALS THAT CAN BE EASILY ABLATED, WHILE STEEP ENTRIES REQUIRE THIN HEAT SHIELDS OF HIGHLY HEAT-RESISTANT MATERIALS.

STEEP ENTRY: HIGH $\dot{\theta}_{\text{MAX}}^\prime$, HIGH $\dot{q}_{\text{NOSE}}^\prime$, LOW $\dot{Q}$

SHALLOW ENTRY: LOW $\dot{\theta}_{\text{MAX}}^\prime$, LOW $\dot{q}_{\text{NOSE}}^\prime$, HIGH $\dot{Q}$

5) THE CRITICAL ALTITUDES FOR MAXIMUM DECELERATION ($z_1$) AND MAXIMUM HEAT FLUXES ($z_2$) AND ($z_3$) ARE SEQUENTIAL

$$z_3 \quad > \quad z_2 \quad > \quad z_1$$

MAXIMUM $\dot{q}_{\text{NOSE}}^\prime$ MAXIMUM $\dot{\theta}^\prime$ 

MAXIMUM $d\dot{Q}/dt$ 

7.4 km 2.7 km.

6) LASTLY, AND MOST IMPORTANTLY, THE EXPRESSION (342) FOR THE TOTAL HEAT INPUT INTO THE SPACECRAFT QUANTIFIES THE FRACTION OF THE INITIAL KINETIC ENERGY THAT HAS BEEN TRANSFORMED INTO HEAT THAT REACHES THE SPACECRAFT. TO SEE THIS, NOTE THAT (342) CAN BE REWRITTEN USING THE IMPACT VELOCITY ($\dot{z}$) AS

$$\dot{Q} = \frac{1}{2} \left( \frac{C_F \dot{z}}{C_D A} \right) \frac{m}{(V_E^2 - U_z^2)} \quad , \quad (342)$$

INDICATING THAT A FRACTION EQUAL TO $\frac{1}{2} \left( \frac{C_F \dot{z}}{C_D A} \right)$ OF THE TOTAL KINETIC ENERGY LOST INTO HEAT HAS BEEN ABSORBED AS HEAT INTO THE SPACECRAFT STRUCTURE. IT IS FOR THIS REASON THAT, FOR THE SAME WETTED AREA AND SAME $C_F$, BLUNT DESIGNS (i.e., $C_D A$ LARGE) ARE EMPLOYED FOR MINIMIZING THE TOTAL AMOUNT OF HEAT ABSORBED, WHILE SLENDER DESIGNS (i.e., $C_D A$ SMALL) ARE USED FOR RAPID PENETRATION AT THE COST OF HIGH THERMAL LOADS (INCLUDING HIGH $\dot{q}_{\text{NOSE}}^\prime$).

THESE CONSIDERATIONS ARE DEPICTED IN FIG. 108 AND SERVE AS CLOSURE FOR THIS COURSE.
Homeworks and Exams
Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. In the calculations, assume a calorically perfect gas unless stated otherwise.

Student’s Name: JAVIER URZAY  Student’s ID: ........................................

Questions (40 pts)

1. Air flows isentropically at \( \text{Ma} = 5 \) at point 1 where the static temperature is \( T = 500 \) K and the stagnation pressure is \( P_0 = 0.5 \) bar. Compute the static temperature and pressure at a downstream point 2 along the same streamline where the Mach number is 8.

2. Sketch the hypersonic flow over a slender two-dimensional wedge of semi-angle \( \delta \) for which the shock is attached at a freestream Mach number \( \text{Ma}_1 \). Briefly describe the main characteristics of the flow and outline the thermochemical effects that may occur as a result of the high temperatures in the shock layer when \( \text{Ma}_1^2 \sin^2 \delta \gg 1 \).
**HW1 Solution**

**Question 1**

\[ H_1 = 5 \]
\[ T_1 = 500 \text{ K} \]
\[ P_0 = 0.5 \text{ bar} \]

\( M_a_2 = 8 \)

\[ T_2 = 2 \]

\( P_2 = \ ? \)

\[ P_0 \]
\[ \frac{P_0}{P_2} = \left(1 + \left(\frac{y-1}{2}\right)M_{a_2}^2\right)^{\frac{y}{y-1}} = 9762 \]

\[ P_2 = 5.12 \times 10^{-5} \text{ bar} \]

\( \frac{T_1}{T_2} = \frac{T_1}{T_0} \cdot \frac{T_0}{T_2} = \frac{1 + \left(\frac{y-1}{2}\right)M_{a_2}^2}{1 + \left(\frac{y-1}{2}\right)M_{a_1}^2} = 2.3 \)

\[ T_2 = 217.3 \text{ K} \]

**Question 2**

See class notes (Chapters 1 and 2)
Problem 1 (60 pts)

Consider the hypersonic waverider depicted below. Designs based on the waverider concept increase the lift-to-drag ratio in the following way. A shock wave emanates from the spatular nose and creates extra compression lift on the underside of the vehicle as it flies powered by a scramjet through the atmosphere. In this case, the flight speed is $U_\infty = 10,000$ km/h at a cruise altitude of 35 km in the stratosphere. The fuselage is made of the high-temperature alloy Inconel-X. The angle of attack is $\alpha_1 = 15^\circ$ with respect to the underside surface, the nose aperture angle is $\Omega = 5^\circ$, the length of the underside is $L_b = 1$ m, and the topside length satisfies $L_t = 3L_b/2$. Assume that the gas is calorically perfect with $\gamma = 1.4$.

![Diagram of hypersonic waverider](image)

a) Compute the lift-to-drag ratio using the shock-jump conditions along with the equations for the isentropic evolution of supersonic flows through expansion fans without making any hypersonic approximations.

b) Determine the maximum static temperature in the flow field and qualitatively discuss whether the fuselage would melt.

c) Redo part (a) employing the shock-jump conditions and expansion-fan relations derived in the limit of hypersonic Mach numbers.

d) Redo part (a) employing the standard Newtonian theory and compare the result to the lift-to-drag ratios obtained in parts (a) and (c).

e) Calculate the position of the center of gravity (CG) in the coordinate system $\{x, z\}$ along the dashed line shown in the figure in such a way that the waverider is aerodynamically trimmed.
Problem 1

At 35 km altitude: \( u_{10} = 300.8 \text{ m/s}, T_{10} = 256.5 \text{ K}, P_0 = 5.74 \text{ mbar} \) (US Standard Atmosphere)

\[ \Rightarrow \frac{u_{10}}{\sqrt{g_0 \rho_{10}}} = 9 \]

Also note the following trigonometric relations:

\[ L_2 \cos \theta = L_1 - L_1' \quad \Rightarrow \quad L_2 = L_1 \left( \frac{3}{2} \cos \theta \right) \]

\[ L_2 \sin \gamma = L_2' \sin \theta = \frac{L_2}{2} \quad \Rightarrow \quad \gamma = 90^\circ \text{, and } \theta = 180^\circ - \gamma - \angle = 165^\circ \]

\[ L_2 \sin \theta = 0.54 L_2 \]

(a) To compute the lift-to-drag ratio one needs the pressure distribution over the body.

From 1 to 4: Shock wave

\[ \begin{align*}
\theta_1 &= 15^\circ \\
M_1 &= 1.4 \quad \Rightarrow \quad \gamma = 75.9^\circ \\
M_2 &= 0.9 \\
\gamma = 9.3^\circ \quad \Rightarrow \quad \theta = 9.3^\circ + 10^\circ = 19.3^\circ \\
\theta_{10} &= 13^\circ \\
\end{align*} \]

Obvious shock chart:

\[ \begin{align*}
\theta_1 &= 15^\circ \\
\gamma &= 75.9^\circ \\
M_2 &= 0.9 \\
\theta &= 9.3^\circ + 10^\circ = 19.3^\circ \\
\theta_{10} &= 13^\circ \\
\end{align*} \]

Normal shock conditions:

\[ \begin{align*}
M_{1n} &= M_{1n}, \sin \gamma = 3.15 \\
\frac{T_1}{T_{10}} &= 2.8 \\
\frac{P_1}{P_{10}} &= 4.9 \\
\end{align*} \]

\[ \frac{C_{PL}}{C_{PW}} = \frac{U_2}{U_{10}} \left( \frac{P_2}{P_{10}} \right) \left( \frac{P_3}{P_2} - 1 \right) = 0.12 \]

Expansion fan relations:

\[ \begin{align*}
M_3 &= 4.9 \quad \Rightarrow \quad \gamma (M_3) = 75.9^\circ \\
\gamma (M_2) &= \gamma (M_{1n}) + \theta = 75.9 + 10^\circ = 85.9^\circ \\
\theta &= 90^\circ \quad \Rightarrow \quad M_3 = \frac{T_3}{T_{10}} \\
\end{align*} \]

Since the expansion is isentropic:

\[ \begin{align*}
\frac{P_3}{P_2} &= \left( \frac{4}{3} \left( \frac{1 + \gamma}{\gamma - 1} \right) \right)^{\gamma - 1} = 0.14 \\
\gamma (P_{1n} - P_3) &= \frac{2}{\gamma - 1} \left( \frac{P_3}{P_2} - 1 \right) = 0.004 \\
\end{align*} \]

Expansion fan relations:

\[ \begin{align*}
M_{1n} &= 9 \\
\gamma (M_{1n}) &= 99.3^\circ \\
\gamma (M_2) &= \gamma (M_{1n}) + \theta = 99.3 + 10^\circ = 109.3^\circ \\
\theta &= 90^\circ \\
\end{align*} \]

Since the expansion is isentropic:

\[ \begin{align*}
\frac{P_3}{P_2} &= \left( \frac{4}{3} \left( \frac{1 + \gamma}{\gamma - 1} \right) \right)^{\gamma - 1} = 0.025 \\
\gamma (P_{1n} - P_3) &= \frac{2}{\gamma - 1} \left( \frac{P_3}{P_2} - 1 \right) = 0.014 \\
\frac{T_3}{T_{10}} &= \left( \frac{P_3}{P_2} \right)^{\gamma - 1} = 0.014 \\
\end{align*} \]

The lift force is

\[ L = \frac{1}{2} C_{PL} L_2 \cos \kappa_1 + \frac{1}{2} C_{PW} L_2 \cos (\kappa_1 + \theta - 180^\circ) \]

The drag force is

\[ D = \frac{1}{2} C_{DG} L_2 \sin \kappa_2 + \frac{1}{2} C_{PW} L_2 \sin (\kappa_2 + \theta - 180^\circ) \]

\[ \Rightarrow \frac{L}{D} = \frac{C_{PL} \cos \kappa_1 - C_{PW} \left( \sin \frac{P_2}{\sin \frac{P_3}{L_2}} \right) \cos (\kappa_1 + \theta) + \frac{3}{2} C_{PL} \cos (\kappa_1 - \theta) - 0.152}{C_{DG} \sin \kappa_2 - C_{PW} \left( \sin \frac{P_2}{\sin \frac{P_3}{L_2}} \right) \sin (\kappa_2 + \theta) + \frac{3}{2} C_{PL} \sin (\kappa_1 - \theta) - 0.042} \]

\[ \Rightarrow \frac{L}{D} = 3.60 \]
b) The maximum temperature is \( T_1 = 2.8T_0 = 862 \, \text{K} \). The melting temperature of inconel-x is approximately 14700 K. Note however that no conclusions can be made about whether the fuselage would heat or not with the information provided in the problem statement. This is because the equilibrium wall temperature is not \( T_1 \), and it might be closer to the stagnation temperature \( T_0 = T_0 \left( \frac{\xi - 1}{\xi + 1} \right) \) of 2000 K multiplied by a recovery factor that is introduced in Chapter III. This would be the case for an adiabatic underside wall, but in reality the wall temperature may be much lower than that due to thermal radiation from the wall to the surroundings. In summary, this part of the exercise cannot be answered solely on the basis of an inviscid flow description.

c) Using the hypersonic limit relations.

From \( M_1 \) to 1: Shock wave

\[
M_{1w} = 9, \quad \frac{B_1}{A_1} = \left( \frac{Y + 1}{2} \right) \Rightarrow \quad M_{1m} = M_{1w} \cdot \frac{\sin \theta}{\cos \theta} = 2.98
\]

\[
\frac{P_1}{P_1} = \frac{2\beta}{\gamma + 1}, \quad \frac{T_1}{T_0} = \frac{2\beta (\gamma - 1) M_{1w}^2}{(\gamma + 1)^2} \Rightarrow \quad \frac{T_1}{T_0} = 0.38
\]

\[
\frac{M_{1m}}{M_{1w}} = \frac{\sin \theta}{\cos \theta} = 0.38
\]

\[
G_{P1} = \frac{2}{\rho M_{1w}^2} \left( \frac{P_1}{P_0} - 1 \right) = 0.44
\]

\[
\frac{P_1}{P_0} = \frac{2}{\gamma - 1} \left( \frac{\rho_1}{\rho_0} \right)^{Y-1} \frac{M_{1m}^2}{M_{1w}^2} = 11.7
\]

\[
G_{P2} = - \frac{2}{\rho M_{2w}^2} \left( \frac{P_2}{P_0} \right) = -0.017
\]

\[
G_{FW} = \frac{2}{\rho M_{FW}^2} \left( \frac{P_1}{P_0} - 1 \right) = -0.016
\]

Substituting \( G_{P1}, G_{P2}, \) and \( G_{FW} \) into (1) \( \frac{L}{D} = \frac{G_{P1} \cos \xi_1}{G_{P2} \sin \xi_1} = 3.85 \)

\[
\frac{L}{D} = \frac{G_{P1} \cos \xi_1}{G_{P2} \sin \xi_1} = \frac{2 \sin^2 \alpha_1 \cos \xi_1}{2 \sin \alpha_1 \cos \xi_1} = \cot \alpha_1 = 3.75 \Rightarrow \frac{L}{D} \text{ is only due to the one computed in parts (a) and (b).}
\]

\[
4.93 \text{ is only a function of the geometry!}
\]
AERODYNAMIC TRIM ~ ALL AERODYNAMIC FORCES YIELD ZERO MOMENT WITH RESPECT TO THE CENTER OF GRAVITY. SINCE THE PRESSURE LOADS ARE UNIFORM ON EACH PANEL OF THE FUZEAGE, THE CENTER OF PRESSURE OF EACH FORCE IS LOCATED AT THE GEOMETRIC CENTER OF EACH PANEL.

\[
\vec{F}_b \wedge (\vec{x}_b - \vec{x}_t) + \vec{F}_w \wedge (\vec{x}_w - \vec{x}_t) + \vec{F}_t \wedge (\vec{x}_t - \vec{x}_t) = 0 , \tag{2}
\]

WHERE

\[
\vec{x}_b = \frac{L_b}{2} \cos \alpha_b \hat{e}_x - \frac{L_b}{2} \sin \alpha_b \hat{e}_z
\]

\[
\vec{x}_w = \left( L_w \cos \alpha - \frac{L_w}{2} \cos (\alpha_1 + \Theta) \right) \hat{e}_x + \left( \frac{L_w}{2} \sin \alpha_1 + \frac{L_w}{2} \sin (\alpha_1 + \Theta) \right) \hat{e}_z
\]

\[
\vec{x}_t = \frac{L_t}{2} \cos (\alpha_1 - \gamma) \hat{e}_x - \frac{L_t}{2} \sin (\alpha_1 - \gamma) \hat{e}_z
\]

ARE THE LOCATIONS OF THE CENTERS OF PRESSURE ON EACH PANEL AND \( \hat{e}_x, \hat{e}_z \) ARE UNIT VECTORS. SIMILARLY:

\[
\begin{align*}
\vec{F}_b &= \frac{1}{2} \rho_0 U_0^2 \left[ \bar{C}_{D_b} L_b \sin \alpha_b \hat{e}_x + \bar{C}_{L_b} \frac{L_b}{2} \cos \alpha_b \hat{e}_z \right] \\
\vec{F}_w &= \frac{1}{2} \rho_0 U_0^2 \left[ \bar{C}_{D_w} L_w \sin (\alpha_1 + \Theta) \hat{e}_x - \bar{C}_{L_w} L_w \cos (\alpha_1 + \Theta) \hat{e}_z \right] \\
\vec{F}_t &= \frac{1}{2} \rho_0 U_0^2 \left[ \bar{C}_{D_t} L_t \sin (\alpha_1 - \gamma) \hat{e}_x + \bar{C}_{L_t} L_t \cos (\alpha_1 - \gamma) \hat{e}_z \right]
\end{align*}
\]

NUMERICALLY:

\[
\begin{align*}
\vec{x}_b &= \begin{pmatrix} 0.48 \hat{e}_x - 0.13 \hat{e}_z \\ 1.22 \hat{e}_x - 0.26 \hat{e}_z \\ 0.73 \hat{e}_x - 0.13 \hat{e}_z \end{pmatrix} \\
\vec{x}_w &= \begin{pmatrix} 0.05 \hat{e}_x + 0.17 \hat{e}_z \\ 1.0 \times 10^{-3} \hat{e}_x + 2.0 \times 10^{-3} \hat{e}_z \\ -4.2 \times 10^{-3} \hat{e}_x - 0.02 \hat{e}_z \end{pmatrix}
\end{align*}
\]

AND THEREFORE (2) BECOMES

\[
\begin{align*}
0.05 \begin{pmatrix} -0.13 \hat{e}_x - 0.17 \hat{e}_z \\ 0.48 - x_b \\ 7.1 \times 10^6 (-0.26 - x_c) + 2.0 \times 10^{-3} (1.22 - x_c) - 4.2 \times 10^{-3} (-0.13 - x_c) \end{pmatrix} - 0.02 \begin{pmatrix} 0.92 - x_b \end{pmatrix} = 0 \Rightarrow -0.10 - 0.05 x_c + 0.19 x_c = 0 \tag{8} \quad \text{WITH } x_c, x_c \text{ IN } [\text{m}]
\end{align*}
\]

ALSO, SINCE THE CG IS LOCATED ALONG THE MIDLINE \( \tilde{Z}_{CG} = -L_c \left( \frac{\pi}{2} - \frac{\Omega}{2} \right) x_c = -0.25 x_c \). \tag{4}

THEN (3) AND (4) YIELD

\[
\begin{align*}
x_c &= 0.50 \text{ m} \\
2c_c &= -0.11 \text{ m}
\end{align*}
\]
Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. In the calculations, assume a calorically perfect gas with $\gamma = 1.4$ unless stated otherwise.

Student's Name: Javier Urzay

Questions (100 pts)

1. A supersonic air stream at $Ma_\infty = 3.0$, $T_\infty = 300$ K and $P = 1$ atm flows over a two-dimensional wedge of semi-angle $\delta = 20^\circ$. Calculate the angle of incidence $\beta$ of the resulting oblique shock and compare it against the one obtained if the wedge is replaced by a cone of semi-angle $\delta = 20^\circ$ (to answer this, you could compute the numerical solution of the conical flow or you could also research the class notes). Explain whether the flow is hypersonic and the reasons that justify your conclusion.

2. In the figure below, the surface pressure coefficient at point Q in the fuselage of projectile #1 is $C_p = 0.3$. Using van Dyke's hypersonic similarity rule studied in class, compute the surface pressure coefficient at point Q in the fuselage of projectile #2. In the calculations, use the aspect ratio $c_1/\lambda = 0.2$ for projectile #1, with $c_2 = c_1/2$ for projectile #2.

3. Describe the character of the Mach-8 flow around a two-dimensional slender wedge of semi-angle $\delta$ as $\delta$ is increased from $2^\circ$ to $50^\circ$, indicating whether the flow is subsonic, supersonic, hypersonic, and whether it can be described using small-disturbance theories for the velocity perturbations.

4. Compare the hypersonic wave drag coefficients on a circular cylinder of radius $R$ at Mach $Ma_\infty = 7.0$ at 35 km altitude obtained using the straight Newtonian theory and the modified Newtonian theory, and provide an estimate of the pressure in the fore stagnation point in each case.

5. Explain the reasons why the straight Newtonian theory is more accurate in three-dimensional flows (i.e. conical flows) than in two-dimensional flows (i.e. wedge flows).
1. From the $\beta = 8$ chart $\Rightarrow \beta = 3^\circ$ for an oblique shock.

2. For a cone, from Fig. 29 in the class notes $\Rightarrow \beta = 30^\circ$

3. In both cases, the flow is not hypersonic since $M_a \sin^2 \beta$ is not a number much larger than unity.

4. Protocol #1: $\gamma = 0.2$, $\varepsilon_1 = 0.2$, $M_a = 6$, $Y = 1.4$

Protocol #2: $\gamma = 0.2$, $\varepsilon_2 = 0.4$, $M_a = 12$, $Y = 1.4$

$\Rightarrow k = \varepsilon_1 M_a = \varepsilon_2 M_a = 1.2$, the similarity rule can be applied.

5. $M_a = \csc \beta (\sqrt{2}) = 7.1^\circ$, $\varepsilon = 2^\circ$ or $5^\circ$

6. Linear supersonic flow

7. Hyperbolic wave (Mach waves)

8. Weak shocks

9. Hyperbolic flow (small disturbances)

10. Hyperbolic flow with detached shock (large disturbances)

11. Straight Newtonian theory:

12. Modified Newtonian theory: $q^* = \frac{q_{\text{max}}}{\cos \beta}$

13. Regular Newtonian theory underpredicts the drag and pressure.

14. Because the cone flow leads to a weaker shock that envelops more closely the body surface due to a 3D revolving effect.
ME 356: Hypersonic Aerothermodynamics, Spring 2018
Stanford University
Homework 3: Hypersonic Viscous Flows
Due Tuesday, May 22, in class.

Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. In the calculations, assume a calorically perfect gas with $\gamma = 1.4$, $R_g = 286$ J/kgK, and $c_p = 1$ kJ/kgK unless stated otherwise.

Student's Name: JAVIER URZAY
Student's ID: 

Questions (50 pts)

1. In his autobiography “The Wind and Beyond” (1967), pp. 342-344, Theodore von Kármán writes:

   Space law has occupied the attention of a number of prominent legal experts, and as a matter of fact the space lawyers now have their own organization known as the International Institute of Space Law. I have been intrigued with their attempts at determining the areas of sovereignty in space, and indeed I played a small part in the early deliberations. Andy Haley has given earnest thought to the subject and has discussed it with me many times.

   On one such occasion — interestingly, it occurred just after I had appropriately received an honorary doctor of laws degree from the University of California at Los Angeles — I made an attempt to assist Haley in defining the possible jurisdiction of space law. The first question naturally is, where does space begin? This is not just a technical question; it has military and political significance. In the analogous case of the ocean, for instance, it took much labor and controversy over the years to establish that the offshore zone of some five to twenty-five kilometers is controlled by the adjacent nation. Beyond that, the seas are free.

   Now in the case of air, how far up should we go? Forty-five years ago, at the birth of flight, I participated in a congress in Frankfurt which met to establish air rights over Europe. A number of jurists present said that the German airspace consisted of a cylindrical volume rising above Germany. No upper limit was set. They forgot that the earth is round. If a cylinder of air exists over Germany, then a similar one exists over France, and who has sovereignty over the conical gap between these cylinders? The jurists finally modified their original language and turned the cylinder into a cone. But they failed to settle the question of how high up the cone of air should go.

   I hope I shall not fall into similar troubles in calculating where space begins. This can actually be determined from the speed of the space vehicle and its altitude above the earth. Consider, for instance, the record flight of Captain Ivan Kurchatov in an X-2 rocket plane. Kurchatov flew 2000 miles per hour at 126,000 feet, or 24 miles up. At this altitude and speed aerodynamic lift still carries 98 per cent of the weight of the plane, and only two per cent is carried by centrifugal force, or the Kepler Force, as the space scientists call it. But at 300,000 feet or 57 miles up this relationship is reversed because there is no longer any air to contribute lift. Only centrifugal force prevails. This is certainly a physical boundary, where aerodynamics stops and astronautics begins. And so I thought why should it not also be a jurisdictional boundary? Haley has kindly called it the Karman Juralistic Line. Below this line space belongs to each country. Above this level there would be free space.

   Not everybody agrees with this division, however. Some think that the line could be put lower than 64 miles above the earth’s surface Major Robert White of the U.S.A.F., who flew the X-15 in a number of tests, reports that at 246,700 feet (47 miles) and 314,750 feet (50 miles) he did not have enough aerodynamic lift to maneuver the aircraft, and was actually maintaining the plane in space. He thinks the boundary should be between 45 and 55 miles.

   Professor John C. Cooper of Princeton, another distinguished space lawyer, has proposed another way of looking at space jurisdiction. Professor Cooper’s view is that instead of arbitrarily selecting 64 miles as the boundary for legal purposes, we can extend the sovereignty of a state upward as far as the scientific progress of that state can take it. He suggests three sovereign zones of airspace: (1) atmospheric space, up to the highest altitude at which aircraft can be operated; this zone would be under the full control of the state under it; (2) 300 miles above the earth’s surface, or what he calls contiguous space; this zone would permit transit of all nonmilitary vehicles, and (3) all space above that, which is free for the "passage of all instrumentalities."

   I am sure that these knotty problems of law will not be solved in my lifetime. In any case I prefer not to speculate in this area but to return to the relative ease and comfort of solving purely scientific questions, or of looking into the future of scientific developments.
In the calculations below, use the following data for the Bell X-2: mass \( m = 6,250 \text{ kg} \), planform wing area \( A/2 = 24 \text{ m}^2 \), and lift coefficient \( C_L = 0.8 \). The sea-level density is \( \rho_0 = 1.22 \text{ kg/m}^3 \), and the scale height is \( 1/\beta_\oplus = 22,000 \text{ ft (6.7 km)} \).

a) Determine the flight speed \( U_\infty \) and Mach number of the X-2 relative to the surface of the Earth for constant-altitude powered flight at 126,000 ft.

b) In the conditions calculated in part a), compute the ratio of the Kepler force to the lift force, and compare it with the estimate provided by von Kármán. Assume an eastward flight direction and note that the peripheral velocity of the surface of the Earth at a latitude corresponding to Edwards AFB (34.92° N) is approximately \( U_\oplus = 380 \text{ m/s} \).

c) At an altitude of 300,000 ft, calculate the flight speed and Mach number relative to the surface of the Earth required for the Kepler force to balance the weight of the aircraft.

d) In the conditions calculated in part c), compute the ratio of the lift force to the Kepler force and and compare it with the estimate provided by von Kármán.

e) More generally, plot in a velocity vs altitude (log-log scale) diagram the dimensionless flight speed \( \bar{U}_\infty \) normalized with the circular orbital velocity \( \sqrt{\mu_\oplus/R_\oplus} \) (with \( \mu_\oplus \) the standard gravitational parameter and \( R_\oplus \) the radius of the Earth) required for constant-altitude flight as a function of the dimensionless altitude \( \bar{z} \) normalized with the terrestrial scale height, and prove that the lift and Kepler forces become of the same order at a dimensionless critical altitude \( \bar{z}_1 \sim \ln(\beta_\oplus R_\oplus A) \) in the mesosphere provided that \( \beta_\oplus R_\oplus > 1 \) and \( U_\oplus \sqrt{R_\oplus/\mu_\oplus} \ll 1 \), where \( \Lambda = \rho_0 C_L A/(2\beta_\oplus m) \). Show that in the present problem \( z_1 \sim 221,000 \text{ ft (67 km)} \).

f) The curve in the velocity vs altitude diagram resulting from your analysis in part e), which should approximately resemble the red line in Figure 1 below, is described by Sänger (1957) as the limit of aerodynamic lifting power, indicating the lowest possible velocity for continuous flight at the corresponding altitude. This curve bends up rapidly at the critical altitude \( z_1 \) and becomes almost vertical near a higher altitude denoted here by \( z_2 \), where the curve attains infinite slope. There, the Kepler force balances the gravitational force and the ambient density is sufficiently small to make the lift force completely inconsequential for the aircraft (now spacecraft) dynamics, which follows a circular orbit around the Earth. At higher altitudes, where \( z \) becomes a significant fraction of \( R_\oplus \), the curve bends back towards lower flight speeds in accordance with the square-root decay of the orbital velocity with distance from Earth. Higher velocities obtained by rocket burns at these orbital altitudes would cause the spacecraft to depart from Earth along an ascending conical trajectory into outer space.

Show that the critical dimensionless altitude \( \bar{z}_2 \) where the curve \( \bar{z} = \bar{z}(\bar{U}_\infty) \) attains infinite slope is given by

\[
\bar{z}_2 \sim \ln \left[ \Lambda (\beta_\oplus R_\oplus)^2 \right],
\]

where use of the approximations \( z_2/R_\oplus \ll 1 \) and \( U_\oplus \sqrt{R_\oplus/\mu_\oplus} \ll 1 \) have been made. At the altitude \( z_2 \), the lift force is smaller than the Kepler force by a factor of order \( (\beta_\oplus R_\oplus)^{-1} \), which in the present calculations is of order 0.1% (in contrast to the 2% arbitrarily delineated by von Kármán as the limit percentage for the jurisdictional line). In addition, \( z_2 \) is shifted above \( z_1 \) by a distance of order \( \beta_\oplus^{-1} \ln(\beta_\oplus R_\oplus) \sim 151,000 \text{ ft (46 km)} \), thereby yielding \( z_2 \sim 372,000 \text{ ft (113 km)} \) in the present problem, as indicated by the blue horizontal line in the figure below. Note that the estimate of \( z_2 \) in Eq. (1) is observed to depend only weakly (i.e., logarithmically) on the aircraft parameters \( m, A \) and \( C_L \), which are accounted for in the non-dimensional parameter \( \Lambda \), thereby making the critical altitude \( \bar{z}_2 \) a relatively robust figure that varies by \( \pm 5 \text{ km} \) upon varying the aircraft parameters by 50%. 

2
Figure 1: A modified version of Sänger's diagram of flight regimes (1957), including the limit of aerodynamic lifting power (red solid line), the limit altitude predicted by von Kármán (red dashed line) along with the values of $z_1$ (black dashed line) and $z_2$ (blue solid line) calculated above. For illustration, the figure also shows the region of hypersonic flight.

Problems (50 pts)

1. Download the Matlab code posted at
   
   http://web.stanford.edu/~jurrzy/ME356_files/HW3code.tar.gz

   for integrating the problem of a compressible laminar boundary layer over a flat plate with zero streamwise gradients in the overriding free stream. In particular, the formulation numerically integrated by the code consists of Eqs. (223)-(226) in the class notes, with the viscosity following a temperature power law with exponent 0.75, while the Prandtl number is set equal to 0.72. Assume that the edge conditions involve a Mach number $Ma_e = 5.0$, a dynamic viscosity $\mu_e = 1.2 \cdot 10^{-5}$ Ns/m², a static temperature $T_e = 220$ K, and a static pressure $P_e = 10$ mbar, which are typical free-stream conditions during stratospheric flight. Use (and modify accordingly) this code to answer the following questions:

   a) Determine the adiabatic wall temperature $T_{aw}$ and the recovery factor $r$. State whether the approximation $r \approx \sqrt{Pr}$ is a good one for these conditions.

   b) For $Re_{e, x} = 10,000$, plot the dimensional values of the static temperature $T$, stagnation temperature $T_o$, and streamwise velocity $u$ versus the dimensional physical wall-normal distance $y$ in the three following cases: $T_u/T_e = 4$, $T_u/T_e = 8$ and $T_u = T_{aw}$, and explain the influences of these choices in the change of direction of the heat flux at the wall. Note: provide the plots of $T$ and $T_o$ vs $y$ in a single figure for the three cases, and the plots of $u$ vs $y$ in another single figure for the same three cases.

   c) For $Re_{e, x} = 10,000$, compute the skin-friction coefficient $C_f$ and the Stanton number $St$ for the same cases $T_u/T_e = 4$, $T_u/T_e = 8$ and $T_u = T_{aw}$ as in part b).

Attach a screenshot of your modified version of the main.m script with your solutions.
1. \( A = 48 \text{ m}^2, \ G_L = 0.8, \ m = 6250 \text{ kg}, \ \beta_0 = 1.22 \text{ kg/s}^3, \ \beta = 22 \text{ kft} \)

\[ L = \frac{1}{2} \beta_0 G_L A \beta_0^2, \ \text{with} \ \beta_0 = \beta_0 \frac{A}{R_E} \]

\[ K = \frac{(\beta_0 + \beta_E)^2}{(R_E + z)^2}, \ W = \frac{\mu_m m}{(R_E + z)^2} \]

\[ \beta_0 = 3.98 \times 10^4 \text{ m}^3/\text{s}^2, \ \beta_E = 380 \text{ m/s} \]

\[ q_0 = \frac{\beta_0}{R_E} = 9.8 \text{ m/s}^2 \]

\[ R_E \approx 6371 \text{ km} \]

\[ L \approx W \Rightarrow \ \beta_0 \approx \left( \frac{q_0 \beta_0}{\beta_0 G_L A} \right)^{\frac{1}{2}} = 898.8 \text{ m/s} \]

\[ \text{AND} \ \beta_{\infty} = \frac{\beta_0}{q_0} = \frac{2.84}{4} \text{ US STANDARD ATMOSPHERE} \]

\[ \beta_{\infty} = 312.7 \text{ m/s} \]

b) \[ \frac{K}{L} = 2.6\% \]

c) \[ \beta_{\infty} = 300 \text{ kft} \text{ (91 km)} \]

\[ \beta_{\infty} \approx \beta_E \Rightarrow \beta_0 \approx \frac{(\beta_0 + \beta_E)^2}{(R_E + z)^2} \]

\[ \beta_0 \approx \left( \frac{\mu_m m}{(R_E + z)^2} \right)^{\frac{1}{2}} \Rightarrow \beta_0 \approx \beta_E \left( \frac{\mu_m m}{(R_E + z)^2} \right)^{\frac{1}{2}} \]

\[ \Rightarrow \beta_0 \approx 7.8 \text{ km/s} \]

\[ \text{AND} \ \beta_{\infty} = \frac{\beta_0}{q_0} = \frac{28}{\text{ US STANDARD ATMOSPHERE}} \]

\[ \beta_{\infty} = 272 \text{ m/s} \]

d) \[ \frac{L}{K} = 2.6\% \]

e) **General Force Balance**

\[ \frac{1}{2} \beta_0 G_L A \beta_0^2 + m (\beta_0 + \beta_E)^2 = \frac{\mu_m m}{(R_E + z)^2} \]

Assuming \( \beta_E < \beta_0 \Rightarrow \beta_0 \approx \beta_0 \approx \left( \frac{\beta_0 + \beta_E}{(R_E + z)^2} \right)^{\frac{1}{2}} \]

\[ \beta_0 \approx \left( \frac{\beta_0 + \beta_E}{(R_E + z)^2} \right)^{\frac{1}{2}} \]
\[
\tilde{U}_\infty = \frac{U_\infty}{\sqrt{\frac{k_\infty}{R_E}}} \quad \tilde{z} = \beta \tilde{x} \quad \lambda = \frac{S_0}{L_0} \frac{A}{A_L} = 25.1 \quad \text{Note also that } \beta R_E \gg 1
\]

\[
\Rightarrow \quad \tilde{U}_\infty (\tilde{z}) = \left( \frac{\lambda}{(\beta R_E + \tilde{z})} \right) \left( \frac{1 + \lambda (\beta R_E + \tilde{z}) e^{-\tilde{z}^2}}{(\beta R_E + \tilde{z})} \right)^{1/2}
\]

Plot:

For \( L \approx L \Rightarrow \lambda e^{-\tilde{z}^2} \approx \frac{1}{(\beta R_E + \tilde{z})} \Rightarrow \tilde{z} = \ln \left( \lambda \beta R_E \right)
\]

\[
\frac{d\tilde{U}_\infty}{d\tilde{z}} = 0 \Rightarrow -1 + \left[ \lambda (\beta R_E)^2 - 2\lambda \beta R_E - 2 \lambda \tilde{z} + 2 \lambda (\beta R_E)^2 \tilde{z} + \lambda \beta R_E \tilde{z}^2 \right] e^{-\tilde{z}^2} = 0
\]

Negligible

\[
\Rightarrow \tilde{z} = \ln \left( \lambda (\beta R_E)^2 \right)
\]

Dimensional: \( \tilde{z} = 372,000 \text{ ft} \quad (113 \text{ km})
\]

Problem 1

a) \( T_{aw} = 5206 \text{ K} \quad T_e = 1445 \text{ K} \quad \Rightarrow \quad \theta = \frac{T_{aw} - T_e}{T_e - T_{aw}} = \frac{2}{(6-1) \alpha_0} (T_{aw}/T_e - 1) = 0.844
\)

Which is similar to \( \theta = (Pr)^{1/3} = 0.848 \)

b) \( \text{Re}_{\theta} = 10^{10} \Rightarrow x = 5.02 \text{ mm} \quad \text{Physical coordinate} \quad \gamma = x \sqrt{\frac{2}{\text{Re}_{\theta}}} \int_0^1 \rho_0 (\beta) d\beta \quad \text{Plots provided below.}
\]

c) For \( T_w = 4 \), \( \xi_f = 0.0057 \), \( S_b = 0.0035 \)

\( T_n = T_{aw} \), \( \xi_f = 0.0056 \), \( S_b = 0 \)

\( \Rightarrow \quad \xi_f = 0.0054 \), \( S_b = 0.0033 \)
solve similarity equations for compressible flow
Here Re is based on distance from leading edge
written by Michael Karp (CTR) 2018
clc;
clear all;
close all;
format long

% code parameters
M = 5.0;       % Edge Mach number
flag_ad = 1;   % Flag for wall thermal boundary condition:
               % 1 = adiabatic wall; 0 = isothermal wall;
Tw = 4;       % dimensionless wall temperature (not used in adiabatic case)
Pr = 0.72;    % Prandtl number
n = 0.75;     % power law coefficient for viscosity f = \frac{f}{f_0} = (T/T_0)^n
gamma = 1.4;  % adiabatic coefficient
etamax = 10;  % maximum eta coordinate for integration

% solve boundary layer equations in similarity form:
[eta, f] = solve_comp(M, flag_ad, Tw, Pr, gamma, n, etamax);

% get variables:

u = f(:, 2);     % dimensionless velocity
T = f(:, 4);     % dimensionless temperature
rho = 1./T;      % dimensionless density

% dimensionless adiabatic temperature (if flag is 1)
Taw = 5.206127541428510;

% plot the selfsimilar solution f'(eta), g(eta)
figure(1);
plot(u, eta, 'r'); hold on;
plot(T, eta, 'b');
xlabel('f' (red) and g (blue)');
ylabel('eta (self-similar variable)');

Re_x = 10000;    % Reynolds based on x
mue = 1.2e-5;    % BL edge viscosity
Rg = 286;        % gas constant
Te = 220;        % BL edge temperature
Pe = 10e-3*101325;  % BL edge pressure
ae = sqrt(1.4*286*Te);  % BL edge speed of sound
Ue = M*ae;       % BL edge velocity
cp = 1e3;        % specific heat
rhoe = Pe/(Rg*Te);  % BL edge density
x = Re_x*mue/(Ue*rhoe);  % dimensional x

% dimensional wall normal distance
y = x*cumtrapz(eta, f(:, 4))*sqrt(2/Re_x);

% plot u, T and To
figure(2);
plot(u*Ue, y, 'r-'); hold on;
xlabel('u');
ylabel('y');

figure(3);
plot(T*Te,y,'b-.'); hold on;
plot(T*Te+(u*Ue).^2/(2*cp),y,'k-.'); hold on;
xlabel('T (blue) and T0 (black)');
ylabel('y');

% Friction coefficient
CW=f(1,4)^(n-1); % Chapman-Rubesin parameter at the wall \( g(0)^{(n-1)} \)
CF=sqrt(2)*CW*f(1,3)/sqrt(Re_x);

% Stanton Number
CW=f(1,4)^(n-1); % Chapman-Rubesin parameter at the wall \( g(0)^{(n-1)} \)
St=(1/sqrt(2))*(CW/Pr)/(Taw-Tw)*f(1,5)/sqrt(Re_x)
ME 356: Hypersonic Aerothermodynamics, Spring 2018  
Stanford University  
Homework 4: Thermochemical effects and re-entry trajectories  
Due Tuesday, June 5, in class.

Guidelines: Please turn in a neat and clean homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. In the calculations, assume a calorically perfect gas with \( \gamma = 1.4 \), \( R_g = 286 \) J/kgK, and \( c_p = 1 \) kJ/kgK unless stated otherwise.

Student’s Name: \textit{Javier Urzay}................. Student’s ID:..............................

Questions (100 pts)

1. What are the main differences between a calorically perfect gas, a thermally perfect gas, and a chemically reacting mixture of ideal gases?

2. Briefly describe the general trends observed in the post-shock thermodynamic variables when thermochemical equilibrium is included in the calculation of normal shock waves in air at hypersonic speeds. Illustrate your response in the case of a pre-shock velocity \( U_1 = 18,000 \) ft/s at an altitude of 200,000 ft in the atmosphere.

3. What are the possible sources of thermodynamic non-equilibrium and chemical non-equilibrium that may exist in the post-shock air behind a normal shock wave at hypersonic velocities? Describe the structure of a normal shock wave in non-equilibrium flow and the flight conditions that exacerbate the occurrence of non-equilibrium phenomena in hypersonics.

4. Estimate the distance required for equilibration of the vibrational molecular motion in pure O\(_2\) gas behind a Mach-8 normal shock in air at a pre-shock pressure \( P_1 = 15 \) mbar and temperature \( T_1 = 230 \) K. How many collisions have the gas molecules undergone along the equilibration distance?

5. Consider the re-entry of a spherical projectile in the terrestrial atmosphere. The projectile has a radius \( R = 20 \) cm and is made of a heat-resistant alloy of density \( \rho_s = 10,000 \) kg/m\(^3\). It enters the atmosphere at a velocity \( V_E = 15 \) km/s at an angle \( \gamma_E = 45^\circ \) with respect to the local horizontal. Calculate the maximum acceleration of the projectile normalized with the acceleration of gravity, the altitude at which the maximum acceleration is attained, the kinetic energy upon collision with the ground, and the percentage of the initial kinetic energy that has been dissipated into heat during the re-entry. In your calculations, use the Newtonian theory of hypersonics to estimate the drag coefficient of the projectile.
1. See pages 90-96 of the class notes.

2. See Fig. 82 on page 108 of the class notes. Use Fig. 83 with $U_1 = 18 \text{ km/s}$ and $2 = 900 \text{ km}$. (The actual post-shock temperature is not stated.)

3. See Fig. 84, and pages 113-114.

4. $M_1 = 8$, normal shock conditions,

\[ \frac{T_2}{T_1} = \frac{P_2}{P_1} = 74.5 \Rightarrow P_2 = 1.1 \text{ bar} \]

\[ \frac{T_2}{T_1} = 13.4 \Rightarrow T_2 = 3082 \text{ K} \]

Then

\[ \frac{C_{vib}}{P} = 0.96 \text{ m/s} \]

\[ C_{vib} = 5.4 \times 10^{-5} \text{ m/s} \]

\[ k_{vib} = 2.9 \times 10^6 \text{ K} \]

From Table V in the class notes.

To estimate the distance, use $C_{vib} \approx U_2^2 v_{vib}$, with

\[ \frac{U_2}{U_1} = 0.18 \text{ at } M_2 = 8 \quad \left( U_1 = 2427.9 \text{ km/s} \right) \]

\[ \Rightarrow \quad l_{vib} \approx 437 \text{ m/s} \cdot 0.4 \text{ mm} = 0.4 \text{ mm} \]

\[ \text{Note that } T_2 = 3082 \text{ K is larger than the actual post-shock temp. in equilibrium and therefore } 0.4 \text{ mm may underestimate the equilibrium distance.} \]

5. \[ 2R = 40 \text{ cm} \]

a) Max. Decceleration:

\[ \frac{p_2}{p_0} = 10^{-3} \Rightarrow \frac{p_2}{p_0} = 0.01 \text{ m} \]

\[ U_2 = 15 \text{ km/s}, \quad Y_2 = 45^\circ \]

**Ballistic Energy**

b) \[ \frac{3}{2} = \frac{\beta M \sin Y E}{\gamma_0 A C_D} = 0.022 \]

\[ \gamma_0 = \frac{4}{3} \pi r^2 = 33 \text{ K} \]

\[ A = \pi r^2 = 0.13 \text{ m}^2 \]

\[ C_D = 1 \text{ (see midterm exam problem #4)} \]

\[ \frac{3}{2} = 1.22 \text{ km/s}^2 \]

**b) Altitude for Max. Decceleration:**

\[ z_1 = \frac{1}{\beta} \ln \left( \frac{A}{J} \right) = 10 \text{ km} \]

\[ \text{c) Velocity at Impact: } \quad U_1 = U_E C \]

\[ \frac{A}{2} = 1577 \text{ m/s} \]

\[ \Rightarrow \quad \frac{\frac{1}{2} m U_1^2}{2} = 0.41 \text{ GJ} \quad \text{Kinetic Energy at Impact} \]

\[ \text{d) } \Delta = \frac{m U_2^2/2 - m U_2^2}{m_0 U_2^2/2} = 1 - \frac{U_2^2}{U_2^2} = 93.9 \% \quad \text{Kinetic Energy Dissipated into Heat.} \]
Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student’s Name: JAVIER UREAY.......................... Student’s ID:..........................

PART I: Closed notes, calculators allowed, compressible-flow tables allowed
Time: 20 mins

Question (40 pts)
1. The velocity/altitude diagram below shows three different re-entry trajectories across the terrestrial atmosphere.

a) (20 pts) Determine what type of aerospace vehicle (intercontinental ballistic missile, manned re-entry winged vehicle, or experimental hypersonic cruise aircraft) is most likely associated with each trajectory and justify your response.

b) (20 pts) Sketch on the diagram the approximate boundaries for the onset of the characteristic thermochemical effects arising at hypersonic flight speeds.

Notes: 1 kft = 304.8 m; 1 kft/s = 304.8 m/s = 681.8 mph = 1097.3 km/h; speed of sound at 35 km altitude ~ 1 kft/s; temperature at 35 km altitude ~ 236 K.

For more details in the solution, see pages 5–6 in the class notes.
Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student’s Name: JAVIER URZAY
Student’s ID: 

PART II: Open notes, calculators allowed, compressible-flow tables allowed
Time: 60 mins

Problem 1 (40 pts)

a) (30 pts) Using the straight Newtonian theory of hypersonic flows, in which the shock layer is infinitesimally thin, obtain expressions for the drag and lift coefficients on the sphere of radius \( R_p \) sketched in Fig. 1 (use the spherical coordinate system shown in the figure to perform the corresponding integrations of the surface pressure coefficient \( C_p \) along the spherical surface).

b) (10 pts) The experimental drag coefficient measured by Hodges (1957) corresponds to \( C_D = 0.91 \). Compare this value to the one obtained in part (a) and identify possible sources of discrepancy between both results.

![Figure 1: Hypersonic flow over a sphere in the Newtonian theory.](image-url)
a) By symmetry, \( \zeta_L = 0 \)

For \( \zeta^l \) one needs to integrate \( \zeta_P = 2\sin^2\theta \) over the element of surface projected on the \( z \) direction and divide it by the frontal area in the \( z \) direction.

\[
\frac{2\pi}{R_L^2} \int_0^{\pi/2} R_P^2 \sin \theta \cos \theta d\theta d\phi = \frac{2\pi}{R_L^2} \int_0^{\pi/2} R_P^2 \sin \theta \cos \theta d\theta d\phi
\]

\[
= \frac{2\pi R_P^2}{R_L^2} \int_0^{\pi/2} \cos \theta \sin \theta d\theta = \frac{2\pi R_P^2}{R_L^2} \left[ \frac{\cos^2 \theta}{2} \right]_0^{\pi/2}
\]

\[
= \frac{\pi R_P^2}{R_L^2}
\]

b) \( \zeta^l \) experimental = 0.91 < \( \zeta^l \) Newton = 1.0 as expected since the straight Newtonian theory overpredicts the pressure in the fore stagnation point (see discussion on pages 17-20 of the class notes).
Problem 2 (20 pts)

Consider the flow schematically illustrated in Fig. 2, where a high-speed gaseous stream at velocity $U_1$, pressure $P_1$ and density $\rho_1$ crosses a two-dimensional stationary curved shock wave. The velocity and thermodynamic variables of the pre-shock gases, including the entropy $s$, are uniform. As a result, in the pre-shock gases, Crocco’s equation

$$ T\nabla s = \omega \times u, \quad (1) $$

requires that the vorticity $\omega$ be zero there. In Eq. (1), $u$ is the velocity vector.

Since the shock wave is curved, different streamlines experience different jumps of entropy across the shock. As a result, the post-shock gas has constant entropy along the streamlines but the entropy varies from streamline to streamline, thereby rendering an isentropic flow. Because of Crocco’s equation (1), however, the post-shock gas is rotational.

The goal of this problem is to obtain an expression for the vorticity $\omega_2$ in the post-shock gas immediately behind the shock in the hypersonic limit. To this end, follow the steps below.

![Diagram of hypersonic flow across a two-dimensional curved shock wave.](image)

Figure 2: Hypersonic flow across a two-dimensional curved shock wave.

a) (4 pts) Obtain an expression for the derivative of the post-shock pressure $P_2$ with respect to the local incidence angle $\beta$ using the Rayleigh line

$$ \frac{P_2 - P_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} = -\dot{m}'' \omega_2, \quad (2) $$

where $\dot{m}'' = \rho_1 U_1 \sin \beta$. 

ME 356 Hypersonic Aerothermodynamics 3
b) (4 pts) Obtain an expression for the derivative of the post-shock static enthalpy \( h_2 \) with respect to the local incidence angle \( \beta \) using the Hugoniot equation

\[
h_2 - h_1 = \frac{P_2 - P_1}{2\rho_1} \left( 1 + \frac{\rho_1}{\rho_2} \right).
\]

(3)

c) (4 pts) Utilize the expressions for \( dP_2/d\beta \) and \( dh_2/d\beta \) derived above, along with first principle of thermodynamics \( T_2ds_2 = dh_2 - dP_2/\rho_2 \) in the post-shock gas, and show that the derivative of the post-shock entropy \( s_2 \) with respect to the local incidence angle \( \beta \) of the shock is given by

\[
T_2 \frac{ds_2}{d\beta} = U_\infty^2(1-\epsilon)^2 \sin \beta \cos \beta,
\]

where \( \epsilon = \rho_1/\rho_2 \) is the density ratio across the shock.

d) (4 pts) Crocco's equation (1) states that the gradient of the post-shock entropy across the streamlines is related to the cross product of the post-shock velocity \( (U_2) \) and vorticity \( (\omega_2) \), namely

\[
T_2 |\nabla s_2| = \omega_2 U_2.
\]

(5)

Equivalently, since the post-shock flow is isentropic, \( \nabla s_2 \) is perpendicular to the streamlines, and correspondingly the derivative of the post-shock entropy along the direction \( \delta \) locally parallel to the shock is

\[
T_2 \frac{ds_2}{d\delta} = T_2 |\nabla s_2| \sin(\beta - \delta),
\]

(6)

where \( \delta \) is the local deflection angle of the streamlines, as shown in Fig. 2.

Using Eqs. (4)-(6), show that the vorticity immediately behind the shock is given by the expression

\[
\omega_2 = U_\infty \frac{(1-\epsilon)^2}{\epsilon} \cos \beta \frac{d\beta}{d\delta}
\]

(7)

where the direction of the vorticity points into the plane in Fig. 2 when \( d\beta/d\delta < 0 \) (as in the upper portion of a bow shock around a blunt body). Note that 1) the derivation of Eq. (7) does not require the gas to be calorically perfect, and 2) the factor \( d\beta/d\delta \) represents the inverse of the radius of curvature of the shock front. In this way, the post-shock vorticity becomes zero either for \( \beta = \pi/2 \) (i.e., normal shock) or for \( d\beta/d\delta = 0 \) (i.e., straight shock), as anticipated in class.

e) (4 pts) Simplify expression (7) for a calorically-perfect gas at hypersonic speeds \( Ma_1^2 \sin^2 \beta \gg 1 \), for which the post-shock density \( \rho_2 \) is uniform and the density ratio \( \epsilon \) becomes a constant that is a sole function of the adiabatic coefficient \( \gamma \).
a) \( \frac{P_2 - P_1}{\frac{1}{S_2} - \frac{1}{S_1}} = -\frac{m''^2}{s_1} \Rightarrow \frac{P_2 - P_1}{S_2} = \frac{m''^2}{S_1} \left( \frac{1}{S_1} - \frac{1}{S_2} \right) = \frac{m''^2}{S_1} \left( 1 - \frac{s_1}{s_2} \right) = \frac{m''^2}{S_1} (1 - \varepsilon) \)

WITH \( \varepsilon = \frac{s_1}{s_2} \) AND \( \ddot{m}'' = \rho_1 v_1 \sin \beta \)

THEN \( \frac{d\rho}{d\beta} = -\frac{\dot{m}''}{S_1} \frac{d\varepsilon}{d\beta} + \frac{2 \dot{m}'' (1 - \varepsilon) d\dot{m}''}{d\beta} = \)

\( = -\rho_1 v_1^2 \sin^2 \beta \frac{d\varepsilon}{d\beta} + 2 \rho_1 v_1^2 (1 - \varepsilon) \sin \beta \cos \beta \)

b) \( h_2 - h_1 = \frac{P_2 - P_1}{2S_1} (1 + \varepsilon) \Rightarrow \frac{d h_2}{d\beta} = \frac{(1 + \varepsilon)}{2S_1} \frac{dP_2}{d\beta} + \frac{(P_2 - P_1)}{2S_1} \frac{d\varepsilon}{d\beta} = \)

\( = -\rho_1 \frac{v_1^2 \sin^2 \beta}{2} (1 + \varepsilon) \frac{d\varepsilon}{d\beta} + \rho_1 \frac{v_1^2 \sin \beta \cos \beta (1 + \varepsilon) (1 - \varepsilon)}{2} \)

\( + \rho_1 \frac{v_1^2 \sin \beta (1 - \varepsilon)}{2} \frac{d\varepsilon}{d\beta} \)

c) \( T_2 \frac{ds_2}{d\beta} = \frac{dh_2}{d\beta} - \frac{1}{S_2} \frac{dp_2}{d\beta} = \frac{dh_2}{d\beta} - \frac{\varepsilon}{S_1} \frac{dp_2}{d\beta} = \rho_1 \frac{v_1^2 \sin \beta \cos \beta (1 - \varepsilon)^2}{2} \)

\( = \rho_1 \frac{v_1 \cos \beta}{\cos (\beta - \delta)} \frac{1}{\sin (\beta - \delta)} \)

AND THEREFORE \( T_2 \left| \frac{ds_2}{d\beta} \right| = \frac{1}{\sin (\beta - \delta)} \frac{ds_2}{d\beta} = \frac{\rho_1 \cos \beta}{\cos (\beta - \delta)} \frac{1}{\sin (\beta - \delta)} \frac{ds_2}{d\beta} \)

ALSO \( \frac{ds_2}{d\beta} = \frac{ds_2}{d\beta} \frac{d\beta}{d\delta} = \rho_1 \frac{v_1^2 \sin \beta \cos \beta (1 - \varepsilon)^2}{2} \frac{d\beta}{d\delta} \)

COMBINING THESE TWO EQUATIONS \( \Rightarrow \frac{\dot{w}_2}{\dot{w}_2} = \rho_1 \frac{v_1 \sin \beta (1 - \varepsilon)^2}{\tan (\beta - \delta)} \frac{d\beta}{d\delta} \)

AND SINCE \( \tan (\beta - \delta) = \varepsilon \tan \beta \), THEN \( \frac{\dot{w}_2}{\dot{w}_2} = \rho_1 \frac{v_1^2 (1 - \varepsilon)^2 \cos \beta \frac{d\beta}{d\delta}}{4} \)

e) IN THE HYPERSONIC LIMIT AND FOR A CALORICALLY PERFECT GAS, \( \varepsilon = (\gamma - 1)/\gamma + 1 \)

\( \Rightarrow \frac{\dot{w}_2}{\dot{w}_2} = \frac{4 \rho_1 v_1^2 \cos \beta \frac{d\beta}{d\delta}}{(\gamma + 1) (\gamma - 1)} \)
Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. In the calculations, assume Pr = 0.7 and a calorically perfect gas with \( \gamma = 1.4 \), \( R_y = 286 \text{ J/kgK} \), and \( c_p = 1 \text{ kJ/kgK} \) unless stated otherwise. Attach these sheets to your solutions. Attach these sheets to your solutions.

Student’s Name: JAVIER URZAY

Student’s ID:..........................

PART I: Closed notes, calculators allowed, compressible-flow tables allowed

Time: 60 mins

Questions (50 pts)

1. (20 pts) Describe under what conditions may a flow be regarded as hypersonic. This is an open-ended question, and therefore both creativity and rigor will be graded positively. **SEE CHAPTER I CLASS NOTES**

2. (10 pts) Describe the general non-equilibrium structure of a hypersonic normal shock wave in air, including the characteristic zones where each degree of freedom of molecular motion, or chemical process, may be in equilibrium or out of equilibrium. In your response, include sketches of typical temperature and density distributions across the shock in non-equilibrium conditions. **SEE PAGES 112-114 CLASS NOTES**

3. (10 pts) Provide a definition for the adiabatic wall temperature \( T_{w,w} \) and describe the shapes of characteristic static temperature profiles that can be encountered in a hypersonic laminar boundary layer depending on whether the wall temperature \( T_w \) is higher, lower or equal to \( T_{w,w} \). **SEE PAGE 68 CLASS NOTES**

4. (10 pts) The temperature of a flat plate is \( T_w = 4T_e \), where \( T_e \) is the static temperature of a high Reynolds-number free stream of gas flowing parallel to the plate. At very small Mach numbers, \( M_{e} \ll 1 \), is the plate cooled or heated by the gaseous free stream? What characteristic minimum value of the free-stream Mach number \( M_a \) needs to be attained for the plate to be heated by the gas, or equivalently, for high-speed aerodynamic heating to become more important than low-speed convective cooling?

\[ q_w \approx (T_e - T_w) \Rightarrow q_w = 0 \text{ when } M_{ad} \text{ is such that } \]
\[ T_{ew} = T_w \Rightarrow \frac{T_{ew}}{T_e} = \frac{T_w}{T_e} = 1 + \left( \frac{\gamma - 1}{2} \right) M_a^2 \]
\[ \Rightarrow M_{ad} \approx \left( \frac{T_w}{T_e} - 1 \right) \frac{1}{\frac{\gamma}{2} - 1} = 3.87 \]
ME 356: Hypersonic Aerothermodynamics, Spring 2018
Stanford University
Final Exam
Friday, June 8

Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. In the calculations, assume Pr = 0.7 and a calorically perfect gas with $\gamma = 1.4$, $R_g = 286$ J/kgK, and $c_p = 1$ kJ/kgK unless stated otherwise. Attach these sheets to your solutions.

Student’s Name: JAVIER URZAY Student’s ID: 

PART II: Open notes, calculators allowed, compressible-flow tables allowed

Time: 120 mins

Problem 1 (50 pts)

A long time before the theory of hypersonics, and during many centuries of history of civilization, the problem of aerodynamic heating, as we call it today, occupied the attention of great physicists, mathematicians, and philosophers such as Aristotle, Seneca, Saint Thomas Aquinas, Averroes, and Galileo. For instance, in 350 BC, Aristotle attempts to explain how the heavenly bodies generate heat and light and transmit them to the terrestrial world by linking heat with the mechanics of motion:

"The warmth and light which proceed from them are caused by the friction set up in the air by their motion. Movement tends to create fire in wood, stone, and iron; and with even more reason should it have that effect on air, a substance which is closer to fire than these. An example is that of missiles (arrows), which as they move are themselves fired so strongly that leaden balls are melted; and if they are fired the surrounding air must be similarly affected. Now while the missiles are heated by reason of their motion in air, which is turned into fire by the agitation produced by their movement, The Upper Bodies are carried on a moving sphere, so that, though they are not themselves fired, yet the air underneath the sphere of the revolving body is necessarily heated by its motion, and particularly in that part where the Sun is attached to it."

Aristotle, in De Caelo (On The Heavens), Book II, Part 7 (350 BC).

A similar topic also caused a heated exchange between Galileo Galilei and Orazio Grassi, a priest best noted as a mathematician, astronomer and architect. In order to support his own explanation of the nature of comets and their bright tails with Aristotle’s theory that motion creates heat, Grassi authoritatively cites ancient wisdom collected in the Suda, a 10th-century Byzantine encyclopedia, to indicate that the Babylonians cooked eggs by whirling them very fast through the air in slings (Libra Astronomica Ac Philosophica, p. 57, 1619). Galileo did not particularly like the work of Grassi and his theory of heat created by motion, and in 1623 he published an extensive treatise in response to Grassi called II Saggiatore (i.e., The Assayer). There, in a literary masterpiece, Galileo tore Grassi’s analysis to pieces for being groundless and solely based on the wisdom of earlier generations$^1$:

$^1$It is thought that this and other clashes alienated many of the Jesuits who had previously been sympathetic to Galileo’s ideas, and that later these might have played a role when he was brought to trial by The Inquisition [e.g., see Koestler A., “The Sleepwalkers: A History of Man’s Changing Vision of the Universe”, Ch. 2 (1959)].
"But it is wrong to say, as Sarsi [Grassi's secret pseudonym] does, that Guiducci [Galileo's friend] and I would laugh and joke at the experiences adduced by Aristotle. We merely do not believe that a cold arrow shot from a bow can take fire in the air; rather, we think that if an arrow were shot when afire, it would cool down more quickly than it would if it were held still. This is not derision; it is simply the statement of our opinion. Sarsi goes on to say that since this experience of Aristotle's has failed to convince us, many other great men also have written things of the same sort. To this I reply that if in order to refute Aristotle's statement we are obliged to represent that no other men have believed it, then nobody on earth can ever refute it, since nothing can make those who have believed it not believe it. But it is news to me that any man would actually put the testimony of writers ahead of what experience shows him. To adduce more witnesses serves no purpose, Sarsi, for we have never denied that such things have been written and believed. We did say they are false, but so far as authority is concerned yours alone is as effective as an army's in rendering the events true or false. You take your stand on the authority of many poets against our experiments. I reply that if those poets could be present at our experiments they would change their views, and without disgrace they could say they had been writing hyperbolically or even admit they had been wrong."

Galileo, in Il Saggiatore (1623).

Coincidentally, in Il Saggiatore Galileo also established the modern scientific method by arguing that mathematics and experimentation, rather than testimonials of famous poets and philosophers, should be the basis of science and nothing else. Galileo was particularly skeptical of the Babylonians way of cooking eggs by whirling them fast in circular orbits in slings through the air, and in this way he sarcastically wrote:

"If Grassi wants me to believe with Suidas that the Babylonians cooked their eggs by whirling them in slings, I shall do so; but I must say that the cause of this effect was very different from what he suggests. To discover the true cause I reason as follows: if we do not achieve an effect which others formerly achieved, then it must be that in our operations we lack something that produced their success. And if there is just one single thing we lack, then that alone can be the true cause. Now we do not lack eggs, nor slings, nor sturdy fellows to whirl them; yet our eggs do not cook, but merely cool down faster if they happen to be hot. And since nothing is lacking to us except being Babylonians, then being Babylonians is the cause of the hardening of eggs, and not friction of the air." And this is what I wished to discover. Is it possible that Sarsi has never observed the coolness produced on his face by the continual change of air when he is riding post? If he has, then how can he prefer to believe things related by other men as having happened two thousand years ago in Babylon rather than present events which he himself experiences?"

Galileo, in Il Saggiatore (1623).

Galileo did not believe in ancient authorities and performed the experiment himself. He took an egg and whirled it in a sling until his arm was tired, and then he asked his graduate students at the University of Padova to whirl the egg for a bit longer. However, despite their efforts, the egg remained raw, but they all ended up with sore arms. Lastly, Galileo boiled an egg and whirled it through the air, and found that the egg was cooler after whirling due to what we call today convective cooling. Galileo therefore concluded that an egg would actually cool down by moving it fast in air.

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3This is a mechanism by which a hot plate can be rapidly cooled by an overriding cold fluid at high Reynolds numbers and low Mach numbers, whereby large negative temperature gradients develop across the resulting boundary layer.
In this problem, we return to the question addressed by Galileo and attempt to calculate whether it would have been possible for the Babylonians to cook eggs by moving them fast in air. However, here we hypothesize that the Babylonians had some sort of advanced technology available to accelerate eggs to hypersonic speeds, and we investigate whether an egg could be cooked by aerodynamic heating.

To this end, follow the steps below.

a) (10 pts) Let us begin by assuming that an egg is an axisymmetric body of revolution as depicted in Fig. 1(a). In particular, the egg consists of two conics, namely, a paraboloid \( x = Br^2 \) with \( r \) in cm and \( B = 1 \) cm\(^{-1}\), and a sphere with radius \( R_s \), which are tangent at a point 4 cm downstream from the leading edge of the sharpest side of the egg. The density of the egg is \( \rho_e = 1000 \) kg/m\(^3\) and its specific heat is \( c_e = 4 \) kJ/kgK.

Using the straight Newtonian theory of hypersonic flows, in which the shock layer is infinitesimally thin, obtain expressions for the hypersonic drag coefficients \( C_D \) on the egg at zero angle of attack depending on whether the free stream approaches from the left (case A) or from the right (case B). In case A, ignore the small portion of the spherical cap that is exposed to the incident hypersonic stream.

b) (10 pts) Which configuration (case A or B) would lead to more intense heating? Justify your response based on elementary heat transfer theory for hypersonics, and disregard any concerns related to longitudinal static stability.

c) (10 pts) The egg is placed at the entrance of a long tunnel filled with still air at pressure \( P_\infty = 1 \)
bar and temperature $T_\infty = 300$ K. The initial temperature of the egg is the same as that of the surrounding air $T_\infty$. As depicted in Fig. 1(b), the egg is then shot horizontally along the tunnel at an initial velocity $U_0$ that is 10 times larger than the speed of sound and is attained by using an advanced railgun\(^4\). Obtain an expression for the velocity of the egg $U$ as a function of the downrange $s$ (the horizontal distance traveled by the egg) by integrating the equation of motion as it moves through the tunnel. In the calculations, assume that the distortion of the trajectory due to gravity is negligible, or equivalently, that $U_0/\sqrt{gL} \gg 1$, where $g$ is the acceleration of gravity and $L$ is the length of the tunnel.

d) (10 pts) Consider the expression of the heat flux [W/m\(^2\)] on a blunt body

$$q_w = 0.763 \Pr^{-0.6} \left( \frac{\rho_e \mu_e A}{c_p(T_{0,\infty} - T_w)} \right)^{1/2},$$

where $A = (dU_e/d\ell)_{\ell=0}$ is the local strain rate, with $\ell$ a curvilinear coordinate emanating from the stagnation point (see Fig. 1a). Additionally, the subindex $e$ represents conditions at the edge of the boundary layer (i.e., post-shock conditions behind a normal shock at pre-shock velocity $U$), $T_{0,\infty}$ is the stagnation temperature of the free stream in the egg reference frame, and $T_w$ is the wall temperature of the egg. Using the Newtonian theory in the vicinity of the egg’s blunt nose (whichever one you have chosen in part b), estimate the strain rate $A$ and derive from (1) the equivalent form

$$q_w = \frac{0.381}{\varepsilon^{1/4}} \Pr^{-0.6} \left[ \frac{\rho_\infty \mu_\infty}{(c_p T_\infty)^{1/2} R_0} \right]^{1/2} U^3$$

by assuming that $T_{0,\infty}/T_w \gg 1$ because of the high Mach numbers, and also that $\mu_e/\mu_\infty = (T_e/T_\infty)^{1/2}$, with $T_e \simeq T_{0,\infty}$ due to the low velocities in the post-shock region. In this formulation, $\varepsilon = \rho_\infty/\rho_e$ is a density ratio that can be obtained from the hypersonic normal-shock jump conditions, $\mu_\infty = 1.8 \cdot 10^{-5}$ Ns/m\(^2\) is the dynamic viscosity of the air in the tunnel, and $R_0$ is the radius of curvature in the vicinity of the axis of symmetry of the blunt side of the egg that you have previously chosen in part b).

e) (10 pts) Equation (2), along with the relation $U(s)$ obtained in part c), provide the dominant aerodynamic heat flux into the egg as a function of time. Let us assume that the heat flux enters the egg uniformly through an effective characteristic area of order $\pi R_0^2$. Assuming also that the temperature $T_b$ inside a volume $V_0 = 4\pi R_0^3/3$ of the egg next to the shell is instantaneously uniform, derive an expression for $T_b$ as a function of the downrange $s$ integrating the conservation of energy, and state whether $T_b$ will ever reach the critical value 338 K for the egg to get cooked in the tunnel\(^5\).

\(^4\)It is assumed that the egg is shielded from any possible aerodynamic pressure loads that could cause it to crack by an advanced resistant armor that has negligible thermal inertia and fits perfectly to the shape of the egg so as to not modify the drag coefficient. Similarly to a railgun, the armor together with the egg slide through an incident electromagnetic field that enables the high accelerations.

\(^5\)Assume $C_D = 1$ if you got stuck in part a).

ME 356 Hypersonic Aerothermodynamics 5
a) Configuration A

\[ L_1 = BR_1^2 \Rightarrow R_1 = 2a \]

At \( t \) (tangency point), the normal vector to the paraboloid is \( \nabla (x_t - Br^2) = (1 - 2Br_t) \). Therefore, \( n_t = \frac{1}{\sqrt{1 + B^2 r_t^2}} \Rightarrow R_5 = \frac{R_4}{\cos \theta_t} = 7.06 \text{ cm} \)

To compute the drag on the paraboloid up to the tangency point (i.e., ignoring the small portion of sphere exposed to the free stream as stated in the text), one needs the projection of the surface element \( ds \) on the \( x \) direction:

\[ \text{Here, } ds = \frac{\rho \, dr \, d\theta}{\sin \theta}, \text{ with } \theta = \frac{\pi}{2}, \text{ and } \tan \Psi = \frac{2Br}{r} \]

Then \( \sin \theta = \cos \Psi = \frac{1}{1 + \frac{4B^2 r^2}{1 + 4B^2 r_t^2}} \)

\[ \Rightarrow \quad q_D = \sqrt{\int_0^{2\pi} \int_0^{2\pi} \frac{2\sin^2 \theta \, dr \, d\theta}{TR_1^2}} = \frac{2\pi \int_0^{R_1} \frac{Zr}{1 + 4B^2 r_t^2} \, dr}{2B^2 R_1^2} = \frac{\pi \arctan (1 + 4B^2 r_t^2)}{2B^2 R_1^2} = 0.34 \]

Configuration B

The drag coefficient is the same as in the problem of a sphere:

\[ C_D = 1 \]

(See details in Problem #4 of the midterm exam)

b) The aerodynamic heating will be larger when the sharpest side of the egg is exposed to the free stream.

* For Configuration \( B \), the radius of curvature is \( R_5 = 2.06 \text{ cm} \)

* For Configuration \( A \),

\[ \frac{1}{R_0} = \frac{1}{\arctan (\frac{dx}{dr})} = \frac{d^2}{d\theta^2} \left( \frac{dx}{dr} \right) = \frac{d^2}{d\theta^2} \left[ \frac{d^2 x / dr^2}{1 + (dx / dr)^2} \right] = \frac{2B}{1 + (dx / dr)^2} \]

AT \( r = 0 \) \( \Rightarrow \frac{1}{R_0} = 2B \Rightarrow R_0 = \frac{1}{2B} = 0.5 \text{ cm} \)
c) \[ \frac{mdU}{dt} = -\frac{1}{2} \rho_0 \sigma D U^2 \] \[ t=0, \quad U = U_0 \]

\[ S = \frac{R_0}{R_{0,s}} = 1.2 \quad R_0/R_{0,s} \]

\[ S = 1.25 \cdot 10^{-3} \text{ m}^2 \]

\[ G = 0.34 \]

Note that it is more convenient to work with distance \( S \) rather than \( t \), that is:

\[ \frac{dU}{dt} = \frac{dS}{dt} \frac{dU}{dS} = \frac{U}{dU/dS} \Rightarrow \frac{mdU}{dS} = -\frac{1}{2} \rho_0 \sigma D S U \]

\[ S = 0, \quad U = U_0 \]

Integrating this equation:

\[ U(S) = U_0 e^{-\frac{\rho_0 \sigma D S}{2m}} \]

Note that \( U \) decays exponentially with distance, \( q_0 \) decays as \( U^2 \) with distance, and \( T_b \) tends to a constant.

Note that the mass of the egg is the sum of the mass of the paraboloid:

\[ m_1 = \int_0^{R_b} \int_0^{2\pi} \int_0^{R_a} \rho \ r^2 \sin \theta \, dr \, d\theta \, d\phi = 2\pi \int_0^{R_b} \rho_0 R_b^3 \, dr = \frac{\pi B R_b^4 \rho_b}{2} \]

And the volume of the spherical cap:

\[ m_2 = \frac{4}{3} \pi R_s^3 \rho_s = \int_0^{\pi/2} \int_0^{R_s} \int_0^{R_b} \rho \ r^2 \sin \theta \, dr \, d\theta \, d\phi \]

\[ = \frac{4}{3} \pi R_s^3 \rho_s \left( 1 - \sin \theta \right)^3 \left( 2 + \sin \theta \right) \]

\[ \Rightarrow m = m_1 + m_2 = 0.90 \pi R_s^3 \rho_s + \frac{\pi B R_b^4 \rho_b}{2} = 0.05 \rho_b \]

The derivations can be found on pages 124, 134 and 135 of the class notes.

d) Energy EQ:

\[ \dot{q}_b \int \rho_b 4\pi R_s^2 \frac{d\theta}{dt} \]

Integrating:

\[ S = 0, \quad T_b = T_{in} \]

\[ \Rightarrow T_b(S) = T_{in} + 3 \cdot 0.38 \cdot Pr^{0.6} \]

\[ = 300 + 54.8 \left( 1 - e^{-S/96.4} \right) \quad [K] \quad \text{and} \quad S \quad [\text{m}] \]

The leading edge portion of the egg is cooled after 114 m (\( T_b = 338 \text{ K} \))