ME 356: Hypersonic Aerothermodynamics, Spring 2018
Stanford University
Midterm Exam
Tuesday, May 8

Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student’s Name: JAVIER UREAY .................................. Student’s ID: ..................................

PART I: Closed notes, calculators allowed, compressible-flow tables allowed
Time: 20 mins

Question (40 pts)
1. The velocity/altitude diagram below shows three different re-entry trajectories across the terrestrial atmosphere.

a) (20 pts) Determine what type of aerospace vehicle (intercontinental ballistic missile, manned re-entry winged vehicle, or experimental hypersonic cruise aircraft) is most likely associated with each trajectory and justify your response.

b) (20 pts) Sketch on the diagram the approximate boundaries for the onset of the characteristic thermochemical effects arising at hypersonic flight speeds.

Notes: 1 kft = 304.8 m; 1 kft/s = 304.8 m/s = 681.8 mph = 1097.3 km/h; speed of sound at 35 km altitude ~ 1 kft/s; temperature at 35 km altitude ~ 236 K.

\[\text{For more details in the solution, see pages 5 and 6 in the class notes.}\]
ME 356: Hypersonic Aerothermodynamics, Spring 2018
Stanford University
Midterm Exam
Tuesday, May 10

Guidelines: Please turn in neat and clean exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student’s Name: TAVIER URBAY Student’s ID: ...........................................

PART II: Open notes, calculators allowed, compressible-flow tables allowed
Time: 60 mins

Problem 1 (40 pts)

a) (30 pts) Using the straight Newtonian theory of hypersonic flows, in which the shock layer is infinitesimally thin, obtain expressions for the drag and lift coefficients on the sphere of radius $R_p$ sketched in Fig. 1 (use the spherical coordinate system shown in the figure to perform the corresponding integrations of the the surface pressure coefficient $C_p$ along the spherical surface).

b) (10 pts) The experimental drag coefficient measured by Hodges (1957) corresponds to $C_D = 0.91$. Compare this value to the one obtained in part (a) and identify possible sources of discrepancy between both results.

![Diagram of hypersonic flow over a sphere](image)

Figure 1: Hypersonic flow over a sphere in the Newtonian theory.
SOLUTION

a) BY SYMMETRY: $\zeta_L = 0$

For $\zeta_D$ one needs to integrate $\zeta_p = 2\sin^2 \theta$ over the element of surface projected on the z direction and divide it by the frontal area in the z direction.

\[\zeta_D = \frac{2 \pi \sin \theta}{4} \int_0^{\pi/2} \frac{R_p^2 \sin \theta \cos \theta d\theta d\psi}{2 \pi R_p^2 \cos \theta} = \frac{4 \pi R_p^2 \cos \theta}{4} \left[ \int_0^{\pi/2} \frac{R_p^2 \sin \theta \cos \theta d\theta}{2 \pi R_p^2 \sin \theta} \right] = \frac{\pi R_p^2}{2 \pi R_p^2} \left[ \frac{\pi/2}{2} \right] = \frac{1}{2}

b) $\zeta_D$ EXPERIMENTAL $= 0.91 < \zeta_D$ NEWTON = D AS EXPECTED SINCE THE STRAIGHT NEWTONIAN THEORY OVERPREDICTS THE PRESSURE IN THE FORE STAGNATION POINT (SEE DISCUSSION ON PAGES 19-20 OF THE CLASS NOTES).
Problem 2 (20 pts)

Consider the flow schematically illustrated in Fig. 2, where a high-speed gaseous stream at velocity $U_1$, pressure $P_1$ and density $\rho_1$ crosses a two-dimensional stationary curved shock wave. The velocity and thermodynamic variables of the pre-shock gases, including the entropy $s$, are uniform. As a result, in the pre-shock gases, Crocco’s equation

$$T\nabla s = \omega \times \mathbf{u},$$

(1)

requires that the vorticity $\omega$ be zero there. In Eq. (1), $\mathbf{u}$ is the velocity vector.

Since the shock wave is curved, different streamlines experience different jumps of entropy across the shock. As a result, the post-shock gas has constant entropy along the streamlines but the entropy varies from streamline to streamline, thereby rendering an isentropic flow. Because of Crocco’s equation (1), however, the post-shock gas is rotational.

The goal of this problem is to obtain an expression for the vorticity $\omega_2$ in the post-shock gas immediately behind the shock in the hypersonic limit. To this end, follow the steps below.

Figure 2: Hypersonic flow across a two-dimensional curved shock wave.

a) (4 pts) Obtain an expression for the derivative of the post-shock pressure $P_2$ with respect to the local incidence angle $\beta$ using the Rayleigh line

$$\frac{P_2 - P_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} = -\dot{m}n^2,$$

(2)

where $\dot{m}n'' = \rho_1 U_1 \sin \beta$. 

ME 356 Hypersonic Aerothermodynamics 3
b) (4 pts) Obtain an expression for the derivative of the post-shock static enthalpy $h_2$ with respect to the local incidence angle $\beta$ using the Hugoniot equation

$$h_2 - h_1 = \frac{P_2 - P_1}{2\rho_1} \left(1 + \frac{\rho_1}{\rho_2}\right).$$

(3)

c) (4 pts) Utilize the expressions for $dP_2/d\beta$ and $dh_2/d\beta$ derived above, along with first principle of thermodynamics $T_2ds_2 = dh_2 - dP_2/\rho_2$ in the post-shock gas, and show that the derivative of the post-shock entropy $s_2$ with respect to the local incidence angle $\beta$ of the shock is given by

$$T_2 \frac{ds_2}{d\beta} = U_\infty^2 (1 - \epsilon^2) \sin \beta \cos \beta,$$

(4)

where $\epsilon = \rho_1/\rho_2$ is the density ratio across the shock.

d) (4 pts) Crocco’s equation (1) states that the gradient of the post-shock entropy across the streamlines is related to the cross product of the post-shock velocity ($U_2$) and vorticity ($\omega_2$), namely

$$T_2 |\nabla s_2| = \omega_2 U_2.$$  

(5)

Equivalently, since the post-shock flow is isentropic, $\nabla s_2$ is perpendicular to the streamlines, and correspondingly the derivative of the post-shock entropy along the direction $\hat{s}$ locally parallel to the shock is

$$T_2 \frac{ds_2}{d\hat{s}} = T_2 |\nabla s_2| \sin(\beta - \delta),$$

(6)

where $\delta$ is the local deflection angle of the streamlines, as shown in Fig. 2.

Using Eqs. (4)-(6), show that the vorticity immediately behind the shock is given by the expression

$$\omega_2 = U_\infty \frac{(1 - \epsilon^2)}{\epsilon} \cos \beta \frac{d\beta}{d\hat{s}}$$

(7)

where the direction of the vorticity points into the plane in Fig. 2 when $d\beta/d\hat{s} < 0$ (as in the upper portion of a bow shock around a blunt body). Note that 1) the derivation of Eq. (7) does not require the gas to be calorically perfect, and 2) the factor $d\beta/d\hat{s}$ represents the inverse of the radius of curvature of the shock front. In this way, the post-shock vorticity becomes zero either for $\beta = \pi/2$ (i.e., normal shock) or for $d\beta/d\hat{s} = 0$ (i.e., straight shock), as anticipated in class.

e) (4 pts) Simplify expression (7) for a calorically-perfect gas at hypersonic speeds $Ma^2_1 \sin^2 \beta \gg 1$, for which the post-shock density $\rho_2$ is uniform and the density ratio $\epsilon$ becomes a constant that is a sole function of the adiabatic coefficient $\gamma$. 

ME 356 Hypersonic Aerothermodynamics  4
\[ a) \quad \frac{P_2 - P_1}{\bar{s}_2 \bar{s}_1} = -\dot{m}^2 \Rightarrow P_2 - P_1 = \dot{m}^2 \left( \frac{1 - \frac{\dot{m}^2}{s_1}}{\bar{s}_1} \right) = \dot{m}^2 \left( 1 - \frac{\dot{m}^2}{\bar{s}_1} \right) = \bar{s}_2 \left( 1 - \varepsilon \right) \]

WITH \( \varepsilon = \frac{s_1}{s_2} \) AND \( \dot{m}^2 = \bar{s}_1 U_1 \sin \beta \)

THEN \[ \frac{d\bar{s}_2}{d\beta} = -\frac{\dot{m}^2}{s_1} \frac{d\varepsilon}{d\beta} + \frac{2\dot{m}^2}{s_1} (1 - \varepsilon) \frac{d\dot{m}^2}{d\beta} = \]

\[ = -\bar{s}_2 (1 - \varepsilon) \frac{d\varepsilon}{d\beta} + 2 \bar{s}_1 U_1^2 (1 - \varepsilon) \sin \beta \cos \beta \]

\[ b) \quad h_2 - h_1 = \frac{P_2 - P_1}{2 \bar{s}_1} (1 + \varepsilon) \Rightarrow \frac{dh_2}{d\beta} = \frac{(1 + \varepsilon)}{2 \bar{s}_1} \frac{dP_2}{d\beta} + \frac{(P_2 - P_1)}{2 \bar{s}_1} \frac{d\varepsilon}{d\beta} = \]

\[ = -\frac{U_1^2 \sin^2 \beta (1 + \varepsilon)}{2} \frac{d\varepsilon}{d\beta} + U_1^2 \sin \beta \cos \beta (1 + \varepsilon) (1 - \varepsilon) + \]

\[ + \frac{U_1^2 \sin^2 \beta (1 - \varepsilon)}{2} \frac{d\varepsilon}{d\beta} \]

\[ c) \quad T_2 \frac{ds_2}{d\beta} = \frac{dh_2}{d\beta} - \frac{1}{s_2} \frac{dP_2}{d\beta} = \frac{dh_2}{d\beta} - \frac{\varepsilon}{\bar{s}_1} \frac{dP_2}{d\beta} = U_1^2 \sin \beta \cos \beta (1 - \varepsilon)^2 \]

\[ d) \quad \text{NOTE THAT} \quad U_2 W_2 = \frac{U_2}{U_1} \frac{L_{12}}{L_{1b}} \frac{1}{W_{1b}} \]

\[ = \frac{U_1 \cos \beta}{\cos (\beta - \delta)} \quad w_2 \]

AND THEREFORE \[ T_2 \frac{ds_2}{d\beta} = \frac{T_2}{\sin (\beta - \delta)} \frac{ds_2}{d\beta} = + \frac{U_1 \cos \beta}{\cos (\beta - \delta)} \quad w_2 \]

ALSO \[ T_2 \frac{ds_2}{d\beta} = T_2 \frac{ds_2}{d\beta} = \frac{U_1^2 \sin \beta \cos \beta (1 - \varepsilon)^2}{ds_2} \]

COMBINING THESE TWO EQUATIONS \[ w_2 = \frac{U_1 \sin \beta (1 - \varepsilon)^2}{\tan (\beta - \delta)} \frac{d\beta}{ds} \]

AND SINCE \[ \tan (\beta - \delta) = \varepsilon \tan \beta \], THEN \[ w_2 = \frac{U_1 (1 - \varepsilon)^2 \cos \beta \frac{d\beta}{ds}}{\varepsilon} \]

\[ e) \quad \text{IN THE HYPERSONIC LIMIT AND FOR A CALORICALLY PERFECT GAS,} \quad \varepsilon = (y - 1)/(y + 1) \]

\[ \Rightarrow w_2 = \frac{4 \frac{U_1^2}{(y + 1)} \cos \beta}{(y - 1)} \frac{d\beta}{ds} \]