I. THE HYPersonic GAS ENVIRONMENT

Hypersonic flows are a special class of flows that arise around aerodynamic shapes moving in gaseous environments at exceedingly high velocities compared to the speed of the sound waves in the gas. In most practical applications related to hypersonics, the velocities associated with aircrafts and spacecrafts piercing through the terrestrial atmosphere are in the range $U_0 \approx 1700 \text{ m/s} - 12600 \text{ m/s}$ (i.e., $3800 \text{ mph} - 28000 \text{ mph}$), although these can be much higher in the context of re-entry flows in other planets of the solar system ($\approx 40000 \text{ m/s}$ in Jupiter). The aforementioned range of velocities in the terrestrial stratosphere and mesosphere translates into Mach numbers in the range $5 \leq M_a \leq 42$. The lower end of this interval corresponds to applications of low-altitude high-speed flight and missile impact, while the upper end represents conditions approached by spacecrafts reentering in the terrestrial atmosphere during return from lunar or inter-planetary missions.

Note that an aircraft flying at Mach 10 can turn around the world along the equator in less than 4 hours.

**FIG. 1:** The gas environment near a notional hypersonic vehicle.

- Atmospheric particulates (fouling, ice, droplets, aerosols)
- Shock/shock interaction and heating
- Hot exhaust plume
- Very short residence time in the scramjet combustor
- Hypersonic boundary-layer transition to turbulence
- Nose shock is located very close to the fuselage at $M_a \approx 1$ (inertial flow)
- Entropy layer (rotational)
- Chemically reacting shock layer (air dissociation and possibly ionization)
- Vibrational non-equilibrium

Very hot, low-speed gas, possible breakdown to the surface
- Surface erosion
- Heat transfer to the surface
- Viscous heating to the surface
The magnitude of the hypersonics problem is perhaps best illustrated by analyzing the associated kinetic energies. For instance, the specific kinetic energy of hypersonic flows around high-speed vehicles are of order $u_2^2 \approx 4 \mathrm{MJ} / \mathrm{kg} = 100 \mathrm{kJ} / \mathrm{kg}$. These energies are comparable or larger than the specific heats of vaporization of water ($2.3 \mathrm{MJ} / \mathrm{kg}$), carbon ($60 \mathrm{MJ} / \mathrm{kg}$), silica ($5.8 \mathrm{MJ} / \mathrm{kg}$), and iron ($7.0 \mathrm{MJ} / \mathrm{kg}$), and they are much higher than the specific thermal energy of air at normal conditions ($300 \mathrm{kJ} / \mathrm{kg}$). At hypersonic Mach numbers, in the presence of shock waves, these kinetic energies are mostly transformed into specific thermal energy behind the shock front, which causes exceedingly large temperatures in the post-shock gases of order:

\[
\begin{align*}
- & \quad T_{2} \approx 3000 \mathrm{K} \quad \text{for } M_{2} \approx 10 \quad \text{hypersonic cruise aircraft (X-43)} \\
- & \quad T_{2} \approx 7000 \mathrm{K} \quad \text{for } M_{2} \approx 25 \quad \text{re-entry spacecraft from low earth orbit (space shuttle orbiter)} \\
- & \quad T_{2} \approx 11000 \mathrm{K} \quad \text{for } M_{2} \approx 40 \quad \text{re-entry spacecraft at escape velocity from lunar or inter-planetary return trajectory (Apollo capsule or Mars-return capsule)}
\end{align*}
\]

The high post-shock temperatures do not translate into similar wall temperatures $T_w$ since radiative equilibrium tends to lower the latter, and additionally, techniques for moderating wall temperatures have been discovered over the years to reject (or conveniently absorb in isolated parts of the structure) the intense aerodynamic heating that the airframe is subjected to at hypersonic speeds.

\[
\begin{align*}
- & \quad T_{w, \text{max}} \approx 1000 \mathrm{K} \quad (X-43) \\
- & \quad T_{w, \text{max}} \approx 600 \mathrm{K} \quad (\text{Shuttle}) \\
- & \quad T_{w, \text{max}} \approx 200 \mathrm{K} \quad (\text{Apollo})
\end{align*}
\]

The heat barrier.

The realm of hypersonics is therefore not one associated with the onset of compressibility effects as passing through the transonic regime into the supersonic regime. That is the focus of transonic and low-supersonic aerodynamics and their challenges were greatly overcome by the breaking of the sound barrier in 1947. Instead, the problem of...
Hypersonics is inexorably related to the everlasting and an ever-increasing thermal barrier that is encountered as the Mach number is increased (Fig. 2), and which poses severe constraints in airframe materials and aerodynamic design. As an example, consider the space shuttle orbiter, \( M = 100 \cdot 10^2 \) kg, orbiting around the Earth at an altitude \( z = 350 \) km, therefore with a circular orbital velocity \( V_c = \sqrt{\frac{GM}{Re^2}} \approx 7.8 \) km/s, with \( Re = 6371 \) km and \( g = 9.8 \) m/s\(^2\) the Earth’s radius and gravitational acceleration. The difference between the mechanical energies of the shuttle in orbit and at rest on the Earth’s surface is

\[
\Delta E = \frac{mV_c^2}{2} + \frac{mgR_e^2}{2} = 3.3 \text{ TJ},
\]

This energy difference must be transformed into heat during atmospheric re-entry. The average gas + electricity consumption for a single-person home in Northern California is of order \( 2.5 \text{ kWh/d} \approx 9 \text{ MJ/d} \text{.} \) As a result, all the heat generated during the space shuttle re-entry as it flies hypersonically through the atmosphere could be used for heating and cooking of a single-person home for \( 360000 \text{ days} \approx 1000 \text{ years} \). However, note that only a small fraction \(< 1\%\) of \( \Delta E \) is transferred as heat to the entry system, the rest being dissipated into heat in the surrounding air. This exact fraction is a solution to the fluid mechanical problem of atmospheric re-entry. An order of magnitude is obtained in Chapter V that indicates

\[
\frac{Q}{\Delta E} \sim 4 \frac{C_f S}{S_d A}
\]

where \( Q \) is the total amount of heat \((J)\) absorbed by the spacecraft, \( C_f \) is the mean friction coefficient, \( S \) is the wetted surface, \( A \) is the frontal area, and \( C_d \) is the drag coefficient. This estimate (originally obtained by Allen and Eggers in 1958), is perhaps one of the most crucial results of the theory of hypersonics, since it establishes that blunt bodies \((\text{large } C_d \text{ and } A)\) with comparatively low skin friction and small wetted areas, are the most practical for re-entry applications, because such aerodynamic shapes enable maximum dissipation of energy in the form of heat into the gas environment \((\text{rather than into the spacecraft structure})\).

The high temperatures associated with hypersonic flight pose severe challenges for the airframe surface materials. Many of the most important refractory materials...
AND ADVANCED ALLOYS USED IN HIGH-TEMPERATURE APPLICATIONS HAVE MELTING POINTS SIMILAR TO SOME OF THE WALL TEMPERATURES CITED ABOVE (2760 K FOR HOLYDENUM, 2000 K FOR ZIRCONIUM, 1700 K FOR INCONEL-X, OR 1560 K FOR BERYLLIUM). THESE HIGH WALL TEMPERATURES IN HYPERSONICS ARE TO BE CONTRASTED WITH THOSE FOUND IN SUPERSONIC AIRCRAFT SUCH AS THE CONCORDE (M\text{a} \approx 2, T_{\text{wall}} \approx 380 K) OR THE SR-71 BLACKBIRD (M\text{a} \approx 3.1, T_{\text{wall}} \approx 600 K).

THIS MAKES NECESSARY THE UTILIZATION OF THERMAL PROTECTION SYSTEMS (TPS) IN HYPERSONICS, INCLUDING REFRACTORY SHIELDS, AND ALSO HEAT SINKS AND ABLATIVE SURFACES. THE STUDY OF TPS IS NOT THE SUBJECT OF THIS COURSE.

HOWEVER, IT IS WORTH MENTIONING THAT THE TPS REQUIREMENTS ARE FUNDAMENTALLY DIFFERENT DEPENDING ON THE APPLICATION. THIS CAN BE SEEN IN THE HEATING PROFILES IN FIG. 3, WHICH ILLUSTRATE THE TRENDS OF HEAT FLUXES ENTERING IN TWO TYPES OF VEHICLES AS A FUNCTION OF ENTRY TIME.

IT IS OBSERVED THAT INTER-CONTINENTAL BALLISTIC MISSILES, WHICH HAVE A SLENDER SHAPE, TEND TO BE SUBJECT TO VERY HIGH HEAT FLUXES, BUT FOR A SHORT AMOUNT OF TIME (∼1 MIN).

IN CONTRAST, MANNED ENTRY CAPSULES ARE MUCH LESS SLENDER AND CONSEQUENTLY ARE SUBJECT TO MUCH SMALLER HEAT FLUXES (∼1/100 SMALLER THAN ICBMS) BUT FOR A MUCH LONGER DURATION (∼10 MIN). IN CHOOSING A TPS, THE MAXIMUM HEAT FLUX IMPOSES THE TYPE OF MATERIAL, WHILE THE TOTAL HEAT LOAD IMPOSES THE VOLUME OF TPS MATERIAL. THE DIFFERENCES IN MISSION TIMES IN FIG. 3 AS WELL AS THE LARGE DISPARITIES IN THE HEAT FLUXES, WILL BE FURTHER EXPLAINED LATER IN THE COURSE.

HIGH-TEMPERATURE EFFECTS IN HYPERSONICS

IN EXPLAINING THE DISTINGUISHED CONDITIONS, IT IS IMPORTANT TO NOTE THAT THE FLIGHT MACH NUMBER

\[ M_{\text{a}} = \frac{U_{\infty}}{a_{\infty}} \]

IS PROPORTIONAL TO THE SQUARE ROOT OF THE RATIO OF THE KINETIC ENERGY AND SPECIFIC THERMAL ENTHALPY OF THE SURROUNDING AIR,

\[ M_{\text{a}}^2 = \frac{U_{\infty}^2}{a_{\infty}^2} = \frac{U_{\infty}^2}{\sqrt{R T}} = \left( \frac{2}{\gamma - 1} \right) \frac{U_{\infty}^2}{C_p T_0} \]
As a result, hypersonic Mach numbers \((M_{\infty} > 5)\) involve kinetic energies larger than the thermal enthalpy of the surrounding gas. At high Mach numbers into the hypersonic range \((M_{\infty} > 5)\), this ratio becomes exceedingly large and it has profound consequences in the structure of the flow field that constitute the core of the problem of hypersonics. Such complex effects, unique to the hypersonics environment, are driven by the high temperatures attained in the gas as a result of transformation of kinetic energy into thermal energy. This transformation occurs ubiquitously in the flow field downstream from shocks, where the flow is decelerated because of the much higher pressures of the post-shock gases in comparison with the free-stream pressure, and near solid surfaces where the kinetic energy is dissipated into heat (Fig. 4). This transformation is, to a good approximation, described by the relation:

\[ h_{\infty} = h_{\infty} + \frac{u^2}{2} = h_2 + \frac{u^2}{2} \]

which establishes that, at hypersonic Mach numbers, the stagnation enthalpy of the free stream is mostly of kinetic origin, and as a result, becomes transformed into local enthalpy in regions where the gas is strongly decelerated (Fig. 4), thus creating enormous increments in the local temperature. The problem is, remarkably, more pronounced in the terrestrial atmosphere where most hypersonic vehicles are designed for, since the free-stream temperatures are relatively high \((220K \sim 300K)\), and therefore become premultiplied across shocks and boundary layers by multiples of the Mach number, the high temperatures generated this way are sufficient to activate complex fluid-mechanical and thermo-chemical effects as follows (Fig. 4):

- Hypersonic boundary layers tend to be thicker and hotter than their low speed counterparts, and they tend to develop chemical conversion in the gas and near surfaces as explained below.
--- AT HIGH MACH NUMBERS \((M_{10} > 20)\), THE GAS DOWNSTREAM OF THE BOW SHOCK EMERGING FROM THE NOSE OF THE SPACECRAFT CAN GET VERY HOT AND GIVE RISE TO RADIATIVE HEAT TRANSFER TO THE FUSELAGE. ---

--- NON-CALORICALLY-PERFECT EFFECTS ARE RAPIDLY ATTAINED AT RELATIVELY LOW FLIGHT SPEEDS. AT TEMPERATURES \(T > 800\) K, CORRESPONDING TO POST-SHOCK CONDITIONS IN THE STRATOSPHERE AT \(M_{\infty} \approx 4-5\) \((\approx 1200 - 1500\) m/s\), THE VIBRATIONAL DEGREES OF FREEDOM ARE EXCITED AND GIVE RISE TO TEMPERATURE DEPENDENCIES IN SPECIFIC HEATS. ADDITIONALLY, AIR DISSOCIATION OF OXYGEN AND NITROGEN MOLECULES ARISE AT HIGHER SPEEDS WHOSE VALUES DEPEND ON THE FLIGHT ALTITUDE. AT NORMAL PRESSURE, OXYGEN BEGINS TO DISSOCATE \((O_2 \rightarrow 2O)\) AT \(T \approx 2000\) K, WHILE NITROGEN BEGINS TO DISSOCATE \((N_2 \rightarrow 2N)\) AT MUCH HIGHER TEMPERATURES \(T > 4000\) K. AT \(T > 9000\) K ALL THE AIR IS VIRTUALLY DISSOCIATED INTO N AND O MOLECULES. THESE EFFECTS HAVE PROFOUND EFFECTS IN THE POST-SHOCK DENSITY THAT ARE CRUCIAL FOR THE COMPUTATION OF TEMPERATURES IN HYPERSONIC FLOWS. IN THE STRATOSPHERE, THE ONSET OF DISSOCIATION BEGINS AT \(M_{\infty} \approx 7\) \((\text{FOR} \; O_2)\) AND \(M_{\infty} \approx 15\) \((\text{FOR} \; N_2)\), AS SHOWN IN FIG. 5. AIR IONIZATION BEGINS AT \(T \approx 9000\) K AT NORMAL PRESSURE \((N \rightarrow N^+ + e^-; \; O \rightarrow O^+ + e^-)\), TRANSLATING INTO \(M_{\infty} \approx 25\) IN THE STRATOSPHERE. THIS PRODUCES A PLASMA SHEATH OF WEAKLY IONIZED GAS AROUND THE AIRCRAFT THAT CAN CAUSE A COMMUNICATIONS BLACKOUT DURING A PORTION OF THE RE-ENTRY. ---

--- IN ADDITION TO THE AFOREMENTIONED EFFECTS, NON-EQUILIBRIUM THERMODYNAMIC AND CHEMICAL PHENOMENA CAN ARISE AT HIGH SPEEDS WHEN THE CHARACTERISTIC FLOW TIME AROUND THE AIRCRAFT IS OF THE SAME ORDER AS THE TIME REQUIRED BY MOLECULAR COLLISIONS TO EQUILIBRATE THE DYNAMICS. ---
II. INVISCID HYPersonic Flows

Despite the wealth of complexities encountered in hypersonic flows, it is instructive to first study the simpler limit in which the flow is inviscid. In this limit, the effects introduced by viscosity and heat conduction are small. This condition is satisfied when the Reynolds number

\[ \text{Re}_x = \frac{\rho u_x L}{\mu} \gg 1 \]  

is moderately large. Inviscid conditions are preferentially attained in free-stream regions away from surfaces, wakes and flow discontinuities (Fig. 6), provided that the free-stream flow does not contain disturbances or these are negligibly small compared to the mean flow. In this way, the flow is slender (i.e., streamlined) and is not subject to energy losses by friction or other dissipation mechanisms except for energy losses across shock waves. Viscous effects are studied in Chapter III.

The hypersonic limit for shock-wave jump conditions:

\[ \beta = \text{streamline deflection angle} \]  
\[ \beta = \text{angle of incidence} \]  
\[ U_1 = \text{pre-shock flow velocity} \]  
\[ U_2 = \text{post-shock flow velocity} \]

In previous courses it was studied that the integral form of the conservation equations across an oblique shock becomes:

\[ \text{Continuity of mass:} \quad S_1 U_1 = S_2 U_2 \]  
\[ \text{Conservation of normal momentum:} \quad S_1 U_1^2 + P_1 = S_2 U_2^2 + P_2 \]  
\[ \text{Conservation of tangential momentum:} \quad U_{1t} = U_{2t} \]  
\[ \text{Conservation of total enthalpy:} \quad \frac{h_{1t} + U_{1t}^2}{2} = \frac{h_{2t} + U_{2t}^2}{2} \]

In Eq. (5), the symbol \( h_0 \) denotes a total or stagnation enthalpy. The same subindex is used throughout these notes to denote other stagnation quantities.
It can be shown that Equations (2)-(5) lead to the Rankine-Hugoniot jump conditions:

\[
\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)M_{a1m}^2}{2 + (\gamma-1)M_{a1m}^2}, \quad \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} \left( M_{a1m}^2 - 1 \right),
\]

\[
\frac{T_2}{T_1} = \frac{\rho_1 S_2}{P_1 S_1} = \frac{2\gamma M_{a1m}^2 - (\gamma - 1)}{\gamma + 1} \left[ \frac{2 + (\gamma-1)M_{a1m}^2}{(\gamma+1)M_{a1m}^2} \right],
\]

\[
\frac{U_{2m}}{U_{1m}} = \frac{\rho_1}{\rho_2} = \frac{2 + (\gamma-1)M_{a1m}^2}{(\gamma+1)M_{a1m}^2} \quad (\gamma) \quad \Rightarrow \quad M_{a2m}^2 = \frac{2 + (\gamma-1)M_{a1m}^2}{2\gamma M_{a1m}^2 + 1 - \gamma}
\]

\[
\left[ \frac{1 + (\gamma-1)M_{a2m}^2}{2} \right]^{\frac{\gamma-1}{\gamma}} \frac{P_2}{P_1} = \frac{\rho_2}{\rho_1} \quad (11)
\]

In this formulation, \( \gamma = \frac{C_p}{C_v} \) is the adiabatic coefficient, with \( C_p \) and \( C_v \) being the specific heats at constant pressure and volume, respectively, which for now are assumed to be constant and related through the expression \( C_p - C_v = R_0\gamma \).

For an ideal gas, where \( R_0 = \frac{R}{\text{mol}} \) is the gas constant, \( R = 8.31 \text{ J/mol K} \), \( R_0 \) is the universal gas constant, and \( \text{mol} \) is the molecular weight.

Correspondingly, the equation of state is

\[ P = \rho R_0 T, \quad (12) \]

where \( P, \rho \), and \( T \) are static values of pressure, density, and temperature.

Static and stagnation quantities are related by the Bernoulli equation

\[ h + \frac{|U|^2}{2} = h_0 \Rightarrow C_p T + \frac{|U|^2}{2} = C_p T_0, \quad (13) \]

or equivalently,

\[ \frac{T_0}{T} = 1 + \frac{|U|^2}{2C_p T} = 1 + (\gamma-1)M_a^2 \]

(14)

where \( M_a = \frac{|U|}{a} \) is the Mach number and

\[ a = \sqrt{\gamma R_0 T} = \sqrt{\frac{\gamma P}{\rho}} \]

(15)

is the speed of the soundwaves in the gas. The stagnation quantities correspond to the state of a gas isentropically brought to rest from its initial velocity \( U \). As a result, the isentropic relations...
\[ \frac{T_0}{T} = \left( \frac{\rho_0}{\rho} \right)^{\gamma-1} = \left( \frac{P_0}{P} \right)^{\frac{\gamma-1}{\gamma}} \]  

Equation (16)

Between stagnation and static quantities are satisfied. Upon combining Eqs. (14) and (16), the expressions

\[ \frac{h_0}{T_0} = \frac{\theta_0}{\theta} = \frac{(\frac{\theta_0}{\theta})^2}{(\frac{\theta_0}{\theta})} = \left( \frac{P_0}{P} \right)^{\frac{\gamma-1}{\gamma}} = 1 + \left( \frac{\gamma-1}{2} \right) \frac{\gamma-1}{\gamma} \]

Equation (17)

As a function of the local Mach number. Note that Eq. (17) is applicable only along each streamline and on each side of the shock wave but never across it, since (17) requires the flow be isentropic. This condition is not satisfied across the shock, which creates a positive entropy jump in the flow because of viscous and heat conduction within the shock.

In the jump conditions (6) - (14), \( M_{a,1m} \) refers to the normal Mach number,

\[ M_{a,1m} = \frac{U_{1m}}{\alpha_1} \]

Equation (18)

where \( U_{1m} = U_1 \sin \beta \) is the flow velocity normal to the shock (see Fig. 2).

Similarly, \( U_{2m} = U_2 \cos \beta \) is the tangential flow velocity, and

\[ U_{2m} = U_2 \cos \left( \frac{\pi}{2} - \beta + \delta \right) = U_2 \sin (\beta - \delta) \]

\[ U_{2t} = U_2 \cos (\beta - \delta) \]

are the corresponding velocity components of the post-shock gas. Combining the two expressions for \( U_{1m} \) and \( U_{2m} \), one obtains

\[ M_{a,1m} = \frac{U_{1m}}{\alpha_1} = \frac{M_1 \sin \beta}{\alpha_1} \]

Equation (19)

\[ M_{a,2m} = \frac{U_{2m}}{\alpha_2} = \frac{M_2 \sin (\beta - \delta)}{\alpha_2} \]

For the Mach numbers in the normal direction. Additionally, a relation between the angle of incidence \( \beta \) and the angle of streamlining deflection \( \delta \) can be obtained by using

\[ \frac{U_{2m}}{U_{1m}} = \frac{U_{2m}/U_{1m}}{U_{1m}/U_{1m}} = \tan (\beta - \delta) \]

Along with the jump condition (9) for \( U_{2m}/U_{1m} \), which gives
\[
\frac{\tan(\beta - \delta)}{\tan \beta} = \frac{Z + (y-1) Ma^2 \sin^2 \beta}{(y+1) Ma^2 \sin^2 \beta} \implies \tan \delta = \frac{2 \cotan \beta \left(\frac{Ma^2 \sin^2 \beta - 1}{Z + Ma^2 \left[\frac{y + \cos(2\beta)}{2}\right]}\right)}{2 + Ma^2}
\] (20)

The contours of Eq. (20) are shown schematically in Fig. 8. Note that, for a given deflection angle, \(\delta = \delta_0\), and a given Mach number, \(Ma_1\), Eq. (20) has 2 solutions, \(\beta_w\) and \(\beta_s\), with \(\beta_w < \beta_s\). The solution \(\beta_s\) corresponds to the strong solution, for which the post-shock flow is subsonic (\(Ma_2 < 1\)). Conversely, the solution \(\beta_w\) corresponds to the weak solution, for which the post-shock flow is supersonic (\(Ma_2 > 1\)).

For normal shock waves (\(\beta = 90^\circ\)) \(\implies \cotan \beta = 0\), which implies \(Ma_2 < 1\).

For compression Mach waves (also called weak shocks - in a particularized sense with respect to that used above) such that \(Ma_1 \sin^2 \beta = 1\), or equivalently, \(\sin \beta = \frac{1}{Ma_1} = \sin \mu_m\), where \(\mu_m\) is the Mach angle, note that \(Ma_1 \sin^2 \beta = 1\) implies \(Ma_{1m} = 1\) because of (14), so that the pre-shock flow in weak shocks is asymptotically sonic.

It is worth mentioning that Eqs. (6) - (14) imply the following inequalities: Across the shock:

\[
\frac{\frac{P_2}{P_1}}{\frac{S_2}{S_1}} > 1, \quad \frac{T_2}{T_1} > 1, \quad \frac{U_{2m}}{U_{1m}} < 1, \quad \frac{P_{02}}{P_{01}} < 1, \quad \frac{\rho_{02}}{\rho_{01}} < 1, \quad \frac{T_{02}}{T_{01}} = 1
\] (21)

The flow behind a shock has higher pressure, temperature, entropy, and density, the same stagnation temperature as the flow upstream, and lower values of stagnation pressure and density. In addition, the normal flow behind a shock is always subsonic.
THE HYPersonic LIMIT OF THE RANKine-HugONIOT JUMP CONDITIONS (6) - (14) IS OBTAINED BY MAKING \( M_{1n}^2 \sin^2 \beta \gg 1 \), OR EQUIVALENTLY, \( M_{1n}^2 \gg 1 \), WHICH YIELDS

\[
\frac{P_2}{P_1} = \frac{2 \gamma}{\gamma + 1} M_{1n}^2 \sin^2 \beta, \quad (22) \quad \frac{T_2}{T_1} = \frac{2 \gamma (\gamma - 1)}{(\gamma + 1)^2} M_{1n}^2 \sin^2 \beta, \quad (23) \\
\frac{\rho_2}{\rho_1} = \frac{\gamma + 1}{\gamma - 1}, \quad (24) \quad \frac{u_{2n}}{u_{1n}} = \frac{\rho_1}{\rho_2} = \frac{\gamma - 1}{\gamma + 1}, \quad (25)
\]

For the Pressure, Temperature, Density and Normal-Velocity Jumps, Similarly, The Relation Between the Incidence Angle \( \beta \) and the Streamwise Deflection Angle \( \delta \) [Eq. (20)] CAN BE SIMPLIFIED FOR SLENDER BODIES \( (\delta \ll 1) \) AND HIGH MACH NUMBERS \( M_{1n}^2 \gg 1 \) AND \( \beta \ll 1 \) By Making \( \delta \ll \delta \), \( \sin \beta \approx \sin \delta \), \( \cos (2\beta) \approx 1 \) AND \( M_{1n}^2 \sin^2 \beta \approx M_{1n}^2 \beta^2 \gg 1 \), Which Gives

\[
S = \frac{2}{\gamma + 1} \frac{\beta^2}{\delta} = D \quad \Rightarrow \quad S = \frac{2 \beta}{\gamma + 1}. \quad (26)
\]

Equation (26) Predicts That the Hypersonic Flow Around a Slender Wedge of Small Semi-Angle \( \beta \) (See Fig. 9), Involves a Shock Wave Whose Inclination Angle \( \beta \) is Just \( \beta = 1.25 \delta \) for \( \delta = 1.4 \), In This Way, in Hypersonic Regimes the Shock Wave Envelops the Wedge at a Very Short Distance \( \delta \ll \frac{(\gamma - 1)}{2} \) \( \delta x \), Where \( x \) is the Horizontal Distance From the Leading Edge. Note That \( \delta \ll \) Becomes Smaller Than the Characteristic Boundary-Layer Thickness \( \delta \lesssim \left( \frac{x}{R_{th}} \right) M_{1n}^2 \), When \( \frac{2 M_{1n}^2}{(\gamma - 1) R_{th}^{1/2}} \delta \gg 1 \), Or Equivalently, For Wedge Angles \( \delta \lesssim \frac{2 M_{1n}^2}{(\gamma - 1) R_{th}^{1/2}} \). In This Regime, A Complex Viscous Interaction Occurs in Which the Hot Shock Layer Merges with the Boundary Layer. However, Viscous Effects Will be Excluded from the Formulation for Now.

An Additional Simplification of Eqs (6) - (14) in the Hypersonic Limit, That IS Useful for Slender-Body Theory, Concerns the Post-Shock Flow-Velocity Components Parallel AND Perpendicular to the Upstream Flow, Which Are Denoted Here by the Lowercase Symbols \( u_2 \) AND \( v_2 \), Respectively (Fig. 5).
IN PARTICULAR, $U_2$ AND $V_2$ REPRESENT FLOW DISTURBANCES ASSOCIATED WITH THE STREAMLINE DEFLECTION INDUCED BY THE OBLIQUE SHOCK. THEIR RATIOS TO THE FREE STREAM VELOCITY, $\frac{U_2}{U_1}$ AND $\frac{V_2}{V_1}$, CAN BE CALCULATED AS FOLLOWS FOR ANY MACH NUMBERS:

$$\frac{U_2}{U_1} = \frac{U_2}{U_2} \frac{U_2}{U_2} \frac{U}{U_2} \frac{U}{U_2} = \cos \delta \left( \frac{Z + (8-1)M_{2m}^2}{\gamma + 1} \frac{\sin (\beta - \delta)}{M_{2m}^2} \right) \sin \beta$$

$$\frac{V_2}{V_1} = \frac{V_2}{V_2} \frac{V_2}{V_2} \frac{V}{V_2} \frac{V}{V_2} = \sin \delta \left( \frac{Z + \gamma - 1}{\gamma + 1} \frac{\sin (\beta - \delta)}{M_{2m}^2} \right) \sin \beta$$

IN SIMPLIFYING THE ABOVE EXPRESSIONS, IT IS USEFUL TO NOTE THAT

$$\frac{\sin \beta \cos \delta}{\sin (\beta - \delta)} = \frac{\sin \beta \cos \delta}{\sin \beta \cos \delta - \cos \beta \sin \delta} = \frac{\tan \beta}{\tan \beta - \tan \delta} = \frac{\tan \beta}{\tan (\beta - \delta) M_{2m}^2} = 1 + \tan \delta \tan \beta$$

$$\frac{\sin \beta \sin \delta}{\sin (\beta - \delta)} = \frac{\sin \beta \cos \delta}{\sin \beta \cos \delta = \cos \beta \sin \delta} = \frac{\tan \beta}{\tan (\beta - \delta) M_{2m}^2} = \frac{\tan \beta \tan \delta}{\tan (\beta - \delta)} = 1 + \tan \delta \tan \beta$$

SUBSTITUTING (29) INTO (27) AND MAKING USE OF (20):

$$\frac{U_2}{U_1} = \frac{1}{M_{2m}^2} \left( \frac{Z + (8-1)M_{2m}^2}{\gamma + 1} \frac{\sin (\beta - \delta)}{M_{2m}^2} \right) \sin \beta$$

SIMILARLY, SUBSTITUTING (30) INTO (28) AND MAKING USE OF (20):

$$\frac{V_2}{V_1} = \frac{\tan \delta \tan \beta}{M_{2m}^2} \left( \frac{Z + (8-1)M_{2m}^2}{\gamma + 1} \frac{\sin (\beta - \delta)}{M_{2m}^2} \right) \sin \beta$$

IN THE HYPERSONIC LIMIT, (31) AND (32) BECOME

$$\frac{U_2}{U_1} = 1 - \frac{2 \sin^2 \beta}{\gamma + 1}$$

$$\frac{V_2}{V_1} = \frac{\sin \beta}{\gamma + 1}$$

NOTE THAT $\frac{V_2}{U_1} = O(\delta)$ FOR $\delta$ SMALL, WHILE $\frac{V_2}{U_1} - 1 = O(\delta^2)$.

AN IMPORTANT QUANTITY IN HIGH-SPEED AERODYNAMICS IS THE PRESSURE COEFFICIENT

$$\frac{q_1}{P_2} = \frac{P_2 - P_1}{q_1}$$

WHERE $q_1 = \frac{1}{2} \frac{U_1^2}{P_1}$ IS THE DYNAMIC PRESSURE UPSTREAM FROM THE SHOCK.

NOTE THAT AT HIGH MACH THE SUM OF STATIC AND DYNAMIC PRESSURES DOES NOT EQUAL THE STAGNATION PRESSURE, I.E., $P_1 + q_1 \neq P_0$. 13
In particular, $q_1$ can be rewritten as $q_1 = \frac{1}{2} \frac{\gamma P_1 U_1^2}{\gamma P_1}$, and the pressure coefficient becomes

$$G_1 = \frac{P_2 - P_1}{q_1} = \frac{2}{\gamma M_1^2} \left( \frac{P_2}{P_1} - 1 \right) = \frac{4}{(\gamma + 1) M_1^2} (M_{1,\infty}^2 - 1) \quad (34)$$

In the hypersonic limit, (34) simplifies to

$$G_1 = \frac{4}{\gamma + 1} \sin^2 \beta \quad (35)$$

Combination of (33) and (35) renders the useful relation

$$G_1 = 2 \left( 1 - \frac{U_2}{U_1} \right) \quad (36)$$

which indicates that the decrease in the parallel component of the velocity across the shock produces an increase in the static pressure and a corresponding positive value of the pressure coefficient. Equation (36) is equivalent to the classic result that can be derived from the aerodynamics of slender bodies (i.e., Eq. (148) in the ME355 notes), where $U = U_2 - U_1$ plays the role of the horizontal disturbance of the free stream velocity. Note that (35) (or (36)) indicates that $G_1$ is a second-order quantity in the stream line deflection angle, i.e., $G_1 \sim O(U_2^2) \sim O(\delta^2)$. Equation (35) is also remarkably similar to the results obtained by a much simpler formulation known as Newtonian theory, in which the aerodynamic forces on a body at hypersonic speeds are a sole function of the inclination angle $\delta$ of its surfaces with respect to the free stream, as discussed further below.

As a summary, the lump conditions (6) - (14) are plotted in Fig. 14 along with their hypersonic limit. It is worth highlighting a couple of aspects of the limiting expressions (22) - (25). First, the pressure and temperature ratios appear to increase unboundedly with the Mach number squared. This behavior, for instance, enables the simplification of the

A BLAST WAVE IS A SPHERICAL PRESSURE WAVE THAT MOVES SUPERSONICALLY OUTWARDS FROM AN EXPLOSIVE CORE. THE ACTUAL MOTION IS HYPERSONIC AS A RESULT OF THE EXCEEDINGLY HIGH SPEEDS INVOLVED IN THE DETONATION OF HIGH EXPLOSIVES (\(n = 0 \text{ km/s}\) for TNT). THAT \(P_2/P_1\) IS PROPORTIONAL TO \(M_0^2\) IN (22) IS USED IN BLAST-WAVE THEORY TO NEGLECT THE EFFECT OF THE AMBIENT PRESSURE, WHICH IS \(1/M_0^2\) TIMES SMALLER THAN THE PRESSURE BEHIND THE SHOCK FRONT. THIS ENABLES THE CONSTRUCTION OF A FAMOUS SELF-SIMILAR THEORY FOR BLAST WAVES (SEE PAGE 57 IN ME 355 NOTES). NOTE HOWEVER THAT THE DENSITY REACHES A MAXIMUM COMPRESSION RATIO OF ORDER \(\frac{\gamma + 1}{\gamma - 1}\).

AN ADDITIONAL CONSEQUENCE OF THE PROPORTIONALITY OF \(T_2/T_1\) WITH \(M_0^2\) AT LARGE \(M_0\) IS THE PREDICTION OF UNREALISTICALLY HIGH TEMPERATURES IN THE SHOCK LAYER. FOR INSTANCE, AT MACH 25, WHICH IS REPRESENTATIVE OF REENTRY FROM A DECAYING LOW-EARTH ORBIT, THE TEMPERATURE JUMP INCREASE ACROSS THE SHOCK IS OF ORDER 100, WHICH WOULD RENDER TEMPERATURES IN THE SHOCK LAYER OF ORDER 30,000 K, CORRESPONDING TO 5 TIMES THE SURFACE TEMPERATURE OF THE SUN. SIMILARLY, IN REENTRY FROM DEFLECTED INTERPLANETARY ORBITS, THE MACH NUMBER IS ABOUT 3.5, AS IN THE APOLLO REENTRY FROM THE MOON, WHICH WOULD LEAD TO EVEN HIGHER SHOCK-LAYER TEMPERATURES OF ORDER 100,000 K ACCORDING TO FIG. 6. THESE VALUES ARE UNREALISTICALLY HIGH BECAUSE THE ANALYSIS THAT LEADS TO (6)-(9) ASSUMES A CALORICALLY PERFECT NON-REACTING GAS. THE REALM OF HYPERSONICS IS THAT, AS \(M_0\) INCREASES, THE HIGH TEMPERATURES IN THE POST-SHOCK GAS.
Along with the comparable timescales of the flow and of the relaxation of gasdynamic effects such as vibrational non-equilibrium of gas molecules, lead to a chemically-reacting non-equilibrium gas in which, to begin with, \( CP \) and \( CV \) are no longer constants, the correct calculation of the shock layer that takes into account these complex effects renders shock-layer temperatures of order 7,000 K for Mach 25 and 11,000 K for Mach 40. These complex effects will be discussed in Chapter IV of these class notes.

The Hypersonic Limit for Expansion Waves

\[ \mathbf{Ma}_2 > 1 \]

The expansion of a supersonic flow can be achieved through a convex corner, as in Fig. 42. This generates an expansion fan through which the flow is accelerated and expanded isentropically.

In contrast to shock waves, the expansion fan occupies a finite-thickness region of space, since sharp discontinuities are forbidden in expansion processes because of the 2nd principle of thermodynamics (see page 24 in the notes). A classic analysis of the stream deflection and associated isentropic expansion leads to the relation

\[ \Theta = \Theta(Ma_2) - \Theta(Ma_1) \]  \hspace{1cm} (37)

between the deflection angle \( \Theta \) and the upstream and downstream Mach numbers \( Ma_1 \) and \( Ma_2 \) (see page 27 in the notes). In Eq. (37), \( \Theta(Ma) \) is the Prandtl-Meyer function

\[
\Theta(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \left( \frac{\gamma-1}{\gamma+1} \frac{Ma^2 - 1}{Ma - 1} \right) - \arctan \sqrt{Ma^2 - 1} ,
\]

for \( Ma >> 1 \), the Prandtl-Meyer function becomes

\[
\Theta(Ma) = \sqrt{\frac{\gamma+1}{\gamma-1}} \arctan \left( Ma \frac{\gamma-1}{\gamma+1} \right) - \arctan Ma ,
\]

which can be further simplified by noting that
\[
\arctan(x) = \frac{\pi}{2} - \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} - \frac{1}{x} - O\left(\frac{1}{x^3}\right) \quad \text{for} \quad x \to \infty
\]

So that

\[
\Theta (M_a) = \sqrt{\frac{y+1}{y-1}} \left[ \frac{\pi}{2} - \frac{\sqrt{y+1}}{y-1} \frac{M_a}{1 + \frac{y}{2} M_a^2} + O\left(\frac{1}{M_a^3}\right) \right] - \left[ \frac{\pi}{2} - \frac{1}{M_a} + O\left(\frac{1}{M_a^3}\right) \right]
\]

Which yields the approximate expression

\[
\Theta (M_a) \approx \sqrt{\frac{y+1}{y-1}} - 1 \quad \frac{\pi}{2} - \frac{2}{y-1} \frac{1}{M_a}
\]

(39)

To order \(1/M_a^3\), substitution of (39) into (37) gives

\[
\Theta = \frac{2}{y-1} \left( \frac{1}{M_a} - \frac{1}{M_a} \right)
\]

(40)

For large \(M_a\), while the isentropic change in pressure through the expansion fan is

\[
\frac{P_2}{P_1} = \frac{P_2}{P_0} = \left( \frac{1 + \frac{y-1}{2} M_a^2}{1 + \frac{y-1}{2} M_a^2} \right)^\frac{y}{y-1} = \left( \frac{M_a}{M_a} \right)^{2y/(y-1)}
\]

WHERE USE OF EQUATION (42) HAS BEEN MADE ALONG WITH THE FACT THAT THE STAGNATION PRESSURE IS CONSTANT ALONG STREAMLINES.

NEWTONIAN THEORY OF HYPERSONIC FLOWS

In his early theory of fluid motion (Principia, 1687), Newton envisioned the force exerted on a body submerged in a fluid as a collective effect of fluid particles initially moving rectilinearly and then colliding with the body surface. As in Fig. 43, at collision, the fluid particles would entirely transfer their normal momentum to the body and they would subsequently move parallel to the body surface. Newton associated this description to a rarefied medium in which he postulates no interaction between individual particles and in which he only accounts for the impact force \(F\) caused by the normal transfer of momentum of the particles impinging on the body surface. The resulting force is the normal component of the velocity,

\[
U_m = U_0 \sin \theta
\]

(41)

Multiplied by the mass flow rate incident on the body surface,

\[
\int_{	heta} U_m \sin \theta = \rho \int_{	heta} U_0 \sin \theta \sin \theta d\theta
\]

(42)
COMBINING (41) AND (42) GIVES

\[ F = \frac{4}{3} h_0 U_0^2 \rho_0 = \frac{8}{3} h_0 U_0^2 \rho_0 \sin^2 \delta \]  

(43)

THE CORRESPONDING PRESSURE ON THE WINDWARD SIDE OF THE BODY IN FIG. 8 IS

\[ P - P_0 = \frac{8}{3} h_0 U_0^2 \rho_0 \sin^2 \delta \]  

(44)

AND THE PRESSURE COEFFICIENT IS

\[ C_p = \frac{P - P_0}{\frac{4}{3} h_0 U_0^2} = 2 \sin^2 \delta \]  

(45)

EQUATIONS (43)-(45) WERE NOT ORIGINALLY OBTAINED BY NEWTON BUT BY EPSTEIN IN "ON THE AIR RESISTANCE OF PROJECTILES," PNAS 1931. IT SHOULD BE NOTED THAT THE THEORETICAL APPROACH PRESENTED ABOVE IS A ROUGH ONE THAT DOES NOT ACCOUNT FOR MULTIPLE EFFECTS THAT TODAY WE KNOW EXIST AND INFLUENCE THE AERODYNAMICS OF BODIES IMMERSED IN FLUIDS. FOR INSTANCE, D'ALEMBERT CORRECTLY FORMULATED THE ZERO FORCE THAT A BODY MUST EXPERIENCE IN AN INVISCID FLUID AS THAT TREATED ABOVE. ADDITIONALLY, BOUNDARY-LAYER EFFECTS WERE INTRODUCED LATER AND WERE FOUND IMPORTANT IN DETERMINING THE FORCE, PARTICULARLY AT MODERATE SPEEDS.

However, despite its shortfalls, the Newtonian theory has proven remarkably useful in hypersonic flows for the following reasons. First is that the flow is immediately upstream from the body remains undisturbed since \( Ma \ll 1 \). As a result, the streamlines are straight and the fluid particles do approach the body surface in the same ballistic way as envisioned by Newton. Second is that, at \( Ma \ll 1 \), the shock wave envelops the body very close to its surface, as suggested by equation (26), so that the ballistic approach perdures until very close to the surface, where the direction of the fluid particles must necessarily change to a near-tangential trajectory. Third is that the wave drag emerging from the interaction of the flow and the body (i.e., the pressure drag \( C_p = C_D \sin \delta = 2 \sin^3 \delta \) emerging from the pressures across the nose shock) tends to be a very important component of the total drag force that, in many situations, supersedes the viscous friction drag because of the high Reynolds numbers involved.

In eq. (45), the angle \( \delta \) is generally the local deflection or inclination angle of the surface, thereby giving rise to a general set of methods for computing aerodynamic...
Forces in hypersonics that are called "local surface inclination methods" include the tangent-cone, tangent-wedge, and shock-expansion methods, along with modifications to the Newtonian theory, some of which are discussed further below.

The Newtonian theory is attractive by its simplicity; however, some important aspects should be taken into consideration when employing this theory:

1) The pressure on any leeward surface is equal to the ambient pressure $P_0$, thereby yielding $C_p = 0$ locally there. This approximation clearly breaks down as $M_0$ decreases.

2) The lift coefficient $C_L = C_p \cos \delta = \frac{2\sin^2 \delta \cos \delta}{\cos \delta}$ on a flat plate becomes zero at zero angle of attack. However, the drag coefficient $C_D = C_p \sin \delta = \frac{2\sin^3 \delta}{\cos \delta}$ approaches zero faster as $\delta \to 0$, such that the lift-to-drag ratio $L/D$ becomes infinite, $L/D \to \infty$ at $\delta \to 0$ (see Fig. 15). This result is clearly counter-intuitive and arises because of neglecting the viscous forces generated by friction, which would lead to finite drag at $\delta \to 0$ and zero lift due to flow symmetric ($L/D \to 0$).

3) The asymptotic behavior of the drag coefficient $C_D \sim \delta^3$ for $\delta$ small is in contrast with the $C_D \sim \delta^2$ behavior obtained from the thin-airfoil theory for supersonic flows (e.g., see Eq. (160) in Page 75 of the Messrs. Notes). This highlights the non-linearities typically associated with hypersonic flows that complicate the analysis of the small-disturbance equations as shown further below.

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Fig. 15: Lift and drag coefficients and lift-to-drag ratio on a flat plate at $M_0=1.1$, computed using the Newtonian theory.

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Fig. 16: Surface pressure coefficient (left) and incidence angle (right) for a 15° cone. Numerical solutions are from Sims, T.L., NASA Tech. Rep. SP-8004, 1964.
iv) The results from the Newtonian theory do not depend on the Mach number. This is a manifestation of the "Mach-number independence principle", which will be studied further below within the context of the small-disturbance equations and states that the non-dimensional variables (and the drag and lift coefficients) become mostly independent of $Ma_{10}$ at $Ma_{10} \gg 1$. This independency has been observed in experimental drag coefficients (Figs. 16a-1), which appear to plateau at sufficiently large values of the Mach number.

v) The Newtonian theory tends to overestimate the drag on blunt bodies (see Figs. 16b and 19a). One relevant modification useful for blunt bodies is the one proposed by Lees (1955),

$$g_p = g_{p,\text{max}} \sin^2 \theta$$

where $g_{p,\text{max}}$ is the maximum surface pressure coefficient evaluated behind the normal shock wave created by a blunt body (Fig. 18), namely

$$g_{p,\text{max}} = \frac{p_{2,\text{stagnation}} - p_0}{\frac{1}{2} \rho_0 u_0^2} \frac{2}{\gamma M_{a,10}^2 - 2} \left( \frac{p_{2,\text{stagnation}} - 1}{p_0} \right)$$

where $p_{2,\text{stagnation}}$ is the stagnation pressure behind the shock (note that a stagnation-point-like flow is formed in between the body and the quasi-normal portion of the nose shock so as to make the flow almost stagnant there). Since

$$p_{2,\text{stagnation}} = p_{2,\text{stagnation}} \frac{p_{2,\text{stagnation}}}{p_{\text{p,0}}} \left( \frac{p_{2,\text{stagnation}}}{p_{\text{p,0}}} \right)$$

and substituting Eq. (11) for $p_{2,\text{stagnation}}/p_{\text{p,0}},$ Eq. (17) for $p_{\text{p,0}}/p_{\text{p,0}},$ and Eq. (10) for $Ma^2_{10},$ one obtains

$$\frac{p_{2,\text{stagnation}}}{p_{\text{p,0}}} = \left[ \frac{(s+1)\frac{1}{2}Ma_{10}^2}{4} \right] \left[ \frac{1}{(s+1)(s-1)} \right]$$

Fig. 17: Experimental measurements of the drag coefficients. Panels (a) and (b) correspond to high-speed photographs of ballistic spheres by Hodges (1957) and Charters and Thomas (1945), with $K = \frac{D}{\frac{1}{2} \rho_0 u_0^2} = \frac{1}{8} g_D$. Panel (c) corresponds to wind-tunnel data from Stevens (1950) in a 60° conic-cylinder.
IN THIS WAY, $G_{p_{\text{max}}}$ IN EQ. (47) BECOMES

$$G_{p_{\text{max}}} = \frac{2}{\gamma M_{\infty}^2} \left[ \frac{(8+1)^2 M_{\infty}^2}{2 (8-1)} \right] \frac{Y}{Y+1} \left[ 1 - \frac{\gamma + 2 M_{\infty}^2}{Y+1} \right] - 1 \left\{ \right. $$


$$P = P_0 + \frac{\rho_0 U_0^2}{2} \sin^2 \alpha \quad (49)$$

AS IMPLIED BY EQ. (44) EVALUATED FOR $\alpha = \frac{\pi}{2}$. AS A RESULT, THE SURFACE PRESSURE COEFFICIENT THERE IS

$$C_P = \frac{P-P_0}{\frac{1}{2} \rho_0 U_0^2} = 2 \quad (50)$$

THE RATIO OF (49) TO THE TRUE STAGNATION PRESSURE IS

$$\frac{P}{P_0} = \frac{P_0 + \frac{\rho_0 U_0^2}{2} \sin^2 \alpha}{P_0} \approx \frac{1 + \gamma M_{\infty}^2}{2} \quad (51)$$

WITH $P_0/P_2$ GIVEN BY THE INVERSE OF (43). AS A RESULT, FOR $M_{\infty} \gg 1$, (51) BECOMES

$$\frac{P}{P_0} \approx \frac{Y M_{\infty}^2}{4Y} \left[ \frac{1}{8} \left( \frac{2 Y M_{\infty}^2}{\gamma + 1} \right) - \frac{\gamma}{8} \right] 
\approx \frac{1}{2} \left( \frac{2 Y M_{\infty}^2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \quad (52)$$

NOT THAT THE DISCREPANCY BETWEEN $P$ AND $P_0$ IN (5) DOES NOT DEPEND ON $M_{\infty}$ AT $M_{\infty} \gg 1$, AND GENERALLY RENDERS $P/P_0 > 1$. AS A RESULT THE STRAIGHT NEWTONIAN THEORY TENDS TO OVERESTIMATE THE SURFACE PRESSURE ON THE NOSE, AND CORRESPONDINGLY, THE DRAG COEFFICIENT ON THE BLUNT BODY. ON THE CONTRARY, THE MODIFIED NEWTONIAN THEORY GIVES A SURFACE PRESSURE ON THE NOSE

$$P = P_0 + \frac{\rho_0 U_0^2}{2} G_{p_{\text{max}}} \sin^2 \alpha \quad (53)$$

AND IT CAN BE TRIVIALLY SEEN THAT

$$\frac{P}{P_0} = \frac{P_0 + \frac{\rho_0 U_0^2}{2} G_{p_{\text{max}}} \sin^2 \alpha}{P_0} \approx 1 \quad (53)$$

CASE. THE IMPROVED PERFORMANCE OF THE MODIFIED NEWTONIAN THEORY IS ILLUSTRATED IN FIG. 19, WHERE THE FLOW OVER A PARABOLOID $x = A y^2 + B$ IS ANALYZED. NOTE THAT THE ENTIRE PRESSURE DISTRIBUTION ALONG THE BODY $P(y)/P_0$ CAN BE DERIVED BY MAKING $\sin^2 \alpha = \left( 1 + 4 A^2 y^2 \right)^{-1}$ IN Eqs. (49) AND (53).
Neither the straight Newtonian Theory (45) nor the modified Newtonian Theory (46) take into account the pressure variations induced by the curvature of the body surface. In particular, centrifugal forces tend to induce pressure gradients in the radial direction, as a result, the pressure on the body surface tends to be smaller than the pressure on the edge of the shock layer in flows over blunt bodies (see Fig. 20). The latter pressure is the nominal one predicted by the straight Newtonian theory (44).

Centrifugal corrections to (45) were first formulated by Adolf Busmann (1933) and are known as the Newton-Busmann Theory, which is summarized below.

Consider the calculation of the surface-pressure coefficient at point "s", where the local deflection angle is $\theta_{s}$ and the local pressure is $P_{s}$. The local normal direction is denoted by $\mathbf{m}$, whereas the local radius of curvature of the surface is $R$ (see Fig. 21). In accordance with the Newtonian theory, the free stream penetrates in the thin shock layer and turns suddenly there along a trajectory following very closely the surface of the body. At every point of entry into the shock layer, the incoming streamtubes form thin laminae along paths enveloping the body, as shown schematically in Fig. 16. Mass continuity through this process requires

$$\int s_{w} U_{o} dy = \int s U d\mathbf{m} \quad (53)$$

where $s$ and $U$ are densities and velocities along the laminae. In addition, the conservation of momentum requires a balance between the centrifugal acceleration and the pressure gradient, namely

$$\frac{2P}{\mathbf{m}} = \frac{sU^{2}}{R} \quad (54)$$

Integrating (54) from a point on the body surface to another one immediately above on the edge of the shock layer, one obtains

$$\int_{P_{s}}^{P} dP = \int_{0}^{s_{s}} \frac{2U^{2}}{R} d\mathbf{m} = \int_{0}^{Y_{3}} \frac{s_{s} U_{o} U}{R} dY \quad (55)$$
Where use of Eq. (53) has been made in the formulation, \( \delta_{SL} \) is the thickness of the shock layer. In the Newtonian limit, \( \delta_{SL} \to 0 \), (55) becomes

\[
P - P_S = \int_0^{Y_S} \frac{f_0 U_0^2}{R} dy 
\]

where \( Y_S \) is the vertical location of the shock-layer edge, since \( R \), which is given by \( R = -\left(\sin \delta \right)^{-1} \frac{dy}{d\delta} \), and represents the radius of curvature of the streamlines, is constant along \( \delta \) (i.e., all streamlines are deflected by an angle \( \delta \) upon entering the shock layer), then (56) becomes

\[
P_S = P + \frac{f_0 U_0^2}{\left(\sin \delta \right) \cos \delta} \int_0^{Y_S} dy 
\]

where the Newtonian relation \( U = U_0 \cos \delta \) has been used indicating that the velocity within the shock layer is parallel to the local surface. Equation (57)

can be easily rewritten as

\[
\frac{P_S}{P} = 2 \sin^2 \delta + 2 \left(\frac{d\delta}{dy}\right) \cos \delta \int_0^{Y_S} dy 
\]

(58)

\( \frac{P_S}{P_0} = \frac{f_0 U_0^2}{2} \frac{1}{R} \frac{\delta}{\sin \delta} \)

which corresponds to the Newton-Busemann formula. In particular, the second term in (58) corresponds to a centrifugal correction to the Newton Eq. (45) that depends on the accumulated history of the streamlines upstream of that point. For slender bodies, \( \sin \delta \approx \delta \) and \( \cos \delta \approx 1 \), so that (58) becomes

\[
\frac{P_S}{P_0} = 2 \delta^2 + 2 \kappa y 
\]

(59)

where \( \kappa = 1/R \) is the local curvature. It is worth remarking, however, that the utilization of (58) generally leads to worse results than the straight and modified Newtonian theories, as shown in Fig. 22(a).

\[\text{Fig. 22, Hypersonic Flow Around a Bi-convex Airfoil (From Anderson 2006, and Eggers et al. NACA1423 (1953), Including Numerical, Straight Newtonian, and Newton-Busemann Solutions for } (a) \gamma = 1.4 \text{ and } (b) \gamma = 1.05.\]
The Newtonian theory tends to be more accurate for three-dimensional bodies of revolution than two-dimensional ones. This aspect is elaborated further below within the context of the Taylor-Maccoll theory for the hypersonic flow over cones.

To summarize, the straight Newtonian theory is preferable for straight bodies such as wedges or cones, and its predictions become increasingly more accurate as the Mach number increases (see Fig. 16 and Fig. 17(c)). On the other hand, the modified Newtonian theory is preferable for blunt bodies (Figs. 17(a) and 19(a)) since it takes into account the overpressure behind the near-normal portion of the nose shock. The centrifugal corrections in the Newton-Busemann theory tend to degrade the predictions in all cases. Remarkably, the latter become more accurate than the straight and modified Newtonian theories in the limit $\gamma \to 1$ (see Fig. 22(b)). The relevance of the $\gamma \to 1$ is described below.

The combined limit $Ma_{0} \to \infty$ and $\gamma \to 1$.

It is interesting to note that the Newton-Busemann theory generally worsens the predictions except for the case in which $\gamma \to 1$, as revealed in Fig. 22(b), where it clearly matches the numerical simulations (also at $\gamma \to 1$) better than the straight Newtonian theory, as expected. As a consequence, two important questions arise from these observations: 1) is the Newtonian theory a simplified representation of the solution to the conservation equations in the limit $\gamma \to 1$, $Ma_{0} \to \infty$, and $2$) why is the modified Newtonian theory accurate for some problems where $\gamma \approx 1.4^{2}$, the answer to the first question is yes, as shown below, and consequently, the answer to the second question is that the agreement appears to be fortuitous.

Consider the oblique-shock jump conditions (6)-(14) along with relation (20) between $\beta$ and $\zeta$. In the limit $Ma_{0}^{2} \sin^{2} \beta \gg 1$, corresponding to very large Mach numbers (particularly, large enough to make the normal Mach number $Ma_{0} \gg Ma_{*} \sin \beta$ much larger than unity), then equations (6)-(9)
BECOME (22) - (25). NOTE THAT THOSE EXPRESSIONS ARE VALID FOR ANY ANGLE OF INCIDENCE INsofar AS $\frac{M_{a_1}^2 \sin^2 \beta}{\Lambda}$, OR EQUIVALENTLY, $\sin \beta \gg \frac{1}{M_{a_1}^2 L}$, so that $\mu_M \ll \beta \ll \frac{\pi}{2}$, WITH $\mu_M = \Delta \csc (\frac{L}{M_{a_1}^2})$ THE MACH ANGLE.

IN THIS LIMIT, THE RELATION (20) BECOMES

$$\tan \delta = \frac{2 \cot \beta \sin^2 \beta}{\gamma + 1 - 2 \sin^2 \beta} \Rightarrow \tan (\beta - \delta) \approx \frac{(\gamma - 1)}{\gamma + 1} \tan \beta.$$  (60)

IN INTERPRETING (22) - (25) AND (60), IT IS IMPORTANT TO NOTE THAT THE STREAMLINE DEFLECTION ANGLE $\delta$ AND THE JUMPS OF DENSITY $\frac{P_2}{P_1}$ AND NORMAL VELOCITY $\frac{U_{2m}}{U_{1m}}$ DO NOT DEPEND ON THE MACH NUMBER, WHEREAS THE PRESSURE AND TEMPERATURE JUMPS, $\frac{P_2}{P_1}$ AND $\frac{T_2}{T_1}$, ARE UNBOUNDED BY THE MACH NUMBER. A SIMILAR QUANTITY THAT BECOMES INDEPENDENT OF THE MACH NUMBER IS THE PRESSURE COEFFICIENT (35). THE FLOW STRUCTURE AT $\frac{M_{a_1}^2 \sin^2 \beta}{\Lambda}$ THEREFORE RESEMBLES THE ONE IN FIG. 23a.

THE ANGLE FOR SHOCK DETACHMENT
SEE NOTE NEXT TO FIG. 39.

FIG. 23

IF, ADDITIONALLY, THE BODY IS SLENDER ($\delta \ll \delta$), THEN (60) BECOMES

$$\beta - \delta \approx \frac{(\gamma - 1)}{\gamma + 1} \beta \Rightarrow \beta \approx \frac{(\gamma + 1)}{2} \delta$$  (61)

WHICH IS THE SAME AS THE RESULT OBTAINED IN EQ. (21). AS STRESSED ABOVE, HOWEVER, $\delta$ AND $\beta$ CANNOT BE ARBITRARILY SMALL, BUT $\beta \gg \mu_M$ FOR THE HYPERSONIC LIMIT (22) - (25) TO HOLD (i.e., $\frac{M_{a_2}^2 \sin^2 \beta}{\Lambda}$), ONLY IN THE LIMIT $M_{a_1} \rightarrow \infty$ CAN $\beta$ AND $\delta$ TEND TO ZERO WHILE SATISFYING $\frac{M_{a_1}^2 \sin^2 \beta}{\Lambda}$. 

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The structure of the flow for the slender case is sketched in Fig. 23(b). In deriving the expressions for the post-shock velocity and Mach number, it is convenient to define the parameter

\[ \varepsilon = \frac{\gamma - 1}{\gamma + 1} \quad (62) \]

Using the definition (62), equations (22)-(25) can be expressed as

\[ \frac{P_2}{P_1} = (1+\varepsilon) \frac{M_a^2 \sin^2 \beta}{1 - \varepsilon} \quad \frac{S_2}{S_1} = \frac{E}{1 + \varepsilon} \quad \frac{T_2}{T_1} = (1+\varepsilon) \frac{M_a^2 \sin^2 \beta}{\gamma - 1} \quad \frac{U_{2m}}{U_{1m}} = E \quad (63) \]

while Eq. (60) becomes

\[ \tan(\beta - \delta) = \frac{E \tan \beta}{1 + \varepsilon \tan^2 \beta} \Rightarrow \sin(\beta - \delta) = \frac{E \tan \beta}{\sqrt{1 + \varepsilon \tan^2 \beta}} \quad (64) \]

In addition, the hypersonic limit \( \frac{M_a^2 \gg 1}{} \) of Eq. (40) is

\[ M_{a2m} = \sqrt{\frac{(\gamma - 1)}{2\gamma}} = \frac{E}{1 + \varepsilon} \quad (65) \]

In these variables, the post-shock velocity in Fig. 18(a) is

\[ U_2 = \frac{U_{2m}}{\sin(\beta - \delta)} = \frac{E U_{1m} \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2}}{\frac{1}{\tan^2 \beta}} = \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2} U_1 \sin \beta \quad (66) \]

while the Mach number of the post-shock flow is

\[ M_{a2} = \frac{M_{a2m}}{\sin(\beta - \delta)} = \frac{E}{1 + \varepsilon} \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2} = \left( \frac{1}{\tan^2 \beta} + \varepsilon^2 \right)^{1/2} \quad (67) \]

For the slender case (\( \beta \ll 1 \), but \( \beta \gg \mu \)), then Eq. (60) becomes

\[ \beta - \delta \approx \varepsilon \beta \Rightarrow \beta \approx \frac{\delta}{1 - \varepsilon} \quad (68) \]

which, after substitution of (62), is exactly the same expression as (26).

In this case, the post-shock velocity is simply

\[ U_2 = U_{2m} \quad \frac{1}{\beta - \delta} \quad (69) \]

while the corresponding Mach number is

\[ M_{a2} = \frac{M_{a2m}}{\beta - \delta} = \sqrt{\frac{E}{1 + \varepsilon}} \frac{1}{\varepsilon \beta} = \sqrt{\frac{1}{(1+\varepsilon)E \varepsilon \beta}} \quad (70) \]

The surface pressure coefficients are

\[ C_p \approx 2 \left(1 - \varepsilon \right) \sin^2 \beta \quad (71) \]
For the case in Fig. 23(a), and
\[ G_p \approx 2(1 - \varepsilon) \beta^2 \] (71)

For the case in Fig. 23(b), it is illustrative to inquire about the limit \( \varepsilon \to 0 \) (i.e., \( \gamma \to 1 \)) in the above equations, which are solutions of the Euler equations in the hypersonic limit, and ask whether the resulting expressions have any resemblance with the Newtonian theory.

In the limit \( \varepsilon \to 0 \), the jump conditions (63) become
\[ \frac{P_2}{P_1} \to M_{\infty}^2 \sin^2 \beta, \quad \frac{\rho_2}{\rho_1} \to \frac{1}{\varepsilon}, \quad \frac{T_2}{T_1} \approx \frac{\varepsilon M_{\infty}^2 \sin^2 \beta}{1} \quad \text{(Indeterminate)} \]
\[ \frac{U_{2m}}{U_{1m}} \approx \varepsilon \to 0, \quad (72) \]

thereby expressing that the post-shock flow becomes infinitely dense, it has zero post-shock normal velocity, and its temperature is indeterminate.

In addition, the normal Mach number of the post-shock flow is
\[ M_{2m} \approx \varepsilon^{1/2} \to 0 \] (73)
indicating that the flow normal to the shock wave is infinitely slow compared to the local speed of sound.

For the case in Fig. 23(b), the relation (64) becomes
\[ \beta \approx \delta \] (74)
so that the shock wave and the surface of the body are coincident, thereby yielding a shock layer of zero thickness through which the fluid moves parallel to the surface at a velocity
\[ U_2 \approx U_1 \cos \delta \] (75)
which corresponds to an infinite Mach number
\[ M_{2m} \approx \frac{1}{\varepsilon^{1/2} \tan \beta} \to \infty \] (76).

The surface pressure coefficient is, however, finite and equals
\[ \Delta p \approx 2 \sin^2 \delta \] (77)
which is exactly the same as the one predicted by the Newtonian theory, [Eq. (45)]. The \( \varepsilon \to 0 \) limit for the slender case in Fig. 23(b) yields...
SIMILAR RESULTS AS THOSE PRESENTED ABOVE, WITH THE SHOCK WAVE AND THE BODY SURFACE BECOMING COINCIDENT IN SPACE, AS SCHEMATICALLY SHOWN IN FIG. 24.

\[ M_2 > 1 \quad (M_2^2 \sin^2 \beta > 1) \]

\[ \eta_2 \approx 2 \sin^2 \beta \]

\[ \delta \approx 0(1) \]
\[ \beta \approx 0(1) \]
\[ \varepsilon \rightarrow 0 \]

(BUT \( \delta < \delta_{MAX} \), 0, ATTACHED SHOCK)

FIG. 24: NEWTONIAN AERODYNAMICS.

THE CONSIDERATIONS GIVEN ABOVE SUGGEST THAT THE NEWTONIAN THEORY CORRESPONDS TO THE COMBINED LIMIT \( M_2 \rightarrow 1 \) AND \( \varepsilon \rightarrow 0 (x \rightarrow 1) \) OF THE INVIScid FLOW EQUATIONS, IN WHICH THE SHOCK AND THE BODY SURFACE BECOME COINCIDENT, THE SHOCK LAYER ATTAINS ZERO THICKNESS AND EMBRACES A FLUID WITH INFINITE DENSITY AND INFINITE MACH NUMBER, AND WHOSE VELOCITY IS JUST THE FREE STREAM VELOCITY MULTIPLIED BY THE COSINE OF THE LOCAL SURFACE INCLINATION ANGLE.

THE ESTIMATES GIVEN BELOW CORRESPOND TO THE LEADING-ORDER APPROXIMATION OF THE NEWTONIAN THEORY, NAMELY \( \varepsilon = 0 \). SECOND-ORDER APPROXIMATIONS PROVIDE USEFUL ESTIMATES, INCLUDING THE STAND-OFF DISTANCE OF A SHOCK WAVE IN THE HYPERSONIC FLOW OVER A BLUNT BODY, WHICH IS OF ORDER \( \Delta \varepsilon R_0 \), WITH \( R_0 \) THE RADIUS OF CURVATURE OF THE NOSE (SEE FIG. 25).

THIS PROBLEM IS INVESTIGATED FURTHER BELOW IN THE LAST SECTION OF THIS CHAPTER.

THE PHYSICAL SIGNIFICANCE OF THE DENSITY RATIO

THE PARAMETER \( \varepsilon \) DEFINED IN (64) PLAYS A FUNDAMENTAL ROLE IN THE LIMITING BEHAVIOR DESCRIBED ABOVE. IN PARTICULAR, \( \varepsilon \) REPRESENTS THE INVERSE OF THE DENSITY RATIO \( \frac{\rho_2}{\rho_1} \) ACROSS
THE SHOCK, WHICH BECOMES INFINITY IN THE LIMIT $\varepsilon \to 0$. BEFORE ELABORATING FURTHER ON THAT LIMIT, IT IS ILLUSTRATIVE TO OBTAIN A GENERAL EXPRESSION FOR $\varepsilon$ THAT DOES NOT INVOLVE THE ASSUMPTION OF A PERFECT GAS. IT IS IMPORTANT TO NOTE THAT THE EQUVALENCE BETWEEN $\varepsilon=0$ AND $\gamma=1$ EMERGES BECAUSE OF THE ASSUMPTION $\gamma = C_P / C_V$ WITH $C_P$ AND $C_V$ BEING CONSTANTS IN THE FORMULATION, AN ASSUMPTION CORRESPONDING TO A (CALORICALLY AND THERMALLY) PERFECT GAS THAT IS REQUIRED TO DERIVE THE RANKINE-HUGONIOT JUMP CONDITIONS (6) - (11). THESE RELATIONS ARE OBTAINED BY COMBINING Eqs. (2) - (5) TO OBTAIN

$$\begin{align*}
\left( P_2 - P_1 \right) &= -\frac{M^2}{a} \quad \text{(RAYLEIGH LINE, WITH } M = \frac{P_1 U_n}{U_2 M} = \frac{P_2 U_2 M}{U_2 M}) \\
\frac{1}{S_2} - \frac{1}{S_1} &= h_2 - h_1 = \frac{1}{Z_0} \left( P_2 - P_1 \right) \left( \frac{\gamma_s}{S_2} + 1 \right) \quad \text{(HUGONIOT CURVE)}
\end{align*}$$

(78) \quad (79)

AND THEN APPLYING THE PERFECT-GAS EQUATIONS $P = S_0 T$ AND $h = C_P T$, WITH $C_P$ A CONSTANT SPECIFIC HEAT AT CONSTANT PRESSURE. NOTE HOWEVER THAT Eqs. (78) AND (79) ARE GENERAL CONSERVATION LAWS VALID ALSO FOR NON-PERFECT GASES. IN SO FAR AS CHEMICAL AND THERMODYNAMIC EQUILIBRIUM CONDITIONS PREVAIL IN THE FLOW (EQUILIBRIUM CONDITIONS ARE DISCUSSED LATER IN THE TEXT), IN THESE GENERAL CONDITIONS, AN EXPRESSION FOR THE DENSITY RATIO CAN BE OBTAINED DIRECTLY FROM (79) AS

$$\varepsilon = \frac{S_1}{S_2} = \frac{P_2 - P_1}{(h_2 - h_1) S_2 - (P_2 - P_1)}$$

(80)

IN THE HIGH-MACH NUMBER LIMIT, $P_2 / P_1 \gg 1$ AND $h_2 / h_1 \gg 1$, AND THE ABOVE EXPRESSION SIMPLIFIES TO

$$\varepsilon = \frac{S_1}{S_2} \approx \frac{P_2 / S_2}{h_2 + \varepsilon_2}$$

WHERE USE OF THE RELATION $h_2 = \varepsilon_2 + P_2 / S_2$ HAS BEEN MADE, IN THIS WAY, $\varepsilon$ IS A SINGLE FUNCTION OF THE POST-SHOCK THERMODYNAMIC STATE. NOTE THAT FOR A PERFECT GAS, $P_2 / S_2 = R T_2$, $h_2 = C_P T_2$, $\varepsilon_2 = C_V T_2$; $\gamma = C_P / C_V$, AND

$$\varepsilon = \frac{\gamma - 1}{\gamma + 1}$$

(81)
The Newtonian limit therefore corresponds to the limit in which the post-shock gases are infinitely dense, which necessarily requires zero cross section of the shock layer by mass conservation. In perfect monoatomic gases, $\gamma = \frac{5}{3}$, and $\delta = \frac{1}{3}$; in perfect diatomic gases, $\gamma = \frac{7}{5}$ and $\varepsilon = \frac{1}{6}$. In both of these cases, $\delta > 1$ and the Newtonian approximation does not necessarily apply. In more realistic conditions typically encountered in the post-shock gases at hypersonic Mach numbers, the gases tend to undergo chemical dissociation and vibrational excitation, both of which tend to decrease $\varepsilon$ in (80) below its perfect value (81), reaching density ratios of order $\varepsilon \approx \frac{1}{15}$. This is to highlight that, despite the fundamental complexities associated with real hypersonic flow effects, these complexities tend to drive $\varepsilon$ to smaller values in the realm of the Newtonian theory. It will be shown later in the course that the perfect-gas-based theory (6)–(11) leads to unrealistically high post-shock temperatures due to the finite and not-so-small density ratio (81) that is obtained by evaluating it with $\delta = 1.4$.

For now this can be understood by using the equation of state and expressions (6)–(7), for a calorically perfect gas,

$$\frac{P_2}{P_1} = \frac{\delta_2}{\delta_1} \frac{T_2}{T_1} \propto \frac{M_0^2}{\varepsilon}$$

which indicates that, for a fixed $\varepsilon$, $T_2/T_1$ increases quadratically with the Mach number, thereby yielding unrealistically high post-shock temperatures of order $T_2 \approx 60,000 \text{ K}$ at $M_0 = 30$ for a pre-shock gas at temperature $T_1 = 300 \text{ K}$.

It will be shown later in the course that non-perfect effects, including chemical and vibrational excitation phenomena, come into play at high Mach numbers to and moderate the temperature of the post-shock gases by influencing the denominator in equation (80).
The supersonic flow field around a cone differs from that around a wedge. In the latter, the velocity and thermodynamic state of the post-shock flow are uniform and equal to the solution prescribed by the shock jump conditions (6)-(11). On the other hand, the flow over a cone is an axisymmetric one where the streamlines converge as the fluid particles move downstream. As a result, the velocity and thermodynamic state of the post-shock gas cannot be uniform. Note, however, that the shock jump conditions (6)-(11) are always locally valid and they have to be matched by the post-shock gas flow at the shock wave.

The solution for the inviscid supersonic flow over a cone was first proposed by Busemann (1929) and later by Taylor and MacCull (Proc. Roy. Soc. A 139, 1933). Assuming a semi-infinite cone, as in Fig. 26, a solution exists in which the flow variables only depend on the angle $\theta$. Equivalently, the flow variables are constant along every ray $\Theta = \text{const.}$ emanating from the vertex, thereby rendering a conical flow field. It is important to note that such conical flow field is compatible with the boundary conditions at the body surface and at infinity if the flow is supersonic and inviscid. In contrast, such conicity is incompatible with flows at finite Reynolds numbers because of the non-conical development of the ensuing boundary layer.

Consider a spherical coordinate system $(r, \theta, \phi)$, where $r$ is the distance from the cone vertex, $\theta$ is the latitude angle measured from the cone axis, and $\phi$ is the longitude angle, which does not play any role in the present configuration at zero angle of attack. The flow velocity components in the radial and latitudinal directions are denoted as $u$ and $v$, respectively.
Because the shock is straight, the vorticity in the $\theta$ direction must vanish,

$$\omega = \frac{1}{r} \left[ \frac{2}{\partial r} (r \nu) - \frac{\partial u}{\partial \theta} \right] = 0 \Rightarrow \frac{du}{d \theta} - v = 0 \quad (82)$$

Similarly, the equation of continuity requires

$$\frac{1}{r^2 \partial r} \left[ \left( \frac{2}{\partial r} (r \nu) \right) + \frac{2}{\partial \theta} (\nu \sin \theta) \right] = 0 \Rightarrow 2 \nu \sin \theta + \frac{d}{d \theta} (\nu \sin \theta) = 0 \quad (83)$$

Combining (82) and (83) leads to the equation

$$\frac{d^2 u}{d \theta^2} + \frac{1}{\sin \theta} \frac{du}{d \theta} + \frac{c_0 \theta}{d \theta} \frac{du}{d \theta} + 2u = 0 \quad (84)$$

which can be further simplified by making use of the Bernoulli equation (13)

written in the form

$$\frac{\gamma}{\gamma-1} \frac{P}{\nu} + \frac{u^2+v^2}{2} = \frac{\gamma}{\gamma-1} \frac{P_{02}}{\nu_{02}} \quad (85)$$

where the enthalpy $h = C_p T$ has been rewritten as $h = \frac{\gamma}{\gamma-1} \left( P \right)^{\frac{\gamma}{\gamma-1}}$ by making use of the relation $C_p - C_v = R_0/\gamma$. The definition of the adiabatic coefficient $\gamma = C_p / C_v$.

And the ideal gas equation of state $P = R_0 T$. Note that this analysis assumes a calorically perfect gas. In Eq. (85), $P_{02}$ and $\nu_{02}$ refer to the stagnation pressure and density of the post-shock gases. Note that these are uniform in the post-shock gases since the flow is homentropic. As a result, the stagnation pressure and density are also related to the corresponding static values through the isentropic trajectory

$$\frac{P}{\gamma} = \frac{P_{02}}{\nu_{02}} \quad (86)$$

As observed from Eq. (17). In this way, (85) can be rewritten as

$$\left( \frac{\nu}{\nu_{02}} \right)^{\gamma-1} = 1 - \left( \frac{\gamma-1}{2} \right) \left( \frac{u^2 + v^2}{a_{02}^2} \right) \frac{u^2 + v^2}{V_{0}^2} \quad \text{with} \quad V_{0} = \left( \frac{2}{\gamma-1} \right)^{\frac{1}{2}} c_{02}, \quad (87)$$

where $a_{02}$ is the stagnation speed of sound of the post-shock gas (also defined in Eq. 17). Note that, according to (87), the velocity $V_{0}$ corresponds to the velocity of the gas if it was flowing in near-vacuum conditions ($\delta_{20}$).

Differentiating the density in (87) and substituting (82) gives

$$\left( \frac{\nu}{\nu_{02}} \right)^{\gamma-1} \frac{d \nu}{d \theta} + \frac{1}{\sin \theta} \frac{du}{d \theta} \frac{du}{d \theta} \frac{d^2 u}{d \theta^2} = 0 \quad (88)$$
However, an expression for \( \frac{dS}{d\theta} \) can be also obtained from (84), namely

\[
\frac{dS}{d\theta} = - \frac{2u + \cot \theta \frac{du}{d\theta} + \frac{d^2 u}{d\theta^2}}{du/d\theta} \quad (84)
\]

Substituting this expression into Eq. (88) yields

\[
- \left( \frac{3}{\rho_{w1}} \right)^{\gamma - 1} \left[ \frac{2u + \cot \theta \frac{du}{d\theta} + \frac{d^2 u}{d\theta^2}}{du/d\theta} \right] + \frac{d}{\rho_{w1}^2} \left\{ \frac{d}{d\theta} \left( \frac{du}{d\theta} \right)^2 \right\} = 0 \quad (90)
\]

Lastly, substituting (82) and (87) into (90) gives

\[
\left( \frac{\gamma - 1}{2} \right) \left[ 1 - \frac{1}{\rho_0^2} \left( \frac{dU}{d\theta} \right)^2 \right] \left\{ \frac{2u + \cot \theta \frac{du}{d\theta} + \frac{d^2 u}{d\theta^2}}{du/d\theta} \right\} = \frac{d}{\rho_0^2} \left( \frac{dU}{d\theta} \right)^2 \quad (91)
\]

which represents a non-linear ordinary differential equation for \( u(\theta) \) given a value of the characteristic velocity \( \rho_0 \). Note that \( \rho_{w1} \) is related to \( \rho_0 \) by the continuity of the stagnation enthalpy across the shock, \( \rho_{w1} = \rho_0^2 \).

As a result, the solution to (94) is specified once the stagnation temperature of the incoming flow is provided to the equation. In addition, the non-penetration boundary condition

\[
V = \frac{dU}{d\theta} = 0 \quad \text{at} \quad \theta = \frac{\pi}{2} \quad (92)
\]

has to be satisfied on the surface of the cone, while the post-shock velocities

\[
U = U_{2t} = U_{0t} = U_{\infty} \cos \beta
\]

\[
V = \frac{dU}{d\theta} = \frac{U_{2m} - U_{0t} \cos \beta}{\rho_{w1} \sin \beta} = - \frac{dU}{d\theta} \sin \beta \quad (93)
\]

have to be recovered immediately downstream of the oblique shock. It is important to note that the incidence angle \( \beta \) in (93) is part of the solution. Note that \( \beta \) and \( \theta \) are not actually related through Eq. (20) in this problem since the local deflection angle of the streamlines is not necessarily \( \theta \) (see Fig. 27) because of the 3D flow displacement effect created by the conical geometry.

As a result, (94) - (93) has to be solved iteratively by first assuming an angle \( \beta \), computing the values
OF \( U \) AND \( \frac{du}{dy} \) AT \( \theta = \beta \) FROM (9.3), INTEGRATING (9.1), AND CHECKING WHETHER THE CONDITION (9.2) IS SATISFIED.

THE STAGNATION SPEED OF SOUND \( a_{20} \) IS RELATED TO THE SPEED OF SOUND IN THE UNDISTURBED GAS \( a_{\infty} \) BY

\[
\frac{a_{20}^2}{a_{\infty}^2} = \frac{a_{\infty}^2}{a_{\infty}^2} = 1 + \frac{(\gamma - 1) M_{\infty}^2}{2} \Rightarrow a_{20}^2 = \frac{a_{\infty}^2}{1 + \frac{(\gamma - 1) M_{\infty}^2}{2}} \tag{9.4}
\]

SIMILARLY, THE DISTRIBUTION OF \( P \) AND \( \rho \) ANYWHERE BEHIND THE SHOCK IS GIVEN BY COMBINING Eqs. (85) AND (86) TO ELIMINATE THE DENSITY, WHICH GIVES

\[
\frac{T}{T_{\infty}} = \left( \frac{P}{P_{\infty}} \right)^{\frac{\gamma - 1}{\gamma}} = \left( \frac{\rho}{\rho_{\infty}} \right)^{\frac{\gamma - 1}{\gamma}} = 1 - \frac{(\gamma - 1)}{2} \left( \frac{U^2 + V^2}{a_{02}^2} \right) \tag{9.5}
\]

WITH \( P_{\infty} \) AND \( \rho_{\infty} \) BEING RELATED TO THE STAGNATION QUANTITIES IN THE UNDISTURBED STREAM, \( P_{\infty} \) AND \( \rho_{\infty} \) THROUGH THE SHOCK JUMP CONDITION (14).

THERE ARE A NUMBER OF IMPORTANT DIFFERENCES BETWEEN THE FLOW OVER A 2D WEDGE AND THE FLOW OVER A CONE. PERHAPS THE MOST IMPORTANT ONES PERTAIN TO THE RELIEVING EFFECT CAUSED BY THE CONICAL GEOMETRY, WHICH, FOR THE SAME MACH NUMBER, LEADS TO A WEAKER SHOCK IN THE CONE CASE (6-8, \( \beta_{\text{cone}} < \beta_{\text{wedge}} \)) AND CORRESPONDINGLY CREATES LOWER VALUES OF \( P, T \) AND \( \rho \) ON THE CONE SURFACE, AS SKETCHED IN FIG. 28(a). FOR THE SAME REASON, THE NOSE SHOCK TENDS TO BECOME DETACHED EARLIER IN THE WEDGE THAN IN THE CONE AS THE SURFACE INCLINATION ANGLE INCREASES AT A GIVEN MACH NUMBER OF THE FREE STREAM, AS DEPICTED IN FIG. 28(b).

FIGURES 29 AND 30 SHOW THE SHOCK ANGLE \( \beta \) AND THE PRESSURE COEFFICIENT \( C_P \) AS A FUNCTION OF THE MACH NUMBER OF THE FREE STREAM AND FOR A NUMBER OF SURFACE INCLINATION ANGLES \( \theta \). SIMILAR TO THE WEDGE CASE IN FIG. 8, THERE EXIST STRONG AND WEAK SOLUTIONS OF THE SHOCK; AS WELL AS A CRITICAL VALUE OF \( M_{\infty} \) BELOW WHICH THERE IS NO SOLUTION FOR AN ATTACHED SHOCK.
AND THE SHOCK MUST BECOME DETACHED AS IN
FIG. 28b. AN EXAMPLE OF A THREAD OF EXPERIMENTAL
PHOTOGRAPHS OF THE SUPersonic FLOW AROUND CONICAL
PROJECTILES IN FLIGHT IS PROVIDED IN FIG. 31
THAT ILLUSTRATES THE TRAJECTORY A→B IN Fig. 29
BY WHICH THE DECREASE IN $M_{0d}$ CAUSES THE SHOCK
TO BE DETACHED FROM THE PROJECTILE. WHILE THE
MAXIMUM SURFACE INCLINATION ANGLE FOR WHICH AN
ATTACHED SHOCK SOLUTION EXISTS IN A WEDGE IS
$\delta_{\text{max}} \approx 45.6^\circ$ (SEE FIG. 8), THE CORRESPONDING
VALUE FOR THE CONE CAN BE OBTAINED NUMERICALLY
AND IS EQUAL TO $\delta_{\text{max}} \approx 57.6^\circ$.

IN FIGS. 24-25, THE HYPERSONIC LIMIT CORRESPONDS
to $M_0^2 \sin^2 \beta \gg 1 \ (M_0^2 \gg 1)$, WHICH IS OBSERVED
TO LEAD TO A PLATEAU IN ALL CURVES WHERE $\beta \approx \delta$,
WITH DIFFERENCES $\beta - \delta = O(0.01, \delta)$ WHERE $\varepsilon = \delta/\delta_2$.

This point can be
formally proven by expanding
the velocities $u$ and $v$
Near the cone surface
$\left(u = u_0 \left(1 - (\beta - \delta)^2\right), v = u_0 (\beta - \delta)\right)$,
with $u_0 \approx u_0 \cos \beta$ THE TANGENTIAL VELOCITY ON THE CONE SURFACE,
and then substituting these
approximations into (93). Note
that the straight Newtonian
theory $q_f = 2 \sin^2 \delta$ provides
a reasonable result in
the hypersonic limit when
compared to the numerical
solution (Fig. 30).

FIG. 31. SUPersonic FLOW AROUND A CONICAL
PROJECTILE WITH $\delta = 20^\circ$ (MACCOLL 1936)
THE MACH-NUMBER INDEPENDENCE PRINCIPLE

Most engineering applications in hypersonics involve cruising or re-entering vehicles at mach numbers in the range $5 \leq M_\infty \leq 30$. However, most hypersonic test facilities are capable of achieving Mach numbers up to approximately $M_\infty = 10-20$ (see page 22, for a discussion on experimental facilities). The results presented in Figs. 16, 17 and 30 suggest that some of the results may become independent as the Mach number increases to large values. This independence tends to be invoked to overcome the deficiencies of experimental facilities in reproducing real flight conditions.

The Mach-number independence principle was derived by Oswatitsch (ZAMP, 24-264 (1951)) for inviscid flow. For blunt bodies, such as the one sketched in Fig. 32, the Mach-number independence is achieved for $M_\infty \gtrsim 1 \quad (M_\infty \gtrsim 5, \text{see Fig. 17(a,b)})$.

For slender bodies, the Mach-number independence is achieved for $M_\infty \sin \theta \gtrsim 1$ so as to make the normal Mach number $M_{\infty, n} \gtrsim 4$. As a result, the Mach number independence is achieved at much larger free-stream Mach numbers ($M_\infty \geq 10$, see Figs. 16 and 17(c)) for slender bodies.

In Oswatitsch's description of this principle, the following variables become independent of the Mach number: the force and moment coefficients, the lift-to-drag ratio, the bow-shock shape, the bow-shock stand-off distance, the flow patterns, the velocity field $\frac{\vec{u}}{U_\infty}$, the density field $\rho/\rho_\infty$, the pressure field $P/\rho_\infty U_\infty^2$, and the temperature field $T/\rho_\infty U_\infty^2$. (Note the appropriate factors used for non-dimensionalization). It is important to note that there is currently no proof that the Mach-number independence principle holds also in the presence of viscous and complex effects inherent to hypersonic flows (i.e., chemical effects, vibrational excitation, radiative processes, etc.). The discussion below is therefore...
Consider the conservation equations for an inviscid, adiabatic flow

\[ \nabla \cdot (\rho \mathbf{v}) = 0 \]  \hspace{1cm} (96)

\[ \nabla \cdot (\rho \mathbf{u} \mathbf{u}) = -\nabla p \]  \hspace{1cm} (97)

\[ \nabla \cdot (\rho \mathbf{u} \mathbf{s}) = 0 \]  \hspace{1cm} (98)

Equations (96) - (98) represent, respectively, the conservation of mass, momentum, and entropy, and are to be applied to solve the flow in Fig. 27 in the region between the body surface and the nose shock. Note that the flow in this region is isentropic since there are no sources of irreversibility other than the shock.

The boundary conditions associated with the integration of (96) - (98) are the no-penetration condition on the body surface

\[ \mathbf{u} \cdot \mathbf{n} = 0 \]  \hspace{1cm} (99)

along with the post-shock jump conditions imposed by Eqs. (6) - (7), (31) and (32), namely

\[ \frac{p}{p_w} = 1 + \frac{2\gamma}{\gamma + 1} \left( \frac{M_w^2 \sin^2 \beta - 1}{\gamma - 1} \right), \quad \frac{\rho}{\rho_w} = \frac{(\gamma + 1) M_w^2 \sin^2 \beta + 2}{(\gamma - 1) M_w^2 \sin^2 \beta + 2} \]  \hspace{1cm} (100)

\[ \frac{U}{U_w} = 1 - \frac{2(M_w^2 \sin^2 \beta - 1)}{(\gamma + 1) M_w^2}, \quad \frac{V}{U_w} = \frac{2(M_w^2 \sin^2 \beta - 1) \cos \beta}{(\gamma + 1) M_w^2} \]

At the shock position \( x = x_s(y) \), equations (96) - (100) are supplemented with the definition of entropy for a calorically-perfect gas

\[ S - S_0 = C_v \ln \left( \frac{p/\rho^\gamma}{p_0/\rho_0^\gamma} \right) \]  \hspace{1cm} (101)

where \( S_0, p_0 \) and \( \rho_0 \) correspond to an arbitrary state. Normalizing the spatial variables with \( R_w \), the velocities with \( U_w \), the pressure with \( p_w U_w^2 \), the density with \( \rho_w \), and the entropy with \( S_0 \), the equations (96) - (98) become

\[ \nabla \cdot (\rho \mathbf{u} \mathbf{v}) = 0 \]  \hspace{1cm} (102)

\[ \nabla \cdot (\rho \mathbf{u} \mathbf{u} \mathbf{v}) = -\nabla p \]  \hspace{1cm} (103)
\[ \nabla \cdot \left( \mathbf{u}^* \right) = 0 \quad (104) \]

While (99) is simply

\[ \mathbf{u}^* \cdot \mathbf{n}^* = 0 \quad (105) \]

Similarly, in the limit \( \mathcal{M}_\infty \rightarrow 0 \), the boundary conditions (100) become

\[
\begin{align*}
W^* &= \frac{Z}{\gamma + 1} \sin^2 \beta, \\
S^* &= \left( \frac{\gamma + 1}{\gamma - 1} \right) \\
U^* &= \frac{Z}{\gamma + 1} \sin^2 \beta, \\
V^* &= \frac{\sin^2 \beta}{\gamma + 1} 
\end{align*}
\quad (106)
\]

with (104) being rewritten as

\[ s^* = 2m \left( \frac{W^*}{S^*} \right) \quad (107) \]

In equations (106), the shock angle is a solution of this free-boundary problem that becomes increasingly independent of the Mach number as observed from the limiting relation (60). As a result, the dimensionless problem (102)-(107) becomes independent of the Mach number as illustrated by the Oswatitsch's principle.

Lastly, it is interesting to note that, as altitude increases, the density decreases and the mean free path increases. For a given Mach number, this effect causes a decrease in the Reynolds number (recall that the Knudsen, Reynolds and Mach numbers are related as \( \mathcal{K}_n \sim \mathcal{M}_\infty / \mathcal{R}_n \)). The decrease in the Reynolds number makes viscous effects to become increasingly important, thereby hindering the type of straightforward discussion presented in this section. Viscous effects are discussed in Chapter III of these notes.

**Small Disturbance Theory of Hypersonic Flows**

Before introducing the conservation equations for small disturbances in hypersonic flows, which are useful to examine high-speed flows over slender bodies, it is illustrative to summarize the characteristics of the flow fields around oblique shocks at supersonic and hypersonic velocities.
SUPersonic Flow

OVER NON-SLender BODIES

\[ u_1 > \sqrt{\frac{n}{2}} \quad \text{BUT NOT LARGE} \]
\[ \frac{\sin \beta}{\sin \delta} \leq 1 \quad \text{BUT NOT SMALL} \]
\[ \beta = 0(1) \]
\[ \sin \delta \leq 1 \quad \text{BUT NOT SMALL} \]
\[ \delta = 0(1) \]
\[ \beta = \delta + 0(1) \]

\[ U_2 = U_1 \cos \beta \left[ 1 + E^2 \tan^2 \beta \right]^{1/2} > U_2 \cos \beta \]
\[ \beta = \frac{\delta}{\sin (\beta - \delta)} \quad \text{BUT NOT LARGE} \]
\[ \sin (\beta - \delta) = E \tan \beta \sqrt{1 + E^2 \tan^2 \beta} \quad \text{BUT NOT SMALL} \]

\[ U_2 = \frac{\cos \delta \sin \beta}{\tan \beta} \sqrt{1 + E^2 \tan^2 \beta} \quad U_1 < U_2 \quad \text{BUT NOT SMALL} \]

\[ \gamma \]

THE SUPersonic Flow OVER NON-SLender BODIES IS CHARACTERIZED

BY ORDER-UNITY DISTURBANCES IN ALL VARIABLES WITH

RESPECT TO THEIR FREE-STREAM VALUES.

SUPersonic Flow

OVER SLender BODIES

\[ u_1 > \sqrt{\frac{n}{2}} \quad \text{BUT NOT NECESSARILY LARGE} \]
\[ \frac{\sin \beta}{\sin \delta} \leq 1 \quad \text{BUT NOT NECESSARILY SMALL} \]
\[ \sin \delta < 1 \quad \text{BUT NOT SLender BODY} \]
\[ \beta = \delta + 0(1) \]

\[ U_2 = U_1 - O(\frac{n}{2}) \]

\[ \gamma \]

THE SUPersonic Flow OVER SLender BODIES RESULTS IN WEAK SHOCKS

THAT ARE CHARACTERIZED BY SMALL FLOW DEFLECTIONS, NEAR-UNITY

NORMAL MACH NUMBERS, AND SMALL DISTURBANCES IN ALL VARIABLES.

\[ \gamma \]
HYPersonic Flow
Over NON-Slender Bodies

\[ M_a > 1 \]
\[ M_a \sin \beta > 1, M_a \sin \delta > 1 \]
\[ \sin \beta > 1 \text{ but not small} \]
\[ \sin \delta < 1 \text{ but not small} \]
\[ \beta \approx \delta \approx O(1) \]

\[ U_2 \approx U_1 \cos \beta \]
\[ M_{a2m} \approx \frac{A}{\epsilon^{1/2}} \approx 1 \]
\[ \sin (\beta-\delta) \approx \epsilon \tan \beta \Rightarrow \beta \approx \delta + O(\epsilon) \]
\[ U_2 \approx \left[1 - O(\sin^2 \beta)\right] U_1 < U_1 \]
\[ V_2 \approx O(\sin \beta \cos \beta) U_1 < U_1 \]
\[ \frac{V_2}{U_1} = O(1) \]
\[ \frac{U_2 - U_1}{U_1} = O(1) \]

THE HYPersonic FLOW OVER NON-Slender BODIES RESULTS IN

LARGE NORMAL MACH NUMBER OF THE INCIDENT STREAM, SMALL NORMAL
MACH NUMBER OF THE POST-SHOCK GASES (ALMOST ZERO POST-SHOCK
NORMAL VELOCITY), LARGE PRESSURE AND DENSITY INCREMENTS, AND
HYPersonic POST-SHock GASES WITH A SHOCK ENVELOPING THE SURFACE
OF THE BODY, (AND BECOMING COINCIDENT WITH IT IN THE LIMIT \( \gamma \rightarrow 1 \)).

HYPersonic Flow
Over slender Bodies

\[ M_a > 1 \]
\[ M_a \sin \beta > 1, M_a \sin \delta > 1 \]
\[ \sin \beta \ll 1 \text{ (note } \beta \approx \delta \text{ in this case)} \]
\[ \sin \delta < 1 \]
\[ \beta \approx \delta \ll 1 \]

\[ U_2 \approx U_1 \]
\[ M_{a2m} \approx \frac{A}{\epsilon^{1/2}} \approx 1 \]
\[ \sin (\beta-\delta) \approx \beta - \delta - \epsilon \beta \Rightarrow \beta \approx \delta/\gamma + O(\epsilon) \ll 1 \]
\[ U_2 \approx \left[1 - 2 \sin^2 \beta \right] U_1 / \gamma + O(\beta^2) U_1 \approx U_1 \]
\[ V_2 \approx \frac{\sin (2\beta^2)}{\gamma + 1} \approx O(\beta^2) U_4 \ll U_1 \]

THE HYPersonic FLOW OVER SLENDER BODIES IS SIMILAR TO THE CASE OVER NON-SLENDER BODIES,
but here the disturbances of the streamwise and spanwise velocity components are small.
The results of these four subdivisions are summarized in Fig. 33 in the form of a $\beta$ vs $\delta$ chart analogous to that sketched in Fig. 8 and resulting from solving equation (20). For a slender body of semiangle $\theta < 1 / \sin \delta$, sufficiently large Mach numbers $M_{\infty} > \Delta / \sin \delta$ are required to enter in the hypersonic range, such that the normal Mach $M_{\infty} = M_{\infty} \sin \beta > \Delta$ and the post-shock temperature is large. All other smaller Mach numbers will tend to yield supersonic flows with $M_{\infty} \sin \beta < \Delta$ and weak shocks.

It is worth mentioning that, whereas the flows over non-slender bodies typically do not admit solutions in terms of small perturbations, both supersonic and hypersonic flows admit solutions in the form of small perturbation theory, the former yielding linear equations for the disturbance and the latter (hypersonic flow) requiring the retention of non-linear terms, as shown below.
In many applications it is desired to have aerodynamic shapes that cause small flow disturbances, as in Fig. 2. In supersonic flows, the assumption of slenderness (1) leads to a useful linearized theory (see pages 67-75 of the ME 355 class notes). The linearized theory, however, breaks down in the transonic range and at hypersonic flow speeds. In both of those regimes, non-linearities play an essential role despite the fact that the velocity disturbances are small compared with $U_\infty$ (note that viscous effects must be necessarily neglected in this context since the non-slip condition leads to no longer small velocity disturbances of order $U_\infty$).

While the linearized supersonic theory requires $M_{\infty} < A$ (or more precisely, $\sqrt{M_{\infty}^2 - 1} < 1$), such that the maximum slope of the body is much smaller than the slope of the free-stream Mach cone ($45^\circ$, $\tan A = 1/\sqrt{M_{\infty}^2 - 1}$), the small-disturbances theory of hypersonics requires $C < A$,

$$M_{\infty} > A$$

so that $M_{\infty} C = 0/4$ or much larger than unity.

Hypersonics

The notation below follows Anderson's book notation, which follows the notation in Van Dyke's "A Study of Hypersonic Small-Disturbance Theory" NACA tech. note 3473, but the formulation here is illustrated only for 2D flows. Consider the slender body in Fig. 34. A non-linear formulation based on small velocity disturbances exists that provides the solution to the rotatic flow behind the nose shock in the following way. Particularizing Eqs. (96-98) for a Cartesian system gives

$$\frac{\partial}{\partial x} (\delta u) + \frac{\partial}{\partial y} (\delta v) = 0 \quad (114)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{1}{S} \frac{\partial p}{\partial y} \quad (115)$$

$$\frac{\partial^2 \delta u}{\partial x^2} + \frac{\partial^2 \delta u}{\partial y^2} = 0 \quad (117)$$
Consider the hypersonic flow over the slender body sketched in Fig. 34. Because of the flow deflection caused by the enveloping shock, streamwise \((u')\) and spanwise \((v')\) velocity disturbances are created with respect to the streamwise pre-shock velocity \(u_{m0}\). As shown in the analysis performed earlier in this section, the post-shock velocities \(u_2\) and \(v_2\) are
\[
\begin{align*}
U_2 &= U_{m0} + u' \quad \text{with } u' = c(\delta^2 V_{a0}) \text{ and negative} \\
v_2 &= v' = O(\delta V_{a0}) \text{ and positive}
\end{align*}
\]

Note that \(u'/u_{m0} \approx \delta\).

Vote that both \(u'\) and \(v'\) are much smaller than \(u_{m0}\), but not necessarily smaller than the local speed of sound:
\[
\begin{align*}
u' &= \frac{\delta^2 U_{m0}}{a_{m0}} = \frac{\delta^2 U_{m0}}{a_{m0}} \quad \text{and} \quad v' = \frac{\delta^2 U_{m0}}{a_{m0}} \left( \frac{L}{M_{a0}} \right) \frac{1}{2} = \frac{L}{E^{1/2}} \\
v' &= \frac{\delta^2 U_{m0}}{a_{m0}} = \frac{\delta^2 U_{m0}}{a_{m0}} \leq \frac{L}{E^{1/2}}
\end{align*}
\]

where \(\delta \ll 1\) and \(E \ll 1\).

Is mentioned earlier, the post-shock gas is deflected at an angle comparable to the local inclination angle of the surface \(\delta\). As a result,
\[
\frac{\delta^2}{u_{m0} + u'} = \frac{d\gamma}{dx} = O\left(\frac{c}{\lambda}\right) \quad (108)
\]

With \(\gamma = \gamma(x)\) the equation of the body surface, \(c\) the width of the body and \(\lambda\) its length. Since the body is slender:
\[
\frac{c}{\lambda} \ll \frac{8}{\delta} = \delta \ll 1 \quad (109)
\]

Here \(\gamma\) is called the aspect or slenderness ratio and is assumed here to be small parameter. In this limit, and since \(u'/u_{m0} \ll 1\), equation (108) becomes
\[
\frac{\delta^2}{u_{m0}} \ll 1 \Rightarrow \frac{\delta^2}{a_{m0}} = O\left(\delta M_{a0} \gamma\right) \quad (110)
\]

Whereas
\[
\frac{\delta^2}{a_{m0}} = O\left(\delta^2 M_{a0}\right) = O\left[2^2 \left(\delta M_{a0} \gamma\right)\right] \quad (111)
\]

It will be shown below that the parameter
\[
K = M_{a0} \gamma \quad (112) \Rightarrow \text{Tsiens' Hypersonic Similarity Parameter (Tsiens, J. Math. Phys. 1946)}
\]

\(K\) is an important similarity parameter to describe hypersonic flows over slender bodies.
With $s$ given by (104). Similarly, the associated boundary conditions are given by (99) (non-penetration on the body surface) and (100) (post-shock gas conditions). In the rescaled (barred) variables

$$
\bar{x} = \frac{x}{\lambda}, \quad \bar{y} = \frac{y}{\lambda}, \quad \bar{U} = \frac{U}{c}, \quad \bar{v} = \frac{\nu}{c},
$$

$$
\bar{P} = \frac{P}{P_0}, \quad \bar{s} = \frac{s}{s_0}, \quad (118)
$$

The conservation equations become

$$
\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\partial (\bar{v} \bar{v})}{\partial \bar{y}} = 0 \quad (119)
$$

$$
\frac{\partial \bar{v}}{\partial \bar{x}} + \frac{\bar{v}}{\bar{s}} \frac{\partial \bar{v}}{\partial \bar{y}} = -\frac{1}{\bar{s}} \frac{\partial \bar{P}}{\partial \bar{y}} \quad (120)
$$

$$
\frac{\partial}{\partial \bar{x}} \left( \frac{\bar{F}}{\bar{s}} \right) + \frac{\bar{v}}{\bar{s}} \frac{\partial}{\partial \bar{y}} \left( \frac{\bar{F}}{\bar{s}} \right) = 0 \quad (122)
$$

Where terms of order $c^2$ have been neglected. Note that $\bar{U}$ and $\bar{V}$ play the role of non-dimensional velocity disturbances scaled with $c$ according to the reasoning explained at the beginning of this section. In contrast, $\bar{P}$ and $\bar{s}$ are not disturbance since it makes little sense to expand these variables because the pressure and density variations are typically large compared to $P_0$ and $S_0$ in hypersonic flows.

In these variables, the boundary condition (99) becomes

$$
\bar{F}_x + \bar{m}_y \bar{V} = 0 \quad (115)
$$

Where terms of order $c^2$ have been neglected. Similarly, the conditions behind the shock become

$$
\bar{F} = \frac{2}{\gamma + 1} \left[ \left( \frac{d\bar{V}}{d\bar{x}} \right)^2 + \frac{1 - \gamma}{2\gamma} \frac{1}{\Ma^2 c^2} \right] \quad (123)
$$

$$
\bar{s} = \frac{\gamma + 1}{\gamma - 1} \left[ \frac{(d\bar{V})^2}{(d\bar{x})^2} + \frac{2}{(\gamma - 1) \Ma^2 c^2} \right] \quad (124)
$$

$$
\bar{W} = -\frac{2}{\gamma + 1} \left[ \left( \frac{d\bar{V}}{d\bar{x}} \right)^2 - \frac{1}{\Ma^2 c^2} \right] \quad (125)
$$

$$
\bar{V} = \frac{2}{\gamma + 1} \left[ \left( \frac{d\bar{V}}{d\bar{x}} \right)^2 - \frac{1}{\Ma^2 c^2} \right] \left( \frac{1}{d\bar{Y}_s/d\bar{x}} \right) \quad (126)
$$

Where $\bar{Y} = \bar{Y}_s(\bar{x})$ is the rescaled body surface such that, for hypersonic speeds and slender bodies,

$$
\sin \beta \v n \beta \approx \frac{d\bar{Y}_s}{d\bar{x}} = \nu \frac{d\bar{Y}_s}{d\bar{x}} \quad (127)
$$
With \( \frac{d^2 \gamma}{d \tau^2} \) a quantity of order unity, from the analysis above it is observed that the problem depends only on 2 parameters: \( \gamma \) and \( \text{Ma}_\infty \). As a result, the solutions are expected to be of the form

\[
\begin{align*}
\bar{P} &= \bar{P}(\bar{x}, \bar{y}, \gamma, \text{Ma}_\infty) \\
\bar{u} &= \bar{u}(\bar{x}, \bar{y}, \gamma, \text{Ma}_\infty) \\
\bar{v} &= \bar{v}(\bar{x}, \bar{y}, \gamma, \text{Ma}_\infty)
\end{align*}
\]

The corresponding surface pressure coefficient is

\[
\bar{Q}_p = \frac{\bar{P} - P_\infty}{\frac{1}{2} \rho_\infty V_{\infty}^2} = \frac{\gamma \text{Ma}_\infty^2 \bar{P} - P_\infty}{\frac{1}{2} \rho_\infty V_{\infty}^2} = 2 \bar{c}^2 \left( \bar{P} - \frac{1}{\gamma \text{Ma}_\infty^2} \right) \quad (128)
\]

And therefore

\[
\bar{Q}_p = F(\bar{x}, \bar{y}, \gamma, \text{Ma}_\infty) \quad (129)
\]

Where \( F \) is a functional obtained by solving the entire problem.

Equation (129) represents an important similarity rule, in that it correlates the surface pressure coefficient of hypersonic flows past slender bodies that differ from one another only by a uniform expansion or contraction of their thickness.

For instance, in Fig. 36, as the thickness is reduced from \( \bar{c}_1 \) to \( \bar{c}_2 \) in hypersonic flow, the similarity rule (129) states that the ratio of the local pressure coefficient at \( Q \) and the square of the thickness \( \bar{c}^2 \) is the same for both bodies as long as Tsiien's similarity parameter \( \text{Ma}_\infty \) is the same in both flows (provided that \( \gamma \) is also the same). This is illustrated in Fig. 36, which shows that provided one knows the surface pressure of a 5° semivertex cone, the utilization of the similarity rule (129) to transform that surface pressure into the one for a 10° semivertex cone (i.e., \( \bar{Q}_p/\bar{c}^2 = \text{const} \)) yields a result.
That is close to the numerical result as long as \( M_{\infty} \gg 1 \). As the Mach number decreases, however, the similarity parameter becomes small, \( M_{\infty} \ll 1 \), and the small-disturbances theory described above is no longer applicable (note that \( \bar{p}, \bar{s}, \bar{u}, \bar{v} \) cease to be of order unity when \( M_{\infty} \ll 1 \) as observed from (123) - (126)). The limit \( M_{\infty} \ll 1 \) corresponds to the linearized theory of supersonic flows over slender bodies, in which the maximum slope of the body is smaller than the Mach angle.

The nonlinear theory of small disturbances in hypersonics cannot, however, be bridged continuously with the linear theory of small disturbances of supersonic flows. Nonetheless, Van Dyke (J. Aeronaut. Sci. 1954) proposed a combined similar rule for supersonic and hypersonic flows up to just above transonic range,

\[
\frac{\bar{p}}{\bar{c}^2} = F \left( \bar{x}, \bar{y}, \bar{z} \sqrt{M_{\infty}^2 - 1} \right) \tag{138}
\]

where \( \bar{y} \) must remain fixed in the transformation, as shown in Fig. 37. The combined rule improves substantially the predictions over the supersonic range. Note that the linearized theory of supersonic flows over slender bodies yields a pressure coefficient of the form \( \bar{p} \sim \frac{z}{2\sqrt{M_{\infty}^2 - 1}} \) with \( \bar{z} \) the local surface incline (\( \bar{z} \sim \bar{s} \) for slender bodies), such that \( \bar{p}/\bar{c}^2 \sim \frac{1}{2\sqrt{M_{\infty}^2 - 1}} \) (see pages 74-75 of the M6355 class notes), which constitutes the lower supersonic limit of the relation (138) proposed by Van Dyke. As \( M_{\infty} \to \infty \) in the hypersonic range, however, the linearized supersonic theory would render \( \bar{p} \to 0 \), which is wrong. As \( M_{\infty} \to \infty \), the similarity parameter becomes large, \( M_{\infty} \chi \gg 1 \), the formulation (119) - (126) ceases to depend on the Mach number in a manner similar to that predicted by O' swathitsch in the Mach-number independence principle, and the similarity relations (129) or (130) become independent of the Mach number, thereby yielding a scaling \( \bar{p} \sim \bar{z}^2 \) that resembles the one predicted by the Newtonian.
THEORY (45).

Similarity relations can be obtained for the drag and lift coefficients on a two-dimensional body,

\[
\begin{align*}
\frac{C_{D}}{C^2} &= \frac{1}{A} \int_{-C}^{C} \left( \frac{C_{P_{U}}}{C^2} - \frac{C_{P_{L}}}{C^2} \right) \, dy = \int_{-1}^{1} \left( \frac{C_{P_{U}}}{C^2} - \frac{C_{P_{L}}}{C^2} \right) \, dy = G \left( \gamma, \text{Ma}_{\infty} \right) \quad (131) \\
\frac{C_{L}}{C^2} &= \frac{1}{A} \int_{0}^{A} \left( \frac{C_{P_{E}}}{C^2} - \frac{C_{P_{U}}}{C^2} \right) \, dx = \int_{0}^{1} \left( \frac{C_{P_{E}}}{C^2} - \frac{C_{P_{U}}}{C^2} \right) \, dx = H \left( \gamma, \text{Ma}_{\infty} \right) \quad (132)
\end{align*}
\]

where \( C_{P_{U}} \) and \( C_{P_{L}} \) are pressure coefficients on the upper and lower surfaces.

Similarly to the discussion above, in the limit \( \text{Ma}_{\infty} \to \infty \), the drag and lift coefficients scale as \( C_{D} \sim C^3 \) and \( C_{L} \sim C^2 \), while their ratio (i.e., the lift to drag ratio \( L/D \)) scales as \( L/D \sim 1/2 \) and becomes a sole function of the geometry, which is typically termed as the \( L/D \) barrier in the literature and represents the fact that there is only so much lift that can be extracted from an aerodynamic shape with a given geometry by just increasing the Mach number (Fig. 38).

These results were already anticipated in the analysis of the Newtonian flow on page 12, where it was highlighted the increased sensitivity of the wave drag to the aspect ratio of the body in hypersonic flow (i.e., \( C_{D} \sim C^3 \)), in comparison with the milder dependency found in supersonic flows (i.e., \( C_{D} \sim C^2 \), see Eq. (160) in the M.B.E.55 class notes).

Lastly, it is worth mentioning that in using the similarity relations (129)-(132) the results tend to be more accurate when \( \gamma \) is taken to be the maximum slope of the surface in bodies with non-uniform surface-inclination angles.
Inviscid Hypersonic Flow Around Blunt Bodies

Blunt bodies are widely employed in hypersonic applications (Fig. 39). It will be shown in the next chapters that a blunt aerodynamic shape of the nose of an aerospace vehicle largely decreases the local rate of heat transfer created by viscous friction, however, the aerodynamics of blunt bodies at high speeds are fundamentally different from the case of sharp pointed bodies such as the projectiles shown in Fig. 39. Two important aspects are treated in this section. The first is that aerodynamic blunt shapes lead to detached or bow shocks, and an analytical solution to the stand-off distance of the shock is derived for a calorically perfect gas. The second aspect is related to the shock curvature, which introduces vorticity in the initially irrotational flow along a relatively thick layer that sits on top of the boundary layer around the body while the first aspect illustrates the usefulness of the Newtonian theory, the second one connects with the next chapter on viscous hypersonic flows.

The Shock Stand-Off Distance in the Newtonian Limit

Consider the problem sketched in Fig. 40. A hypersonic stream at Mach $Ma$, density $\rho_0$, pressure $P_0$, and velocity $U_0$. The gas is assumed to be calorically perfect with a constant adiabatic coefficient $\gamma$. An axisymmetric coordinate system $\{\Gamma, x\}$ is used, where $x = x_0(\Gamma)$ denotes the surface of the body and $x = 0$ is centered at the shock tip. Similarly, the axis $\Gamma = 0$ runs through the axis of symmetry of the body, which is impinged by the free stream with a zero angle of attack. It is known from the theory of oblique shocks that no solution exists when the deflection angle exceeds a critical value $\Theta_{max}$ that
Increase with the Mach number to a maximum of \(45.6^\circ\) in plane wedges and \(57.6^\circ\)
cones at infinite Mach number. Instead, the solution to the problem must involve
a detached shock standing off at a distance \(x_o\) from the blunt nose (see Fig. 40) that
needs to be calculated as part of the solution.

Fluid flow variables are normalized as

\[
U = \frac{u}{u_{\infty}}, \quad V = \frac{v}{u_{\infty}}, \quad P = \frac{p}{p_{\infty}u_{\infty}^2}, \quad \rho = \frac{\rho}{\rho_{\infty}}, \quad x = \frac{x}{x_{\infty}}, \quad \gamma = \frac{\gamma}{\gamma_{\infty}},
\]

where \(x_{\infty}\) is the characteristic radius of curvature of the blunt nose, and \(\rho_{\infty}\) is the
fundamental dimensionless parameter \(\alpha\) emerging from the Newtonian theory, namely

\[
\rho_{\infty} = \frac{V - 1}{\gamma + 1},
\]

in such a way that the density in \(\rho_{\infty}\) is normalized with the post-shock density.

In these variables, the conservation equations for the inviscid isentropic flow behind
the bow shock are

\[
\frac{2}{\partial x} (S u U) + \frac{2}{\partial r} (S v V) = 0 \quad (135) \quad \frac{u}{\partial x} + \frac{v}{\partial r} = -\frac{\rho}{\gamma} \frac{\partial p}{\partial r} \quad (137)
\]

\[
\frac{2}{\partial x} (S u U) + \frac{2}{\partial r} (S v V) = -\frac{\rho}{\gamma} \frac{\partial p}{\partial r} \quad (136) \quad \frac{2}{\partial x} (S x U) + \frac{2}{\partial r} (S x V) = 0 \quad (138)
\]

where the asterisks indicating dimensionless variables are omitted.

Equation \(135\) indicates that there exists a stream function \(\psi\) such that

\[
\frac{\partial \psi}{\partial r} = S u U \quad \text{and} \quad \frac{\partial \psi}{\partial x} = -S v V,
\]

which automatically satisfies Eq. \(135\). Upstream from the bow shock, in the
uniform flow, \(\psi = \frac{1}{2} r^2\). Additionally, on the surface of the body the stream function
is chosen to be \(\psi = 0\).

The analysis is facilitated when equations \(135\) - \(138\) are transformed to
an equivalent but more insightful form using the von Mises transformation \(\xi, \eta\)
where \(\xi = \psi(x, r)\) and \(\eta = r\) are chosen as independent variables.

Note that

\[
\frac{2}{\partial x} (S x U) = \frac{2}{\partial x} (S x U) \quad \text{and} \quad \frac{2}{\partial r} (S x V) + \frac{2}{\partial r} = \frac{S v V}{\gamma} \frac{\partial \psi}{\partial x} + \frac{2}{\partial r} \psi.
\]
so that the relation

\[
\frac{\partial U}{\partial x} + \frac{\partial \theta}{\partial y} = \frac{\partial \theta}{\partial y}
\]  

(141)

is satisfied. Physically, the Von-Mises transformation rewrites the material derivative which is an operator acting along a streamline, in the form (141) by taking advantage of changing one of the independent variables \(x\) by one that is constant along the streamlines. Using these new variables, it is not difficult to show that Eqs. (135) - (138) become

\[
\frac{\partial}{\partial \xi} \left( \frac{U}{\sqrt{V}} \right) + \frac{\xi}{\partial \eta} \left( \frac{1}{\sqrt{V}} \right) = 0, 
\]

(142)

\[
\frac{z + \sqrt{\xi} \left( 1 + \xi \right)}{\xi} = 1 + \sqrt{\xi}, 
\]

(143)

\[
\frac{\partial \theta}{\partial \xi} = f(\xi), 
\]

(145)

where (142) is continuity, (143) is the conservation of streamwise momentum, (144) is the conservation of radial momentum along with the conservation of energy (i.e., Bernoulli equation along a streamline), and (145) is the conservation of entropy. While (142), (143), and (145) are straightforward to derive, Eq. (144) requires some treatment of the pressure gradient in (139). Note that, upon transforming (137), the equation

\[
\frac{\partial U}{\partial \xi} = \frac{\xi}{\partial \eta} \left( \frac{\partial P}{\partial \eta} + \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} \right) = \frac{\xi}{\partial \eta} \frac{\partial P}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta}
\]

(146)

is obtained, which represents the momentum equation across streamlines, while the left-hand side of (146) can be written as an exact differential of the kinetic energy, the right-hand side requires introduction of the density in the pressure gradient, which can be made by noticing that the flow is isentropic \(\xi = \xi, \) Eq. (138) and as a result of the first principle of thermodynamics

\[
\frac{\partial h}{\partial \eta} = 0 = \frac{T \partial s}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial P}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta} = \frac{\xi}{\partial \eta} \frac{\partial \theta}{\partial \eta}
\]

(146)

and therefore (146) becomes

\[
\frac{\partial}{\partial \xi} \left( \frac{U^2}{2} + \frac{\xi}{2} + \frac{\xi}{\eta} \right) = 0 \Rightarrow \frac{U^2}{2} + \frac{\xi}{2} + \frac{\xi}{\eta} \frac{\partial P}{\partial \eta} = \xi \left( \frac{U^2}{2} \right)
\]

(147)

with \(\xi\) a stagnation enthalpy evaluated in the freestream \(\xi = 1 + \frac{\xi}{\eta} \left( \frac{1}{\xi} \right), \) where \(U = V = \xi, \frac{\xi}{\eta} = 1, \) and \(\frac{\xi}{\eta} = 1 + \frac{2}{\xi} \frac{\xi}{\eta} \left( \frac{1}{\xi} \right) \) is where \(U = V = \xi, \) \(\frac{\xi}{\eta} = 1, \) and \(\frac{\xi}{\eta} = 1 + \frac{2}{\xi} \frac{\xi}{\eta} \left( \frac{1}{\xi} \right) \) is the conversion of
(147) INTO (144) IS MADE EASILY BY USING THE RELATIONS \( \frac{1}{\gamma-1} = \frac{1-\varepsilon}{2\varepsilon} \) AND \( \frac{\varepsilon}{\gamma-1} = \frac{1+\varepsilon}{2\varepsilon} \) EMERGING FROM (134). LASTLY, IN EQ. (145), \( f(\varepsilon) \) IS AN ARBITRARY FUNCTION THAT RESULTS FROM INTEGRATING (138) IN THE TRANSFORMED VARIABLES, NAMELY \( \partial \{ P/\varepsilon \}/\partial \varepsilon = 0 \).


EQUATIONS (142) - (145) ARE SUBJECT TO THE NON-PEENETRATION CONDITION AT \( \psi = \frac{\varepsilon}{\gamma} = 0 \),

\[
\vec{u} \cdot n = u m_x + v m_y = 0 \Rightarrow u + v x_{b\psi} = 0 \quad (146)
\]
WHERE \( x_{b\psi} = \partial x_b / \partial \psi \), AND TO THE DENSITY \( \rho \), PRESSURE \( P \), STREAMWISE VELOCITY \( u \)
AND SPANWISE VELOCITY \( v \) JUMP CONDITIONS AT THE BOW SHOCK, \( x = x_s(\psi) \) (WITH \( x_s(0) = 0 \)) WHICH, IN DIMENSIONLESS FORM, CAN BE WRITTEN AS

\[
P = (1-\varepsilon) \left( \frac{1}{1 + x_{s\psi}^2} \right) - \frac{\varepsilon}{(1+\varepsilon) \rho_{\infty}^2} \quad (149)
\]

\[
\eta = \frac{1}{1 - (1-\varepsilon) \varepsilon^{-1} \rho_{\infty}^2 (1 + x_{s\psi}^2)} \quad (150)
\]

\[
u = (1-\varepsilon) \left( \frac{1}{1 + x_{s\psi}^2} \right) - \frac{1}{\rho_{\infty}^2} \quad (151)
\]

\[
v = (1-\varepsilon) \left( \frac{1}{1 + x_{s\psi}^2} \right) - \frac{1}{\rho_{\infty}^2} \quad (152)
\]

WHERE \( x_{s\psi} = \partial x_s / \partial \psi \) IS THE INVERSE OF THE LOCAL SLOPE OF THE SHOCK, SO THAT \( \eta \frac{\partial x_s}{\partial \psi} = \frac{1}{\tan \beta} \) IN THE SHOCK JUMP CONDITIONS. IN THE TRANSFORMED VARIABLES, EQUATIONS (149) - (152) ARE APPLIED AT \( \psi = \frac{\pi}{2} \).

INTEGRATION OF THE SECOND EQUATION IN (149) GIVES A RELATION BETWEEN THE EQUATION IF THE BODY AND THE EQUATION OF THE SHOCK, NAMELY

\[
\int_{x_b}^{x_s} dx = - \int_{\frac{1}{2} \pi}^{\frac{1}{2} \pi} \frac{1}{\psi_{\infty}} d\psi \Rightarrow x_b = x_s + \varepsilon \int_{\frac{1}{2} \pi}^{\frac{1}{2} \pi} \left( \frac{1}{\psi_{\infty}} \right) d\psi \quad (153)
\]

IN SUCH A WAY THAT THE STAND-OFF DISTANCE IS GIVEN BY

\[
x_0 = x_b(0) - x_s(0) = \varepsilon \lim_{r \to 0} \int_0^{\frac{1}{2} \pi} \left( \frac{1}{\psi_{\infty}} \right) d\psi. \quad (154)
\]
AT THE SHOCK, NOTE THAT \( \psi = \frac{r}{2} \), AND CONSEQUENTLY
\[
X_{Sr} = \frac{\partial \psi}{\partial \theta} X_{S} = r X_{S\psi} = (2 \psi)^{1/2} X_{S\psi}, \quad (155)
\]

IN EQ. (145), \( f(\psi) \) CAN BE OBTAINED BY USING (149), (150) AND (155), NAMELY
\[
f(\psi) = (1 - \epsilon) \left[ \frac{1}{1 + 2 \psi X_{S\psi}^2} - \frac{\epsilon \mathcal{M}_{\infty}^{-2}}{1 + \epsilon} \right] \left[ 1 + (1 - \epsilon) \frac{\epsilon \mathcal{M}_{\infty}^{-2}}{1 + \epsilon} \left( 1 + 2 \psi X_{S\psi}^2 \right) \right] \frac{1 + \epsilon}{1 - \epsilon} \cdot (1)
\]

WHICH IS VALID EVERYWHERE BEHIND THE SHOCK SINCE \( f(\psi) \), THE EXPONENTIAL OF THE ENTROPY, IS CONSTANT ALONG STREAMLINES.

OF PARTICULAR RELEVANCE IN THE ABOVE EQUATIONS IS THE DIMENSIONLESS GROUP
\[
\frac{1}{\epsilon \mathcal{M}_{\infty}^{-2}} \cdot (157)
\]

NOTE THAT THE NEWTONIAN LIMIT IS \( \epsilon = 0 \) (i.e., \( \mathcal{M} = 1 \)), WHILE HYPERSONIC SPEEDS REQUIRE \( \mathcal{M}_{\infty}^2 \to \infty \). AS A RESULT, THE NEWTONIAN LIMIT CONSISTS OF THE SIMULTANEOUS LIMIT
\[
\epsilon \to 0 \quad \text{AND} \quad \mathcal{M}_{\infty}^{-2} \to 0 \quad (158)
\]

SUCH THAT
\[
\epsilon^{-1} \mathcal{M}_{\infty}^{-2} = 0 \quad (1) \quad (159)
\]

TAKING THE LIMIT (158)-(159), EQUATIONS (142)-(145) REDUCE TO
\[
\frac{\partial}{\partial \theta} \left( \frac{U}{V} \right) = 0, \quad (160)
\]
\[
\frac{2}{V} U + \frac{2}{V} P = 0, \quad (161)
\]
\[
P = \frac{1}{1 + 2 \psi X_{S\psi}^2} \quad (162)
\]
\[
\mathcal{G} = \frac{A}{1 + \mathcal{M}_{\infty}^{-2} (1 + X_{Sr}^2)} \quad (163)
\]

WHILE EQ. (158) ESTABLISHES \( X_b = X_S \), NAMELY THAT THE SHOCK AND THE BODY SURFACE ARE COINCIDENT AND THEREFORE THE SHOCK STAND-OFF DISTANCE IS ZERO,
\[
X_0 = 0 \quad (164)
\]

SIMILARLY, THE BOUNDARY CONDITIONS (149)-(152) BECOME
\[
P = \frac{A}{1 + X_{Sr}} \quad (165)
\]
\[
U = A - \frac{A}{1 + X_{Sr}^2} \quad (166)
\]
\[
V = \frac{X_{Sr}}{1 + X_{Sr}^2} \quad (167)
\]

NOTE THAT (164), (165), (166) AND (167) COINCIDE WITH THE NEWTONIAN LIMIT OF THE OBLIQUE SHOCK RELATIONS ANTICIPATED IN PREVIOUS SECTIONS. IN THESE EQUATIONS,
The incidence angle of the bow shock is related to $X_{st}$ as $\tan \beta = \frac{1}{X_{st}}$, so that $(1 + X_{st}^2)^{-1} = \sin^2 \beta$ and $X_{st} (1 + X_{st}^2)^{-1} = \frac{1}{2} \sin (2\beta)$.

Equations (160) and (162) provide the leading-order approximations for the velocities

$$u = X_{st}, v = X_{st} \left( \frac{2 \psi}{(X_{st}^2 + 1)^{1/2}} X_{st}^2 (\psi) \right)^{1/2} \frac{1}{(X_{st}^2 + 1)^{1/2}} \right) \left( 1 + 2 \psi X_{st}^2 (\psi) \right)^{1/2}.$$  

Substitution of this result into (161) yields

$$P = \frac{1}{1 + X_{st}^2} + \left( \int_{r_0}^{r} \frac{1}{2} \frac{\partial u}{\partial \beta} ds \right) - \frac{1}{1 + X_{st}^2} + \frac{1}{1 + X_{st}^2} \left( \frac{X_{st}^2 (\psi)}{(1 + X_{st}^2)^{1/2}} \right) \left( 1 + (2 \psi X_{st}^2 (\psi)) \right)^{1/2} \phi \psi = \frac{P_2}{r} \sin \beta \frac{d \beta}{dr} \int_{r_0}^{r} r \cos \beta \, dr$$

(170)

which represents the Busemann centrifugal correction for axisymmetric bodies, with $P_2$ the post-shock pressure, despite the heuristics involved in the derivation of the Busemann correction earlier in this chapter; it is remarkable that it emerges from the conservation equations in the limit (158)-(159).

The rest of the analysis is quite extensive and will be omitted here. It consists of successive approximations to Eqs. (142)-(145) in the expansion parameter $d = \varepsilon + M_0^{-2}$. (171)

The analysis is greatly facilitated by assuming a parabolic bow shock $X_5 = r^{2/3}$ and then calculating the equation of the body surface $X = X_b(r)$ that is consistent with $X_5$ and with the conservation equations. Such method is typically referred to as "inverse method." It is therefore important to notice that the shock stand-off distance at hypersonic speeds ceases to be zero in the second approximation to the Newtonian limit, i.e., when $\varepsilon > 0$ ($\gamma > 1$) and correspondingly the post-shock density becomes finite. After lengthy calculations that can be found in Chester, J. Fluid Mech. A, 490-496 (1956), the asymptotic expansion

$$\frac{X}{R_0} = \varepsilon - \left( \frac{8}{3} \right)^{1/2} \varepsilon^{3/2} + \frac{13}{5} \varepsilon^2 - \frac{463}{168} \left( \frac{8}{3} \right)^{1/2} \varepsilon^{5/2} + \ldots$$  

(172)
IS OBTAINED FOR THE STANDOFF DISTANCE OF A PARABOLOIDAL SHOCK AT INFINITELY HIGH MACH NUMBERS. UNFORTUNATELY, THE EXPANSION (172) IS FOUND TO CONVERGE EXTREMELY SLOWLY AND WAS LATER GREATLY IMPROVED BY VAN DYKE, J. FLUID MECH. 3, 515-522 (4).

* THE GENERATION OF VORTICITY IN CURVED SHOCKS

CONSIDER THE MOMENTUM EQUATION

$$\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u} = -\nabla P,$$  \hspace{1cm} (173)

WHERE THE CONVECTIVE TERM CAN BE WRITTEN IN TERMS OF THE VORTICITY \( \omega = \nabla \times \vec{u} \) BY MAKING USE OF THE VECTOR IDENTITY

$$\vec{u} \cdot \nabla \vec{u} = \nabla \left( \frac{1}{2} |\vec{u}|^2 \right) - \vec{u} \cdot \nabla \vec{u},$$  \hspace{1cm} (174)

SO THAT (173) BECOMES

$$\frac{\partial \vec{u}}{\partial t} - \vec{u} \cdot \nabla \vec{u} = -\nabla P - \nabla \left( \frac{1}{2} |\vec{u}|^2 \right).$$  \hspace{1cm} (175)

THE FIRST PRINCIPLE OF THERMODYNAMICS PROVIDES THE ENERGY-CONSERVATION RELATION

$$TdS = de + Pd\left( \frac{1}{\rho} \right),$$  \hspace{1cm} (176)

WHERE \( e = h - \frac{P}{\rho} \) IS THE INTERNAL ENERGY. IN TERMS OF THE ENTHALPY \( h \), EQ (176) CAN BE REWRITTEN AS

$$TdS = dh - \frac{P}{\rho} d\rho,$$  \hspace{1cm} (177)

WHICH CAN BE TRANSFORMED INTO

$$T \nabla S = \nabla h - \nabla \left( \frac{P}{\rho} \right).$$  \hspace{1cm} (178)

SUBSTITUTING (178) INTO (175), THE MODIFIED MOMENTUM EQUATION

$$\frac{\partial \vec{u}}{\partial t} + \omega \times \vec{u} = T \nabla S - \nabla \left( h + \frac{1}{2} |\vec{u}|^2 \right),$$  \hspace{1cm} (179)

IS OBTAINED, WHICH, IN THE STEADY LIMIT BECOMES

$$\omega \times \vec{u} = T \nabla S - \nabla \left( h + \frac{1}{2} |\vec{u}|^2 \right)$$  \hspace{1cm} (180)

_CROCCO'S EQUATION_

WHERE \( h_0 = h + \frac{1}{2} |\vec{u}|^2 \) IS THE STAGNATION ENTHALPY AS DEFINED IN EQ. (43). IF \( h_0 \) IS UNIFORM IN THE FLOW, OR EQUIVALENTLY, WHEN THE FLOW IS ADIABATIC (i.e., THERE IS NO EXTERNAL HEAT ADDITION OR HEAT LOSSES), THEN (180) BECOMES

$$\omega \times \vec{u} = T \nabla S$$  \hspace{1cm} (181)

WHICH ESTABLISHES A DIRECT RELATION BETWEEN VORTICITY AND ENTROPY GRADIENTS.
At this point, recall that flows with uniform entropy, $\nabla s = 0$, are referred to as **homentropic**. In homentropic flows, $s = s_0$ everywhere, but $s_0$ can vary with time (Fig. 41a). In contrast, inviscid compressible flows satisfying

$$\frac{Ds}{Dt} = \nabla \cdot \mathbf{u} = 0$$

(182)

are referred to as **isentropic**, in that the initial entropy of a fluid particle, $s = s_p$, is conserved along pathlines, although $s_p$ can vary in space and time. In steady flows, (182) simplifies to

$$\nabla \cdot \mathbf{u} = 0$$

(183)

which corresponds to an isentropic flow in which the entropy is constant along streamlines (Fig. 41b).

The implications of Eq. (183) in hypersonics are important. Consider, for instance, the flow over a two-dimensional wedge or an axisymmetric cone, in which the nose shock is attached as in Fig. 42a. The flow upstream is inviscid, irrotational and adiabatic, and therefore by Eq. (183) it is also homentropic. As it passes through the oblique shock, an entropy jump is generated that increases the entropy behind the shock. However, since the shock is straight, its strength is uniform, and consequent

**FIG. 41**

**FIG. 42**
The entropy jump is the same along the entire shock. As a result, the entropy is uniform behind the shock and the flow is homentropic there. By equation (184), the flow must be necessarily irrotational behind the shock as well.

In contrast, in the case of a hypersonic flow over a blunt body, as in Figure 36, the shock strength is not uniform because of the variable incidence angle. A streamline passing through the near-normal portion of the shock experiences a larger entropy increase than the streamlines passing through the shock away from the nose. The result is the occurrence of a gradient of entropy normal to the stream, with fluid particles near the wall having larger entropy than those away from the wall. An entropy layer is then formed that contains the transverse entropy gradient within the shock layer, and which involves rotational flow as prescribed by Eq. (184). The flow in the entropy layer is however isentropic in the absence of external heating deposition or heat losses by radiation. The adiabatic assumption is accurate unless the post-shock temperature is large enough to cause the gas to radiate through the shock layer, which is rare in applications, only occurring significantly for flight speeds in excess of 12 km/s (Bertin, 1993).

In practical applications, there is always a thin boundary layer close to the wall where viscous effects are important (Fig. 44). The boundary layer displaces the inviscid stream around the body, which causes a change in the shape of the bow shock predicted by an inviscid analysis. The growth of the boundary layer around the body is influenced by the pressure gradient above and by the velocity 1D temperature at the edge of the boundary layer. The displacement created by the boundary layer can cause an order-unity increase of the post-shock gas pressure near the nose, where the percentage of the shock-layer thickness occupied by the boundary layer is largest. In addition, as the boundary layer grows,
IT ENRAINTS THE ENTRALY LAYER ABOVE IT AND ITS CORRESPONDING VORTICITY. THESE COMPLEX EFFECTS CAUSED BY THE FINITE REYNOLDS NUMBERS INVOLVED, ARE TYPICALLY REFERRED TO AS VISCOUS/INVISCID INTERACTIONS AND BECOME IMPORTANT AT HIGH ALTITUDES IN HYPERSONIC FLOWS, WHERE THE EFFECTS OF VISCOSITY MUST ENTER IN THE DESCRIPTION. VISCOUS EFFECTS ARE DISCUSSED IN THE NEXT CHAPTER.

III. VISCOUS HYPERSONIC FLOWS

IN THE PREVIOUS CHAPTER, THE FOCUS HAS BEEN ON FLOWS WHERE THE EFFECTS OF VISCOITY ARE NEGLIGIBLE, OR MORE PRECISELY, ON THE CALCULATION OF QUANTITIES THAT DO NOT FUNDAMENTALLY DEPEND ON VISCOSITY. EXAMPLES OF SUCH QUANTITIES ARE THE SHOCK ANGLES IN THE FREE STREAM AROUND AN AERODYNAMIC SHAPE, WHICH DO NOT DEPEND ON VISCOSITY UP TO THE VISCOUS/INVISCID INTERACTIONS BRIEFLY MENTIONED ABOVE, OR THE PRESSURE, DRAG AND LIFT COEFFICIENTS, WHICH, AT HIGH MACH NUMBERS, ARE MOSTLY DETERMINED BY THE PATTERN OF SHOCK WAVES INSTEAD OF FRICTION (U.E., RECALL THE CONCEPT OF SUPersonic Wave DRAG IN PAGE 28 OF THE ME 355 CLASS NOTES).

IT SHOULD HOWEVER BE MENTIONED THAT VISCOUS EFFECTS ARE INEXORABLY PRESENT IN ALL PRACTICAL APPLICATIONS, EVEN THOUGH THEY MAY NOT CRUCIALLY INFLUENCE SOME QUANTITIES.

THEY ARE NONTHELESS ESSENTIAL FOR THE CALCULATION OF A NUMBER OF RELEVANT ENGINEERING METRICS OF PERFORMANCE FOR AN AEROSPACE VEHICLE. THESE INCLUDE FOR INSTANCE, THE HEAT FLUX INTO THE VEHICLE STRUCTURE AT ALL FLIGHT ALTITUDES, THE FRICTION DRAG AND LIFT FORCES AT HIGH ALTITUDES, OR THE ABLATION RATE OF THERMAL PROTECTION SYSTEMS SHIELDING THE VEHICLE FROM THE HIGH THERMAL LOADS UPON REENTRY. THEY ALSO PLAY A ROLE IN ADVANCED AERODYNAMIC ASPECTS RELATED TO BOUNDARY-LAYER TRANSITION TO TURBULENCE AND THE ASSOCIATED THERMAL LOADS CAUSED BY THIS EFFECT.
THE ROLE OF FLIGHT ALTITUDE

The emergence of viscous effects is perhaps more critical at high altitudes for the following reasons. Figure 45 provides the Earth's atmosphere strata definitions. A change in strata coincides with approximate altitudes where the temperature remains constant with altitude (i.e., zero thermal lapse $\frac{dT}{dz} = 0$). The region of the atmosphere between sea level and 2.4 km is the troposphere, the density is observed to decay by 63% (i.e., by a factor $e^{-1}$) within this layer (see Fig. 46) in the atmosphere model where it is assumed that the temperature, composition, and gravitational acceleration are constant with height, the density can be expressed as

$$\rho \approx \rho_0 e^{-\frac{z}{h}} \quad (183)$$

where $\rho_0$ and $h$ are adjustable parameters and $z$ is the altitude, while $\rho_0 \approx 1.22 \text{ kg/m}^3$ is the sea-level density, and $h \approx 6700 \text{ m} = 22000 \text{ ft}$ is a scale height that measures the e-folding of the density above the troposphere.

The stratosphere and mesosphere, the latter ends at an altitude of 86 km and is considered approximately the limit height above which aerodynamic loads become negligible fractions of the vehicle weight. Correspondingly, above the mesosphere, the motion of the vehicle is almost always suborbital and is to a good approximation solely described by gravitational mechanics. Note that the atmosphere above 86 km is highly dynamic and highly rarefied, and its composition and physical properties are very influenced by solar radiation. Figure 47 provides temperature variations in each strata concurrent with the decrease in density is an increase in the kinematic viscosity, $\nu = \mu/\rho$, with altitude, as shown in Fig. 48. Although it is difficult to define a physically significant Reynolds number that can illustrate the character of the aerodynamic field over the entire vehicle, since there are large changes in temperature in the flow around the vehicle and near the surface because of compressibility effects, it is clear
That a vehicle characteristic length $L$ flying at a speed $U_i$ will experience a Reynolds number $\approx 10^5$ times smaller in the mesosphere relative to that at sea level. In re-entry applications, however, the Reynolds number increases along the re-entry trajectory but not by a factor of order $10^5$. It increases instead by a smaller factor of order $10^3$ because at high altitudes the flight velocity is much larger than close to the ground, which somewhat compensates for the larger kinematic viscosity at high altitudes so as to not render too small Reynolds numbers.

Boundary-layer transition

The increase of the Reynolds number along re-entry is greatly illustrated by the in-flight measurements of the shuttle orbiter reported by Goodrich et al., AIAA-83-0485 (1983). There, a series of thermocouples along the spine of the orbiter's underbelly were employed, along with on-board and telemetry data, to determine characteristic Reynolds and Mach numbers upon re-entry, and most importantly, to determine the location of boundary-layer transition to turbulence along the orbiter's underbelly. (Fig. 4.9).

In standard missions, the space shuttle re-enters the atmosphere from low-earth orbits at altitude of order $123\text{ km}$, angles of attack $\alpha \approx 40^\circ$, and Mach numbers $M_a \approx 2.5$, which correspond to circular orbit velocities of order $U_0 \approx 7.4\text{ km/s}$. The velocity/altitude curve of the shuttle STS-2 mission...
IS PROVIDED IN FIGURE 50 (DOTTED LINE), WHILE ITS TRANSLATION INTO FREESTREAM REYNOLDS NUMBERS, MACH NUMBERS AND ENTRY TIME ARE PROVIDED IN FIGURE 51.

NOTE THAT THE REYNOLDS NUMBER OF THE ORBITER VARIES FROM $R_{90, L} \approx 10^5$ AT THE ENTRY INTERFACE ($\approx 123$ km) TO $R_{90, L} \approx 10^8$ NEAR THE GROUND AFTER APPROXIMATELY $1800$ s = $30$ min. OF TOTAL MANEUVERING FLIGHT. MOST REMARKABLY, WITHIN THAT RANGE OF REYNOLDS NUMBERS, THERE OCCURS THE CRITICAL PHENOMENON OF BOUNDARY-LAYER TRANSITION FROM LAMINAR TO TURBULENT FLOW NEAR THE SURFACE OF THE FUSELAGE. THIS BEGINS NEAR AN ALTITUDE OF 50 km WHEN THE ORBITER IS DESCENDING AT $M_{90} \approx 10$ ($\approx 3800$ km/s), AND IT IS SPATIALLY LOCALIZED NEAR THE AFTMOST PART OF THE UNDERBELLY, WHICH CREATES A SIGNIFICANT THERMAL LOADING (FIG. 52). THE REGION OF TRANSITION THEN MOVES RAPIDLY UPSTREAM ALONG THE ORBITER’S UNDERBELLY AND REACHES THE FOREBODY AFTER APPROXIMATELY $300$ s = $5$ min. AFTER TRANSITION ONSET (FIG. 52a AND TABLE I), WHEN THE ORBITER IS DESCENDING AT $M_{90} \approx 6$ ($\approx 1900$ km/s) AT 40 km ALTITUDE. IN THE FOREBODY, TRANSITION TO TURBULENCE CAUSES A LESS SEVERE THERMAL LOADING COMPARED TO PEAK.

FIG. 51. (a) HISTORY OF FREESTREAM REYNOLDS NUMBERS; (b) FREESTREAM FLOW PARAMETERS DURING VOYAGE DESCENT.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$M_{90}$</th>
<th>$T_{exo}$</th>
<th>$P_{exo}$</th>
<th>$R_{90, L}$</th>
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<tr>
<td>0</td>
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<td>0</td>
<td>0</td>
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<tr>
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<td>24000</td>
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<td>4000</td>
<td>40</td>
<td>8000</td>
<td>16000</td>
<td>32000</td>
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</table>

TABLE I: FREESTREAM FLOW CONDITIONS AT TIMES OF BOUNDARY-LAYER TRANSITION ALONG THE ORBITER WINDWARD PITCH PLANE (GOODRICH et al, 1983).
Conditions, which are attained well before at higher altitudes where the flow is laminar (Fig. 52a). In general, transition in lifting re-entry vehicles such as the Shuttle orbiter occurs well past peak heating and therefore contributes very little (but causes locally large heat fluxes) to the total heat load during re-entry (Goodrich et al., 1983). In the case of the Shuttle orbiter, transition was mostly induced by 4 mm roughness in the tiles of the thermal protection system (TPS) covering the underbelly.

Boundary-layer transition is a complex phenomenon that is subject of research in high-speed flows and is beyond the scope of this course. Note however that transition is linked to viscous effects and involves growth of perturbations in the laminar boundary layer, which can be caused by roughness or free-stream disturbances and destabilize the boundary layer downstream, creating first large-scale streamwise vortices that increase friction and are effective at heating the surface, and which then become a fully turbulent, mostly disorganized forest of vortices downstream (Figs. 53 and 54).

At hypersonic velocities, the problem described above is greatly complicated by several other factors, including the delay of transition caused by increasing Mach numbers (Fig. 55) and cold wall temperatures, both of which represent stabilizing effects, and the influences of thermochemical effects, recession of ablative surfaces of thermal protection systems, and freestream disturbances, the latter becoming important for the transition of boundary layers along smooth surfaces. Additional effects, such as viscous/inviscid interactions caused by vorticity in the entropy layer overriding the boundary layer are open research questions.

**Fig. 52:**

**Fig. 53:**

**Fig. 54:** Shadowgraph of boundary-layer transition at Mach 3 (Cerros, NASA RMA5685)

**Fig. 55:** Transition ReL data on sharp cones from wind tunnel and free flight (Stetson, 1987), with flight data: 77 points.
DEPENDING ON THE PARTICULAR TARGETED APPLICATION, HYPERSONIC FLIGHT TRAJECTORIES CAN BE SUBDIVIDED INTO:

1) TRANS-ATMOSPHERIC TRAJECTORIES, WHICH INVOLVE ASCENT AND CROSSING OF THE MESOSPHERE INTO LOW- EARTH ORBIT, AS WELL AS ATMOSPHERIC RE-ENTRY FROM CIRCULAR OR PARABOLIC ORBITS.

2) ENDO-ATMOSPHERIC TRAJECTORIES, WHICH INVOLVE ASCENT, CRUISE FLIGHT (MOST OFTEN IN THE STRATOSPHERE) AND DESCENT, ALL WITHOUT LEAVING THE CONFINES OF THE ATMOSPHERE.

WHILE VEHICLES UNDERGOING THE TRANS-ATMOSPHERIC TRAJECTORIES, SUCH AS THE APOLLO CAPSULE OR THE SHUTTLE ORBITER, TYPICALLY ENCOUNTER LAMINAR, TRANSITIONAL, AND TURBULENT BOUNDARY LAYERS, VEHICLES UNDERGOING ENDO-ATMOSPHERIC TRAJECTORIES ARE MOST LIKELY TO ENCOUNTER TURBULENT BOUNDARY LAYERS FOR THE MOST PART OF THE FLIGHT TIME. IN FACT, EACH TYPE OF MISSION TRAJECTORY DEMANDS A SPECIFIC TYPE OF HYPERSONIC VEHICLE (FIGS. 36-57). THE REMARKABLE ADAPTATION OF THE FUSELAGE TO INDUCE HIGH-ALTITUDE BREAKING WITH MINIMUM HEAT FLUX (BLUNT BODIES; SPACE CAPSULES AND LIFTING RE-ENTRY BODIES) IS IN STARK CONTRAST WITH THE SHARP AERODYNAMIC SHAPES PREVAILING IN ENDO-ATMOSPHERIC VEHICLES (SCRAMJETS, ROCKET-POWERED AIRCRAFTS) AND ALSO UNMANNED TRANS-ATMOSPHERIC VEHICLES (BOOST-Glide WEAPONS, INTERCONTINENTAL BALLISTIC MISSILES). THIS REMARKABLE ADAPTATION IS ABSENT IN MOST LOW-SPEED AIRCRAFT DESIGN.
Since the hypersonic flow environment imposes severe constraints on the aircraft structure, aerodynamic trim, thermal protection system, radio communications, and propulsion system, in particular, endo-atmospheric vehicles are designed so as to minimize wave drag (which is inviscid in origin) and heating loads, the latter being closely connected with viscous friction. On the other hand, trans-atmospheric vehicles have blunt shapes to increase wave drag, decrease deceleration forces harmful to crews, and decrease local heat fluxes in the nose area (exceptions that are less blunted are missiles and boost glide weapons for reasons that will become more evident in the analysis of trajectories performed in Chapter V). Table II and Figure 50 summarize some of these aspects.

<table>
<thead>
<tr>
<th>TABLE II</th>
<th>ENDO-ATMOSPHERIC VEHICLES (X-15, X-51, etc.)</th>
<th>TRANS-ATMOSPHERIC VEHICLE</th>
<th>WINGED RE-ENTRY</th>
<th>NON-WINGED</th>
<th>UNWIRED</th>
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<td>0.25</td>
</tr>
<tr>
<td>CONFIGURATION</td>
<td>SLENDER</td>
<td>BLUNT NOSE SLENDER LIFTING BODY</td>
<td>BLUNT NOSE, NON-SLENDER LIFTING BODY</td>
<td>BLUNT NOSE, NON-SLENDER LIFTING BODY</td>
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<tr>
<td>FLIGHT TIME</td>
<td>LONG (&gt; 1 hr)</td>
<td>SHORT (20 min)</td>
<td>SHORT (30 min)</td>
<td>SHORT (30 min)</td>
<td></td>
</tr>
<tr>
<td>FLIGHT ALTITUDE</td>
<td>0 - 80 km (UP TO WITHIN THE STRATOSPHERE)</td>
<td>0 - 300 km (LOW-EARTH ORBIT)</td>
<td>0 - 400 km (INTERPLANETARY ORBIT)</td>
<td>0 - 1,000 km (INTERPLANETARY ORBIT)</td>
<td></td>
</tr>
<tr>
<td>ANGLE OF ATTACK</td>
<td>SMALL (k = 0 - 2°)</td>
<td>LARGE (k = 4°)</td>
<td>k = 0° (MERKURY)</td>
<td>k = 2° (GEMINI, APOLLO)</td>
<td></td>
</tr>
<tr>
<td>BOUNDARY LAYERS</td>
<td>MORE TURBULENT THAN LAMINAR; IN SCRAMJETS, TRANSITION IS TRIPPED PURPOSEFULLY IN THE FOREBODY</td>
<td>LAMINAR + TRANSITIONAL +</td>
<td>LAMINAR + TRANSITIONAL +</td>
<td>LAMINAR + TRANSITIONAL +</td>
<td></td>
</tr>
<tr>
<td>WAVE DRAG</td>
<td>IMPORTANT</td>
<td>VERY IMPORTANT</td>
<td>VERY IMPORTANT</td>
<td>IMPORTANT</td>
<td></td>
</tr>
<tr>
<td>SKIN DRAG</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
<td></td>
</tr>
<tr>
<td>LIFT-TO-DRAG RATIO</td>
<td>~ 0.1</td>
<td>~ 1</td>
<td>~ 0.3</td>
<td>~ 0.3</td>
<td></td>
</tr>
<tr>
<td>ALTITUDE OF MAX. DECELERATION</td>
<td>—</td>
<td>—</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
<td></td>
</tr>
<tr>
<td>MAX. DECELERATION/ACCELERATION</td>
<td>2g (ACCEL. X-15)</td>
<td>3 - 6g</td>
<td>3 - 6g</td>
<td>&gt;150g</td>
<td></td>
</tr>
<tr>
<td>ALTITUDE OF MAX. HEAT FLUX (W/m²)</td>
<td>—</td>
<td>—</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
<td></td>
</tr>
<tr>
<td>TOTAL HEAT LOAD (J)</td>
<td>HIGH</td>
<td>HIGH</td>
<td>VERY HIGH</td>
<td>VERY HIGH</td>
<td></td>
</tr>
<tr>
<td>RAREFACTION EFFECTS</td>
<td>NOT IMPORTANT</td>
<td>MILD</td>
<td>MILD</td>
<td>MILD</td>
<td></td>
</tr>
<tr>
<td>AIR DISSOCIATION EFFECTS</td>
<td>MILD / IMPORTANT (O₂)</td>
<td>VERY IMPORTANT (O₂ + N₂)</td>
<td>VERY IMPORTANT (O₂ + N₂)</td>
<td>IMPORTANT</td>
<td></td>
</tr>
<tr>
<td>AIR IONIZATION EFFECTS</td>
<td>NOT IMPORTANT</td>
<td>MILD / IMPORTANT</td>
<td>MILD / IMPORTANT</td>
<td>IMPORTANT</td>
<td></td>
</tr>
<tr>
<td>VIBRATIONAL NON-EQUILIBRIUM</td>
<td>MILD / IMPORTANT</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
<td>IMPORTANT</td>
<td></td>
</tr>
</tbody>
</table>
FIG. 57: TRAJECTORIES OF HYPERSONIC VEHICLES

RAREFACTION EFFECTS

In close connection with viscosity are rarefaction effects arising at high altitudes as a result of the exponential decrease in density \( N = 183 \). The Navier-Stokes equations are based on a "field formulation" of the fluid variables \( \frac{d}{dx} \), where \( \phi \) and its gradients are continuous in space. This is a good approximation when the mean free path between molecular collisions is much smaller than the characteristic length of the flow \( L \), or equivalently, when \( \frac{K_m}{L} \ll 1 \), where \( K_m \) is the Knudsen number. For instance, consider a gas of molecules of diameter \( d \) with uniform number density \( n_0 \). A molecule labeled "A" in Fig. 58 will then collide with other molecules if their centers penetrate into a sphere of radius \( d \) centered at "A". Assuming that the molecule moves at a mean thermal speed \( \bar{v} = \left( \frac{8kT_m}{\pi m} \right)^{\frac{1}{2}} \), with \( k \) the Boltzmann constant and \( m \) the mass of a molecule, then the "A" molecule sweeps a volume \( \frac{4}{3} \pi d^3 \bar{v} \) per unit time. The corresponding collision frequency is

\[ \bar{\omega} = \frac{1}{n_0} \frac{4}{3} \pi d^3 \bar{v} m_0 \]  \( (184) \)

\( v \) collisions per unit time, whereas
\[ \lambda_{\infty} = \frac{c_{\infty}}{\Theta_{\infty}} = \frac{1}{\sqrt{2 \pi m T_{\infty}}} = \frac{m}{\sqrt{2 \pi}} \]  
(185)

is the mean free path. In Eq. (185), the factor \( \frac{1}{2} \) arises from the fact that \( c_{\infty} \) is not the relevant velocity to consider, but rather \( \sqrt{\frac{2}{3}} \) \( c_{\infty} \), which is the mean relative speed of the molecules (e.g., see rigorous derivation of (185) on pages 47-54 of Vincenti and Kruger [201, "Introduction to Physical Gas Dynamics"]). At sea level,

\[
\begin{align*}
T_{\infty} & = 298 \text{ K} \\
P_{\infty} & = 101325 \text{ Pa} \\
m & = \frac{w_{\text{air}}}{N_{A}} = \frac{28.9 \times 10^{-3} \text{ kg/m}^3}{6.022 \times 10^{23} \text{ molecules/mol}} = 4.79 \times 10^{-26} \text{ kg/molecule} \\
d & = d_{N_{A}} = 340 \times 10^{-12} \text{ m} \\
R_{g} & = R_{g} \sqrt{N_{A}} = 8.31 \text{ J/mol K} \left( 28.9 \times 10^{-3} \text{ kg/m}^3 \right)^{-1} = 286.8 \text{ J/kg K} \\
\Theta_{\infty} & = \frac{P_{\infty}}{R_{g} T_{\infty}} = 1.2 \text{ kg/s}^{-1} \Rightarrow m_{\Theta} = \frac{3m}{m} = 2.4 \times 10^{25} \text{ molecules/m}^3 \\
\lambda_{\infty} & = \left( \frac{8kT_{\infty}}{P_{\infty}} \right)^{1/2} \approx 467 \text{ m/s} \\
\end{align*}
\]

At sea level, \( \lambda_{\infty} \) is always small compared to most characteristic lengths of interest, so that all relevant high-speed flows around bodies tend to satisfy the continuum assumption that \( k_{m} = \lambda_{\infty} / L \ll 1 \). However, \( \lambda_{\infty} \) is a strong function of altitude because of its dependence on \( \Theta_{\infty} \); as shown in Fig. 5.10. (Note that \( \Theta_{\infty} \) is additionally dependent on \( c_{\infty} \), whose variations are similar to those of the speed of sound \( c_{\infty} \) and are not as strong as those of \( \Theta_{\infty} \), as observed in Fig. 5.8b).

As a result, the mean free path increases with altitude as

\[ \frac{\lambda_{\infty}}{\lambda_{\infty}(29.5)} \approx e^{38} \]  
(186)

At a critical altitude \( R_{c} = \frac{1}{P \Theta_{\infty} \left( \frac{L}{\lambda_{\infty}(29.5)} \right)} \), the Knudsen number becomes unity and \( L = \lambda_{\infty} \). In this limit, the flow is said to be strongly rarefied, in that every molecule on average — collides once or less with other molecules in distances of order \( L \). Note that for the radius of the nose of the Shuttle Orbiter (\( L \approx 1.2 \text{ m} \)), the critical height is \( R_{c} \approx 100 \text{ km} \), which is similar to the altitude of the entry interface of its flight.

![Graph](https://via.placeholder.com/150)

Fig. 5.8b, US Standard Atmosphere 1976
For this reason, rarefaction effects are not crucial in re-entry applications; except for inter-planetary space capsules or ballistic missiles, whose descent begins in the outermost skins of the atmosphere. Even in the latter cases, the bulk of the aerodynamic interactions, including aerobreaking and heating, occurs below the mesopause where the density is sufficiently large to enable continuum effects of friction and wave drag. In addition, note that, for aerodynamic bodies flying at velocities much higher than the speed of sound \( \lambda_0 \) is not the appropriate mean free path but something of order \( \sqrt{\left( \frac{\bar{c}_m}{U_m} \right) \lambda_0} \), since the volume swept by the molecules between collisions is \( \Pi_0 \bar{U}_m^2 \). Instead (note that \( \lambda_0 \) is measured in the gas frame but the relevant mean free path in the flow around a hypersonic vehicle must be measured in the vehicle frame), and \( \lambda_0 \) the number density of the gas around the hypersonic vehicle varies largely in space and cannot be univocally characterized by a single number. For these reasons, and many others that are described thoroughly by Probst in "Shock Wave and Flow Field Development in Hypersonic Re-Entry," Air Journal (1964), great caution should be used when using \( \lambda_0 \) or \( \sqrt{\frac{\bar{c}_m}{U_m} \lambda_0} \) to characterize the altitude \( z_0 \) for the onset of free-molecule flow \( (\lambda_m = L) \). In fact, the correction factor \( \sqrt{\left( \frac{\bar{c}_m}{U_m} \right) \lambda_0} \) to \( \lambda_0 \), described above, already suggests that the true altitude for free-molecule flow is actually higher than the altitude for which \( \lambda_0 = L \) in hypersonic flight (6-8) \( z_0 = \frac{1}{\beta} \left( \frac{L}{\lambda_0} \right)^{1/2} \).

The flow field around a blunted body as it descends into the atmosphere is qualitatively sketched in Fig. 59. Note that viscous effects at high altitudes in the continuum limit first as a thick fully-viscous (and laminar) shock layer, which, as \( R_e \) increases...

---

**Fig. 59**

**Sonic flow**

1. Rarefied flows
2. To be directly solved
3. Boltzmann equation

**Molecular flow**

1. Rarefied flows
2. To be directly solved

**Decreasing altitude**

- Shock forms
- Transitional regime in which molecules begin to collide with each other and form a thick shock \( \Theta_{\text{shock}} \)

**Merged layer**

- The shock is thinner but not too thin (\( \Theta_{\text{shock}} \) is 4)
- Can be used to calculate shock structure with free-stream conditions

**Viscous layer**

- The shock is thin (\( \Theta_{\text{shock}} < \Delta \))
- The shock layer is fully viscous; the Navier-Stokes equations are used

**Boundary layer**

- The shock is thin (\( \Theta_{\text{shock}} < \Delta \))
- A viscous boundary layer is formed (\( \Theta_{\text{shock}} < \Delta \)) with a rotational layer above it.
BECOMES A SHOCK LAYER OF THE TYPE IN Fig. 64, WITH A THIN VISCOUS BOUNDARY LAYER NEAR THE SURFACE (WHICH MAY BE LAMINAR, TRANSITIONAL OR TURBULENT DEPENDING ON THE VALUE OF $Re_{x0}$, $Ma_{x0}$ AND OTHER PARAMETERS AS DISCUSSED IN THE PREVIOUS SECTION).

ALONG WITH A ROTATIONAL ENTROPY LAYER RIGHT ABOVE YET WITHIN THE SHOCK LAYER (Fig. 60)


BEFORE FINISHING THIS SECTION, IT IS WORTH MENTIONING THAT THE VISCOSITY AND MEAN FREE PATH ARE APPROXIMATELY RELATED BY THE KINETIC THEORY AS

$$\mu \approx \frac{1}{2} \frac{d_0^2 \lambda}{U}, \quad (187)$$

AS A RESULT, RAREFACTION EFFECTS TYPICALLY INDUCE AN INCREASE IN THE SKIN DRAG RELATIVE TO THE PRESSURE DRAG (SEE Fig. 64).

THE REST OF THIS CHAPTER IS DEDICATED TO COMPRESSIBLE LAMINAR BOUNDARY LAYERS INCLUDING THE NEAR STAGNATION REGION AT THE NOSE OF BLUNT HYPERSONIC VEHICLES.
BOUNDARY LAYERS ARE THIN REGIONS CLOSE TO SOLID SURFACES WHERE VISCOUS EFFECTS BECOME IMPORTANT AT HIGH REYNOLDS NUMBERS. ACROSS THE BOUNDARY LAYER, THE AEROTHERMAL VARIABLES TRANSIT FROM THEIR VALUES IN THE FREE STREAM (WHICH HENCEFORTH WILL BE REFERRED TO AS EDGE VALUES WITH A SUBINDEX "E") TO EMPHASIZE THAT THESE MAY BE DIFFERENT FROM THOSE IN THE FREE STREAM UPSTREAM FROM THE AEROSPACE VEHICLE BECAUSE OF THE PRESENCE OF SHOCKS) TO THEIR VALUES AT THE WALL. AT HIGH REYNOLDS NUMBERS, \( R_e^L = \frac{\rho_0 U_0 L}{\mu_0} \gg 1 \), THE BOUNDARY LAYER IS SLENDER IN THE SENSE

\[
\frac{\delta}{L} \ll 1
\]

(188)

WHERE \( L \) IS A CHARACTERISTIC SIZE OF THE VEHICLE. THIS SLENDERNESS CAN BE EXPLOITED IN SIMPLIFYING THE NAVIER-STOKES EQUATIONS, AS DISCUSSED IN CLASSICAL TEXTS.

HIGH-SPEED BOUNDARY LAYERS DIFFER FROM THEIR INCOMPRESSIBLE COUNTERPARTS IN THREE MAIN ASPECTS:

1. THE MOMENTUM AND THERMAL FIELDS ARE STRONGLY COUPLED THROUGH THE DENSITY.
2. SUCH COUPLING IS AGGRAVATED BY THE DEPENDENCE OF VISCOSITY AND THERMAL CONDUCTIVITY ON TEMPERATURE.

\* THE RECOVERY FACTOR AND ADIABATIC WALL TEMPERATURE:

AN EXAMPLE OF THE COUPLING DESCRIBED ABOVE IS ILLUSTRATED SCHEMICALLY IN FIG. 62 FOR A HIGH-SPEED BOUNDARY LAYER OVER AN ADIABATIC WALL. WHILE IN THE OVERRIDING INVIScid STREAM THE STATIC TEMPERATURE IS \( T_0 \), THE VISCOUS DISSIPATION IN THE BOUNDARY LAYER CAUSES AN INCREASE OF TEMPERATURE THERE. IN THIS EXAMPLE, STEADY-STATE CONDITIONS ARE ACHIEVED BY BALANCING THAT HEAT WITH CONVECTION AND CONDUCTION AWAY FROM THE WALL. THE RESULTING TEMPERATURE AT THE WALL IS REFERRED TO AS THE ADIABATIC WALL TEMPERATURE \( T_{aw} \), WHICH GENERALLY SATISFIES

\[
T_{aw} \leq T_{oe}
\]

(189)
where \( T_{eq} \) is the stagnation temperature in the inviscid stream. Note that, although the fluid is necessarily brought to rest at the wall, the deceleration is by no means isentropic, and therefore the stagnation temperature \( T_{eq} \) is not recovered at the wall. Instead, a recovery factor

\[
t = \frac{T_{aw} - T_{0}}{T_{aw} - T_{in}} = \frac{T_{aw} - T_{0}}{U_0^2/2c_p} = \frac{Z}{(\gamma-1)M_0^2} \left( \frac{T_{aw} - 1}{T_{aw}} \right)
\]

(189)

can be defined that is smaller than unity. A typical profile of stagnation temperature to across the boundary layer is also provided in Fig. 62 and will be discussed further below.

With non-adiabatic walls, the temperature profile arrives at the wall with non-zero slope and can attain the shapes sketched in Fig. 63 depending on whether the wall temperature is lower or higher than the adiabatic wall temperature. In the former case, the temperature in the boundary layer needs not be bounded by \( T_0 \) or \( T_{aw} \), but it can develop an internal maximum due to the heat created by the viscous dissipation, which is commonly referred to as "aerodynamic heating." (Here, this term should be understood to be limited to viscous effects, since other aerodynamically-derived heating phenomena can occur, for instance, by shock/shock interactions). In general, the adiabatic wall temperature is a solution of the boundary-layer equations, which are described below.

**The boundary-layer equations**

The slenderness condition (188) enables the simplification of the Navier-Stokes equations by neglecting second-order flow phenomena related to \( \partial^2 \). The streamwise viscous stress of the two velocity components, \( u \) & \( v \), streamwise heat conduction, \( \partial \), viscous dissipation generated by all gradients of the transverse velocity, \( \partial u \), viscous dissipation generated by the streamwise gradients of the streamwise velocity. In addition, if the Mach number is not too large compared to \( \frac{L}{e_B} \), the pressure variations across the boundary layer can be safely neglected. With these simplifications, the Navier-Stokes equations become.
\[ \frac{\partial}{\partial x} \left( \rho U \right) + \frac{\partial}{\partial y} \left( \rho V \right) = 0 \quad \text{Continuity (190)} \]

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = - \frac{dP}{dx} + \frac{\rho V}{\partial y} \left( \frac{\partial U}{\partial y} \right) \quad \text{Streamwise Momentum Conservation (191)} \]

\[ \frac{\partial P}{\partial y} = 0 \quad \Rightarrow \quad P = P_0(x) \quad \text{Transverse Momentum Conservation, (192)} \]

\[ \frac{\partial h}{\partial x} + \frac{\partial h}{\partial y} = \frac{1}{\rho} \left( \frac{\partial}{\partial y} \left( \frac{\rho V}{\partial y} \right) + U \frac{dP}{dx} + \mu \left( \frac{\partial U}{\partial y} \right)^2 \right) \quad \text{Enthalpy Conservation (193)} \]

\[ \text{(Non-Radiating Gas)} \]

EQUATIONS (190)–(193) ARE THE BASIC BOUNDARY-LAYER EQUATIONS FOR COMPRESSIBLE FLOWS IN THERMODYNAMIC AND CHEMICAL EQUILIBRIA. (THIS CONCEPT WILL BE DEFINED IN MORE DEPTH IN CHAPTER IV). IN EQ. (193), $h$ IS A STATIC ENTHALPY GENERALLY DEFINED AS $h = h_0 + \int_{T_0}^{T} C_p(T) \, dT$, WHERE $T_0$ IS AN ARBITRARY REFERENCE TEMPERATURE AND $C_p$ IS A CONSTANT-$T$ PRESSURE SPECIFIC HEAT THAT MAY GENERALLY VARY WITH TEMPERATURE DEPENDING ON THE DEGREE OF VIBRATIONAL OR ELECTRONIC EXCITATION (SEE CHAPTER IV). EVEN IF $C_p$ VARIES WITH TEMPERATURE, NOTE THAT $\frac{\partial h}{\partial x} = C_p \frac{\partial T}{\partial x}$ AND $\frac{\partial h}{\partial y} = C_p \frac{\partial T}{\partial y}$ BY THE LIENZER INTEGRAL RULE, AND THEREFORE (193) CAN BE WRITTEN AS

\[ \frac{\partial}{\partial x} \left( \rho U \right) + \frac{\partial}{\partial y} \left( \rho V \right) = \frac{1}{\rho} \left( \frac{\partial}{\partial y} \left( \frac{\rho V}{\partial y} \right) + U \frac{dP}{dx} + \mu \left( \frac{\partial U}{\partial y} \right)^2 \right) \]

WHERE $k$ IS THE THERMAL CONDUCTIVITY, IN CALORICALLY PERFECT GASES ($\rho = \text{const.}$), (194) REMAINS THE SAME, BUT A USEFUL COMBINATION OF (191) AND (194) CAN BE MADE BY NOTICING THAT

\[ T_0 = T + \frac{U^2}{2C_p} \quad (195) \]

AND SUMMING THE KINETIC-ENERGY EQUATION

\[ \frac{\partial}{\partial x} \left( \rho \frac{U^2}{2} \right) + \frac{\partial}{\partial y} \left( \rho \frac{V^2}{2} \right) = - \frac{dP}{dx} \frac{\partial U}{\partial y} + \frac{\rho}{\partial y} \left( \frac{\mu}{\partial y} \left( \frac{U^2}{2} \right) \right) - \mu \left( \frac{\partial V}{\partial y} \right)^2 \]

TO EQ. (194) (NOTE THAT (196) IS OBTAINED BY MULTIPLYING (191) BY $U$), WHICH GIVES

\[ \frac{\partial}{\partial x} \left( \rho C_p U \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho C_p V \frac{\partial T}{\partial y} \right) = \frac{1}{\rho} \left( \frac{\partial}{\partial y} \left( \frac{\rho V}{\partial y} \right) \right) + \frac{1}{\rho} \left[ \frac{\partial}{\partial y} \left( \frac{\mu}{\partial y} \left( \frac{U^2}{2} \right) \right) \right] \]

AND SINCE $C_p$ IS CONSTANT, THEN THE ABOVE EQUATION BECOMES

\[ \frac{\partial}{\partial x} \left( \rho C_p U \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left( \rho C_p V \frac{\partial T}{\partial y} \right) = \frac{1}{\rho} \left[ \frac{\partial}{\partial y} \left( \frac{\mu}{\partial y} \left( \frac{U^2}{2} \right) \right) \right] \]

WHERE $Pr = \mu C_p / k$ IS THE PRANDTL NUMBER AND $D_T = k / C_p$ IS THE THERMAL DIFFUSIVITY. THE PROBLEM OF A NON-REACTION, COMPRESSIBLE LAMINAR BOUNDARY LAYER...
In local thermodynamic equilibrium is defined by Eqs. (190)-(192) along with (194) or (197), the latter being only valid if $C_p = \text{const}$. These equations need to be supplemented with the equation of state

$$P_0 = \varrho R_0 T$$  \hspace{1cm} (198)

along with an expression for the thermodynamic pressure at the boundary-layer edge as a function of $x$. Note that gradients $dP/dx$ can be externally imposed or result from the inviscid solution over curved bodies obtained, for instance, by using the Newtonian hypersonic theory (45).

The boundary conditions at the wall may include

$$\begin{align*}
U &= 0 \quad \text{(non-slip)}, \\
V &= V_w \quad \text{(non-slip $V_w = 0$ or suction/blowing $V_w \neq 0$)}, \\
T &= T_w \quad \text{(thermal contact with uniform or variable wall temperature)}, \\
\frac{\partial T}{\partial y} &= 0 \quad \text{(adiabatic wall)}
\end{align*}$$

and other mixed thermal boundary conditions that can be used with radiative or ablative walls.

In addition, constitutive laws for the variations of $C_p$, $\mu$, $K$, and $P_r$ must be incorporated in these equations in the most general case. In what follows, the variations of $C_p$ and $P_r$ with temperature will be neglected, so that $\mu$ and $K$ become proportional to each other and to a prescribed power of $T$.

Before solving the set of equations outlined above, it is worth analyzing first a simplified case that nonetheless is conceptually useful and provides grounds for analogies between the velocity and temperature fields.

# The Case of Unity Prandtl Number

Equation (197) is reminiscent of the momentum equation (194), but not quite the same. Consider first making $P_r = 1$ in (197), which gives

$$\varrho U \frac{\partial T_0}{\partial x} + \varrho V \frac{\partial T_0}{\partial y} = \frac{\partial}{\partial y} \left( \mu \frac{\partial T_0}{\partial y} \right)$$  \hspace{1cm} (199)

or equivalently,

$$\frac{\partial}{\partial y} \left[ K \frac{\partial T_0}{\partial y} + \mu U \frac{\partial U}{\partial y} \right] = \frac{\partial}{\partial y} \left( \mu \frac{\partial T_0}{\partial y} \right)$$  \hspace{1cm} (200)

As observed from the equation immediately above equation (197), an important solution to (199) that satisfies the boundary condition $T_0 = T_0(x)$ at $y \to 0$ is the one in
IN WHICH THE STAGNATION TEMPERATURE IS UNIFORM IN THE BOUNDARY LAYER,
\[ T_0 = T_{\infty} \text{ everywhere}, \] (201)

IN SUCH A WAY THAT (200) GIVES
\[ k \frac{\partial T}{\partial y} + \mu \frac{\partial u}{\partial y} = \text{const.} = 0, \]
IN THE INVIScid STREAM BOTH \( \frac{\partial T}{\partial y} \) AND \( \frac{\partial u}{\partial y} \) ARE ASSUMED TO BE ZERO (202)

THIS CONSTRAINT INDICATES THAT THE HEAT CONDUCTION AND THE WORK DONE BY THE VISCOUS STRESS ARE IN BALANCE EVERYWHERE, IN PARTICULAR, AT THE WALL
\[ q_w = -k \frac{\partial T}{\partial y} \bigg|_{y=0} = -\mu \frac{\partial u}{\partial y} \bigg|_{y=0} = 0 \] (203)

SINCE \( U = 0 \) AT THE WALL, AS A RESULT, A COMPRESSIBLE BOUNDARY LAYER OF A GAS WITH UNITY PRANDTL NUMBER OVER AN ADIABATIC WALL NECESSARILY REQUIRES THE STAGNATION TEMPERATURE (OR ENTHALPY) TO BE UNIFORM AND EQUAL TO \( T_{\infty} \).

BECAUSE OF (45), THE ADIABATIC WALL TEMPERATURE EQUALS THE STAGNATION TEMPERATURE
\[ T_{\infty} \approx T_{\infty w} + \frac{U^2}{2c_p} = T_{\infty w} \] (204)

THEREBY INDICATING A UNITY RECOVERY FACTOR, \( q = 1 \). NOTE THAT THE SAME CONCLUSION CAN BE INFERRED FROM (202),
\[ \frac{1}{\rho} \left( T + \frac{P_r U^2}{2c_p} \right) = 0 \Rightarrow T + \frac{U^2}{2c_p} = T_{\infty w} \text{ everywhere} \] (205)

EQUATIONS (204)-(205) SUGGEST THAT FOR GASES WITH \( P_r < 1 \), IT IS EXPECTED THAT \( q < 1 \), BUT SINCE \( P_r \approx 0.72 \) IN PRACTICAL APPLICATIONS, THE ADIABATIC WALL TEMPERATURE IS ALWAYS MUCH CLOSER TO THE STAGNATION TEMPERATURE OF THE OVERRIDING INVIScID STREAM THAN TO ITS STATIC TEMPERATURE. AT HYPERSONIC MACH NUMBERS, \( T_{\infty}/T_{\infty w} >> 1 \) AND AS A CONSEQUENCE, \( T_{\infty w} \) IS VERY HIGH, AND THE SURFACE OF THE ADIABATIC WALL GETS VERY HOT (NOTE THAT IN PRACTICE RADIATION FROM THE SURFACE ESTABLISHES LOWER WALL TEMPERATURES ALTHOUGH THEY ARE SUFFICIENTLY HOT TO MELT THERMAL-RESISTANT ALLOYS). IT IS FOR THESE REASONS THAT SOME OF THE MOST IMPORTANT CHALLENGES OF HYPERSONIC FLIGHT ARE NOT ASSOCIATED WITH BREAKING THE SOUND BARRIER, BUT RATHER WITH BREAKING THIS "THERMAL BARRIER".

CONSIDER NOW AN ADDITIONAL SIMPLIFICATION OF THE PROBLEM BY MAKING A ZERO PRESSURE GRADIENT \( (dP/dx = 0) \) IN (191), WHICH NOW BECOMES
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{2}{\beta y} \left( \frac{\partial \vartheta}{\partial y} \right). \]  

(206)

This equation is very similar to Eq. (199) for the stagnation temperature at \( P_T = 1 \). (Recall also that \( \mu = k \) since \( \varphi = \text{const} \), if the wall is isothermal. It is straightforward to see that the similarity relation

\[ \frac{T_0 - T_w}{T_{oe} - T_w} = \frac{U}{U_0} \]  

(207)

must be necessarily satisfied, since (199) and (206) along with the boundary conditions \( U = U_0 \), \( T_0 = T_{oe} \) \((y = 0)\), and \( U = 0 \), \( T_0 = T_w \) \((y = 0)\) are formally equivalent problems when formulated in the variables \( U/U_0 \) and \( (T_0 - T_w)/(T_{oe} - T_w) \). Equation (207) indicates that isolevels of velocity are also isolevels of stagnation temperature (Fig. 64), and as a result the thickness of the thermal and momentum boundary layers have equal thicknesses.

Upon substituting (195) into (207), the relation between the velocity and static temperature

\[ T - T_w = (T_{oe} - T_w) \left( \frac{U}{U_0} \right) - \left( \frac{y - a}{z} \right) \frac{Ma^2}{Re} \frac{T_o}{U_0} \left( \frac{U}{U_0} \right)^2 \]  

(208)

is obtained, which enables the calculation of the local heat flux

\[ q = \frac{-k}{\beta} \frac{\partial T}{\partial y} = \frac{k}{U_0} \frac{\partial u}{\partial y} \left[ h_{oe} - (y - a) \frac{Ma^2}{Re} \frac{T_o}{U_0} \right]. \]  

(209)

It is interesting to note that, since \( \partial u/\partial y > 0 \) for attached boundary layers, then \( q > 0 \) (i.e. \( \partial T/\partial y < 0 \)) everywhere if \( T_w > T_{oe} \), thereby resembling the blue curve in Fig. 63 in which heat is always conducted away from the hot plate. On the other hand, if \( T_w < T_{oe} \), then \( q \) may be positive or negative depending on position, indicating a non-monotonic behavior in the temperature as in the red curve in Fig. 63. (Note that \( T_{oe} = T_{oe} \) when \( P_T = 1 \) as elucidated above).

In all cases, the wall heat flux is given by

\[ q_w = \frac{-k}{\beta} \frac{\partial T}{\partial y} \bigg|_{y = 0} = \frac{C_T}{U_0} \left( T_w - T_{oe} \right). \]  

(210)
WHERE \( \gamma_w = \frac{M}{\partial U / \partial y} \bigg|_{y=0} \) is the shear stress at the wall. Equation (210) can be recast into the familiar form

\[
S_t = \frac{C_f}{2}
\]

Reynolds Analogy (211)

Where
\[
S_t = \frac{q_w}{\frac{\rho e U e C_p}{(T_{aw} - T_w)}},
\]

is the Stanton number, and
\[
C_f = \frac{q_w}{\frac{1}{2} \frac{\rho e U e^2}{}}
\]

is the skin friction coefficient.

Equation (211), the Reynolds analogy, establishes an analogy between heat and momentum transfer, or between heat transfer and viscous friction. In deriving such analogy recall that it has been assumed the following: i) the gas is calorically perfect and non-radiating, ii) the Prandtl number is \( Pr = 1 \), and iii) the wall is isothermal and its temperature is constant along the wall, and iv) the pressure gradient is zero, \( \partial p_e / \partial x = 0 \). Note that no assumptions have been made regarding \( \rho_e \) or \( U_e \), which can vary arbitrarily with temperature.

It is not difficult to reformulate the Reynolds analogy (211) in terms of the Nusselt number, \( Nu_x = \frac{h_x}{k_e} \), with \( h = \frac{q_w}{\frac{T_{aw} - T_w}} \) the convective coefficient, and the Reynolds number, \( Re_x = \frac{\rho e U e x}{\mu_e} \), which gives
\[
Nu_x = \frac{C_f}{2} Re_x \quad (212)
\]

For practical purposes, the analysis above suggests that the direction of the heat flux at the wall depends on the difference \( T_w - T_{aw} \), with \( T_{aw} \) being of order \( T_{aw} \), rather than on the difference \( T_w - T_e \), which instead becomes relevant for heat transfer only in low-speed boundary layers.

4. Self-similar boundary-layer equations

The considerations given above only pertain to integral relations between \( T \) and \( U \). Instead, this section focuses on deriving a self-similar system of equations equivalent to the general one given by (190)-(193). For this purpose, the following change of variables is introduced, which was proposed by Lees in "Laminar heat transfer over blunt-nosed bodies at hypersonic speeds", JET Propulsion 26 (1956), and which
COMBINES THE LEVY AND MANGER AND HOWARTH-DORODNITZYN TRANSFORMS. THE CHANGE
OF INDEPENDENT VARIABLES IS GIVEN BY
\[ \xi = \xi(x) = \int_0^x \frac{3}{8} \rho \omega_0 \omega_0 \, dx, \quad \eta = \eta(x, y) = \frac{3}{125} \int_y^\infty \xi \, dy \]  \hspace{1cm} (213)

IN SUCH A WAY THAT
\[ \frac{\partial \xi}{\partial x} = \frac{3}{8} \rho \omega \omega_0, \quad \frac{\partial \eta}{\partial y} = \frac{\rho \omega_0}{125} \]  \hspace{1cm} (214)

THE ANALYSIS IS ALSO FACILITATED BY INTRODUCING THE STREAM FUNCTION \( \psi \), WITH
\[ \frac{\partial \psi}{\partial y} = \psi \quad \text{AND} \quad \frac{\partial \psi}{\partial x} = -\frac{\partial \psi}{\partial x}, \]  \hspace{1cm} (215)

AND BY TRANSFORMING THE DEPENDENT VARIABLES \( \psi, u \) AND \( h \) AS
\[ \psi = \sqrt{25} f(\xi), \quad u = U_0(x) f'(\xi) \quad \text{AND} \quad h = h_0(x) g(\xi), \]  \hspace{1cm} (216)

WHERE \( U_0(x) \) AND \( h_0(x) \) ARE DISTRIBUTIONS AT THE EDGE OF THE BOUNDARY LAYER BY
THE OVERRIDING INVISID STREAM, AND \( f \) AND \( g \) ARE SOLE FUNCTIONS OF \( \xi \) THAT ARE
TO BE SOLVED FOR, INTRODUCING THIS CHANGE IN EQUATIONS \((190)-(193)\) THE SELF-SIMILAR
FORMS

\[ \left( \frac{d^2 \psi}{dx^2} \right)' + \frac{f''}{f} = \frac{25}{U_0} \left( \frac{f'^2}{f} - \frac{\rho \omega_0}{\rho \omega} \frac{dU_0}{dx} \right) \]  \hspace{1cm} \text{MOMENTUM CONSERVATION (217)}

\[ \left( \frac{d^2 \psi'}{dx^2} \right)' + \frac{f'''}{f} = 25 \left( \frac{f''}{f} \frac{d\rho}{dx} + \frac{\rho \omega_0}{\rho \omega} f' \frac{dU_0}{dx} \right) - \frac{\rho \omega_0}{\rho \omega} \frac{dU_0}{dx} \]  \hspace{1cm} \text{ENHYPHY CONSERVATION (218)}

ARE OBTAINED ALONG WITH THE UNIFORMITY OF THE PRESSURE \( \frac{\partial P}{\partial y} = 0 \) ACROSS THE BOUNDARY
LAYER, IN DERIVING THESE EQUATIONS, USE OF THE EULER EQUATION IN THE INVISID
STREAM

\[ -\frac{dP}{dx} = \frac{3}{8} \rho \omega_0 \frac{dU_0}{dx} \]  \hspace{1cm} (219)

HAS BEEN MADE. IN ADDITION, THE PARAMETER \( \xi' \) IS GIVEN BY
\[ \xi' = \frac{8}{3} \frac{\rho}{\rho \omega_0} \]  \hspace{1cm} "CHAPMAN-RUBESIN PARAMETER" \hspace{1cm} (220)

THE TRANSVERSE VELOCITY COMPONENT \( v \) CAN BE OBTAINED FROM THE SECOND EQUATION IN \((215),
\[ \frac{\partial v}{\partial x} = \frac{3}{8} \frac{\rho \omega_0}{\rho \omega} \frac{dU_0}{dx} \]  \hspace{1cm} (221)

WHERE \( \frac{\partial v}{\partial x} \) CAN BE COMPUTED THE FOLLOWING WAY FOR UNIFORM FREE STREAM CONDITIONS, THE
INTEGRATION OF \((214) \) YIELDS \( \gamma = \chi \sqrt{\frac{2}{R_e \omega_0 \rho \omega_0}} \int_0^x \xi(\gamma) d\xi \), SO THAT \( \frac{\partial v}{\partial x} = -\frac{1}{2 \sqrt{R_e \omega_0 \rho \omega_0}} \). SUBSTITUTING THIS
INTO \((221) \) YIELDS \( v/U_0 = (\frac{1}{2} \chi f) f - \frac{3}{8} \frac{\rho \omega_0}{\rho \omega} \), WHERE \( R_e = \rho \omega_0 \sqrt{\frac{\rho \omega_0}{\rho \omega}} \) IS THE LOCAL REYNOLDS NUMBER.
THE BOUNDARY CONDITIONS ASSOCIATED WITH \((217) \) AND \((218) \) ARE
\( f' = f'' = 0 \) AT THE WALL \( z = 0 \), CORRESPONDING TO NON-SLIP AND NON-PERMEABLE WALL,

- \( q' = 0 \) AT THE WALL \( z = 0 \), CORRESPONDING TO ADIABATIC WALL,

OR \( q = q_w = \frac{h_w}{\rho_0} \) AT THE WALL \( z = 0 \), CORRESPONDING TO ISOTHERMAL WALL,

- \( q'' = \frac{q}{\rho} \) AT THE BOUNDARY LAYER EDGE, CORRESPONDING TO THE INVIScid STREAM VELOCITY AND ENTHALPY.

It is important to note that (217) - (218) are generally valid for a non-reacting, non-radiating gas at local thermodynamic equilibrium. In the formulation, \( C, \mu, K, \Phi \) and \( q' \) may vary with temperature, and the edge conditions at the wall \( T_w, \rho_w, \rho_0, h_0 \) and \( P_0 \), as well as the wall temperature \( T_w \), may vary with \( x \) (or \( z \)). As a consequence, very rarely in hypersonic flows the equations (217) - (218) are fully self-similar, and do not in that they only depend on \( \beta \). For such self-similar conditions to be satisfied, the following conditions must be simultaneously satisfied:

1) The right-hand side of (217) must be a function of \( \beta \) or a constant. This condition is trivially satisfied in flat-plate boundary layers with zero pressure gradient in the streamwise direction, which, by equation (219), implies that \( d (\beta x) / dx = 0 \).

2) The Chapman-Rubesin parameter must be a function of \( \beta \), i.e., \( \beta = \beta (\beta) \), or equal to a constant. It is anticipated that this occurs when \( \beta = \beta_0 + \beta(T) \).

- \( \beta \) varies as \( \beta \) \( \rightarrow \) \( \beta = \beta_0 + \beta(T) \)

- The gas is calorically perfect and \( \beta \) varies with a power of the temperature \( \beta = \beta_0 + \beta(T) = \beta_0 + \beta(T) \)

\( \beta = \beta_0 + \beta(T) \)

- The gas is calorically perfect, \( \beta \) varies with \( T \) according to the Sutherland law, and \( T_0 \) is uniform in \( x \):

\[ \beta = \beta_0 + \beta(T) = \beta_0 + \beta_0 \left( \frac{T}{T_0} \right)^{3/2} \left( \frac{T_0 + A}{T + A} \right) = \beta_0 \left( \frac{T_0}{T} \right)^{3/2} \left( \frac{T_0 + A}{T + A} \right) \]

(\( \beta = \beta_0 \left( \frac{T}{T_0} \right)^{3/2} \left( \frac{T_0 + A}{T + A} \right) \))

3) The Prandtl number must be constant or a function of \( \beta \) alone. As with the \( \beta \) parameter, this condition is satisfied when the gas is calorically perfect, and \( \beta \) varies with a power of the temperature, or with Sutherland's...
4) **The wall temperature, if imposed by boundary conditions, must be a constant.**

5) **The right-hand side of equation (248) must be a constant or a function of \( \frac{y}{L} \) alone. This condition is trivially satisfied in flat-plate boundary layers under zero pressure gradient, since this implies zero enthalpy gradient in the streamwise direction along the edge**

\[
\varrho_0 \frac{d\vartheta}{dx} = \alpha_0 \frac{d\varrho}{dx} = 0. \quad (223)
\]

In these conditions, the aerodynamic heating term in (248) scales as

\[
\frac{U_0^2}{\varrho_0} \frac{d\vartheta}{dL} = \frac{\varrho_0^2}{\alpha_0} \frac{d\varrho}{dL} = (8-1) M_0^2 \quad (224)
\]

with \( M_0 \) the Mach number at the boundary-layer edge.

**The case of a boundary layer over a flat plate with \( d\varrho/dx = 0 \)**

For a flat-plate boundary layer with zero pressure gradient, \( d\varrho/dx = dU_0/dx = d\vartheta/dx = 0 \), and \( d\varrho_0/dx = d\vartheta_0/dx = 0 \) correspondingly, Eqs. (219) - (218) become

\[
\begin{align*}
\left( \frac{Q}{Pr} \right)' + (f')' & = 0 \quad (225) \\
\left( \frac{Q_0}{Pr} \right)' + f_0' + \left( 8-1 \right) \frac{M_0^2}{2} f_0'' & = 0 \quad (226)
\end{align*}
\]

Subject to \( f(0) = (f_0)' = 0 \), \( \varrho_0'(0) = 0 \) (or \( \varrho_0(0) = \varrho_0 \)), and \( f'(\infty) = (f_0)'(\infty) = 1 \),

where a calorically perfect gas has been assumed so as to express the last term on (218) using (224). Note that an equation for the stagnation temperature

\[
M = \frac{T_0}{T_0e} = \frac{T_0 - \alpha_0 x}{T_0e} + \frac{U_0^2}{C_p T_0e} \frac{f'}{2} = M \left( \frac{L}{x} \right) \quad (227)
\]

Can be easily derived by multiplying (225) by \( f' U_0^2 / C_p T_0e \) and adding the resulting equation to (226) multiplied by \( T_0 / T_0e \), which gives

\[
\left( \frac{Q}{Pr} \right)' + f_0' + \left( 8-1 \right) \frac{M_0^2}{2} \left[ 1 + \left( \frac{U_0^2}{C_p T_0e} \right) \right]^{-1} \left( 1 + \frac{M_0^2}{2} \right) \varrho_0' \left( \frac{1 - \frac{1}{Pr}}{1 - \frac{1}{Pr}} \right) \left( f'' \right)' = 0 \quad (228)
\]

Or equivalently

\[
\left( \frac{Q}{Pr} \right)' + f_0' + \left( 8-1 \right) M_0^2 \left[ 1 + \left( \frac{U_0^2}{C_p T_0e} \right) \right]^{-1} \left( 1 - \frac{1}{Pr} \right) (f'' \left( f' \right)' = 0 \quad (229)
\]
In particular, (228) is the self-similar form equivalent to (197), and it also leads to the constant stagnation-temperature result $\frac{m}{P_r} = 1$ for adiabatic walls and $P_r = 1$, in such a way that the adiabatic wall temperature in self-similar form is

$$q_{w} = \frac{T_{w}}{T_e} = 1 + \left( \frac{y-1}{2} \right) Ho_e^2$$

which is the same as the stagnation temperature of the inviscid stream (i.e., unity recovery factor $q = 1$).

Profiles of $U/U_e$ and $T/T_e$ are shown in Fig. 65 for adiabatic and isothermal walls and varying edge Mach numbers. The vertical axis refers to $y/s_{Bo}$ with

$$s_{Bo}(x) = \frac{x}{\sqrt{Re_{e,x}}} = \left( \frac{8e U_e x}{\mu e} \right)^{1/2}$$

The classic boundary-layer thickness estimates are also provided for $Ho_e = 0$, no that in this limit, (225) does not collapse to the Blasius problem unless $q_{w} = 1$ or $q = 1$, and

$$\beta = \frac{1}{12} \frac{y}{s_{Bo}(x)}$$

becomes the classic self-similar variation for incompressible boundary layers, and (225) is completely decoupled from (226).

The skin friction coefficient distribution on the flat plate is given by

$$C_f = \frac{\gamma_w}{2} \frac{U_w}{s_{Bo} U_e^2} = \frac{2}{8e U_e^2} \left[ \frac{\mu dU/dy}{y=0} \right]$$

$$= \frac{2kW}{8e U_e^2} \left[ \frac{\nu^2}{8} \frac{y}{y=0} \right] = \frac{2kW}{8e U_e^2} \left[ \frac{\nu^2}{8} \frac{y}{y=0} \right] = \frac{2kW}{8e U_e^2} \left[ \frac{\nu^2}{8} \frac{y}{y=0} \right]$$

$$= 2^{-1} \frac{\nu^2}{8e U_e^2} \left( 0 \right) = \frac{1}{2} \frac{\gamma_w}{s_{Bo} U_e^2} \left( 0 \right) = \frac{1}{2} \frac{\gamma_w}{s_{Bo} U_e^2} \left( 0 \right)$$

(234)

Where $q_{w} = \frac{8e U_w}{s_{Bo} U_e}$ is the Chapman-Rubesin parameter at the wall, and

$$Re_{e,x} = \frac{s_{Bo} U_e x}{\mu e}$$

(232)
IS THE LOCAL REYNOLDS NUMBER BASED ON THE EDGE CONDITIONS, IN GENERAL, NOTE FROM
\[ (225) \] - \[ (226) \], THE ASSOCIATED BOUNDARY CONDITIONS, AND \[ (281) \], THAT \( \zeta' \) IS A COMPLEX FUNCTION
OF SEVERAL PARAMETERS IN COMPRESSIBLE BOUNDARY LAYERS, \( \zeta' = F(M_{\infty}, P, T, T_N/T_e) \).

THIS IN CONTRAST WITH THE SIMPLER INCOMPRESSIBLE CASE, \( \zeta' = 0.664/\sqrt{Re_x} \).

A SIMILAR DERIVATION CAN BE MADE FOR THE STANTON NUMBER,
\[
St = \frac{q}{\frac{\theta}{\varepsilon_0 U_e (h_w-h_0)}} = \frac{1}{\frac{\theta}{\varepsilon_0 U_e (h_w-h_0)}} \left[ \frac{K_{OT}}{\theta Y} \right]_{Y=1} = \frac{K_w}{\frac{\theta}{\varepsilon_0 U_e C_p (T_w-T_N)}} \left[ \frac{\theta U_e C_p q'(0)}{\frac{1}{2} T_w} \right] = \frac{1}{\sqrt{2}} \frac{G_w}{P_r} \frac{T_e}{(T_w-T_N)} \frac{q'(0)}{\sqrt{R_e x}} \quad (283)
\]

FOR \( P_r = 1 \), THE ADIABATIC WALL TEMPERATURE IS EQUAL TO THE STATION TEMPERATURE OF
THE INVIScid STREAM, \( T_w = T_{1,c} \). ADDITIONALLY

FOR \( P_r = 1 \) AND IZOTHERMAL WALL, THE
EQUATIONS \[ (225) \] AND \[ (226) \] ARE SIMILAR
TO EACH OTHER IN TERMS OF THE VARIABLES
\[
\frac{m'(t)}{q_w/q_{aw}} = f(t) \quad (284)
\]
\[ (89) \quad \text{SEE Eq. 207} \] WITH \( m'(t) \) GIVEN
BY \[ (227) \]. AS A RESULT, FOR \( P_r = 1 \) AND

ISOHERMAL WALLS, THE LEYNOLES ANALOG TO \[ (214) \] IS RECOVERED.

WHEN \( P_r \neq 1 \), THE ADIABATIC WALL TEMPERATURE MUST BE COMPUTED NUMERICALLY. HOWEVER, IT
IS WORTH MENTIONING THAT, IN ALL CASES, THE RECOVERY FACTOR \( 189 \) TENDS TO
SCALES VIRTUALLY ACCURATELY AS
\[
\zeta' = \sqrt{P_r} \quad (285)
\]

THIS RELATION TENDS TO HOLD FOR WIDE RANGES OF \( P_r \) AND \( M_{\infty} \), AND FOR CONSTANT AND
VARIABLE VISCOSITIES, AS SHOWN IN FIG. 67.

The boundary-layer profiles shown in Fig. 65, along with the variations of $G_1$ and $S_0$ in Fig. 66 indicate a number of important results that can be summarized as follows:

1) The boundary-layer thickness increases rapidly with $M_c$. This is clearly observed in Fig. 65a, suggesting that $S_{Bo}$ is no longer an appropriate scale at high Mach numbers. In contrast, when

$$S_B = \frac{X}{(\frac{S_w}{U_w} \frac{U_e}{U_w})^{1/2}} \quad (236)$$

is employed to scale the results, an improved collapse is observed (Fig. 68). Note that (236) can be written in terms of the Mach number by doing

$$S_B = \frac{X}{(\frac{S_w}{U_w} \frac{U_e}{U_w})^{1/2}} = (\frac{S_w}{S_w})^{1/2} (\frac{M_w}{M_e})^{1/2} S_{Bo} = \left(\frac{S_w}{S_w}\right)^{1/2} (\frac{T_w}{T_e})^{1/2} \frac{1}{\sqrt{\frac{\gamma - 1}{\gamma}}}$$

with $S_{Bo}$ the incompressible scaling given by (236). Assuming $T_w = T_{aw}$ and $\gamma = 1$ then $T_w/T_e = T_{aw}/T_e = T_{ew}/T_e = 1 + (\frac{S_{Bo}}{2}) M_a^2$. Introducing this approximation in (237) and taking the limit $M_{aw} \to 1$ gives

$$S_B \approx M_{aw}^2 S_{Bo} = \frac{M_{aw}^2}{R_e_{aw}}$$

where the temperature exponent $\gamma$ for the viscosity is assumed to be close to unity as is usually the case.

6) The boundary layer becomes thinner as the wall temperature decreases.

This effect is clearly illustrated by comparing the adiabatic and isothermal cases in Fig. 65 and noting that the wall temperature in the former is much higher. A decrease in $T_w$ increases the density in the boundary layer, which therefore renders a thinner
Boundary layer that can carry the same amount of momentum deficit as a thicker boundary layer with a higher wall temperature. This scaling is also ratified by (237).

66) The peak temperature in the boundary layer increases rapidly with $M_a$.

This is an important effect visualized in Fig. 65b that is caused by the increase of the viscous dissipation term in Eq. (226). As a result, as $M_a$ increases, an increasing amount of heat is transferred to the surface that is usually referred to as hypersonic aerodynamic heating, and which plays an important role in the design of aerospace vehicles.

The aerodynamic heating rate is typically of the same order as the flux of stagnation enthalpy impinging on the vehicle. This can be seen by taking Eq. (233) and assuming that $h_{ow} \gg h_w$ as is usually the case at $M_a \gg 1$, and approximating $h_{ow}$ by $h_{oo}$ (i.e., $\gamma \approx 1$), which gives

$$q_w = \frac{\rho e U e h_{oo} S f}{2}$$  \hspace{1cm} (239)

Where use of the definition $h_{oo} = h_e + \frac{1}{2} U_e^2 \approx \frac{1}{2} U_e^2$ has been made in the limit $M_a \gg 1$. Note that the aerodynamic heating rate (239) is, in the first approximation proportional to $U_e^3$ (i.e., up to other dependencies in the stagnation number $S_f = F(M_a, Pr, \gamma, Tw/10)/\sqrt{Re_c}$). This cubic dependence on $U_e$ is in contrast with the quadratic dependence of the friction drag on the velocity

$$D = \frac{1}{2} \rho_o U_\infty^2 S_f C_f$$

(where "$\rho_o$" indicates true free-stream conditions) and highlights the importance of aerodynamic heating at high speeds.

Note: Eq. (239) is one form of aerodynamic heating created by viscous friction. The term "aerodynamic heating" is also widely employed to refer to other aerodynamically derived heating mechanisms that are not necessarily linked to viscous dissipation and which may involve shock/shock interactions, shock/boundary-layer interactions, shock-induced radiative heating, or low-speed heat transfer in stagnation flow regions. Remarkably, however, (239) already elucidates the importance of minimizing friction ($S_f \approx C_f$) on vehicles flying at high speeds in denser (lower) parts of the atmosphere ($\rho_o$ large).
The Skin Friction Coefficient and the Stanton Number Decrease As $M_{\infty}$ Increases

This is clearly observed in Fig. 66 but should not be misinterpreted as a decrease in the wall shear stress or wall heat flux, since these two increase quadratically and cubically with velocity.

The Skin Friction Coefficient and the Stanton Number Increase As the Wall is Cooled

This is also clearly observed in Fig. 66 and is expected from the fact that the boundary layer becomes thinner as $T_w$ decreases.

The considerations above illustrate the peculiarities found in compressible laminar boundary layers on flat plates. Attention is now diverted to a different type of flow that can also be solved with Eqs. (217)-(218) and which is of technological relevance for hypersonic flows over blunt bodies.

The Case of the Stagnation-Region Flow Field Around a Blunt Body

As mentioned previously, the hypersonic flow over a blunt body generates a bow shock that sits close to the body surface. Near the normal portion of the shock, a stagnation streamline appears that divides the flow into upwards and downwards turning portions (Fig. 69).

It is important to note, however, that this dividing streamline, which terminates at a stagnation point, does not generally pass through the normal portion of the shock under non-zero angles of attack, but instead tends to drift towards the side of the body with the largest curvature (see Fig. 70). Focus here will nevertheless be limited to the simpler symmetric case depicted in Fig. 69.
The flow field around the stagnation region does not fulfill the conditions for self-similarity, since \( U_0, T_0, \phi \), and \( \rho_0 \) generally vary in the streamwise direction along the blunt body surface as specified by the corresponding inviscid description of the post-shock flow (see Chapter II). However, one can always use "local self-similarity" to describe the flow locally in the stagnation point region assuming that such streamwise variations are slow. In this way, one can still use Eqs. (217) - (218) by letting \( \frac{dU_0}{ds} \) and \( \frac{d\rho_0}{ds} \) be functions of \( s \) that nonetheless vary slowly, and in this particular problem, render again a formulation that only depends on \( \frac{d}{d} \). To see this, consider the following approximations:

1) Near the stagnation point, the edge velocity \( U_0 \) is small. In particular, the flow in this region is subsonic, and therefore the viscous heating term does not play any role in the enthalpy equation (218). In addition, because of the associated small values of \( M_{ae} \), the enthalpy along the edge is assumed to be locally uniform and equal to \( h_{0e} = h_{00} \), so that the term involving \( \frac{d\rho_0}{ds} \) in (218) vanishes.

2) Near the stagnation point, the velocity field in the inviscid stream can be approximated by a local strain rate \( \frac{dU_0}{dx} |_{x=0} \) multiplied by the distance \( x \) from the stagnation streamline along the curved body, namely

\[
U_0 \approx \left( \frac{dU_0}{dx} \right) |_{x=0} x
\]

(240)

(This flow is typically referred to as Higgen's after Higgen 1944 and represents a particular case of the Falkner-Skan family of solutions). An important consequence of (240) is that

\[
f = \int_0^x \frac{1}{\rho_0 U_0} \frac{dU_0}{dx} dx \Rightarrow \frac{dU_0}{dx} |_{x=0} = \frac{1}{\frac{d}{d} x \frac{dU_0}{dx}}
\]

Substituting from (240)

\[
\frac{dU_0}{dx} |_{x=0} = \frac{dU_0}{dx} |_{x=0} \frac{U_0}{U_0} = \frac{1}{\rho_0 (U_0)^2} \Rightarrow \frac{2f}{U_0} \frac{dU_0}{dx} = 1
\]

(242)
EQUATION (242) HELPS IN GETTING RID OF THE $\xi$ DEPENDENCE OF THE RIGHT-HAND SIDE OF THE MOMENTUM EQUATION (247). ADDITIONALLY, IT IS NOT DIFFICULT TO SEE THAT THE COEFFICIENT \( \frac{2}{9} \frac{\partial u_e}{\partial \xi} \) ON THE RIGHT-HAND SIDE OF THE ENERGY EQUATION (248) CAN BE REWRITTEN AS

\[
\frac{2}{9} \frac{\partial u_e}{\partial \xi} \frac{\partial u_e}{\partial \xi} = 2 \left( \frac{\partial u_e}{\partial x} \right)^2 \frac{1}{(\frac{\partial u_e}{\partial x})^2} = \frac{2}{9} \frac{\partial u_e}{\partial x} \frac{\partial u_e}{\partial x} \frac{1}{(\frac{\partial u_e}{\partial x})^2}
\]

WHICH BECOMES VANISHINGLY SMALL NEAR $x=0$ AND CAN BE NEGLECTED IN THE FIRST APPROXIMATIONS.

(a) FOR CALORICALLY PERFECT GASES, $C_p = \text{const.}$ AND THE TERM $\frac{\partial g}{\partial \xi}$ ON THE RIGHT-HAND SIDE OF THE MOMENTUM EQUATION (247) CAN BE REWRITTEN AS

\[
\frac{\partial g}{\partial \xi} = \frac{1}{T_e} \frac{\partial h}{\partial \xi} = \frac{g}{T_e}
\]

WITH THESE $i)$ - $iv)$ APPROXIMATIONS, THE SELF-SIMILAR CONSERVATION EQUATIONS FOR THE STAGNATION-POINT FLOW BECOME

\[
\begin{align*}
\left( g \frac{\partial f}{\partial \xi} \right)' + \frac{\partial f}{\partial x} & = f' - g \\
\left( g \frac{\partial q}{\partial \xi} \right)' + \frac{\partial q}{\partial x} & = 0
\end{align*}
\]

(243) \hspace{1cm} (244)

SUBJECT TO $f(0) = f'(0) = 0$, $q'(0) = 0$, (OR $g(0) = g_w$), AND $f'(\infty) = q'(\infty) = 1$.

IT IS INTERESTING TO COMPARE (243) - (244) WITH THEIR COUNTERPARTS FOR THE FLAT-PLATE PROBLEM (225) - (226) AND REALIZE THAT HEREB THE STREAMWISE CONVECTION OF MOMENTUM AND THE STREAMWISE PRESSURE GRADIENT DO PLAY A ROLE, WHILE VISCOUS DISSIPATION DOES NOT PLAY ANY ROLE, ALL THE AERODYNAMIC HEATING IS DUE HERE TO THE THERMAL ENERGY RECOVERY BY THE GAS DECELERATION BEHIND THE SHOCK, WHERE THE TEMPERATURE INCREASES SIGNIFICANTLY (I.E. $T_e \approx T_{\infty} \rightarrow T_e \approx T_{\infty}$), AND THE HEAT IS TRANSFERRED BY CONDUCTION TO THE BODY SURFACE (FIG. 74) ACROSS THE BOUNDARY LAYER.

THE WALL HEAT FLUX NEAR THE STAGNATION REGION CAN BE EXPRESSED AS

\[
q_w = S_b \rho_0 \frac{u_e^2}{2} \frac{C_p}{T_e} (T_w - T_{\infty})
\]

WHERE $T_w = T_e + \frac{u_e^2}{2C_p}$, $T_e \approx T_{\infty}$, WHILE $S_b$ IS

$\text{Ma}_{\text{b,c}} \rightarrow \text{T}_o = \text{const.}$ ACROSS $\text{Ma}_{\text{b,c}}$.

\[\text{Fig. 74} \]
A STANTON NUMBER THAT NEEDS TO BE OBTAINED BY NUMERICAL INTEGRATION OF Eqs. (243)-(244). FOR CYLINDERS AT ZERO MACH NUMBERS

$$S_t = 0.570 \frac{Pr^{-0.2}}{\sqrt{Re_{e,l,x}}} \quad (246)$$

WHILE FOR SPHERES IT IS BEST CORRELATED WITH

$$S_t = 0.763 \frac{Pr^{-0.6}}{\sqrt{Re_{e,l,x}}} \quad (247)$$

IN BOTH CASES, (245) ATTAINS THE FORM

$$\frac{q_w}{\text{const.}} \propto \frac{Pr^{-0.6}}{\sqrt{Re_{e,l,x}}} \left( \frac{dU_e}{dx} \right)_{x=0}^x \left( \frac{C_p}{T_{x=0} - T_w} \right) \quad (248)$$

WHICH IS TYPICALLY REFERRED TO AS "FAY-RIDDELL" CORRELATION AFTER FAY AND RIDDELL

"THEORY OF STAGNATION POINT HEAT TRANSFER IN DISSOCIATED AIR" J. AERONAUT. SCI. 25 (1958),

ALTHOUGH FAY AND RIDDELL'S EXPRESSION IS A MORE GENERAL ONE THAT CONTAINS CHEMICAL EFFECTS RESULTING FROM AIR DISSOCIATION.

IN COMPUTING (248), THE LOCAL STRAIN RATE IN THE VICINITY OF THE STAGNATION POINT IS A FUNCTION OF THE GEOMETRY OF THE BODY. AT HYPERSONIC VELOCITIES, AN ESTIMATE OF THIS STRAIN RATE CAN BE OBTAINED BY USING THE EULER EQUATION (249) ALONG THE BOUNDARY-LAYER EDGE

$$\frac{dP_e}{dx} = -\rho_e U_e \left( \frac{dU_e}{dx} \right)_{x=0}^x - \rho_e \left( \frac{dU_e}{dx} \right)^2_{x=0}^x \times \left( \frac{P_e}{T_{x=0} - T_w} \right) \quad (249)$$

ALONG WITH THE NEWTONIAN APPROXIMATION $P_e \approx P_o + \rho_0 U_{x=0}^2 \sin^2 \theta = P_o + \rho_0 U_{x=0}^2 \cos^2 \theta$, WHICH GIVES

$$\sin \theta = \frac{x}{R_o} \quad \text{FOR} \quad \theta \leq \pi$$

$$\frac{dP_e}{dx} = -2\rho_0 U_{x=0}^2 \cos \theta \sin \theta \frac{d\theta}{dx} \approx -2\rho_0 U_{x=0}^2 x \frac{d\theta}{dx} \quad (249)$$

AND THEREFORE

$$\left( \frac{dU_e}{dx} \right)_{x=0}^x \propto \frac{1}{R_o} \left( \frac{2\rho_0 U_{x=0}^2}{\rho_e} \right)^{1/2} \quad (249)$$

INTRODUCING THIS EXPRESSION INTO (248), T IS OBSERVED THAT THE WALL HEAT FLUX IN THE STAGNATION REGION SCALES AS

$$\frac{q_w}{\text{const.}} \propto \frac{Pr^{-0.6}}{\sqrt{Re_{e,l,x}}} \left( \frac{dU_e}{dx} \right)_{x=0}^x \left( \frac{C_p}{T_{x=0} - T_w} \right) \quad (249)$$

WHICH REVEALS A CUBIC DEPENDENCY ON $U_{x=0}$ SIMILARLY TO (239), AND MOST IMPORTANTLY,

THAT THE STAGNATION POINT HEATING VARIES INVERSELY WITH THE SQUARE ROOT OF

THE RADIUS OF CURVATURE OF THE NOSE. THE RESULT (249) IS OF PARAMOUNT IMPORTANCE

FOR THE DESIGN OF HYPERSONIC VEHICLES AND ESTABLISHES THAT SHARP EDGES, ALbeit

LEADING
RELEVANT FOR REDUCING DRAG, ARE IMPractical IN HYPERSONICS SINCE THE MATERIAL WOuLD NOT Be CapABLE OF WithSTANDING THE IndUCED HeATING AND wOULD QuIckLY MELT. NOtE ThAT THE HeAT FLUX DECREASES RAPIDLy AWAY FROM THE StAGNATION REgION AS $\phi$ AND $\mu$ DECREASE, AND THEREFORE THE HEAT TRANSFER IS MAXIMUM AT ThE StAGNATION POINT.

IT IS FOR THESE REASONS THAT HYPERSONIC VEHICLES FoNdAMENTALLY CONSIST OF BLuNT NOSES AND BLuNT LEADING EDGES.

**AERODYNAmIC HEATING**

ThE ABOVE CONSIDERATIONS ILLUSTRATE THE IMPORTANCE OF VIScOUS EFFECTS IN CAUSING LARGE THERMAL LOADS AT HYPERSONIC SPEEDS. A5 MENTIONED IN THE FOOTNOTE BELOW EQu. 123. However, NOT all THE GENERATION MECHANISMS OF AERODYNAMIC HEATING ARE VIScOSITY RELATe

ThE MoSt COMMON MECHANISMS OF AERODYNAMIC HEATING ARE ILLUSTRATED IN FIG. 72 BELOW, AND A SURFACE TEMPERATURE DISTRIBUTION IS GIVEN IN FIG. 73 FoR THE X-15 HYPERSONIC CRUISE AIRC...
A practical example of aerodynamic heating by shock-shock interactions, which have not been studied here, is provided in Fig. 74. For the X-15A2 last flight, shock-shock interactions of type IV are particularly critical for aerodynamic heating (Figs. 74d and 72d) and occur at intersections between bow shocks from canopies, rudders, wings or engine pylons, with incident shocks from the aircraft nose or inlet spikes. The interaction produces a hot supersonic jet directed to the fuselage.
IV HIGH-SPEED THERMOCHEMICAL EFFECTS

As mentioned in Chapter I, the high temperatures involved in hypersonic phenomena resulting from partial conversion of kinetic into thermal energy lead to a number of thermochemical processes that increase the complexity of the hypersonic flow theory in a multi-fold manner. These processes are the subject of this chapter.

The Role of Thermochemical Effects in Hypersonics

As discussed within the context of Fig. 5, a mostly sequential onset of complex processes occur as the flow velocity is increased that are related to vibrational excitation of the air molecule, air dissociation and ionization. It is important to understand that the mapping of these different regions in Fig. 5 is obtained by solving a normal shock problem subject to the velocity $U_i$ of the pre-shock gases indicated along the abscissa, along with pre-shock gas conditions $P_i$ and $T_i$ at the corresponding altitude given by a standard model atmospheric analyses of the aforementioned normal shock problem supplemented with vibrational excitation, air dissociation and ionization.

Are provided further below and constitute a central part of this chapter in contrast with the perfect gas calculations that have been performed in Chapters II and III.

Before introducing the details of these complex thermochemical effects, it is worth emphasizing that they have a profound quantitative impact on the solution, as shown in Fig. 7.5 by comparing the post-shock temperature at an altitude of $z = 52$ km obtained using the perfect gas model (i.e., $C_p = \text{const.}$ with $Y = 1.4 = \text{const.}$, and governed by the ideal gas equation of state) and an equilibrium chemically reacting (non-perfect, dissociating, and ionizing) gas.

---

**Fig. 6.5:** Static temperature behind a normal shock wave for a vehicle flying at speed $U_i$ at an altitude of $z = 52$ km ($T_i = 269$ K, $P_i = 5.9$ mbar, $q_{sh} = (\gamma r_0 T_i)^{1/\gamma} m/s^2$). Source: J.D. Anderson, AIAA (2006).
The post-shock temperature predicted by the calorically-perfect gas model is systematically much higher than that predicted by including some of the thermochemical effects enabled at high temperatures, as observed in Eq. 75. This disparity can be qualitatively explained by the following considerations:

As mentioned in Chapter 1 and ratified in subsequent chapters, at hypersonic velocities the kinetic energy of the pre-shock gases is mostly transformed into thermal energy in the post-shock gas. In a calorically-perfect gas, this excess of thermal energy is dedicated to increasing the energy of the translational and rotational motion of the molecules.

In a non-calorically perfect gas, the excess of thermal energy is dedicated to increasing the energy of the translational, rotational, and vibrational motion of the molecules, to exciting the electrons, and, in chemically-reacting gases, to provide chemical energy for the chemical conversion of the air molecules. Since only the translational motion contributes to the temperature in an ideal gas, it is clear that a calorically-perfect gas would always have a much higher temperature behind the shock that a non-calorically perfect one for the same kinetic energy of the pre-shock gases. For instance, at $U_* \approx 14 \text{ km/s}$ in Fig. 75 (corresponding to $M_\infty \approx 33$ at $r = 52 \text{ km}$), the shock temperature jumps are

\[
\begin{align*}
T_2 / T_1 & \approx 210 \\
T_2 / T_3 & \approx 41
\end{align*}
\]

Using the shock jump condition (Eq. (8)) for calorically-perfect gases including vibrational excitation, and air dissociation and ionization in chemical equilibrium.

2) Dissociation and ionization reactions are endothermic, i.e., energy is always required to break the bonds of nitrogen and oxygen molecules and to strip electrons from their orbits. This energy cannot be but drained from the gas itself, thereby decreasing its temperature with respect to the case in which these effects are ignored.

3) The post-shock pressure—in contrast to the temperature—is mostly insensitive to the aforementioned thermochemical effects. This is due to conservation of momentum (Eq. (3)) across the shock, which, at hypersonic speeds, simplifies
approximately to \( P_2 \approx s_1 u_1^2 \) since \( P_1 \ll s_1 u_1^2 \) (i.e., \( M_{a1} \gg 1 \)) and \( P_2 \gg s_2 u_2^2 \) (i.e., \( M_{a2} \ll 1 \)). In this way, the post-shock static pressure is mostly imposed by the dynamic pressure of the free stream, which, at typical free-stream temperatures 220-300 K of flight in the mesosphere and stratosphere, is mostly independent of the above-mentioned thermochemical effects since these do not play any role in the free-stream gases. For instance, at \( U_0 \approx 11 \text{ km/s} \) in Fig. 7.5 (corresponding to \( M_{a0} \approx 33 \) at \( z = 52 \text{ km} \)), the shock pressure jumps are

\[
\begin{align*}
\frac{P_2}{P_1} & \approx 1290 \quad \text{using the shock-jump condition (Eq. (7)) for calorically perfect gases} \\
\frac{P_2}{P_1} & \approx 1390 \quad \text{including vibrational excitation, and air dissociation and ionization in chemical equilibrium.}
\end{align*}
\]

4) Concurrent with Points 1) and 2) above is a large increase in the post-shock density \( s_2 \) when the thermochemical effects are included, as prescribed by the ideal gas equation of state

\[
P_2 = s_2 R_0 \frac{T_2}{s_2} \quad \Rightarrow \quad s_2 \quad \text{increases a lot when thermo-chemical effects are accounted for.}
\]

As a result, the density ratio \( \varepsilon = \frac{s_1}{s_2} \) is much smaller than that predicted by the shock theory for calorically-perfect gases (note that \( \varepsilon \to \frac{1}{6} \) for \( M_{a1} \to \infty \) in Eq. (6)) is the maximum compression ratio allowed by the theory). For instance, at \( U_0 \approx 11 \text{ km/s} \) in Fig. 7.5 (corresponding to \( M_{a0} \approx 33 \) at \( z = 52 \text{ km} \)), the shock density jumps are

\[
\begin{align*}
\frac{s_2}{s_1} & \approx 5.97 \quad (\varepsilon \approx 0.17) \quad \text{using the shock-jump condition (Eq. (6)) for calorically-perfect gases.} \\
\frac{s_2}{s_1} & \approx 15 \quad (\varepsilon \approx 0.07) \quad \text{including vibrational excitation, and air dissociation and ionization in chemical equilibrium.}
\end{align*}
\]

It is worth highlighting at this point the resemblance of the above results, when thermo-chemical effects are accounted for, with the Newtonian theory in
In Chapter II, which requires vanishingly small density ratios $\beta \to 0$ (i.e., $\gamma \to 1$ in the realm of the theory for calorically-perfect gases) to hold, to some extent, the incorporation of thermochemical effects in the solution makes it more "Newtonian-like," which remarkably stresses the importance of the Newtonian theory. Despite its heuristic origins, in what follows, basic thermodynamic and chemical concepts are introduced to assist in quantifying these effects.

Basic Concepts in Thermodynamics and Physical Chemistry

**Ideal Gas**

Most applications in hypersonics are well described by the ideal-gas equation of state

$$P = \frac{\rho R^\beta}{W} T \quad (250)$$

which pertains to gases in which intermolecular forces are negligible (as opposed to real gases in which such interactions are important). The ideal-gas equation (250) can be derived from statistical mechanics under the assumptions that

1. The different degrees of freedom of the molecules (e.g., translation, rotation, vibration, and electronic excitation) are energetically decoupled from each other, and
2. Each degree of freedom is fully equilibrated so as to yield only one temperature in the system.

These concepts will be explained further below. Equation (250) tends to be a good approximation for gases at not too large pressures ($P < 100 \text{ bar}$) and not too low temperatures ($T > 100 \text{ K}$), but also for not too low pressures where there are sufficient collisions in the system to render a meaningful temperature, and for not too high temperatures to not cause high levels of ionization and different temperatures for the charged particles (i.e., thermodynamic non-equilibrium).

**Calorically-Perfect Single-Component Gas**

A calorically-perfect gas is one in which (250) is satisfied and the thermal energy is linearly proportional to the temperature,

$$e = CvT \quad (251)$$

specific internal energy

with $C_v = \frac{\partial e}{\partial T}$ is constant. A specific heat at constant volume that remains constant, because of the first principle of thermodynamics,
\[ \frac{\Delta q}{dT} = \frac{\partial (v)}{\partial T} T \backslash \rho + \frac{\partial (\tilde{V})}{\partial T} \tilde{V} + \frac{\partial (\tilde{V})}{\partial \tilde{V}} \frac{d \tilde{V}}{d \tilde{V}} \Delta \tilde{V} \]  

with the heat into the system \((\Delta q > 0)\) or into the surroundings \((\Delta q < 0)\), then

\[ \frac{\partial q}{\partial T} = \frac{\partial (v)}{\partial T} T \backslash \rho + \frac{\partial (\tilde{V})}{\partial T} \tilde{V} + \frac{\partial (\tilde{V})}{\partial \tilde{V}} \frac{d \tilde{V}}{d \tilde{V}} \Delta \tilde{V} \]

\[ \Rightarrow \quad \frac{\partial q}{\partial T} \bigg|_P = \frac{\rho}{s^2} \frac{d s}{d \rho} \frac{d \rho}{d T} = C_V - R_{g} \]

\[ \Rightarrow \quad C_P - C_V = R_{g} \]

\[ \text{(253)} \]

where \( \frac{\partial q}{\partial T} \bigg|_P \) is the specific heat at constant pressure, which remains constant with temperature. Note that the enthalpy \( h = e + \frac{p}{\rho} \) can be written as

\[ h = C_V T + R_{g} T = C_P T \]

\[ \text{(254)} \]

Similarly, the adiabatic coefficient

\[ \gamma = \frac{C_P}{C_V} = \text{constant} \]

\[ \text{(255)} \]

remains constant. It is also not difficult to show that an adiabatic trajectory \((\Delta q = 0)\) in thermodynamic space is represented by

\[ \frac{\partial q}{\partial T} = \frac{\partial (v)}{\partial T} T \backslash \rho + \frac{\partial (\tilde{V})}{\partial T} \tilde{V} + \frac{\partial (\tilde{V})}{\partial \tilde{V}} \frac{d \tilde{V}}{d \tilde{V}} \Delta \tilde{V} \]

\[ \Rightarrow \quad \frac{\partial q}{\partial T} = \frac{\rho}{s^2} \frac{d s}{d \rho} \frac{d \rho}{d T} \]

\[ \text{FROM} \quad h = e + \frac{p}{\rho} \]

\[ \text{AND} \quad (254) \]

which can be integrated to give

\[ \int \frac{d p}{P} \bigg|_{P_0} = \int \gamma \frac{d \rho}{\rho} \bigg|_{\rho_0} \Rightarrow \quad \frac{P}{P_0} \gamma = \frac{P_0}{P_0} \gamma = \text{const.} \]

\[ \text{(256)} \]

The adiabatic trajectory \((256)\) coincides with an isentropic when the process is reversible, namely

\[ d s \text{ UNIVERSE} = d s \text{ SYS} + d s \text{ SURROUNDINGS} \]

\[ = 0 \quad \text{(} = 0 \text{ FOR REVERSIBLE PROCESSES )} \]

\[ \Rightarrow \quad d s = \frac{d e + \frac{P}{T} d \left( \frac{1}{T} \right)}{T} \quad \text{d ssys} = C_V d T - R_{g} \frac{d \rho}{\rho} \]

\[ \text{d s} = \frac{d h - \frac{1}{T} d p}{T} \quad \text{d s} = C_P d T - R_{g} \frac{d p}{p} \]

\[ \text{integrating:} \]

\[ S - S_0 = C_V \ln \left( \frac{T}{T_0} \right) - R_{g} \ln \left( \frac{\rho}{\rho_0} \right) \quad \text{S - S_0 = C_P \ln \left( \frac{T}{T_0} \right) - R_{g} \ln \left( \frac{P}{P_0} \right)} \]

And eliminating \( \ln \left( \frac{T}{T_0} \right) \) from the two equations, one obtains the definition

\[ S - S_0 = C_V \ln \left( \frac{p}{p_0} \frac{\gamma}{\gamma_0} \right) \]

\[ \text{SPECIFIC ENThALPY} \]

\[ \text{SPECIFIC ENTROPY} \text{(258)} \]

For a calorically perfect gas, note that \( S = \text{const.} \) (isentropic) when \((256)\) and \((257)\) are satisfied simultaneously. In summary, a calorically perfect gas is one...
N which \( P, \rho \), and \( T \) are related by the ideal equation of state (250) and \( C_v \) = constant, high automatically leads to all other relations (253), (254), (255), (256), and (258). Note that the shock-jump relations (7)-(11), as well as the definition of the stagnation temperature (14) and other stagnation variables (17), are all based in a calorically-perfect gas. As a matter of fact, almost all mathematical developments in Chapters II and III (with exception of Eqs. (217)-(218), which are generally valid) are based on a calorically-perfect gas, since the formulation becomes much more involved otherwise and requires the numerical solution of the original set of conservation equations.

**Thermally-Perfect Single-Component Gas**

A thermally perfect gas is one in which (250) is satisfied and the internal energy is a sole — but non-linear — function of temperature, namely

\[
\frac{dE}{dT} = C_v(T) \Rightarrow E - E_0 = \int_{T_0}^{T} C_v(T) \, dT \quad (259)
\]

With \( C_v = C_v(T) = \frac{\partial E}{\partial T} \), varying with temperature. Note that Eq. (253) still holds, and as a consequence,

\[
\frac{dh}{dT} = \frac{\partial h}{\partial T} + R_0 \frac{\partial T}{\partial T} = C_p(T) \, dT \Rightarrow h = \int_{T_0}^{T} C_p(T) \, dT \quad (260)
\]

With \( C_p = C_p(T) = \frac{\partial h}{\partial T} \bigg|_p = \frac{\partial p}{\partial T} \bigg|_p \), varying with temperature so as to satisfy

\[
C_p - C_v = R_0 \quad \text{const.} \quad \text{Correspondingly, the adiabatic coefficient}
\]

\[
\gamma = \frac{C_p(T)}{C_v(T)} = 1 + \frac{R_0}{C_v(T)} = \frac{1}{1 - \frac{R_0}{C_p(T)}} \quad (261)
\]

\( \gamma \) is not constant and tends to decrease with \( T \), since in many gases \( C_p \) increases with \( T \) (see Fig. 76). Note also that the integrations leading to the adiabatic trajectories (256) and to the entropy definition (258) are impeded now and can only be addressed numerically. For the same reasons, the jump relations (7)-(11), as well as the definition of the stagnation temperature (14) and other stagnation variables (17), cannot be used. However, the general conservation relations across the shock (2)-(5) can be used even if the gas is thermally perfect. In fact, it can be shown
Fig. 7. Specific heat at constant pressure for selected gases (Poirier and Veinante 2006).

Which represent the Rankine-Hugoniot relations. In particular, the iterative resolution of (262)–(263) together with (250) and (260), provides the complete solution of the shock wave problem for a given value of the mass flow rate per unit area $m$ and for pre-shock values $h_1, P_1$ and $P_4$. A normalized version of the problem can be easily written by defining the dimensionless variables $\hat{P} = P / \rho_1 U_1^2$, $\hat{V} = \frac{V}{U_1}$, $\Theta = \frac{\hat{C}_p(T_{1}) T_1}{U_1^2}$, $H = \frac{h_2}{U_1^2}$, $\hat{C}_p = \frac{C_p(T)}{C_p(T_1)}$, and $\hat{Y}_1 = \frac{C_p(T_{1})}{C_v(T_1)}$, which gives

$$\hat{P} - \frac{1}{\hat{Y}_1 \hat{M}_a^2} = - \hat{V}$$

and

$$H - \frac{\hat{P} \hat{M}_a^2}{U_1^2} = (1 + \hat{U}) \left( \frac{\hat{P} - 1}{\hat{Y}_1 \hat{M}_a^2} \right) - \frac{1}{(\hat{Y}_1 - 1) \hat{M}_a^2}$$

As dimensionless versions of (262)–(263) and

$$\hat{P} \hat{V} = (\hat{Y}_1 - 1) \Theta$$

and

$$H = \int_0^\Theta \hat{C}_p(\Theta) d\Theta$$

As dimensionless versions of (250) and (260). In these equations, $\hat{M}_a = U_1 / a_1$ is a Mach number of the pre-shock gases based on the speed of sound for a single-component thermally perfect gas.

$$\left[ a_1^2 = \frac{\partial P}{\partial S} \right] = \left[ a_1^2 = \frac{\partial P}{\partial S} \right] + \frac{\partial P}{\partial T} \frac{\partial T}{\partial S} = a_1^2 T + \frac{\partial a_1^2}{\partial S} \left[ \frac{\partial P}{\partial T} \frac{\partial T}{\partial S} \right] = \frac{R_8 T}{S_2} + \frac{5}{2} \frac{R_8}{S_2} \left( \frac{P}{\rho_1 c_v} \right) = \frac{R_8 T}{S_2} + \frac{5}{2} \frac{R_8}{S_2} \left( \frac{P}{\rho_1 c_v} \right) = \frac{R_8 T}{S_2} + \frac{5}{2} \frac{R_8}{S_2} \left( \frac{P}{\rho_1 c_v} \right) = \frac{R_8 T}{S_2} \left( \frac{P}{\rho_1 c_v} \right)$$

(250) with $R_8 = \text{const}$.

This is an inverse of an Eckert number.

A useful principle is $T_d = \frac{d e}{S_2} = \frac{d e}{S_2} = \frac{\partial e}{\partial S_2} \frac{\partial T}{\partial S_2} = \frac{P}{S_2} \frac{\partial T}{\partial S_2}$.

Which resembles the one for calorically perfect gases but with variable $S_2$. Note that the parameter $Z h_1 / U_1^2$ arises in the Hugoniot equation in addition to $M_a$. For typical pre-shock temperatures, $T_1 \approx 220-300 \text{K}$ in air, $h_1 \approx C_p(T_1) T_1$, and therefore $Z h_1 / U_1^2 \approx \frac{Z}{(\kappa_x-1) M_a^2}$. At high Mach numbers, the formulation above becomes...
\[ P = 1 - \sqrt{\frac{\gamma}{\gamma - 1}} \frac{P}{\gamma}, \quad T = \frac{T^*}{\gamma - 1}, \quad H = \int_{0}^{\gamma} \phi(\theta) d\theta, \quad \text{which remarkably does not depend on the high number as expected from the independence principle.} \]

For calorically-perfect gases, the above formulation collapses to

\[ \frac{P}{\gamma U_{1/2}^2} = \frac{2}{\gamma + 1}, \quad \frac{P}{\gamma U_{1/2}^2} = \frac{4/\gamma}{(\gamma + 1)^2}, \quad \frac{\gamma}{\gamma - 1} = \frac{\gamma + 1}{\gamma - 1}, \quad \text{consistent with the shock-tube relations (22)-(25).} \]

It will be further explained below that the temperature variations of the specific heats in ideal gases is caused by the excitation of vibrational energy in the molecules and by the excitation of electrons in their orbits within the atoms of the molecules.

### Chemically-Reacting Multi-Component Ideal Gas

A multi-component ideal gas \((i = 1, \ldots, N \text{ components})\) that is chemically reacting is characterized by satisfying (25), namely

\[ P = \frac{\gamma}{\gamma - 1} \frac{R_0}{\overline{\nu}} T, \quad \text{where} \quad R_0 = R^0 \overline{\nu}, \quad \overline{\nu} = \text{mean molecular weight}, \quad (264) \]

And it also satisfies that each of the components in the mixture is thermally perfect, namely

\[ e_i^0 - e_0^0 = \int_{T_0}^{T} C_{V_i}(T) dT \quad (265) \]

\[ \text{partial specific internal energy,} \quad \psi_i = \int_{T_0}^{T} C_{V_i}(T) dT \]

with \( C_{V_i} = \frac{\partial e_i}{\partial T} \). The specific heat of species \( \psi_i \), which generally varies with temperature.

For convenience, it is useful to define the molar fraction \( X_i = \frac{m_i}{m}\), where \( m = \sum_{i=1}^{N} m_i \) is the number of moles of species \( \psi_i \), \( m_0 \) is the corresponding mass, and consequently \( X_i = \frac{m_i}{m} = \frac{m_i}{\sum_{i=1}^{N} m_i} = \frac{1}{N} \).

The mean molecular weight of the mixture is \( \overline{\nu} = \frac{\sum_{i=1}^{N} m_i \overline{\nu}_i}{m} \), where use of the relations \( \sum_{i=1}^{N} X_i = \sum_{i=1}^{N} \overline{\nu}_i = 1 \) has been made.

In Eq. (265), \( \phi^0 \) is a formation energy. Note that, for every species, the ideal gas equation is

\[ P_i = \frac{\phi_i^{0}}{R_i^{0} T / \overline{\nu}_i}, \quad \text{with} \quad P_i = X_i P \quad \text{the partial pressure} \quad (266) \]
In such a way that (264) is recovered upon summing over all species,

\[ \sum_{i=1}^{N} P_i = P \sum_{i=1}^{N} X_i = P = \sum_{i=1}^{N} Y_i \frac{R^0}{W_i} = \sum_{i=1}^{N} Y_i \frac{R^0}{W_i} = \frac{R^0}{W} \]

Similarly, the first principle for every species is

\[ Tds_i = dE_i + P_i d\left(\frac{1}{\theta_i}\right) = \frac{\partial E_i}{\partial T} dt + P_i d\left(\frac{1}{\theta_i}\right) \]

so that

\[ T \frac{\partial s_i}{\partial T} = \frac{\partial E_i}{\partial T} + P_i \frac{d s_i}{d T} = c_v_i - \frac{R^0}{W_i} = \frac{\partial h_i}{\partial T} \]

which indicates that Meyer's relation

\[ c_{p_i} - c_{v_i} = \frac{R^0}{W_i} \]

(267)

is satisfied per component (but not globally). Using (266) and (267), Eq. (26) can be easily rewritten in terms of the partial specific enthalpy \( h_i = h_i^0 + P_i / s_i \) as

\[ h_i = h_i^0 = \int_{T_0}^{T} C_{p_i}(T) dT \]

(268)

Partial specific enthalpy

\( h_i = h_i^0(T) \)

where \( C_{p_i} = \frac{\partial h_i}{\partial T} \) and \( h_i^0 \) are, respectively, the specific heat and formation enthalpy of species \( \text{species} \).

In terms of mixture quantities, the formulation above leads to

\[ h = \sum_{i=1}^{N} Y_i h_i^0 + \sum_{i=1}^{N} Y_i \int_{T_0}^{T} C_{p_i} dT \]

(269) \[ \Rightarrow \] \( h = h(Y_i, T) \)

Formation

\[ \text{enthalpy} \]

\[ \text{thermal enthalpy} \]

\[ \text{enthalpy} \]

(Thermal

Energy

Formation

Enthalpy

Enthalpy

Enthalpy

\[ e = \sum_{i=1}^{N} Y_i e_i^0 + \sum_{i=1}^{N} Y_i \int_{T_0}^{T} C_{v_i} dT \]

(270) \[ \Rightarrow \] \( e = e(Y_i, T) \)

Formation

Energy

\[ e \]

(Thermal

Thermal

Enthalpy

Enthalpy

Enthalpy

\[ T \]

Where \( T_0 \approx 298 \text{K} \) is an arbitrary reference temperature at which the formation values are typically tabulated. Note that the formation component represents enthalpy or energy variations due to chemical conversion. Similarly, the first principle for each species, \( Tds_i = dh_i - dP_i / \theta_i \)

(271)

Can be transformed into the first principle for the chemically-reacting mixture
By multiplying by $Y_i$ and summing over all components, namely

$$T \sum_{i=1}^{N} Y_i \, d\varepsilon_i = \sum_{i=1}^{N} Y_i \, dh_i - \sum_{i=1}^{N} \frac{\partial}{\partial Y_i} d\xi_i,$$

and since $dh = \sum_{i=1}^{N} Y_i \, dh_i + \sum_{i=1}^{N} h_i \, dY_i$ and $ds = \sum_{i=1}^{N} Y_i \, ds_i + \sum_{i=1}^{N} s_i \, dY_i$, then the above expression becomes

$$Td\sigma = dh - \frac{d\tau}{\tau} - \sum_{i=1}^{N} q_i \, dY_i \quad (272)$$

where $q_i = h_i - T \sigma_i$ is the partial specific Gibbs free energy (also called the specific chemical potential). Equivalently, in terms of $\sigma$, (278) can be written as

$$Td\sigma = d\varepsilon + P \, d(\frac{4}{3}) + \sum_{i=1}^{N} q_i \, dY_i \quad (273)$$

Writing down the conservation equations for chemically-reacting flows, additional terms are sometimes required to account for molecular diffusion and chemical reactions:

$$\left\{ \begin{align*}
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= -\nabla \cdot (\mathbf{v} \theta) + \nabla \cdot \left( \mathbf{V} \cdot \mathbf{u} \right) - \nabla \cdot \mathbf{q} \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla \mathbf{p} + \nabla \left( \mathbf{V} \cdot \mathbf{u} \right) - \nabla \cdot \mathbf{q} \\
\frac{\partial \mathbf{V}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{V} &= -\nabla \left( \mathbf{V} \cdot \mathbf{u} \right) + \mathbf{w}
\end{align*} \right. \quad (274)$$

Stagnation (total)

Energy conservation,

$$\frac{\partial \varepsilon}{\partial t} + \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) = \frac{\partial}{\partial t} \left( \mathbf{r} \cdot \mathbf{u} \right) + \int_{V} \mathbf{J} \cdot \mathbf{e} \, dV \quad (275)$$

Stagnation (total)

Enthalpy conservation,

$$\frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial}{\partial t} \left( \frac{\mathbf{V} \cdot \mathbf{V}}{2} \right) = \frac{\partial}{\partial t} \left( \mathbf{r} \cdot \mathbf{u} \right) + \int_{V} \mathbf{J} \cdot \mathbf{e} \, dV \quad (276)$$

Heat flux

$$\mathbf{q} = -\kappa \nabla T + \sum_{i=1}^{N} h_i Y_i \mathbf{V}_{i} \quad (277)$$

Viscous stress tensor

$$\mathbf{\tau} = \mu \left( \nabla \mathbf{u} + \nabla \mathbf{u}^{T} \right) + \left( \kappa - \frac{2}{3} \mu \right) \mathbf{\nabla} \cdot \mathbf{u} \mathbf{I} \quad (278)$$

Species conservation

$$\frac{\partial \mathbf{Y}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{Y} = -\nabla \left( \mathbf{V} \cdot \mathbf{Y} \right) + \mathbf{w} \quad (279)$$

Species production rate

where $\mathbf{w} = \mathbf{w} \sum_{j=1}^{M} \left( \mathbf{v}_{j} - \mathbf{v}_{i} \right) \mathbf{w}_{j}$

In $\left[ \frac{\text{kmol}}{\text{s m}^{3}} \right]$ for a mechanism of $M$ elementary reactions

$$\sum_{i=1}^{N} \mathbf{r}_{i} = \sum_{i=1}^{N} \mathbf{r}_{i}$$

Chemical species

where $r_{i}$ are stoichiometric coefficients, the chemical rate of reaction $i$ is

$$\frac{d[Y_{i}]}{dt} = \mathbf{K}_{i} \sum_{i=1}^{N} \mathbf{r}_{i} = \mathbf{K}_{i} \sum_{i=1}^{N} \mathbf{r}_{i} \mathbf{w}_{i}$$

Units

$$\left[ \text{mol/s m}^{3} \right]$$

Here $[c] = \frac{\text{mol}}{\text{m}^{3}}$ indicates molar concentration, and $K_{f_i}$ and $K_{b_i}$ are forward and backward reaction constants of reaction step "i".
\[ K_i \text{ or } K_{ni} \text{ are usually written in the Arrhenius form} \]

\[ K_i = A_i T^{m_i} e^{-E_{aci}/RT} \quad (281) \]

\[ A_i \text{ = pre-exponential factor} \]

\[ m_i \text{ = temperature exponent} \]

\[ E_{aci} \text{ = activation energy}. \]

Note that the chemical heat release rate is absent from (274) and (275) because \( h \) and \( e \) already incorporate the chemical energy through the formation terms. As a result, Eqs. (2)–(5) are for instance also valid for chemically-reacting shock waves (although an additional equation for conservation of species may be necessary in situations of chemical non-equilibrium as explained further below). Additional closures are required for the diffusion velocity \( \vec{V}_d \) in (276), but these are beyond the scope of this course. The simplest form of \( \vec{V}_d \), which neglects thermal diffusion and barodiffusion is

\[ \vec{V}_d = -\frac{1}{W} \left( \sum_{k=1}^{N} D_k W_k \nabla X_k - \sum_{k=1}^{N} D_{ki} W_k Y_k \nabla X_k \right). \quad (282) \]

**Holar Entropy and Gibbs Free Energy**

Reformulating the 1st Principle (274) applied to species \( b \) in terms of holar entropy

\[ \tilde{N}_i = S_i W_i, \text{ holar enthalpy } \tilde{h}_i = \tilde{h}_i W_i, \text{ and specific volume } \tilde{V}_i = W_i / S_i \text{ yields} \]

\[ \tilde{S}_i - \tilde{S}^{0} = \int_{T_0}^{T} \tilde{C}_p \frac{dT}{T} - R \ln \frac{p}{p_0} \]

\[ \quad \text{or} \quad \tilde{S}^{0} = \tilde{S}_i \text{ of } \tilde{S}^{0} \text{ at } p_0 \text{ and } T_0 \quad (283) \]

Where \( \tilde{C}_p = C_p W_i \) is the holar specific heat at constant pressure. The holar entropy of the mixture is then given by

\[ \tilde{S} = \sum_{i=1}^{N} X_i \tilde{S}_i = \tilde{S}(P, T, X_i). \quad (284) \]

Note that (283)–(284) become (258) when the gas is calorically perfect.

The partial holar Gibbs free energy, \( \tilde{\gamma}_i = \tilde{h}_i - T \tilde{S}_i \) can be expressed as

\[ \tilde{\gamma}_i - \tilde{\gamma}_i^{0} = \int_{T_0}^{T} \tilde{C}_p \frac{dT}{T} - \int_{T_0}^{T} \tilde{C}_p \frac{dT}{T} + R \ln \frac{p}{p_0} \quad (285) \]

More compactly, letting \( T^{0} = T \), then

\[ \tilde{\gamma}_i^{0}(P, T) = \tilde{\gamma}_i^{0}(T) + R \ln \frac{p}{p_0} \quad (286) \]

Which provides an explicit dependence of \( \tilde{\gamma}_i \) on the partial pressure of component \( \tilde{\gamma}_i \).

Similarly to (284), the holar Gibbs free energy of the mixture is

\[ \tilde{\gamma} = \sum_{i=1}^{N} X_i \tilde{\gamma}_i = \tilde{\gamma}(P, T, X_i). \quad (287) \]
THE EXACT DIFFERENTIAL OF THE MOLAR GIBBS FREE ENERGY \( \partial G \) OF THE MIXTURE CAN BE EXPRESSED AS

\[
d\tilde{G} = \left( \frac{\partial \tilde{G}}{\partial T} \right)_{P, x_i, i = 1, \ldots, N = 1} dT + \left( \frac{\partial \tilde{G}}{\partial P} \right)_{P, x_i, i = 1, \ldots, N = 1} dP + \sum_{i=1}^{N} \left( \frac{\partial \tilde{G}}{\partial x_i} \right)_{P, x_i, i = 1, \ldots, N = 1} dX_i,
\]

AND SINCE \( \frac{\partial \tilde{G}}{\partial x_i} = \tilde{G}_i \), THEN THE ABOVE EXPRESSION BECOMES

\[
d\tilde{G} = -\tilde{S} dT + \tilde{V} dP + \sum_{i=1}^{N} \tilde{G}_i dX_i.
\]

(287)

AT CONSTANT PRESSURE AND TEMPERATURE, EQUATION (287) REDUCES TO

\[
d\tilde{G} = \sum_{i=1}^{N} \tilde{G}_i dX_i,
\]

(288)

BUT SINCE \( d\tilde{G} = \sum_{i=1}^{N} x_i d\tilde{G}_i + \sum_{i=1}^{N} \tilde{G}_i dx_i \), THEN \( \sum_{i=1}^{N} x_i d\tilde{G}_i = 0 \) IN THESE CONDITIONS. IN FACT, IT IS EASY TO SEE FROM (287) THAT THE RELATION

\[
\sum_{i=1}^{N} x_i d\tilde{G}_i = -\tilde{S} dT + \tilde{V} dP
\]

(289) (GIBBS-DUHEIM RELATION)

IS SATISFIED. EXPRESSION (288) IS PARTICULARLY USEFUL IN SYSTEMS AT CONSTANT PRESSURE AND TEMPERATURE. IN PARTICULAR, EQUATION (282) WRITTEN ON A MOLAR BASIS YIELDS

\[
Td\tilde{S} + \sum_{i=1}^{N} \tilde{G}_i dx_i = d\tilde{H} - \tilde{V} dP
\]

AND SINCE \( d\tilde{S} = d\tilde{S}_{\text{surr}} + d\tilde{S}_{\text{universe}} \), WITH \( d\tilde{S}_{\text{surr}} = \frac{dq}{T} \), THEN

\[
Td\tilde{S}_{\text{universe}} = \sum_{i=1}^{N} \tilde{G}_i dx_i = d\tilde{G}
\]

(290)

AT CONSTANT PRESSURE AND TEMPERATURE. EQUIVALENTLY, (290) STATES THAT IN REVERSIBLE EQUILIBRIUM, \( Td\tilde{G}_{\text{universe}} = 0 \), THE GIBBS FREE ENERGY SATISFIES

\[
d\tilde{G} = \sum_{i=1}^{N} \tilde{G}_i dX_i = 0 \quad \text{AT CONSTANT } P \text{ AND } T. \quad (291)
\]

TO THAT, FOR A REVERSIBLE REACTION \( \sum_{i=1}^{N} v_i \rho_i = \sum_{i=1}^{N} v''_i \rho_i \), THE PROGRESS RATE (280) DICATES \( d[\rho_i]/(v''_i - v_i) \) IS THE SAME FOR ALL REACTANTS, AND SINCE \( d[\rho_i] = \frac{P}{\partial T} dx_i \)

CONSTANT \( P \) AND \( T \), THEN (291) REQUIRES THAT \( \sum_{i=1}^{N} (v''_i - v_i) \tilde{G}_i = 0 \) USING (286) IN THIS EXPRESSION LEADS TO

\[
-\sum_{i=1}^{N} (v''_i - v_i) \tilde{G}_i (T) = R^0T \ln \left\{ \prod_{i=1}^{N} \left( \frac{P_i}{P^0} \right) \right\} (\rho''_i - \rho_i)^{v''_i - v_i}
\]
or equivalently,

\[
K_F = K_F(T) = \exp \left( - \sum_{i=1}^{N} \frac{\Delta G_i^0}{RT} \right) = \prod_{i=1}^{N} \left( \frac{P_i}{P_0} \right)^{\gamma_i} = \frac{P_0}{\Pi_{i=1}^{N} \gamma_i} \prod_{i=1}^{N} X_i \gamma_i
\]

EQUATION (292) ESTABLISHES A RELATION \( F(P, T, \{X_i\}_{i=1}^{N}) = 0 \) THAT THE MIXTURE COMPOSITION MUST SATISFY IN EQUILIBRIUM. IN ADDITION TO THIS CONSTRAINT AND TO \( \sum_{i=1}^{N} X_i = N-2 \) OTHER EQUATIONS PERTAINING TO THE CONSERVATION OF ATOMS MUST BE SPECIFIED IN ORDER TO DETERMINE THE COMPOSITION IN CHEMICAL EQUILIBRIUM (SEE WORKUP PROBLEM SET FOR AN EXAMPLE).

TWO ASPECTS ARE WORTH EMPHASIZING IN THE ABOVE FORMULATION:

1) AIR DISSOCIATION AND IONIZATION REACTIONS ARE TYPICALLY \( \text{MOLE}^{-1} \) PRODUCING REACTIONS

\( \text{N}_2 \rightarrow \text{N} + \text{N}, \quad \text{O}_2 \rightarrow \text{O} + \text{O}, \quad \text{N} \rightarrow \text{N}^+ + \text{e}^- \)

IN THAT \( \sum_{i=1}^{N} \gamma_i = 0 \), SO THAT THE EXPONENT OF THE PRESSURE RATIO IN (292) IS POSITIVE. AS A RESULT, A DECREASE IN PRESSURE (FOR INSTANCE BY INCREASING ALTITUDE) IS ACCOMPANIED BY A DISPLACEMENT OF EQUILIBRIUM CONCENTRATIONS TOWARDS THE PRODUCTS, WHICH PARTLY EXPLAINS THE NEGATIVE SLOPES OF THE BOUNDARIES FOR DISSOCIATION AND IONIZATION IN THE VELOCITY-ALTITUDE MAP IN FIG. 5 (6-8). THESE THERMOCHEMICAL EFFECTS BECOME MORE PROMINENT AT INCREASING ALTITUDES.

2) CHEMICAL EQUILIBRIUM ENTAILS ZERO NET RATE OF REACTION, SINCE THE FORWARD AND BACKWARD RATES BECOME EQUAL. THIS IS A GOOD APPROXIMATION WHEN THERE EXIST SUFFICIENT COLLISIONS TO WARRANT EQUILIBRIUM. AT HIGH ALTITUDES, HOWEVER, THE COLLISIONS ARE NOT SO NUMEROUS AND IT BECOMES NECESSARY TO RETAIN FINITE-RATE CHEMISTRY EFFECTS, WHICH ALSO ENTER IN COMPETITION WITH FLUID MECHANICAL TRANSFER THE LATTER PHENOMENON REQUIRES THE STUDY OF CHEMICAL NON-EQUILIBRIUM. IN OPPOSITION TO CHEMICAL EQUILIBRIUM IS FROZEN FLOW, IN WHICH THE NUMBER OF COLLISIONS IS NOT SUFFICIENT TO TRIGGER ANY CHEMICAL PROCESSES AND THE MIXTURE FLOWS WITHOUT UNDERGOING ANY CHEMICAL CONVERSION; SUCH IS THE CASE AT LOW TEMPERATURES \( (T \lesssim 1500-2000 \text{ K}) \) WHERE AIR CAN BE CONSIDERED AS NON-REACTING.

HIGH-TEMPERATURE AIR IN CHEMICAL EQUILIBRIUM

AN IMPORTANT APPLICATION OF CHEMICAL EQUILIBRIUM IN HYPERSONICS IS THE ANALYSIS OF DISSOCIATION AND IONIZATION THAT OCCURS IN AIR AT HIGH TEMPERATURES. HOWEVER, AS MENTIONED ABOVE, EQUILIBRIUM CONDITIONS ARE ONLY REACHED IN SITUATIONS WHEN FORWARD AND BACKWARD REACTION RATES ARE VERY FAST COMPARED TO ANY TRANSPORT PHENOMENA. IN THESE
 Conditions, a typical chemical-equilibrium configuration is described by the reaction set in Table II. At room temperature the molar composition of air is approximately 78% N₂, 21% O₂, 1% Ar, along with trace amounts of CO₂. At higher temperatures, however, other chemical species emerge as a result of dissociation and ionization. In particular, the description in Table II employs 30 species:

\[
\begin{align*}
O₂, N₂, O, N, O^-, O₂^-, O₂^+, O^+, N^+; N^++, N₂^+; C, C^+, C^{++}; NO₂, NO, N₂O, NO^+, Ar^+, Ar^{++}; N₆, N₆^+, \text{ and } e^-.
\end{align*}
\]

The solution to the chemical-equilibrium problem supplied with the many species listed above is cumbersome and needs to be obtained numerically.

One such solution is provided in Fig. 76 for pressure conditions resembling 31 km altitude (stratosphere). There, it is worth noting that

1) For \( T < 2500K \), the air composition is mostly the same as at ambient temperature, with only minute amounts of O (\( \sim 1\% \)) and NO (\( \sim 1\% \)).

4) For \( 2500K < T < 4000K \), oxygen dissociates significantly (\( X_O \sim 0.3 \)) and NO reaches a peak concentration (\( 4\% \) at 3000K). At 4000K, the O₂ has dissociated almost completely.

6) For \( 4000K < T < 8000K \), N₂ dissociates significantly (\( X_N \sim 0.7 \)) and NO⁺ ions and electrons are present in trace amounts (\( \leq 0.1\% \)).

At \( T = 8000K \) the gas is mainly composed of O.

6) For \( T > 8000K \), ionization of atomic species is important. At \( T = 10000K \), the gas is mainly composed of N, O, N⁺, O²⁻, N₂. At
NON-CALORICALLY-PERFECT EFFECTS INDUCED BY EQUILIBRIUM VIBRATIONAL EXCITATION

In order to analyze basic effects of the high temperatures encountered in hypersonic flows on the behavior of the air environment, it is necessary to introduce some fundamental concepts of statistical mechanics. These, for instance, describe departures of the specific heat $C_p$ from the calorically perfect value as the temperature increases, and are useful in discussing the corresponding decreas in postshock temperatures as compared to their counterparts in calorically perfect gases. The explanations here, however, cannot be but superficial and limited to summarizing the main results. Further discussions and more elaborated expositions can be found, for instance, in the book "Physical Gas Dynamics" by Vincenti and Kruger (2002). The discussion here will be focused on vibrational-excitation effects that arise at low to intermediate temperatures ($T > 800$ K) and are responsible for thermochemical effects acting at $M_a \approx 4$ that tend to lower the post-shock temperature, yet without involving chemical reactions.

Fig. 77: Modes in which energy can be stored in a diatomic molecule (trans = translational motion, rot = rotational motion, vib = vibrational motion).

The characterization of the thermodynamic state of a gas requires the specification of the state of each one of the molecular constituents. This, in practice, is a futile endeavor given the large number of molecules (e.g., $10^{25}$ molecules in $1 m^3$ at normal conditions) typically present. Only an "average" thermodynamic state can be often calculated using concepts from statistical mechanics understanding the word average here as an statistical number. That exists and manifests itself with most probability the calculation (and existence) of this average state is the realm of equilibrium thermodynamics and requires consideration of the statistics of the energetics of the constituent molecules (see schematics in Fig. 77).
THE MOLECULAR STATES CAN BE CALCULATED BY SOLVING THE SCHRODINGER EQUATION, WHICH PROVIDES THE ASSOCIATED WAVE FUNCTION. HOWEVER, THE CHARACTER OF THE SCHRODINGER EQUATION IS SUCH THAT SOLUTIONS (i.e., STATES) ONLY EXIST FOR DISCRETE VALUES (i.e., EIGENVALUES) OF THE TOTAL ENERGY OF THE MOLECULE $E_i$, THE LATTER BEING COMPOSED OF TRANSLATIONAL, ROTATIONAL, VIBRATIONAL, AND ELECTRONIC COMPONENTS, NAMLY,  

$$E_i = E_{\text{trans}} + E_{\text{rot}} + E_{\text{vib}} + E_{\text{elec}}.$$  

IN EQUATION (293), THE ENERGY COMPONENTS ALSO ONLY EXIST AT DISCRETE VALUES, WHICH ARE THE LEAST SPACED IN THE TRANSLATIONAL MOTION, AND THE MOST SPACED IN THE ELECTRONIC MOTION. IN THIS WAY, THE ENERGY LEVELS ARE QUANTIZED, THE QUANTUM NUMBERS $(n, \ell, k, m)$ ARE EIGENVALUES OF THE SCHRODINGER EQUATION, AND THE TOTAL ENERGIES $E_i$ CAN ONLY EXIST AT DISCRETE LEVELS. IT IS NONTHELESS FOUND THAT A NUMBER OF DIFFERENT STATES, OR EQUIVALENTLY, A NUMBER OF DIFFERENT COMBINATIONS OF MOLECULAR MOTION, CAN RENDER THE SAME ENERGY EIGENVALUE $E_0$. THIS MULTIPLICITY IS QUANTIFIED BY THE DEGENERACY $g_i$ OF EACH ENERGY LEVEL $E_i$. 

IN A GAS OF $N$ MOLECULES, IN WHICH $N_0$ MOLECULES HAVE AN ENERGY $E_0$, THE TOTAL ENERGY OF THE SYSTEM IS $E = \sum_{i=1}^{N} N_i E_i$, WITH $\sum_{i=1}^{N} N_i = N$. THE DISTRIBUTION $N_i$ ACROSS THE ENERGY LEVELS IS CALLED THE MACROSTATE OF THE SYSTEM. THE MACROSTATE THAT OCCURS WITH MOST PROBABILITY, OR EQUIVALENTLY, THE MACROSTATE THAT OCCURS IN THERMODYNAMIC EQUILIBRIUM IS THE ONE THAT CONSISTS OF THE MAXIMUM NUMBER OF MICROSTATES $W_{\text{max}}$, HERE, A MICROSTATE REFERS TO ONE OF THE MULTIPLE COMBINATIONS ENABLED BY THE DEGENERACIES $g_i$ THAT THE POPULATIONS OF MOLECULES $N_i$ AT EACH ENERGY LEVEL HAVE FOR POPULATING THE DIFFERENT ENERGY MODES IN EQUATION (293) WHILE CONSERVING THE SAME MACROSTATE CONFIGURATION $N_i (E_i)$. 

(102)
It can be shown from statistical mechanics that in thermodynamic equilibrium the most probable macrostate satisfies the Boltzmann distribution

\[ N_i = N \sum_{q} \frac{q_i e^{-\varepsilon_i/kT}}{Q} \]  \hspace{1cm} (294)

with \( Q = \sum_{q} \frac{q_i e^{-\varepsilon_i/kT}}{kT} \) \hspace{1cm} (295)

the partition function. Note that (294) establishes that \( N_i \) decreases with \( \varepsilon_i \), in such a way that the population of molecules decreases as the energy level increases, as does so in a monotonic manner (Fig. 78). Similarly, at a given temperature \( T \), most of the contribution to the sum in the partition function (295) is provided by energy levels such that \( \varepsilon_i \ll kT \), the population of molecules has decreased multi-fold at energy levels \( \varepsilon_i \gg kT \). In this way, increasing the temperature leads to a wider range of energy levels that can be activated (Fig. 78). Since the vibrational and electronic levels are much more spaced than the translational and rotational levels, such an increase in temperature typically enables the population of vibrational (\( T \approx 800 \text{ K} \)) and electronic (\( T \approx 9000 \text{ K} \)) energy levels.

It is important to note that the population distribution is monotonic in thermodynamic equilibrium, but that non-monotonicity may arise in situations concerning thermodynamic non-equilibrium, when, for instance, a vibrational energy level is suddenly populated by the passage of a gas through a shock. Provided that a sufficient number of collisions occurs, however, thermodynamic equilibrium is eventually achieved after a transient in which the population distribution is not given by (284) (note that this also means that during this transient, the internal energy and pressure of the gas are not given by the equilibrium expressions that are introduced right below).
THE PARTITION FUNCTION

THE PARTITION FUNCTION \( Q \) IN (294) IS GENERALLY A FUNCTION OF THE TEMPERATURE AND VOLUME OF THE SYSTEM. IMPORTANT QUANTITIES SUCH AS THE INTERNAL ENERGY AND PRESSURE IF THE GAS CAN BE EXPRESSED AS A FUNCTION OF THE PARTITION FUNCTION AS

\[
e = R_b T^2 \frac{\partial \ln Q}{\partial T} \bigg|_V, \quad P = R_b T \frac{\partial \ln Q}{\partial V} \bigg|_T,
\]  

(296)

HERE \( V \) IS THE SPECIFIC VOLUME. ONE IMPORTANT PROPERTY OF THE PARTITION FUNCTION (295) IS THAT IT EQUALS THE PRODUCT OF THE PARTITION FUNCTIONS OF EACH ENERGY MODE INASMUCH AS EACH ENERGY MODE IS SEPAREABLE AS IN (293), NAMELY

\[
Q = \sum_l q_l e^{-E_l / kT} = \sum_{m} \sum_{i} \sum_{k} q_m q_i q_k q_m e^{-E_m / kT} = \sum_{m} q_m e^{-E_m / kT} \sum_{i} q_i e^{-E_i / kT} \sum_{k} q_k e^{-E_k / kT} = Q_{\text{trans}} Q_{\text{rot}} Q_{\text{vib}} Q_{\text{elec}}
\]  

(297)

THE SEPARATION ESTABLISHED BY (297) IS WELL JUSTIFIED FOR THE TRANSLATIONAL DEGREE OF FREEDOM IN IDEAL GASES. ELECTRONIC EXCITATIONS CAN ALSO BE SEPARATED FROM ROTATION AND VIBRATION AT CONVENTIONAL TEMPERATURES. HOWEVER, THE LATTER TWO CAN BE SEPARATED IN A LESS ACCURATE APPROXIMATION (ROTATION + CELEBRAL OF THE KINETIC THEORY OF IDEAL GASES THAT ENABLES THE CALCULATION OF MACROSCOPIC QUANTITIES AS IN (296), UNDER THE SEPARATION ASSUMED IN (297), THE INDIVIDUAL PARTITION FUNCTIONS CAN BE DETERMINED BY SOLVING THE ASSOCIATED SCHRÖDINGER EQUATION PER DEGREE OF FREEDOM, WHICH, FOR A DIATOMIC MOLECULE OF AN IDEAL GAS GIVES:

\[
Q_{\text{trans}} = \left( \frac{2\pi m kT}{\hbar^2} \right)^{3/2} \sqrt{\frac{\hbar}{2\pi}} \left(298\right) \quad \text{Q}_{\text{vib}} = \frac{1}{1 - e^{-\hbar \nu / kT}} \left(300\right)
\]

\[
Q_{\text{rot}} = \frac{8\pi^2 I_0 kT}{\hbar^2} \left(299\right) \quad \text{Q}_{\text{elec}} = q_0 + q_1 e^{-E_1 / kT} + q_2 e^{-E_2 / kT} \left(301\right)
\]

WHERE \( \hbar \) IS THE PLANCK CONSTANT, \( I \) IS THE MOMENT OF INERTIA OF THE MOLECULE, \( \nu \) IS A VIBRATION FREQUENCY, AND \( E_1 \) AND \( E_2 \) ARE THE FIRST TWO ENERGY LEVELS OF THE ELECTRONIC MODE, WHICH ARE THE ONLY IMPORTANT ONES FOR \( T \leq 15000 \) K.

IT IS IMPORTANT TO NOTE THAT ONLY \( Q_{\text{trans}} \) DEPENDS ON THE VOLUME, AND AS A CONSEQUENCE ONLY THE TRANSLATIONAL MODE CONtributes TO THE PRESSURE IN (296), NAMELY
\[ P = R_0 T \left( \frac{\partial Q}{\partial V} \right)_T = R_0 T \left( \frac{\partial V}{\partial T} \right)_P \] 

Note that in real gases the molecules interact through potential energy and the partition functions need to be recalculated, which corresponds to the ideal-gas equation of state. On the other hand, all modes contribute to the internal energy of the gas in (296), this can be seen by substituting (298)-(304) into (296), which gives

\[ e = \frac{3}{2} R_0 T + R_0 \frac{e^\frac{\hbar v}{kT}}{e^\frac{\hbar v}{kT} - 1} R_0 T + \frac{\partial e_d(t)}{\partial T} \]

(302)

\[ e = \frac{3}{2} R_0 T + \frac{e^\frac{\hbar v}{kT} R_0 e_d(t)}{(e^\frac{\hbar v}{kT} - 1)^2} \]

(303)

\[ C_v = \frac{3}{2} R_0 + \frac{e^\frac{\hbar v}{kT}}{(e^\frac{\hbar v}{kT} - 1)^2} R_0 \frac{\partial e_d(t)}{\partial T} \]

(304)

The associated specific heats are

\[ C_p = C_v + R_0. \]

Note that, for atoms, there are not vibrational or rotational energy modes, and (302)-(303) simplify to

\[ e = \frac{3}{2} R_0 T + \frac{e^\frac{\hbar v}{kT} R_0 e_d(t)}{(e^\frac{\hbar v}{kT} - 1)^2} \]

(305)

\[ \Theta_{\text{vib}} = \frac{\hbar v}{k}, \]

(306)

which delineates the limit where vibrational motion is fully excited. A similar rotational temperature \( \Theta_{\text{rot}} = \frac{\hbar^2}{8k} \) can be defined, as well as electronic temperature \( \Theta_{\text{elec1}} = \frac{e_1}{k} \) and \( \Theta_{\text{elec2}} = \frac{e_2}{k} \). Table provides some values of relevance for Hig. temperature air.

It is important to note that

1) At conventional temperatures for hyper sonic flight, the rotational mode is always fully excited.
2) Up to temperatures of order \( T \approx 600 \) K, the gas behaves calorically perfect with \( \gamma = 1.4 \) and \( C_v \approx 5R_0/2 \).
3) For \( T \approx 600 \) K, the vibrational mode becomes progressively more excited, with \( C_v \) increasing with temperature and \( \gamma \) correspondingly decreasing. In such a way...
That the gas ceases to be a calorically perfect one. The decreasing trend of γ with temperature makes the gas more amenable for Newtonian-type of analysis as anticipated on page 28. Note however that the dependence of Cv and y on T is less trivial in monatomic species or at higher temperatures T >> \( \Theta_{vib} \) in diatomic species because of the subsequent excitation of electronic modes (note that \( C_v \rightarrow R_0 T \) and \( C_v \rightarrow R_0 T \) for \( T >> \Theta_{vib} \), indicating a clear separation of the vibrational degree of freedom, but by then the electronic excitations may have taken over).

**High-Temperature Effects on Equilibrium Inviscid Flows**

The general form of the conservation equations for high-temperature inviscid flows in chemical and thermodynamic equilibrium is given by:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) &= 0 \quad \text{(Continuity, 306)} \\
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} &= -\nabla P \quad \text{(Momentum, 309)} \\
\frac{\partial E}{\partial t} + \nabla \cdot (E \mathbf{u}) &= \frac{\partial P}{\partial t} \quad \text{(Total Enthalpy, 308)}
\end{align*}
\]

**Fig. 80:**

The latter being obtained by simplifying Eq. (275) neglecting molecular-diffusion terms. Note that \( h \) in (308) generally includes both thermal and chemical enthalpies as in (259),

\[
h = h(T, P, T) = \sum_{i=1}^{N} Y_i h_i^0 + \sum_{i=1}^{N} Y_i \int c_p(T) dT \quad (309)
\]

Additionally, the pressure, temperature and density are related by the equation of state

\[
P = \gamma R_0 T \quad \text{with} \quad R_0 = R_0 \frac{W_i}{W_i} = R_0 \left( \sum_{i=1}^{N} \frac{Y_i}{W_i} \right) \quad (310)
\]

The composition in chemical equilibrium is obtained from the equilibrium constant (292)

\[
\frac{\sum_{i=1}^{N} \frac{(Y_i^0 - Y_i)}{Y_i^0}(T)}{R_0 T} = \left( \frac{P}{P_0} \right)^{\frac{\sum_{i=1}^{N} (Y_i^0 - Y_i)}{N Y_i^0 (Y_i^0 - Y_i)}} \quad (311)
\]

Which can be applied to the \( M \) chemical steps in equilibrium. The remaining \( N-M-1 \) equations needed to obtain the molar fractions \( X_i = Y_i \frac{W_i}{W_i} \) correspond to atom conservations. The \( N \)-th molar fraction is obtained from the constraint \( \sum_{i=1}^{N} X_i = 1 \). These relations, written symbolically, along with (311), provide the mixture composition in chemical equilibrium as a function of the local pressure and temperature,

\[
Y_i = Y_i(P, T) \quad (312)
\]

The simultaneous resolution of (306)-(310) and (312), subject to appropriate
Boundary conditions and supplemented with relations for $c_0(T)$ and $\gamma_0(T)$ (obtained for instance from Jannaf tables or similar databases), provides the solution to the problem in the case of planar shocks, the above formulation simplifies to the integral conservation laws (2), (3) and (5), which in turn become (262) and (263), where

$$n^2 = \dot{q}, U, \sum \beta,$$

the solution is then direct, but in most cases requires numerical integration. Before describing qualitative aspects of the ensuing solutions, it is worth emphasizing the following:

(i) Equations (306)-(316) and (312) represent an appropriate formulation for systems in thermodynamic equilibrium, meaning that the distribution of molecules in the system across energy levels is approximately given by the Boltzmann distribution (294). This is achieved when a sufficiently large number of collisions occur to statistically relax the distribution of molecules to (294). In this limit the different energy modes are equilibrated and their corresponding energy are given by the different components appearing in (302). Non-equilibrium phenomena, resulting from a deficitary number of collisions, are addressed further below.

(ii) Similarly, equations (306)-(316) and (312) represent an appropriate formulation for systems in chemical equilibrium, meaning that the chemical conversion times scales are infinitesimally small compared to the fluid-mechanical time scale. As a consequence, the distribution of chemical species has reached equilibrium values that only depend on the local pressure and temperature, in a way that the associated distribution of molecules of each species also follows a Boltzmann distribution. Note that chemical equilibrium also requires a sufficiently high number of collisions (in turn, it requires a much high number of collisions than the one required for thermodynamic equilibrium.

The above considerations suggest that thermo-chemical equilibrium is an appropriate assumption when the fluid-mechanical time scales are much larger than the characteristic time scales of thermodynamic and chemical relaxation.
The aforementioned approximations breakdown as the flight speed increases (i.e., as the residence time of the air around the spacecraft decreases), as the pressure decreases (i.e., as the vibrational relaxation time increases and as the rate of recombination decreases), and as the temperature decreases (i.e., as the vibrational relaxation time increases). It is therefore clear that all processes that lead to a deficit of collisions will ultimately lead to non-equilibrium phenomena. As a result, increasing altitude and flight speeds degrade the thermo-chemical equilibrium built into Eqs. (306)-(316) and (312) (see Fig. 84).

**Fig. 84**

**Thermo-Chemical Equilibrium Effects on Shock Waves**

As mentioned on pages 87-89, thermo-chemical effects arising at high temperatures in hypersonic regimes have profound consequences in the post-shock gases. Firstly, the temperature \( T_2 \) is much smaller than that predicted by the theory of calorically-perfect gases (Eq. (8)). This can be understood by noticing that the increment of static enthalpy across the shock, described by the Hugoniot equation (263), has to be dedicated to dissociate the air molecules and to excite molecular degrees of freedom of vibration and electronic motion (Fig. 82). This is illustrated in the numerical calculations shown in Fig. 83, where the departure from the calorically-perfect gas values is observed to increase as the flight speed (or Mach number) and altitude increase. In particular, the increase in altitude translates into a decrease in pressure, which favors air dissociation and ionization.
FIG. 83: VARIATION OF POST-SHOCK GAS TEMPERATURE WITH FLIGHT VELOCITY AND ALTITUDE FOR A NORMAL SHOCK WAVE IN EQUILIBRIUM AIR (HUBER P.W., NASA TR R-163, 1963)

FIG. 84: VARIATION OF THE DENSITY RATIO ACROSS A NORMAL SHOCK IN EQUILIBRIUM AIR FOR DIFFERENT FLIGHT SPEEDS AND ALTITUDES (HUBER P.W., NASA TR R-163, 1963)

FIG. 85: VARIATION OF THE RATIO OF THE POST-SHOCK PRESSURE $P_2$ TO THE CALORICALLY-PERFECT VALUE $P_2^{\infty}$ FOR DIFFERENT FLIGHT SPEEDS AND ALTITUDES IN A NORMAL SHOCK (HUBER P.W., NASA TR R-163)
I contrast with the large effects on temperature, the post-shock pressure $P_2$ is only slightly affected by high-temperature thermo-chemical phenomena, as shown in the calculations in Fig. 85. As mentioned above, the post-shock pressure is mostly imposed by the pre-shock momentum, and therefore remains mostly insensitive up to departures of order $20\%$ from the value predicted by the theory of calorically-perfect gases (Eq. (7)).

Inconcurrent with the decrease in $T_2$, the mostly invariant $P_2$, and a slight increase in mean molecular weight $\mu_2$, is a large increase in the post-shock density $\rho_2$ in such a way as to render density ratios $\epsilon = \frac{\rho_1}{\rho_2}$ much smaller than the value $1/6$ predicted by the theory of calorically-perfect gases. As shown in Fig. 84, the density ratio may be as low as $\epsilon \sim 1/20$ at high altitudes in a way that resembles the limiting behavior $\epsilon \to 0$ that is foundational to the Newtonian theory of inviscid hypersonic flows.

The increase in the post-shock density $\rho_2$ caused by thermo-chemical equilibrium effects leads to modifications in the shock patterns. For instance, since the shock standoff scales as $\Delta \sim \epsilon R_0$, and $\epsilon$ decreases in equilibrium air, then a decrease in $\Delta$ is observed when equilibrium effects are incorporated in the calculations. For the same reasons, since the angle of incidence $\beta$ and the streamline deflection angle are related through the kinematic constraint $\tan(\beta - \delta) = \epsilon \tan \beta$, at high Mach numbers, the decrease in $\epsilon$ by equilibrium effects translates into a decrease of $\beta$ in oblique shocks with respect to the value predicted by the theory of calorically-perfect gases and also into an increase of the maximum $\delta$ supported by an attached shock (see McEckel, NASA TN 3845, and Fig. 8).

As a general rule in hypersonic flow over cones, at $M_{\infty} \sin \theta \sim 10$, the surface density $\rho$ and temperature $T_2$ from the calorically-perfect gas model are off by...
FACTORS OF ORDER 2 \((i.e., \ T^c = \frac{1}{2} T^c_{\text{air}}, \ \bar{c}^c = \frac{1}{2} \bar{c}^c_{\text{air}})\),

AND THE DISCREPANCIES INCREASE WITH DECREASING PressURE (\(\text{i.e., increasing altitudes}\))

AS SHOWN IN FIG. 87. THE DISCREPANCIES

HOWEVER VANISH AT \(\frac{M_a \sin \delta}{\theta} = 4\) FOR ALL

PRACTICAL FLIGHT ALTITUDES, WHERE THE POST-

SHOCK TEMPERATURE \(T_2\) CEASES TO BE

SUFFICIENTLY LARGE TO ENABLE SIGNIFICANT

VIBRATIONAL EXCITATIONS AND AIR DISSOCIATION

IN THAT LIMIT, WHICH BELONGS TO THE SUPERSONIC

RANGE, THE THEORY OF CALORICALLY PERFECT

GASES PROVIDES AN ACCURATE RESULT.

THE EQUILIBRIUM SPECIES DISTRIBUTION

BEHIND AN OBLIQUE SHOCK WAVE AROUND AN AIRCRAFT FOLLOWING A TYPICAL

ENTRY TRAJECTORY IS SHOWN IN FIG. 88.

AT RELATIVELY LOW MACH (\(\text{i.e., decreasing altitudes}\))

GAS COMPOSITION RESEMBLES THAT OF NORMAL AIR, WITH

MINOR TRACES OF NO BEING FORMED AS MACH NEARS 7.

ABOVE MACH 7, THE FOLLOWING STAGES ARE DISCERNED:

\[7 \leq M_a \leq 15\]: THE POST-SHOCK GAS IS FORMED BY (IN ORDER OF IMPORTANCE)

\[N_2, O_2, O, NO \text{ and } Ar\], WITH TRACES OF \(N\).

\[15 \leq M_a \leq 20\]: THE POST-SHOCK GAS IS FORMED BY (IN ORDER OF IMPORTANCE)

\[N_2, O, N, Ar, NO\], WITH TRACES OF \(O, e^-\) AND \(NO^+\).

\[20 \leq M_a \leq 25\]: \(N, N_2, O, Ar, NO\), WITH TRACES OF \(e^-\) AND \(NO^+\).
A classic effect arising in hypersonic flows is the production of electrons across shock waves because of the partial ionization of the air. This is illustrated in Fig. 88(b), where trace amounts of electrons, \( X_e^- \approx 10^{-4} \) are observed in equilibrium conditions in the post-shock gas. That maximum amount is attained in the portion of the flight envelope (Fig. 88a) related to high altitudes (\( z \approx 170,000 \) ft, i.e., above the stratopause) and high Mach numbers (\( M_a \approx 20 \)).

Note however that \( X_e^- \approx 10^{-4} \) translates into electron number densities

\[
N_e = \frac{p_e}{W_e} = 0 \left(10^{-10} \right) \text{cm}^{-3}
\]

(\( p_e \approx 18 \), \( W_e \approx 5 \times 10^4 \text{Kg/m}^2 \), \( W_e \approx 20 \text{g/m}^2 \) at \( z \approx 200,000 \) ft, and \( N_e = 6 \times 10^{22} \text{molecules/m}^3 \)), corresponding to a partially ionized fluid (ionization degree = \( X_e^- \approx 10^{-5} - 10^{-4} \)). The associated plasma frequencies are

\[
f_p = \left( \frac{m_e - e^2}{2 \pi \epsilon_0 m_e} \right)^{1/2} \approx 0 \left(10^{-100} \right) \text{GHz}
\]

The resulting plasma sheath plays an important role in the radio communications of the hypersonic vehicle (Fig. 90) in the following way. Only waves at frequency \( f_p \) larger than the plasma frequency can penetrate the sheath. As a result, VLF \((1-10 \text{kHz})\) or VHF \((10-100 \text{MHz})\) radiowaves will tend to be cut off.

The more general map of variations of \( f_p \) with altitude and speed is provided in Fig. 89.
From the discussions in the previous sections it is evident that the establishment of thermodynamic and chemical equilibria requires a sufficiently large number of collisions as the fluid particle moves in the spatiotemporally varying flow field around a hypersonic vehicle. This implies the competition of two time scales:

\[ \frac{t_R}{t_T} = \text{residence (fluid-mechanical)} \]

\[ \frac{t_T}{t_{\text{chem}}} = \text{thermodynamic/chemical relaxation time} \]

The assumption of equilibrium requires that the readjustment time by collisions is negligible compared to \( t_R \). In many cases of interest for hypersonics, however, this requirement is not met either locally or globally around the aircraft.

**Fig. 94**

In the notation above,

\[ \frac{t_R}{t_T} \gg 1 \rightarrow \text{thermo-chemical equilibrium flow} \]

\[ \frac{t_R}{t_T} < 1 \rightarrow \text{thermo-chemically frozen flow} \]

\[ \frac{t_R}{t_T} = O(1) \rightarrow \text{thermo-chemical non-equilibrium} \]

The time scale \( t_T \) has different components representing molecular motion (translation \( t_{\text{trans}} \), rotation \( t_{\text{rot}} \), vibration \( t_{\text{vib}} \), or electronic \( t_{\text{elec}} \)) and chemical conversion \( t_{\text{chem}} \). The readjustment of translation and rotation requires only a few collisions, and the corresponding relaxation times are of the order of the collision time \( \Theta^{-1} \), with \( \Theta \) the collision frequency defined in Eq. (184). Non-equilibrium phenomena in these processes occur when the ratios \( \frac{t_R}{t_{\text{trans}}} \) and \( \frac{t_R}{t_{\text{rot}}} \) are finite, which typically occurs in flow regions with large gradients (e.g., shocks, and to a lesser extent boundary layers), and involve finite molecular transport effects (viscosity, thermal conductivity, and mass...)
Diffusivity for translational non-equilibrium, and bulk viscosity for rotational non-equilibrium. As a result, translational and rotational non-equilibrium are representative of non-inviscid effects, which do not necessarily require high temperatures to exist.

In most practical situations, for hypersonics, with exception of highly rarefied flows for which $M_r \rightarrow \infty$, the gas is in translational and rotational equilibrium, with small departures from equilibrium being noticed in boundary layers and much more severe departures existing in shock waves (note that $\frac{\nu_{\text{eq}}}{\nu_r} \approx \frac{\lambda}{L} = \frac{U_L}{L} = \left( \frac{A}{L} \right) \approx Re L K_m^2$).

In contrast, vibrational and chemical processes take much longer to readjust. As a result, extended regions in the flow field may occur where vibrational and chemical non-equilibrium exist, with these regions not being necessarily confined to regions with large gradients. In general, vibrational motion readjusts earlier than the chemical composition, although both are intricately coupled during the non-equilibrium stage. It is important to emphasize that both vibrational and chemical processes require high temperatures to occur, and are therefore of high relevance in the post-shock gases of hypersonic shock waves (Fig. 92).

Based on the above considerations, the following definitions can be made with regards to the type of gas:

1) **Calorically-Perfect Gas**: Translationally and rotationally equilibrated, vibrationally frozen (unexcited), and chemically frozen.

2) **Thermally-Perfect Gas**: Translationally, rotationally, and vibrationally equilibrated, and chemically frozen.

3) **Chemically-Reacting Equilibrium Ideal Gas**: Translationally, rotationally, vibrationally, and chemically equilibrated (see Fig. 91).

Note that the degree of chemical equilibrium is measured by the time-scale ratio

$$Da = \frac{t_r}{t_{\text{chem}}} \quad (314)$$

**Dannköhler Number**

With $Da = 0$ and $Da = \infty$ indicating, respectively, frozen and equilibrium conditions.
In situations where vibrational non-equilibrium prevails, $e_{\text{vib}}$ has not yet relaxed to the equilibrium value in (315). Instead, $e_{\text{vib}}$ is obtained from a supplementary equation that describes the relaxation process and which is given generally by

$$\frac{de_{\text{vib}}}{dt} = e_{\text{vib}}^\infty - e_{\text{vib}} - F(K_{\text{diss}})$$  (317)

where $e_{\text{vib}}^\infty = \frac{\Theta}{\sqrt{T}}$ is the equilibrium value, $\tau_{\text{vib}}$ is a characteristic vibrational relaxation time, and $F(K_{\text{diss}})$ is a function that depends on reaction rates of dissociation (see Treanor and Markows, Phys. Fluids (1962)). Note that there exists a coupling between the chemical reaction rates and the vibrational relaxation rate, in that molecules that are highly excited vibrationally are more readily dissociated by collisions, thus increasing the rate of dissociation (Hammerling et al., Phys. Fluids (1959)), while dissociation causes an additional drainage of mean vibrational energy given by the second term in (211).

The relaxation time $\tau_{\text{vib}}$ is usually obtained by the Landau-Teller (1936) theory

$$\tau_{\text{vib}} = \frac{\Theta_{\text{vib}} e^{\left(\frac{k_{\text{vib}}}{T}\right)}}{P}$$  (318)

where $\Theta_{\text{vib}}$ and $k_{\text{vib}}$ are constants that depend on the vibrating molecule and on the bath of translational molecules on which they collide (see Table V). The characteristic values of $\tau_{\text{vib}}$ are of order $10^{-5} - 10^{-2}$ s in atmospheric pressures and high temperature. For instance, consider pure $N_2$ at $T=3000 K$ and $P=1$ atm, so that $\tau_{\text{vib}} \approx 38 \mu s$ according to the values in Table V. Note that the table V: values of $\Theta_{\text{vib}}$ and $k_{\text{vib}}$ in (319).

<table>
<thead>
<tr>
<th>Species</th>
<th>Heat-Bath</th>
<th>$C_{\text{vib}}$ atm-microsec</th>
<th>$K_{\text{vib}}^\infty$ K</th>
<th>Approximate Range of $T$, K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_2$</td>
<td>$O_2$</td>
<td>$5.42 \times 10^{-3}$</td>
<td>$2.95 \times 10^6$</td>
<td>$800-3200$</td>
</tr>
<tr>
<td>$O_2$</td>
<td>Ar</td>
<td>$3.58 \times 10^{-4}$</td>
<td>$2.95 \times 10^6$</td>
<td>$1300-4300$</td>
</tr>
<tr>
<td>$N_2$</td>
<td>$N_2$</td>
<td>$7.12 \times 10^{-3}$</td>
<td>$1.91 \times 10^6$</td>
<td>$800-6000$</td>
</tr>
<tr>
<td>NO</td>
<td>NO</td>
<td>$4.86 \times 10^{-3}$</td>
<td>$1.37 \times 10^6$</td>
<td>$1500-3000$</td>
</tr>
<tr>
<td>NO</td>
<td>Ar</td>
<td>$6.16 \times 10^{-1}$</td>
<td>$1.37 \times 10^6$</td>
<td>$1500-4600$</td>
</tr>
</tbody>
</table>


That during the vibrational relaxation period, approximately $\tau_{\text{vib}} \approx 10^{-3}$ collisions have taken place, at this pressure and temperature, which are representative of the gas behind a normal shock wave at Mach 10 in the stratosphere ($T_2 \approx 3200 K$, $S_2/S_1 \approx 8.5$ at $U_2 \approx 1040 m/s$ and $R_2 \approx 120 K$ in Fig. 5.3 (a,b)), the vibrational relaxation of $N_2$ lasts for $(211)/(\tau_{\text{vib}}) \approx 1.3 cm$ downstream of the normal shock. This distance over which vibrational non-equilibrium...
VIBRATIONAL AND CHEMICAL NON-EQUILIBRIUM

NON-EQUILIBRIUM PHENOMENA RELATED TO CHEMICAL AND VIBRATIONAL PROCESSES ARE IMPORTANT IN HYPERSONICS DUE TO THE FOLLOWING FACTORS:

i) CHEMICAL AND VIBRATIONAL PROCESSES TEND TO REQUIRE MANY COLLISIONS TO EQUILIBRATE COMPARED TO TRANSLATIONAL AND ROTATIONAL PROCESSES (THE LATTER ONLY REQUIRE A FEW COLLISIONS TO EQUILIBRATE). AT HYPERSONIC VELOCITIES, THE COLLISIONS AROUND THE AIRCRAFT ARE HINDERED BECAUSE OF THE SMALL RESIDENCE TIME OF THE FLUID PARTICLES AND BECAUSE OF THE LOW AMBIENT PRESSURES INVOLVED IN HIGH ALTITUDE FLIGHT.

ii) HYPERSONIC FLOWS TYPICALLY INVOLVE HIGH TEMPERATURES THAT ARE NECESSARY FOR THE ACTIVATION OF VIBRATIONAL AND CHEMICAL PROCESSES. NOTE HOWEVER THAT HIGH TEMPERATURES TEND TO EQUILIBRATE THE FLOW BECAUSE OF THE CORRESPONDING DECREASE IN THE MEAN FREE PATH, THEREBY TENDING TO COUNTERACT THE NON-EQUILIBRATING EFFECT OF THE LOW PRESSURE AND HIGH SPEED.

THE SUBJECTS OF CHEMICAL AND VIBRATIONAL NON-EQUILIBRIUM ARE DISCIPLINES ON THEIR OWN AND CANNOT BE TREATED IN DEPTH HERE. THE DISCUSSION HERE WILL BE LIMITED TO BASIC RESULTS. THE READER IS REFERRED TO VINCENTI AND KRUGER'S "INTRODUCTION TO PHYSICAL GASDYNAMICS," CH. VII-VIII (1965) FOR FURTHER DETAILS.

THE EXPRESSION FOR THE INTERNAL ENERGY (302) CONTAINS DIFFERENT TERMS THAT PERTAIN TO DIFFERENT MODES OF MOLECULAR MOTION. MOMENTARILY DISREGARDING ELECTRONIC EXCITATIONS (WHICH GIVE RISE TO A PLETHORA OF RADIATIVE PHENOMENA BEYOND THE SCOPE OF THIS COURSE), THE INTERNAL ENERGY IN THERMODYNAMIC EQUILIBRIUM CAN BE WRITTEN AS

\[
E = E_{\text{trans}} + E_{\text{rot}} + E_{\text{vib}}, \quad \text{with} \quad \begin{cases} 
E_{\text{trans}} = \frac{3}{2} \frac{R_0}{N} T \\
E_{\text{rot}} = \frac{R_0}{N} T \\
E_{\text{vib}} = \frac{\Omega V}{T} \frac{R_0}{N} T
\end{cases} (315)
\]

WITH \( \Omega V = k_{\text{B}} \), A CHARACTERISTIC VIBRATIONAL TEMPERATURE. IT IS IMPORTANT TO NOTE THAT THE TEMPERATURE \( T \) IN (315) IS THE SAME FOR ALL COMPONENTS WHEN THE SYSTEM IS IN THERMODYNAMIC EQUILIBRIUM, AND IS EQUAL TO THE TRANSLATIONAL KINETIC TEMPERATURE DEFINED BY

\[
E_{\text{trans}} = \frac{5}{2} \frac{R_0}{N} T = \frac{1}{2} c^2 (316)
\]

WHERE \( c_0 = \sqrt{3R_0 T} = \sqrt{\frac{3P}{\gamma}} \) IS THE ROOT-MEAN SQUARE SPEED OF THE GAS MOLECULES.
Persists is actually comparable to characteristic shock standoff distances \( \Delta y \left( \frac{y}{s_3} \right) R_e \) from a nose of curvature radius \( R_0 = 10 \) cm (Fig. 9.3).

In the calculation of flows subject to vibrational non-equilibrium, Equations (306) - (308) are integrated with \( h \) including the vibrational enthalpy, namely

\[
h = \sum_{i=1}^{N} Y_i h_{i0} + \sum_{i=1}^{N} Y_i (h_{i_{\text{trans}}} + h_{i_{\text{rot}}} + h_{i_{\text{vib}}})
\]

where \( h_{i_{\text{vib}}} = e_{i_{\text{vib}}} \) and \( e_{i_{\text{vib}}} \) is obtained by integrating (317), while \( h_{i_{\text{trans}}} \) and \( h_{i_{\text{rot}}} \) are given by their equilibrium expressions.

Note that vibrational non-equilibrium in the post-shock gases typically coexists with chemical non-equilibrium whereby finite-rate chemistry becomes important in dissociating the air molecules (Fig. 9.3). As a result, Eqs. (306) - (308) need to be supplemented with the equation for species transport (278) which in the inviscid limit becomes

\[
\frac{\partial Y_i}{\partial t} + \mathbf{u} \cdot \nabla Y_i = \dot{W}_i, \quad (319)
\]

where \( \dot{W}_i \) is a chemical production term that competes with the advection terms when the characteristic Damköhler number (314) is of order unity, as specified in (280) - (281).

However, \( \dot{W} \) depends on temperature. In situations where chemical and vibrational equilibrium coexist, as in the zone immediately behind a normal shock wave at hypersonic velocities (Fig. 9.3), the character of the temperature participating in the definition of \( \dot{W} \) is unclear because of the two-way coupling effect between dissociation and vibration relaxation mentioned above. In fact, the successive relaxation of the different degrees of freedom motivates the definition of alternative temperatures,

\[
\begin{align*}
E_{tr} &= R_g T_{\text{trans}}, \\
E_{rot} &= R_g T_{\text{rot}}, \\
e_{\text{vib}} &= \frac{\Theta V_{\text{vib}}}{e^{\Theta V_{\text{vib}}}} R_g T_{\text{vib}}
\end{align*}
\]

Fig. 9.4. Distribution of temperatures behind a normal shock.
where $T_{kin}$ is the kinetic translational temperature, $T_{rot}$ is the rotational temperature, and $T_{vib}$ is the vibrational temperature. As shown in Fig. 9.4, the translational and rotational temperatures equilibrate fast across the shock since only a few collisions are required to fully excite translational and rotational degrees of freedom. Note that the translational temperature is the one that participates in the equation of state, which can be considered still valid for small departures from thermodynamic equilibrium. On the other hand, the vibrational temperature takes much longer distances to equilibrate as implied by the large parameter $\frac{T_{vib}}{T_0} \gg 1$ which suggests that many collisions are needed for $T_{vib}$ to become the equilibrium value $T_{eq}^{\infty}$.

During the non-equilibrium period, $T \neq T_{vib}$ in the conservation equations, and it is customary to use a combined temperature $T_0 = \left( \frac{T_{vib}}{T} \right)^{\frac{1}{2}}$ to evaluate the rates of the chemical reactions participating in the dissociation process (see the treatise "Non-Equilibrium Hypersonic Flows" by Park about this subject). Here, $T$ is the translational kinetic temperature ($i.e., T_{vib} \to T$ at distances much larger than $U_2/2v_{vib}$ behind the shock as shown in Fig. 9.4).

**Non-Equilibrium Flow Across a Shock**

![Diagram showing temperature profiles across a shock wave]

The characteristic temperature and density profiles across a normal shock are provided in Fig. 9.5 for $M_{0} = 10$ and $\approx 120$ Ks. It is important to note that, because of the only few collisions across the shock, the values $T_{eq}$ and $\rho_{eq}$ immediately downstream of the shock are those corresponding to frozen flow, or equivalently, those that would be obtained by applying the shock jump conditions (6) and (8). It is only after a downstream distance of order $\rho_{eq} \approx U_2 t_{chem}$ (which incorporates both vibrational and chemical non-equilibrium since $t_{chem} \gg t_{vib}$ in conventional situations) that the flow variables relax.
The shock wave demands the continuity of the frozen composition across:

\[
\frac{d}{dt} \int Y_i dV + \int \nabla \cdot \mathbf{U}_i d\mathbf{x} = \int \mathbf{F}_i dV
\]

\[
0 = \frac{\partial}{\partial t} Y_i + \frac{1}{c_s^2} \nabla \cdot \mathbf{U}_i = \frac{1}{c_s^2} \mathbf{F}_i \quad \Rightarrow \quad Y_i = \frac{Y_0}{c_s^2} \tag{320}
\]

As a result, the composition immediately downstream of the shock is that of the fresh (non-dissociated) gas and only after a downstream distance of order \( l_{ne} \) the equilibrium composition (i.e., the one shown in Fig. 96) is achieved (see Fig. 96).

Before closing this chapter, it is worth emphasizing that the considerations given above are in many cases oversimplified with respect to real non-equilibrium phenomena that may be encountered in flows around hypersonic vehicles. Real conditions may include flow unsteadiness, pre-shock chemically reacting gases passing through additional shocks, non-equilibrium effects in expansion waves, molecular recombination in walls, boundary-layer effects, and non-equilibrium radiative effects in the gas and in the wall surfaces. Most importantly, the non-equilibrium character of the flow is often set by the local flow conditions rather than by global flow indicators such as those depicted in Fig. 5, thereby making the prediction of thermo-chemical phenomena around hypersonic vehicles a challenging problem that remains subject of research.
V. RE-ENTRY AERO-MECHANICS

The previous chapters have been devoted to the study of hypersonic flow fields in several forms. The design of a hypersonic vehicle, however, heavily relies on the global characteristics of the flight trajectory. A problem that has received much attention over the years is the atmospheric re-entry from space, which involves hypersonic flows at exceedingly high temperatures that set severe constraints in the design of the spacecraft.

General Considerations

The problem of re-entry involves the penetration of a body (meteor, space capsule, glider, warhead, satellite, asteroid, etc.) into the atmosphere at high velocities ranging from the circular orbital velocity ($\sim 7.900 \text{ m/s} = 28500 \text{ km/h}$ for Earth) to the escape velocity ($\sim 11200 \text{ m/s} \approx 40200 \text{ km/h}$ for Earth) and beyond. Outside the planetary atmosphere, the motion of the body is described by Kepler's laws of celestial mechanics, as the body enters the atmosphere and more particularly after crossing the mesopause, the aerodynamic forces become increasingly important because of the increasingly larger value of the density as the ground is approached. The aerodynamic forces lead to a large deceleration and intense friction that more often than not cause disintegration of the body in the case of meteoroids and risk the structural integrity in the case of spacecrafts. In manned spaceflight, the re-entry stage becomes much more crucial, since the objective turns into slowing down the spacecraft from entry velocities of order of tens of thousands of miles per hour to zero velocity at landing, and doing it safely. An obvious but surely impractical solution to the problem is to employ retro-rockets to slow down the spacecraft through the atmosphere. However, it is characteristic of rocket propulsion that for every ton of payload lifted to space, many tons of fuel are required in the booster. As a result, the extra weight that retro-rocketing renders...
This alternative highly unattractive. To date, the alternative is to let the atmosphere slow down the spacecraft by viscous friction. A typical re-entry trajectory is sketched in Fig. 97, where

\[
\begin{align*}
\gamma_E &= \text{Entry Flight Path Angle (with respect to the local horizontal)} \\
U_e &= \text{Entry Velocity} \\
Z_e &= \text{Entry Interface Height (on Earth, } Z_e \text{ ~86 km)}
\end{align*}
\]

The passive deceleration caused by friction may look daunting at first, since most reentry objects are completely vaporized during re-entry due to the intense heating generated. However, throughout the years, solutions have been created that warrant decelerations within human tolerances and preserve the structural integrity of the spacecraft, although such solutions still lead to excessive mission costs because of the added weight necessary to fulfill these tolerances.

* Types of Re-entry Trajectories

The problem of atmospheric entry takes on different forms depending on the mission trajectory under investigation. Perhaps the three most common situations are re-entries produced as a result of (i) ballistic or boost-glide trajectories of rocket payloads from one to another point on the Earth surface, (ii) deflected orbital trajectories of spacecrafts initially orbiting along circular or elliptical orbits around the Earth, and (iii) hyperbolic trajectories of spacecrafts returning from the moon or other planets. Examples of these spaceflight trajectories are provided in Fig. 98. In that figure, the following trajectories are highlighted:

- **Ballistic Entry**: Non-lifting bodies (\(L/D = 0\)) enter the atmosphere ballistically from space. The characteristic flight path angle \(\gamma_e\) is large, and as a result the deceleration and heat flux are large (see next section). The gravitational and centrifugal forces tend to be negligible once the spacecraft enters the atmosphere in ballistic mode.
Fig. 98: Main types of re-entry trajectories in a planetary atmosphere.
- **GLIDING ENTRY:** LIFTING BODIES \((L/D = O(1))\) CAN SMOOTHLY ENTER THE ATMOSPHERE AT ALMOST ZERO INITIAL FLIGHT-PATH ANGLE. THIS TENDS TO REDUCE THE ACCELERATION BUT INCREASES THE TOTAL HEAT LOAD BECAUSE OF AN INCREASED AMOUNT OF ENTRY TIME AND BECAUSE OF THE ADDITIONAL WEIGHT REQUARED FOR A LIFT-SUPPORTING STRUCTURE WITH WINGS OR OTHER MEANS. IN THIS TYPE OF TRAJECTORY, THE VERTICAL COMPONENT OF THE ACCELERATION AND THE VERTICAL COMPONENT OF THE DRAG FORCE ARE NEGLIGIBLE.

- **SKIPPING ENTRY:** VARIABLE VALUES OF THE LIFT-TO-DRA4 RATIO CAN BE USED IN LIFTING BODIES TO LIFT-UP THE VEHICLE ONCE IT HAS ENTERED THE ATMOSPHERE, THE RESULTING UPWARDS MOTION MAY CAUSE THE VEHICLE TO HOSTLY EXIT THE ATMOSPHERE, AFTER WHICH A NEGATIVE \(L/D\) NEEDS TO BE APPLIED FOR IT TO PENETRATE AGAIN INTO THE ATMOSPHERE. IN SKIPPING ENTRIES THE INITIAL FLIGHT-PATH ANGLE IS LARGE, AND THE GRAVITATIONAL AND CENTRIFUGAL FORCES ARE NEGLIGIBLE.

OTHER GENERAL TRAJECTORIES MAY INVOLVE SITUATIONS NOT DESCRIBED ABOVE, SUCH AS NON-LIFTING BODY AT A VERY SMALL ANGLE AS FROM A DECAYING ORBIT OR IN AN AEROBRAKING MANEUVER, REQUIRE SIMULTANEOUS CONSIDERATION OF GRAVITATIONAL, CENTRIFUGAL, AND LIFT FORCES, VERTICAL ACCELERATION, VERTICAL COMPONENT OF THE DRAG, ALONG WITH THE HORIZONTAL ACCELERATION AND HORIZONTAL COMPONENT OF THE DRAG (SEE CHAPMAN D.R., "AN APPROXIMATE ANALYTICAL METHOD FOR STUDYING ENTRY INTO PLANETARY ATMOSPHERES", NASA TN 4296 (1958), FOR A GENERAL TREATISE ON THIS TOPIC).

![Diagram of Entry Types](image-url)
ENTRY VELOCITY:

The characteristic entry velocity also depends on the trajectory under investigation. For satellites or manned spacecrafts orbiting in circular orbits around the Earth, the entry velocity $U_e$ is of the order of the circular velocity,

$$ U_e = \left( \frac{\mu}{R^3} \right)^{\frac{1}{2}} = 7.9 \text{ km/s} = 28495 \text{ km/h} $$

"FIRST COSMIC VELOCITY"

with $\mu = 3.98 \times 10^{14} \text{ m}^3/\text{s}^2$ the standard gravitational parameter, and $R = 6371 \text{ km}$.

The radius of the Earth. Other trajectories involve supercircular entries at the parabolic velocity

$$ U_e = \left( \frac{2 \mu}{R^2} \right)^{\frac{1}{2}} = 11.2 \text{ km/s} = 40269 \text{ km/h} $$

"SECOND COSMIC VELOCITY"

as in the case of a spacecraft returning from the Moon, or at hypersonic velocities

$$ U_e > \left( \frac{2 \mu}{R^2} \right)^{\frac{1}{2}} $$

for spacecrafts returning from other planets. In all cases, $U_e$ is much larger than the peripheral velocity of the Earth as it rotates around its axis (e.g., the absolute velocity of the atmosphere), which is approximately $400 \text{ m/s} \approx 1600 \text{ km/h} \approx 1000 \text{ mph}$ along the Earth's equator, and in this way direct and retrograde orbits tend to pose similar challenges for re-entry.

The entry velocity discussed above is not generally the true velocity at the entry interface, since the spacecraft must typically execute trajectory control on approach to Earth. One characteristic maneuver is the deboost from a circular orbit, which involves the firing of retro-rockets to deflect the spacecraft into a descent ellipse that intercepts the surface of the Earth (Fig. 100).
In general, however, the change in altitude along the deflected orbit is small compared to the distance of the deboost point to the center of the Earth, and the difference between the actual entry velocity $V_e$ and the circular orbital velocity is small. In most practical deboost maneuvers, the retrorockets need to provide a velocity increment $\Delta V$ that is of order $1\% - 10\%$ the circular orbital velocity $V_c$. A complete analysis of deboost maneuvers from circular and supercircular orbits is provided in the seminal report by Low G. (1959).

* ENTRY CORRIDOR

A problem in spaceflight is the safe re-entry of spacecrafts from supercircular orbits. For instance, in their return from the Moon or other planets, in particular, an overshoot or an undershoot approach with respect to the established return trajectory can have fatal consequences as a result of guidance errors.

In an undershoot approach, the spacecraft enters the atmosphere at a large flight-path angle, which causes decelerations beyond human tolerances. In contrast, in an overshoot approach, the atmosphere is too little dense to provide the drag force needed to slow down the spacecraft, which will traverse the outer layer of the atmosphere and will continue on an orbit that will not allow it to re-enter until most likely the power and oxygen are exhausted. Although return passes would eventually slow down the vehicle, this would also involve repeated crossings of the Van Allen belts and therefore too much exposure to heavy radiation (see Fig. 98).

The entry-corridor depth is defined as the difference between the altitudes of the perigees of the undershoot and overshoot orbits resulting from vacuum trajectories. In a lunar return trajectory, for instance, the entry-corridor depth is of order 10 miles.
Lift force and downrange precision landing

The utilization of lift does not only increase the entry corridor depth with respect to a purely ballistic entry, but it also lengthens the path to the ground, decreases the maximum deceleration and heating rate, provides control capability for correcting too steep entry angles, and most interestingly, provides capabilities for landing within much wider areas on the surface of the Earth. Lift can be generated by changing the attitude of the spacecraft (Fig. 102). In non-lifting bodies, such as space capsules, lift is generated by shifting the center of gravity above the axis of symmetry. Negative lift can be produced by rotating the capsule around its axis of symmetry using small rockets pointed in the azimuth direction, in such a way as to shift the center of gravity downwards. This control authority was employed in the reentry of the Apollo command module (Fig. 103) for a "pull up" at about 180,000 ft and a "skip" to increase the maximum range of the spacecraft and to reduce the aerodynamic heating on the thermal shield. The control period during "pull-up" is employed to reduce the landing footprint to less than 2,000 miles length.

**Fig. 103: Apollo 14 re-entry trajectory (adapted from Hill, NASA TN D5399 [1969]).**
LANDING FOOTPRINT BY REDUCING ITS LENGTH TO 690 MILES. NOTE THAT AN ERROR IN THE ATTITUDE OF THE CAPSULE DURING THE FIRST CONTROL PERIOD, FOR INSTANCE INVOLVING TOO MUCH LIFT, COULD SEND THE SPACECRAFT UP AWAY FROM THE ATMOSPHERE LIKE A STONE SKIPPING ON A POND, AND INTO AN ORBIT THAT WOULD LAST BEYOND THE LIFETIME OF THE COMMAND MODULE SYSTEMS.

AN ILLUSTRATION OF A LANDING FOOTPRINT FOR A LIFTING VEHICLE IS PROVIDED IN FIG. 184. IN GENERAL, A VEHICLE WITH A LIFT-TO-Drag RATIO $L_D = 2$ CAN BE LANDED ANYWHERE WITHIN AN AREA EXTENDING APPROXIMATELY 5,000 MILES. THE CAPABILITY FOR A HYPERSONIC GLIDER TO CONTROL THE LANDING ZONE BY BANKING ITSELF AND VARYING ITS FLIGHT ATTITUDE, IS COMMONLY KNOWN AS "DYNAMIC SOARING" (HENCE THE PROJECT "DYNASAIL", 1957-1963 AIMED AT DEVELOPING A HYPERSONIC DYNAMIC SOARING VEHICLE FOR ACCESS AND RETURN FROM LOW-EARTH ORBIT).

BALLISTIC RE-ENTRY AEROMECHANICS

Perhaps the simplest re-entry trajectory is the ballistic type (Allen J.H., Egers A.J. NASA TR 1384, 1959). Based on the schematic shown in FIG. 105, the acceleration of the spacecraft in the polar coordinate system

$\dot{r}, \dot{\phi}$ is $\dot{r} = \left( \frac{dv}{dt} + \frac{v^2}{r} \right) \hat{r} + \left( \frac{dv - v\omega r}{dt} \right) \hat{\phi}$,

where $\hat{r}$ and $\hat{\phi}$ are unit vectors in the radial and polar directions, respectively, and $v$ and $u$ are the corresponding velocity.
components, \( V = \frac{dV}{dt} = U \sin \gamma \), and \( U = \frac{dU}{dt} = U \cos \gamma \), with \( U \) the velocity magnitude.

The second Newton's law for this planar motion corresponds to

\[
-m \frac{dV}{dt} = +m \frac{d^2 \gamma}{dt^2} = m \frac{d^2 \gamma}{dt^2} = \frac{m U^2}{r} \cos \gamma - m \frac{U^2}{r} \sin \gamma + D \sin \gamma,
\]

\[
-m \frac{dU}{dt} = m \frac{d \gamma}{dt} \frac{d^2 \gamma}{dt^2} = -m \frac{U^2}{r} \cos \gamma + L \sin \gamma - D \cos \gamma.
\]

In the ballistic entry, the following approximations are made: 1) the lift is zero, 2) the drag coefficient is constant, 3) the centrifugal force is negligible, 4) the Coriolis force is negligible, 5) the weight of the vehicle is negligible, and 6) the flight-path angle is constant and equal to the entry value \( \gamma = \gamma_E \). In this way, (321) - (322) simplify to

\[
m \frac{dV}{dt} = -\frac{1}{2} \frac{80 A C_d}{\rho_0} U^2 \sin \gamma_E \quad \text{(323)}, \quad \text{where} \quad \frac{1}{2} \frac{80 A C_d}{\rho_0} U^2 \sim \beta^2 \quad \text{(exponential atmosphere)}
\]

An equation for the acceleration magnitude \( m \frac{dU}{dt} \) is easily obtained by multiplying the first of the equations in (323) by \( U \) and the second one by \( V \), and by making use of \( U^2 = U^2 + V^2 \) upon summing both equations, which gives

\[
m \frac{dU}{dt} = -\frac{1}{2} \frac{80 A C_d}{\rho_0} U^2, \quad \text{subject to} \quad U = U_E (\text{entry velocity}) \quad \text{at} \quad t = 0 \quad (324)
\]

Note that the approximations made above are accurate for not too small entry angles \( \gamma_E \), whereby the centrifugal, Coriolis and gravitational forces are all much smaller than the drag force, which is responsible for the large decelerations of order 1000 g's experienced by a vehicle in ballistic entry.

The solution to (324) is direct and involves the replacement of time by altitude by using the relation \( \frac{dU}{dt} = \frac{dU}{dz} \frac{dz}{dt} = -U \sin \gamma_E \frac{du}{dz} \), which transforms (324)

\[
\int \frac{U_E}{U} \left( -\frac{1}{2} \frac{80 A C_d}{\rho_0} \frac{e^{-\beta^2}}{m \sin \gamma_E} \right) dz = -\frac{1}{2} \frac{80 A C_d}{\rho_0} \frac{e^{-\beta^2}}{m \sin \gamma_E} U
\]

which simplifies to

\[
U = U_E e^{-\frac{1}{2} \frac{80 A C_d}{\rho_0} \frac{e^{-\beta^2}}{m \sin \gamma_E}}, \quad (325)
\]
WHERE $\zeta$ IS A DIMENSIONLESS PARAMETER GIVEN BY

$$\zeta = \frac{B_m \sin \gamma E}{\varphi_0 A D} \quad \text{(326)}$$

WHICH CAN BE REWRITTEN AS \( \zeta = (B L) \left( \frac{\rho_s}{\rho_0} \right) \left( \frac{\sin \gamma E}{C_D} \right) \), WHERE $\rho_s$ IS THE MEAN STRUCTURAL DENSITY OF THE VEHICLE, AND $L = \sqrt{V/A}$ IS A CHARACTERISTIC LENGTH OF THE VEHICLE BASED ON ITS VOLUME $V = \frac{m}{\rho_s}$. ALSO PART OF $\zeta$ IS THE BALLISTIC COEFFICIENT $C = \frac{m}{C_D A \left[ \frac{\rho_s}{m} \right]}$ SUCH A WAY THAT $\zeta = (B L / \rho_0) \sin \gamma E$.

NOTE THAT $U$ IN (325) IS A SUPER-EXponentIALLY DECReasing FUNCTION OF THE ALTITUDE SUCH RAPID DECREASE IS CAUSED BY THE ATMOSPHERIC DRAG, AND IT DECREASES INCREASING MORE RAPID AS $\zeta$ BECOMES SMALLER. THIS CAN BE SEEN BY CALCULATING THE DECELERATION NORMALIZED BY THE GRAVITATIONAL ACCELERATION, NAMELY

$$-\frac{A}{\varphi} \frac{dU}{dt} = \frac{U \sin \gamma E}{\varphi} \frac{dU}{dt} = \frac{\sin \gamma E}{\gamma} \left( \frac{U_E^2}{\gamma} - \frac{1}{2} \right) e^{-\frac{p^2}{2}} e^{-\frac{p^2}{2}} = \left( \frac{U_E^2}{\gamma} \right) \frac{\gamma}{2} \frac{\sin \gamma E}{\gamma} e^{-\frac{p^2}{2}} e^{-\frac{p^2}{2}} = \frac{8}{7} \gamma \quad \text{(327)}$$

THE EXPRESSION FOR THE ACCELERATION PROVIDED ABOVE HAS A MAXIMUM AT THE CRITICAL ALTITUDE

$$z_L = \frac{4}{\beta} \ln \left( \frac{4}{5} \right) = \frac{4}{\beta} \ln \left( \frac{\varphi_0 A D}{B m \sin \gamma E} \right) \quad \text{(328)}$$

WHERE THE MAXIMUM DECELERATION IS

$$-\frac{A}{\varphi} \frac{dU}{dt} \mid_{\text{max}} = \left( \frac{U_E^2}{\gamma} \right) \frac{\beta \sin \gamma E}{2e^4} \quad \text{(329)}$$

\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig106.png}
\caption{Fig. 106.}
\end{figure}
A Schematic of the solution represented by (325)-(329) is provided in Fig. 105. For the case \( \zeta < \lambda \) \( (\zeta \approx 0.5) \). A number of important aspects of the above formulation are worthy of discussion:

1) The value of the maximum deceleration is independent of the mass of the vehicle \( m \).

2) The value of the maximum deceleration decreases with decreasing entry angles \( \gamma_E \). It is for this reason that shallow entries are preferable for they enable maximum decelerations within human tolerances.

3) For entries at the first cosmic velocity, \( U_E = \sqrt{\gamma_R \lambda} \), the maximum deceleration is proportional to \( \sqrt{\gamma_R \lambda} \). For Earth, \( \beta R = 950 \), and as a result \( \alpha_{max} / g_0 = (\beta R \lambda / 2 \sin \gamma_E \approx 1.74 \). For 100% of maximum acceleration, the entry angle must be \( \gamma_E \approx 3.3^{\circ} \). Note that in other planets \( \gamma_E \),

\[
\alpha_{max} / g_0 = \frac{\beta R \lambda / 2 \sin \gamma_E}{g_0}
\]

For instance, in Mars \( g_0 / g_0 \approx 0.38 \) and \( (\beta R) \approx 309 \). So that \( \alpha_{max} / g_0 \approx 21 \sin \gamma_E \). As a consequence, a much steeper entry angle \( \gamma_E \approx 28^{\circ} \) can be used to attain the same 100% of maximum acceleration.

4) The altitude for maximum deceleration \( z_1 \) is independent of the entry velocity.

5) The altitude for maximum deceleration \( z_1 \) increases as the entry angle \( \gamma_E \) becomes increasingly smaller. As a consequence, shallow entries lead to smaller maximum decelerations and to higher altitudes where the maximum deceleration occurs.

6) The velocity at the altitude of maximum deceleration can be easily obtained by substituting (328) into (325), which gives

\[
U_4 = U_E e^{-\frac{1}{2}} \approx 0.61 U_E
\]

Equivalently, maximum deceleration occurs when 63% of the initial kinetic energy has dissipated into heat, and this is independent of all parameters.
7) The impact velocity of the vehicle with the ground (in the absence of retro-rockets or drogue parachutes to slow it down near the ground) is given by equation (325) particularized for $z = 0$, namely

$$u^i = u_e e^{-\frac{1}{2}z} \quad (331) \quad \text{Impact Velocity}$$

Note that only a fraction $e^{-\frac{1}{2}z}$ of the initial kinetic energy survives till impact. Equivalently, a fraction $1 - e^{-\frac{1}{2}z}$ of the initial kinetic energy is dissipated into heat during re-entry. In practical application, $1 - e^{-\frac{1}{2}z} \approx 99\%$, so that typically only $1\%$ of the initial kinetic energy remains at impact.

8) The maximum deceleration represented by (329) occurs above the ground only when $\frac{x_0}{z} < 1$ so as to make $2 \alpha_1$ (328) defined. In re-entries with $\frac{x_0}{z} > 1$, the maximum acceleration occurs near impact, $2 = 2_{\alpha} = 0$, where $\alpha_{\text{max}} = \left(\frac{u_e^2}{a_0}\right) \frac{3 \sin \delta}{z} e^{-\frac{1}{2}z}$. Note that $\alpha_0$ is not defined at impact. That this value is smaller than (329) by a factor $\int e^{-\frac{1}{2}z} < 1$, indicating that a projectile of high ballistic coefficient experiences lower deceleration prior to impact albeit the much deeper atmospheric penetration and the higher impact velocity.

**Re-entry Heating**

At the first cosmic speed, the kinetic energy per unit mass of a spacecraft is $q_R \frac{R_0}{2} \approx 31 \text{ MJ/kg}_0$. As mentioned above, approximately 99% of this kinetic energy is dissipated into heat during re-entry. If 100% of this heat was to be absorbed by the spacecraft structure, the re-entry would be catastrophic. Note that a kg of steel (specific heat $C = 0.48 \text{ kJ/kg}_0 K$) would heat up by $\Delta T = \frac{q_R}{2C} \approx 65,000$ K if it absorbed that large amount of heat, and would vaporize upon re-entry. Fortunately,
Most of that heat is dissipated into the air flowing around the spacecraft. This heat diversion away from the spacecraft is mainly achieved by the action of strong shock waves, as shown below. For this reason, re-entry vehicles have blunt shapes that generate strong bow shocks. Despite this, a small portion of heat does always reach the spacecraft by heat conduction and radiation in shock layers and boundary layers, which can be however enough to destroy the spacecraft if it is not dealt with appropriately.

One way of absorbing the heat reaching the spacecraft is by using thick metal plates (heat sinks) that store the heat by increasing their temperature. As for instance, the beryllium panels used for the recovery canister of the Mercury space capsule (Fig. 107).

Other methods include heat rejection by materials with high emissivities that re-radiate the heat away from the spacecraft, or ablating materials that vaporize as they absorb the heat, such as the Avcoat 5026-39 (epoxy resin in a fiberglass matrix) that was used for the thermal shield of the Apollo Command Module.

Generally three figures of merit are important for characterizing the thermal load on a spacecraft: (i) the total heat input \( Q [J] \) absorbed during re-entry, (ii) the maximum time rate of spatially averaged heat input per unit area \( \dot{q}_M [W/m^2] \), and (iii) the maximum time rate of the local heat input in a sensitive location (e.g., nose region) \( \dot{q}_{local}[W/m^2] \).

These quantities are estimated in what follows for ballistic re-entry (Allen and Eggers, NASA TR 1381, 1958) and neglecting radiative heating (a good approximation for not too high velocities \( M < 25-30 \)).
THE LOCAL HEAT FLUX INTO THE SPACECRAFT CAN BE WRITTEN BY USING THE DEFINITION
OF THE STANTON NUMBER,

\[
\overline{q}'' = \frac{1}{S} \int q'' \, dS' = \frac{U^2}{S} \int \overline{\epsilon e} \, \overline{q_f} \, dS' = \frac{\rho_0 U^3}{4} \overline{q_f},
\]

WHERE \( \overline{q_f} \) IS A RECOVERY TEMPERATURE THAT FOR \( Pr = 1 \) IS GIVEN BY

\[
T_{aw} = T_{oe} = T_v \left( 1 + \left( \frac{x-1}{2} \right) M_{aw}^2 \right)
\]

(HERE THE SUBINDEX "oe" REFERS TO EDGE QUANTITIES). SINCE THE STAGNATION TEMPERATURE
IS CONSTANT ACROSS SHOCKS, IT IS INFERRED FROM (333) THAT \( Pr \) IS EQUAL TO THE
FREE-STREAM STAGNATION TEMPERATURE \( T_{aw} \) TO A GOOD APPROXIMATION,

\[
T_{aw} = T_{oo} = T_v \left( 1 + \left( \frac{x-1}{2} \right) M_{aw}^2 \right)
\]

(HERE THE SUBINDEX "oo" REFERS TO FREE-STREAM CONDITIONS). SUBTRACTING THE
LOCAL WALL TEMPERATURE \( T_w \) FROM (334) AND TAKING THE LIMIT \( M_{aw} \to 1 \) YIELDS

\[
\overline{T_{aw} - T_{oo}} = \frac{U^2}{2 \rho_c} \left( \frac{x-1}{2} \right) M_{aw}^2
\]

WHERE \( U = U_{oo} \) IS THE MAGNITUDE OF THE FLIGHT VELOCITY GIVEN BY (325).

SUBSTITUTING (335) INTO (332) GIVES

\[
\overline{q}'' = \frac{\overline{\epsilon e}}{2} \frac{U^2}{S} \overline{q_f}/2
\]

AND USING THE REYNOLDS ANALOGY \( \overline{S_t} = \overline{q_f}/2 \),

\[
\overline{q}'' = \frac{\overline{q_f}}{4} \frac{\overline{\epsilon e}}{2} \frac{U^2}{S} \overline{q_f}
\]

WHICH IS ANALOGOUS TO THE SCALING OBTAINED IN EQ. (229). THE SURFACE-AVERAGED
HEAT FLUX OVER THE TOTAL WETTED SURFACE \( S \) OF THE SPACECRAFT IS

\[
\overline{q}'' = \frac{1}{S} \int \overline{q}'' \, dS = \frac{U^2}{4S} \int \overline{\epsilon e} \, \overline{q_f} \, dS = \frac{\rho_0 U^3}{4} \overline{q_f}
\]

WHERE \( \overline{q_f} = \frac{1}{S} \int \overline{\epsilon e} \, \overline{q_f} \, dS \) IS A SURFACE AVERAGED SKIN FRICTION.

SUBSTITUTION OF THE SPACECRAFT VELOCITY (325) INTO (338) GIVES THE
EVOLUTION OF THE SURFACE-AVERAGED HEAT FLUX WITH ALTITUDE,
\[ \bar{q}'' = \frac{g_{0} e^{-\beta^{2}}}{4} \bar{q} f U_{E}^{3} e^{-\frac{3}{25}} e^{-\beta^{2}} \]  

\[ \text{SURFACE-AVERAGED HEAT FLUX INTO THE SPACECRAFT} \]

\[ Z_{2} = \frac{1}{\beta} \ln \left( \frac{3}{25} \right) = \frac{1}{\beta} \ln \left( \frac{3 g_{0} A_{1d}}{\beta m \sin \gamma_{E}} \right) = Z_{1} + \frac{1}{\beta} \ln \left( \frac{3}{25} \right) \]  

\[ Z \approx 2 + \text{km on Earth} \]

\[ \bar{q}''_{\text{MAX}} = \frac{g_{0} e^{-\beta^{2}}}{4 \varepsilon^{1}} \frac{Z_{2}}{3} = \frac{\beta}{6 \varepsilon^{1}} \left( \frac{\bar{q} f}{A_{1d}} \right) m U_{E}^{3} \sin \gamma_{E} \]

WHERE USE OF THE DEFINITION OF \( \varepsilon \) IN (326) HAS BEEN MADE.

THE SURFACE-AVERAGED HEAT FLUX (339) IS LINKED TO THE TOTAL HEAT INPUT RATE

\[ \frac{dQ}{dt} = \int_{s} \bar{q}'' \, ds = \bar{q}'' s = \frac{g_{0} e^{-\beta^{2}}}{4} \bar{q} f U_{E}^{3} e^{-\frac{3}{25}} e^{-\beta^{2}} \left[ \text{W} \right] \]

WHICH IS ALSO MAXIMUM AT \( Z = Z_{2} \). THE TOTAL HEAT INPUT CORRESPONDS TO THE INTEGRAL OF (342) DURING THE ELLIPSICAL TIME. IT IS HOWEVER MORE CONVENIENT TO PERFORM THE INTEGRATION IN \( Z \) INSTEAD BY USING \( \frac{dQ}{dt} = \frac{dQ}{dz} \frac{dz}{dt} \)

\[ Q = -\frac{g_{0} e^{-\beta^{2}}}{4 \sin \gamma_{E}} U_{E}^{2} \int_{0}^{+\infty} e^{-\frac{1}{5}} e^{-\beta^{2}} e^{-\beta^{2}} \, dz = \frac{1}{4} \left( \frac{\bar{q} f}{A_{1d}} \right) m U_{E}^{2} (1 - \varepsilon_{S}^{1}) \]

TOTAL HEAT INPUT INTO THE SPACECRAFT DURING BALLISTIC RE-ENTRY

\[ \int_{0}^{+\infty} e^{-\frac{1}{5}} e^{-\beta^{2}} \, dz = -\frac{5}{\beta} \left( 1 - \varepsilon_{S}^{1} \right) \]

LASTLY, THE HEAT FLUX ON THE BLUNT NOSE OF THE SPACECRAFT IS GIVEN BY

\[ (249) \text{rewritten as} \]

\[ \bar{q}''_{\text{NOSE}} = \frac{2^{3/4} \text{const.} \left( \frac{M_{\infty}}{R_{0}} \right)^{1/2} U_{S/2}^{5/2}}{\varepsilon^{1/4}} \frac{\text{const.} \left( \frac{g_{0} R_{0}}{M_{\infty}} \right)^{1/2}}{\varepsilon^{1/4}} U_{S/2}^{5/2} = \frac{1}{2 \varepsilon^{1/4}} \left( \frac{M_{\infty}}{R_{0}} \right)^{1/2} \]

\[ \text{WHERE} \quad M_{\infty} \text{HAS BEEN APPROXIMATED AS} \quad \frac{M_{\infty}}{M_{\infty}} \approx \left( \frac{1}{T_{\infty}} \right)^{1/2} \quad \text{WITH} \quad T_{\infty} \quad \text{THE APPROXIMATE POST-SHOCK STATIC TEMPERATURE} \quad T_{\infty}, \quad \text{AND} \quad \frac{T_{\infty}}{T_{\infty}} \approx \frac{U_{E}^{2}}{2 c_{p E}} \text{AT HIGH MACH NUMBERS} \quad M_{\infty} \gg 1 \]
SUBSTITUTING THE SPACECRAFT VELOCITY \((325)\) AND \(g_0\) INTO \((343)\) YIELDS

\[
q_{\text{nose}} = \frac{\text{Const}}{2\varepsilon^4} \left[ \frac{L^2}{(Cp T_0)^{1/2} R_0} \right] \left( \frac{E^2}{U_E^2} \right)^{3/2} e^{-\frac{3}{2} \frac{E}{U_E}} e^{-\frac{1}{2} \frac{E}{U}} \]  

\[\text{[W/m}^2]\text{] AT (344) HEAT FLUX ON THE BLUNT NOSE

\[2 = \frac{1}{\beta} \ln \left( \frac{3}{5} \right) = \frac{1}{\beta} \ln \left( \frac{3.8G D A}{\beta m \sin \gamma_E} \right) = 2.1 + \frac{1}{\beta} \ln 3 \]  

\[(345)\]

\[\approx 7.4 \text{ km} \text{ ON EARTH}

WHERE \(q_{\text{nose}}\) BECOMES MAXIMUM AND EQUAL TO

\[
q_{\text{nose max}} = \frac{\text{Const}}{2\varepsilon^4} \left( \frac{\beta m \kappa_0}{3EAG D (Cp T_0)^{1/2} R_0} \right)^{1/2} U_E^3
\]

\[(346)\]  

IN EXAMINING THE ABOVE RESULTS IT IS WORTH HIGHLIGHTING THE FOLLOWING ASPECTS:

1) THE ALTITUDES FOR MAXIMUM SURFACE-AVERAGED HEAT FLUX \((\bar{Q}_2)\) AND MAXIMUM HEAT FLUX ON THE BLUNT NOSE \((\bar{Q}_3)\) DO NOT DEPEND ON THE ENTRY VELOCITY. HOWEVER, THEY BOTH INCREASE AS THE ENTRY ANGLE BECOMES SMALLER.

2) THE MAXIMUM SURFACE-AVERAGED HEAT FLUX \(q_{\text{max}}\) AND THE MAXIMUM HEAT FLUX ON THE BLUNT NOSE \(q_{\text{nose max}}\) DECREASE AS THE ENTRY ANGLE DECREASES. THIS, TOGETHER WITH A SIMILAR OBSERVATION MADE FOR THE MAXIMUM DECELERATION \(a_{\text{max}}/g\), SUGGEST THAT SHALLOW ENTRIES ARE PREFERABLE FOR MANNED MISSIONS ON BALLISTIC TRAJECTORIES, FOR WHICH LOW DECELERATION AND HEATING RATES ARE DESIRABLE.

3) IN CONTRAST, SHALLOW ENTRIES TEND TO DECREASE \(\bar{Q}\), AND AS A RESULT THE TOTAL HEAT INPUT \(Q\) INCREASES DUE TO THE LONGER RE-ENTRY TIME.

4) WHILE \(\bar{Q}_{\text{max}}\) AND \(q_{\text{nose max}}\) PROVIDE INFORMATION ABOUT THE TYPE OF MATERIAL NECESSARY FOR THE TYPE OF MATERIAL OF THE THERMAL PROTECTION SYSTEM, THE TOTAL HEAT INPUT \(Q\) PROVIDES INFORMATION ABOUT THE MATERIAL VOLUME REQUIRED, THEREBY INDICATING THAT SHALLOW ENTRIES REQUIRE BULKY THERMAL PROTECTION SYSTEM.
If relatively heat unresistant materials that can be easily ablated, while steep entries require thin heat shields of highly heat-resistant materials.

Steep entry: high \( \dot{q} \)\(_{\text{max}} \), high \( \dot{q} \)\(_{\text{nose}} \), low \( \dot{q} \)\(_{\text{MAX}} \)

Shallow entry: low \( \dot{q} \)\(_{\text{max}} \), low \( \dot{q} \)\(_{\text{nose}} \), high \( \dot{q} \)\(_{\text{MAX}} \)

5) The critical altitudes for maximum deceleration \((z_1)\) and maximum heat fluxes \((z_2)\) and \((z_3)\) are sequential

\[
\begin{align*}
z_3 & > z_2 & > z_1 \\
\text{Maximum } \dot{q} \text{\(_{nose}\)} & > \text{Maximum } \dot{q} \text{\(_{max}\)} & > \text{Maximum } \dot{q} \text{\(_{dt}\)}
\end{align*}
\]

\[
\begin{align*}
7.4 \text{ km} & > 2.7 \text{ km}
\end{align*}
\]

6) Lastly, and most importantly, the expression (342) for the total heat input into the spacecraft quantifies the fraction of the initial kinetic energy that has been transformed into heat that reaches the spacecraft. To see this, note that (342) can be rewritten using the impact velocity (331) as

\[
Q = \frac{1}{2} \left( \frac{c_{E} S}{g_{oA}} \right) M \left( \frac{U^{2}_{E} - U^{2}_{Z}}{2} \right)
\]

(342)

indicating that a fraction equal to \( \frac{1}{2} \left( \frac{c_{E} S}{g_{oA}} \right) \) of the total kinetic energy lost into heat has been absorbed as heat into the spacecraft structure. It is for this reason that, for the same wetted area and same \( \frac{c_{E}}{g_{oA}} \), blunt designs (i.e., \( g_{oA} \) large) are employed for minimizing the total amount of heat absorbed, while slender designs (i.e., \( g_{oA} \) small) are used for rapid penetration at the cost of high thermal loads (including high \( \dot{q} \)\(_{nose}\)). These considerations are depicted in Fig. 108 and serve as closure for this course.

![Diagram showing the comparison between blunt and sharp noses in terms of heat dissipation and flow conditions.](Fig. 108)