STANFORD 2017
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ME 451C: COMPRESSIBLE TURBULENCE

I. FUNDAMENTAL ASPECTS OF COMPRESSIBLE FLOWS

1. CONSERVATION EQUATIONS

THIS FIRST SECTION INTRODUCES CONCEPTS USED THROUGHOUT THE REST OF THESE NOTES AND IMPORTANT HYPOTHESES FOR THE DERIVATION OF THE CONSERVATION EQUATIONS. THE DESCRIPTION BEGINS BY INTRODUCING A SUMMARY OF THE CONTINUUM HYPOTHESIS TO THEN TRANSITION INTO THE THERMODYNAMICS OF IDEAL GASES AND CONSERVATION BALANCE EQUATIONS THAT INCORPORATE TRANSPORT AS REQUIRED FOR TURBULENT FLOWS.

THE CONTINUUM LIMIT

THE NAVIER-STOKES EQUATIONS ARE BASED ON A "FIELD FORMULATION" OF THE FLUID VARIABLES $\phi(\vec{x},t)$, WHERE ϕ AND ITS GRADIENTS ARE CONTINUOUS IN SPACE. THIS IS A GOOD APPROXIMATION WHEN THE MEAN FREE PATH BETWEEN MOLECULAR COLLISIONS λ IS MUCH SMALLER THAN THE CHARACTERISTIC LENGTH OF THE FLOW L, OR EQUIVALENTLY, WHEN

$$Kn = \frac{\lambda}{L} \ll 1 \qquad (1)$$

WHERE Kn IS THE KNUDSEN NUMBER. FOR INSTANCE, CONSIDER A GAS OF MOLECULES OF DIAMETER d. A MOLECULE LABELLED "A" WILL THEN COLLIDE WITH OTHER MOLECULES IF THEIR CENTERS PENETRATE INTO A SPHERE OF RADIUS d CENTERED AT "A". ASSUMING THAT THE MOLECULE MOVES AT MEAN THERMODYNAMIC SPEED $\bar{c}$, IT SWEEPS OUT A VOLUME $\pi d^2 \bar{c}$ PER UNIT TIME. IF n IS THE NUMBER OF MOLECULES PER UNIT VOLUME, THEN $\mathcal{N} = \pi d^2 \bar{c} n$ IS THE NUMBER OF COLLISIONS PER UNIT TIME AND

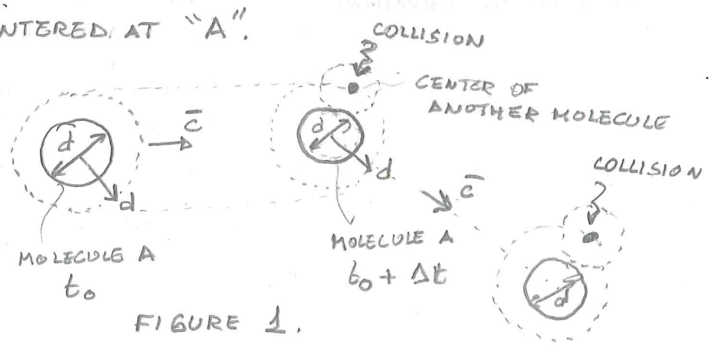


FIGURE 1.

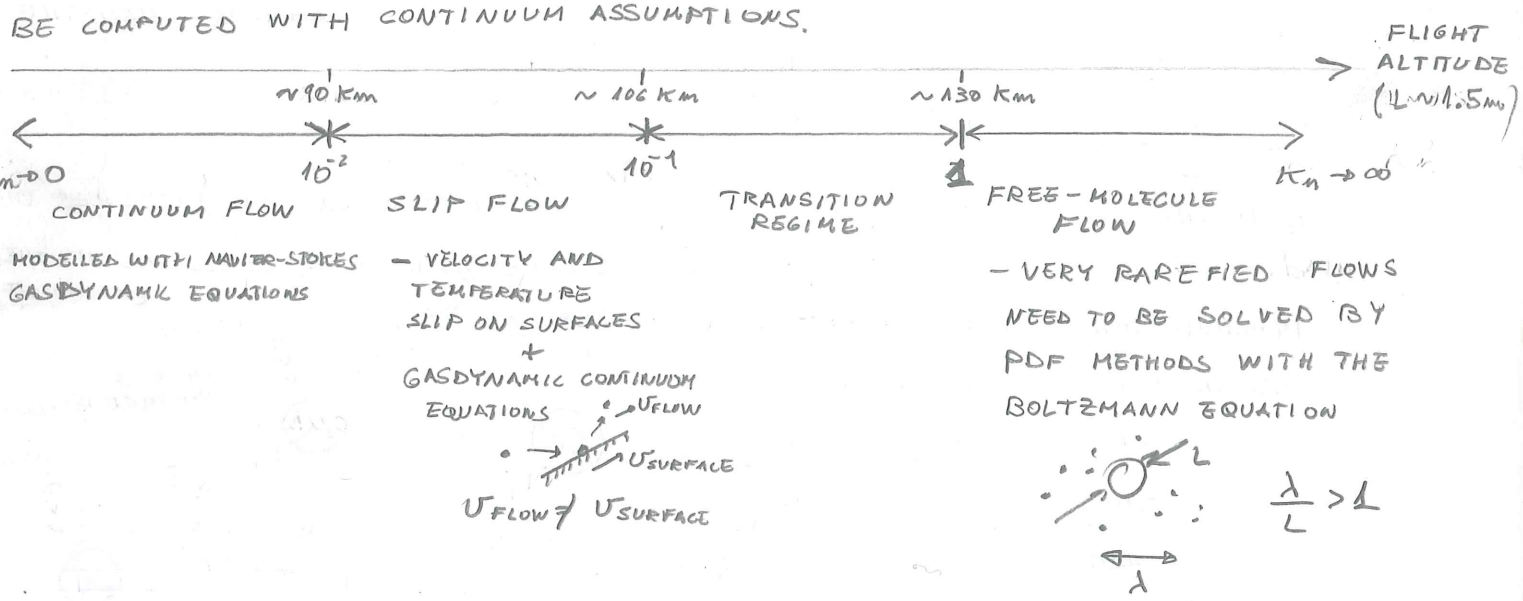
$$\lambda = \frac{\bar{c}}{\mathcal{N}} = \frac{1}{\sqrt{2}\pi n d^2} \qquad (2)$$

IS THE MEAN FREE PATH, WHERE THE FACTOR $\sqrt{2}$ IN THE DENOMINATOR ARISES FROM THE FACT THAT $\bar{c}$ IS NOT THE RELEVANT SPEED TO CONSIDER, BUT RATHER $\sqrt{2}\bar{c}$ WHICH IS THE MEAN RELATIVE SPEED OF THE MOLECULES (e.g., SEE EXACT DERIVATION IN PP. 48-54 OF VINCENTI + KRUGER'S (2002) "INTRO TO PHYSICAL GAS DYNAMICS". SINCE $n = \rho/m$, WHERE ρ IS THE GAS DENSITY AND m IS THE MASS OF 1 MOLECULE, THEN $\lambda = m/(\sqrt{2}\pi d^2 \rho)$, SHOWING THAT λ IS INVERSELY PROPORTIONAL TO DENSITY. FOR INSTANCE, FOR AIR AT SEA LEVEL

$T = 298\ K$
$P = 101325\ Pa$
$m = \frac{W}{N_A} = \frac{28.9 \cdot 10^{-3}\ kg/mol}{6.022 \cdot 10^{23}\ molecules/mol} = 4.79 \cdot 10^{-26}\ kg/molecule$
$d \sim d_{N_2} \sim 390 \cdot 10^{-12}\ m$
$R_g = R^0/W = 8.31\ J/mol\,K\ /\ 28.9 \cdot 10^{-3}\ kg/mol = 286.8\ J/kg\,K$
$\rho = P/R_g T = 1.18\ kg/m^3 \rightarrow n = \rho/m = 2.4 \cdot 10^{25}\ molecules/m^3$

$\lambda \approx 60 \cdot 10^{-9}\ m$
$\mathcal{N} \approx 10^{10}$ COLLISIONS/s
$\delta_I \sim n_0^{-1/3} \sim 3.4 \cdot 10^{-9}\ m$ (MEAN INTERMOLECULAR DISTANCE)

THE DEPENDENCE OF λ ON ρ MAKES IT A STRONG FUNCTION OF THE ALTITUDE h. SINCE $\rho/\rho_A \sim e^{-h/\kappa}$, WITH $\rho_A = 1.18\ kg/m^3$ THE SEA-LEVEL DENSITY AND $\kappa \sim 7200\ m$, THEN $\lambda/\lambda_A \sim e^{h/\kappa}$, WITH $\lambda_A = 60\cdot10^{-9}\ m$ THE SEA-LEVEL MEAN FREE PATH. AS A RESULT, $Kn = \frac{\lambda_A}{L} e^{h/\kappa}$ INCREASES EXPONENTIALLY WITH ALTITUDE. IN PARTICULAR, THERE IS A CRITICAL ALTITUDE $h_0 = \kappa \ln\left(\frac{L}{\lambda_A}\right)$ FOR WHICH $Kn = 1$ AND THE FLOW BECOMES STRONGLY RAREFIED ABOVE IT. FOR INSTANCE, FOR THE SPACE SHUTTLE ORBITER, THE RADIUS OF THE NOSE CONE IS $L/2 \sim 0.7926\ m$. CORRESPONDINGLY, THE FLOW AROUND IT BECOMES STRONGLY RAREFIED FOR FLIGHT ALTITUDES ABOVE $h_0 = 7200 \ln\left(\frac{2\cdot0.7926}{60\cdot10^{-9}}\right) \approx 123\ km$ AND CANNOT BE COMPUTED WITH CONTINUUM ASSUMPTIONS.



NOTE ALSO THAT $\mu \sim \frac{1}{2}\rho \bar{c} \lambda$ FROM KINETIC THEORY, SO THAT (a = SPEED OF SOUND)

$$Kn = \frac{\lambda}{\ell} \sim \frac{\mu}{\rho \bar{c} \ell} \sim \frac{\mu}{\rho u_\ell \ell} \frac{u_\ell}{\bar{c}} \sim \frac{Ma_\ell}{Re_\ell},$$

WHERE $Ma_\ell = u_\ell/a$ IS THE TURBULENT MACH NUMBER, AND $Re_\ell = \frac{\rho u_\ell \ell}{\mu}$ IS THE REYNOLDS NUMBER OF THE LARGE EDDIES OF SIZE ℓ. AS A RESULT,

$$\frac{\lambda}{\ell} \sim \frac{\lambda}{\ell_K} \underbrace{\frac{\ell_K}{\ell}}_{Re_\ell^{-3/4}} = \frac{Ma_\ell}{Re_\ell} \Rightarrow \boxed{\frac{\lambda}{\ell_K} \sim \frac{Ma_\ell}{Re_\ell^{1/4}}} \quad (3)$$

IS THE RATIO OF THE MEAN FREE PATH TO THE SIZE OF THE KOLMOGOROV EDDIES ℓ_K. MORE OFTEN THAN NOT, Re_ℓ IS LARGE ENOUGH AND Ma_ℓ IS SMALL ENOUGH SUCH THAT $Re_\ell \gg Ma_\ell^4$ AND $\lambda/\ell_K \ll 1$ IN MOST PRACTICAL APPLICATIONS.

THERMODYNAMICS OF IDEAL GASES

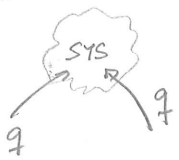
* FIRST LAW OF THERMODYNAMICS: $\boxed{de = dq - P d\left(\frac{1}{\rho}\right)}$ (4)

$\frac{1}{\rho} = v$ SPECIFIC VOLUME

e = SPECIFIC INTERNAL ENERGY

q = HEAT $\begin{cases} dq > 0 & \text{(INTO THE SYSTEM)} \\ dq < 0 & \text{(TO THE SURROUNDINGS)} \end{cases}$

SURROUNDINGS



IF ρ=CONST, AND $dq > 0$ (HEAT FLOWS INTO THE SYSTEM, THEN $de > 0$ (THE INTERNAL ENERGY OF THE SYSTEM INCREASES)

* SPECIFIC HEAT CAPACITIES: - AT CONSTANT PRESSURE: $c_p = \left(\frac{dq}{dT}\right)_p$

- AT CONSTANT VOLUME: $c_v = \left(\frac{dq}{dT}\right)_\rho$

} "THERMODYNAMIC COEFFICIENTS"

* IDEAL GAS EQUATION: $\boxed{P = \rho R_g T}$ (5), $R_g = \frac{R^0}{W} \rightarrow 8{,}31\ J/mol\,K$ (UNIVERSAL GAS CONSTANT)

R_g: GAS CONSTANT IN $J/kg\,K$; W: MOLECULAR WEIGHT

⇒ VALID AS LONG AS THE PRESSURE IS NOT TOO LARGE AND THE TEMPERATURE IS NOT TOO LOW, SUCH THAT THE ATTRACTION BETWEEN MOLECULES IS NEGLIGIBLE. EQUIVALENTLY, AN IDEAL GAS IS ONE IN WHICH THE DEGREES OF FREEDOM OF THE MOLECULES ARE UNCOUPLED, WITH THE TRANSLATIONAL PARTITION FUNCTION BEING THE ONLY CONTRIBUTOR TO THE PRESSURE. NOTE THAT AN IDEAL GAS MAY STILL BEAR MOLECULES WITH INTERNAL STRUCTURE SUCH AS DIATOMIC OR NON ELECTRONEUTRAL MONOATOMIC MOLECULES. IN THAT CASE, ENERGY DIFFERENT FROM PURELY TRANSLATIONAL ONE COMES INTO PLAY. THIS INCLUDES ROTATIONAL, VIBRATIONAL AND ELECTRONIC EXCITATION ENERGIES, WHICH MUST BE DECOUPLED FOR IDEALITY TO PERSIST. THE LAST TWO ARE TYPICALLY MANIFESTED AS NON-CALORICALLY-PERFECT BEHAVIOR BY WHICH THE SPECIFIC HEATS DEPEND ON TEMPERATURE AND THE INTERNAL ENERGY IS NO LONGER A LINEAR FUNCTION OF THE TEMPERATURE.

FOR AN IDEAL GAS, THE INTERNAL ENERGY IS A SOLE FUNCTION OF THE TEMPERATURE, $e = e(T)$, AND THEREFORE $\left(\frac{\partial e}{\partial \rho}\right)_T = 0$, OR EQUIVALENTLY, $de = \left(\frac{\partial e}{\partial T}\right)_\rho dT$, SUBSTITUTING THIS INTO (4): $\left(\frac{\partial e}{\partial T}\right)_\rho dT = dq - P d\left(\frac{1}{\rho}\right)$, AND SIMPLIFYING FOR CONSTANT ρ:

$\left(\frac{\partial e}{\partial T}\right)_\rho = \left(\frac{dq}{dT}\right)_\rho = c_v \Rightarrow \boxed{e = c_v T}$ (6) INTERNAL ENERGY OF AN IDEAL GAS (c_v = CONST AS IN A CALORICALLY PERFECT GAS)

SIMILARLY, AT CONSTANT P: $\underbrace{\left(\frac{\partial e}{\partial T}\right)_\rho}_{c_v} = \underbrace{\left(\frac{dq}{dT}\right)_P}_{c_p} + \underbrace{\frac{P}{\rho^2}\left(\frac{d\rho}{dT}\right)_P}_{\frac{R_g T}{\rho}\left(-\frac{P}{R_g T^2}\right)} \Rightarrow \boxed{c_p - c_v = R_g}$ (7)

$\hookrightarrow \boxed{\gamma = \frac{c_p}{c_v} = \frac{1 + \frac{R_g}{c_v}}{1 - R_g/c_p}}$ (8) ADIABATIC COEFFICIENT

UNLESS STATED OTHERWISE, THE REST OF THIS COURSE WILL CONSIDER ONLY IDEAL AND CALORICALLY PERFECT GASES:

- FOR MONOATOMIC GASES (Ar, He, Ne): $C_v = \frac{3}{2} R_g$, $C_p = \frac{5}{2} R_g$, $\gamma = \frac{5}{3}$
- FOR DIATOMIC GASES (N_2, O_2): $C_v = \frac{5}{2} R_g$, $C_p = \frac{7}{2} R_g$, $\gamma = \frac{7}{5}$

} OBTAINED FROM THE KINETIC THEORY OF GASES.

⇒ FOR AIR: $\gamma = 1.4$, $C_p = 1\ kJ/kg$

* ADIABATIC PROCESSES:

USING THE EQUATIONS ABOVE, THE FIRST LAW CAN BE WRITTEN AS $\dot{d}q = C_v dT - \frac{P}{\rho^2} d\rho = C_p dT - \frac{dP}{\rho}$

FOR AN ADIABATIC PROCESS, THERE IS NO EXCHANGE OF HEAT WITH THE SURROUNDINGS, SO THAT

$\dot{d}q = 0$ AND $\left.\begin{aligned} C_v dT &= \frac{P}{\rho^2} d\rho \\ C_p dT &= \frac{dP}{\rho} \end{aligned}\right\} \quad \frac{C_p}{C_v} = \frac{\rho}{P}\frac{dP}{d\rho} = \gamma \Rightarrow \frac{dP}{d\rho} = \frac{\gamma P}{\rho} \Rightarrow \int_{P_0}^{P} \frac{dP}{P} = \int_{\rho_0}^{\rho} \frac{\gamma\, d\rho}{\rho}$

WHICH GIVES $\boxed{\frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} = \text{CONST.}}$ ADIABATIC TRAJECTORY. (9)

* ENTROPY

$ds = \frac{\dot{d}q}{T} \Rightarrow s_2 - s_1 = \int_1^2 \frac{\dot{d}q}{T}$ ENTROPY (WHEN THE PROCESS IS REVERSIBLE)

SUBSTITUTING THE EXPRESSIONS FOR $\dot{d}q$ IN AN IDEAL GAS:

$$ds = C_v \frac{dT}{T} - R_g \frac{d\rho}{\rho} \Rightarrow s - s_0 = C_v \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{\rho}{\rho_0}\right)$$

$$ds = C_p \frac{dT}{T} - R_g \frac{dP}{P} \Rightarrow s - s_0 = C_p \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{P}{P_0}\right)$$

NOTE THAT $s = s_0$ IN THE ABOVE EQUATIONS IMPLIES $\ln\left(\frac{P}{P_0}\right) = \gamma \ln\left(\frac{\rho}{\rho_0}\right) \Rightarrow \frac{P}{\rho^\gamma} = \frac{P_0}{\rho_0^\gamma} = \text{CONST.}$

IS ALSO AN ISENTROPIC TRAJECTORY

IN GENERAL: $s - s_0 = C_v \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{\rho}{\rho_0}\right) = C_p \ln\left(\frac{T}{T_0}\right) - R_g \ln\left(\frac{P}{P_0}\right)$

OR EQUIVALENTLY: $\boxed{s - s_0 = C_v \ln\left(\frac{P}{\rho^\gamma} \Big/ \frac{P_0}{\rho_0^\gamma}\right)}$ (10)

CONSERVATION EQUATIONS OF COMPRESSIBLE FLOWS

THE SET OF CONSERVATION EQUATIONS THAT WILL BE CONSIDERED IN THIS COURSE IS

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \qquad \text{MASS CONSERVATION} \quad (11)$$

$$\rho\frac{\partial \vec{v}}{\partial t} + \rho\vec{v}\cdot\nabla\vec{v} = -\nabla P + \nabla\cdot\bar{\bar{\tau}}' \qquad \text{MOMENTUM CONSERVATION} \quad (12)$$

$$\rho\frac{\partial}{\partial t}\left(h + \frac{|\vec{v}|^2}{2}\right) + \rho\vec{v}\cdot\nabla\left(h + \frac{|\vec{v}|^2}{2}\right) = \frac{\partial P}{\partial t} + \nabla\cdot(\bar{\bar{\tau}}'\cdot\vec{v}) - \nabla\cdot\vec{q}'' \qquad \text{TOTAL ENTHALPY CONSERVATION} \quad (13)$$

WITH $h = C_p T$

OR ALTERNATIVELY, (13) CAN BE REPLACED BY ANY OF THE FOLLOWING EQUATIONS:

$$\rho\frac{\partial}{\partial t}\left(e + \frac{|\vec{v}|^2}{2}\right) + \rho\vec{v}\cdot\nabla\left(e + \frac{|\vec{v}|^2}{2}\right) = -\nabla\cdot(P\vec{v}) + \nabla\cdot(\bar{\bar{\tau}}'\cdot\vec{v}) - \nabla\cdot\vec{q}'' \qquad \text{TOTAL ENERGY CONSERVATION} \quad (14)$$

$$\rho\frac{\partial e}{\partial t} + \rho\vec{v}\cdot\nabla e = -P\nabla\cdot\vec{v} + \bar{\bar{\tau}}' : \nabla\vec{v} - \nabla\cdot\vec{q}'' \qquad \text{INTERNAL ENERGY CONSERVATION} \quad (15)$$

$$\rho\frac{\partial h}{\partial t} + \rho\vec{v}\cdot\nabla h = \frac{DP}{Dt} + \vec{v}\cdot\nabla P + \bar{\bar{\tau}}' : \nabla\vec{v} - \nabla\cdot\vec{q}'' \qquad \text{ENTHALPY CONSERVATION} \quad (16)$$

NOTE: THE PRODUCT $\bar{\bar{A}} : \bar{\bar{B}}$ IS A TENSOR CONTRACTION

THE CONSERVATION EQUATIONS ARE SUPPLEMENTED WITH THE EQUATION OF STATE FOR AN IDEAL GAS:

$$P = \rho R_g T \,. \qquad (17)$$

NOTE THAT $\mu \sim \frac{1}{2}\rho\lambda\bar{c} \propto T^{1/2}$, $\bar{c} = \sqrt{\frac{8RT}{\pi}}$

IN THIS FORMULATION, $\bar{\bar{\tau}}'$ IS THE VISCOUS STRESS TENSOR

$$\bar{\bar{\tau}}' = 2\mu\bar{\bar{S}} + \left(\mu_B - \frac{2}{3}\mu\right)(\nabla\cdot\vec{U})\,\bar{\bar{I}} \qquad (18)$$

WHERE $\bar{\bar{S}} = \frac{1}{2}\left(\nabla\vec{U} + \nabla\vec{U}^T\right)$ IS THE STRAIN-RATE TENSOR, AND μ AND μ_B ARE THE MOLECULAR AND BULK VISCOSITIES, RESPECTIVELY. μ IS IN GENERAL A FUNCTION OF THE TEMPERATURE, WITH

$$\left\{\begin{array}{l} \mu = \mu_{REF}\left(\dfrac{T}{T_{REF}}\right)^{\sigma} \quad (\sigma \sim 0.7) \\ \mu = \left(\dfrac{T}{T_{REF}}\right)^{1/2}\dfrac{1 + \chi/T_{REF}}{1 + \chi/T} \quad \text{(SUTHERLAND'S)} \end{array}\right. \qquad (19)$$

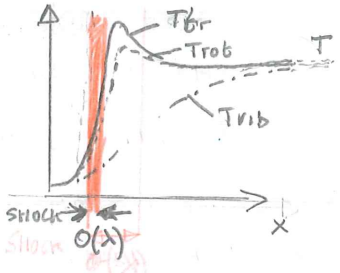
TWO TYPICAL MODELS FOR THE MOLECULAR VISCOSITY.

SIMILARLY, $\mu_B = 0$ FOR IDEAL MONOATOMIC DILUTE GASES, BUT $\mu_B \neq 0$ FOR DIATOMIC OR POLYATOMIC GASES. SETTING $\mu_B = 0$ YIELDS THE STOKES APPROXIMATION,

$$\bar{\bar{\tau}}' = 2\mu\bar{\bar{S}} - \frac{2}{3}\mu(\nabla\cdot\vec{U})\,\bar{\bar{I}} \,. \qquad (20)$$

($\bar{\bar{I}}$ = IDENTITY TENSOR)

THE BULK VISCOSITY ENTERS THE FORMULATION IN TWO SEPARATE SITUATIONS. THE FIRST IS WHEN THE GAS IS DENSE (HIGH PRESSURES → EVEN IF MONOATOMIC) AND THE STRESS TENSOR NEEDS TO BE MODIFIED TO ACCOUNT FOR INTERMOLECULAR FORCES, WHICH WILL NOT BE ADDRESSED HERE. THE SECOND INVOLVES DEPARTURES FROM LOCAL EQUILIBRIUM IN THE INTERNAL ENERGY OF DIATOMIC OR POLYATOMIC GASES. FOR INSTANCE, WHEN A GAS FLOWS THROUGH A SHOCK WAVE, THE TRANSLATIONAL ENERGY INCREASES AND EQUILIBRATES BEHIND THE SHOCK (WHICH TAKES ONLY A FEW COLLISIONS → $t \sim \frac{1}{\omega} \sim 10^{-10}$ s). THE ROTATIONAL INTERNAL ENERGY ALSO INCREASES BUT IT TAKES SOMEWHAT LONGER FOR IT TO EQUILIBRATE, WHILE THE VIBRATIONAL INTERNAL ENERGY TAKES MUCH LONGER TO EQUILIBRATE ($t \sim 10^{-6}$–10^{-7} s). AS A RESULT, THE SUM OF THE INTERNAL ENERGIES BEHIND THE SHOCK IS SMALLER THAN FARTHER DOWNSTREAM WHERE ALL ENERGIES HAVE EQUILIBRATED: THIS TRANSLATES INTO A PRESSURE P^* LARGER THAN THE EQUILIBRATED ONE FARTHER DOWNSTREAM.



THE DIFFERENCE BETWEEN THAT AND THE EQUILIBRIUM KINETIC PRESSURE IS

$$P^* - \underbrace{\frac{1}{3}\rho\overline{c^2}}_{\substack{\text{KINETIC EQUILIBRIUM PRESSURE} \\ = \rho R_g T}} = + \underbrace{\mu_B \nabla\cdot\vec{U}}_{\substack{\text{PRESSURE DECREMENT DUE TO ROT/VIB RELAXATION} \\ \mu_B \propto \text{RELAXATION TIME}}} .$$

FOR N_2, O_2: $\mu/\mu_B \sim 1$

FOR H_2: $\mu/\mu_B \sim 30$

THE EFFECT OF THE BULK VISCOSITY TENDS TO BE NEGLECTED IN CALCULATIONS.

THE HEAT FLUX $\vec{q}''$ IS COMPUTED FROM

$$\vec{q}'' = -K\nabla T \qquad (21)$$

, WITH K = THERMAL CONDUCTIVITY THAT GENERALLY DEPENDS ON TEMPERATURE IN A MANNER THAT CAN BE COMPUTED BY, FOR INSTANCE, PRESETTING →

A CONSTANT VALUE OF THE PRANDTL NUMBER, $Pr = \frac{\mu c_p}{\lambda}$ ($Pr = 0.7$ FOR AIR).

IMPORTANT REMARKS ABOUT THE CONSERVATION EQUATIONS

- IN COMPRESSIBLE INVISCID FLOWS, THE TRANSPORT TERMS INVOLVING $\bar{\bar{\tau}}'$ AND $\vec{q}''$ IN EQS. (11)-(13) ARE NEGLECTED AS IS APPROPRIATE FOR THE IDEAL, STREAMLINED CHARACTER OF THE FLOW. CONVERSELY, IN COMPRESSIBLE TURBULENT FLOWS THESE TERMS NEED TO BE RETAINED AND ARE RESPONSIBLE FOR MOLECULAR TRANSPORT IN THE SMALL SCALES OF THE TURBULENCE. IT IS IMPORTANT TO NOTE THAT THE LIMIT $Re = \frac{UL}{\nu} \to \infty$ IS A SINGULAR ONE, IN THAT THE PREFACTORS MULTIPLYING $\nabla \cdot \bar{\bar{\tau}}$, $\nabla \cdot \vec{q}'$ AND $\nabla \cdot (\bar{\bar{\tau}}' \cdot \vec{u})$ (OR $\bar{\bar{\tau}}' : \nabla \vec{u}$), WHICH ARE OF ORDER $\frac{1}{Re_L}$ WHEN THE INTEGRAL QUANTITIES ARE USED FOR NONDIMENSIONALIZATION, TEND TO ZERO WHILE THE VELOCITY AND TEMPERATURE GRADIENTS TEND TO INFINITY. THESE TRANSPORT TERMS ARE OF PARAMOUNT IMPORTANCE IN THE SMALL SCALES OF THE TURBULENCE.

- AT ORDER-UNITY OR HIGHER MACH NUMBERS, THE KINETIC AND THERMAL ENERGIES ARE COMPARABLE:

$$\frac{\frac{|\vec{u}|^2}{2}}{c_v T} \sim \frac{|\vec{u}|^2}{\frac{2 R_g}{\gamma - 1} T} = \frac{(\gamma-1)\gamma}{2} \frac{|\vec{u}|^2}{a^2} = \frac{(\gamma-1)\gamma}{2} M_a^2 = O(1), \qquad M_a = \frac{|\vec{u}|}{a} \to \#\,\text{MACH}$$

$a = \sqrt{\gamma R_g T}$ SPEED OF SOUND.

NOTE THAT THIS IS NOT THE CASE IN LOW-MACH FLOWS, WHERE THE THERMAL ENERGY IS ALWAYS MUCH LARGER THAN THE KINETIC ENERGY.

- AT ORDER-UNITY OR HIGHER MACH NUMBERS, THE FLOW DILATATION $\Delta_V = \nabla \cdot \vec{u}$ PLAYS AN IMPORTANT ROLE IN CONTINUITY, IN THE VISCOUS STRESS TENSOR, AND IN THE PRODUCTION OR DESTRUCTION OF KINETIC ENERGY. THE ASSOCIATED LARGE-SCALE DENSITY VARIATIONS ARE OF ORDER $\frac{\Delta \rho}{\rho_0} \sim \frac{\Delta P}{P_0} \sim \frac{\rho_0 U^2}{P_0} \sim \gamma M_a^2 \sim O(1)$, SO THAT THE LARGE-SCALE DILATATION IS OF ORDER $\Delta_V \sim \frac{\Delta \rho}{\rho_0 t_0} \sim \gamma M_a^2 U/L$, WHERE U, L AND $t_0 \sim L/U$ ARE A SUITABLE LARGE-SCALE VELOCITY, LENGTH AND RESIDENCE TIME OF THE PROBLEM. AS $M_a \to 0$, AND IN ABSENCE OF OTHER DILATATIONAL MECHANISMS, $\Delta_V \to 0$.

- AT ORDER-UNITY OR HIGHER MACH NUMBERS, THE VISCOUS DISSIPATION OF ENERGY, $\rho \varepsilon = \bar{\bar{\tau}}' : \nabla \vec{u}$, BECOMES AN IMPORTANT MECHANISM FOR INCREASE OF THERMAL ENERGY IN EQ. (15). THIS VISCOUS DISSIPATION IS A SINK OF KINETIC ENERGY AND A SOURCE OF THERMAL ENERGY, AND TOGETHER WITH THE PRESSURE-DILATATION WORK $-P \nabla \cdot \vec{u}$ PLAYS A FUNDAMENTAL ROLE IN EXCHANGING BOTH TYPES OF ENERGIES.

- LOW-MACH NUMBER FLOWS ARE CHARACTERIZED BY SMALL PRESSURE VARIATIONS RELATIVE TO THE BACKGROUND THERMODYNAMIC PRESSURE, WHICH REMAINS UNIFORM EVERYWHERE TO LEADING ORDER IN THE MACH NUMBER. CONVERSELY, AT ORDER-UNITY OR HIGHER MACH NUMBERS, THE CHARACTERISTIC VARIATIONS IN THE THERMODYNAMIC PRESSURE ARE COMPARABLE TO THE DYNAMIC PRESSURE: $\frac{\Delta p}{p_0} \sim \frac{\rho_0 U^2}{p_0} \sim \gamma M_\infty^2 \sim O(1)$, AS A RESULT, THE GRADIENT OF THE THERMODYNAMIC PRESSURE PLAYS AN IMPORTANT ROLE IN BALANCING AND COMPETING AGAINST THE OTHER TERMS IN THE MOMENTUM EQUATION (12).

- THE ABOVE CONSIDERATIONS ARE BETTER ILLUSTRATED BY NONDIMENSIONALIZING THE CONSERVATION EQUATIONS. HOWEVER, BEFORE DOING SO, IT IS IMPORTANT TO NOTE THAT, IN CONTRAST TO COMPRESSIBLE INVISCID FLOWS, COMPRESSIBLE TURBULENT FLOWS ARE MULTI-SCALE, IN THAT EACH TERM INVOLVED IN THE CONSERVATION EQUATIONS ACTS UPON A LARGE RANGE OF SPATIAL AND TIME SCALES. THE EXTENT OF THIS RANGE INCREASES AS THE REYNOLDS NUMBER INCREASES, AS A RESULT, THE NON-DIMENSIONAL VERSION OF THE CONSERVATION EQUATIONS IS OFTEN NOT AS INFORMATIVE SINCE THERE ARE MULTIPLE INTERNAL SCALES INVOLVED IN THE PROBLEM. NONETHELESS, CONSIDER FOR INSTANCE A TURBULENT FLOW EXITING AN ORIFICE OF CHARACTERISTIC LENGTH L, AS SHOWN IN FIG. 2. DENOTE U AS THE MEAN VELOCITY, U_ℓ AS THE VELOCITY FLUCTUATIONS OF THE LARGE EDDIES OF SIZE ℓ, AND U_K THE FLUCTUATION VELOCITIES OF THE SMALLEST (OR KOLMOGOROV) EDDIES OF SIZE ℓ_K. CONSIDER FOR SIMPLICITY THAT $U \sim U_\ell$ AND $L \sim \ell$ (A GOOD APPROXIMATION FOR JET FLOWS, AND DEFINE THE DIMENSIONLESS VARIABLES

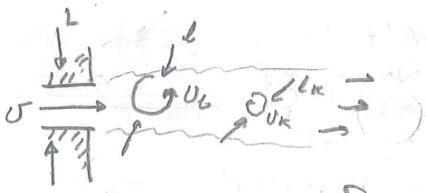


FIG 2.

$$\vec{U}^* = \frac{\vec{U}}{U}, \quad \vec{x}^* = \frac{\vec{x}}{L}, \quad p^* = \frac{p}{p_\infty}, \quad \rho^* = \frac{\rho}{\rho_\infty}, \quad T^* = \frac{T}{T_\infty}, \quad \mu^* = \frac{\mu}{\mu_\infty}, \quad K^* = \frac{K}{K_\infty}, \quad t^* = \frac{t}{L/U}.$$

IN THESE VARIABLES, THE CONSERVATION EQUATIONS (11)-(13) BECOME

$$\frac{\partial \rho^*}{\partial t^*} + \nabla \cdot (\rho^* \vec{U}^*) = 0 \qquad (21)$$

$$\rho^* \frac{\partial \vec{U}^*}{\partial t^*} + \rho^* \vec{U}^* \nabla \vec{U}^* = -\frac{1}{\gamma M_\infty^2} \nabla p^* + \frac{1}{Re_L} \nabla \cdot \bar{\bar{\sigma}}' \qquad (22)$$

$$\rho^* \frac{\partial}{\partial t^*}\left(h^* + \frac{(\gamma-1)}{2} M_\infty^2 |\vec{U}^*|^2\right) + \rho^* \vec{U}^* \nabla\left(h^* + \frac{(\gamma-1)}{2} M_\infty^2 |\vec{U}^*|^2\right) = \frac{(\gamma-1)}{\gamma} \frac{\partial p^*}{\partial t^*} + (\gamma-1) M_\infty^2 \nabla \cdot (\bar{\bar{\tau}}^* \vec{U}^*) - \frac{1}{Pr Re_L} \nabla \cdot \vec{q}^* \qquad (23)$$

THE OTHER FORMS OF THE ENERGY EQUATION BECOME

$$\rho^* \frac{\partial}{\partial t^*}\left(e^* + \frac{(\gamma-1)\gamma M_\infty^2}{2}|\vec{u}|^2\right) + \rho \vec{u}^* \nabla\left(e^* + \frac{(\gamma-1)\gamma M_\infty^2}{2}|\vec{u}|^2\right) = (\gamma-1)\nabla\cdot(p^*\vec{u}^*) +$$

$$\frac{(\gamma-1)\gamma M_\infty^2}{Re_L}\nabla\cdot(\bar{\bar{\tau}}'^*\vec{u}^*) - \frac{\gamma}{Pr Re_L}\nabla\cdot\vec{q}''^* \qquad (24)$$

FOR THE TOTAL INTERNAL ENERGY,

$$\rho^* \frac{\partial e^*}{\partial t^*} + \rho^* \vec{u}^* \nabla e^* = -(\gamma-1)p^*\nabla\cdot\vec{u}^* + \frac{(\gamma-1)\gamma M_\infty^2}{Re_L}\bar{\bar{\tau}}'^* : \nabla\vec{u}^* - \frac{\gamma}{Pr Re_L}\nabla\cdot\vec{q}''^* \qquad (25)$$

FOR THE INTERNAL ENERGY, AND

$$\rho^* \frac{\partial h^*}{\partial t^*} + \rho^* \vec{u}^* \nabla h^* = \frac{(\gamma-1)}{\gamma}\frac{Dp^*}{Dt^*} + \frac{(\gamma-1)M_\infty^2}{Re_L}\bar{\bar{\tau}}'^* : \nabla\vec{u}^* - \frac{1}{Pr Re_L}\nabla\cdot\vec{q}''^* \qquad (26)$$

FOR THE ENTHALPY. IN THIS FORMULATION, M_∞, Re_L AND Pr ARE GIVEN BY

$$M_\infty = \frac{U}{a_\infty} = \frac{U}{\sqrt{\gamma R_g T_\infty}} \quad \text{\# MACH NUMBER} \sim O(1)$$

$$Re_L = \frac{\rho_\infty U L}{\mu_\infty} \quad \text{\# REYNOLDS NUMBER} \gg 1$$

$$Pr = \frac{\mu_\infty C_p}{K_\infty} = \frac{\mu_\infty}{\rho_\infty}\frac{\rho_\infty C_p}{K_\infty} = \frac{\nu_\infty}{\alpha_\infty} \quad \text{\# PRANDTL NUMBER} \sim 0.7$$

(ν_∞: KINEMATIC VISCOSITY; α_∞: THERMAL DIFFUSIVITY)

IN EQS. (21)-(26), THE TERMS PREMULTIPLIED BY $1/Re_L$ ARE MOLECULAR TRANSPORT TERMS THAT ARE IMPORTANT IN THE SMALL SCALES OF TURBULENCE. THESE INCLUDE THE VISCOUS FORCE IN (22), THE DIVERGENCE OF THE HEAT FLUX IN (23)-(26), AND THE VISCOUS TERMS IN (23)-(26). IN PARTICULAR, NOTE THAT THE VISCOUS DISSIPATION TERM ($\bar{\bar{\tau}}' : \nabla\vec{u}$) IS PROPORTIONAL TO $\frac{M_\infty^2}{Re_L}$. THE FACTOR $1/Re_L \ll 1$ DOES NOT IMPLY THAT THE DISSIPATION IS SMALL. IF $M_\infty \sim O(1)$, THE VISCOUS DISSIPATION BECOMES IMPORTANT AS A PRODUCTION MECHANISM OF THERMAL ENERGY, ACTING PRIMARILY AT THE SMALL SCALES OF TURBULENCE WHERE IT BECOMES OF ORDER M_∞^2 INSTEAD OF M_∞^2/Re_L. SIMILARLY, THE PRESSURE-DILATATION WORK ($p\nabla\cdot\vec{u}$) IS AN ORDER-UNITY CONTRIBUTION IN THE TRANSFER OF THERMAL ENERGY. IT WILL BE SHOWN FURTHER BELOW THAT BOTH VISCOUS DISSIPATION AND PRESSURE-DILATATION WORK ARE FUNDAMENTAL MECHANISMS OF TRANSFER BETWEEN KINETIC AND THERMAL ENERGIES.

LOW-MACH NUMBER LIMIT

ALTHOUGH THE LOW-MACH LIMIT IS OF LITTLE SIGNIFICANCE FOR COMPRESSIBLE TURBULENT FLOWS, IT IS EXPEDIENT TO ANALYZE IT HERE AS THE MOST COMMON PATHWAY OF SIMPLIFICATION OF EQS. (11)-(13) FOR THE NEARLY-INCOMPRESSIBLE FLOWS COMMONLY FOUND IN LOW-SPEED AERODYNAMICS. NOTE THAT THE EXPLICIT NUMERICAL RESOLUTION OF EQS. (11)-(13) AT LOW MACH NUMBERS IS TOO COSTLY, SINCE THE TIME STEP REQUIRED IS OF ORDER $\Delta t \lesssim \Delta x / a$ IN ORDER TO SOLVE THE NECESSARY SOUND WAVES (FIG. 3). IN THIS WAY, IT TAKES $\frac{\Delta x}{v} = \frac{a}{v}\frac{\Delta x}{a} = \frac{1}{Ma}\Delta t$ TIME STEPS TO ADVANCE ONE FLUID PARTICLE OVER ONE GRID CELL, WITH $Ma \ll 1$ YIELDING AN UNREASONABLY LARGE NUMBER OF STEPS. AN ALTERNATIVE TO CIRCUMVENT THIS IS TO EMPLOY IMPLICIT OR SEMI-IMPLICIT INTEGRATION OF THE FOLLOWING MACH EQUATIONS WHICH DO NOT RESOLVE THE ACOUSTIC WAVES.

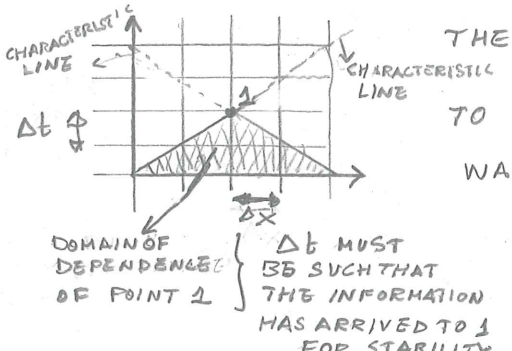


FIG. 3.

AT LOW MACH NUMBERS, THE DEPARTURES OF THE THERMODYNAMIC PRESSURE RELATIVE TO ITSELF ARE OF THE SAME ORDER AS THE DYNAMIC PRESSURE:

$$\frac{P}{P_\infty} \sim 1 + O\left(\frac{\rho_\infty v^2}{P_\infty}\right) \sim 1 + O(Ma^2) \quad , \quad Ma \ll 1$$

WHERE P_∞ IS THE THERMODYNAMIC PRESSURE AT AN ARBITRARY POINT OF THE FLOW DOMAIN. IN GENERAL, THE EVOLUTION OF THE PRESSURE CAN BE DECOMPOSED AS

$$\frac{P}{P_\infty} = P^* \sim \Pi + \gamma Ma^2 \Pi + \dots$$

Π = DIMENSIONLESS LEADING-ORDER (THERMODYNAMIC) PRESSURE FIELD

Π = DIMENSIONLESS SECOND-ORDER (DYNAMIC) PRESSURE FIELD

SUBSTITUTING THIS EXPRESSION INTO THE DIMENSIONLESS MOMENTUM EQUATION (22) YIELDS

$$\rho^* \frac{\partial \vec{v}^*}{\partial t^*} + \rho^* \vec{v}^* \nabla \vec{v}^* = -\frac{1}{\gamma Ma^2}\left(\nabla \Pi + \gamma Ma^2 \cdot \nabla \Pi\right) + \frac{1}{Re_L}\nabla \cdot \bar{\bar{\tau}}^* ,$$

WHICH, AT $Ma \ll 1$, SIMPLIFIES TO

$$\nabla \Pi = 0 \Rightarrow P = P_\infty \text{ , UNIFORM THERMODYNAMIC PRESSURE}$$

TO LEADING ORDER, AND

$$\rho^* \frac{\partial \vec{v}^*}{\partial t^*} + \rho^* \vec{v}^* \nabla \vec{v}^* = -\nabla \Pi + \frac{1}{Re_L}\nabla \cdot \bar{\bar{\tau}}^* \qquad (27).$$

TO SECOND ORDER. IN ADDITION TO (27), THE CONTINUITY EQUATION (21) AND THE THE THERMAL ENERGY EQUATION

$$\rho^* \frac{\partial e^*}{\partial t^*} + \rho^* \vec{v}^* \nabla e^* = -(\gamma - 1)\underbrace{\frac{P}{P_\infty}}_{1 + O(Ma^2)} \nabla \cdot \vec{v}^* - \frac{1}{Pr\, Re_L}\nabla \cdot \vec{q}''^* = -(\gamma - 1)\nabla \cdot \vec{v}^* - \frac{1}{Pr\, Re_L}\nabla \cdot \vec{q}''^* \qquad (28)$$

ARE INTEGRATED IN THE LOW-MACH APPROXIMATION FOR DENSITY-VARYING FLOWS, SUCH AS THOSE ENCOUNTERED FOR INSTANCE IN LOW-SPEED COMBUSTION SYSTEMS LIKE FURNACES OR LAND-BASED GAS TURBINE COMBUSTORS. NOTE THAT THE SYSTEM OF EQUATIONS (21), (27) AND (28) MUST BE SUPPLEMENTED WITH THE EQUATION OF STATE

$$\rho^* T^* = 1 \qquad (29)$$

AND WITH THE DEFINITION OF THE INTERNAL ENERGY OF AN IDEAL GAS,

$$e^* = T^*, \qquad (30)$$

THE DYNAMIC PRESSURE Π IS OFTEN OBTAINED BY COMBINING CONTINUITY (21) AND MOMENTUM (27) CONSERVATION EQUATIONS. FOR INSTANCE, TAKING THE DIVERGENCE OF (27) AND USING (21), ONE OBTAINS

$$\nabla^2 \Pi = + \frac{1}{Re_L} \nabla \cdot (\nabla \cdot \bar{\bar{\tau}}'^*) - \frac{\partial^2 \rho^*}{\partial t^{*2}} - \nabla \cdot \left[\nabla \cdot (\rho^* \vec{u}^* \vec{u}^*) \right] .$$

WHICH CORRESPONDS TO A POISSON EQUATION FOR THE DYNAMIC PRESSURE. NOTE THAT Π DOES NOT BEAR ANY PHYSICAL SIGNIFICANCE BY ITSELF, BUT ITS GRADIENTS ARE OF PARAMOUNT IMPORTANCE IN MOTORIZING THE FLOW.

FOR CONSTANT-DENSITY FLOWS, THE FORMULATION SIMPLIFIES GREATLY,

$$\nabla \cdot \vec{u}^* = 0$$

$$\frac{\partial \vec{u}^*}{\partial t} + \vec{u}^* \nabla \vec{u}^* = - \nabla \Pi + \frac{1}{Re_L} \nabla \cdot \bar{\bar{\tau}}'^* \quad \text{WITH} \quad \bar{\bar{\tau}}'^* = 2 \mu^* \bar{\bar{S}}^*$$

$$\frac{\partial T^*}{\partial t} + \vec{u}^* \nabla T^* = - \frac{1}{Pr Re_L} \nabla \cdot \vec{q}''^* , \quad \text{WITH} \quad \vec{q}''^* = -K^* \nabla T^*$$

WHERE THE TEMPERATURE AND VELOCITY FIELDS BECOME UNCOUPLED UP TO VISCOSITY VARIATIONS WITH T (IF ANY). THE CORRESPONDING POISSON EQUATION BECOMES

$$\nabla^2 \Pi = \frac{1}{Re_L} \nabla \cdot (\nabla \cdot \bar{\bar{\tau}}'^*) - (\nabla \vec{u}^*)^T \cdot (\nabla \vec{u}^*) .$$

IT WILL BE SHOWN FURTHER BELOW THAT IN COMPRESSIBLE TURBULENT FLOWS THE EQUATION FOR THE PRESSURE IS A WAVE EQUATION RATHER THAN A POISSON EQUATION, SO THAT THE PRESSURE DISTURBANCES ARE RADIATED HYPERBOLICALLY AT FINITE SPEED RATHER THAN BEING TRANSMITTED ELLIPTICALLY AND INSTANTANEOUSLY IN THE FLOW FIELD AS PREDICTED IN THE LOW-MACH LIMIT.

2. SPEED OF SOUND AND THE MACH CONE

THE SPEED OF SOUND IS DEFINED AS

$$a = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \,, \qquad (31)$$

FOR AIR AT AMBIENT TEMPERATURE $T = 298\,K$:
$a = 346\ m/s$
($\gamma = 1.4$, $W = 28.9\ g/mol$, $R_g = 286\ J/kgK$)

WHICH, FOR AN IDEAL GAS BECOMES

$$\left[a = \left(\frac{\partial}{\partial \rho}\left(\frac{p_0}{\rho_0^\gamma}\rho^\gamma\right)\right)^{1/2} = \left(\frac{p_0}{\rho_0^\gamma}\gamma\rho^{\gamma-1}\right)^{1/2} = \left(\frac{p}{\rho^\gamma}\gamma\rho^{\gamma-1}\right)^{1/2} = \left(\frac{\gamma p}{\rho}\right)^{1/2} = \underline{\sqrt{\gamma R_g T} = \sqrt{(\gamma-1)h}}\right] \qquad (32)$$

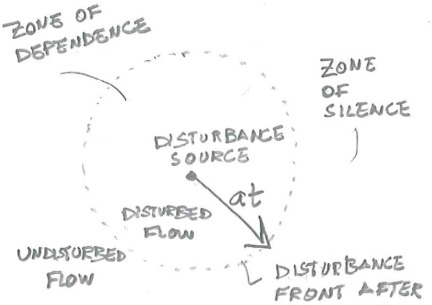
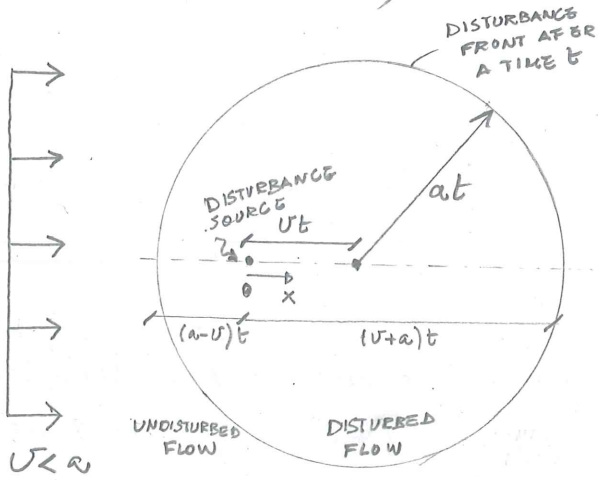


FIG. 4

IN PARTICULAR, THE SPEED OF SOUND IS THE PROPAGATION VELOCITY OF THE DISTURBANCES GENERATED IN A FLUID AT REST (THIS WILL BE SHOWN LATER WHEN DESCRIBING WAVE MOTION IN COMPRESSIBLE FLOWS). THEREFORE, A DISTURBANCE IN A FLUID AT REST WILL BE PROPAGATED AT A RADIAL DISTANCE at FROM THE SOURCE AFTER A TIME t (SEE FIG. 4).

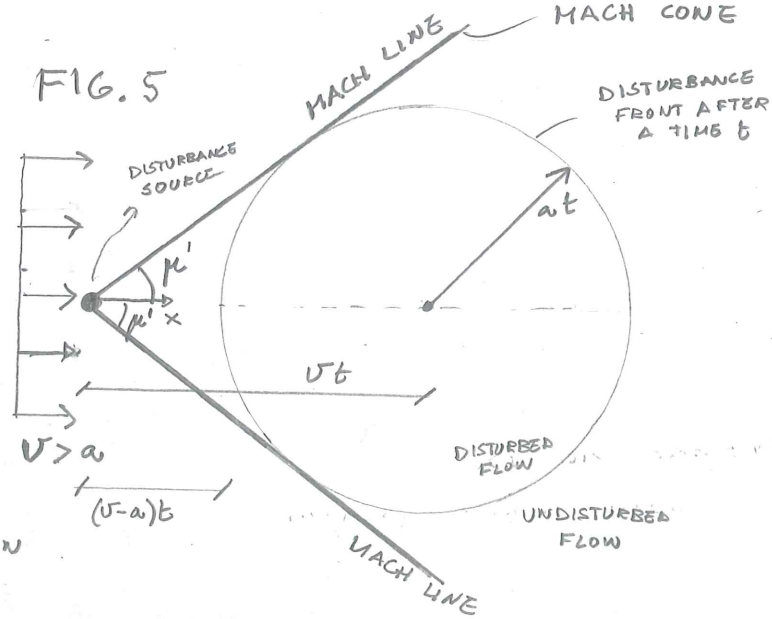
IN A FLUID MOVING WITH VELOCITY U, A DISTURBANCE PRODUCED AT THE ORIGIN AT $t=0$ WILL BE AT $t>0$ ON A SPHERE OF RADIUS at CENTERED AT A DISTANCE Ut FROM THE ORIGIN (SEE FIG. 5)

FIG. 5



(a) DISTURBANCE PROPAGATION IN A SUBSONIC FLOW

$Ma = U/a < 1$



(b) DISTURBANCE PROPAGATION IN A SUPERSONIC FLOW

$Ma = U/a > 1$

A FLOW STREAM IS CALLED <u>SUBSONIC</u> WHEN ITS VELOCITY U IS SMALLER THAN a, OR EQUIVALENTLY, WHEN THE MACH NUMBER

$$\boxed{Ma = U/a \quad \text{\# MACH NUMBER}} \qquad (33)$$

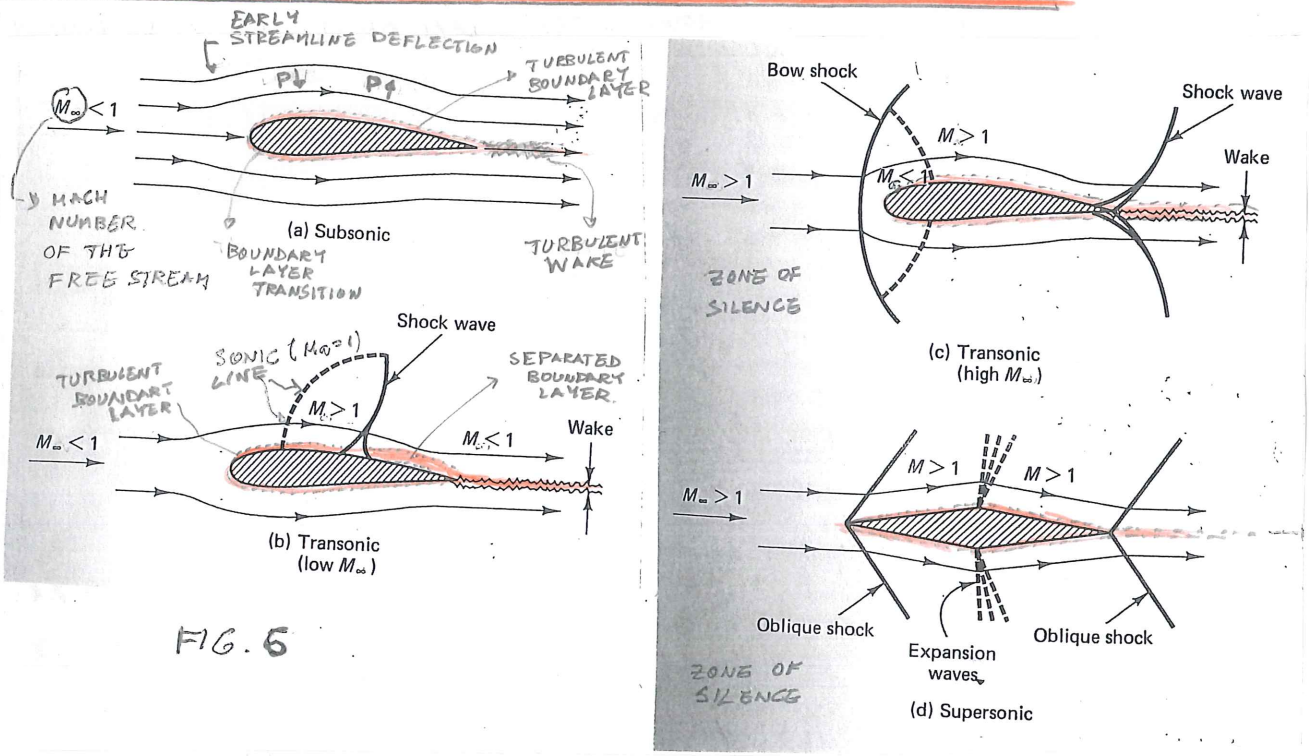
IS SMALLER THAN ONE, $Ma < 1$, IN SUCH A WAY THAT THE EFFECT OF A DISTURBANCE PRODUCED AT $x=0$ AT $t=0$ IS NOTICED AT A DISTANCE $(a-U)t$ <u>UPSTREAM</u> FROM THE SOURCE AT TIME t (FIG. 5a).

ON THE OTHER HAND, A FLOW STREAM IS CALLED SUPERSONIC WHEN $Ma > 1$, IN SUCH A WAY THAT THE EFFECT OF THE DISTURBANCE CANNOT BE NOTICED UPSTREAM FROM OR AT THE POINT WHERE IT WAS PRODUCED. THE ZONE INFLUENCED BY THE EFFECTS OF THE DISTURBANCE IS THE INTERIOR OF A CONE (i.e., THE MACH CONE) BOUNDED BY MACH LINES TILTED AT AN ANGLE

$$\mu' = \arcsin\left(\frac{a\,t}{v\,t}\right) = \arcsin\left(\frac{1}{Ma}\right) \qquad (34)$$

MACH ANGLE

FROM SUBSONIC TO SUPERSONIC FLOWS AROUND SLENDER BODIES



(b) FIG. 6

AT SMALL MACH NUMBERS (FIG 6a) THE PRESENCE OF THE BODY IS NOTICED FAR UPSTREAM AS EVIDENCED BY THE EARLY TURNING OF THE STREAMLINES (FIG. 6a). IN MOST PRACTICAL CASES, THE LENGTH OF THE BODY IS SUFFICIENTLY LARGE THAT BOUNDARY-LAYER TRANSITION TO TURBULENCE OCCURS EARLY NEAR THE LEADING EDGE, GIVING RISE TO TURBULENT BOUNDARY LAYERS ALONG THE UPPER AND LOWER SURFACES THAT MERGE INTO A SINGLE TURBULENT WAKE. IF THE ANGLE OF ATTACK REMAINS SHALLOW, THE TURBULENT BOUNDARY LAYERS REMAIN ATTACHED. OTHERWISE, THE BOUNDARY LAYERS MAY SEPARATE ON THE LEEWARD SIDE AND PREFERENTIALLY CLOSER TO THE DOWNSTREAM SIDE OF THE MAXIMUM BODY WIDTH POINT, WHERE THE ADVERSE PRESSURE GRADIENTS ARE THE LARGEST. NOTE THAT THE EARLY DEFLECTION OF THE STREAMLINES IS CAUSED BY THE ELLIPTICITY

WITH WHICH THE PRESSURE IS TRANSMITTED IN SUBSONIC FLOWS. AT HIGHER BUT STILL SUBSONIC MACH NUMBERS, THE FLOW EXPANSION ALONG THE LATERAL SURFACES LEADS TO SONIC AND LATER SUPERSONIC CONDITIONS THERE, WITH A SHOCK WAVE BEING FORMED THERE THAT CREATES A SUDDEN JUMP IN PRESSURE AND VELOCITY (FIG. 6b). THESE CORRESPOND TO THE TYPICAL CHARACTERISTICS OF A TRANSONIC FLOW WITH $Ma_\infty \sim 0.8-0.9$. THE INTERACTION OF THE TURBULENT

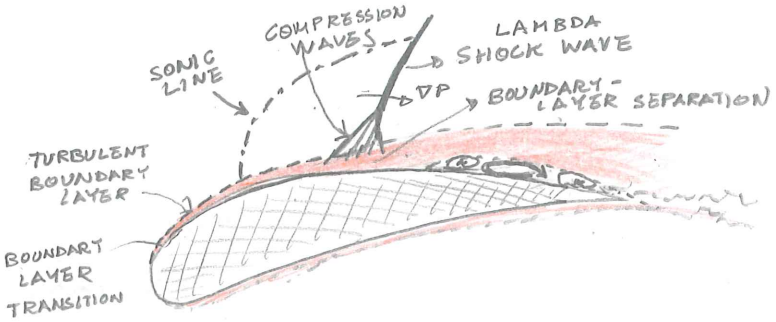
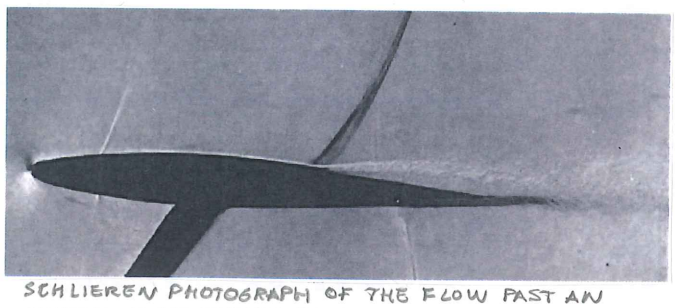


FIG. 7



SCHLIEREN PHOTOGRAPH OF THE FLOW PAST AN AIRFOIL WITH SEVERE SEPARATION AT THE FOOT OF THE UPPER-SURFACE SHOCK. $Ma_\infty = 0.88$, 2 DEG. INCIDENCE. CREDIT: HH PEARCEY (1959).

BOUNDARY LAYER WITH THE SHOCK CAN CAUSE BOUNDARY-LAYER SEPARATION AND A CORRESPONDING INCREASE IN DRAG (FIG. 7). AT HIGHER SPEEDS ENTERING IN THE SUPERSONIC RANGE, A DETACHED SHOCK WAVE (OR BOW SHOCK) IS CREADED UPSTREAM FROM THE BODY IF ITS FOREBODY IS BLUNT. THE SHOCK WAVE STRONGLY INCREASES THE PRESSURE AND TEMPERATURE OF THE FLOW WHILE SLOWING IT DOWN (FIG. 6c). THE PART OF THE FUSELAGE IMMEDIATELY DOWNSTREAM FROM THE FRONTAL PART OF THE BOW SHOCK IS THEREFORE HIGHLY SUSCEPTIBLE TO AERODYNAMIC HEATING AND SUBSEQUENT DETERIORATION OF THE MATERIAL SURFACES. SUPERSONIC FLOW CONDITIONS ARE RESTORED ALONG THE LATERAL SURFACES AS THE FLOW EXPANDS THERE. TRAILING SHOCK WAVES DECELERATE AND TURN THE FLOW IN THE STREAMWISE DIRECTION. AN IMPORTANT EFFECT OF THE BOW SHOCK IS TO PRODUCE AN ENTROPY LAYER AROUND THE BODY, WHERE GRADIENTS OF ENTROPY OCCUR. IT WILL LATER BE SHOWN THAT THE GRADIENTS OF ENTROPY ARE SOURCES OF VORTICITY (SEE SECTION ON THE CROCCO'S EQUATION). IF INSTEAD OF A BLUNT NOSE, THE BODY HAS A WEDGE-TYPE OF NOSE, THE LEADING SHOCK BECOMES ATTACHED (FIG. 6d), WITH EXPANSION FANS OCCURRING ON THE LATERAL SURFACES ALONG WITH RECOMPRESSION-SHOCKS NEAR THE TRAILING EDGE.

3. ISENTROPIC FLOWS

A TRANSPORT EQUATION FOR THE ENTROPY CAN BE OBTAINED BY COMBINING THE CONSERVATION EQUATION FOR THE INTERNAL ENERGY (15) AND THE FIRST LAW (4). SPECIFICALLY, TAKING THE MATERIAL DERIVATIVE IN EQ. (4) LEADS TO

$$\rho T \frac{Ds}{Dt} = \rho \frac{De}{Dt} - \frac{p}{\rho} \frac{D\rho}{Dt} \quad , \quad \text{WITH} \quad \frac{D\rho}{Dt} = -\rho \nabla \cdot \vec{u} .$$

SUBSTITUTING (15) INTO THE ABOVE EXPRESSION LEADS TO

$$\boxed{\frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = \frac{\bar{\bar{\tau}}' : \nabla \vec{u}}{\rho T} + \frac{\nabla \cdot (k \nabla T)}{\rho T}} \qquad (35)$$

WHICH SHOWS THAT ENTROPY IS PRODUCED BY VISCOUS DISSIPATION AND HEAT CONDUCTION. AS MENTIONED ABOVE, THESE TWO MECHANISMS ARE IMPORTANT IN THE SMALL SCALES OF TURBULENCE AND, AS A RESULT, THOSE ARE EXPECTED TO BE HIGHLY NON-ISENTROPIC.

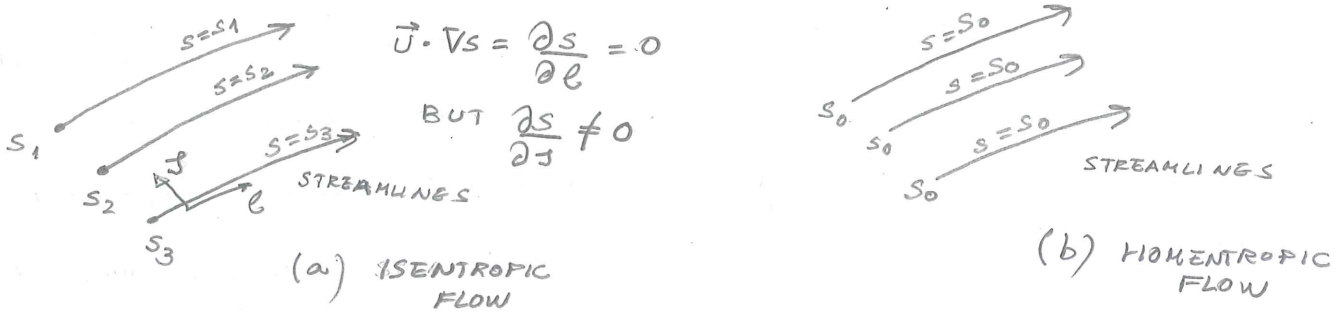
FOR ILLUSTRATION, CONSIDER AN INVISCID COMPRESSIBLE FLOW SUCH THAT EQ. (35) SIMPLIFIES TO

$$\frac{Ds}{Dt} = \frac{\partial s}{\partial t} + \vec{u} \cdot \nabla s = 0 \quad , \quad \Rightarrow \underline{\text{ISENTROPIC FLOW}}$$

WHICH CAN BE INTEGRATED TO GIVE $s = s_p$, WHERE s_p IS THE INITIAL ENTROPY OF A FLUID PARTICLE (NOTE THAT s_p MAY GENERALLY IN SPACE AND TIME). IN STEADY FLOWS, THE PROBLEM SIMPLIFIES TO

$$\vec{u} \cdot \nabla s = 0 ,$$

WHICH CORRESPONDS TO AN ISENTROPIC FLOW IN WHICH THE ENTROPY IS CONSTANT ALONG STREAMLINES (FIG 8a).



(a) ISENTROPIC FLOW

(b) HOMENTROPIC FLOW

FIG. 8

IF, IN ADDITION, ALL FLUID PARTICLES HAVE THE SAME ENTROPY, THEN $\nabla s = 0$, AND THE FLOW IS REFERRED TO AS "HOMENTROPIC". IN THIS LIMIT, THE ENTROPY IS UNIFORM ALTHOUGH IT COULD BE A FUNCTION OF TIME. AN ISENTROPIC MOTION WITH UNIFORM INITIAL ENTROPY FOR ALL PARTICLES IS ALSO AN HOMENTROPIC FLOW.

THE BERNOULLI EQUATION FOR COMPRESSIBLE FLOWS

A FUNDAMENTAL RELATION BETWEEN THE ENTHALPY AND THE VELOCITY OF A HIGH-SPEED INVISCID FLOW CAN BE DERIVED BY SIMULTANEOUSLY CONSIDERING MOMENTUM AND ENERGY CONSERVATION. THIS RELATION IS CALLED THE BERNOULLI EQUATION AND IS IMPORTANT IN ESTABLISHING DIFFERENCES BETWEEN STAGNATION AND STATIC QUANTITIES INTRODUCED BELOW. CONSIDER FIRST THE MOMENTUM EQUATION (12),

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} = -\frac{\nabla p}{\rho} \qquad (36)$$

WHERE THE VISCOUS TRANSPORT HAS BEEN DROPPED. NOTE THAT THE CONVECTIVE TERM CAN BE EXACTLY REWRITTEN AS

$$\vec{v}\cdot\nabla\vec{v} = \nabla\left(\frac{|\vec{v}|^2}{2}\right) - \underbrace{\vec{v}\wedge\nabla\wedge\vec{v}}_{\text{ALWAYS PERPENDICULAR TO } \vec{v}}, \qquad (37)$$



MULTIPLICATION OF THIS EXPRESSION BY AN UNIT VECTOR $\vec{e}_l$ TANGENT TO THE STREAMLINES LEADS TO

$$\vec{e}_l\cdot(\vec{v}\cdot\nabla\vec{v}) = \vec{e}_l\cdot\nabla\left(\frac{|\vec{v}|^2}{2}\right) - \vec{e}_l\cdot(\vec{v}\wedge\nabla\wedge\vec{v})$$

THEREFORE, MULTIPLICATION OF (36) BY $\vec{e}_l$ GIVES

$$\frac{\partial}{\partial l}\left(\frac{|\vec{v}|^2}{2}\right) + \frac{1}{\rho}\frac{\partial p}{\partial l} = 0 \qquad (38)$$

WHERE $\frac{\partial}{\partial l} = \vec{e}_l\cdot\nabla$ IS THE DIRECTIONAL DERIVATIVE, AND THE UNSTEADY TERM HAS BEEN NEGLECTED AS NECESSARY FOR THE DERIVATION. IN INCOMPRESSIBLE FLOWS, ρ = CONST. AND EQ. (38) REDUCES TO

$$\frac{\partial}{\partial l}\left(\frac{|\vec{v}|^2}{2} + \frac{p}{\rho}\right) = 0 \Rightarrow \frac{p}{\rho} + \frac{|\vec{v}|^2}{2} = C, \qquad (39)$$

WHERE C IS A CONSTANT THAT IN PRINCIPLE DEPENDS ON THE STREAMLINE. IN PARTICULAR, EQ. (39) REPRESENTS THE CLASSIC EXCHANGE BETWEEN PRESSURE AND KINETIC ENERGY PER UNIT VOLUME. NOTE THAT THE PARTICIPATING PRESSURE IN (39) IS THE DYNAMIC COMPONENT OF THE PRESSURE, WHICH UNDERGOES VARIATIONS PROPORTIONAL TO THE KINETIC ENERGY PER UNIT VOLUME.

IN COMPRESSIBLE FLOWS, THE CHARACTERISTIC VARIATIONS OF THE DENSITY ARE OF THE SAME ORDER AS THE DENSITY ITSELF, $\Delta\rho/\rho_0 = O(1)$, AND EQ. (39) IS NO LONGER VALID. CONVERSELY, ONE CAN USE IN (38) THE FIRST PRINCIPLE OF THERMODYNAMICS

PARTICULARIZED FOR ISENTROPIC FLOWS, NAMELY

$$T\cancel{\frac{\partial s}{\partial t}}^{\,0} = \frac{\partial e}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t}.$$

AND SINCE $h = e + p/\rho \Rightarrow \frac{\partial h}{\partial t} = \frac{\partial e}{\partial t} + \frac{1}{\rho}\frac{\partial p}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} \Rightarrow \frac{\partial e}{\partial t} - \frac{p}{\rho^2}\frac{\partial \rho}{\partial t} = \frac{\partial h}{\partial t} - \frac{1}{\rho}\frac{\partial p}{\partial t}$,

THEN $$+\frac{1}{\rho}\frac{\partial p}{\partial t} = \frac{\partial h}{\partial t} \qquad (40).$$

SUBSTITUTING (40) INTO (38) YIELDS THE BERNOULLI EQUATION FOR COMPRESSIBLE FLOWS

$$\frac{\partial}{\partial t}\left(\frac{|\vec{v}|^2}{2} + h\right) = 0 \Rightarrow \boxed{h + \frac{|\vec{v}|^2}{2} = \mathcal{C} = h_0} \qquad (41)$$

WHICH REPRESENTS A EXCHANGE BETWEEN ENTHALPY AND SPECIFIC KINETIC ENERGY. IN PRINCIPLE, AT ORDER-UNITY MACH NUMBERS BOTH TERMS IN (41) ARE OF THE SAME ORDER OF MAGNITUDE, WITH THEIR RATIO BEING PROPORTIONAL TO THE MACH NUMBER ITSELF,

$$\frac{|\vec{v}|^2/2}{h} = \frac{|\vec{v}|^2/2}{c_p T} = \frac{(\gamma-1)}{\gamma}\frac{|\vec{v}|^2/2}{R_g T} = \frac{(\gamma-1)}{2}Ma^2.$$

IN EQ. (41) THE CONSTANT $\mathcal{C}$ IS EQUAL TO THE STAGNATION ENTHALPY h_0, WHOSE VALUE GENERALLY DEPENDS ON THE THERMODYNAMIC CONDITIONS UPSTREAM ALONG THE STREAMLINE IN QUESTION. AT POINTS WHERE THE FLOW VELOCITY IS ZERO IN THE PARTICULAR REFERENCE FRAME CHOSEN, THE ENTHALPY IS EQUAL TO THE STAGNATION ENTHALPY, $h = h_0$.

STAGNATION (OR TOTAL) AND STATIC QUANTITIES.

IN EQ. (41), h AND h_0 ARE REFERRED TO STATIC AND STAGNATION (ALSO CALLED "TOTAL") ENTHALPIES, RESPECTIVELY. STAGNATION QUANTITIES CORRESPOND TO THE THERMODYNAMIC STATE OF A GAS ISENTROPICALLY BROUGHT TO REST FROM ITS INITIAL VELOCITY $\vec{v}$. THE STAGNATION QUANTITIES h_0, T_0, ρ_0, p_0, a_0 AND e_0 ARE RELATED AMONG THEMSELVES BY USING THE IDEAL GAS RELATIONS (5)-(7) AND (32), NAMELY:

$$h_0 = c_p T_0 = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0} = \frac{a_0^2}{\gamma-1} = \gamma e_0. \qquad (42)$$

IN GENERAL, HOWEVER, JUST TWO OF THESE VARIABLES SUFFICE TO COMPLETELY CHARACTERIZE THE STAGNATION THERMODYNAMIC STATE OF THE MIXTURE.

CONVERSELY, THE VARIABLES h, T, ρ, P, a AND e ARE CALLED STATIC, IN THAT THEY DO NOT ACCOUNT FOR THE POTENTIAL RECOVERY OF KINETIC ENERGY OF THE FLOW. IT IS STRAIGHTFORWARD TO SEE THAT

$$h < h_0, \quad P < P_0, \quad \rho < \rho_0, \quad a < a_0 \text{ AND } e < e_0,$$

WHICH ILLUSTRATES THAT IN COMPRESSIBLE FLOWS PART OF THE INITIAL THERMAL ENERGY CAN BE TRANSFORMED AND CARRIED ALONG IN THE FORM OF KINETIC ENERGY. IT IS INSTRUCTIVE TO REWRITE THE BERNOULLI EQ. (27) IN A MORE COMPACT FORM AS

$$h + \frac{|\vec{U}|^2}{2} = h_0 \xrightarrow[h = c_p T]{} \boxed{\frac{T_0}{T} = 1 + \frac{\gamma - 1}{2} M_a^2} \qquad (42).$$

COMBINING THIS RELATION WITH THE THERMODYNAMIC RELATIONS FOR AN ISENTROPIC PROCESS, NAMELY

$$\frac{T_0}{T} = \left(\frac{\rho_0}{\rho}\right)^{\gamma - 1} = \left(\frac{P_0}{P}\right)^{\frac{\gamma - 1}{\gamma}} \qquad (43)$$

NOTE THAT THE STAGNATION OR TOTAL PRESSURE IS NOT $P + \frac{\rho U^2}{2}$ IN COMPRESSIBLE FLOWS, BUT P_0 IN THE EQUATION BELOW.

THE EXPRESSIONS

$$\boxed{\frac{h_0}{h} = \frac{T_0}{T} = \frac{e_0}{e} = \left(\frac{a_0}{a}\right)^2 = \left(\frac{\rho_0}{\rho}\right)^{\gamma - 1} = \left(\frac{P_0}{P}\right)^{\frac{\gamma - 1}{\gamma}} = 1 + \frac{(\gamma - 1)}{2} M_a^2} \qquad (44)$$

ARE OBTAINED, WHICH RELATE THE STAGNATION AND STATIC QUANTITIES WITH THE LOCAL MACH NUMBER $M_a = |\vec{U}|/a$.

NOTE THAT THE STAGNATION QUANTITIES DEPEND ON THE REFERENCE FRAME (FIG. 9)

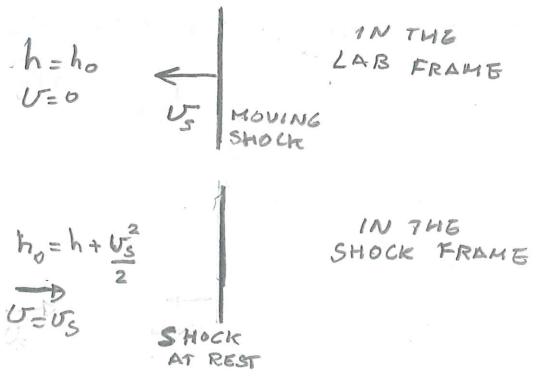


FIG. 9

AS SUCH, THE STAGNATION QUANTITIES BEAR NO REAL SIGNIFICANCE FROM THE THERMODYNAMIC STANDPOINT. ON THE CONTRARY, THE STATIC VARIABLES CARRY THE INFORMATION OF THE THERMODYNAMIC TEMPERATURE, PRESSURE, AND DENSITY, ALL THESE QUANTITIES NECESSARILY BEING INDEPENDENT OF THE REFERENCE FRAME.

3. GENERATION OF VORTICITY IN COMPRESSIBLE FLOWS

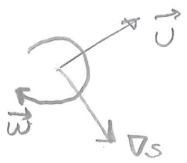
CONSIDER THE MOMENTUM EQUATION (12) AND MAKE USE OF THE IDENTITY (37), WHICH GIVES

$$\rho \frac{\partial \vec{U}}{\partial t} + \rho \vec{\omega} \wedge \vec{U} = -\nabla P - \rho \nabla \left(\frac{|\vec{U}|^2}{2}\right) + \nabla \cdot \bar{\bar{\tau}}', \quad \text{WITH } \boxed{\vec{\omega} = \nabla \wedge \vec{U}} \text{ THE VORTICITY.} \qquad (45)$$

FROM THE FIRST PRINCIPLE, THE PRESSURE GRADIENT IS RELATED TO THE ENTROPY GRADIENT AS

$$T \nabla s = \nabla h - \nabla P / \rho, \qquad (46)$$

SUBSTITUTING (46) INTO (45) ONE OBTAINS



$$\frac{\partial \vec{U}}{\partial t} + \vec{\omega} \wedge \vec{U} = T\nabla s - \nabla\left(h + \frac{|\vec{U}|^2}{2}\right) + \frac{\nabla \cdot \bar{\bar{\tau}}'}{\rho} \quad . \qquad (47)$$

WHEN THE FLOW IS INVISCID AND STEADY, EQ. (47) BECOMES

$$\boxed{\vec{\omega} \wedge \vec{U} = T\nabla s - \nabla\left(h + \frac{|\vec{U}|^2}{2}\right)} \quad , \qquad (48)$$

CROCCO'S EQUATION

WHERE $h + \frac{|\vec{U}|^2}{2} = h_0$ IS THE STAGNATION ENTHALPY. IF THE LATTER IS UNIFORM IN THE FLOW (FOR INSTANCE, IF THERE IS NO HEAT ADDITION), EQ. (48) SIMPLIFIES TO

→ ADIABATIC FLOWS

$$\boxed{\vec{\omega} \wedge \vec{U} = T\nabla s} \quad . \qquad (49)$$

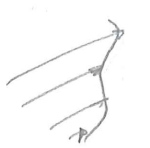
ALSO NOTE THAT THE GRADIENTS OF ENTROPY ARE RELATED TO THE GRADIENTS OF STAGNATION PRESSURE IN ADIABATIC FLOWS IN THE FOLLOWING MANNER. CONSIDER EQ. (10) AND REWRITE IT IN TERMS OF STAGNATION QUANTITIES,

$$s - s_0 = c_v \ln\left(\frac{p/\rho^\gamma}{p_0/\rho_0^\gamma}\right) \overset{\text{USE (43)}}{=} c_v \ln\left(\frac{p}{p_0}\left(\frac{T}{T_0}\right)^{-\frac{\gamma}{\gamma-1}}\right) \overset{\text{USE (44)}}{=} c_v \ln\left(\frac{p}{p_0}\left(\frac{h}{h_0}\right)^{-\frac{\gamma}{\gamma-1}}\right) =$$

$$\overset{\text{USE (44) AGAIN}}{=} c_v \left(\frac{p_0'}{p_0}\left(1 + \frac{(\gamma-1)}{2}Ma^2\right)^{-\frac{\gamma}{\gamma-1}} \frac{h_0'}{h_0}\left(1 + \frac{(\gamma-1)}{2}Ma^2\right)^{\frac{\gamma}{\gamma-1}}\right) = c_v \ln\left(\frac{p_0'}{p_0}\right), \qquad (50)$$

↑ h_0 IS UNIFORM

WHERE p_0' AND h_0' ($h_0 = h_0'$ IS ASSUMED) ARE STAGNATION VALUES ALONG AN ARBITRARY STREAMLINE DIFFERENT FROM THE ONE TO WHICH p_0 AND h_0 BELONG. NOTE THAT EQ. (50) IMPLIES



$$\nabla s = c_v \nabla(\ln p_0) \quad . \qquad (51)$$

SUBSTITUTING THIS EXPRESSION INTO (49), ONE OBTAINS

$$\vec{\omega} \wedge \vec{U} = c_v T \nabla(\ln p_0) \qquad (52)$$

AS AN ALTERNATIVE FORM.

THE IMPLICATIONS OF EQUATION (49) ARE DEEP. CONSIDER, FOR INSTANCE, AN INVISCID IRROTATIONAL FLOW PASSING THROUGH A SHOCK WAVE. IT WILL BE SHOWN LATER THAT h_0 = CONSTANT THROUGH A SHOCK. IF THE STRENGTH OF THE SHOCK IS UNIFORM ALONG ITS LENGTH, THAT IS, IF THE ANGLE OF INCIDENCE IS CONSTANT (SO AS TO MAKE THE RATIO OF POST-SHOCK TO PRE-SHOCK QUANTITIES UNIFORM ALONG THE SHOCK FRONT), THEN THERE IS A JUMP IN ENTROPY ACROSS THE SHOCK BUT THIS JUMP

IS CONSTANT ALONG THE SHOCK. CORRESPONDINGLY, $\nabla s = 0$ UPSTREAM AND DOWNSTREAM OF THE SHOCK (FIG. 10a). AS A RESULT, EQ. (49) INDICATES THAT $\vec{\omega} \wedge \vec{v} = 0$ UPSTREAM AND DOWNSTREAM OF THE SHOCK. IN 2D FLOWS, THIS IMPLIES THAT THE INITIALLY IRROTATIONAL FLOW REMAINS IRROTATIONAL DOWNSTREAM OF THE SHOCK.

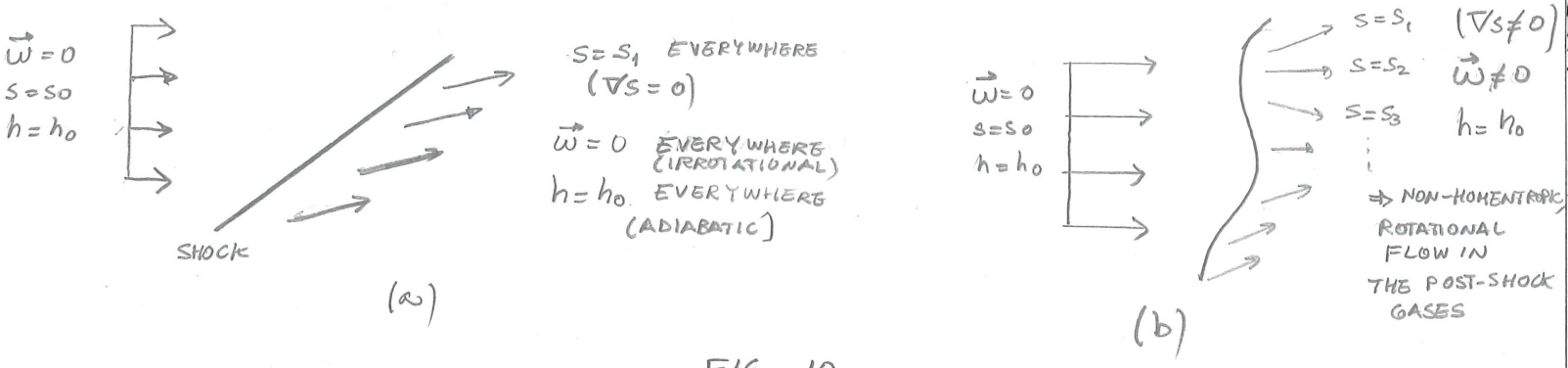


FIG. 10

ON THE OTHER HAND, THE FLOW DOWNSTREAM FROM A CURVED SHOCK IS GENERALLY ROTATIONAL SINCE THE SHOCK STRENGTH VARIES ALONG ITS FRONT DUE TO THE VARYING ANGLE OF INCIDENCE. AS A RESULT, THE FLOW IS NON-HOMENTROPIC ($\nabla s \neq 0$) DOWNSTREAM OF THE SHOCK (EVEN THOUGH IT IS ISENTROPIC IN THAT THE ENTROPY IS CONSTANT ALONG STREAMLINES IN THE POSTSHOCK GAS), AND BECAUSE OF (49) $\vec{\omega} \wedge \vec{v} \neq 0$, INDICATING THAT THE FLOW IS ROTATIONAL (FIG. 10b).

THE VORTICITY EQUATION

A VORTICITY EQUATION FOR COMPRESSIBLE FLOWS CAN BE EASILY DERIVED BY TAKING THE CURL OF EQ. (47), WHICH GIVES

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{v} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{v} - \vec{\omega} (\nabla \cdot \vec{v}) + \nabla T \wedge \nabla s + \nabla \wedge \left(\frac{\nabla \cdot \bar{\bar{\tau}}'}{\rho} \right) \quad (53)$$

WHERE USE OF THE IDENTITIES

$$\nabla \wedge (\nabla h_0) = 0 \quad , \quad \nabla \wedge (T \nabla s) = \nabla T \wedge \nabla s \quad ,$$

$$\nabla \wedge (\vec{\omega} \wedge \vec{v}) = \vec{v} \cdot \nabla \vec{\omega} - \vec{v} \cancelto{0}{(\nabla \cdot \vec{\omega})} + \vec{\omega} (\nabla \cdot \vec{v}) - \vec{\omega} \cdot \nabla \vec{v}$$

HAS BEEN MADE. IN EQ. (53), THE FIRST TERM ON THE RIGHT-HAND SIDE CORRESPONDS TO THE PRODUCTION OF VORTICITY THROUGH VORTEX STRETCHING ($\vec{\omega} \cdot \nabla \vec{v}$). THIS IS THE MOST IMPORTANT MECHANISM OF CREATION OF VELOCITY GRADIENTS AND ONLY OCCURS IN THREE-DIMENSIONAL FLOWS (NOTE THAT $\vec{v}$ AND THEREFORE $\nabla \vec{v}$ TOO ARE PERPENDICULAR TO $\vec{\omega}$ IN TWO-DIMENSIONAL FLOWS). SINCE $\vec{\omega} \cdot \nabla \vec{v} = \vec{\omega} \cdot \bar{\bar{S}} + \frac{1}{2} \cancelto{0}{\vec{\omega} \wedge \vec{\omega}}$, THIS TERM UNDERSCORES THE IMPORTANCE OF THE STRAIN-RATE FIELD IN STRETCHING THE VORTICITY AND GENERATING ENSTROPHY IN THE FLOW. THE SECOND TERM ON THE RIGHT-HAND SIDE OF (53) REFERS

TO THE PRODUCTION OF VORTICITY BY COMPRESSIBILITY EFFECTS $\left(\vec{\omega}(\nabla\cdot\vec{v})\right)$. THE CROSS-PRODUCT TERM $\nabla T\wedge\nabla S$ IS A BAROCLINIC TORQUE THAT CAN BE RECASTED IN THE MORE FAMILIAR FORM $\nabla P\wedge\nabla\left(\frac{1}{\rho}\right)$ BY EXPRESSING ∇T AS

$$\nabla T = \left(\frac{\partial T}{\partial P}\right)_S \nabla P + \left(\frac{\partial T}{\partial S}\right)_P \nabla S$$

SO THAT $\nabla T\wedge\nabla S = \left(\frac{\partial T}{\partial P}\right)_S \nabla P\wedge\nabla S = \left(\frac{\partial T}{\partial P}\right)_S \nabla P\wedge\left[\left(\frac{\partial S}{\partial v}\right)_P \nabla v + \left(\frac{\partial S}{\partial P}\right)_v \nabla P\right] =$

$$= \left(\frac{\partial T}{\partial P}\right)_S \left(\frac{\partial S}{\partial v}\right)_P (\nabla P\wedge\nabla v) = \nabla P\wedge\nabla v \,, \quad \text{WIT } v = \frac{1}{\rho}$$

($\left(\frac{\partial T}{\partial P}\right)_S \left(\frac{\partial S}{\partial v}\right)_P = 1$ FROM MAXWELL RELATIONS)

THIS TERM GENERATES VORTICITY THROUGH THE MISALIGNMENT BETWEEN PRESSURE AND DENSITY GRADIENTS, OR EQUIVALENTLY, BETWEEN TEMPERATURE AND ENTROPY GRADIENTS. LASTLY, THE TERM $\nabla\wedge(\nabla\bar{\bar{\tau}}')$ REPRESENTS THE PRODUCTION OF VORTICITY BY VISCOUS FORCES, ALTHOUGH AT HIGH REYNOLDS NUMBERS THIS TERM IS ONLY IMPORTANT AS A MOLECULAR DIFFUSION OF VORTICITY IN THE SMALL SCALES OF TURBULENCE.

NOTE THAT AN INVISCID COMPRESSIBLE FLOW THAT IS HOMENTROPIC OR HAS A UNIFORM DISTRIBUTION OF A THERMODYNAMIC VARIABLE, WILL REMAIN IRROTATIONAL IF IT WAS INITIALLY IRROTATIONAL SINCE (53) BECOMES

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{v}\cdot\nabla\vec{\omega} = \vec{\omega}\cdot\nabla\vec{v} - \vec{\omega}(\nabla\cdot\vec{v}),$$

WHICH DOES NOT CONTAIN ANY SOURCE INDEPENDENT OF $\vec{\omega}$.

AN ALTERNATIVE FORM OF THE VORTICITY EQUATION, CALLED THE VAZSONYI'S EQUATION, CAN BE DERIVED IN TERMS OF THE SPECIFIC VORTICITY $\vec{\omega}/\rho$. CONSIDER FOR INSTANCE EQ (53) DIVIDED BY ρ:

$$\frac{1}{\rho}\frac{D\vec{\omega}}{Dt} = \frac{\vec{\omega}}{\rho}\cdot\nabla\vec{v} - \frac{\vec{\omega}}{\rho}(\nabla\cdot\vec{v}) + \frac{\nabla\rho}{\rho^3}\wedge\nabla P + \frac{1}{\rho}\nabla\wedge(\nabla\cdot\bar{\bar{\tau}}')$$

$\frac{\vec{\omega}}{\rho}$ PLAYS A DYNAMIC ROLE IN COMPRESSIBLE FLOWS SIMILAR TO THE ROLE OF $\vec{\omega}$ IN INCOMPRESSIBLE FLOWS.

AND USE CONTINUITY, $\nabla\cdot\vec{v} = -\frac{1}{\rho}\frac{D\rho}{Dt}$, THEN THE EQUATION ABOVE SIMPLIFIES TO

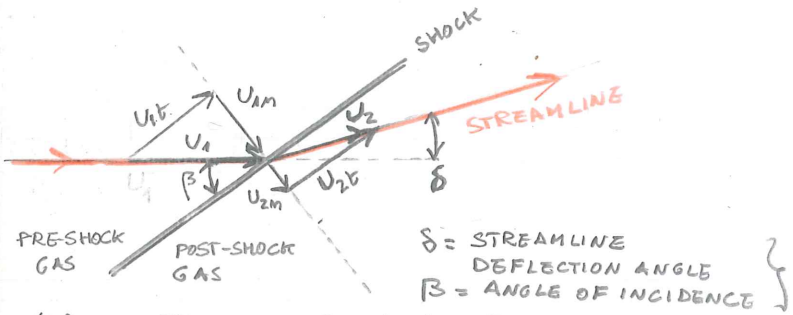
$$\boxed{\frac{\partial}{\partial t}\left(\frac{\vec{\omega}}{\rho}\right) + \vec{v}\cdot\nabla\left(\frac{\vec{\omega}}{\rho}\right) = \frac{\vec{\omega}}{\rho}\cdot\nabla\vec{v} + \frac{\nabla\rho}{\rho^3}\wedge\nabla P + \frac{1}{\rho^2}\nabla\wedge\left(\frac{\nabla\cdot\bar{\bar{\tau}}'}{\rho}\right)} \quad \text{VAZSONYI'S EQUATION} \quad (54)$$

WHICH CLEARLY ILLUSTRATES THAT THE VORTEX-STRETCH MECHANISM ACTUALLY ACTS UPON $\vec{\omega}/\rho$ RATHER THAN $\vec{\omega}$, IN THAT THE STRETCHING ACTION OF $\vec{\omega}\cdot\nabla\vec{v}$ IN (53) COULD BE COMPENSATED BY $\vec{\omega}(\nabla\cdot\vec{v})$ IN COMPRESSIBLE FLOWS, WHEREAS THE STRETCHING $\vec{\omega}/\rho\cdot\nabla\vec{v}$ CANNOT BE COMPENSATED BY ANYTHING ELSE IN ABSENCE OF OTHER NEUTRALIZING MECHANISMS SUCH AS THE BAROCLINIC TORQUE.

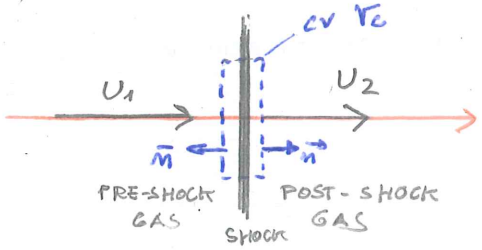
II. SHOCK WAVES

THE CONSERVATION EQUATIONS FOR COMPRESSIBLE FLOWS IN THE INVISCID FORM ADMIT SOLUTIONS IN THE FORM OF DISCONTINUITIES IN THE AEROTHERMAL VARIABLES AND THEIR GRADIENTS. THESE DISCONTINUITIES ARE REFERRED TO AS SHOCK WAVES AND ARE INFINITESIMALLY THIN IN THE INVISCID LIMIT. THE CLASSIC DESCRIPTION DOES NOT NECESSITATE INCORPORATION OF MOLECULAR TRANSPORT, WHOSE EFFECT IS TO BROADEN THE DISCONTINUITIES IN A MANNER THAT THEY STILL REMAIN THINNER THAN ANY HYDRODYNAMIC SCALE OF INTEREST, AND TO PROVIDE AN EXPLANATION OF THEIR INTERNAL STRUCTURE THAT IS NOT ESSENTIAL FOR THE CROSSING GASES. THIS CHAPTER IS FOCUSED ON THE GASDYNAMICS OF SHOCK WAVES IN THE INVISCID LIMIT AND WHEN THE PRESHOCK AND POSTSHOCK GASES REMAIN IRROTATIONAL. THE EFFECTS OF VORTICITY DISTURBANCES, WHICH OCCUR UBITIQUOSLY IN TURBULENT FLOWS, IS DEFERRED TO CHAPTER III.

CONSERVATION LAWS ACROSS SHOCK WAVES



(a) OBLIQUE SHOCK WAVE



(b) NORMAL SHOCK WAVE

FIG. 11

SHOCK WAVES GENERALLY OCCUR OBLIQUELY TO THE FLOW, AS IN FIG. 11a, OR NORMAL TO THE FLOW, AS IN FIG 11b. AT THE SAME MACH NUMBER, NORMAL SHOCKS ARE STRONGER THAN OBLIQUE SHOCKS, IN THAT THEY LEAD TO HIGHER PRESSURES, TEMPERATURES AND DENSITIES IN THE POST-SHOCK GASES. THE DESCRIPTION HERE BEGINS BY CONSIDERING NORMAL SHOCKS FIRST, WITH THE OBLIQUE ONES BEING TREATED LATER IN THE TEXT SINCE THEY ARE MORE COMPLEX.

* NORMAL SHOCK WAVES

CONSIDER THE CONTROL VOLUME V_c IN FIG 11b. THE INTEGRATION OF THE STEADY INVISCID FORMS OF CONTINUITY (11), MOMENTUM (12) AND TOTAL ENTHALPY (13) LEAD TO

$$\bullet\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{v}) = 0 \Rightarrow \frac{d}{dt}\int_{V_c}\rho\, d\vec{V} + \int_{S_c}\rho\vec{v}\cdot d\vec{S} = 0 \Rightarrow \boxed{\rho_1 U_1 = \rho_2 U_2} = \underbrace{\dot{m}''}_{\text{MASS FLOW RATE PU SURFACE}} = \text{CONST.} \quad (55)$$

- $\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = -\nabla p \Rightarrow \frac{d}{dt} \int_{V_c} \rho \vec{v} \, dV + \int_{S_c} \rho \vec{v} \vec{v} \cdot d\vec{S} = -\int_{S_c} p \, d\vec{S}$

$$\Rightarrow -\rho_1 u_1^2 + \rho_2 u_2^2 = +p_1 - p_2 \Rightarrow \boxed{p_1 + \rho_1 u_1^2 = p_2 + \rho_2 u_2^2} \quad (56)$$

CONTINUITY OF MOMENTUM FLUX

- $\rho \frac{\partial}{\partial t}\left(h + \frac{|\vec{v}|^2}{2}\right) + \rho \vec{v} \cdot \nabla\left(h + \frac{|\vec{v}|^2}{2}\right) = \frac{\partial p}{\partial t} \Rightarrow \int_{S_c} \rho \vec{v} \cdot \left(h + \frac{|\vec{v}|^2}{2}\right) \cdot d\vec{S} = 0$

$$\Rightarrow \boxed{h_1 + \frac{u_1^2}{2} = h_2 + \frac{u_2^2}{2}} \quad (57)$$

CONTINUITY OF STAGNATION ENTHALPY.

EQUATIONS (55) - (57) REPRESENT A COMPLETE SET OF BOUNDARY CONDITIONS AT A SHOCK WAVE. THEY CAN BE SIMPLIFIED AS FOLLOWS IN ORDER TO OBTAIN THE APPROPRIATE JUMP CONDITIONS ACROSS THE SHOCK. SUBSTITUTING (55) INTO (56) YIELDS

$$\frac{\dot{m}''^2}{\rho_1} + p_1 = \frac{\dot{m}''^2}{\rho_2} + p_2 \Rightarrow \boxed{\frac{p_2 - p_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}} = -\dot{m}''^2} \quad (58)$$

"RAYLEIGH LINE"

SIMILARLY, SUBSTITUTION OF (55) INTO (57) GIVES

$$h_2 - h_1 = \frac{u_1^2}{2} - \frac{u_2^2}{2} = \frac{u_1^2}{2}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right) = \frac{\dot{m}''^2}{2\rho_1^2}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right)$$

AND USING (58)

$$\frac{\dot{m}''^2}{2\rho_1^2}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right) = -\frac{1}{2\rho_1^2} \frac{p_2 - p_1}{\frac{1}{\rho_2} - \frac{1}{\rho_1}}\left(1 - \frac{\rho_1^2}{\rho_2^2}\right) = -\frac{1}{2\rho_1\rho_2} \frac{p_2 - p_1}{\rho_1 - \rho_2}\left(\rho_2^2 - \rho_1^2\right) = +\frac{1}{2}(p_2 - p_1)\frac{(\rho_1 + \rho_2)}{\rho_1\rho_2}$$

SO THAT

$$\boxed{h_2 - h_1 = \frac{1}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)(p_2 - p_1)} \quad (59)$$

"HUGONIOT EQUATION"

EQUATIONS (58) - (59) ARE FUNDAMENTAL TO THE TREATMENT OF SHOCK WAVES, IN PARTICULAR, EQ. (58) SHOWS THAT SOLUTIONS OF THE TYPE

$$p_2 > p_1 \text{ AND } \rho_2 > \rho_1 \quad \text{(COMPRESSION WAVE)}$$

OR

$$p_2 < p_1 \text{ AND } \rho_2 < \rho_1 \quad \text{(EXPANSION WAVE)}$$

ARE BOTH POSSIBLE IN PRINCIPLE, ALTHOUGH IT IS SHOWN BELOW THAT ONLY THE FORMER CASE CAN OCCUR IN THE FORM OF A DISCONTINUITY, SINCE A DISCONTINUOUS EXPANSION WAVE VIOLATES THE LAW OF INCREASING ENTROPY.

SIMILAR TO THE OTHER AEROTHERMODYNAMIC VARIABLES, THE ENTROPY IS ALSO DISCONTINUOUS ACROSS A SHOCK WAVE. BECAUSE OF THE LAW OF INCREASING ENTROPY IN ADIABATIC SYSTEMS, THE ENTROPY OF THE GAS CAN ONLY INCREASE AS IT FLOWS ACROSS THE SHOCK.

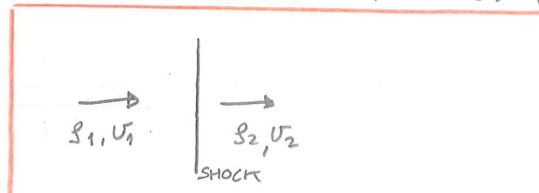
THEREFORE, THE ENTROPY S_2 OF THE POSTSHOCK GAS MUST EXCEED S_1. RECALLING EXPRESSION (10) FOR THE ENTROPY OF A PERFECT GAS:

$$S_2 - S_1 = C_v \ln\left(\frac{P_2/\rho_2^\gamma}{P_1/\rho_1^\gamma}\right) \Rightarrow \frac{P_2}{\rho_2^\gamma} > \frac{P_1}{\rho_1^\gamma} \Rightarrow \frac{P_2}{P_1} > \left(\frac{\rho_2}{\rho_1}\right)^\gamma . \quad (60)$$

THE ENTROPY-INCREASE CONSTRAINT (60), TOGETHER WITH THE HUGONIOT EQUATION (59), SHOW THAT NORMAL DISCONTINUITIES MUST BE COMPRESSION - RATHER THAN EXPANSION - WAVES ($P_2 > P_1$, $\rho_2 > \rho_1$), AND THAT DISCONTINUOUS EXPANSION WAVES CANNOT EXIST (AT LEAST IN NON-REACTING FLOWS). TO SEE THIS, REWRITE EQ. (59) IN TERMS OF THE PRESSURE AND SPECIFIC-VOLUME RATIOS, $\mathcal{P} = \frac{P_2}{P_1}$ AND $\nu = \frac{\rho_1}{\rho_2}$, WHICH GIVES

$$\mathcal{P} = \frac{\frac{\gamma+1}{\gamma-1} - \nu}{\frac{\gamma+1}{\gamma-1}\nu - 1} \quad (61)$$

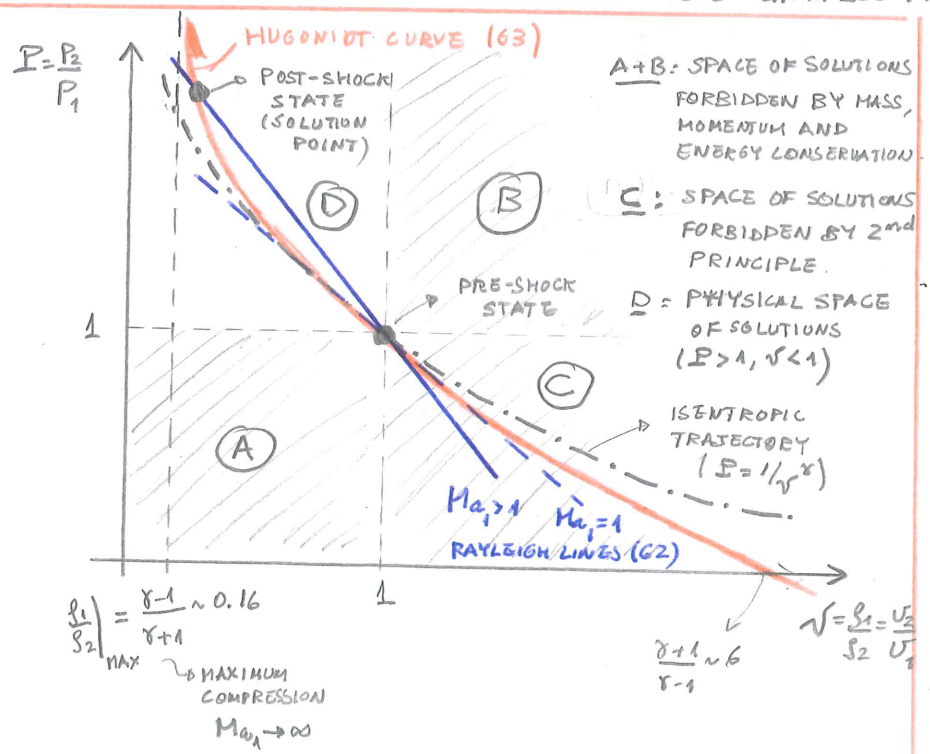
IT IS EASY TO SEE, AS ILLUSTRATED IN THE FIGURE BELOW, THAT THE HUGONIOT CURVE (61) IS ALWAYS BELOW THE ISENTROPIC CURVE $\mathcal{P} = 1/\nu^\gamma$ IN THE CASE OF EXPANSION WAVES ($\mathcal{P} < 1$, $\nu > 1$), AND THEREFORE DISCONTINUOUS EXPANSION WAVES ARE FORBIDDEN BY THE SECOND PRINCIPLE SINCE THEY WOULD LEAD TO A DECREASE IN ENTROPY IN AN ISOLATED SYSTEM. THE COMPLETE SPACE OF SOLUTIONS OF THE RANKINE-HUGONIOT EQUATIONS IS SUMMARIZED IN THE DIMENSIONLESS CHART BELOW.



DIMENSIONLESS RANKINE-HUGONIOT EQS. FOR NORMAL SHOCK WAVES:

$$\left.\begin{aligned} &\mathcal{P} = \frac{P_2}{P_1}\ , \quad \nu = \frac{\rho_1}{\rho_2} = \frac{U_2}{U_1} \\ &\mu'' = \frac{(\dot{m}'')^2}{P_1\rho_1} = \frac{U_1^2}{P_1/\rho_1} = \gamma Ma_1^2 \end{aligned}\right\}$$

$$\Rightarrow \boxed{\begin{aligned} &\frac{\mathcal{P}-1}{\nu-1} = -\mu'' \quad (62) \\ &\mathcal{P} = \frac{\frac{\gamma+1}{\gamma-1} - \nu}{\frac{\gamma+1}{\gamma-1}\nu - 1} \quad (63) \end{aligned}}$$



THE UPSTREAM / DOWNSTREAM STATES OF THE GAS ARE DETERMINED BY THE INTERSECTION POINT BETWEEN THE RAYLEIGH LINE AND THE HUGONIOT CURVE.

THE SHOCK JUMP CONDITIONS ARE OBTAINED AS A FUNCTION OF THE UPSTREAM MACH NUMBER BY SUBSTITUTING (63) INTO (62), WHICH GIVES

$$\frac{1}{\nu} = \frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,Ma_1^2}{2+(\gamma-1)\,Ma_1^2} \qquad (64)$$

DENSITY JUMP

AND CORRESPONDINGLY,

$$\frac{U_2}{U_1} = \frac{\rho_1}{\rho_2} = \frac{2+(\gamma-1)\,Ma_1^2}{(\gamma+1)\,Ma_1^2} \qquad (65)$$

VELOCITY JUMP

SUBSTITUTING (64) INTO (62), THE JUMP CONDITION FOR THE PRESSURE

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_1^2 - 1\right) \qquad (66)$$

PRESSURE JUMP

IS OBTAINED. SIMILARLY, USING THE EQUATION OF STATE (5) ALONG WITH (64) AND (66),

$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\cdot\frac{\rho_1}{\rho_2} = \frac{\left[2\gamma Ma_1^2 - (\gamma-1)\right]\left[2+(\gamma-1)\,Ma_1^2\right]}{(\gamma+1)^2\,Ma_1^2} \qquad (67)$$

TEMPERATURE JUMP

WHICH CORRESPONDS TO THE TEMPERATURE JUMP ACROSS. THE MACH NUMBER DOWNSTREAM FROM THE SHOCK IS GIVEN BY

$$Ma_2^2 = \left(\frac{U_2}{a_2}\right)^2 = \left(\frac{U_1}{a_1}\,\frac{a_1}{a_2}\,\frac{U_2}{U_1}\right)^2 = Ma_1^2\,\frac{T_1}{T_2}\left(\frac{U_2}{U_1}\right)^2 = \frac{2+(\gamma-1)\,Ma_1^2}{2\gamma Ma_1^2+1-\gamma} \qquad (68)$$

WHERE USE OF (65) AND (67) HAS BEEN MADE. THE JUMP IN STAGNATION PRESSURE AND DENSITY IS

$$\frac{P_{02}}{P_{01}} = \frac{\rho_{02}}{\rho_{01}} = \left(\frac{1+\frac{(\gamma-1)}{2}Ma_2^2}{1+\frac{(\gamma-1)}{2}Ma_1^2}\right)^{\frac{\gamma}{\gamma-1}}\frac{P_2}{P_1} \qquad (69)$$

STAGNATION PRESSURE AND STAGNATION DENSITY JUMPS

EQUATIONS (64)-(69) ARE PLOTTED BELOW IN FIG. 12 AS A FUNCTION OF Ma_1:

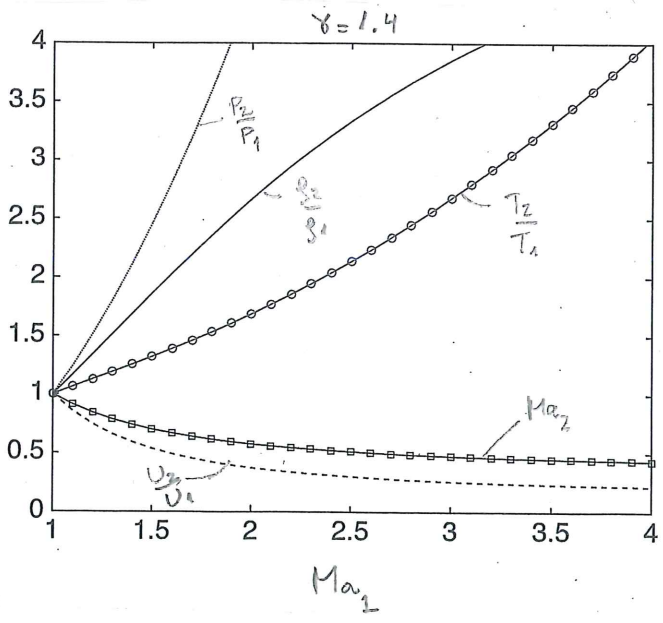


FIG. 12

NOTE: IN A SHOCK WAVE, THE FOLLOWING INEQUALITIES ARE SATISFIED:

$$\frac{P_2}{P_1} > 1, \quad \frac{P_{02}}{P_{01}} < 1,$$

$$\frac{\rho_2}{\rho_1} > 1, \quad \frac{\rho_{02}}{\rho_{01}} < 1,$$

$$\frac{T_2}{T_1} > 1, \quad \frac{T_{02}}{T_{01}} = 1,$$

$$\frac{U_2}{U_1} < 1, \quad \frac{S_2}{S_1} > 1,$$

$Ma_2 < 1 \Rightarrow$ THE FLOW DOWNSTREAM OF A SHOCK IS ALWAYS SUBSONIC IN THE SHOCK FRAME OF REFERENCE.

* OBLIQUE SHOCK WAVES

CONSIDER FIG. 11a FOR OBLIQUE WAVES ($\beta \neq \frac{\pi}{2}$). FROM THE TRIANGLES OF VELOCITIES IT IS EASY TO VERIFY THAT

$$U_{1m} = U_1 \sin\beta \qquad U_{1t} = U_1 \cos\beta$$
$$U_{2m} = U_2 \cos\left(\frac{\pi}{2} - \beta + \delta\right) = U_2 \sin(\beta - \delta) \qquad U_{2t} = U_2 \cos(\beta - \delta) \tag{70}$$

FOR THE NORMAL AND TANGENTIAL COMPONENTS OF THE VELOCITY. CORRESPONDINGLY,

$$Ma_{1m} = \frac{U_{1m}}{a_1} = Ma_1 \sin\beta$$
$$Ma_{2m} = \frac{U_{2m}}{a_2} = Ma_2 \sin(\beta - \delta) \tag{71}$$

FOR THE MACH NUMBERS OF THE FLOW NORMAL TO THE SHOCK. AN INTEGRAL ANALYSIS SIMILAR TO THE ONE IN THE PREVIOUS SECTION LEADS TO

$$\rho_1 U_{1m} = \rho_2 U_{2m} \qquad \text{(CONSERVATION OF NORMAL MASS FLUX)}$$
$$P_1 + \rho_1 U_{1m}^2 = P_2 + \rho_2 U_{2m}^2 \qquad \text{(CONSERVATION OF NORMAL MOMENTUM)}$$
$$U_{1t} = U_{2t} \qquad \text{(CONSERVATION OF TANGENTIAL MOMENTUM)}$$
$$h_1 + \frac{U_{1m}^2}{2} = h_2 + \frac{U_{2m}^2}{2} \qquad \text{(CONSERVATION OF STAGNATION ENTHALPY)}$$

WHICH ARE IDENTICAL TO (55)-(57) IF U_1 IS REPLACED BY U_{1m} AND U_2 IS REPLACED BY U_{2m}. THEREFORE, THE RANKINE-HUGONIOT EQUATIONS (62)-(63) AND THE JUMP CONDITIONS (64)-(69) CAN ALSO BE USED FOR OBLIQUE SHOCKS IF Ma_1 IS REPLACED BY Ma_{1m}, NAMELY

JUMP CONDITIONS FOR OBLIQUE SHOCKS

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)\,Ma_{1m}^2}{2 + (\gamma-1)\,Ma_{1m}^2}, \qquad \frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1}\left(Ma_{1m}^2 - 1\right),$$
$$\frac{T_2}{T_1} = \frac{P_2}{P_1}\frac{\rho_1}{\rho_2} = \frac{[2\gamma Ma_{1m}^2 - (\gamma-1)]\,[2 + (\gamma-1)Ma_{1m}^2]}{(\gamma+1)^2 Ma_{1m}^2}$$
$$\frac{U_{2m}}{U_{1m}} = \frac{\rho_1}{\rho_2} = \frac{2 + (\gamma-1)Ma_{1m}^2}{(\gamma+1)Ma_{1m}^2}, \qquad Ma_{2m}^2 = \frac{2 + (\gamma-1)Ma_{1m}^2}{2\gamma Ma_{1m}^2 + 1 - \gamma}$$
$$\frac{P_{02}}{P_{01}} = \frac{\rho_{02}}{\rho_{01}} = \left(\frac{1 + \frac{(\gamma-1)}{2}Ma_2^2}{1 + \frac{(\gamma-1)}{2}Ma_1^2}\right)^{\frac{\gamma}{\gamma-1}} \frac{P_2}{P_1}. \tag{72}$$

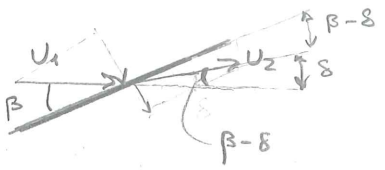
NOTE THAT $\frac{P_{02}}{P_{01}}$ AND $\frac{\rho_{02}}{\rho_{01}}$ ARE INDEPENDENT OF THE CHOICE Ma_1 VS Ma_{1m} AND THE RESULT REMAINS THE SAME.

→ THE FLOW DOWNSTREAM OF AN OBLIQUE SHOCK IS NOT NECESSARILY SUBSONIC, BUT THE NORMAL COMPONENT THERE MUST BE SUBSONIC.

A RELATION BETWEEN THE INCIDENCE ANGLE β AND THE ANGLE OF STREAMLINE DEFLECTION δ CAN BE OBTAINED BY USING THE GEOMETRICAL RELATION

$$\frac{U_{2m}}{U_{1m}} = \frac{U_{2m}/U_{1t}}{U_{1m}/U_{1t}} = \frac{\tan(\beta-\delta)}{\tan\beta}$$

U_{2m}/U_{1t}: CONSERVATION OF TANGENTIAL MOMENTUM



ALONG WITH THE JUMP CONDITION IN (72) FOR U_{2m}/U_{1m}, WHICH GIVES

$$\frac{\tan(\beta-\delta)}{\tan\beta} = \frac{2+(\gamma-1)Ma_1^2\sin^2\beta}{(\gamma+1)Ma_1^2\sin^2\beta} \Rightarrow \boxed{\tan\delta = \frac{2\cot\beta\,(Ma_1^2\sin^2\beta-1)}{2+Ma_1^2\left[\gamma+\cos(2\beta)\right]}} \quad (73)$$

$\tan(\beta-\delta) = \frac{\tan\beta-\tan\delta}{1+\tan\beta\tan\delta}$, $\sin^2\beta = \frac{1-\cos(2\beta)}{2}$

THE CONTOURS OF THE SOLUTIONS $\beta(\delta)$ OF EQ. (73) ARE PROVIDED IN FIG. 13

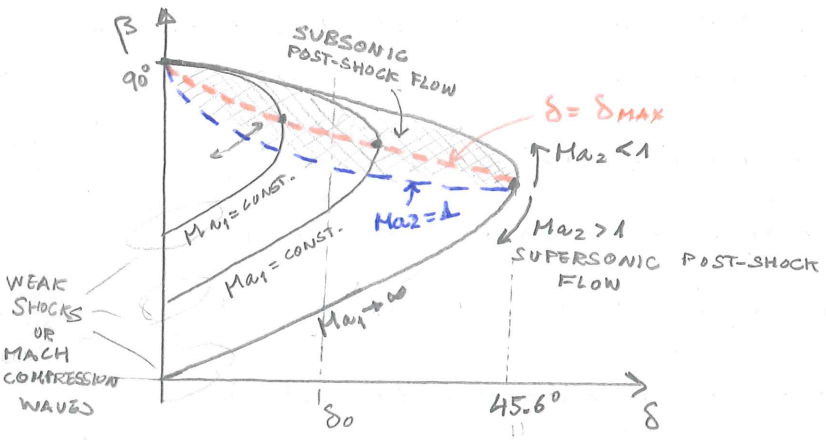


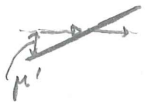
FIG. 13

COMMENTS:

- FOR A GIVEN DEFLECTION ANGLE, e.g. $\delta=\delta_0$, AND A MACH NUMBER Ma_1, THERE EXIST TWO SOLUTIONS FOR β: ONE WITH β LARGE (STRONG SHOCK WAVE), AND ANOTHER ONE FOR SMALLER β (WEAK SHOCK WAVE). STRONG SHOCKS ARE CHARACTERIZED BY HAVING SUBSONIC FLOWS DOWNSTREAM ($Ma_2<1$).

- THE CASE OF ZERO STREAMLINE DEFLECTION OCCURS IN TWO DIFFERENT SITUATIONS:

i) IN NORMAL SHOCK WAVES ($\beta=90°$) $\Rightarrow \cot\beta = 0$



ii) IN WEAK SHOCK WAVES, THAT ARE CHARACTERIZED BY $Ma_1^2\sin^2\beta = 1$, OR EQUIVALENTLY, $\sin\beta = \pm 1/Ma_1 = \sin\mu'$ (μ' = MACH ANGLE)

NOTE THAT $Ma_1^2\sin^2\beta = 1$ IMPLIES $Ma_{1m} = 1$. IN THIS LIMIT, THESE WAVES ARE COMPRESSION MACH WAVES OF VANISHING STRENGTH.

- CONSIDER THE WEDGE IN FIG. 14. FOR $\delta > \delta_{MAX}$, THERE IS NO SOLUTION WITH A SHOCK ATTACHED TO THE CORNER. INSTEAD, THE SHOCK MUST BE DETACHED FROM THE WEDGE, WITH THE SOLUTION BECOMING TWO-DIMENSIONAL AND DEPENDENT ON THE GEOMETRY AND BOUNDARY CONDITIONS.

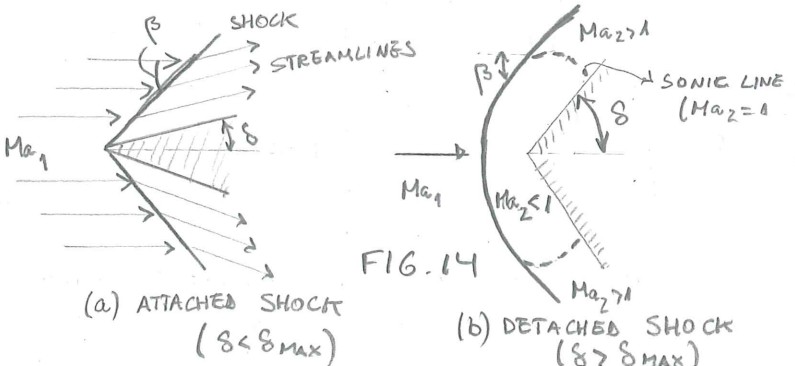
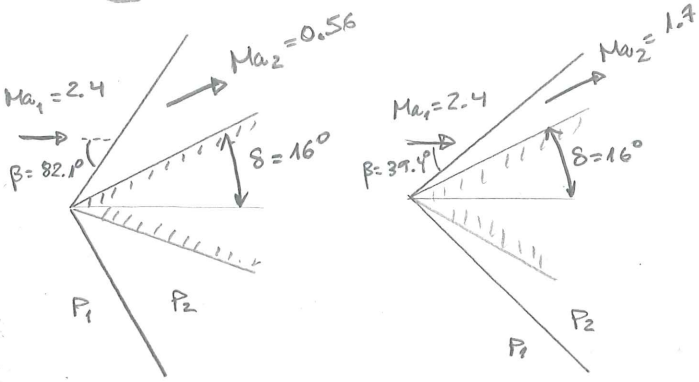


FIG. 14

• AS MENTIONED ABOVE, THERE ARE 2 SOLUTIONS FOR β, ONE REPRESENTING A STRONG WAVE WITH $Ma_2 < 1$, AND ANOTHER ONE CORRESPONDING TO A WEAK SHOCK WAVE WITH $Ma_2 > 1$.

EXAMPLES: 16° WEDGE AT $Ma_1 = 2.4$



$\frac{P_2}{P_1} = 6.3$
STRONG-SHOCK SOLUTION

$\frac{P_2}{P_1} = 2.52$
WEAK-SHOCK SOLUTION.

THE STRONG-SHOCK SOLUTION YIELDS A LARGER BACKPRESSURE P_2/P_1 THAN THE WEAK-SHOCK SOLUTION. IN EXTERNAL AERODYNAMICS, STRONG SHOCKS OCCUR RARELY EXCEPT FOR $\delta > \delta_{MAX}$, WHERE THEY ARE OBSERVED NEAR THE WEDGE NOSE WITH SUBSONIC FLOW DOWNSTREAM (FIG. 14 (b)). FOR ATTACHED SHOCKS THE WEAK SOLUTION IS THE ONE TYPICALLY OBSERVED DUE TO THE TENDENCY OF THE BACKPRESSURE TO BECOME EQUALIZED WITH THE FAR-FIELD PRESSURE IN OPEN ENVIRONMENTS. THIS MIGHT NOT BE SO IN INTERNAL FLOWS, WHERE A BACKPRESSURE CAN BE MORE READILY SUSTAINED AND, AS A RESULT, COULD BE ORIGINATED BY A STRONG SHOCK.

WEAK SHOCKS OR MACH COMPRESSION WAVES

"WEAK SHOCKS" IN THIS SECTION REFER TO SOLUTIONS OF EQ. (73) IN CONDITIONS WHERE THE MACH NUMBER OF THE INCOMING FLOW IN THE NORMAL DIRECTION IS CLOSE TO 1, WHICH LEAD TO SHOCK JUMPS OF VANISHING STRENGTHS AND VANISHING DEFLECTION ANGLES. THESE SOLUTIONS CORRESPOND TO THE $\delta \to 0$, SMALL-β ZONES OF THE DIAGRAM IN FIG. 13 AND ARE ALSO CALLED MACH WAVES. TO SEE THIS, REARRANGE (73) IN THE FORM

$$Ma_1^2 \sin^2\beta - 1 = \left(\frac{\gamma+1}{2}\right) Ma_1^2 \frac{\sin\beta\cos\delta}{\cos(\beta-\delta)},$$

WHICH, AT $\delta \to 0$, BECOMES

$$Ma_1^2 \sin^2\beta - 1 \simeq \left(\frac{\gamma+1}{2} Ma_1^2 \tan\beta\right)\delta \qquad (74)$$

SHOWING THAT THE INCIDENCE ANGLE β NECESSARILY APPROACHES THE MACH ANGLE μ' AS $\delta \to 0$, i.e. $\sin\beta \sim \sin\mu' = 1/Ma_1$, OR EQUIVALENTLY, $Ma_{1N} \sim 1$, MEANING THAT WEAK SHOCKS ARE MACH LINES (SEE FIG. 5b) IN THE FIRST APPROXIMATION.

THIS CAN BE EXPRESSED EQUIVALENTLY AS

$$\tan\beta = \tan\mu' = \frac{1}{\sqrt{Ma_1^2 - 1}},$$

WHICH CAN BE SUBSTITUTED IN (74) TO GIVE

$$Ma_1^2 \sin^2\beta - 1 \simeq \frac{\gamma+1}{2} \frac{Ma_1^2}{\sqrt{Ma_1^2-1}} \delta \qquad (75)$$

SUBSTITUTING AGAIN THIS RELATION INTO THE JUMP CONDITION (72) FOR THE PRESSURE YIELDS

$$\frac{P_2 - P_1}{P_1} = \frac{2\gamma}{\gamma+1}\left(Ma_1^2 \sin^2\beta - 1\right) \simeq \frac{\gamma Ma_1^2}{\sqrt{Ma_1^2-1}} \delta, \qquad (76)$$

WHICH SHOWS THAT THE SHOCK STRENGTH Π IS SMALL,

$$\Pi = \frac{P_2 - P_1}{P_1} \simeq O\left(Ma_1^2 \sin^2\beta - 1\right) = O(\delta) \ll 1. \qquad (77)$$

IT IS INSTRUCTIVE TO ANALIZE THE JUMP OF ENTROPY ACROSS A WEAK SHOCK. RECALL EQ. (10) FOR AN IDEAL GAS

$$S_2 - S_1 = c_v \ln\left(\frac{P_2/\rho_2^\gamma}{P_1/\rho_1^\gamma}\right). \qquad (78)$$

A SHOCK IS CHARACTERIZED BY A RISE OF ENTROPY AND DROP IN STAGNATION PRESSURE, BUT THESE JUMPS ARE VERY SMALL FOR A WEAK SHOCK. NOTE THAT THE DENSITY JUMP CAN BE GENERALLY WRITTEN AS

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma+1)Ma_{1n}^2}{2+(\gamma-1)Ma_{1n}^2} = \frac{(\gamma+1)Ma_1^2\sin^2\beta}{2+(\gamma-1)Ma_1^2\sin^2\beta} = \frac{(\gamma+1)\left[\frac{(\gamma+1)}{2\gamma}\Pi+1\right]}{2+(\gamma-1)\left[\frac{(\gamma+1)}{2\gamma}\Pi+1\right]} = \frac{1+\frac{(\gamma+1)}{2\gamma}\Pi}{1+\frac{(\gamma-1)}{2\gamma}\Pi}.$$

SUBSTITUTING THIS EXPRESSION INTO (78) YIELDS

$$\frac{S_2 - S_1}{c_v} = \ln\left(1+\Pi\right) - \gamma\ln\left(1+\frac{(\gamma+1)}{2\gamma}\Pi\right) + \gamma\ln\left(1+\frac{(\gamma-1)}{2\gamma}\Pi\right),$$

$\ln(1+\varepsilon) \sim \varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} + O(\varepsilon^4)$

WHICH CAN BE EXPANDED FOR $\Pi \ll 1$ AS

$$\frac{S_2 - S_1}{c_v} \approx \Pi - \frac{\Pi^2}{2} + \frac{\Pi^3}{3} - \gamma\left[\frac{\gamma+1}{2\gamma}\Pi - \frac{1}{2}\left(\frac{\gamma+1}{2\gamma}\right)^2\Pi^2 + \frac{1}{3}\left(\frac{\gamma+1}{2\gamma}\right)^3\Pi^3\right] + \gamma\left[\frac{\gamma-1}{2\gamma}\Pi - \frac{1}{2}\left(\frac{\gamma-1}{2\gamma}\right)^2\Pi^2 + \frac{1}{3}\left(\frac{\gamma-1}{2\gamma}\right)^3\Pi^3\right]$$

TO THIRD ORDER. SINCE THE COEFFICIENTS MULTIPLYING Π AND Π^2 VANISH WHEN ADDED TOGETHER, THE ENTROPY JUMP BECOMES A THIRD-ORDER QUANTITY IN Π^3,

$$\boxed{\frac{S_2 - S_1}{c_v} \approx \frac{\gamma^2-1}{12\gamma^2}\Pi^3 + O(\Pi^4)} \qquad (79)$$

THIS SHOWS THAT WEAK SHOCKS ARE NEARLY ISENTROPIC, AND CORRESPONDINGLY, THE DROP IN STAGNATION PRESSURE IS SMALL. FIGURE 13 SHOWS THAT THE COMPRESSION ACROSS A SHOCK IS NEARLY ISENTROPIC UNTIL DENSITY RATIOS $\rho_2/\rho_1 \sim 2$, CORRESPONDING TO $Ma_{1n} \sim 1.6$, BEYOND WHICH THE COMPRESSION STARTS TO SIGNIFIGNILY DEPART FROM THE ISENTROPIC TRAJECTORY. FIGURE 14 SUMMARIZES THE SCALING OF AEROTHERMAL VARIABLES ACROSS AN OBLIQUE WEAK SHOCK AS A FUNCTION OF THE SHOCK STRENGTH.

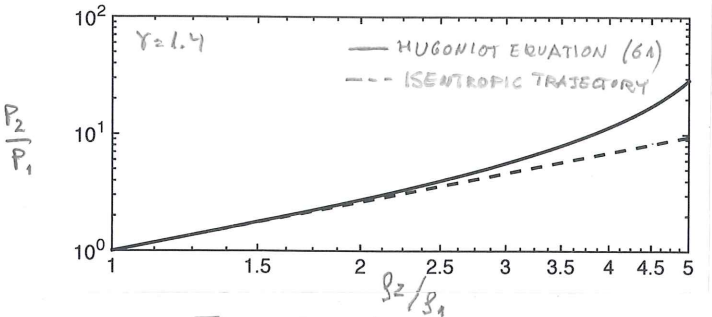


FIG. 13: COMPARISON BETWEEN PRESSURE JUMP ACROSS A SHOCK AND AN ISENTROPIC COMPRESSION.

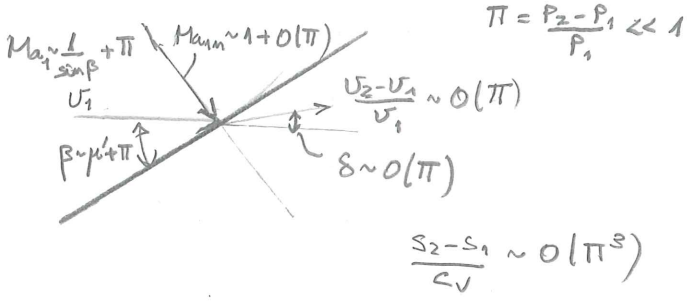


FIG. 14: SCALING ACROSS A WEAK SHOCK

OBLIQUE WEAK SHOCKS ARISE, FOR INSTANCE, IN GRADUAL SUPERSONIC COMPRESSION, AS IN THE RAMP FLOW SKETCHED IN FIG. 15. THERE, THE TURNING OF THE FLOW IS ACHIEVED THROUGH A LARGE NUMBER OF WEAK OBLIQUE SHOCKS. AWAY FROM THE WALL, THE OBLIQUE WEAK SHOCKS MAY COALESCE AND FORM A STRONGER SHOCK WITH AN ORDER-UNITY PRESSURE VARIATION Π ACROSS.

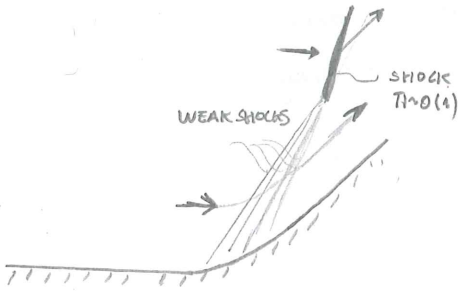


FIG. 15.

MOVING NORMAL SHOCKS

IN MANY APPLICATIONS THE SHOCK WAVES MOVE IN THE LABORATORY REFERENCE FRAME. EXAMPLES OF THESE ARE SHOCK TUBES, BLAST WAVES, AND ACOUSTIC WAVES. IN THE LABORATORY REFERENCE FRAME, THE VELOCITY FIELD RESEMBLES THAT IN FIG. 16, WITH $U_S = U_1$ THE SHOCK SPEED AND $U_1 - U_2$ THE VELOCITY OF THE POST-SHOCK GASES IN THE SAME DIRECTION AS THE ONE OF THE SHOCK PROPAGATION. IN THIS REFERENCE FRAME, IT IS STRAIGHTFORWARD TO COMPUTE THE SPEED OF THE SHOCK AND POST-SHOCK GAS:

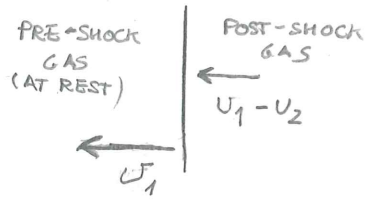


FIG. 16.
SHOCK MOTION IN THE LABORATORY FRAME

USING THE PRESSURE JUMP EQUATION (66),

$$Ma_1^2 = Ma_S^2 = \left(\frac{U_1}{a_1}\right)^2 = \left(\frac{U_S}{a_1}\right)^2 = \frac{\gamma+1}{2\gamma}\frac{P_2}{P_1} + \frac{\gamma-1}{2\gamma}$$

AND THEREFORE

$$\boxed{U_S = a_1\sqrt{\frac{(\gamma+1)}{2\gamma}\frac{P_2}{P_1} + \frac{(\gamma-1)}{2\gamma}}}, \quad (80)$$

THE SPEED OF THE POST-SHOCK GASES IS

$$U_1 - U_2 = U_1\left(1 - \frac{U_2}{U_1}\right) = U_S\left(1 - \frac{U_2}{U_1}\right) = \frac{a_1}{\gamma}\left(\frac{P_2}{P_1} - 1\right)\left\{\frac{\frac{2\gamma}{\gamma+1}}{\frac{P_2}{P_1} + \frac{\gamma-1}{\gamma+1}}\right\}^{1/2} \quad (81)$$

WHERE USE OF (63) AND (80) HAS BEEN MADE.

III ACOUSTICS

THE SPEED OF SOUND IS THE PROPAGATION VELOCITY OF DISTURBANCES IN THE GASDYNAMIC EQUATIONS. IN THIS CHAPTER, THE FOCUS IS ON THE PROPAGATION OF THOSE DISTURBANCES, WHICH ARE FIRST ASSUMED TO BE SMALL, ACROSS THE FLOW FIELD. IT WILL BE DESCRIBED THAT AFTER APPROPRIATE ASSUMPTIONS THE GASDYNAMIC EQUATIONS CAN BE WRITTEN AS WAVE EQUATIONS. THE LAST PART OF THIS CHAPTER BRIEFLY ADRESSES THE SOUND GENERATED BY TURBULENCE.

THE VELOCITY POTENTIAL EQUATION

THE PROPAGATION OF LINEAR ACOUSTIC WAVES IS ESSENTIALLY AN IRROTATIONAL INVISCID PHENOMENON AMENABLE TO BEING DESCRIBED BY USING THE VELOCITY POTENTIAL ϕ DEFINED AS

$$\vec{U} = \nabla\phi. \quad (82)$$

CONSIDERING THE EULER EQUATION

$$\frac{\partial\vec{U}}{\partial t} + \vec{U}\cdot\nabla\vec{U} = -\frac{\nabla P}{\rho}$$

AND ASSUMING THE FLOW IS ISENTROPIC, $-\nabla P = -a^2\nabla\rho$, THEN MULTIPLICATION OF THE EULER EQUATION BY $\vec{U}$ YIELDS

$$\frac{\partial}{\partial t}\left(\frac{|\vec{U}|^2}{2}\right) + \vec{U}\cdot\nabla\left(\frac{|\vec{U}|^2}{2}\right) = -\frac{a^2}{\rho}\vec{U}\cdot\nabla\rho = +a^2\nabla\cdot\vec{U} + \frac{a^2}{\rho}\frac{\partial\rho}{\partial t}$$

(CONTINUITY)

SUBSTITUTING (82) INTO THIS EXPRESSION, ONE OBTAINS

$$\frac{\partial}{\partial t}\left(\frac{|\nabla\phi|^2}{2}\right) + \nabla\phi\cdot\nabla\left(\frac{|\nabla\phi|^2}{2}\right) - a^2\nabla^2\phi - \frac{a^2}{\rho}\frac{\partial\rho}{\partial t} = 0. \quad (83)$$

IN ORDER TO ELIMINATE ρ FROM THIS EQUATION, DIFFERENTIATE THE EULER EQUATION WITH RESPECT TO TIME

$$\frac{\partial^2}{\partial t^2}(\nabla\phi) + \frac{\partial}{\partial t}\left[\nabla\left(\frac{|\nabla\phi|^2}{2}\right)\right] = -\frac{\partial}{\partial t}\left(\frac{\nabla P}{\rho}\right) = -\frac{\partial}{\partial t}\left(\frac{a^2}{\rho}\nabla\rho\right),$$

AND SINCE $\frac{\partial}{\partial t}\left(\frac{a^2}{\rho}\nabla\rho\right) = \nabla\left(\frac{a^2}{\rho}\frac{\partial\rho}{\partial t}\right)$ WHEN THE FLOW IS ISENTROPIC, THEN ($a^2 = \frac{\gamma p}{\rho}$ AND $p = \text{const.}\,\rho^\gamma$)

$$\nabla\left(\frac{\partial^2\phi}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{|\nabla\phi|^2}{2}\right) + \frac{a^2}{\rho}\frac{\partial\rho}{\partial t}\right) = 0 \quad , \quad \text{OR EQUIVALENTLY,}$$

$$\frac{a^2}{\rho}\frac{\partial\rho}{\partial t} = -\frac{\partial^2\phi}{\partial t^2} - \frac{\partial}{\partial t}\left(\frac{|\nabla\phi|^2}{2}\right) \quad , \quad \text{WHERE THE CONSTANT OF INTEGRATION HAS}$$

BEEN ABSORBED INTO ϕ. SUBSTITUTING THIS EXPRESSION INTO (83), THE GENERAL VELOCITY-POTENTIAL EQUATION

$$\boxed{\frac{\partial}{\partial t}\left(|\nabla\phi|^2\right) + \nabla\phi\cdot\nabla\left(\frac{|\nabla\phi|^2}{2}\right) - a^2\nabla^2\phi + \frac{\partial^2\phi}{\partial t^2} = 0} \qquad (84)$$

IS OBTAINED. THE STEADY VERSION OF (84) IS

$$a^2\nabla^2\phi - \nabla\phi\cdot\left(\nabla\phi\cdot\nabla\right)\nabla\phi = 0$$

WHERE THE IDENTITY $\frac{1}{2}(\nabla\phi)\cdot\nabla\left(\frac{|\nabla\phi|^2}{2}\right) = \nabla\phi\cdot\left(\nabla\phi\cdot\nabla\right)\nabla\phi$ HAS BEEN USED. FOR ILLUSTRATION, CONSIDER THE TWO-DIMENSIONAL VERSION OF THE STEADY VELOCITY-POTENTIAL EQUATION:

$$\left(a^2 - \phi_x^2\right)\phi_{xx} + \left(a^2 - \phi_y^2\right)\phi_{yy} = 2\phi_x\phi_y\phi_{xy} \quad , \qquad (85)$$

WHERE $\phi_x = \partial\phi/\partial x$ AND $\phi_y = \partial\phi/\partial y$. IT CAN BE SHOWN THAT IN PROBLEMS CONSISTING OF UNIFORM STREAMS AT VELOCITIES U_∞ FLOWING AROUND THIN AIRFOILS OR OTHER SLENDER GEOMETRICAL OBSTACLE THAT CREATES ONLY A SMALL VELOCITY DISTURBANCE IN THE FLOW (FIG. 15), EQ. (85) BECOMES

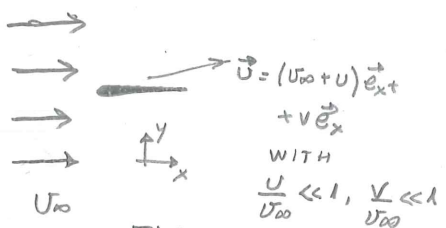


FIG. 15

$$\left\{\begin{array}{l} \left(1 - Ma_\infty^2\right)\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = 0 \qquad (86). \\ \text{SUBJECT TO } \dfrac{\partial\phi}{\partial x} \to \dfrac{\partial\phi}{\partial y} \to 0 \text{ AT } \vec{x} \to \infty, \text{ AND } \dfrac{\partial\phi}{\partial n} = 0 \text{ ON THE BODY.} \end{array}\right.$$

EQ. (86) IS A LINEAR POTENTIAL EQUATION VALID FOR SUBSONIC AND SUPERSONIC FLOWS AWAY FROM TRANSONIC CONDITIONS. (ie., TRANSONIC REGIMES Ma_∞ REQUIRE RETENTION OF HIGHER-ORDER APPROXIMATIONS). IT CAN BE SHOWN THROUGH STANDARD ANALYSIS THAT THE EQUATION OF THE CHARACTERISTICS CORRESPONDING TO (86) IS

$$\frac{dy}{dx} = \pm\tan\mu' = \pm\sqrt{\frac{1}{Ma_\infty^2 - 1}} \quad . \qquad (87)$$

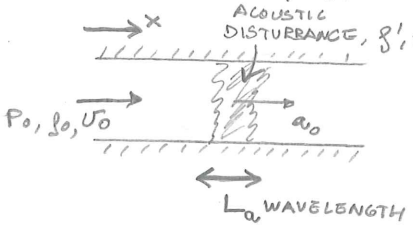
FOR $Ma_\infty < 1$, THE CHARACTERISTICS ARE COMPLEX AND EQ. (86) IS ELLIPTIC. ON THE OTHER HAND, FOR $Ma_\infty > 1$, THE CHARACTERISTICS ARE REAL AND EQ. (87) IS HYPERBOLIC

LINEAR SOUND WAVES

CONSIDER THE LIMITING CASE IN WHICH CHANGES IN DENSITY AND PRESSURE ACCOMPANYING THE FLUID MOTION ARE VERY SMALL IN COMPARISON WITH THE AVERAGE VALUES ρ_0 AND P_0,

$$\frac{\rho'}{\rho_0} = \frac{\rho - \rho_0}{\rho_0} = O(M_a) \ll 1, \quad \text{AND} \quad \frac{P'}{P_0} = \frac{P - P_0}{P_0} = O(M_a) \ll 1 . \tag{88}$$

THE ANALYSIS IS BETTER ILLUSTRATED BY CONSIDERING, FOR INSTANCE, THE PROPAGATION OF THESE DISTURBANCES IN A GAS FLOWING IN A PIPE AT A CHARACTERISTIC VELOCITY U_0, WITH $U_0/a_0 = M_a \ll 1$. IN ADDITION TO (88), THE RELEVANT LIMIT IS ONE IN WHICH THE PRESSURE AND DENSITY DISTURBANCES CREATE A VELOCITY DISTURBANCE u' OF THE SAME ORDER AS U_0 AND SMALL COMPARED WITH THE SPEED OF SOUND,



$$\frac{u'}{U_0} = \frac{u - U_0}{U_0} \sim O(1), \quad \frac{u'}{a_0} \ll 1 \tag{89}$$

IN THE ABSENCE OF VISCOUS OR HEAT-CONDUCTION EFFECTS (SEE COMMENTS AT THE END OF THIS SECTION ON THE CORRESPONDING ORDERS OF MAGNITUDE), THE PROPAGATION OF THESE DISTURBANCES IS DESCRIBED BY THE ISENTROPIC, INVISCID EQUATIONS OF MOTION

$$\left.\begin{aligned} &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0 \\ &\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{a^2}{\rho}\frac{\partial \rho}{\partial x} \end{aligned}\right\} \tag{90}$$

WHERE USE OF $a^2 = \left(\frac{\partial P}{\partial \rho}\right)_s$ HAS BEEN MADE IN ORDER TO ELIMINATE THE PRESSURE.

EQS. (90) CAN BE FURTHER SIMPLIFIED IN THE "LINEAR ACOUSTICS" LIMIT BY USING THE CONSTRAINTS (88)-(89). FOR INSTANCE, IN THE CONTINUITY EQUATION:

$$\underbrace{\frac{\partial \rho}{\partial t}}_{O(\rho'/t_0)} + \underbrace{u\frac{\partial \rho}{\partial x}}_{O(U_0\rho'/L_a)} + \underbrace{\rho\frac{\partial u}{\partial x}}_{O(\rho_0 U_0/L_a)} = 0 \quad \Rightarrow \quad \boxed{\frac{\partial \rho'}{\partial t} + \rho_0\frac{\partial u'}{\partial x} = 0} \tag{91}$$

SINCE $\frac{U_0\rho'}{u'\rho_0} \sim \frac{\rho'}{\rho_0} \ll 1$

NOTE ALSO THAT $\frac{\rho'}{t_0} \sim \frac{U_0\rho_0}{L} \Rightarrow \boxed{t_0 \sim \frac{\rho'}{\rho_0}\frac{L_a}{U_0} \sim \frac{L_a}{a_0}}$ ACOUSTIC WAVE PERIOD

ALSO, IN THE MOMENTUM EQUATION

$$\underbrace{\frac{\partial u}{\partial t}}_{O(U_0/t_0)} + \underbrace{u\frac{\partial u}{\partial x}}_{O(U_0^2/L_a)} + \underbrace{\frac{a^2}{\rho}\frac{\partial \rho}{\partial x}}_{O(a_0^2\rho'/L_a\rho_0)} \quad \Rightarrow \quad \boxed{\frac{\partial u'}{\partial t} + \frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} = 0} \tag{92}$$

WHERE $a_0^2 = \frac{\gamma P_0}{\rho_0}$

SINCE $\frac{U_0^2/L}{a_0^2\rho'/\rho_0 L} = \frac{U_0^2}{a_0^2}\left(\frac{\rho_0}{\rho'}\right) \ll 1$

NOTE THAT (91) AND (92) ARE CONSISTENT WITH (88), IN THAT

$$\frac{\partial u'}{\partial t} \sim -\frac{a_0^2}{\rho_0}\frac{\partial \rho'}{\partial x} \Rightarrow \frac{U_0}{t_0} \sim \frac{a_0^2}{\rho_0}\frac{\rho'}{L_a} \Rightarrow \frac{\rho'}{\rho_0} \sim \frac{U_0}{a_0} = M_a \ll 1 .$$

AND $\frac{\partial \rho'}{\partial t} \sim -\rho_0 \frac{\partial u'}{\partial x} \Rightarrow \frac{\rho'}{t_0} \sim \rho_0 \frac{U_0}{L_a} \Rightarrow \frac{\rho'}{\rho_0} \sim \frac{U_0}{a_0} = Ma \ll 1.$

SIMILARLY, SINCE $dP = a_0^2 d\rho$ FOR ISENTROPIC FLOWS, THEN $P' = a_0 \rho'$ AND

$$\frac{P'}{P_0} = \frac{a_0^2 \rho'}{P_0} = \frac{a_0^2}{P_0} \rho_0 \frac{\rho'}{\rho_0} = \frac{\gamma \rho'}{\rho_0} = O(Ma) \ll 1.$$

NOTE ALSO THAT $\frac{T'}{T_0} \sim \frac{P'}{P_0} - \frac{\rho'}{\rho_0}$ BECAUSE OF THE EQUATION OF STATE $P = \rho R_g T$, AND THEREFORE $\frac{T'}{T_0} = O(Ma) \ll 1$ IN THIS LIMIT. IN SUMMARY, EQUATIONS (91)-(92) REPRESENT THE DESCRIPTION OF LINEAR ACOUSTIC WAVES. FURTHERMORE, DIFFERENTIATION OF (91) WITH RESPECT TO TIME YIELDS

$$\frac{\partial^2 \rho'}{\partial t^2} + \rho_0 \frac{\partial}{\partial x}\left(\frac{\partial u'}{\partial t}\right) = 0$$

WHICH, IN CONJUNCTION WITH (92) GIVES THE WAVE EQUATION

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{\rho'}{\rho_0}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{\rho'}{\rho_0}\right)} \qquad (93)$$

FOR THE DENSITY CHANGES, SIMILAR MANIPULATION OF EQ (92) YIELDS

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{u'}{a_0}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{u'}{a_0}\right)} \qquad (94)$$

FOR THE VELOCITY CHANGES. ADDITIONALLY, THE ISENTROPIC RELATION $P' = a_0^2 \rho'$ MAKES POSSIBLE TO EXPRESS (93) IN TERMS OF THE PRESSURE CHANGES AS

$$\boxed{\frac{\partial^2}{\partial t^2}\left(\frac{P'}{\rho_0 a_0^2}\right) = a_0^2 \frac{\partial^2}{\partial x^2}\left(\frac{P'}{\rho_0 a_0^2}\right)}. \qquad (95)$$

LASTLY, TAKING THE CURL OF (92) SHOWS THAT $\vec{\omega}' = \nabla \wedge u' = 0$, AND THEREFORE $u' = \frac{\partial \phi}{\partial x}$. SUBSTITUTING THIS RESULT INTO (92) YIELDS

$$\frac{\partial}{\partial t}\left(\frac{\partial \phi}{\partial x}\right) + a_0^2 \frac{\partial}{\partial x}\left(\frac{\rho'}{\rho_0}\right) = 0 \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial t} + \frac{a_0^2 \rho'}{\rho_0}\right) = 0 \Rightarrow \frac{\partial \phi}{\partial t} + a_0^2 \frac{\rho'}{\rho_0} = 0$$

WHERE THE CONSTANT OF INTEGRATION HAS BEEN ABSORBED INTO ϕ. SUBSTITUTING THIS RESULT INTO (91) GIVES

$$\boxed{\frac{\partial^2 \phi}{\partial t^2} = a_0^2 \frac{\partial^2 \phi}{\partial x^2}} \qquad (96)$$

WHICH REPRESENTS THE LINEAR-ACOUSTICS LIMIT OF EQ. (84). EQUATIONS (93)-(95) DESCRIBE THE PROPAGATION OF LINEAR SOUND WAVES.

THE GENERAL SOLUTION OF (93)-(95) CORRESPONDS TO THE SUM OF TWO WAVES $F_{(\rightarrow)}$ AND $G_{(\leftarrow)}$ PROPAGATING IN THE $+x$ AND $-x$ DIRECTIONS, RESPECTIVELY:

$$\left.\begin{aligned} \frac{\rho'}{\rho_0} &= F_{(\rightarrow)}(x-a_0t) + G_{(\leftarrow)}(x+a_0t) \\ \frac{p'}{\rho_0 a_0^2} &= F_{(\rightarrow)}(x-a_0t) + G_{(\leftarrow)}(x+a_0t) \\ \frac{u'}{a_0} &= F_{(\rightarrow)}(x-a_0t) - G_{(\leftarrow)}(x+a_0t) \end{aligned}\right\} \quad (96)$$

$$\Rightarrow \boxed{\begin{aligned} F &= \frac{1}{2}\left(\frac{u'}{a_0} + \frac{\rho'}{\rho_0}\right) \\ G &= \frac{1}{2}\left(\frac{\rho'}{\rho_0} - \frac{u'}{a_0}\right) \\ \rho'/\rho_0 &= p'/\rho_0 a_0^2 \end{aligned}}$$

$F_{(\rightarrow)} \equiv$ RIGHT-RUNNING WAVE

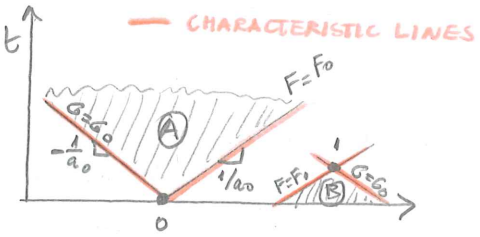
$G_{(\rightarrow)} \equiv$ LEFT-RUNNING WAVE.

NOTE: $\frac{\partial}{\partial t}\left(\frac{u'}{a_0}\right) = -a_0 F + a_0 G$; $a_0\frac{\partial}{\partial x}\left(\frac{u'}{a_0}\right) = a_0 F - a_0 G$; $\frac{\partial}{\partial t}\left(\frac{\rho'}{\rho_0}\right) = -a_0 F + a_0 G$; $a_0\frac{\partial}{\partial x}\left(\frac{\rho'}{\rho_0}\right) = a_0 F + a_0 G$

WHERE THE PARTICULAR FORM OF F AND G DEPENDS ON THE INITIAL CONDITIONS OF THE DISTURBANCE, INCLUDING ITS SPATIAL DISTRIBUTION. IT IS HOWEVER IMPORTANT TO REALIZE THAT (93)-(95) CAN BE REWRITTEN IN CHARACTERISTIC FORM

$$\left(\frac{\partial}{\partial t} \pm a_0\frac{\partial}{\partial x}\right)\left(\frac{\partial \varphi}{\partial t} \mp a_0\frac{\partial \varphi}{\partial x}\right) = 0 \qquad (97)$$

FOR $\varphi = \rho'/\rho_0$, p'/p_0 OR u'/a_0, WHERE $\frac{dx}{dt} = \pm a_0$ ARE THE "CHARACTERISTIC LINES".



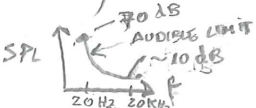
(A): DOMAIN OF INFLUENCE OF POINT "0"

(B): DOMAIN OF DEPENDENCE OF POINT "1".

FIG. 16

IN PARTICULAR, THE FLOW INFORMATION PROPAGATES ALONG THE CHARACTERISTICS, WITH THE HATCHED REGION (A) IN THE SIDE FIGURE REPRESENTING THE DOMAIN OF INFLUENCE WHERE THE INFORMATION AT POINT "0" AT $t=0$ IS FELT AT A LATER TIME. SIMILARLY, ANY POINT IN THE SPATIO TEMPORAL PLANE $\{x,t\}$ HAS BEEN INFLUENCED BY ALL POINTS LYING IN ITS DOMAIN OF DEPENDENCE (B), WHICH IS ENCLOSED BY THE THE TWO INTERSECTING CHARACTERISTICS PASSING THROUGH THE POINT. (e.g., POINT "1" IN THE SIDE FIGURE).

* CHARACTERISTIC ORDERS OF MAGNITUDE



THE TYPICAL RANGE OF FREQUENCIES AUDIBLE BY THE HUMAN EAR IS $f \sim 20 - 20\,kHz$ WITH MINIMUM SOUND PRESSURE LEVELS ($SPL = 20\log_{10}(p'/p_{ref})$) OF ORDER 10-70 dB. THE REFERENCE PRESSURE p_{ref} IN THE SPL IS USUALLY TAKEN TO BE $p_{ref} = 2\cdot10^{-5}\,Pa$. THE THRESHOLD OF EAR PAIN IS IN 140 dB, WHICH CORRESPONDS TO $p' = 200\,Pa$, OR EQUIVALENTLY, $p'/p_0 = 2\%$ OF THE ATMOSPHERIC PRESSURE. THIS CORRESPONDS TO A CHARACTERISTIC DENSITY VARIATION $\rho'/\rho_0 = \frac{p'}{\rho_0 a_0^2} = 0.15\%$ AND A VELOCITY

$u' \sim \frac{p'}{\rho_0 a_0} \sim 0.5$ mm/s, WITH AN ASSOCIATED MACH NUMBER $Ma \sim 0.001$. THE CORRESPONDING AMPLITUDE OF THE FLUID PARTICLE DISPLACEMENT AS THE ACOUSTIC WAVE PASSES BY IS OF ORDER $u'/f \sim 0.025 - 2.5\,10^{-5}$ m FOR FREQUENCIES $f \sim$ 20Hz - 20 kHz. THIS DISPLACEMENT IS SMALL COMPARED TO THE CHARACTERISTIC WAVELENGTH $L_a = a_0/f \sim 17 - 0.017$ m, WITH THEIR RATIO BEING PROPORTIONAL TO THE INVERSE OF THE MACH NUMBER, $\frac{L_a f}{u'} \sim \frac{1}{Ma} \gg 1$. NOTE THAT MOLECULAR TRANSPORT PLAYS A NEGLIGIBLE ROLE IN THE PROPAGATION OF THE ACOUSTIC WAVE. FOR INSTANCE, CONSIDER THE RATIO OF UNSTEADY TO VISCOUS TERMS IN THE MOMENTUM EQUATION

$$\frac{\rho_0 \frac{\partial u'}{\partial t}}{\mu_0 \frac{\partial^2 u'}{\partial x^2}} \sim \frac{\frac{\rho_0 u'}{t_0}}{\mu_0 \frac{u'}{L_a^2}} \sim \frac{a_0 L_a}{\nu} \sim \frac{a_0 L_a}{\lambda a_0} \sim \frac{L_a}{\lambda} \gg 1 .$$

(λ: MEAN FREE PATH.)

SOUND GENERATION BY TURBULENCE

AT HIGH REYNOLDS NUMBERS, TURBULENT VELOCITY FLUCTUATIONS PRODUCE SOUND IN THE SURROUNDING FLUID. AN IMPORTANT THEORY OF THIS PHENOMENON WAS FORMULATED BY LIGHTHILL (PROC. ROYAL SOC. LONDON, 1952, 1954), WHICH IS SUMMARIZED BELOW. CONSIDER THE CASE OF A TURBULENT FLOW OCCUPYING A FINITE REGION OF SPACE V_0 SURROUNDED BY AN INFINITE VOLUME OF FLUID AT REST. LIGHTHILL'S FORMULATION OF THIS PROBLEM CONSISTS OF A PARTICULAR REARRANGEMENT OF THE CONSERVATION EQUATIONS THAT REVEALS SOME IMPORTANT CHARACTERISTICS. TAKING THE PARTIAL DERIVATIVE OF THE CONTINUITY EQUATION (11) WITH RESPECT TO TIME GIVES

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left[\nabla\cdot(\rho \vec{u})\right] = 0 , \qquad (98)$$

WHILE THE DIVERGENCE OF THE MOMENTUM EQUATION (12) YIELDS

$$\frac{\partial}{\partial t}\left(\nabla\cdot(\rho\vec{u})\right) + \nabla\cdot\left(\nabla\cdot(\rho\vec{u}\vec{u})\right) = -\nabla^2 P + \nabla\cdot\left(\nabla\cdot\bar{\bar{\tau}}'\right) . \qquad (99)$$

SUBSTRACTING (98) FROM (99),

$$\frac{\partial^2 \rho}{\partial t^2} - \nabla^2 P = +\nabla\cdot\left(\nabla\cdot(\rho\vec{u}\vec{u})\right) - \nabla\cdot\left(\nabla\cdot\bar{\bar{\tau}}'\right) , \qquad (100)$$

WHICH CAN BE REWRITTEN AS

$$\boxed{\frac{\partial^2 \rho'}{\partial t^2} - a_0^2 \nabla^2 \rho' = \nabla\cdot\left(\nabla\cdot\bar{\bar{\mathcal{T}}}\right)} \qquad (101)$$

WHERE USE OF THE DEFINITION $\rho' = \rho - \rho_0$, HAS BEEN MADE, WHERE ONLY ρ' VARIES

ON TIME, AND a_0 IS A UNIFORM SPEED OF SOUND IN THE SURROUNDING FLUID. IN EQUATION (100), THE FORCING TENSOR $\bar{\bar{\mathcal{T}}}$ IS GIVEN BY

$$\bar{\bar{\mathcal{T}}} = \rho \vec{v}\vec{v} - \bar{\bar{\tau}}' + (p' - a_0^2 \rho')\bar{\bar{I}}, \quad (102)$$

NOTE: $s' \sim \frac{c_v}{p_0}(p' - a_0^2 \rho')$ WHEN $p', \rho' \ll 1$ AS OBSERVED FROM EQ. (10)

WHERE IT HAS BEEN ADDITIONALLY ASSUMED THAT ρ_0 IS UNIFORM, AS IF THE TURBULENT FLOW WAS MOVING AT LOW MACH NUMBERS. IN THAT LIMIT, THE DENSITY IN THE CONVECTIVE FLUX IS APPROXIMATELY ρ_0, i.e., $\rho \vec{v}\vec{v} \sim \rho_0 \vec{v}\vec{v}$ IN (102). THREE CHARACTERISTIC SOURCES OF SOUND CAN BE DISTINGUISHED IN (102): i) THE NON-LINEAR CONVECTIVE FLUX $\rho_0 \vec{v}\vec{v}$, ii) THE VISCOUS STRESS $\bar{\bar{\tau}}'$, AND iii) THE DEVIATIONS FROM ISENTROPIC FLOW $p' - a_0^2 \rho'$.

NOTE THAT EQ. (100) CAN ALSO BE REWRITTEN AS

$$\frac{1}{a_0^2}\frac{\partial^2 p'}{\partial t^2} + \nabla^2 p' = \nabla \cdot (\nabla \cdot \bar{\bar{\mathcal{T}}}) + \frac{\partial^2}{\partial t^2}\left(\frac{p'}{a_0^2} - \rho'\right) \quad (103)$$

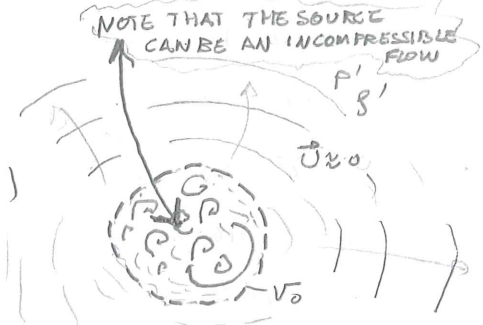
NOTE THAT THIS EQUATION SIMPLIFIES TO THE POISSON EQUATION $\nabla^2 p' = \nabla \cdot (\nabla \cdot \bar{\bar{\mathcal{T}}})$ FOR INCOMPRESSIBLE FLOWS FOR $Ma \ll 1$!

FOR p' INSTEAD OF ρ', WHERE $\bar{\bar{\mathcal{T}}}$ IS GIVEN BY

$$\bar{\bar{\mathcal{T}}} = \rho \vec{v}\vec{v} - \bar{\bar{\tau}}', \quad (103)$$

NOTE THAT THE DEVIATIONS FROM ISENTROPICITY IN (103) ARE MANIFESTED BY $\frac{\partial^2}{\partial t^2}\left(\frac{p'}{a_0^2} - \rho'\right) \neq 0$ INSTEAD OF $\nabla^2(p' - a_0^2 \rho') \neq 0$ IN (101). IN THE ABSENCE OF THESE EFFECTS, (104) AND (103) BECOME

$$\left.\begin{aligned} \frac{\partial^2 \rho'}{\partial t^2} - a_0^2 \nabla^2 \rho' &= \nabla \cdot (\nabla \cdot (\rho_0 \vec{v}\vec{v} - \bar{\bar{\tau}})) \quad (Re \gg 1) \\ \frac{1}{a_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' &= \nabla \cdot (\nabla \cdot (\rho_0 \vec{v}\vec{v} - \bar{\bar{\tau}})) \quad (Re \gg 1) \end{aligned}\right\} \quad (104)$$



WHICH CORRESPOND TO INHOMOGENEOUS VERSIONS OF EQS. (93) AND (95), AND DESCRIBE THE PROPAGATION OF DENSITY AND PRESSURE FLUCTUATIONS AROUND THE TURBULENT FLOW AND ACROSS THE SURROUNDING FLOW AT REST. NOTE THAT THE RIGHT-HAND SIDE OF (104) IS OF SECOND ORDER OUTSIDE V_0 AND MAY BE NEGLECTED, SO THAT (104) BECOME THERE EQUAL TO THEIR HOMOGENEOUS COUNTERPARTS (93) AND (94) CORRESPONDING TO A SOURCELESS PROPAGATION OF THE SOUND WAVES. THE RIGHT-HAND SIDE SOURCE IN (104) ACTS AS A SOURCE OF SOUND IN V_0, WHERE THE TURBULENT FLOW MOVES AT VELOCITY $\vec{v}$. EQUATIONS (104) ARE THE BASIS OF THE LIGHTHILL ANALOGY: "THE DENSITY AND PRESSURE FLUCTUATIONS IN A LOW-MACH FLOW MAY BE IDENTIFIED WITH THOSE OCCURRING IN A UNIFORM ACOUSTIC MEDIUM AT REST PRODUCED BY THE SOURCE $\nabla \cdot (\nabla \cdot \bar{\bar{\mathcal{T}}})$, WHERE $\bar{\bar{\mathcal{T}}} \approx \rho_0 \vec{v}\vec{v}$ IS THE EFFECTIVE FORCING TENSOR."

IT IS IMPORTANT TO NOTE THAT MOST OF THE TURBULENCE ENERGY IS AT THE INTEGRAL SCALES U_ℓ AND ℓ. THE CHARACTERISTIC TURNOVER TIME OF THESE EDDIES IS $t_\ell \sim \ell/U_\ell$. AS A RESULT, THESE ARE ALSO IMPORTANT FREQUENCIES OF THE RADIATED SOUND, WHICH HAS A CORRESPONDING CHARACTERISTIC WAVELENGTH

$$\left[L_a \sim a_0 t_\ell \sim \frac{a_0 \ell}{U_\ell} = \frac{\ell}{Ma_\ell} \right] \quad (105)$$

WITH $Ma_\ell = U_\ell / a_0$ THE TURBULENT MACH NUMBER. AS Ma_ℓ INCREASES, THE SOUND SOURCE BECOMES INCREASINGLY LESS COMPACT, IN THAT L_a BECOMES OF THE SAME ORDER AS, OR SMALLER THAN, THE LARGE-EDDY SIZE ℓ. EXCEPT FOR HIGHLY SUPERSONIC TURBULENCE, IN MOST PRACTICAL SITUATIONS $Ma_\ell \lesssim 0.4$, AND A SIMPLIFIED THEORY EXISTS TO INTEGRATE (104) IN THE LIMIT OF COMPACT SOURCES, AS DESCRIBED BELOW.

THE PROPAGATION EQUATION FOR p' IN (104) IS OF THE RETARDED-POTENTIAL TYPE. RETARDED POTENTIALS ARE UBIQUITOUSLY FOUND IN ELECTRODYNAMICS WHEN THE ELECTRIC CURRENT OR THE ELECTRIC CHARGE VARY IN TIME. SINCE THE ELECTROMAGNETIC FIELD PROPAGATES AT FINITE SPEED, IT TAKES A FINITE TIME FOR THE INFORMATION TO PROPAGATE FROM THE TIME-VARYING CAUSE (THE CHARGE) TO ANOTHER POINT IN SPACE WHERE THE EFFECT IS OBSERVED. IN SOUND RADIATION, THE PRESSURE AND DENSITY DISTURBANCES PROPAGATE AT THE SPEED OF SOUND, AND THE CAUSE OF THE SOUND IS THE TURBULENT FLOW WITHIN THE VOLUME V_0. EXPRESSING THE SECOND EQUATION IN (104) AS

$$\Box(p') = \nabla\cdot(\nabla\cdot\bar{\bar{\mathcal{T}}}) \,, \quad \text{WHERE } \Box = \frac{1}{a_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \text{ IS THE D'ALEMBERT OPERATOR}$$

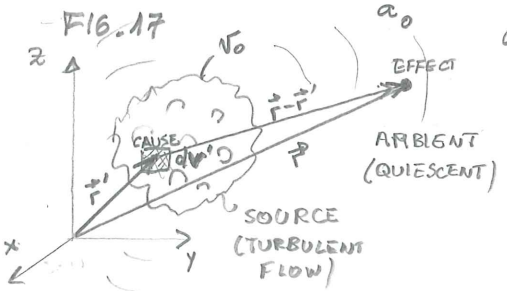
AND $\bar{\bar{\mathcal{T}}} = \rho_0 \vec{u}\vec{u}$ IS THE FORCING TENSOR, AND PERFORMING THE INTEGRATION IN 3D FREE SPACE, THE CORRESPONDING SOLUTION IS

$$p'(\vec{r},t) = \frac{1}{4\pi}\int_{V_0} \nabla\cdot\{\nabla\cdot\bar{\bar{\mathcal{T}}}(\vec{r}',t_R)\}\frac{dV'}{|\vec{r}-\vec{r}'|} \quad (106)$$

→ SEE FIG. 17 FOR INTERPRETATION OF $\vec{r}$ AND $\vec{r}'$.

WHERE $t_R = t - \frac{|\vec{r}-\vec{r}'|}{a_0}$ IS THE RETARDED TIME. CONSIDERATION IS GIVEN HERE TO CASES WHERE THE SOURCE IS COMPACT, i.e. $L_a/V_0^{1/3} \sim \frac{L_a}{\ell} \gg 1$, WHICH REQUIRES SMALL TURBULENT MACH NUMBERS, $Ma_\ell \ll 1$, AS IMPLIED BY (105). IN THIS LIMIT, $\vec{r}-\vec{r}' \sim \vec{r}$ (SEE FIG. 17), AND (106) BECOMES

FIG. 17



$$p'(\vec{r},t) = \frac{1}{4\pi r}\int_{V_0} \nabla\cdot\{\nabla\cdot\overline{\overline{\mathcal{T}}}(\vec{r}',t_R)\}\,dV' \qquad (107)$$

$t_R = t - \frac{|\vec{r}-\vec{r}'|}{a_0} \approx t - \frac{r}{a_0} = \tau$ ($r \gg r'$)

CHAIN RULE $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial \tau}\frac{\partial \tau}{\partial x_i}\frac{\partial r}{\partial r}$, $m_i \downarrow$, $\downarrow \frac{1}{a_0}$

WHICH FURTHER BECOMES

$$p'(\vec{r},t) \approx \frac{1}{4\pi r}\nabla_{\vec{r}}\left\{\nabla_{\vec{r}}\cdot\int_{V_0}\overline{\overline{\mathcal{T}}}\left(\vec{r}',t-\frac{r}{a_0}\right)dV'\right\} = \frac{\rho_0}{4\pi r a_0^2}\vec{m}\vec{m}:\int_{V_0}\overline{\overline{\ddot{\mathcal{T}}}}\,(\vec{r}',\tau)\,dV' \qquad (107)$$

$\frac{\partial}{\partial x_i} \sim \frac{\partial}{\partial x_i}$ FOR $r \gg r'$

WHERE $\tau = t - \frac{r}{a_0}$ AND $\vec{m}$ IS A UNIT VECTOR ALONG $\vec{r}$. THE DOUBLE-DOTTED TENSOR $\overline{\overline{\ddot{\mathcal{T}}}}$ REPRESENTS THE SECOND DERIVATIVE OF $\overline{\overline{\mathcal{T}}}$ WITH RESPECT TO τ. SUBSTITUTING THE DECOMPOSITION

$$\ddot{\mathcal{T}}_{ij} = \left(\ddot{\mathcal{T}}_{ij} - \frac{1}{3}\ddot{\mathcal{T}}_{kk}\delta_{ij}\right) + \frac{1}{3}\ddot{\mathcal{T}}_{kk}\delta_{ij} = Q_{ij} + Q\,\delta_{ij}$$

INTO (107), WITH $Q_{ij} = \ddot{\mathcal{T}}_{ij} - \frac{1}{3}\ddot{\mathcal{T}}_{kk}\delta_{ij}$ A TENSOR WITH ZERO TRACE AND $Q = \frac{1}{3}\ddot{\mathcal{T}}_{kk}$, THE REPRESENTATION

$$p'(\vec{r},t) = \frac{1}{4\pi a_0^2 r}\left\{\int_{V_0} Q(\vec{r}',\tau)\,dV' + \vec{m}\vec{m}:\int_{V_0}\overline{\overline{Q}}(\vec{r}',\tau)\,dV'\right\} \qquad (108)$$

IS OBTAINED, WHICH CORRESPONDS TO THE SOUND RADIATED FROM MONOPOLE (FIRST TERM ON THE RHS) AND QUADRUPOLES (SECOND TERM ON THE RHS). BASED ON (108) THE TOTAL INTENSITY OF SOUND EMITTED IS OF ORDER $I \sim \frac{\langle p'^2\rangle}{\rho_0 a_0} V_0^{2/3} \sim \frac{\langle Q\rangle^2 V_0^2}{\rho_0 a_0^5}$, WHICH HAS UNITS OF KINETIC-ENERGY SURFACE FLUX. CORRESPONDINGLY, THE AMOUNT OF ENERGY EMITTED AS SOUND PER UNIT TIME AND PER UNIT MASS OF THE TURBULENT FLOW IS OF ORDER

$$\varepsilon_s = \frac{I}{\rho_0 V_0} \sim \frac{\langle Q\rangle^2 V_0}{\rho_0^2 a_0^5} \sim \frac{1}{\rho_0^2 a_0^5}\underbrace{\left[\rho_0\frac{\ddot{u}_\ell^2}{(\ell/u_\ell)^2}\right]^2}_{\text{ORDER OF MAGNITUDE OF } \langle Q\rangle}\underbrace{\ell^3}_{\text{ORDER OF MAGNITUDE OF } V_0} \sim \frac{u_\ell^8}{a_0^5\ell}. \qquad (109)$$

COMPARING THIS RATE OF SOUND ENERGY TRANSFER WITH THE RATE OF KINETIC ENERGY TRANSFERRED ACROSS THE CASCADE IN THE TURBULENT FLOW, $\varepsilon \sim \frac{u_\ell^2}{t_\ell} \sim \frac{u_\ell^3}{\ell}$, GIVES

$$\boxed{\frac{\varepsilon_s}{\varepsilon} \sim \frac{u_\ell^8/a_0^5\ell}{u_\ell^3/\ell} \sim \left(\frac{u_\ell}{a_0}\right)^5 = Ma_\ell^5} \qquad (110)$$

WHICH INDICATES THAT THE RATE OF ENERGY RADIATED BY SOUND IS A VERY SMALL FRACTION OF THE ENERGY DISSIPATED BY THE TURBULENT FLOW MOTION, SINCE $Ma_\ell \lesssim 0.4$ IN MOST PRACTICAL APPLICATIONS.

IV. FLUCTUATION DYNAMICS IN COMPRESSIBLE TURBULENCE

INCOMPRESSIBLE TURBULENT FLOWS UNDERGO FLUCTUATIONS IN THE VELOCITY AND PRESSURE, THE LATTER BEING SMALL AND ENSURING THAT THE VELOCITY FIELD IS DIVERGENCE-FREE. ON THE OTHER HAND, IT WAS OBSERVED IN THE PREVIOUS CHAPTER THAT SOUND IS RADIATED FROM TURBULENT ZONES WHEN THE FLOW IS COMPRESSIBLE. IN PARTICULAR, THE PRESSURE DISTURBANCES PROPAGATE AT A FINITE SPEED AND INFLUENCE A FINITE-SIZE PORTION OF THE FLOW DOMAIN IN A MANNER THAT DEPENDS ON THE STATE OF THE SOURCE AT AN EARLIER TIME (e.g., SEE EQ. (104)). THESE CONCEPTS ARE FURTHER GENERALIZED IN THIS CHAPTER TO INCLUDE ADDITIONAL FLUCTUATION MODES THAT ARE OF INTEREST FOR MODELLING COMPRESSIBLE TURBULENT FLOWS. THE ANALYSIS BEGINS WITH THE LINEARIZED CONSERVATION EQUATIONS, SINCE THEY PROVIDE A USEFUL PLATFORM FOR SEPARATING THE DIFFERENT FLUCTUATION MODES, WITH NON-LINEAR COUPLING EFFECTS BEING TREATED BY THE END OF THIS CHAPTER. APPLICATIONS OF THE FORMULATION TO COMPRESSIBLE FLOW PROBLEMS ARE DEFERRED TO CHAPTER V.

THE LINEARIZED CONSERVATION EQUATIONS

IN THIS SECTION, THE CONSERVATION EQUATIONS (11), (12) AND (35), SUPPLIED WITH THE EQUATION OF STATE (17), ARE LINEARIZED AROUND A BASE STATE THAT CONSISTS OF A MEAN (i.e. ENSEMBLE AVERAGE) UNIFORM FLOW. THROUGHOUT THE CALCULATIONS, THE TERMS INVOLVING PRODUCTS OF DISTURBANCES ARE NEGLECTED ON THE BASIS THAT THE DISTURBANCES ARE SMALL.

CONSIDER THE EQUATION OF STATE $P = \rho R_g T$, AND DECOMPOSE THE VARIABLES AS $\varphi = \bar{\varphi} + \varphi'$, WITH $\varphi' \ll \bar{\varphi}$.

$$\Rightarrow \cancel{\bar{P}} + P' = (\bar{\rho} + \rho')\, R_g\, (\bar{T} + T') = \cancel{\bar{\rho} R_g \bar{T}} + \rho' R_g \bar{T} + \bar{\rho} R_g T' + \cancel{\rho' R_g T'}$$

SO THAT

$$\boxed{\frac{P'}{\bar{P}} = \frac{\rho'}{\bar{\rho}} + \frac{T'}{\bar{T}}} \quad , \quad (111)$$

WHICH RELATES THE PRESSURE, DENSITY AND TEMPERATURE DISTURBANCES.

SIMILARLY FROM THE DEFINITION OF ENTROPY (10),

$$S - \bar{S} = C_p \ln\left(\frac{T'}{\bar{T}} + 1\right) - R_g \ln\left(\frac{P'}{\bar{P}} + 1\right) ,$$

AND SINCE THE DISTURBANCES ARE SMALL,

$$\boxed{\frac{s'}{c_p} = \frac{T'}{\bar{T}} - \frac{\gamma-1}{\gamma}\frac{p'}{\bar{p}}} \qquad (112)$$

WHERE USE OF $c_p = \gamma R_g/(\gamma-1)$ HAS BEEN MADE. THE CORRESPONDING LINEARIZATION OF THE CONTINUITY EQUATION (11) IS

$$\cancel{\frac{\partial \bar{\rho}}{\partial t}} + \cancel{\bar{\vec{u}}\cdot\nabla\bar{\rho}} + \cancel{\bar{\rho}\nabla\cdot\bar{\vec{u}}} + \frac{\partial \rho'}{\partial t} + \bar{\vec{u}}\cdot\nabla\rho' + \vec{u}'\cdot\nabla\bar{\rho} + \bar{\rho}\nabla\cdot\vec{u}' + \rho'\nabla\cdot\bar{\vec{u}} = 0$$

WHERE THE NON-LINEAR DISTURBANCE TERMS HAVE BEEN NEGLECTED. THIS EQUATION CAN BE REARRANGED AS

$$\boxed{\frac{\partial \rho'}{\partial t} + \bar{\vec{u}}\cdot\nabla\rho' = -\vec{u}'\cdot\nabla\bar{\rho} - \bar{\rho}\nabla\cdot\vec{u}' - \rho'\nabla\cdot\bar{\vec{u}}} \quad . \qquad (113)$$

IN EQ. (113) THE SECOND TERM ON THE LEFT-HAND SIDE CORRESPONDS TO THE CONVECTION OF DENSITY DISTURBANCES BY THE MEAN FLOW. THE FIRST TERM ON THE RIGHT-HAND SIDE REPRESENTS STIRRING OF MEAN DENSITY GRADIENTS BY THE VELOCITY DISTURBANCES. THE SECOND AND THIRD TERMS REFER TO THE PRODUCTION OF DENSITY DISTURBANCES FROM DILATATION OF THE DISTURBED AND MEAN VELOCITY FIELDS, RESPECTIVELY.

PERFORMING A SIMILAR LINEARIZATION OF THE MOMENTUM EQUATION YIELDS

$$\cancel{\bar{\rho}\frac{\partial \bar{\vec{u}}}{\partial t}} + \rho'\frac{\partial \bar{\vec{u}}}{\partial t} + \bar{\rho}\frac{\partial \vec{u}'}{\partial t} + \rho'\bar{\vec{u}}\cdot\nabla\bar{\vec{u}} + \bar{\rho}\vec{u}'\cdot\nabla\bar{\vec{u}} + \bar{\rho}\bar{\vec{u}}\cdot\nabla\vec{u}' + \cancel{\bar{\rho}\bar{\vec{u}}\cdot\nabla\bar{\vec{u}}} = \cancel{-\nabla\bar{p}} - \nabla p' +$$
$$+ \bar{\mu}\left(\nabla\vec{u}' + \nabla\vec{u}'^{T}\right) - \frac{2}{3}\bar{\mu}\nabla\cdot\vec{u}'\bar{\bar{I}} + \mu'\left(\nabla\bar{\vec{u}} + \nabla\bar{\vec{u}}^{T}\right) - \frac{2}{3}\mu'\nabla\cdot\bar{\vec{u}}\bar{\bar{I}}$$
$$+ \cancel{\bar{\mu}\left(\nabla\bar{\vec{u}} + \nabla\bar{\vec{u}}^{T}\right)} - \cancel{\frac{2}{3}\bar{\mu}\nabla\cdot\bar{\vec{u}}\bar{\bar{I}}}$$

$$\Rightarrow \boxed{\bar{\rho}\left(\frac{\partial \vec{u}'}{\partial t} + \bar{\vec{u}}\cdot\nabla\vec{u}' + \vec{u}'\cdot\nabla\bar{\vec{u}}\right) + \rho'\bar{\vec{u}}\cdot\nabla\bar{\vec{u}} + \rho'\frac{\partial \bar{\vec{u}}}{\partial t} = -\nabla p' + \qquad (114)$$
$$+ \nabla\cdot\left\{\bar{\mu}\left(\nabla\vec{u}' + \nabla\vec{u}'^{T}\right) - \frac{2}{3}\bar{\mu}\nabla\cdot\vec{u}'\bar{\bar{I}} + \mu'\left(\nabla\bar{\vec{u}} + \nabla\bar{\vec{u}}^{T}\right) - \frac{2}{3}\mu'\nabla\cdot\bar{\vec{u}}\bar{\bar{I}}\right\}}$$

NOTE THAT THE TERMS INVOLVING VISCOSITY FLUCTUATIONS, WHICH ARE OF ORDER $\mu' \sim \sigma T'$ AS ARISING FROM THE LINEARIZATION OF THE POWER LAW IN (19), $\mu \sim T^{\sigma}$, ARE ACTIVE IN FLOWS WHERE THE MEAN VELOCITY IS NON-UNIFORM, AND BECOME OF THE SAME ORDER AS THE VISCOUS TERMS INVOLVING THE MEAN VISCOSITY WHEN $\frac{\mu'}{\bar{\mu}} \sim \sigma\frac{T'}{\bar{T}} \sim \frac{u'}{\bar{U}}$, WHICH WILL BE SHOWN TO BE THE CASE IN FLOWS OF PRACTICAL INTEREST. IN THAT LIMIT, THE FLUCTUATIONS OF VISCOSITY PLAY A ROLE IN THE VISCOUS STRESSES.

LASTLY, THE LINEARIZED VERSION OF THE ENTROPY EQUATION (35) IS

$$\frac{\bar{P}}{R_g}\frac{\partial \bar{s}}{\partial t} + \frac{\bar{P}}{R_g}\frac{\partial s'}{\partial t} + \frac{P'}{R_g}\frac{\partial \bar{s}}{\partial t} + \frac{\bar{P}}{R_g}\vec{\bar{u}}\cdot\nabla\bar{s} + \frac{\bar{P}}{R_g}\vec{u}'\cdot\nabla\bar{s} + \frac{\bar{P}}{R_g}\vec{\bar{u}}\cdot\nabla s' + \frac{P'}{R_g}\vec{\bar{u}}\cdot\nabla\bar{s} =$$

$$= \nabla\cdot(\bar{k}\nabla\bar{T}) + \nabla\cdot(\bar{k}\nabla T') + \nabla\cdot(k'\nabla\bar{T}) + 2\bar{\mu}\,\bar{\bar{S}}:\bar{\bar{S}} - \frac{2}{3}\bar{\mu}(\nabla\cdot\vec{\bar{u}})^2 + 4\bar{\mu}\,\bar{\bar{S}}:\bar{\bar{S}}' - \frac{4}{3}\bar{\mu}(\nabla\cdot\vec{\bar{u}})(\nabla\cdot\vec{u}') + 2\mu'\bar{\bar{S}}:\bar{\bar{S}} - \frac{2}{3}\mu'(\nabla\cdot\vec{\bar{u}})^2$$

$$\Rightarrow \boxed{\frac{\bar{P}}{R_g}\left(\frac{\partial s'}{\partial t} + \vec{\bar{u}}\cdot\nabla s' + \vec{u}'\cdot\nabla\bar{s}\right) + \frac{P'}{R_g}\left(\frac{\partial \bar{s}}{\partial t} + \vec{\bar{u}}\cdot\nabla\bar{s}\right) = \nabla\cdot(\bar{k}\nabla T') + \nabla\cdot(k'\nabla\bar{T}) + 4\bar{\mu}\,\bar{\bar{S}}':\bar{\bar{S}} - \frac{4}{3}\bar{\mu}(\nabla\cdot\vec{\bar{u}})(\nabla\cdot\vec{u}') + 2\mu'\bar{\bar{S}}:\bar{\bar{S}} - \frac{2}{3}\mu'(\nabla\cdot\vec{\bar{u}})^2, \quad (115)}$$

WHERE USE OF $\rho T = P/R_g$ HAS BEEN MADE ALONG WITH THE RELATION

VISCOUS DISSIPATION $= \varepsilon = \bar{\bar{\tau}}':\nabla\vec{u} = \bar{\bar{\tau}}':\left\{\frac{1}{2}(\nabla\vec{u}+\nabla\vec{u}^T)+\frac{1}{2}(\nabla\vec{u}-\nabla\vec{u}^T)\right\} = \bar{\bar{\tau}}':\bar{\bar{S}}$,

THE CONTRACTION OF A SYMMETRIC TENSOR ($\bar{\bar{\tau}}'$) WITH ANY ANTISYMMETRIC TENSOR IS ZERO

WITH $\bar{\bar{S}} = \frac{1}{2}(\nabla\vec{u}+\nabla\vec{u}^T)$ THE STRAIN-RATE TENSOR, AND $\bar{\bar{\tau}}'$ THE VISCOUS STRESS TENSOR GIVEN BY EQ. (18). SIMILAR TO THE DISCUSSION ABOUT THE IMPORTANCE OF μ' GIVEN ABOVE, IN EQ. (115) THE FLUCTUATIONS OF k',

$$\frac{k'}{\bar{k}} = \frac{\mu'}{\bar{\mu}} = \sigma\frac{T'}{\bar{T}} \quad (c_p = \text{const.})$$

BECOME IMPORTANT IN THE HEAT FLUX.

* PARTICULARIZATION TO UNIFORM MEAN FLOW

CONSIDER EQS. (111)-(115) IN THE CASE THAT THE MEAN FLOW IS UNIFORM AND STEADY, WITH $\vec{\bar{u}} = U_\infty\vec{e}_x$. THEN EQS. (111)-(112) REMAIN THE SAME, BUT (113)-(115) BECOME

$$\frac{\partial\rho'}{\partial t} + U_\infty\frac{\partial\rho'}{\partial x} = -\bar{\rho}\nabla\cdot\vec{u}', \quad (116)$$

$$\bar{\rho}\left(\frac{\partial\vec{u}'}{\partial t} + U_\infty\frac{\partial\vec{u}'}{\partial x}\right) = -\nabla p' + \bar{\mu}\nabla^2\vec{u}' + \frac{1}{3}\bar{\mu}\nabla(\nabla\cdot\vec{u}'), \quad (117)^*$$

$$\frac{\bar{P}}{R_g}\left(\frac{\partial s'}{\partial t} + U_\infty\frac{\partial s'}{\partial x}\right) = \bar{k}\nabla^2 T'. \quad (118)$$

* NOTE: $\nabla\cdot(\nabla\vec{u}+\nabla\vec{u}^T) = \nabla^2\vec{u} + \nabla(\nabla\cdot\vec{u})$

EQS. (111)-(112) ALONG WITH (116)-(118) REPRESENT A CLOSED SYSTEM OF EQUATIONS FOR THE AEROTHERMAL DISTURBANCES ρ', $\vec{u}'$, T', p' AND s'.

KOVASZNAY'S MODAL DECOMPOSITION

AN IMPORTANT CONTRIBUTION TO THE STRUCTURE OF THE FLUCTUATIONS IN COMPRESSIBLE TURBULENCE WAS MADE BY L.S. KOVASZNAY (JOURNAL OF THE AERONAUTICAL SCIENCES 20, 1953), WHO SUBDIVIDED THE FLUCTUATIONS INTO THREE DIFFERENT MODES USING LINEARIZED EQUATIONS OF MOTION SIMILAR TO THE ONES INTRODUCED IN THE PREVIOUS SECTION.

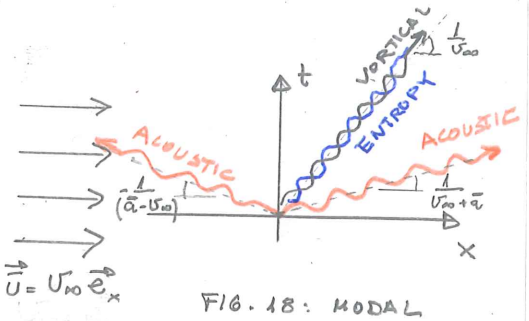


FIG. 18: MODAL REPRESENTATION IN THE INVISCID LIMIT

IN PARTICULAR, KOVASZNAY CONSIDERED A UNIFORM (OR LOCALLY UNIFORM FLOW) OF A VISCOUS, HEAT-CONDUCTING PERFECT GAS AWAY FROM WALLS. THE MEAN STATE COULD BE THEN CONSIDERED AS THE SPATIAL AVERAGE OF THE AEROTHERMAL VARIABLES OVER A FINITE-SIZE DOMAIN (FIG. 18). AN ADDITIONAL ASSUMPTION REGARDING THE PRANDTL NUMBER IS NECESSARY, AS EXPLAINED BELOW. KOVASZNAY'S ANALYSIS OF THE FLUCTUATIONS ABOUT THE MEAN UNIFORM STATE YIELDED THREE DISTINCT MODES (ENTROPY, VORTICITY AND SOUND-WAVE OR ACOUSTIC) THAT BECAME INDEPENDENT IN THE INVISCID LIMIT. THE REMAINDER OF THIS SECTION FOCUSES ON THE DESCRIPTION OF THESE MODES.

BEFORE PROCEEDING FARTHER, IT IS CONVENIENT TO ILLUSTRATE SOME ASPECTS OF THE COUPLING OF EQS. (111)-(112) AND (116)-(118). FOR INSTANCE, THE CURL OF THE MOMENTUM EQ. (117) GIVES

$$\bar{\rho}\left(\frac{\partial \vec{\omega}'}{\partial t} + U_\infty \frac{\partial \vec{\omega}'}{\partial x}\right) = \bar{\mu}\nabla^2\vec{\omega}' \qquad (119)$$

FOR THE VORTICITY FLUCTUATION $\vec{\omega}'$, WHICH SHOWS THAT $\vec{\omega}'$ EVOLVES INDEPENDENTLY FROM ALL OTHER VARIABLES. SIMILARLY, THE CONTINUITY EQUATION (116) CAN BE REWRITTEN AS

$$\frac{D\rho'}{Dt} = -\bar{\rho}\theta' , \qquad (120)$$

WHERE $\frac{D(\,)}{Dt} = \frac{\partial(\,)}{\partial t} + U_\infty \frac{\partial(\,)}{\partial x}$ IS THE MATERIAL DERIVATIVE AND $\theta' = \nabla\cdot\vec{u}'$ IS THE DILATATION. TAKING THE DIVERGENCE OF THE MOMENTUM EQ. (117) YIELDS

$$\frac{D\theta'}{Dt} = -\frac{1}{\bar{\rho}}\nabla^2 p' + \frac{4}{3}\bar{\nu}\nabla^2\theta' , \qquad (121)$$

WHERE $\bar{\nu} = \bar{\mu}/\bar{\rho}$, WHICH ILLUSTRATES THAT THE PRESSURE FLUCTUATIONS ARE RELATED TO THE DILATATION FLUCTUATIONS. THE LATTER CAN BE REWRITTEN IN TERMS OF p' AND s' BY TAKING THE MATERIAL DERIVATIVE OF (112) AND USING THE EQUATION OF STATE (111).

$$\frac{D}{Dt}\left(\frac{s'}{c_p}\right) = \frac{D}{Dt}\left(\frac{T'}{\bar{T}}\right) - \frac{\gamma-1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right) = -\frac{D}{Dt}\left(\frac{\rho'}{\bar{\rho}}\right) + \frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right) = \theta' + \frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)$$

SO THAT $$\boxed{\theta' = \frac{D}{Dt}\left(\frac{s'}{c_p}\right) - \frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)} \quad (122).$$

NOTE: $\frac{D^2}{Dt^2} = \frac{\partial^2}{\partial t^2} + 2U_\infty\frac{\partial^2}{\partial x \partial t} + U_\infty^2\frac{\partial^2}{\partial x^2}$

SUBSTITUTING (122) INTO (121) ONE OBTAINS

$$\frac{D^2}{Dt^2}\left(\frac{s'}{c_p}\right) - \frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{p'}{\bar{p}}\right) = -\frac{1}{\bar{\rho}}\nabla^2 p' + \frac{4}{3}\bar{\nu}\nabla^2\left[\frac{D}{Dt}\left(\frac{s'}{c_p}\right)\right] - \frac{4}{3\gamma}\bar{\nu}\nabla^2\left[\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)\right], \quad (123)$$

WHICH CAN BE FURTHER SIMPLIFIED BY TAKING THE MATERIAL DERIVATIVE OF (118), NAMELY

$$\frac{\bar{p}}{R_g}\frac{D^2 s'}{Dt^2} = \bar{k}\nabla^2\left(\frac{DT'}{Dt}\right) \overset{\text{EQ.(112)}}{=} \bar{T}\bar{k}\nabla^2\left[\frac{D}{Dt}\left(\frac{s'}{c_p}\right)\right] + \left(\frac{\gamma-1}{\gamma}\right)\bar{T}\bar{k}\nabla^2\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)$$

OR EQUIVALENTLY,

$$\frac{D^2}{Dt^2}\left(\frac{s'}{c_p}\right) = \frac{\bar{k}}{\bar{\rho}c_p}\nabla^2\left[\frac{D}{Dt}\left(\frac{s'}{c_p}\right)\right] + \left(\frac{\gamma-1}{\gamma}\right)\frac{\bar{k}}{\bar{\rho}c_p}\nabla^2\left[\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)\right].$$

AND SUBSTITUTING THIS RESULT INTO THE FIRST TERM OF (123), WHICH GIVES THE PROPAGATION EQUATION FOR THE PRESSURE

$$\frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{p'}{\bar{p}}\right) - \frac{\bar{\nu}}{\gamma}\left[\frac{4}{3} + \frac{(\gamma-1)}{P_r}\right]\nabla^2\left[\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)\right] = \frac{1}{\bar{\rho}}\nabla^2 p' + \bar{\nu}\left(\frac{1}{P_r} - \frac{4}{3}\right)\nabla^2\left[\frac{D}{Dt}\left(\frac{s'}{c_p}\right)\right] \quad (124)$$

WHICH SIMPLIFIES TO

$$\boxed{\frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{p'}{\bar{p}}\right) - \frac{1}{\bar{\rho}}\nabla^2 p' = \frac{4}{3}\bar{\nu}\nabla^2\left[\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)\right]} \quad (125)$$

WHEN $P_r = 3/4$. FOR THIS PARTICULAR VALUE OF THE PRANDTL NUMBER, WHICH IS NOT TOO DIFFERENT FROM THE ONE IN AIR, THE PRESSURE FLUCTUATIONS ARE ALSO DECOUPLED FROM ALL OTHER FLUCTUATIONS. NONETHELESS, ALTHOUGH (119) AND (125) INDICATE THAT $\vec{\omega}'$ AND p' EVOLVE INDEPENDENTLY, THE VISCOUS DIFFUSION OF PRESSURE FLUCTUATIONS GENERATES ENTROPY FLUCTUATIONS AS A RESULT OF THE ASSOCIATED TEMPERATURE AND DENSITY FLUCTUATIONS EMERGING FROM THE COUPLING THROUGH THE EQUATION OF STATE (111). THIS CAN BE QUANTIFIED EXPLICITLY BY SUBSTITUTING T' FROM (112) INTO THE RIGHT-HAND SIDE OF THE ENTROPY EQUATION (118), WHICH GIVES

$$\boxed{\frac{D}{Dt}\left(\frac{s'}{c_p}\right) = \frac{4\bar{\nu}}{3}\nabla^2\left[\frac{s'}{c_p} + \left(\frac{\gamma-1}{\gamma}\right)\frac{p'}{\bar{p}}\right]} \quad (126)$$

SHOWING THAT p' ACTS AS A VISCOSITY-DEPENDENT SOURCE OF ENTROPY.

IT IS ILLUSTRATIVE TO SIMPLIFY THESE EQUATIONS IN THE INVISCID LIMIT.

IN PARTICULAR, (117)-(119) AND (125) BECOME

$$\left| \frac{D\vec{u}'}{Dt} = -\frac{\nabla p'}{\bar{\rho}}, \quad \frac{Ds'}{Dt} = 0, \quad \frac{D\vec{\omega}'}{Dt} = 0 \quad \text{AND} \quad \frac{D^2}{Dt^2}\left(\frac{p'}{\bar{p}}\right) = \bar{a}^2 \nabla^2 \left(\frac{p'}{\bar{p}}\right) \right| \quad (127)$$

WHERE $\bar{a}^2$ IS DEFINED AS $\bar{a}^2 = \gamma \bar{p}/\bar{\rho}$. IN THE INVISCID LIMIT, THE VORTICITY, ENTROPY, AND PRESSURE FLUCTUATIONS EVOLVE INDEPENDENTLY WHEN THE MEAN FLOW IS UNIFORM, WITH THE PROPAGATION EQ. FOR p' RESEMBLING THAT DESCRIBING LINEAR ACOUSTICS IN A SOURCELESS MEDIUM (i.e., EQ. (95)).

* MODAL DECOMPOSITION

THE LINEARITY OF THE CONSERVATION EQUATIONS GIVEN ABOVE AND THE INDEPENDENT EVOLUTION OF PRESSURE AND VORTICITY FLUCTUATIONS SUGGEST THAT THE FLUCTUATIONS OF THE AEROTHERMAL VARIABLES CAN BE DECOMPOSED INTO THREE MODES AS

$$\varphi' = \varphi^{(\omega)} + \varphi^{(p)} + \varphi^{(s)} \qquad (128) \qquad \Rightarrow \text{LINEAR SUPERPOSITION}$$

WHERE (ω), (p) AND (s) CORRESPOND, RESPECTIVELY, TO VORTICITY, PRESSURE AND ENTROPY MODES. IN WHAT FOLLOWS, IT IS ASSUMED THAT $P_r = 3/4$, SO THAT THE PRESSURE FLUCTUATIONS EVOLVE INDEPENDENTLY.

THE VORTICITY MODE IS DEFINED BY THE RELATIONS

$$\frac{D\vec{\omega}^{(\omega)}}{Dt} = \bar{\nu} \nabla^2 \vec{\omega}^{(\omega)} \qquad (129)$$

$$p^{(\omega)} = \rho^{(\omega)} = T^{(\omega)} = s^{(\omega)} = 0 \qquad (130)$$

$$\Theta^{(\omega)} = \nabla \cdot \vec{u}^{(\omega)} = 0 \qquad (131)$$

$$\vec{\omega}^{(\omega)} = \nabla \wedge \vec{u}^{(\omega)} \qquad (132)$$

WHICH CORRESPOND TO A SOLENOIDAL VELOCITY COMPONENT THAT HAS NO THERMODYNAMIC FLUCTUATIONS. THIS MODE RESEMBLES "INCOMPRESSIBLE TURBULENCE". PRESSURE AND ENTROPY FLUCTUATIONS ARE ABSENT FROM THE VORTICITY MODE. THE FORMER ONES ARE RULED OUT BY EQ. (121), IN THAT THE SOLENOIDAL FIELD IN ABSENCE OF MEAN VELOCITY GRADIENTS CANNOT GENERATE PRESSURE DISTURBANCES. SIMILARLY, IN ABSENCE OF DENSITY AND PRESSURE FLUCTUATIONS, THE EQUATION OF STATE (111) REQUIRES THE TEMPERATURE FLUCTUATIONS TO BE ZERO, AND AS A RESULT THERE ARE NO ENTROPY FLUCTUATIONS IN THE VORTICITY MODE (SEE EQ. (112)). NOTE THAT THESE CONSIDERATIONS ARE VALID TO THE FIRST EXPANSION ORDER, IN THAT NON-LINEAR TERMS INVOLVING FLUCTUATION PRODUCTS MAY CAUSE DEPARTURES FROM (130)-(131)

THE ACOUSTIC MODE IS DEFINED BY THE EXPRESSIONS

$$\frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{p^{(p)}}{\bar{p}}\right) - \frac{1}{\bar{\rho}}\nabla^2 p^{(p)} = \frac{4}{3}\bar{\nu}\nabla^2\left[\frac{D}{Dt}\left(\frac{p^{(p)}}{\bar{p}}\right)\right] \qquad (133)$$

$$\frac{D}{Dt}\left(\frac{s^{(p)}}{c_p}\right) = \frac{4}{3}\bar{\nu}\nabla^2\left[\frac{s^{(p)}}{c_p} + \left(\frac{\gamma-1}{\gamma}\right)\frac{p^{(p)}}{\bar{p}}\right] \qquad (134)$$

$$\theta^{(p)} = \nabla\cdot\vec{u}^{(p)} \neq 0, \quad \text{WITH} \quad \theta^{(p)} = \frac{D}{Dt}\left(\frac{s^{(p)}}{c_p}\right) - \frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p^{(p)}}{\bar{p}}\right) \qquad (135)$$

$$\vec{w}^{(p)} = \nabla\wedge\vec{u}^{(p)} = 0 \Rightarrow \vec{u}^{(p)} = \nabla\phi^{(p)} \qquad (136)$$

WHICH REPRESENT THE PROPAGATION OF (LINEAR) ACOUSTIC WAVES. IN PARTICULAR, EQ. (133) IS SIMILAR TO EQ. (95) SUPPLEMENTED WITH MEAN CONVECTION AND VISCOUS EFFECTS. EQ. (133) RENDERS PRESSURE FLUCTUATIONS THAT RESULT IN DENSITY AND TEMPERATURE FLUCTUATIONS. THE FORMER LEAD TO A COMPRESSIBLE VELOCITY FIELD THAT IS NONETHELESS IRROTATIONAL. THE VISCOUS DIFFUSION OF PRESSURE FLUCTUATIONS GENERATE ENTROPY FLUCTUATIONS, WHICH ARE SMALL SINCE VISCOSITY TENDS TO PLAY A SECONDARY ROLE IN TURBULENT FLOWS, THE RESULTING ACOUSTIC WAVES ARE MOSTLY ISENTROPIC.

THE ENTROPY MODE IS DEFINED AS

$$\frac{D}{Dt}\left(\frac{s^{(s)}}{c_p}\right) = \frac{4}{3}\bar{\nu}\nabla^2\left(\frac{s^{(s)}}{c_p}\right) \qquad (137)$$

$$p^{(s)} = 0 \Rightarrow \frac{T^{(s)}}{\bar{T}} = -\frac{\rho^{(s)}}{\bar{\rho}} = \frac{s^{(s)}}{c_p} \qquad (138)$$

$$\vec{w}^{(s)} = 0 \Rightarrow u^{(s)} = \nabla\phi^{(s)} \qquad (139)$$

$$\theta^{(s)} = \nabla\cdot\vec{u}^{(s)} = \frac{D}{Dt}\left(\frac{s^{(s)}}{c_p}\right) \qquad (140)$$

WHICH CORRESPOND TO ISOBARIC FLUCTUATIONS OF TEMPERATURE AND DENSITY LINKED THROUGH THE EQUATION OF STATE. THESE ARE HOT OR COLD SPOTS (e.g., COMBUSTION PRODUCTS) THAT ARE NOT ASSOCIATED TO ACOUSTIC WAVES. FINITE-THERMAL CONDUCTIVITY EFFECTS MAKE THE ENTROPY TO DIFFUSE (i.e., EQ. (137)), THEREBY GENERATING A DENSITY FLUCTUATIONS THROUGH THE FIRST PRINCIPLE (i.e., EQ. (138)), WHICH PRODUCE COMPRESSIBILITY EFFECTS IN THE VELOCITY FIELD THROUGH $\theta^{(s)}$ IN (140). HOWEVER, THIS FLOW DIVERGENCE IS PROPORTIONAL TO THE VISCOSITY, AS OBSERVED FROM (137)

AND IS THEREFORE SMALL. THE RESULTING VELOCITY FIELD IS WEAKLY COMPRESSIBLE AND IS ALSO IRROTATIONAL, AS OBSERVED FROM (140), SINCE

$$\nabla\cdot\vec{U}^{(s)} = \frac{D}{Dt}\left(\frac{s^{(s)}}{c_p}\right) = \frac{4}{3}\bar{\nu}\nabla^2\left[\frac{s^{(s)}}{c_p}\right] \Rightarrow \vec{U}^{(s)} = \nabla\phi^{(s)} = \nabla\left(\frac{4}{3}\bar{\nu}\frac{s^{(s)}}{c_p}\right).$$

THE CONSIDERATIONS GIVEN ABOVE MAY BE SUMMARIZED IN THE FOLLOWING DECOMPOSITION OF AEROTHERMAL VARIABLES:

$$\vec{\omega}' = \omega^{(\omega)} \quad \text{(VORTICAL MODE)}, \qquad p' = p^{(p)} \quad \text{(ACOUSTIC MODE)},$$
$$s' = s^{(s)} + \overbrace{s^{(p)}}^{\text{SMALL}} \quad \text{(ENTROPY AND ACOUSTIC MODES)},$$
$$\vec{u}' = \underbrace{\vec{u}^{(\omega)}}_{\text{SOLENOIDAL}} + \underbrace{\vec{u}^{(s)} + \vec{u}^{(p)}}_{\text{IRROTATIONAL}} \quad \text{(VORTICAL, ENTROPY AND ACOUSTIC MODES)}, \qquad (141)$$
$$\rho' = \rho^{(p)} + \rho^{(s)} \quad \text{(ACOUSTIC AND ENTROPY MODES)}$$
$$\theta' = \theta^{(p)} + \theta^{(s)} \quad \text{(ACOUSTIC AND ENTROPY MODES)}$$

IN THE PREVIOUS FORMULATION, EQUATIONS (129) AND (137) PLAY A FUNDAMENTAL ROLE IN THE TRANSPORT OF VORTICAL AND ENTROPY FLUCTUATIONS, RESPECTIVELY, WHICH ARE OBSERVED TO MOVE AT THE FLOW SPEED U_∞. ON THE OTHER HAND, THE WAVE EQUATION (133) DESCRIBES THE PROPAGATION OF ACOUSTIC FLUCTUATIONS AT THE MEAN SPEED OF SOUND $\bar{a} = \left(\frac{\gamma\bar{p}}{\bar{\rho}}\right)^{1/2}$ RELATIVE TO THE MEAN FRAME MOVING AT THE FLOW VELOCITY U_∞. THE EFFECTS OF FINITE MOLECULAR TRANSPORT ARE SMALL AT HIGH REYNOLDS NUMBERS, BUT HAVE DIFFERENT CONSEQUENCES ON EACH MODE. FOR INSTANCE, THEY MAKE $\vec{\omega}'$ AND s' TO DIFFUSE AS OPPOSED TO BEING PASSIVELY ADVECTED AT INFINITE REYNOLDS NUMBERS, (AS IMPLIED BY EQS. (129)-(137)). SIMILARLY, THEY CAUSE DIFFUSION OF ACOUSTIC PRESSURE, WHICH LEADS TO FINITE NON-ISENTROPIC EFFECTS, AS SHOWN IN EQ. (134). ADDITIONALLY, THEY LEAD TO ENTROPY DIFFUSION AND DILATATIONAL EFFECTS, AS OBSERVED FROM EQS. (137)-(140). THE FORMULATION IS SIMPLIFIED IN THE FOLLOWING SECTION FOR INVISCID FLOWS.

BEFORE CONCLUDING THIS SECTION, IT IS WORTH EMPHASIZING THAT, DUE TO THE LINEARIZATION ABOUT THE UNIFORM MEAN FLOW, THE THREE MODES DESCRIBED ABOVE DO NOT INTERACT. INTERACTIONS MAY HOWEVER OCCUR DUE TO NON-LINEARITIES, AT WALLS, OR WHEN THE MEAN FLOW VARIATIONS ARE SIGNIFICANT, AS IN NOZZLES, BOUNDARY LAYERS, OR SHOCK WAVES.

* THE MODAL DECOMPOSITION IN THE INVISCID LIMIT

RETAINING THE ASSUMPTION OF MEAN UNIFORM FLOW AND NEGLECTING VISCOUS EFFECTS, KOVASZNAY'S MODAL DECOMPOSITION BECOMES

$$\left.\begin{aligned} &\frac{D\vec{\omega}^{(\omega)}}{Dt} = 0 \\ &p^{(\omega)} = \rho^{(\omega)} = T^{(\omega)} = s^{(\omega)} = \\ &\quad = \theta^{(\omega)} = 0 \\ &\vec{\omega}^{(\omega)} = \nabla \wedge \vec{u}^{(\omega)} \end{aligned}\right\} \text{VORTICITY MODE (142)}$$

$$\left.\begin{aligned} &\frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{p^{(p)}}{\bar{p}}\right) = \frac{1}{\rho}\nabla^2 p^{(p)} \\ &s^{(p)} = 0 \\ &\theta^{(p)} = -\frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p^{(p)}}{\bar{p}}\right) \\ &\vec{\omega}^{(p)} = 0 \end{aligned}\right\} \text{ACOUSTIC MODE (143)}$$

AND

$$\left.\begin{aligned} &\frac{D}{Dt}\left(\frac{s^{(s)}}{c_p}\right) = 0, \\ &p^{(s)} = 0, \quad \vec{\omega}^{(s)} = 0 \\ &\frac{T^{(s)}}{\bar{T}} = -\frac{\rho^{(s)}}{\bar{\rho}} = \frac{s^{(s)}}{c_p} \\ &\theta^{(s)} = 0 \Rightarrow u^{(s)} = 0 \end{aligned}\right\} \text{ENTROPY MODE (144)},$$

AS ANTICIPATED IN EQS. (127), AND VISUALIZED HERE IN (142) - (144), THE VORTICITY, PRESSURE AND ENTROPY FLUCTUATIONS EVOLVE INDEPENDENTLY IN THE INVISCID LIMIT WHEN THE MEAN FLOW IS UNIFORM. IN PARTICULAR, THE ENTROPY AND VORTICITY (FLUCTUATIONS) ARE PASSIVELY ADVECTED WITHOUT DIFFUSION WITH THE MEAN FLOW, AS INDICATED BY THE FIRST EQUATIONS IN (142) AND (144), WHILE THE PRESSURE FLUCTUATIONS ARE PROPAGATED BY THE LOSELESS WAVE EQUATION IN (143) AT THE SPEED OF SOUND RELATIVE TO THE MEAN FLOW. SIMILARLY, THE LAST EQUATION IN (144) INDICATES THAT THE ENTROPY MODE BECOMES SOLENOIDAL, AND SINCE IT WAS ALSO IRROTATIONAL, THE CORRESPONDING VELOCITY FLUCTUATIONS OF THIS MODE VANISH. THE VORTICITY MODE BEING SOLENOIDAL, THE ONLY SOURCE OF DILATATION IS THAT FROM THE ACOUSTIC MODE, IN THE THIRD EQUATION IN (143). IN THIS LIMIT, THE ACOUSTIC MODE BECOMES ISENTROPIC. THE RESULTING DECOMPOSITION OF AEROTHERMAL VARIABLES IS

$$\vec{\omega}' = \vec{\omega}^{(\omega)} \quad \text{(VORTICAL MODE)}, \qquad p' = p^{(p)} \quad \text{(ACOUSTIC MODE)},$$

$$s' = s^{(s)} \quad \text{(ENTROPY MODE)}, \qquad \vec{u}' = \underbrace{\vec{u}^{(\omega)}}_{\text{ROTATIONAL SOLENOIDAL}} + \underbrace{\vec{u}^{(p)}}_{\text{IRROTATIONAL COMPRESSIBLE}} \quad \text{(VORTICITY AND ACOUSTIC MODES)},$$

$$\rho' = \rho^{(s)} + \rho^{(p)} \quad \text{(ACOUSTIC AND ENTROPY MODES)},$$

$$\theta' = \theta^{(p)} \quad \text{(ACOUSTIC MODE)} \qquad (145)$$

* HOT-WIRE MEASUREMENTS OF KOVASZNAY'S MODES

KOVASZNAY'S ORIGINAL MOTIVATION FOR THE MODAL DECOMPOSITION WAS THE ANALYSIS OF HOT-WIRE MEASUREMENTS, SINCE EACH FLUCTUATION MODE APPEARS TO PRODUCE A DIFFERENT OUTPUT IN THE HOT WIRE. A HOT WIRE ANEMOMETER PLACED NORMAL TO A SUPERSONIC STREAM HAS THE FOLLOWING OPERATIONAL CHARACTERISTIC (KOVASZNAY, 1953): THE HOT WIRE HEATED BY A CONSTANT ELECTRIC CURRENT RESPONDS TO A LINEAR COMBINATION OF MASS FLOW FLUCTUATIONS AND STAGNATION TEMPERATURE FLUCTUATIONS. IN MATHEMATICAL FORM, THIS IS EQUIVALENT TO

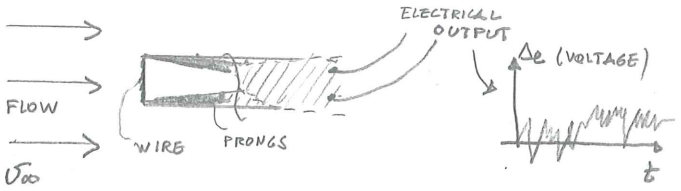


FIG. 19 SCHEMATICS OF A HOT WIRE ANEMOMETER

$$\Delta e(t) = \Delta e_T(t)\, v(t) - \Delta e_M(t)\, m(t), \qquad (146)$$

WHERE Δe_T AND Δe_M ARE THE SENSITIVITIES OF THE OUTPUT TO STAGNATION TEMPERATURE AND MASS FLOW VARIATIONS, WHILE $v(t) = T_0'/\bar{T}_0$ AND $m(t) = 100\,(\rho U)'/\bar{\rho} U_\infty$ ARE THE CORRESPONDING DIMENSIONLESS FLUCTUATIONS. NOTE THAT $m(t) = 100\,\rho'/\bar{\rho} + 100\,u'/U_\infty$, AND SINCE $T_0 = T + U^2/2c_p \Rightarrow T_0' = T' + u' U_\infty / c_p$, THEN

$$v(t) = \frac{T_0'}{\bar{T}_0} = \frac{T' + U u'/c_p}{\bar{T} + U_\infty^2/2c_p} = \frac{T'}{\bar{T}}\alpha + \frac{u'}{U_\infty}\beta, \qquad (147)$$

WHERE $\alpha = \left(1 + (\gamma-1) M_\infty^2/2\right)^{-1}$ AND $\beta = (\gamma-1) M_\infty^2 / \left[1 + (\gamma-1) M_\infty^2/2\right]$, WITH $M_\infty = \dfrac{U_\infty}{\sqrt{\gamma R_g \bar{T}}}$.

EQUATION (146) CAN BE REWRITTEN IN TERMS OF THE OUTPUT VARIANCE AS

$$V_v^2 = \frac{\overline{\Delta e^2}}{\Delta e_T^2} = \overline{v^2} - 2r\overline{mv} + r^2\overline{m^2}, \qquad (148)$$

WITH $r = \Delta e_M / \Delta e_T$ A QUANTITY THAT IS FOUND TO DEPEND SOLELY ON THE WIRE TEMPERATURE. EQUATION (148) THEREFORE SHOWS THE RESPONSE OF THE WIRE IN THE FORM OF A VIRTUAL STAGNATION-TEMPERATURE FLUCTUATION V_v AS A FUNCTION OF THE WIRE TEMPERATURE THROUGH THE PARAMETER r. THE DIAGRAM $\{r, V_v\}$ IS CALLED THE "FLUCTUATION DIAGRAM (SEE FIG. 20).

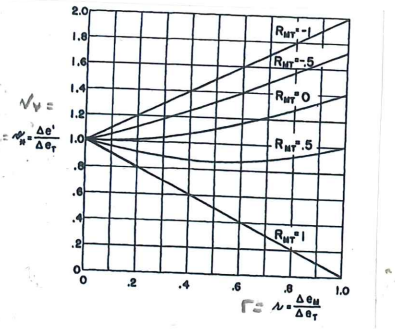


FIG. 20. FLUCTUATION DIAGRAM FOR 1% m AND 1% v FLUCTUATIONS WITH VARYING DEGREES OF CORRELATION $R_{MT} = \overline{mv}/m'v'$ (ADAPTED FROM KOVASZNAY 53)

ALTHOUGH THE HOT WIRE DOES NOT RESPOND DIRECTLY TO THE FLUCTUATION MODES DESCRIBED IN THE PREVIOUS SECTION, IT DOES SO INDIRECTLY THROUGH m AND V_v, WITH EACH MODE ATTAINING A DIFFERENT FLUCTUATION DIAGRAM.

FOR INSTANCE, IN THE VORTICAL MODE, $m = 100\,u'/U_\infty$ ($\rho' = 0$) AND $v = \beta u'/U_\infty$ ($T' = 0$), AND (146) BECOMES

$$\frac{\Delta e}{\Delta e_T} = \beta\frac{u'}{U_\infty} - r\,100\frac{u'}{U_\infty} = (\beta - r\cdot 100)\frac{u'}{U_\infty},$$

OR EQUIVALENTLY, $V_*^2 = (\beta - 100\tau)^2 \overline{U'^2}/U_\infty^2$. NOTE THAT THIS FLUCTUATION DIAGRAM APPEARS AS A BROKEN LINE WITH A ZERO AT $\beta = 100\tau$ (SEE FIG. 21a). NOTE THAT AT THAT PARTICULAR WIRE TEMPERATURE, THE SENSOR IS SOLELY SENSITIVE TO THE ACOUSTIC AND ENTROPY MODES. IN THE ENTROPY MODE, $v(t) = \alpha T'/\bar{T}$ SINCE $U' = 0$ NEGLECTING THE POTENTIAL FLOW SET UP BY THE VISCOUS DECAY OF THE TEMPERATURE SPOTS, AND ALSO $m = 100\,\rho'/\bar{\rho} = -100\,T'/\bar{T}$ SINCE $P' = 0$. AS A RESULT, (146) BECOMES

$$\frac{\Delta e}{\Delta e_T} = \alpha\frac{T'}{\bar{T}} + \tau\cdot 100\frac{T'}{\bar{T}} = (\alpha + 100\tau)\frac{T'}{\bar{T}}$$

SO THAT $V_*^2 = (\alpha + 100\tau)^2\overline{T'^2}/\bar{T}^2$, WHICH CORRESPONDS TO A FLUCTUATION DIAGRAM WITH A ZERO AT $\tau = -\alpha/100$ (SEE FIG. 21b). SIMILAR CONSIDERATIONS ENABLE THE OBTENTION OF THE FLUCTUATION DIAGRAM FOR THE ACOUSTIC MODE (FIG. 21c), WHICH RESEMBLES A STRAIGHT LINE PASSING THROUGH THE ORIGIN. (COMPARISONS OF THESE DIAGRAMS AGAINST SUPERSONIC WIND-TUNNEL DATA IS PROVIDED IN KOVASZNAY 1953.

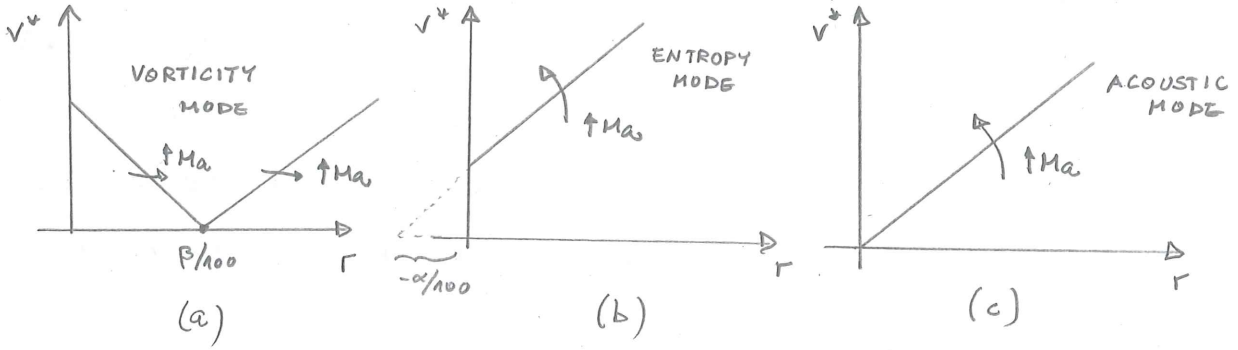


FIG. 21 : FLUCTUATION DIAGRAMS FOR VORTICITY, ENTROPY, AND ACOUSTIC MODES IN HOT-WIRE ANEMOMETRY.

NON-UNIFORM MEAN-FLOW EFFECTS

IN MANY APPLICATIONS OF INTEREST, SUCH AS BOUNDARY LAYERS OR SHEAR LAYERS, THE MEAN FLOW IS NOT UNIFORM. THIS, IN TURN, COMPLICATES THE MODAL ANALYSIS PROVIDED ABOVE AND AMOUNTS TO COUPLING BETWEEN MODES. FOR INSTANCE, CONSIDER THE LINEARIZED VERSION OF THE VORTICITY EQUATION (53) FOR NON-UNIFORM MEAN FLOWS,

$$\frac{\partial\vec{\omega}'}{\partial t} + \vec{U}\cdot\nabla\vec{\omega}' = +\vec{\omega}'\cdot\nabla\vec{U} - \vec{\omega}'(\nabla\cdot\vec{U}) - \vec{U}'\cdot\nabla\vec{\omega} - \vec{\omega}(\nabla\cdot\vec{U}') + \vec{\omega}\cdot\nabla\vec{U}'$$
$$+\frac{1}{\bar{\rho}^2}(\nabla P'\wedge\nabla\bar{\rho}) + \frac{1}{\rho^2}(\nabla P + \nabla\rho'), \qquad (149)$$

WHERE THE VISCOUS TERMS HAVE BEEN NEGLECTED. IN PARTICULAR, EQ. (149) INCLUDES THE EFFECTS OF VORTEX STRETCHING INDUCED BY THE MEAN FLOW ON THE DISTURBANCES, AS WELL AS DILATATION-INDUCED VORTICITY AND BAROCLINIC-TORQUE FLUCTUATIONS. NEITHER ONE OF THESE EFFECTS ARE PRESENT IN THE SIMPLER VORTICITY EQUATION

(149) LINEARIZED AROUND UNIFORM MEAN FLOWS. NOTE THAT IN (149) THE DENSITY AND PRESSURE FLUCTUATIONS PRODUCE VORTICITY WHEN COUPLED WITH THE NON-UNIFORM MEAN FLOW. SIMILARLY, THE DILATATION OF THE MEAN AND FLUCTUATING VELOCITY FIELDS LEAD TO VORTICITY PRODUCTION. THESE EXTRA EFFECTS ARE ABSENT WHEN THE MEAN FLOW IS UNIFORM, AND INDICATE THAT FOR NON-UNIFORM MEAN FLOWS THE MECHANISMS GENERATING PRESSURE, DENSITY AND VELOCITY DISTURBANCES INVARIABLY LEAD TO VORTICAL VELOCITY FIELDS, THEREBY COUPLING THE FLUCTUATION MODES AND MAKING THEIR SEPARATION NOT STRAIGHTFORWARD. A PRACTICAL EXAMPLE IN WHICH COUPLINGS OF THIS TYPE OCCUR ARE COMPRESSIBLE HOMOGENEOUS SHEAR FLOWS (FIG. 122), WHERE THE MEAN THERMODYNAMIC QUANTITIES ARE CONSTANT BUT THE MEAN STREAMWISE VELOCITY IS LINEAR. IN THIS CASE, THE BAROCLINIC TORQUE TERMS IN (149) ARE ZERO, BUT THE STIRRING $\left(-\vec{U}'\cdot\nabla\vec{\omega}\right)$, VORTEX STRETCHING $\left(\vec{\omega}'\cdot\nabla\vec{U}+\vec{\omega}\cdot\nabla\vec{U}'\right)$ AND DILATATIONAL $\left(-\vec{\omega}\Theta'\right)$ TERMS PLAY A ROLE IN PRODUCING VORTICITY. IN PARTICULAR, THE LATTER CAN BE SHOWN TO BE RELATED TO ACOUSTIC PRESSURE

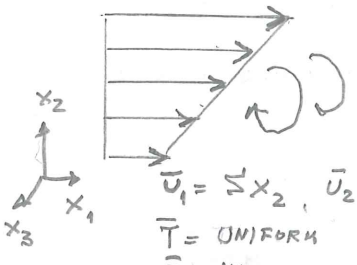


FIG. 122: COMPRESSIBLE HOMOGENEOUS SHEAR FLOW

FLUCTUATIONS, AS EVIDENCED BY USING (122), NAMELY

$$\Theta' = \frac{D}{Dt}\left(\frac{s'}{c_p}\right) - \frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right) \qquad (150)$$

AND SUBSTITUTING THE LINEARIZED ENTROPY EQUATION (115) IN THE INVISCID LIMIT,

$$\frac{D}{Dt}\left(\frac{s'}{c_p}\right) = -\vec{U}'\cdot\nabla\left(\frac{\bar{s}}{c_p}\right) = -\vec{U}'\left(\frac{1}{\gamma\bar{p}}\nabla\bar{p} - \frac{1}{\bar{\rho}}\nabla\bar{\rho}\right) \qquad (151)$$

WHICH IN THIS CASE LEADS TO $\frac{D}{Dt}\left(\frac{s'}{c_p}\right) = 0$ AND $\Theta' = -\frac{1}{\gamma}\frac{D}{Dt}\left(\frac{p'}{\bar{p}}\right)$.

THE CONSIDERATIONS GIVEN ABOVE INDICATE THAT THE PRESENCE OF NON-UNIFORMITIES IN THE MEAN FLOW TENDS TO COUPLE THE MODES DERIVED IN THE PREVIOUS SECTION. VORTICAL MOTIONS CAN BE GENERATED FROM PRESSURE, DENSITY AND DILATATIONAL DISTURBANCES. SIMILARLY, (151) SUGGESTS THAT STIRRING OF MEAN ENTROPY GRADIENTS GENERATES ENTROPY FLUCTUATIONS, WHICH IN TURN PRODUCE DILATATION FLUCTUATIONS THROUGH (150) AND SUBSEQUENT VORTICITY BY EQ. (149).

NON-LINEAR MODAL INTERACTIONS

HIGHER-ORDER TERMS IN THE MODAL DECOMPOSITION DESCRIBED ABOVE LEAD TO NON-LINEAR INTERACTIONS AND COUPLING BETWEEN MODES. THESE EFFECTS WERE ANALYZED BY CHU AND KOVASZNAY (JOURNAL OF FLUID MECHANICS, 1958), CONSIDER THE EXPANSIONS

$$\begin{aligned} p &= \bar{p} + p' + p'' + \cdots \\ \rho &= \bar{\rho} + \rho' + \rho'' + \cdots \\ T &= \bar{T} + T' + T'' + \cdots \\ \vec{u} &= \vec{u}' + \vec{u}'' + \cdots \end{aligned} \qquad (152) \qquad (\bar{p}, \bar{\rho}, \bar{T} \text{ ARE UNIFORM})$$

WHERE $\bar{u} = 0$ HAS BEEN TAKEN AS IF THE ANALYSIS WAS TO BE PERFORMED IN THE REFERENCE FRAME MOVING AT THE MEAN SPEED U_∞ (SEE SCHEMATICS IN FIG. 18). IN THIS FORMULATION $\alpha = p'/\bar{p}$ IS A CHARACTERISTIC DISTURBANCE AMPLITUDE THAT IS A SMALL PARAMETER, $\alpha \ll 1$, WITH ALL PRIMES IN (152) BEING OF ORDER α AND AND ALL DOUBLE PRIMES BEING OF ORDER α^2 WHEN NORMALIZED WITH THE CORRESPONDING MEAN VALUE. THE ANALYSIS IS THEN BASED ON SUBSTITUTING (152) INTO THE CONSERVATION EQUATIONS (11), (12), (35) ALONG WITH (10) AND (17), AND THEN EQUATE TERMS MULTIPLYING THE SAME POWER OF α, TO ORDER α, THE EQUATIONS ARE THE SAME AS THOSE USED IN THE LINEAR ANALYSIS, NAMELY (111), (112), (116) - (118). THIS SET OF EQUATIONS DO NOT CONTAIN CROSSED TERMS AND THEREFORE LEAD TO SEPARATION OF THE THREE (VORTICAL, ACOUSTIC AND ENTROPY) MODES DESCRIBED IN THE PREVIOUS SECTIONS. ON THE OTHER HAND, THE CONSERVATION EQUATIONS TO ORDER α^2 NECESSARILY LEAD TO CROSSED TERMS INVOLVING MULTIPLICATIONS OF SINGLE-PRIMED $O(\alpha)$ VARIABLES. THESE TERMS ACT AS MASS, MOMENTUM AND ENTROPY SOURCES, WHICH, UPON DECOMPOSITION INTO THE THREE MODES, LEAD TO COMBINED MODAL DYNAMICS, AS SHOWN BELOW.

TO ORDER α^2, EQUATIONS (11), (12), (35), (10), (17) BECOME

$$\frac{\partial \rho''}{\partial t} + \bar{\rho} \nabla \cdot \vec{u}'' = m \qquad (153)$$

$$\bar{\rho} \frac{\partial \vec{u}''}{\partial t} = -\nabla p'' + \bar{\mu} \nabla^2 \vec{u}'' + \frac{\bar{\mu}}{3} \nabla (\nabla \cdot \vec{u}'') + \bar{\rho} \vec{f} \qquad (154)$$

$$\frac{\bar{p}}{R_g} \frac{\partial s''}{\partial t} = \bar{k} \nabla^2 T'' + Q \qquad (155)$$

$$\frac{p''}{\bar{p}} = \frac{\rho''}{\bar{\rho}} + \frac{T''}{\bar{T}} \qquad (156)$$

$$\frac{s''}{c_p} = \frac{T''}{\bar{T}} - \left(\frac{\gamma - 1}{\gamma} \right) \frac{p''}{\bar{p}} \qquad (157)$$

WHERE

$$m = -\bar{\rho}\nabla\left[\left(\frac{p'}{\gamma\bar{p}} - \frac{s'}{c_p}\right)\vec{u}'\right] + \bar{\rho}\frac{\partial}{\partial t}\left[(\gamma-1)\left(\frac{p'}{\gamma\bar{p}}\right)^2 + \left(\frac{s'}{c_p}\right)^2 + \frac{p'}{\gamma\bar{p}}\frac{s'}{c_p}\right] \quad (158)$$

IS A MASS SOURCE,

$$\vec{f} = -\left(\frac{p'}{\gamma\bar{p}} - \frac{s'}{c_p}\right)\frac{\partial\vec{u}'}{\partial t} - \vec{u}'\cdot\nabla\vec{u}' - \frac{3}{2}\nabla\left[\frac{\mu'}{\bar{\rho}}(\nabla\cdot\vec{u}')\right] + \nabla\cdot\left(\frac{\mu'}{\bar{\rho}}\nabla\vec{u}'\right) \quad (159)$$

IS A BODY FORCE, AND

$$Q = \bar{\rho}c_p\bar{T}\left\{-\frac{p'}{\bar{p}}\frac{\partial}{\partial t}\left(\frac{s'}{c_p}\right) - \vec{u}'\cdot\nabla\left(\frac{s'}{c_p}\right) + \frac{4}{3}\nabla\cdot\left[\frac{\mu'}{\bar{\rho}}\nabla\left(\frac{s'}{c_p} + \frac{(\gamma-1)}{\gamma}\frac{p'}{\bar{p}}\right)\right]\right.$$
$$+ \frac{\bar{\nu}}{c_p\bar{T}}\left[\frac{1}{2}\nabla\vec{u}' : \nabla\vec{u}' - \frac{2}{3}(\nabla\cdot\vec{u}')^2\right] - \frac{2}{3}\left(\frac{\gamma-1}{\gamma}\right)\bar{\rho}\nabla^2\left[\frac{1}{\gamma}\left(\frac{p'}{\bar{p}}\right)^2\right.$$
$$\left.\left. - 2\frac{p'}{\bar{p}}\frac{s'}{c_p} - \frac{\gamma}{\gamma-1}\left(\frac{s'}{c_p}\right)^2\right]\right\} \quad (160)$$

→ ASSUMES $Pr = 3/4$

IS A HEAT SOURCE. NOTE THAT m, $\vec{f}$ AND Q ACT AS FORCING TERMS IN THE $O(\kappa^2)$ EQUATIONS AND INVOLVE PRODUCTS OF $O(\kappa)$ FLUCTUATIONS. THEREFORE, WHEN THE $O(\kappa)$ FLUCTUATIONS ARE DECOMPOSED INTO VORTICAL, ACOUSTIC AND ENTROPY MODES, THE SOURCES (158)-(160) GIVE RISE TO MULTIPLICATIONS OF VARIABLES FROM ONE MODE BY VARIABLES FROM ANOTHER MODE, NAMELY

$$\left\{\begin{aligned} m &= m^{(ww)} + m^{(pp)} + m^{(ss)} + m^{(wp)} + m^{(ps)} + m^{(sw)}, \\ \vec{f} &= \vec{f}^{(ww)} + \vec{f}^{(pp)} + \vec{f}^{(ss)} + \vec{f}^{(wp)} + \vec{f}^{(ps)} + \vec{f}^{(sw)} \\ Q &= Q^{(ww)} + Q^{(pp)} + Q^{(ss)} + Q^{(wp)} + Q^{(ps)} + Q^{(sw)}. \end{aligned}\right. \quad (161)$$

INTERACTION TYPE ⇒	VORTICITY -VORTICITY	ACOUSTIC- ACOUSTIC	ENTROPY- ENTROPY	VORTICITY -ACOUSTIC	ACOUSTIC- ENTROPY	ENTROPY -VORTICITY

THE SECOND-ORDER MODAL EQUATIONS BECOME

$$\left.\begin{aligned} &\frac{\partial\vec{\omega}^{(ww)}}{\partial t} - \bar{\nu}\nabla^2\vec{\omega}^{(ww)} = \nabla\wedge\vec{f} \\ &p^{(ww)} = s^{(ww)} = \rho^{(ww)} = 0 \\ &\vec{\omega}^{(ww)} = \nabla\wedge\vec{u}^{(ww)} \neq 0, \quad \theta^{(ww)} = \nabla\cdot\vec{u}^{(ww)} = 0 \end{aligned}\right\} \text{VORTICITY MODE} \quad (162)$$

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}\left(\frac{p^{(pp)}}{\bar{p}}\right) + \bar{a}^2\nabla^2\left(\frac{p^{(pp)}}{\bar{p}}\right) - \frac{4}{3}\gamma\bar{\nu}\frac{\partial}{\partial t}\left[\nabla^2\left(\frac{p^{(pp)}}{\bar{p}}\right)\right] = \frac{\partial}{\partial t}\left(\frac{m}{\bar{\rho}}\right) - \frac{4}{3}\bar{\nu}\nabla^2\left(\frac{m}{\bar{\rho}}\right) - \nabla\cdot\vec{f} + \frac{\partial}{\partial t}\left(\frac{Q}{\bar{\rho}c_p\bar{T}}\right) \\ &\frac{\partial}{\partial t}\left(\frac{s^{(pp)}}{c_p}\right) - \frac{4}{3}\bar{\nu}\nabla^2\left(\frac{s^{(pp)}}{c_p}\right) = \frac{4}{3}\frac{(\gamma-1)}{\gamma}\bar{\nu}\nabla^2\left(\frac{p^{(pp)}}{\bar{p}}\right) \\ &\theta^{(pp)} = \nabla\cdot\vec{u}^{(pp)} = -\frac{1}{\gamma}\frac{\partial}{\partial t}\left(\frac{p^{(pp)}}{\bar{p}}\right) + \frac{\partial}{\partial t}\left(\frac{s^{(pp)}}{c_p}\right) + \frac{m}{\bar{\rho}} \\ &\nabla\wedge\vec{u}^{(pp)} = \vec{\omega}^{(pp)} = 0 \end{aligned}\right\} \begin{array}{l}\text{ACOUSTIC}\\ \text{MODE}\end{array} \quad (163)$$

AND $$\frac{\partial}{\partial t}\left(\frac{s''^{(s)}}{c_p}\right) = \frac{4}{3}\bar{\nu}\nabla^2\left(\frac{s''^{(s)}}{c_p}\right) + \frac{Q}{\bar{\rho}c_p\bar{T}}$$

$$p''^{(s)} = 0 \Rightarrow \frac{T''^{(s)}}{\bar{T}} = -\frac{\rho''^{(s)}}{\bar{\rho}} = \frac{s''^{(s)}}{c_p}$$

$$\vec{w}''^{(s)} = 0\,, \quad \Theta''^{(s)} = \nabla\cdot\vec{v}''^{(s)} = \frac{\partial}{\partial t}\left(\frac{s''^{(s)}}{c_p}\right)$$

} ENTROPY MODE (164)

IT IS INTERESTING TO NOTE THAT THE EFFECT OF MASS ADDITION FROM THE FIRST-ORDER FLUCTUATION FIELD IS TO GENERATE SOUND (SEE FIRST EQ. IN (163)) BY DISPLACING FLUID. SIMILARLY, THE IRROTATIONAL COMPONENT OF THE BODY FORCE PRODUCES SOUND, WHILE ITS SOLENOIDAL COMPONENT PRODUCES VORTICITY. THE HEAT ADDED FROM THE FIRST-ORDER FLUCTUATION FIELD PRODUCES ENTROPY VARIATIONS AND SOUND IN THE SECOND-ORDER FLUCTUATION FIELD.

SUBSTITUTION OF (161) INTO (162)-(164) PRODUCES A MYRIAD OF CROSSED PHENOMENA. FOR INSTANCE, THE VORTICITY-VORTICITY INTERACTION IN THE ACOUSTIC MODE (163) IS MAINLY MANIFESTED BY THE BODY FORCE

$$\vec{f}^{(ww)} = -\vec{v}'^{(w)}\cdot\nabla\vec{v}'^{(w)} = -\nabla\cdot\left(\vec{v}'^{(w)}\vec{v}'^{(w)}\right)$$

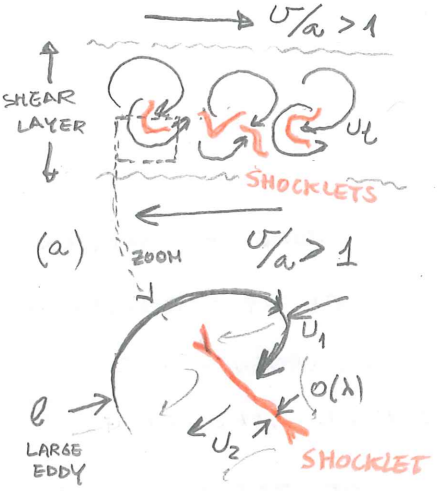
WHICH, WHEN SUBSTITUTED INTO THE RIGHT-HAND SIDE OF THE PRESSURE-PROPAGATION EQUATION IN (163) RESEMBLES THE MECHANISM OF SOUND GENERATED BY TURBULENCE IN CHAPTER III, EQS (104). IN FACT, $\vec{f}^{(ww)}$ IS REMINISCENT OF THE DIVERGENCE OF LIGHTHILL'S FORCING TENSOR (103). MANY OTHER PHENOMENA CAN BE INFERRED FROM SECOND-ORDER CROSSED INTERACTIONS THAT ARE SUMMARIZED IN THE TABLE BELOW (SEE CHU AND KOVASZNAY 1958 FOR DETAILS):

MODE → / ↓INTERACTION	ACOUSTIC	VORTICITY	ENTROPY
ACOUSTIC - ACOUSTIC	SELF-SCATTERING AND STEEPENING OF SOUND	VERY SMALL	VERY SMALL
VORTICITY - VORTICITY	SOUND GENERATION BY TURBULENCE	SELF-CONVECTION AND SELF-STRETCHING OF $\vec{w}$	VERY SMALL
ENTROPY - ENTROPY	VERY SMALL	VERY SMALL	VERY SMALL
ACOUSTIC - VORTICITY	SOUND SCATTERING BY TURBULENCE	CONVECTION, STRETCHING AND DILATATION OF $\vec{w}$ BY SOUND	VERY SMALL
ACOUSTIC - ENTROPY	SOUND SCATTERING BY ISOBARIC T INHOMOGENEITIES	BAROCLINIC VORTICITY GENERATION	CONVECTION OF ENTROPY BY SOUND
VORTICITY - ENTROPY	VERY SMALL	VERY SMALL	CONVECTION OF ENTROPY BY TURBULENCE

TABLE: EFFECTS OF FIRST-ORDER FLUCTUATION PRODUCTS ON THE SECOND-ORDER FLUCTUATION MODES. ADAPTED FROM CHU AND KOVASZNAY (1958).

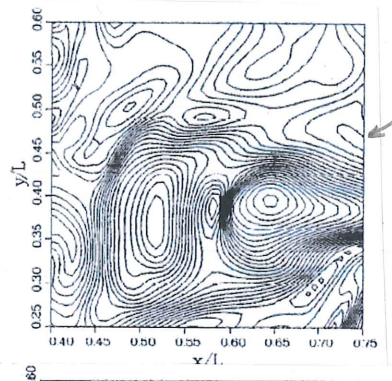
FULLY NON-LINEAR PHENOMENA IN COMPRESSIBLE TURBULENCE: SHOCKLETS.

TURBULENT FLOWS UNDERGO VELOCITY FLUCTUATIONS THAT REPRESENT A SIGNIFICANT PERCENTAGE OF THE MEAN VELOCITY. IN COMPRESSIBLE FLOWS, RAPID CHANGES IN VELOCITY, ACCOMPANIED BY ORDER-UNITY VARIATIONS IN PRESSURE AND TEMPERATURE, OCCUR ACROSS THIN REGIONS CORRESPONDING TO SHOCK WAVES, AS DESCRIBED IN CHAPTER II. SHOCK WAVES BY OBSTACLES OR OTHER GEOMETRICAL CONSTRAINTS MEDIATED IN SUPERSONIC FLOWS, AND HAVE CHARACTERISTIC THICKNESSES THAT ARE OF THE SAME ORDER AS FEW MEAN FREE PATHS λ, SUCH THAT $\lambda \Delta U / \nu \sim O(1)$, WITH λ GIVEN BY EQ. (2) AND $\Delta U = U_1 - U_2 = O(a)$ A VELOCITY DECREMENT THROUGH THE SHOCK. IN COMPRESSIBLE TURBULENT FLOWS, SHOCK WAVES CAN BE GENERATED INTERNALLY IN THE BULK AS A RESULT OF THE INTERACTION BETWEEN FAST AND SLOW EDDIES EVEN AT SUBSONIC TURBULENT MACH NUMBERS $Ma_\ell = U_\ell / a < 1$, FOR AS LONG AS THERE IS SUPERSONIC RELATIVE MOTION. THESE SHOCK WAVES ARE CALLED "EDDY SHOCKLETS", SINCE THEY TEND TO BE SMALL FEATURES ASSOCIATED WITH THE SMALL TURBULENCE SCALES. SHOCKLETS HAVE BEEN OBSERVED IN COMPRESSIBLE ISOTROPIC TURBULENCE AT TURBULENT MACH NUMBERS $Ma_\ell \sim 0.3-0.6$ (e.g. SEE LEE, LELE AND MOIN, POF 1991; SAMTANEY, PULLIN AND KOSOVIC, POF 2001; ALSO FIG. 21a), AND IN COMPRESSIBLE SHEAR LAYERS AT CONVECTIVE MACH NUMBERS $Ma_c \sim 1.2-2.0$ (SEE VREMAN 1996, AND FIG. 21)



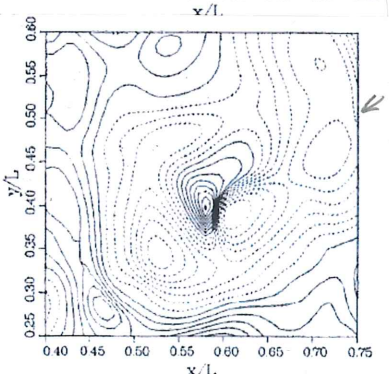
(b) EDDY-SHOCKLET SCHEMATICS

FIG. 20

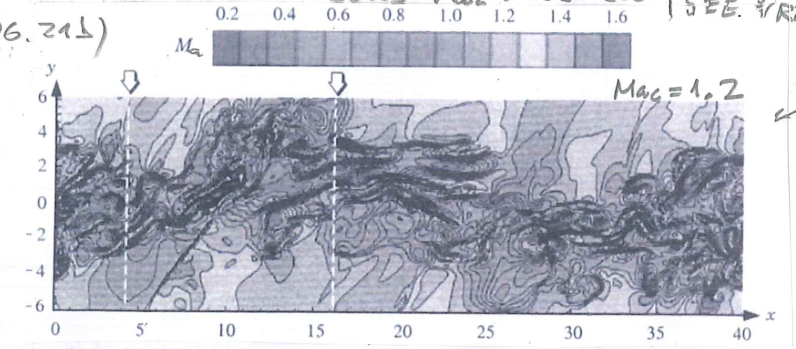


MACH CONTOURS, HOMOGENEOUS ISOTROPIC TURBULENCE (LEE et al, POF 1991)

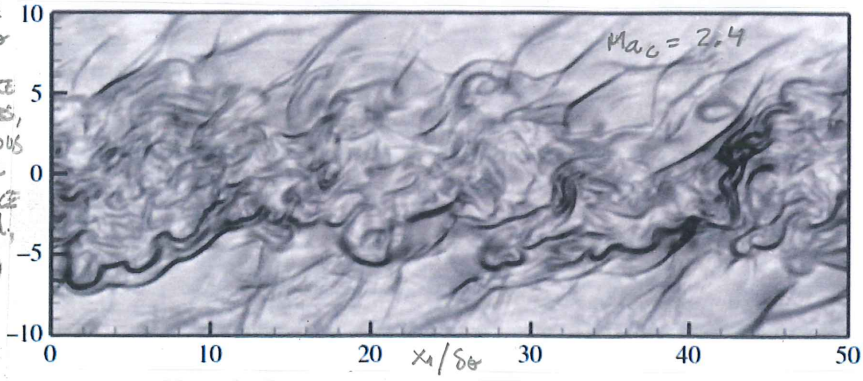
$Ma_\ell = 0.61$



(a) PRESSURE CONTOURS, HOMOGENEOUS ISOTROPIC TURBULENCE (LEE et al, POF 1991)



MACH CONTOURS, TEMPORAL SHEAR LAYER (SAVAGE, 1994)



(b) DENSITY GRADIENT CONTOURS (O'BRIEN et al, 2014)

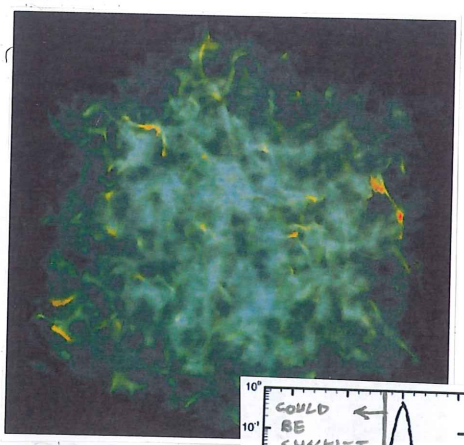
FIG. 21

THE IDENTIFICATION OR UNIVOCAL CHARACTERIZATION OR DEFINITION OF AN EDDY SHOCKLET IS NOT STRAIGHTFORWARD AND CONTINUES TO BE A TOPIC OF ACTIVE RESEARCH. EDDY SHOCKLETS ARE ASSOCIATED WITH RAPID VARIATIONS IN VELOCITY, MACH NUMBER, PRESSURE, TEMPERATURE, DENSITY AND ENTROPY, AS SCHEMATICALLY SHOWN IN FIG. 20 AND IN THE SIMULATIONS IN FIG. 21. SIMILAR TO SHOCK WAVES, SHOCKLETS CORRESPOND TO REGIONS OF STRONG COMPRESSIONAL DILATATION (θ LARGE AND NEGATIVE) AND STRONG DISSIPATION. AS THE MACH NUMBER INCREASES, THE DILATATION FIELD BECOMES HIGHLY INTERMITTENT (SEE FIG. 22), IN SUCH A WAY THAT A MYRIAD OF SHOCKS (i.e., EDDY SHOCKLETS) AND EXPANSION WAVES DEVELOP IN THE TURBULENT FLOW IN THE SMALL SCALES. FROM THE RANKINE-HUGONIOT CONDITION (65), THE CHARACTERISTIC VELOCITY JUMP ACROSS THE EDDY SHOCKLET (SEE FIG. 20b) IS $\frac{\Delta u}{U_1} = \frac{U_1 - U_2}{U_1} = \frac{2}{\gamma+1}\left(\frac{Ma_1^2 - 1}{Ma_1^2}\right)$, WHERE $Ma_1 = U_1/a_1$ REFERS TO THE LOCAL MACH NUMBER RELATIVE TO THE SHOCKLET. SINCE $\lambda \Delta u/\nu \sim O(1)$, THEN

$$\frac{\lambda}{\ell} \sim \frac{\nu}{\ell \Delta u} \sim \frac{\theta}{U_1 \ell}\frac{U_1}{\Delta u} \sim \frac{1}{Re_\ell}\frac{Ma_1^2}{Ma_1^2 - 1}$$

THUS SHOWING THAT INCREASINGLY TURBULENT FLOWS CREATE INCREASINGLY DISPARATE TURBULENCE SCALES WITH RESPECT TO THE EDDY SHOCKLETS.

IDENTIFICATION OF EDDY SHOCKLETS IS TYPICALLY MADE THROUGH THE INSPECTION OF INSTANTANEOUS CONTOURS OF DENSITY OR LOCAL MACH NUMBERS (SEE FIGS. 21 AND 23). MORE SOPHISTICATED METHODS ARE BASED ON THE LOCALIZATION OF HIGHLY COMPRESSED REGIONS ($\theta < 0$, $\theta < -|\theta_{threshold}|$) CONDITIONED ON DENSITY INFLECTION POINTS ($\nabla^2 \rho = 0$), AS ARE CHARACTERISTIC OF SHOCK WAVES. THESE CRITERIA ISOLATE REGIONS OF THE FLOW THAT CAN BE FURTHER FILTERED AND ANALYZED USING COMPUTER-VISION TECHNIQUES (AS DESCRIBED, FOR INSTANCE, IN SAMTANEY et al., POF 2001).



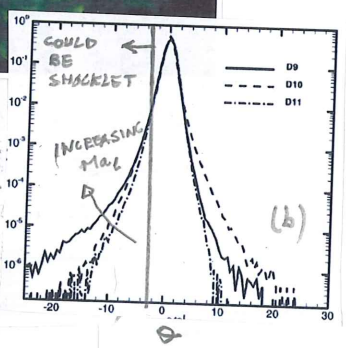


FIG. 22. (a) DILATATION CONTOURS IN COMPRESSIBLE HOMOGENEOUS TURBULENCE ($Ma_t = 0.48$). (b) PDF OF DILATATION. (ADAPTED FROM SAMTANEY et al., POF 2001)

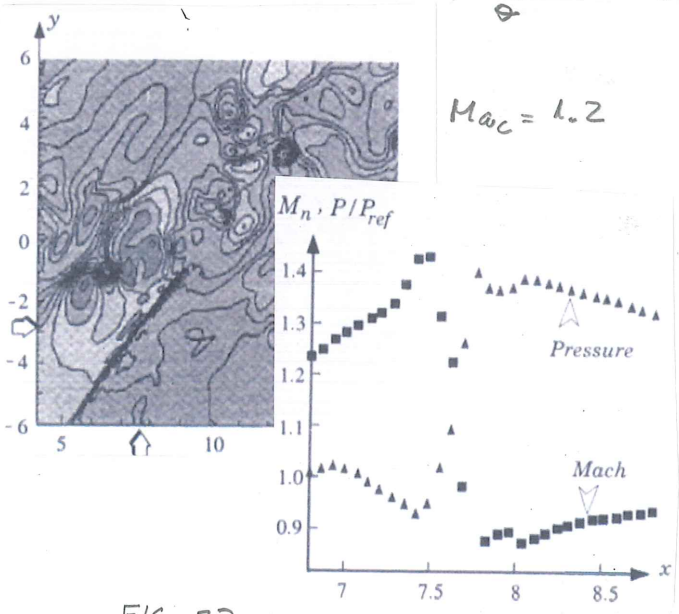


FIG. 23. IDENTIFICATION OF A SHOCKLET IN A COMPRESSIBLE SHEAR LAYER, SHOWING NORMAL VARIATIONS ACROSS (ADAPTED FROM SAVAGE, 1994).

V. THE INTERACTION OF SHOCK WAVES WITH PLANE DISTURBANCES

MANY PROBLEMS IN COMPRESSIBLE TURBULENCE ARE ASSOCIATED TO SUDDEN CHANGES IN THE EDDY FLUCTUATIONS AS THEY MOVE WITH THE FAST MEAN FLOW. THESE MAY OCCUR AS THE TURBULENCE FLOWS RAPIDLY THROUGH A WIRE MESH DAMPING SCREEN OR A SUDDEN CONTRACTION IN A WIND TUNNEL. IF TURBULENCE CONSTITUTES A SMALL FRACTION OF THE MEAN FLOW, THE DISTORTION PROCESS MAY CAUSE CHANGES IN THE FLUCTUATIONS THAT FOLLOW LINEAR LAWS. SPECIFICALLY, IF $t_d \sim L/\bar{U}$ IS A CHARACTERISTIC TIME SCALE FOR EDDY DISTORTION OR EDDY STRAINING, WITH L THE CHARACTERISTIC LENGTH OVER WHICH THE DISTORTION OCCURS AND $\bar{U}$ THE MEAN FLOW SPEED, THEN THE CHANGES IN THE FLUCTUATIONS MAY BE CONSIDERED LINEAR IF t_d IS MUCH SMALLER THAN $t_\ell \sim \ell/u_\ell$, WHERE ℓ IS THE CHARACTERISTIC WAVELENGTH OF THE FLUCTUATIONS (WHICH COULD BE ASCRIBED TO AN INTEGRAL LENGTH) AND u_ℓ IS THE MAGNITUDE OF THE VELOCITY FLUCTUATIONS (WHICH COULD CORRESPOND TO THE TURNOVER VELOCITY OF THE LARGE EDDIES), WITH $u_\ell/\bar{U} \ll 1$ AS IN WEAK TURBULENCE (SEE FIG. 24a). NOTE THAT THIS IS IN CONTRAST TO OTHER SLOWER PROCESSES INVOLVING STRAINING OR DISTORTION OF TURBULENCE ALONG PROLONGED REGIONS OF SPACE (FIG 24b), WHICH MAY GIVE RISE TO NON-LINEAR INTERACTIONS ALONG WITH SIGNIFICANT TURBULENCE DECAY. THESE OCCUR WHEN $t_d \sim t_\ell$, WITH $t_\ell \sim \ell/u_\ell$ BEING BOTH THE CHARACTERISTIC FLUCTUATION TIME AND THE DECAY TIME (NOTE THAT WHEN $u_\ell/\bar{U}$ IS NOT TOO SMALL, TURBULENT DISSIPATION CAUSED BY NON-LINEAR INTERACTIONS LEAD TO DECAY TIMES MUCH FASTER THAN THE MOLECULAR VISCOUS SCALE ℓ^2/ν.

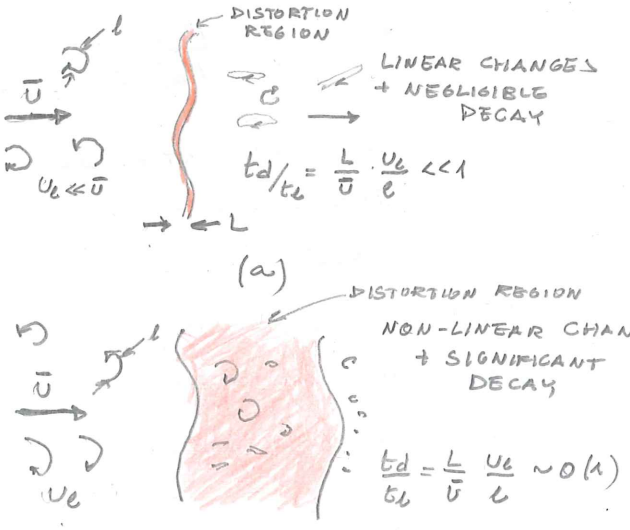


FIG. 24
INTERACTION OF TURBULENCE WITH EDDY-DISTORTING REGIONS. (a) RAPID AND (b) SLOW INTERACTIONS

IN THIS CHAPTER, THE FOCUS IS ON THE TREATMENT OF WEAK TURBULENCE AS IT IS CONVECTED THROUGH A SHOCK WAVE. GIVEN THE SUPERSONIC SPEEDS OF THE INCOMING WEAK TURBULENCE AND THE SMALL THICKNESS OF THE SHOCK, THIS CORRESPONDS TO AN ILLUSTRATIVE CASE WHERE LINEAR THEORY CAN BE APPLIED.

THIS PROBLEM WAS FIRST SOLVED BY H.S. RIBNER (NACA TN 2864, 1953) USING LINEAR INTERACTION ANALYSIS. THE DESCRIPTION BELOW FOLLOWS CLOSELY RIBNER'S FORMULATION OF THE PROBLEM. SIMILAR ANALYSES CAN BE FOUND IN MOORE (NACA TN 2879, 1953).

WEAK TURBULENCE INTERACTING WITH A SHOCK WAVE

CONSIDER THE TWO-DIMENSIONAL INTERACTION OF A SHOCK WAVE WITH A PLANE VORTICITY-ENTROPY WAVE TILTED AT AN ANGLE Ψ WITH RESPECT TO THE MEAN FLOW, AS SHOWN SCHEMATICALLY IN FIG. 25. THE INCIDENT FIELD IS REPRESENTED BY THE SUPERPOSITION OF A UNIFORM MEAN FLOW $\bar{u} = U_1$ ALONG WITH INVISCID ENTROPY AND VORTICITY MODES (SEE EQS. 142-144), WITH

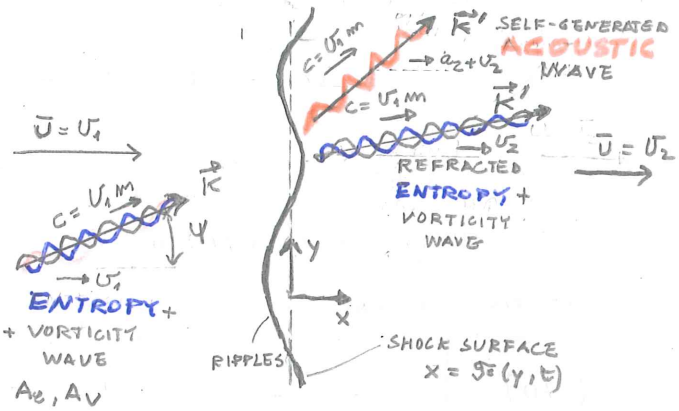


FIG. 25 : ENTROPY-VORTICITY WAVE INTERACTION WITH A SHOCK.

$$\frac{u_1'}{U_1} = \ell A_v e^{j|\vec{k}|(mx+\ell y - U_1 m t)} \qquad (165)$$

$$\frac{v_1'}{U_1} = -m A_v e^{j|\vec{k}|(mx+\ell y - U_1 m t)}$$

NOTE THAT THE PROPAGATION SPEED IS $U_1 m$ ALONG $\vec{k}$ AND U_1 ALONG THE x AXIS

THE CORRESPONDING DIVERGENCE-FREE FLUCTUATION VELOCITY FIELD ($\nabla \cdot \vec{u}' = j\vec{k}\cdot\vec{u}' = j k_x u_1' + j k_y v_1' = 0$), WITH $m = \vec{k}\cdot\vec{e}_x/|\vec{k}| = k_x/|\vec{k}| = \cos\Psi$ AND $\ell = \vec{k}\cdot\vec{e}_y = k_y/|\vec{k}| = \sin\Psi$ THE PROJECTIONS OF THE INCIDENT WAVENUMBER ALONG THE x AND y AXES. SIMILARLY, THE INCIDENT DENSITY FLUCTUATIONS ARE OF ENTROPIC ORIGIN AND ARE GIVEN BY

$$\rho_1'/\bar{\rho}_1 = A_e e^{j|\vec{k}|(mx+\ell y - U_1 m t)} \qquad (166)$$

WHICH ARE RELATED ISOBARICALLY TO THE TEMPERATURE AS

$$\rho_1'/\bar{\rho}_1 = -T_1'/\bar{T}_1 \qquad (167).$$

LASTLY, THE INCIDENT PRESSURE FLUCTUATIONS ARE ZERO SINCE THERE IS NO MECHANISM FOR THEM TO BE GENERATED IN THE VORTICITY OR ENTROPY MODES,

$$p_1' = 0 \qquad (168).$$

NOTE THAT THE INCOMPRESSIBLE VELOCITY FIELD (165) IS SOLELY CARRIED BY THE VORTICITY MODE, WITH CORRESPONDING OUT-OF-PLANE VORTICITY GIVEN BY

$$\frac{\vec{\omega}_1'}{|\vec{k}|U_1} = j\frac{\vec{k}\wedge\vec{u}'}{|\vec{k}|U_1} = -j|A_v| e^{j|\vec{k}|(mx+\ell y - U_1 m t)}\vec{e}_z \qquad (169)$$

($|A_v|$ = VORTICITY AMPLITUDE) ⇒ TRANSVERSAL WAVE ($\perp$ TO $\vec{k}$)

ALL FLUCTUATIONS (165) - (169) PROPAGATE AT THE MEAN VELOCITY OF THE PRE-SHOCK GASES $U_1 > a_1$. NOTE THAT A_v AND A_e CAN BE GENERALLY COMPLEX AND OUT OF PHASE.

THE PROBLEM IS FORMULATED IN A REFERENCE FRAME WHERE THE MEAN POSITION OF THE SHOCK REMAINS AT REST AT $X=0$. BECAUSE OF THE INTERACTIONS WITH THE INCIDENT FLUCTUATIONS (165)-(169), RIPPLES ARE PRODUCED ON THE SHOCK SURFACE IN SUCH A WAY THAT THE POINTS ON THE SURFACE ARE DISPLACED IN THE STREAMWISE DIRECTION BY A SMALL AMOUNT $X = F(y,t)$. THESE RIPPLES LEAD TO DISTORTION OF THE FLOW DOWNSTREAM OF THE SHOCK $(x>0)$. THE FLOW UPSTREAM OF THE SHOCK $(x<0)$ CANNOT BE PERTURBED BY THE RIPPLES SINCE IT IS SUPERSONIC. THE ANALYSIS THEN REDUCES TO A BOUNDARY-VALUE PROBLEM FOR THE POST-SHOCK FLOW SUBJECT TO APPROPRIATE BOUNDARY CONDITIONS AT THE SHOCK SURFACE, WHOSE DISPLACEMENT $F(y,t)$ IS ALSO A SOLUTION OF THE PROBLEM.

CONSIDER THE INVISCID LIMIT OF THE CONTINUITY (11), MOMENTUM (12) AND ENTROPY (35) CONSERVATION EQUATIONS LINEARIZED AROUND THE MEAN POST-SHOCK FLOW,

$$\frac{\partial \rho_2'}{\partial t} + U_2 \frac{\partial \rho_2'}{\partial x} = -\bar{\rho}_2 \frac{\partial u_2'}{\partial x} - \bar{\rho}_2 \frac{\partial v_2'}{\partial y} \qquad (170)$$

$$\frac{\partial u_2'}{\partial t} + U_2 \frac{\partial u_2'}{\partial x} = -\frac{1}{\bar{\rho}_2} \frac{\partial p_2'}{\partial x} \qquad (171)$$

$$\frac{\partial v_2'}{\partial t} + U_2 \frac{\partial v_2'}{\partial x} = -\frac{1}{\bar{\rho}_2} \frac{\partial p_2'}{\partial y} \qquad (172)$$

$$\frac{\partial s_2'}{\partial t} + U_2 \frac{\partial s_2'}{\partial x} = 0 \,, \qquad (173)$$

ALONG WITH THE LINEARIZED EQUATION OF STATE AND THE LINEARIZED EXPRESSION FOR THE ENTROPY, NAMELY

$$\frac{p_2'}{\bar{p}_2} = \frac{\rho_2'}{\bar{\rho}_2} + \frac{T_2'}{\bar{T}_2} \qquad (174)$$

AND

$$\frac{s_2'}{c_p} = \frac{p_2'}{\gamma \bar{p}_2} - \frac{\rho_2'}{\bar{\rho}_2} \,. \qquad (175)$$

EQUATIONS (170)-(175) ARE SOLVED IN THE POSTSHOCK REGION $(x \geq 0)$ SUBJECT TO LINEARIZED RANKINE HUGONIOT BOUNDARY CONDITIONS AT THE MEAN SHOCK POSITION $(x=0)$ AS FOLLOWS. SPECIFICALLY, FOUR BOUNDARY CONDITIONS AT $x=0$ ARE REQUIRED. THESE HAVE TO BE NECESSARILY FORMULATED WITH RESPECT TO THE MOVING RIPPLE (e.g. SEE FIG. 26), SINCE THE RANKINE-HUGONIOT CONDITIONS ARE WITH RESPECT TO THE MOVING SHOCK.

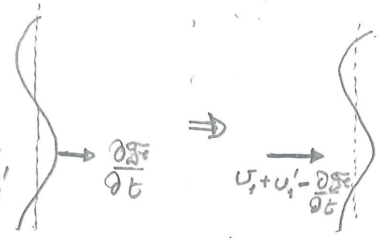


FIG. 26.

* BOUNDARY CONDITIONS

CONSIDER FIRST THE CONTINUITY JUMP CONDITION (55), WHICH FOR THIS PROBLEM CAN BE REWRITTEN AS

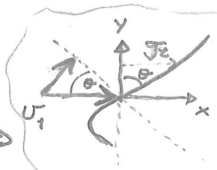
$$(\bar{\rho}_1 + \rho_1')\left(U_1 + u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) = (\bar{\rho}_2 + \rho_2')\left(U_2 + u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right). \quad (176)$$

SIMILARLY, THE MOMENTUM CONSERVATION JUMP CONDITION (56) BECOMES

$$(\bar{P}_1 + P_1') + (\bar{\rho}_1 + \rho_1')\left(U_1 + u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right)^2 = (\bar{P}_2 + P_2') + (\bar{\rho}_2 + \rho_2')\left(U_2 + u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right)^2. \quad (177)$$

WHILE THE TANGENTIAL VELOCITY MUST REMAIN CONSTANT,

$$\underbrace{U_1 \frac{\partial \mathcal{F}_e}{\partial y}}_{\text{MEAN VELOCITY TANGENTIAL TO RIPPLE (PRESHOCK)}} + v_1' = \underbrace{U_2 \frac{\partial \mathcal{F}_e}{\partial y}}_{\text{MEAN VELOCITY TANGENTIAL TO RIPPLE (POSTSHOCK)}} + v_2'. \quad (178)$$



$\tan\theta = \partial F/\partial y \approx \theta$

MEAN VELOCITY TANGENTIAL TO THE RIPPLE: $V_1 = U_1 \sin\theta = \approx U_1 \frac{\partial \mathcal{F}_e}{\partial y}$

LASTLY, THE JUMP VELOCITY CONDITION (65) CAN BE EXPRESSED AS

$$U_2 U_1 = \frac{U_2}{U_1} \cdot U_1^2 = \frac{2 + (\gamma-1) Ma_1^2}{(\gamma+1) Ma_1^2} U_1^2 = 2\left(\frac{\gamma-1}{\gamma+1}\right)\left\{\frac{U_1^2}{2} + \frac{\gamma R_g T_1}{\gamma-1}\right\}.$$

FOR THE PRESENT PROBLEM, THE ABOVE RELATION CAN BE WRITTEN AS

$$\left(U_2 + u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right)\left(U_1 + u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) = 2\left(\frac{\gamma-1}{\gamma+1}\right)\left[\frac{1}{2}\left(U_1 + u_1' - \frac{\partial \mathcal{F}_e}{\partial y}\right)^2 + \frac{\gamma R_g}{\gamma-1}\left(\bar{T}_1 + T_1'\right)\right] \quad (179).$$

THE JUMP CONDITIONS (176) - (179) CAN BE LINEARIZED TO FIRST ORDER AS

$$\bar{\rho}_1\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + \rho_1' U_1 = \bar{\rho}_2\left(u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + \rho_2' U_2 \quad (180) \quad \text{CONTINUITY}$$

$$2\bar{\rho}_1\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right)U_1 + \rho_1' U_1^2 = P_2' + 2\bar{\rho}_2\left(u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right)U_2 + \rho_2' U_2^2 \quad (181) \quad \text{MOMENTUM}$$

$$U_2\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + U_1\left(u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right) = 2\left(\frac{\gamma-1}{\gamma+1}\right)\left[U_1\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial y}\right) + \frac{\gamma R_g}{\gamma-1} T_1'\right] \quad (182) \quad \text{ENERGY}$$

WHILE THE CONTINUITY OF THE TANGENTIAL VELOCITY (178) IS ALREADY LINEAR, EQUATIONS (180) - (182) AND (178) CAN BE REARRANGED TO GIVE THE FINAL EXPRESSIONS

$$\frac{1}{U_1}\left(u_2' - \frac{\partial \mathcal{F}_e}{\partial t}\right) = \frac{B_1}{U_1}\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + B_2 \frac{T_1'}{\bar{T}_1}, \quad (183) \qquad \frac{\rho_2'}{\bar{\rho}_2} = \frac{C_1}{U_1}\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + C_2 \frac{T_1'}{\bar{T}_1}, \quad (184)$$

$$\frac{P_2'}{\bar{P}_2} = \frac{D_1}{U_1}\left(u_1' - \frac{\partial \mathcal{F}_e}{\partial t}\right) + D_2 \frac{T_1'}{\bar{T}_1}, \quad (185) \qquad \frac{v_2'}{U_1} = \frac{v_1'}{U_1} + E_1 \frac{\partial \mathcal{F}_e}{\partial y}, \quad (186)$$

AT $x = 0$

WHERE $B_1, B_2, C_1, C_2, D_1, D_2$ AND E_1 ARE COEFFICIENTS THAT DEPEND ON THE INCIDENT MACH NUMBER AND ARE GIVEN BY

$$\begin{cases} B_1 = \dfrac{(\gamma-1)\,Ma_1^2 - 2}{(\gamma+1)\,Ma_1^2}, \qquad B_2 = \dfrac{2}{(\gamma+1)\,Ma_1^2} \\ C_1 = \dfrac{4}{(\gamma-1)\,Ma_1^2 + 2}, \qquad C_2 = -\dfrac{(\gamma-1)\,Ma_1^2 + 4}{(\gamma-1)\,Ma_1^2 + 2}, \\ D_1 = \dfrac{4\gamma Ma_1^2}{2\gamma Ma_1^2 - (\gamma-1)}, \qquad D_2 = -\dfrac{2\gamma Ma_1^2}{2\gamma Ma_1^2 - (\gamma-1)}, \quad E_1 = \dfrac{2}{\gamma+1}\left(\dfrac{Ma_1^2-1}{Ma_1^2}\right) \end{cases} \tag{187}$$

THE FORMULATION GIVEN ABOVE IS WRITTEN IN TWO SPATIAL DIMENSIONS. EXTENSION TO THREE DIMENSIONS IS STRAIGHTFORWARD. INTEGRATION OF THE PROBLEM (170-175) SUBJECT TO (183)-(186), ALONG WITH (165)-(168) AND (187), PROVIDES THE SOLUTION AS DESCRIBED BELOW.

* PROPAGATING AND DECAYING REGIMES DOWNSTREAM OF THE SHOCK

IT IS ILLUSTRATIVE TO EXAMINE THE PROPAGATION OF PRESSURE FLUCTUATIONS DOWNSTREAM OF THE SHOCK, WHICH GIVES RISE TO TWO DIFFERENT REGIMES AS A FUNCTION OF THE ANGLE OF INCIDENCE Ψ OF THE UPSTREAM PERTURBATIONS (165)-(168). TO SEE THIS, TAKE THE DIVERGENCE OF THE MOMENTUM EQUATIONS (171)-(172),

$$\frac{\partial}{\partial t}\left(\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y}\right) + U_2 \frac{\partial}{\partial x}\left(\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y}\right) = -\frac{1}{\bar{\rho}_2}\left(\frac{\partial^2 p_2'}{\partial x^2} + \frac{\partial^2 p_2'}{\partial y^2}\right), \tag{188}$$

AND SINCE

$$\frac{\partial u_2'}{\partial x} + \frac{\partial v_2'}{\partial y} = -\frac{1}{\bar{\rho}_2}\left(\frac{\partial \rho_2'}{\partial t} + U_2 \frac{\partial \rho_2'}{\partial x}\right) \overset{\text{USE EQS. (173) AND (175)}}{=} -\frac{1}{\gamma \bar{p}_2}\left(\frac{\partial p_2'}{\partial t} + U_2 \frac{\partial p_2'}{\partial x}\right),$$

THEN (188) BECOMES

$$\boxed{\frac{\partial^2 p_2'}{\partial t^2} + 2U_2 \frac{\partial^2 p_2'}{\partial x \partial t} = (\bar{a}_2^2 - U_2^2)\frac{\partial^2 p_2'}{\partial x^2} + \bar{a}_2^2 \frac{\partial^2 p_2'}{\partial y^2}} \tag{189}$$

WHERE $\bar{a}_2^2 = \gamma \bar{p}_2 / \bar{\rho}_2$ IS THE SQUARE OF THE MEAN SPEED OF SOUND IN THE POST-SHOCK GASES. NOTE THAT EQ. (189) IS ANALOGOUS TO THE INVISCID VERSION OF THE PRESSURE PROPAGATION EQUATION (125) DERIVED PREVIOUSLY, AND, SIMILAR TO (125), IT IS RESPONSIBLE FOR THE GENERATION OF AN ACOUSTIC MODE DOWNSTREAM OF THE SHOCK.

NOTE THAT EQS. (183)-(186) DEMAND THAT THE SOLUTION DOWNSTREAM OF THE SHOCK MUST BE OF OSCILLATORY TYPE IN BOTH y (WITH THE SAME TRANSVERSE WAVENUMBER $k_y = |\vec{k}| \sin\Psi = |\vec{k}|/b$ AS THE INCIDENT PERTURBATIONS) AND t (WITH THE SAME FREQUENCY $\omega = k_x U_1 = \|\vec{k}\|/m\,U_1$ AND PROPAGATION SPEED $c = \omega/|\vec{k}| = m U_1$ AS THE INCIDENT PERTURBATIONS).

NOT THE SAME TOTAL WAVENUMBER

FOR INSTANCE EQUATIONS (165), (166), (167) AND (188) IMPOSE A SOLUTION TO (189) OF THE FORM $P_2' = G(x)\, e^{j|\vec{k}|(ly - m U_1 t)}$, SUBSTITUTING THIS EXPRESSION INTO (189), THE EQUATION

$$\left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right) G'' + 2j|\vec{k}| m \frac{U_2}{U_1} G' + |\vec{k}|^2 \left(m^2 - \frac{l^2 \bar{a}_2^2}{U_1^2}\right) G = 0 \qquad (190)$$

THE CHARACTERISTIC SOLUTIONS OF THIS EQUATION ARE OF THE FORM $G(x) = K e^{jbx}$, WHICH, UPON BEING SUBSTITUTED INTO (190) LEADS TO THE SECOND-ORDER EQUATION

$$\left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right) b^2 + 2|\vec{k}| m b \frac{U_2}{U_1} - |\vec{k}|^2 \left(m^2 - \frac{l^2 \bar{a}_2^2}{U_1^2}\right) = 0 \qquad (191)$$

WHERE THE INTEGRATION CONSTANT K HAS BEEN CANCELLED OUT. THE SOLUTIONS TO (191) ARE GIVEN BY

$$\frac{b}{|\vec{k}|} = \frac{-m\frac{U_2}{U_1} \pm \frac{l \bar{a}_2}{U_1}\sqrt{\frac{m^2}{l^2} - \left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right)}}{\left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right)} . \qquad (192)$$

IN ORDER TO SIMPLIFY THIS EXPRESSION, NOTE THAT THE JUMP CONDITION (68) FOR THE MACH NUMBER CAN BE REWRITTEN AS

$$Ma_1^2 = \frac{2 + Ma_2^2(\gamma - 1)}{2\gamma Ma_2^2 + (1 - \gamma)} , \qquad (193)$$

SO THAT THE VELOCITY JUMP CONDITION (65) BECOMES

$$\frac{U_2}{U_1} = \frac{(\gamma+1) Ma_2^2}{2 + (\gamma - 1) Ma_2^2} \qquad (194)$$

AS A FUNCTION OF THE POST-SHOCK MACH NUMBER Ma_2. IN THIS WAY, (192) BECOMES

$$\frac{b}{|\vec{k}|} = \frac{U_1}{U_2} \frac{Ma_2}{1 - Ma_2^2} \left[-m Ma_2 \pm l \sqrt{\left(\frac{m}{l}\right)^2 - \frac{U_2^2}{U_1^2}\left(\frac{1 - Ma_2^2}{Ma_2^2}\right)} \right] =$$

$$= \frac{2 + (\gamma - 1) Ma_2^2}{(\gamma+1) Ma_2 (1 - Ma_2)} \left[-m Ma_2 \pm l \sqrt{\left(\frac{m}{l}\right)^2 - \frac{(\gamma+1)^2 Ma_2^2 (1 - Ma_2^2)}{[2 + (\gamma - 1) Ma_2^2]^2}} \right] . \qquad (195)$$

IN GENERAL, (195) CAN BE EXPRESSED AS $b = b_r + j b_I$, WHERE b_r AND b_I ARE THE REAL AND IMAGINARY COMPONENTS. NOTE THAT IF $b_I = 0$, i.e., THERE IS NO IMAGINARY COMPONENT, THEN THE PRESSURE FLUCTUATION IS A PLANE PROPAGATING WAVE,

$$P_2' = K e^{j|\vec{k}|(b_r^+ x + ly - m U_1 t)} , \qquad (196)$$

WITH b^- ($b < 0$) BEING RULED OUT AS IT REPRESENTS A REFLECTED WAVE ONTO THE INCIDENT SUPERSONIC STREAM. AS A RESULT, THE PRESSURE FLUCTUATION PROPAGATES ALONG A WAVENUMBER DIRECTION $b_r^+ = \cos\phi$ GENERALLY DIFFERENT FROM THE UPSTREAM INCIDENCE ANGLE ψ.

ON THE OTHER HAND, IF THE IMAGINARY PART IS NONZERO, $b_I \neq 0$, THEN THE PRESSURE FLUCTUATION BECAMES A WAVE THAT DECAYS (FOR THE ROOT b_I^+, $b_I > 0$) OR INCREASES (FOR THE ROOT b_I^-, $b_I < 0$) EXPONENTIALLY. RULING OUT THE LATTER AS UNPHYSICAL, THE PRESSURE FLUCTUATION IN THIS CASE BECOMES AN EXPONENTIALLY DECAYING WAVE

$$p_2' = K e^{-b_I^+ |\vec{k}| x} e^{i|\vec{k}|(b_r x + l y - m U_1 t)} \qquad (197)$$

WHICH IS USUALLY REFERRED TO AS AN EVANESCENT WAVE AS A RESULT OF TOTAL INTERNAL REFLECTION.

FROM THE DISCUSSION ABOVE, IT IS CLEAR THAT THE PRESENCE OR NOT OF IMAGINARY ROOTS OF EQ. (191) HAS GREAT IMPACT ON THE DYNAMICS. SINCE $m/l = \cot\Psi$, THIS DEPENDS ON ANGLE OF INCIDENCE OF THE UPSTREAM DISTURBANCES, AS SUMMARIZED BELOW.

- FOR $\left(\frac{m}{l}\right)^2 = \cot^2\Psi > \left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right) \Rightarrow$ b IS REAL AND HAS TWO SOLUTIONS, b^+ AND b^-, WITH b^+ BEING THE PHYSICAL SOLUTION. THE PRESSURE FLUCTUATION IS A PLANE PROPAGATING WAVE.
  $(\Psi < \Psi_c)$

- FOR $\left(\frac{m}{l}\right)^2 = \cot^2\Psi < \left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right) \Rightarrow$ b HAS A REAL COMPONENT b_r AND TWO IMAGINARY COMPONENTS b_I^+ AND b_I^-, THE LATTER BEING RULED OUT AS IT LEADS TO EXPONENTIAL INCREASE. THE PRESSURE FLUCTUATION IS AN EVANESCENT WAVE IN THE X DIRECTION.
  $(\Psi > \Psi_c)$

THE CRITICAL ANGLE Ψ_c IS GIVEN BY

$$\Psi_c = \arctan\left\{\sqrt{\left(\frac{\bar{a}_2^2}{U_1^2} - \frac{U_2^2}{U_1^2}\right)^{-1}}\right\} = \arctan\left\{\sqrt{\frac{\left(2+(\gamma-1)Ma_2^2\right)^2}{(\gamma+1)^2 Ma_2^2 (1-Ma_2^2)}}\right\} =$$

$$= \arctan\left\{\sqrt{\frac{(\gamma+1)^2 Ma_1^4}{\left(2+(\gamma-1)Ma_1^2\right)^2}\,\frac{Ma_2^2}{1-Ma_2^2}}\right\} = \arctan\left\{\sqrt{\frac{(\gamma+1)Ma_1^2}{\left(2+(\gamma-1)Ma_1^2\right)(Ma_1^2-1)}}\right\} \qquad (198)$$

FOR INSTANCE, $\Psi_c = 43.3^\circ$ FOR $Ma_1 = 2.0$. THEREFORE $\Psi > 43.3^\circ$ LEADS TO AN EVANESCENT PRESSURE WAVE.

* THE VORTICITY DOWNSTREAM OF THE SHOCK

THE VORTICITY DOWNSTREAM OF THE SHOCK IS DESCRIBED BY THE CURL OF THE MOMENTUM EQUATIONS (171)-(172), WHICH GIVES

$$\frac{\partial \omega_2'}{\partial t} + U_2 \frac{\partial \omega_2}{\partial x} = 0 \qquad (199)$$

WHICH HAS A PROPAGATING WAVE SOLUTION OF THE FORM

$$\omega_2' = W e^{i\delta x} e^{i|\vec{k}|(ly - U_1 m t)}, \qquad (200)$$

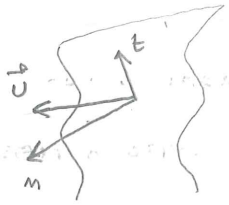
(DIRECTED IN THE +Z DIRECTION)

WHERE δ IS OBTAINED BY SUBSTITUTION BACK IN EQ. (199),

$$-i|\vec{k}|U_1 m + U_2 i\delta = 0 \Rightarrow \delta = |\vec{k}| m U_1/U_2 \quad (201).$$

→ NOTE: THE SAME RESULT (201) APPLIES TO THE DENSITY FLUCTUATION FROM THE ENTROPY MODE, WHICH FROM (170) MUST OBEY $\frac{\partial \rho_2'}{\partial t} + U_2 \frac{\partial \rho_2'}{\partial x} = 0$

NOTE THAT, IN GENERAL THREE DIMENSIONAL FLOWS, THE COMPONENT OF THE VORTICITY NORMAL TO THE SHOCK FRONT MUST NECESSARILY REMAIN CONSTANT ACROSS THE SHOCK DUE TO THE ZERO JUMP OF TANGENTIAL VELOCITY ACROSS THE SHOCK:



m, t : NORMAL AND TANGENTIAL VECTORS TO THE FRONT

⇒ VELOCITY COMPONENT NORMAL TO THE FRONT: $U_m = \vec{U}\cdot\vec{m}$

TANGENTIAL COMPONENT NORMAL TO THE FRONT: $U_t = \vec{U}\cdot\vec{t} = \vec{m}\wedge\vec{U}\wedge\vec{m}$

VORTICITY COMPONENT NORMAL TO THE FRONT: $\omega_m = (\nabla\wedge\vec{U}_t)_m$

SINCE $U_{1t} = U_{2t}$ ACROSS A SHOCK ⇒ $\omega_{1m} = \omega_{1t}$ (202)

* THE SOLUTION COEFFICIENTS

THE SOLUTION DOWNSTREAM FROM THE SHOCK IS COMPRISED OF ACOUSTIC, VORTICITY AND ENTROPY MODES. THE CORRESPONDING FLUCTUATIONS CAN BE EXPRESSED AS

$$\frac{u_2'}{U_2} = F e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)} + G e^{i|\vec{k}|\left(m\frac{U_1}{U_2}x + ly - mU_1 t\right)} \quad (203)$$

NOTE THAT THE PROPAGATION SPEED IS mU_1 ALONG $\vec{k}$ AND $\frac{mU_1}{mU_1/U_2} = U_2$ ALONG x (204)

$$\frac{v_2'}{U_2} = \underbrace{H e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}}_{\text{ACOUSTIC MODE, } u_2'^{(p)}, v_2'^{(p)}} + \underbrace{I e^{i|\vec{k}|\left(m\frac{U_1}{U_2}x + ly - mU_1 t\right)}}_{\text{VORTICITY MODE, } u_2'^{(\omega)}, v_2'^{(\omega)}}$$

$$\frac{p_2'}{\bar{p}_2} = \underbrace{K e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}}_{\text{ACOUSTIC MODE, } p_2'^{(p)}} \quad (205)$$

NOTE THAT THE WAVENUMBER OF THE REFRACTED WAVE IS $|\vec{k}'| = \left[m^2\left(\frac{U_1}{U_2}\right)^2 + l^2\right]^{1/2} |\vec{k}| > |\vec{k}|$ BUT $k_y' = k_y = |\vec{k}| l$ IS CONSERVED.

$$\frac{\rho_2'}{\bar{\rho}_2} = \underbrace{L e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}}_{\text{ACOUSTIC MODE, } \rho_2'^{(p)}} + \underbrace{Q e^{i|\vec{k}|\left(m\frac{U_1}{U_2}x + ly - mU_1 t\right)}}_{\text{ENTROPY MODE, } \rho_2'^{(s)}} \quad (206)$$

$$\frac{T_2'}{\bar{T}_2} = \underbrace{X e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}}_{\text{ACOUSTIC MODE, } T_2'^{(p)}} + \underbrace{Y e^{i|\vec{k}|\left(m\frac{U_1}{U_2}x + ly - mU_1 t\right)}}_{\text{ENTROPY MODE, } T_2'^{(s)}} \quad (207)$$

ALSO NOTE THAT THE FRONT DISPLACEMENT, ACCORDING TO CONDITIONS (183)-(186), IS NECESSARILY OF THE FORM $\mathcal{F} \sim e^{i|\vec{k}|(ly - mU_1 t)}$, SO THAT

$$\frac{1}{U_1}\frac{\partial \mathcal{F}}{\partial t} = C e^{i|\vec{k}|(ly - mU_1 t)} \quad \text{AND} \quad \frac{\partial \mathcal{F}}{\partial y} = D e^{i|\vec{k}|(ly - mU_1 t)} \quad (208)$$

THE 11 COEFFICIENTS GIVEN ABOVE (F, G, H, I, K, L, Q, X, Y, C AND D) REPRESENT THE UNKNOWNS OF THE PROBLEM.

SOLUTIONS TO THESE COEFFICIENTS ARE LENGTHY AND CAN BE OBTAINED IN THE FOLLOWING FORM. FIRST NOTE THAT THE ACOUSTIC MODE IS ISENTROPIC AND AS A RESULT SATISFIES

$$\frac{s_2'^{(p)}}{c_p} = \frac{p_2'^{(p)}}{\gamma \bar{p}_2} - \frac{\rho_2'^{(p)}}{\bar{\rho}_2} = \frac{T_2'^{(p)}}{\bar{T}_2} - \left(\frac{\gamma-1}{\gamma}\right)\frac{p_2'^{(p)}}{\bar{p}_2} = 0$$

AND THEREFORE

$$\boxed{L = K/\gamma \quad \text{AND} \quad X = \left(\frac{\gamma-1}{\gamma}\right)K,} \qquad (209)$$

SIMILARLY, THE ENTROPY MODE IS ISOBARIC, WHICH IMPLIES $\rho_2'^{(s)}/\bar{\rho}_2 = -T_2'^{(s)}/\bar{T}_2$, AND

$$\boxed{Y = -Q}. \qquad (210)$$

IN ADDITION, C AND D ARE NECESSARILY RELATED, SINCE

$$\frac{\partial^2 F}{\partial t \partial y} = -i D |\vec{k}| m U_1 e^{i|\vec{k}|(ly - mU_1 t)} = \frac{\partial^2 F}{\partial y \partial t} = i|\vec{k}| l C e^{i|\vec{k}|(ly - mU_1 t)}$$

AND THEREFORE

$$\boxed{D = -lC/mU_1}. \qquad (211)$$

EQUATIONS (209)-(211) REDUCE THE NUMBER OF UNKNOWNS FROM 11 TO 7 (i.e, F, G, H, I, K, Q AND C): SUBSTITUTION OF (203) AND (205) INTO THE x-MOMENTUM EQUATION (171) GIVES

NOTE CANCELLATION OF VORTICAL PARTS

$$-i|\vec{k}| m U_1 F e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)} - i|\vec{k}| m U_1 G e^{i|\vec{k}|(ly - mU_1 t)} + ibU_2 F e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}$$
$$+ i|\vec{k}| U_2 m \frac{U_1}{U_2} G e^{i|\vec{k}|(ly - mU_1 t)} = -\frac{p_2}{\bar{\rho}_2}\frac{K}{U_2} ib e^{ibx} e^{i|\vec{k}|(ly - mU_1 t)}$$

AND SIMPLIFYING, $-i|\vec{k}| m F U_1 + ibU_2 F = -\frac{1}{\bar{\rho}_2} i K \frac{p_2}{U_2} \Rightarrow \boxed{F = \alpha K}$ (212)

WITH $\alpha = \frac{b}{|\vec{k}|}\frac{1}{\gamma Ma_2^2} / \left(U_1 U_2 m - b/|\vec{k}| U_2^2\right)$. SIMILARLY, SUBSTITUTION OF (204) INTO THE y-MOMENTUM EQUATION LEADS TO

$$\boxed{H = \beta K} \qquad (213)$$

WITH $\beta = l/\gamma Ma_2^2 / \left(U_1 U_2 m - b/|\vec{k}| U_2^2\right)$. ADDITIONAL CONSTRAINTS BETWEEN COEFFICIENTS CAN BE OBTAINED BY SUBSTITUTING (203), (204) AND (206) INTO THE CONTINUITY EQUATION (120), WHICH GIVES

$$iG|\vec{k}| m \left(\frac{U_2}{U_1}\right) + I|\vec{k}| i l = 0 \Rightarrow \boxed{I = -\frac{m}{l}\frac{U_2}{U_1} G}, \qquad (214)$$

NOTE THAT (214) EMERGES FROM THE ZERO DILATATION OF THE VORTICAL MODE OF THE VELOCITY, i.e., $\frac{\partial u_2'^{(\omega)}}{\partial x} + \frac{\partial v_2'^{(\omega)}}{\partial y} = 0$. THE ACOUSTIC MODE OF THE DENSITY TRIVIALLY

SATISFIES THAT $\frac{\partial \rho_2'^{(p)}}{\partial t} + U_2 \frac{\partial \rho_2'^{(p)}}{\partial x} = -\bar{\rho}_2 \frac{\partial u_2'^{(p)}}{\partial x} - \bar{\rho}_2 \frac{\partial v_2'}{\partial y}$, WHILE THE ENTROPY MODE OF THE DENSITY ALSO SATISFIES $\frac{\partial \rho_2'^{(s)}}{\partial t} + U_2 \frac{\partial \rho_2'^{(s)}}{\partial t} = 0$. THESE RELATIONS GIVE NO USEFUL INFORMATION AS THEY ARE SATISFIED BY CONSTRUCTION OF (203), (204) AND (206).

SUBSTITUTION OF THE SOLUTIONS (203) - (207) INTO THE BOUNDARY CONDITIONS (183)-(186) AT THE MEAN SHOCK LOCATION YIELDS THE ADDITIONAL RELATIONS

$$(F+G)\frac{U_2}{U_1} - C = B_1(\ell A_V - C) - B_2 A_e, \qquad (215) \quad \text{(FROM 183)}$$

$$Q + \frac{K}{\gamma} = C_1(\ell A_V - C) - C_2 A_e, \qquad (216) \quad \text{(FROM 184)}$$

$$K = D_1(\ell A_V - C) - D_2 A_e, \qquad (217) \quad \text{(FROM 185)}$$

$$(H+I)\frac{U_2}{U_1} = -m A_V - E_1 \ell C/m \qquad (218) \quad \text{(FROM 186)}$$

WHERE USE OF (211) HAS BEEN MADE. EQUATIONS (212) - (218) CORRESPOND TO A SYSTEM OF 7 EQUATIONS FOR 7 COEFFICIENTS (F, G, C, Q, K, H AND I) THAT CAN BE SOLVED AS A FUNCTION OF THE KNOWN VALUES U_2/U_1, ℓ, A_V (VORTICITY AMPLITUDE), A_e (DENSITY AMPLITUDE), m, α, β, AND THE COEFFICIENTS B_1, B_2, C_1, C_2, D_1, D_2, AND E_1. FOR INSTANCE, IT CAN BE SHOWN THAT THE AMPLITUDE OF THE VELOCITY OF THE RIPPLES IS

$$\boxed{\frac{C}{A_V} = \frac{-m - \beta\left(D_1\ell - D_2 A_e/A_V\right) + \left(\frac{m}{\ell}\right)\left(\frac{U_2}{U_1}\right)\left[\alpha\left(D_2\frac{A_e}{A_V} - D_1\ell\right) + B_1\ell - B_2\frac{A_e}{A_V}\right]}{E_1\left(\frac{\ell}{m}\right) - \beta D_1 - \left(\frac{m}{\ell}\right)\left(\frac{U_2}{U_1}\right)\left(1 - B_1 + \alpha D_1\right)}} \qquad (219)$$

SUBSTITUTION OF (219) INTO (217) LEADS TO THE SOLUTION TO K AND SUBSEQUENTLY TO ALL OTHER COEFFICIENTS. THIS REPRESENTS THE FINAL SOLUTION OF AN ENTROPY-VORTICITY WAVE INTERACTING WITH A SHOCK AS GIVEN BY LINEAR INTERACTION ANALYSIS.

FLOW STATISTICS DOWNSTREAM OF THE SHOCK WAVE

CONSIDER THE KINETIC ENERGY DOWNSTREAM OF THE SHOCK (* = COMPLEX CONJUGATE)

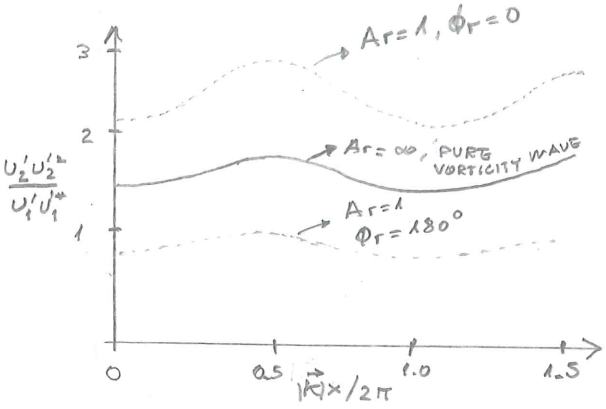
$$\frac{U_2' U_2'^*}{U_2^2} = \frac{U_2^2}{U_1^2}\left\{ |F|^2 e^{i(b-b^*)x} + |G|^2 + FG^* e^{i\left(b - |\vec{k}|m\frac{U_1}{U_2}\right)x} + F^* G e^{-i\left(b^* - |\vec{k}|m\frac{U_1}{U_2}\right)x} \right\}$$

THE TWO REGIMES BEHIND THE SHOCK LEAD TO TWO DIFFERENT BEHAVIORS OF THE KINETIC ENERGY.

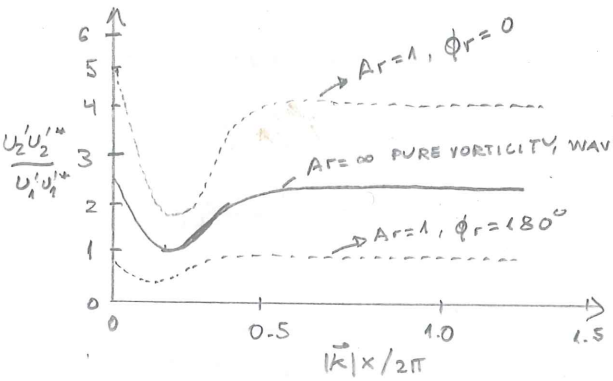
IN THE PROPAGATING REGIME ($\Psi < \Psi_c$), b IS REAL AND THE KINETIC ENERGY BECOMES

$$\frac{U_2' U_2'^*}{U_1^2} = \frac{U_2^2}{U_1^2} \left\{ |F|^2 + |G|^2 + \Re e\left(2FG^* e^{i\left(b - |\vec{k}| m \frac{U_1}{U_2}\right)x} \right) \right\} \qquad (220)$$

WHICH CONTAINS SPATIALLY UNIFORM CONTRIBUTIONS FROM THE ACOUSTIC AND VORTICITY MODES, ALONG WITH AN OSCILLATORY COMPONENT. IN GENERAL, THE KINETIC ENERGY (220) IS AMPLIFIED THROUGH THE SHOCK, AS SCHEMATICALLY SHOWN IN FIG. 26a AS A FUNCTION OF THE VORTICITY-TO-ENTROPY INCIDENT AMPLITUDE RATIO $A_r = |A_v|/|A_e|$ AND THE RELATIVE PHASE BETWEEN THEM ϕ_r, AND FOR A CRITICAL $\Psi_c = 61.4°$ CORRESPONDING TO AN UPSTREAM MACH NUMBER $Ma_1 = 1.5$. HOWEVER, STRONG OUT-OF-PHASE BETWEEN THE INCIDENT VORTICITY AND ENTROPY WAVES (OR EQUIVALENTLY, STRONG CORRELATION BETWEEN U_1' AND T_1') TENDS TO SUPPRESS THE AMPLIFICATION OF KINETIC ENERGY THROUGH THE SHOCK. NOTE THAT THE ENERGY ASSOCIATED TO THE PRESSURE FLUCTUATIONS IN THE POSTSHOCK GAS REMAINS UNIFORM IN THIS REGIME, $P_2' P_2'^* / \bar{P}_2^2 = |K|^2$



(a) KINETIC-ENERGY RATIO FOR $\Psi = 45°$ AND $Ma_1 = 1.5$.



(b) KINETIC-ENERGY RATIO FOR $\Psi = 75°$ AND $Ma_1 = 1.5$.

FIG. 26 : RESULTS FROM LINEAR INTERACTION ANALYSIS

IN THE DECAYING REGIME ($\Psi > \Psi_c$) b HAS BOTH REAL AND IMAGINARY PARTS, AND THE KINETIC ENERGY BECOMES

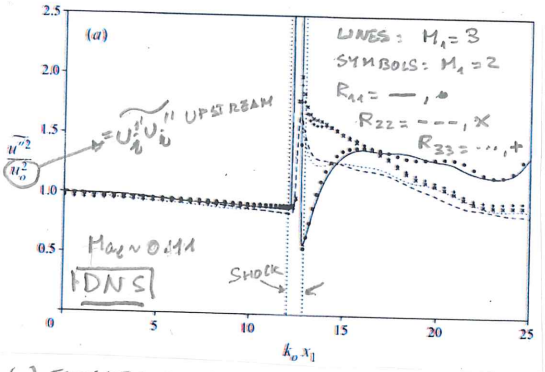
$$\frac{U_2' U_2'^*}{U_1^2} = \frac{U_2^2}{U_1^2} \left\{ |F|^2 e^{-2b_I^+ x} + |G|^2 + 2 e^{-2b_I^+ x} \Re e\left(FG^* e^{i\left(b_r - |\vec{k}| m \frac{U_1}{U_2}\right)x} \right) \right\} \qquad (221)$$

WHICH CONTAINS AN ACOUSTIC-DEPENDENT PORTION THAT DECAYS EXPONENTIALLY, ALONG WITH A FAR-FIELD VORTICAL COMPONENT THAT REMAINS SPATIALLY UNIFORM AT $x \to 0$. THE ESTABLISHMENT OF THE FAR-FIELD COMPONENT OCCURS AT DOWNSTREAM DISTANCES OF THE SAME ORDER AS THE INCIDENT WAVELENGTH $\lambda = 2\pi/|\vec{k}|$ AFTER A PRECIPITUOUS DROP OF THE ACOUSTIC MODE, AS SHOWN IN FIG. 26. NOTE THAT THE ENERGY ASSOCIATED WITH PRESSURE FLUCTUATIONS DECAYS ABRUPTLY,

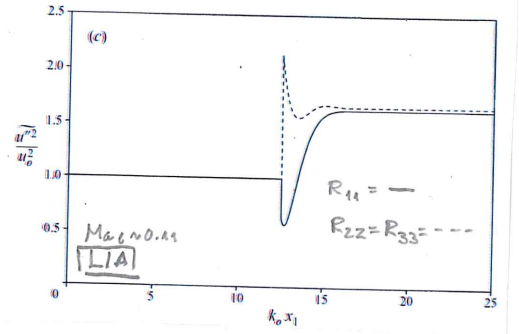
$$\frac{P_2' P_2'^*}{P_2^2} = |K|^2 e^{-2b_I^+ x} .$$

* COMPARISONS BETWEEN DNS AND LINEAR INTERACTION ANALYSIS (LIA)

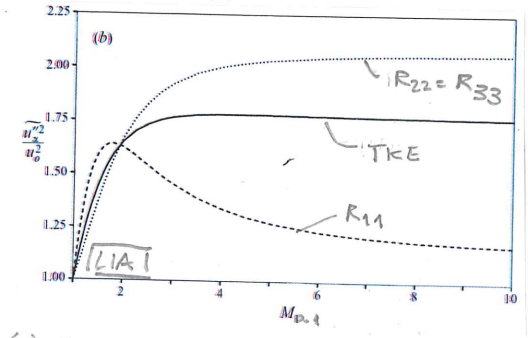
THE TRENDS PREDICTED BY LIA TEND TO BE CORROBORATED BY DNS. EXTENSIVE COMPARISONS ARE AVAILABLE IN THE LITERATURE (e.g., SEE LEE, LELE AND MOIN, JFM 1993-1997. FOR DETAILS). A COMPARISON BETWEEN THE STREAMWISE DISTRIBUTIONS OF REYNOLDS STRESSES $R_{ij} = \widetilde{u_i'' u_j''}$ IS SHOWN IN FIG. 26. ALL VELOCITY COMPONENTS TEND TO GET AMPLIFIED THROUGH THE SHOCK. THE AMPLIFICATION DEPENDS ON THE MACH NUMBER OF THE FLOW UPSTREAM THE SHOCK (FIG. 26c). SPECIFICALLY, THE AMPLIFICATION OF THE TRANSVERSE COMPONENT IS MONOTONIC WITH THE MACH NUMBER, WHILE THE AMPLIFICATION OF THE LONGITUDINAL COMPONENT IS MAXIMUM FOR $Ma_1 \simeq 2$. THE RAPID EVOLUTION OF THE KINETIC ENERGY BEHIND THE SHOCK CAN BE MOSTLY EXPLAINED AS A LINEAR PROCESS. SIMILARLY, THE TRANVERSE COMPONENTS OF THE VORTICITY ARE AMPLIFIED (FIG. 26d), WHILE THE STREAMWISE COMPONENTS REMAIN MOSTLY UNAFFECTED BECAUSE OF THE REASONS EXPLAINED EARLIER IN THE TEXT CONCERNING THE CONTINUITY OF THE VELOCITY COMPONENT TANGENTIAL TO THE FRONT. THE VORTICITY FLUCTUATIONS ARE ALSO AMPLIFIED ACROSS THE SHOCK IN A MONOTONIC MANNER WITH RESPECT TO Ma_1 (FIG. 26e). THE MAIN MECHANISMS RESPONSIBLE FOR THE AMPLIFICATION ARE VORTEX-COMPRESSION ($\sim \vec{\omega}\cdot\nabla\vec{u}$) AND VORTEX-STRETCHING ($\sim \vec{\omega}\cdot\nabla\vec{u}$), WITH BAROCLINIC TERMS BEING OF SECOND ORDER. FOR $Ma_0^2 \gtrsim 0.1\,(Ma_1^2 - 1)$, THE SHOCK WAVE NO LONGER HAS A WELL-DEFINED FRONT.



(a) EVOLUTION OF NORMAL COMPONENTS OF THE REYNOLDS STRESS $R_{ii} = \widetilde{u_i'' u_i''}$, (WHERE ~ IS A DENSITY-WEIGHTED PLANAR AVERAGE)

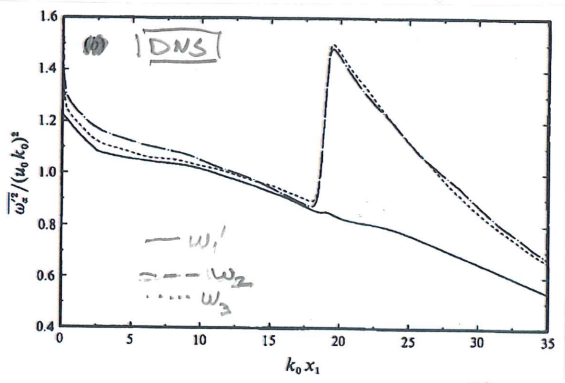


(b) AMPLIFICATION OF R_{ii} AS PREDICTED BY LIA AT $Ma_1 = 2$.



(c) FAR-FIELD ($\vec{x} \to \infty$) AMPLIFICATION OF R_{ii} AND TURBULENT KINETIC ENERGY (TKE = $R_{kk}/2$) AS A FUNCTION OF Ma_1.

d) EVOLUTION OF THE VORTICITY VARIANCE FOR $Ma_1 = 1.2$



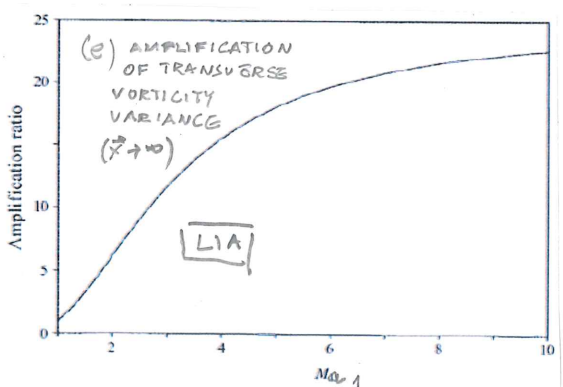


FIG. 26. SHOCK-TURBULENCE NUMERICAL SIMULATIONS (FROM LEE et al., JFM 1993, 1997)

VI ENERGETICS AND SUBGRID-SCALE MODELING FOR COMPRESSIBLE TURBULENCE

THIS CHAPTER DESCRIBES THE AVERAGED CONSERVATION EQUATIONS FOR COMPRESSIBLE TURBULENCE, INCLUDING FULLY-NONLINEAR EFFECTS THAT CAN NO LONGER BE ADDRESSED BY MODAL DECOMPOSITIONS. AS A RESULT THE DENSITY VARIATIONS CANNOT BE SPLIT IN DIFFERENT CONTRIBUTIONS ALTHOUGH THEY ARE JOINTLY ACCOUNTED FOR IN THE FORMULATION THROUGH A SPECIAL TYPE OF AVERAGING. THE AVERAGED CONSERVATION EQUATIONS NECESSARILY CONTAIN UNCLOSED TERMS THAT REQUIRE MODELLING.

FAVRE AVERAGING.

THE REYNOLDS AVERAGING $\overline{\varphi} = \int_t^{t+T} \varphi(\vec{x},t)\,dt$ TYPICALLY USED IN INCOMPRESSIBLE FLOWS DOES NOT LEAD TO USEFUL EQUATIONS WHEN DENSITY VARIATIONS ARE IN EFFECT. FOR INSTANCE, CONSIDER THE CONTINUITY EQUATION (11) AND APPLY THE REYNOLDS AVERAGING, WHICH LEADS TO THE REYNOLDS-AVERAGED EQUATION

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\overline{\rho u_i}\right) = 0 \qquad (222)$$

DECOMPOSING THE DENSITY AND VELOCITY AS $\rho = \bar{\rho} + \rho'$ AND $u_i = \bar{u}_i + u_i'$, WHERE PRIMES INDICATE REYNOLDS FLUCTUATIONS, AND NOTICING THAT $\overline{\bar{\varphi}} = \bar{\varphi}$ AND $\overline{\varphi'} = 0$, THEN THE ABOVE EQUATION BECOMES

$$\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\rho}\bar{u}_i\right) + \frac{\partial}{\partial x_i}\left(\overline{\rho' u_i'}\right) = 0 \qquad (224)$$

WHERE THE LAST TERM IS A TURBULENT FLUX THAT IS UNCLOSED. IN ORDER TO AVOID UNCLOSED TERMS APPEARING AS EARLY AS IN THE CONTINUITY EQUATION, THE VARIABLES ARE FAVRE-AVERAGED INSTEAD,

$$\tilde{\varphi} = \frac{1}{\bar{\rho}} \int_t^{t+T} \rho(\vec{x},t)\,\varphi(\vec{x},t)\,dt \qquad (225)$$

NOTE THAT (223) IS A DENSITY-WEIGHTED AVERAGE, WITH $\varphi'' = \varphi - \tilde{\varphi}$ THE CORRESPONDING FLUCTUATIONS. SINCE $\tilde{\varphi} = \overline{\rho\varphi}/\bar{\rho}$, EQUATION (222) BECOMES

$$\boxed{\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\right) = 0} \qquad (226)$$

WHICH DOES NOT REQUIRE DIRECT CLOSURE. THE FLUCTUATIONS FROM THE FAVRE AVERAGE ARE NOT TIME OR SPACE-CENTERED, IN THAT $\overline{\varphi''} \neq 0$. ON THE OTHER HAND,

$$\overline{\rho\varphi} = \overline{\rho(\tilde{\varphi}+\varphi'')} = \bar{\rho}\tilde{\varphi} + \overline{\rho\varphi''} \Rightarrow \boxed{\overline{\rho\varphi''} = 0} \qquad (227)$$

SO THAT THE DENSITY-WEIGHTED AVERAGE OF THE FAVRE FLUCTUATION IS WHAT INSTEAD REMAINS TIME- OR SPACE-CENTERED. NOTE THAT IN FLOWS WITH HOMOGENEOUS DIRECTIONS (225) IS TYPICALLY SUBSTITUTED BY SPACE AVERAGES ALONG THOSE DIRECTIONS. IN PRINCIPLE, THE DIFFERENCE BETWEEN REYNOLDS- AND FAVRE AVERAGES IS AN UNCLOSED QUANTITY. THIS CAN BE SHOWN BY NOTICING THAT $\overline{\varphi} = \overline{\widetilde{\varphi} + \varphi''} = \widetilde{\varphi} + \overline{\varphi''} \Rightarrow \overline{\varphi} = \widetilde{\varphi} + \overline{\varphi''}$, AND THAT $\overline{\rho\varphi} = \bar{\rho}\overline{\varphi} + \overline{\rho'\varphi'}$. SO THAT $\overline{\varphi} - \widetilde{\varphi} = \overline{\varphi''} = -\overline{\rho'\varphi'}/\bar{\rho}$. AS A RESULT, IN SYSTEMS WHERE ρ AND φ ARE HIGHLY CORRELATED, THE DIFFERENCE BETWEEN $\overline{\varphi}$ AND $\widetilde{\varphi}$ IS SIGNIFICANT.

AVERAGED CONSERVATION EQUATIONS

THE DECOMPOSITION TYPICALLY USED TO EXPRESS THE AVERAGED CONSERVATION EQUATIONS IN COMPRESSIBLE FLOWS IS

$$\left.\begin{aligned} u_i &= \widetilde{u}_i + u_i'' \\ h &= \widetilde{h} + h'' \\ e &= \widetilde{e} + e'' \\ T &= \widetilde{T} + T'' \end{aligned}\right\} \text{VARIABLES DECOMPOSED INTO FAVRE AVERAGE + FAVRE FLUCTUATIONS} \qquad \left.\begin{aligned} \rho &= \bar{\rho} + \rho' \\ p &= \bar{p} + p' \end{aligned}\right\} \text{VARIABLES DECOMPOSED INTO REYNOLDS AVERAGE AND REYNOLDS FLUCTUATIONS} \qquad (228)$$

THAT (228) IS CONVENIENT CAN BE UNDERSTOOD BY REYNOLDS-AVERAGING THE MOMENTUM EQUATION (12)

$$\frac{\partial}{\partial t}(\overline{\rho u_i}) + \frac{\partial}{\partial x_j}(\overline{\rho u_i u_j}) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j}$$

(τ_{ij}: VISCOUS STRESS TENSOR $= 2\mu S_{ij}$, $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}$)

AND USING THE DEFINITION OF THE FAVRE AVERAGE

$$\frac{\partial}{\partial t}(\bar{\rho}\widetilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\widetilde{u}_i\widetilde{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j} \qquad (229)$$

NOTE THAT $\bar{\rho}\widetilde{u_i u_j} = \bar{\rho}\widetilde{(\widetilde{u}_i + u_i'')(\widetilde{u}_j + u_j'')} = \bar{\rho}\widetilde{u}_i\widetilde{u}_j + \widetilde{u}_i\cancel{\bar{\rho}\widetilde{u_j''}}^{\,0} + \widetilde{u}_j\cancel{\bar{\rho}\widetilde{u_i''}}^{\,0} + \bar{\rho}\widetilde{u_i''u_j''}$.

THEREFORE, (229) BECOMES

$$\boxed{\frac{\partial}{\partial t}(\bar{\rho}\widetilde{u}_i) + \frac{\partial}{\partial x_j}(\bar{\rho}\widetilde{u}_i\widetilde{u}_j) = -\frac{\partial \bar{p}}{\partial x_i} + \frac{\partial \overline{\tau_{ij}}}{\partial x_j} - \frac{\partial}{\partial x_j}(\bar{\rho}\widetilde{u_i''u_j''})} \qquad (230)$$

MEAN MOMENTUM EQ.

WHERE THE LAST TERM IS THE DIVERGENCE OF THE REYNOLDS STRESS TENSOR

$$R_{ij} = \bar{\rho}\widetilde{u_i''u_j''}, \qquad (231)$$

WHICH IS AN UNCLOSED QUANTITY THAT REQUIRES MODELLING AND PLAYS A FUNDAMENTAL ROLE IN TURBULENT FLOWS. NOTE THAT THE DIVERGENCE OF THE REYNOLDS STRESSES IN (230) IS MUCH MORE IMPORTANT THAN THE DIVERGENCE

OF THE MEAN VISCOUS STRESSES WHEN THE REYNOLDS NUMBER IS LARGE. TO SEE THIS, CONSIDER THE ORDERS OF MAGNITUDE

$$\frac{\partial \bar{\tau}_{ij}}{\partial x_j} \sim \mu \frac{U_\ell}{\ell^2} \Rightarrow \text{MEAN STRESSES (ACTING ON INTEGRAL SCALES)}$$

$$\frac{\partial R_{ij}}{\partial x_j} \sim \rho U_\ell^2 \Rightarrow \text{REYNOLDS STRESSES}$$

$$\left\{ \quad \frac{\mu \frac{U_\ell}{\ell^2}}{\rho U_\ell^2} \simeq \frac{\mu}{\rho U_\ell \ell} = \frac{1}{Re_\ell} \ll 1 \right.$$

AS A RESULT, THE MEAN MOLECULAR TRANSPORT OF MOMENTUM TENDS TO BE NEGLIGIBLE COMPARED TO THE TURBULENT TRANSPORT ENABLED BY THE REYNOLDS STRESSES.

AN EQUATION FOR THE MEAN KINETIC ENERGY CAN BE EASILY DERIVED BY MULTIPLYING (230) BY $\tilde{U}_i$, WHICH GIVES

$$\bar{\rho}\tilde{U}_i \frac{\partial \tilde{U}_i}{\partial t} + \bar{\rho}\tilde{U}_i \tilde{U}_j \frac{\partial \tilde{U}_i}{\partial x_j} = -\tilde{U}_i \frac{\partial \bar{P}}{\partial x_i} + \tilde{U}_i \frac{\partial \bar{\tau}_{ij}}{\partial x_j} - \tilde{U}_i \frac{\partial}{\partial x_j}\left(\bar{\rho}\widetilde{U_i'' U_j''}\right)$$

WHICH CAN BE REARRANGED TO GIVE

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{K}\right) + \frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{U}_i \tilde{K}\right) = \underbrace{-\frac{\partial}{\partial x_j}\left(\bar{P}\tilde{U}_j - \bar{\tau}_{ij}\tilde{U}_i + \bar{\rho}\widetilde{U_i'' U_j''}\tilde{U}_i\right)}_{\text{"FLUX" OR "REDISTRIBUTION" TERMS}} + \underbrace{\bar{P}\frac{\partial \tilde{U}_i}{\partial x_i}}_{\text{LARGE-SCALE PRESSURE-DILATATION WORK}} - \underbrace{\bar{\tau}_{ij}\frac{\partial \tilde{U}_i}{\partial x_j}}_{\text{LARGE-SCALE DISSIPATION (NEGLIGIBLE)}} + \underbrace{\bar{\rho}\widetilde{U_i'' U_j''}\frac{\partial \tilde{U}_i}{\partial x_j}}_{\text{TURBULENT DESTRUCTION OF ENERGY (-TURBULENT PRODUCTION)}} \quad (231)$$

EQUATION FOR THE LARGE-SCALE (MEAN) KINETIC ENERGY

WHERE $\tilde{K} = \frac{1}{2}\tilde{U}_i \tilde{U}_i$ IS THE MEAN OR LARGE-SCALE KINETIC ENERGY. IN (231) THE REDISTRIBUTION TERMS ARE NAMED LIKE THAT BECAUSE THEY VANISH WHEN INTEGRATED ALONG PERIODIC DIRECTIONS, MEANING THAT THEIR NET EFFECT IS NOT TO PRODUCE OR DESTROY ENERGY BUT TO REDISTRIBUTE IT. THE MOST IMPORTANT TERMS ON THE RIGHT-HAND SIDE OF (231) ARE THE MEAN PRESSURE DILATATION WORK, WHICH IS BASED ON THE LARGE-SCALE DILATATION $\partial \tilde{U}_i / \partial x_i$, AND THE TURBULENT DESTRUCTION, WHICH IS A PSEUDO DISSIPATION THAT CAN TRANSFER ENERGY FROM OR TO THE FLUCTUATIONS.

THE EQUATION FOR THE MEAN INTERNAL KINETIC ENERGY IS OBTAINED BY REYNOLDS-AVERAGING EQ. (15),

$$\frac{\partial}{\partial t}\left(\overline{\rho e}\right) + \frac{\partial}{\partial x_i}\left(\overline{\rho U_i e}\right) = -\overline{P\frac{\partial U_i}{\partial x_i}} + \overline{\tau_{ij}\frac{\partial U_i}{\partial x_j}} - \frac{\partial \bar{q}_i}{\partial x_i}$$

MOLECULAR HEAT FLUX $= -\kappa \frac{\partial T}{\partial x_i}$

EXPANDING THE CONVECTING TERM, $\overline{\rho U_i e} = \bar{\rho}\tilde{U}_i\tilde{e} - \bar{\rho}\widetilde{U_i'' e''}$, THE ABOVE EQUATION CAN

BE REWRITTEN AS

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{e}\right)+\frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{e}\tilde{u}_i\right)=-\overline{p\frac{\partial u_i}{\partial x_i}}+\overline{\tau_{ij}\frac{\partial u_i}{\partial x_j}}-\frac{\partial \bar{q}_i}{\partial x_i}-\frac{\partial}{\partial x_i}\left(\bar{\rho}\widetilde{e''u_i''}\right) \qquad (232)$$

WHERE THE LAST TERM IS A TURBULENT TRANSPORT OF THERMAL ENERGY. IN ORDER TO DERIVE CONCLUSIONS ABOUT THE ENERGETICS OF COMPRESSIBLE TURBULENCE, IT IS CONVENIENT TO EXPAND THE TERMS IN (232) INVOLVING PRESSURE AND VISCOUS STRESSES BY MAKING $p=\bar{p}+p'$, $\tau_{ij}=\bar{\tau}_{ij}+\tau_{ij}'$ AND $u_i=\tilde{u}_i+u_i''$, WHICH GIVES

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{e}\right)+\frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{e}\tilde{u}_i\right)=\underbrace{-\bar{p}\frac{\partial\tilde{u}_i}{\partial x_i}}_{\text{LARGE-SCALE}}\underbrace{-\left(\overline{p\frac{\partial u_i''}{\partial x_i}}+\overline{p'\frac{\partial u_i''}{\partial x_i}}\right)}_{\text{TURBULENT}}\Bigg\}\text{PRESSURE-DILATATION WORK}+\underbrace{\bar{\tau}_{ij}\frac{\partial\tilde{u}_i}{\partial x_j}}_{\text{LARGE-SCALE}}+\underbrace{\left(\overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}}+\overline{\tau_{ij}'\frac{\partial u_i''}{\partial x_j}}\right)}_{\text{TURBULENT}}\Bigg\}\text{DISSIPATION}$$

$$\underbrace{-\frac{\partial\bar{q}_i}{\partial x_i}}_{\substack{\text{LARGE-SCALE}\\ \text{MOLECULAR}\\ \text{HEAT CONDUCTION}}}\underbrace{-\frac{\partial}{\partial x_i}\left(\bar{\rho}\widetilde{e''u_i''}\right)}_{\substack{\text{TURBULENT FLUX OF}\\ \text{THERMAL ENERGY}}} \qquad (233)$$

LEQUATION FOR THE LARGE-SCALE (MEAN) THERMAL ENERGY

NOTE THAT $\bar{p}$ AND $\bar{\rho}$ ARE RELATED THROUGH THE EQUATION OF STATE WITH TEMPERATURE AS

$$\bar{p}=\overline{\rho R_g T}=\overset{R_g=\text{const}}{\downarrow}\bar{\rho}R_g\tilde{T}\ ,\quad\text{WITH}\quad\tilde{T}=\overset{c_v\ \text{const}}{\tilde{e}/c_v}\ . \qquad (234)$$

AN ILLUSTRATIVE DESCRIPTION OF THE ENERGETICS CAN BE OBTAINED BASED ON (231) AND (234) AN EQUATION FOR THE TURBULENT KINETIC ENERGY, WHICH CAN BE DERIVED AS FOLLOWS. CONSIDER THE AVERAGE OF THE KINETIC ENERGY EQUATION,

$$\frac{\partial}{\partial t}\left(\overline{\rho k}\right)+\frac{\partial}{\partial x_i}\left(\overline{\rho u_i k}\right)=-\overline{u_i\frac{\partial p}{\partial x_i}}+\overline{u_i\frac{\partial\tau_{ij}}{\partial x_j}}\ ,\quad\text{WITH}\quad k=u_k u_k/2\ ,$$

WHICH CAN BE REWRITTEN AS

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{k}\right)+\frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\tilde{k}\right)=-\tilde{u}_i\frac{\partial\bar{p}}{\partial x_i}-\left(\overline{u_i''\frac{\partial\bar{p}}{\partial x_i}}+\overline{u_i''\frac{\partial p'}{\partial x_i}}\right)+\tilde{u}_i\frac{\partial\bar{\tau}_{ij}}{\partial x_j}+\left(\overline{u_i''\frac{\partial\bar{\tau}_{ij}}{\partial x_j}}+\overline{u_i''\frac{\partial\tau_{ij}'}{\partial x_j}}\right)$$
$$+\frac{\partial}{\partial x_i}\left(\bar{\rho}\widetilde{u_i k}-\bar{\rho}\tilde{u}_i\tilde{k}\right)\ ,$$

WHERE THE LAST TERM CAN BE EXPANDED AS

$$\bar{\rho}\widetilde{u_i k}-\bar{\rho}\tilde{u}_i\tilde{k}=\bar{\rho}\left(\widetilde{\tilde{u}_i+u_i''}\right)\left(\widetilde{\tilde{u}_k+u_k''}\right)\left(\widetilde{\tilde{u}_k+u_k''}\right)-\bar{\rho}\left(\widetilde{\tilde{u}_i+u_i''}\right)\left(\widetilde{\tilde{u}_k+u_k''}\right)\left(\widetilde{\tilde{u}_k+u_k''}\right)=$$
$$=\bar{\rho}\tilde{u}_i\frac{\tilde{u}_k\tilde{u}_k}{2}+\bar{\rho}\widetilde{u_i''}\frac{\tilde{u}_k\tilde{u}_k}{2}+\tilde{u}_i\bar{\rho}\widetilde{u_k''}\tilde{u}_k+\frac{\tilde{u}_i\bar{\rho}\widetilde{u_k''u_k''}}{2}+\tilde{u}_i\widetilde{u_k}\bar{\rho}\widetilde{u_k''}+\bar{\rho}\tilde{u}_k\widetilde{u_i''u_k''}+\frac{\bar{\rho}\tilde{u}_k\widetilde{u_k''u_i''}}{2}$$
$$+\bar{\rho}\widetilde{u_i''u_k''u_k''}/2-\bar{\rho}\tilde{u}_i\frac{\tilde{u}_k\tilde{u}_k}{2}-\bar{\rho}\tilde{u}_k\widetilde{u_i''u_k''}=\bar{\rho}\tilde{u}_k\widetilde{u_i''u_k''}+\bar{\rho}\widetilde{u_i''}\frac{\widetilde{u_k''u_k''}}{2}.$$

AS A RESULT, THE AVERAGED KINETIC ENERGY EQUATION BECOMES

$$\frac{\partial}{\partial t}\left(\bar{\rho}\tilde{k}\right)+\frac{\partial}{\partial x_i}\left(\bar{\rho}\tilde{u}_i\tilde{k}\right)=+\bar{P}\frac{\partial\tilde{u}_i}{\partial x_i}+\left(\bar{P}\frac{\partial\overline{u_i''}}{\partial x_i}+\overline{P'\frac{\partial u_i''}{\partial x_i}}\right)-\bar{\tau}_{ij}\frac{\partial\tilde{u}_i}{\partial x_j}+\left(\bar{\tau}_{ij}\frac{\partial\overline{u_i''}}{\partial x_j}+\overline{\tau_{ij}'\frac{\partial u_i''}{\partial x_j}}\right)$$

AVERAGED KINETIC ENERGY EQUATION.

(235)

$$-\frac{\partial}{\partial x_i}\left(\bar{P}\tilde{u}_i+\bar{P}u_i''+\overline{P'u_i''}-\bar{\tau}_{ij}\tilde{u}_j-\bar{\tau}_{ij}\overline{u_j''}-\overline{\tau_{ij}'u_j''}+\bar{\rho}\tilde{u}_k\widetilde{u_i''u_k''}+\bar{\rho}\widetilde{u_i''\frac{u_k''u_k''}{2}}\right)$$

BEFORE PROCEEDING FURTHER, AN IMPORTANT CONCEPT EMERGING FROM THIS FORMULATION IS THAT THE DIFFERENCE BETWEEN THE AVERAGE OF THE MULTIPLICATION OF TWO VARIABLES AND THE MULTIPLICATION OF THEIR AVERAGES IS ALWAYS PROPORTIONAL TO THEIR CORRELATION COEFFICIENT,

$$\bar{\rho}\left(\widetilde{\phi\psi}-\tilde{\phi}\tilde{\psi}\right)=\bar{\rho}\left(\widetilde{(\tilde{\phi}+\phi'')(\tilde{\psi}+\psi'')}\right)-\bar{\rho}\left(\widetilde{(\tilde{\phi}+\phi'')(\tilde{\psi}+\psi'')}\right)=$$

$$=\bar{\rho}\tilde{\phi}\tilde{\psi}+\bar{\rho}\widetilde{\phi''}\tilde{\psi}+\bar{\rho}\widetilde{\psi''}\tilde{\phi}-\bar{\rho}\tilde{\phi}\tilde{\psi}+\bar{\rho}\widetilde{\phi''\psi''}-\bar{\rho}\tilde{\phi}\widetilde{\psi''}$$

$$-\bar{\rho}\widetilde{\phi''}\tilde{\psi}-\bar{\rho}\widetilde{\phi''}\widetilde{\psi''}=\bar{\rho}\widetilde{\phi''\psi''}\qquad(236)$$

IN PARTICULAR, IF $\phi=\psi$, THEN

$$\bar{\rho}\left(\widetilde{\phi^2}-\tilde{\phi}^2\right)=\bar{\rho}\widetilde{\phi''^2}\Rightarrow\boxed{\widetilde{\phi^2}-\tilde{\phi}^2=\widetilde{\phi''^2}\equiv\text{VARIANCE OF }\phi}\qquad(236)$$

(NOTE THAT IN LES $\widetilde{\phi^2}-\tilde{\phi}^2$ IS NOT EQUAL TO THE VARIANCE OF ϕ SINCE THE FAVRE OPERATOR CONTAINS A FILTER KERNEL THAT DOES NOT SATISFY THE CONDITIONS NECESSARY FOR ARRIVING AT (236); NONETHELESS, IT IS USUAL TO REFER TO $\widetilde{\phi^2}-\tilde{\phi}^2$ TO THE "SUBGRID VARIANCE" OF ϕ IN THAT CONTEXT). WHEN (236) IS PARTICULARIZED FOR THE VELOCITY, ONE OBTAINS

$$\underbrace{\frac{\widetilde{u_k u_k}}{2}}_{\substack{\text{FAVRE AVERAGE}\\ \text{OF THE KINETIC}\\ \text{ENERGY } \tilde{k}}}-\underbrace{\frac{\tilde{u}_k\tilde{u}_k}{2}}_{\substack{\text{LARGE-SCALE}\\ \text{KINETIC ENERGY}\\ \mathbb{K}}}=\underbrace{\widetilde{\frac{u_k''u_k''}{2}}}_{\substack{\text{TURBULENT}\\ \text{KINETIC ENERGY}\\ k}}\qquad(237)$$

BASED ON (237), AN EQUATION FOR THE TURBULENT KINETIC ENERGY CAN BE OBTAINED BY SUBSTRACTING (233) FROM (235), WHICH YIELDS

$$\frac{\partial}{\partial t}(\bar{\rho}K) + \frac{\partial}{\partial x_i}(\bar{\rho}\tilde{u}_i K) = \underbrace{\left(\bar{p}\frac{\partial \widetilde{u_i''}}{\partial x_i} + \overline{p'\frac{\partial u_i''}{\partial x_i}}\right)}_{\text{TURBULENT PRESSURE-DILATATION}} - \underbrace{\left(\overline{\tau_{ij}\frac{\partial u_i''}{\partial x_j}} + \overline{\tau_{ij}'\frac{\partial u_i''}{\partial x_j}}\right)}_{\text{TURBULENT DISSIPATION}}$$

$$- \underbrace{\frac{\partial}{\partial x_i}\left(\bar{p}u_i'' + \overline{p'u_i''} - \overline{\tau_{ij}u_j''} - \overline{\tau_{ij}'u_j''} + \bar{\rho}\widetilde{u_i''\frac{u_k''u_k''}{2}}\right)}_{\text{REDISTRIBUTION TERMS}} \quad \leftarrow \text{TRIPLE CORRELATION}$$

$$- \underbrace{\bar{\rho}\widetilde{u_i''u_j''}\frac{\partial \tilde{u}_i}{\partial x_j}}_{\text{TURBULENT PRODUCTION OF ENERGY}} \qquad (238)$$

EQUATION FOR THE TURBULENT KINETIC ENERGY

$$K = \widetilde{\frac{u_k''u_k''}{2}}$$

IN VIEW OF (231), (233) AND (238), AN ENERGY DIAGRAM (FIG. 27) CAN BE ENVISIONED THAT DESCRIBES THE TRANSFER OF ENERGY BETWEEN LARGE SCALES AND SMALL SCALES.

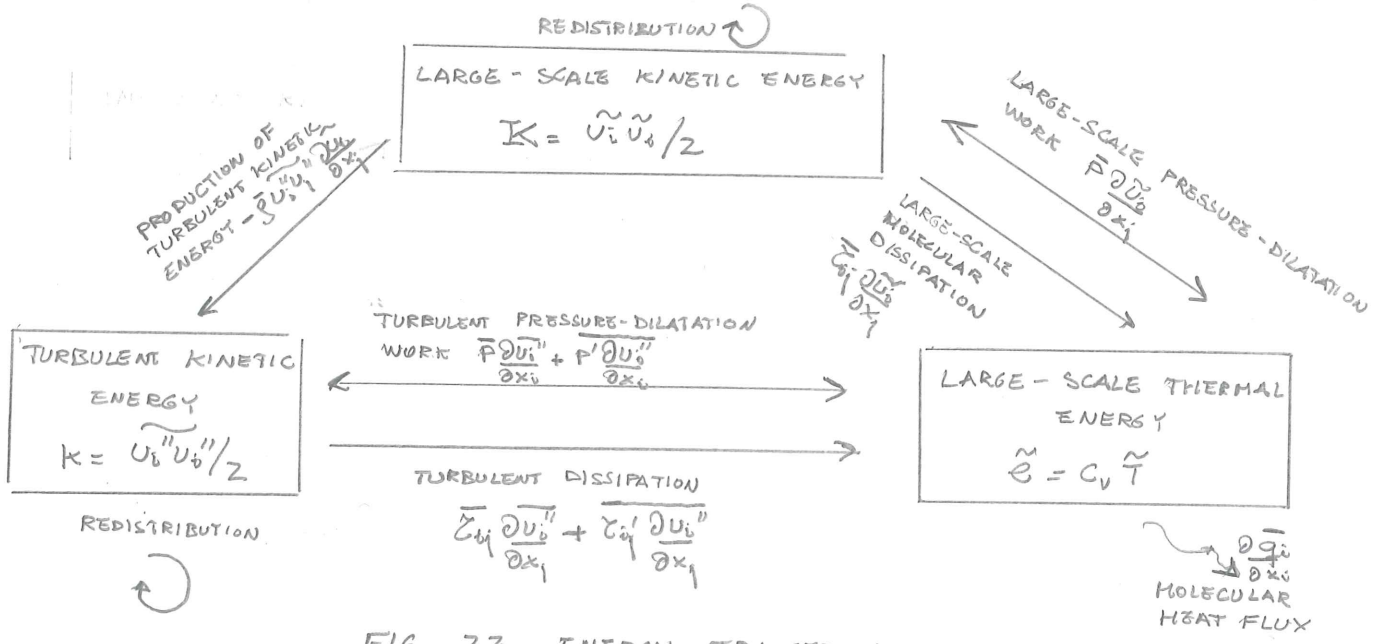


FIG. 27. ENERGY-TRANSFER DIAGRAM

NOTE THAT THE TURBULENT PRODUCTION IS THE MAIN MECHANISM FOR TRANSFORMATION OF LARGE-SCALE ENERGY INTO TURBULENT KINETIC ENERGY. IN TRIPLY PERIODIC FLOWS WITH ZERO MEAN (AS IN ISOTROPIC TURBULENCE), THE REDISTRIBUTION TERMS VANISH UPON SPACE INTEGRATION AS PRESCRIBED BY THE GAUSS THEOREM. IT IS ALSO ILLUSTRATIVE TO OBSERVE THE CHANNELS FOR CONVERSION OF KINETIC ENERGY INTO THERMAL ENERGY, WHICH CAN OCCUR THROUGH BOTH LARGE SCALES (RIGHT-HAND SIDE OF FIG. 27) AND SMALL SCALES (LEFT-HAND SIDE OF FIG. 27), AND WHICH INVOLVE PRESSURE DILATATION WORK AND FLOW DISSIPATION. AT LOW MACH NUMBERS, THE DILATATION IS NEGLIGIBLE AND THE DISSIPATION PLAYS NO ROLE ON AUGMENTING THE THERMAL ENERGY, WHICH IS TOO LARGE COMPARED TO THE KINETIC ENERGY, AND AS A RESULT THE MOMENTUM AND THERMAL FIELDS BECOME DECOUPLED.

ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Homework 1: Fundamental Aspects of Compressible Flows and Shock Waves

Due Thursday, April 27, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student's Name: JAVIER URZAY **Student's ID:**.....

1pt 1. Air flows isentropically through a constant-area duct at $\mathrm{Ma} = 0.7$. The stagnation pressure at point 1 is $P_0 = 2$ bar. What is the stagnation pressure at a downstream point 2?

1pt 2. In addition to the conditions described in the previous question, it is also known that the stagnation density at point 1 is $\rho_0 = 2.0$ kg/m^3. What is the static enthalpy at the downstream point 2?

1pt 3. Starting from Crocco's equation, derive Bernoulli's equation $h + |\mathbf{u}|^2/2 = C$, where h is the enthalpy, $\mathbf{u}$ is the velocity vector, and C is a constant that depends on the streamline chosen for integration.

1pt 4. Show that the entropy variation across two streamlines 1 and 2, i.e., $s_2 - s_1 = c_v \ln[(P_2/\rho_2^\gamma)/(P_1/\rho_1^\gamma)]$, can be written solely in terms of the ratio of stagnation pressures P_{02}/P_{01} if the stagnation enthalpy is uniform everywhere.

2pt 5. A supersonic air flow at $\mathrm{Ma}_1 = 4$ and $P_1 = 2$ bar encounters a normal shock wave. Compute the Mach number in the post-shock flow Ma_2 as well as the relative variation in the stagnation pressure $(P_{01} - P_{02})/P_{01}$ and specific entropy $(s_1 - s_2)/c_v$.

2pt 6. A supersonic inviscid mixture of H_2 and air flows parallel to a wall at $\mathrm{Ma}_1 = 5.0$, $T_1 = 300$ K and $P_1 = 1$ bar, and encounters a compression ramp that deflects the stream upwards at an angle δ creating an oblique shock wave that emanates from the ramp corner. Compute the angle δ required to increase the temperature of the gas to the crossover value $T_2 = 950$ K required for autoignition. What is the associated post-shock Mach number Ma_2? Assume that the properties of the mixture are similar to those of air.

2pt 7. Sketch i) the inviscid supersonic flow at $\mathrm{Ma}_1 = 4.5$ around a symmetric wedge of semi-angle $\delta = 15°$ at pressure $P_1 = 1$ bar, ii) the same flow when the semi-angle is increased to $\delta = 70°$. Comment on whether the flow downstream of the shock in both cases is rotational or irrotational and give appropriate analytical justifications to support your explanations.

SOLUTIONS:

① : ISENTROPIC FLOW $\Rightarrow$ $P_{02} = 2\ \mathrm{bar} = P_{01}$

② : $h_{01} = h_{02} = \dfrac{P_{01}}{\rho_{01}}\dfrac{\gamma}{\gamma-1} = h_2 + \dfrac{U_2^2}{2} \Rightarrow h_2 = \dfrac{h_{01}}{1+\frac{(\gamma-1)}{2}\mathrm{Ma}_1^2} = \dfrac{\frac{P_{01}}{\rho_{01}}\frac{\gamma}{\gamma-1}}{1+\frac{(\gamma-1)}{2}\mathrm{Ma}_1^2} = 318\ \mathrm{kJ/kg}$

↳ CONSTANT AREA, ISENTROPIC

③ $\vec{\omega} \wedge \vec{v} = T\nabla s - \nabla\left(h + \frac{|\vec{v}|^2}{2}\right)$ CROCCO'S EQUATION

MULTIPLYING THIS EQUATION BY A UNIT VECTOR $\vec{e}_\ell$ TANGENT TO THE STREAMLINES:

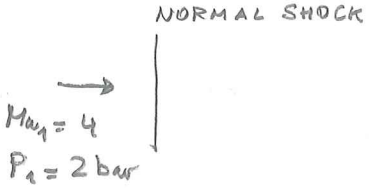
$$(\vec{\omega} \wedge \vec{v}) \cdot \vec{e}_\ell = T \vec{e}_\ell \cdot \nabla s - \vec{e}_\ell \cdot \nabla\left(h + \frac{|\vec{v}|^2}{2}\right) \Rightarrow \frac{\partial}{\partial \ell}\left(h + \frac{|\vec{v}|^2}{2}\right) = 0$$

(0 SINCE $\vec{\omega} \wedge \vec{v} \perp \vec{v}$ AND $\perp \vec{e}_\ell$; ISENTROPIC $(\vec{v} \cdot \nabla s = 0)$)

AND THEREFORE $h + \frac{|\vec{v}|^2}{2} = C$

④ SEE PAGE 18 OF CLASS NOTES.

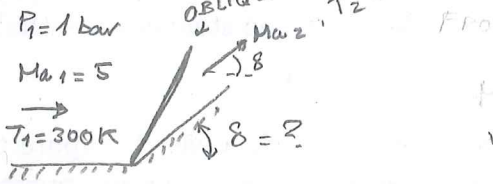
⑤



- POST-SHOCK MACH: $Ma_2 = \left(\frac{2 + (\gamma-1) Ma_1^2}{2\gamma Ma_1^2 + 1 - \gamma}\right)^{1/2} = \underline{\underline{0.43}}$
- STAGNATION PRESSURE: $\frac{P_{02}}{P_{01}} = \left(\frac{1 + \frac{\gamma-1}{2} Ma_2^2}{1 + \frac{\gamma-1}{2} Ma_1^2}\right)^{\frac{\gamma}{\gamma-1}} \left[1 + \frac{2\gamma}{\gamma+1}(Ma_1^2 - 1)\right] = 0.138$
- ENTROPY INCREASES THROUGH THE SHOCK $\Rightarrow \frac{P_{02} - P_{01}}{P_{01}} = \underline{\underline{0.862}}$

$$\frac{s_2 - s_1}{c_v} = \ln\left(\frac{P_2/\rho_2^\gamma}{P_1/\rho_1^\gamma}\right) = \ln\left(\frac{P_2}{P_1}\left(\frac{\rho_1}{\rho_2}\right)^\gamma\right) \overset{\text{JUMP CONDITIONS (64)-(66) IN CLASS NOTES}}{=} \ln\left(\frac{18.5}{4.57^{1.4}}\right) = \underline{\underline{0.79}}$$

⑥



TEMPERATURE JUMP: $\frac{T_2}{T_1} = \frac{1000}{300} = 3.33$,

WHICH CORRESPONDS TO A NORMAL MACH NUMBER THAT HAS TO BE OBTAINED FROM THE JUMP CONDITION

$$\frac{T_2}{T_1} = \frac{[2\gamma Ma_{1n}^2 - (\gamma-1)][2 + (\gamma-1) Ma_{1n}^2]}{(\gamma+1)^2 Ma_{1n}^2} = 3.33$$

$\Rightarrow Ma_{1n} \sim 3.5$, THEREFORE $\beta = \arcsin\left(\frac{Ma_{1n}}{Ma_1}\right) = 44.4^\circ$

ENTERING IN THE β-δ DIAGRAM WITH $\beta = 44.4^\circ$ AND $Ma_1 = 5$ ONE OBTAINS $\delta \sim \underline{\underline{33^\circ}}$ (WEAK SOLUTION)

⑦ NOTE THAT δ_{MAX} AT $Ma_1 = 4.5$ IS $\underline{\underline{40^\circ}}$ (FROM THE δ-β CHART). FOR $\delta < \delta_{MAX}$ THE SHOCK IS ATTACHED. FOR $\delta > \delta_{MAX}$ THE SHOCK IS DETACHED AND FORMS A BOW SHOCK. SKETCHES OF THE CORRESPONDING FLOWS ARE PROVIDED IN PAGE 26 OF THE CLASS NOTES. THE POST-SHOCK GAS IS IRROTATIONAL WHEN $\delta < \delta_{MAX}$, AND ROTATIONAL WHEN $\delta > \delta_{MAX}$, THE REASONS BEING EXPLAINED IN PAGES 18-19 OF THE CLASS NOTES.

ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Homework 2: Fluctuation Modes in Compressible Turbulence

Due Thursday, May 18, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student's Name: JAVIER URZAY ..... **Student's ID:** .....

In uniform mean flows with density $\overline{\rho}$, pressure $\overline{P}$ temperature $\overline{T}$, and velocity U in the $+x$ direction, the linearization of the equation of state, the entropy equation, and the entropy transport equation lead, respectively, to the relations [1]

$$\frac{P'}{\overline{P}} = \frac{\rho'}{\overline{\rho}} + \frac{T'}{\overline{T}}, \tag{1}$$

$$\frac{s'}{c_p} = \frac{T'}{\overline{T}} - \left(\frac{\gamma-1}{\gamma}\right)\frac{P'}{\overline{P}}, \tag{2}$$

$$\frac{D}{Dt}\left(\frac{s'}{c_p}\right) = \frac{4}{3}\overline{\nu}\nabla^2\left[\frac{s'}{c_p} + \left(\frac{\gamma-1}{\gamma}\right)\frac{P'}{\overline{P}}\right], \tag{3}$$

where $\overline{\nu}$ is the mean kinematic viscosity and $D/Dt = \partial/\partial t + U\partial/\partial x$ is the material derivative. Similarly, using the linearized momentum equation

$$\frac{D\boldsymbol{u}'}{Dt} = -\frac{\nabla P'}{\overline{\rho}} + \overline{\nu}\nabla^2\boldsymbol{u}' + \frac{\overline{\nu}}{3}\nabla(\nabla\cdot\boldsymbol{u}') \tag{4}$$

along with (1)-(3), the conservation equation for the pressure fluctuations

$$\frac{1}{\gamma}\frac{D^2}{Dt^2}\left(\frac{P'}{\overline{P}}\right) = \frac{1}{\overline{\rho}}\nabla^2 P' + \frac{4}{3}\overline{\nu}\nabla^2\left[\frac{D}{Dt}\left(\frac{P'}{\overline{P}}\right)\right] \tag{5}$$

is obtained for a Prandtl number equal to 3/4. Lastly, the curl of the momentum equation yields

$$\frac{D\boldsymbol{\omega}'}{Dt} = \overline{\nu}\nabla^2\boldsymbol{\omega}' \tag{6}$$

for the vorticity fluctuations. This homework problem focuses on the structure of Kovazsnay's fluctuation modes in Fourier space. Assume that $\overline{\nu} = 1.5\cdot 10^{-5}$ m^2/s, $\gamma = 1.4$, $U = 600$ m/s, and $\overline{a} = 340$ m/s for all your calculations. Additionally, assume that the disturbance wavelength is 1 mm, as being created for instance by a small grid in a supersonic wind tunnel (Fig. 1).

1. In Kovasznay's analysis, the pressure fluctuations P' are ascribed to an acoustic mode, which is denoted by a superindex $^{(p)}$ and corresponds to the solution of Eq. (5). In particular, consider P' expressed as a single Fourier wave as

$$\frac{P'}{\overline{P}} = \hat{P}^{(p)}\exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r} - ct)], \tag{7}$$

[1] See class notes for details.

where j is the imaginary unit, $\boldsymbol{\kappa}$ is the wavenumber vector, and c is generally a complex eigenvalue whose imaginary component represents the propagation velocity of the mode. Similarly, $\boldsymbol{n}$ is the unit vector in the direction of the wavenumber vector, and $\boldsymbol{r}$ is the position vector in the laboratory frame.

4/4 a) Show that the modes (7) can be redefined in terms of $C = c - U\kappa_x/|\boldsymbol{\kappa}|$ when the problem is formulated in the reference frame moving at the mean velocity U in the $+x$ direction, where κ_x is the projection of $\boldsymbol{\kappa}$ in that direction.

In the remainder of this homework, we will assume for simplicity that $\boldsymbol{\kappa}$ is aligned with the x axis so that $C = c - U$. We will also use the reference frame moving at speed U, where the position vector is now $\boldsymbol{r}'$, so that all the material derivatives become local time variations $\partial/\partial t$.

8/8 b) Substituting (7) into (5), show that C is given by

$$C = -\frac{2}{3}\overline{\nu}\gamma|\boldsymbol{\kappa}|j \pm \overline{a}\sqrt{1 - \left(\frac{2\overline{\nu}\gamma|\boldsymbol{\kappa}|}{3\overline{a}}\right)^2}, \tag{8}$$

where $\overline{a} = \sqrt{\gamma\overline{P}/\overline{\rho}}$ is the mean speed of sound. Note that (8) indicates the presence of two modes of P' corresponding to the positive and negative signs preceding the square root.

8/8 c) How small must the characteristic disturbance wavelength be for the second (dispersive) term in the square root in Eq. (8) to become important?

8/8 d) What is the characteristic decay time of the acoustic pressure disturbance and how long have the two modes traveled along the x direction in the laboratory frame before significant decay by viscosity occurs?

8/8 e) Using Eq. (4), find an expression for the amplitude of the acoustic mode of the velocity fluctuation as a function of $\hat{P}^{(p)}$ and show that the field is irrotational. This indicates that the acoustic mode is a longitudinal wave, in that it is parallel to the wavenumber.

2. The entropy conservation equation (3) suggests that the Fourier representation of the entropy fluctuations can be written as

$$\frac{s'}{c_p} = \hat{s}^{(p)}\exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - Ct)] + \hat{s}^{(s)}\exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - C_s t)], \tag{9}$$

where the first and second terms on the right-hand side correspond, respectively, to the particular and homogeneous solutions of (3). In particular, the super-index $^{(s)}$ indicates the entropy mode.

8/8 a) The first term on the right-hand side of (9) represents an entropy disturbance generated by the the viscous diffusion of the acoustic pressure fluctuation (7) and can therefore be ascribed to the acoustic mode. Obtain a relation between the fluctuation amplitudes $\hat{s}^{(p)}$ and $\hat{P}^{(p)}$ by making use of Eq. (3), and show that $\hat{s}^{(p)}$ is proportional to the viscosity.

b) On the other hand, the second term on the right-hand side of (9) represents an entropy mode that does not have an acoustic origin, and which is necessarily related to temperature fluctuations as

suggested by Eq. (2) and described below. Prove that the value of C_s is a purely imaginary one given by

$$C_s = -\frac{4}{3}\overline{\nu}|\boldsymbol{\kappa}|j. \tag{10}$$

8/8 At what speed relative to the laboratory frame does this entropy fluctuation propagate? Compute the characteristic decay time of this entropy disturbance and the distance it has traveled along the x direction in the laboratory frame before significant decay by viscosity occurs.

3. In view of Eqs. (1)-(2) and (9), it is convenient to express the temperature and density fluctuations as a sum of entropic and acoustic components, namely

$$\frac{T'}{\overline{T}} = \hat{T}^{(s)} \exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - C_T t)] + \hat{T}^{(p)} \exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - Ct)], \tag{11}$$

$$\frac{\rho'}{\overline{\rho}} = \hat{\rho}^{(s)} \exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - C_\rho t)] + \hat{\rho}^{(p)} \exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - Ct)]. \tag{12}$$

8/8 a) Compute C_T, the corresponding decay time of the entropic component of the temperature fluctuations, and the distance traveled by them along the x direction in the laboratory frame before significant decay by viscosity occurs. At what speed relative to the laboratory frame does this does this component of the temperature fluctuations propagate? Obtain relations between $\hat{T}^{(s)}$ and $\hat{s}^{(s)}$, and between $\hat{T}^{(p)}$ and $\hat{P}^{(p)}$.

8/8 b) Using Eq. (1), compute C_ρ, the corresponding decay time of the entropic component of the density fluctuations, and the distance traveled by them along the x direction in the laboratory frame before significant decay by viscosity occurs. At what speed relative to the laboratory frame does this component of the density fluctuations propagate? Derive relations between $\hat{\rho}^{(s)}$ and $\hat{s}^{(s)}$, and between $\hat{\rho}^{(p)}$ and $\hat{P}^{(p)}$.

8/8 c) The entropy mode does not include any pressure fluctuations. As a result, Eqs. (1)-(2) suggest that in this mode the entropy fluctuations are related to the density fluctuations,

$$\frac{s'}{c_p} = -\frac{\rho'}{\overline{\rho}}. \tag{13}$$

Using (13) along with continuity, $D\rho'/Dt = -\overline{\rho}\nabla\cdot\boldsymbol{u}'$, obtain an expression for the amplitude of the entropy mode of the velocity fluctuations as a function of $\hat{s}^{(s)}$, show that the resulting field is irrotational and that the velocities and their divergence induced by the entropy mode are proportional to $\overline{\nu}$.

8/8 4. As indicated above, the acoustic and entropy modes are irrotational. As a result, the vorticity conservation equation (6) suggests that a separate fluctuation mode for the vorticity exists, which in wave form may be written as

$$\boldsymbol{\omega}' = \hat{\omega}^{(\omega)} \exp[j|\boldsymbol{\kappa}|(\boldsymbol{n}\cdot\boldsymbol{r}' - C_\omega t)], \tag{14}$$

where the super-index $^{(\omega)}$ indicates the vortical mode.

a) Substituting (14) into (6) find an expression for C_ω, compute the corresponding decay time of the vorticity fluctuations, and the distance traveled by them along the x direction in the laboratory frame before significant decay by viscosity occurs. At what speed relative to the laboratory frame does

the vorticity fluctuation propagate?

b) Since the vortical mode does not have any associated fluctuations in pressure, entropy or temperature, prove that the induced velocity field is incompressible and corresponds to a transversal wave (i.e., a wave that is orthogonal to the wavenumber vector).

5. The effect of viscosity can be neglected at distances from the grid much smaller than the smallest one of the characteristic streamwise distances computed above for decay of the fluctuations. Describe how the results obtained above change when $\overline{\nu} = 0$.

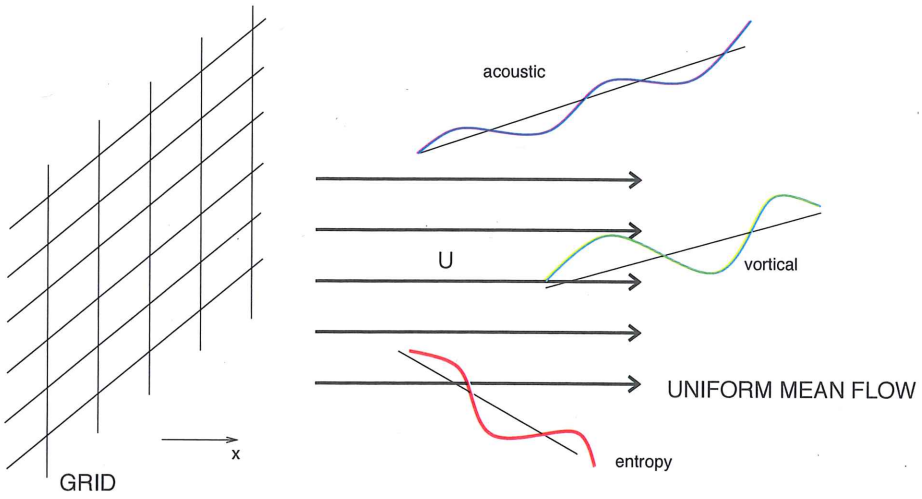
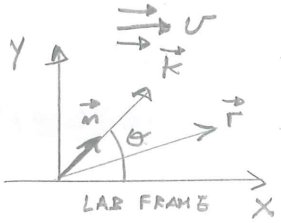


Figure 1: Schematics.

SOLUTIONS

1. $\frac{P'(P)}{\bar{P}} = \hat{P}(P)\, e^{j|\vec{k}|(\vec{n}\cdot\vec{r} - ct)}$



$\vec{r}$ = POSITION VECTOR
$\vec{k}$ = WAVENUMBER VECTOR

a) IN THE MOVING FRAME, $\vec{r}' = \vec{r} - Ut\vec{e}_x$, THEREFORE $j|\vec{k}|(\vec{n}\cdot\vec{r} - ct) = j|\vec{k}|(\vec{n}\cdot(\vec{r}' + Ut\vec{e}_x) - ct)$

$= j|\vec{k}|(\vec{n}\cdot\vec{r}' - \zeta t)$, WITH $\zeta = c - U\cos\theta = c - U\frac{k_x}{|\vec{k}|}$ //, WHICH BECOMES $\zeta = c - U$ WHEN $\vec{k} \parallel \vec{e}_x$

b) NOTE THAT IN FOURIER SPACE,

$$\frac{\partial}{\partial t} \to -j|\vec{k}|\zeta\,, \quad \frac{\partial^2}{\partial t^2} = -|\vec{k}|^2\zeta^2\,, \quad \nabla^2 \to -|\vec{k}|^2\,, \quad \nabla \to -j\vec{k}$$

THEN (5) BECOMES $-|\vec{k}|^2\zeta^2 = -\bar{a}^2|\vec{k}|^2 + |\vec{k}|^3\frac{4}{3}\bar{\nu}\gamma j\zeta$, WITH $\bar{a}^2 = \frac{\gamma\bar{P}}{\bar{\rho}}$.

THIS CAN BE REWRITTEN AS $\zeta^2 - \bar{a}^2 + \frac{4}{3}\bar{\nu}\gamma j|\vec{k}|\zeta = 0$, WHICH HAS THE ROOTS CORRESPONDING TO

$$\zeta = -\frac{2}{3}\bar{\nu}\gamma j|\vec{k}| \pm \bar{a}\sqrt{1 - \left(\frac{2\bar{\nu}\gamma|\vec{k}|}{3\bar{a}}\right)^2}$$ //

c) IT MUST SATISFY $\frac{2\bar{\nu}\gamma|\vec{k}|}{3\bar{a}} \sim 1 \Rightarrow |\vec{k}| \sim \frac{3\bar{a}}{2\bar{\nu}\gamma}$, SO THAT THE WAVELENGTH $\boxed{\lambda = \frac{2\pi}{|\vec{k}|} \lesssim \frac{4\pi\bar{\nu}\gamma}{3\bar{a}}}$

WHICH GIVES $\lambda \lesssim \underline{2.58\cdot10^{-7}\,m}$. FOR LARGER WAVELENGTHS, THE DISPERSIVE TERM IS NEGLIGIBLE. SINCE $\underline{\lambda = 1\,mm}$ IN THIS PROBLEM, THEN THAT TERM CAN BE NEGLECTED.

d) SUBSTITUTING (8) INTO (7) GIVES

$$\frac{P'(P)}{\bar{P}} = \hat{P}(P)\, e^{j|\vec{k}|(\vec{n}\cdot\vec{r} - \frac{2}{3}\bar{\nu}\gamma j|\vec{k}|t \pm \bar{a}t)} = \hat{P}(P)\underbrace{e^{|\vec{k}|^2\frac{2}{3}\bar{\nu}\gamma t}}_{\text{DECAYING}}\underbrace{e^{j|\vec{k}|(\vec{n}\cdot\vec{r} \pm at)}}_{\text{PROPAGATING}}$$

THE CHARACTERISTIC DECAY TIME OF THE ACOUSTIC MODE IS

$$t_D^{(P)} \sim \frac{3}{2}\frac{1}{|\vec{k}|^2\bar{\nu}\gamma} = \frac{3}{8\pi^2}\frac{\lambda^2}{\bar{\nu}\gamma} = \underline{1.8\cdot10^{-3}\,s}$$

THE CORRESPONDING DISTANCES TRAVELED BY THE ACOUSTIC MODES BEFORE DECAY ARE

$$L_D^{(P)} \sim (U \pm \bar{a})\,t_D^{(P)} = 0.46\ m \ \text{(LEFT RUNNING MODE)}$$
$$= 1.67\ m \ \text{(RIGHT RUNNING MODE)}$$

e) SUBSTITUTING (7) INTO (4) AND ASSUMING THAT $\frac{\vec{U}^{(P)}}{U} = \hat{U}^{(P)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r} - \zeta t)}$, THEN

$$-j|\vec{k}|\zeta U\hat{U}^{(P)} = -j\frac{\bar{P}}{\bar{\rho}}\vec{k}\hat{P}^{(P)} - \frac{4}{3}\bar{\nu}|\vec{k}|^2 U\hat{U}^{(P)} \Rightarrow$$

$$\boxed{\hat{U}^{(P)} = \frac{\bar{a}^2}{\gamma U\zeta}\frac{\vec{k}}{|\vec{k}|}\frac{\left(1 + j\frac{4\bar{\nu}|\vec{k}|}{3\zeta}\right)\hat{P}^{(P)}}{\left(1 + \left(\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{\zeta}\right)^2\right)}}$$

WHICH SHOWS THAT $\hat{U}^{(P)} \parallel \vec{k}$, AND THEREFORE $\nabla\wedge\vec{U}^{(P)} = 0$ //

2. CONSIDER THE ENTROPY FLUCTUATIONS

$$\frac{s'}{c_p} = \underbrace{\hat{s}^{(p)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c t)}}_{\text{ACOUSTIC MODE}} + \underbrace{\hat{s}^{(s)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c_s t)}}_{\text{ENTROPY MODE}}$$

a) SUBSTITUTING THE FIRST TERM INTO (2) GIVES

$$-j|\vec{k}| c\, \hat{s}^{(p)} = -\frac{4}{3}\bar{\nu}|\vec{k}|^2 \left(\hat{s}^{(p)} + \left(\frac{\gamma-1}{\gamma}\right) \hat{p}^{(p)} \right)$$

SO THAT

$$\hat{s}^{(p)} = - \frac{\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c} \left(\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c} + j \right) \left(\frac{\gamma-1}{\gamma}\right) \hat{p}^{(p)}}{1 + \left(\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c}\right)^2}$$

, WHICH SHOWS THAT $\hat{s}^{(p)}$ IS PROPORTIONAL TO THE VISCOSITY.

b) SUBSTITUTING THE SECOND TERM INTO (2) WITH $\tau' = 0$ GIVES

$$-j|\vec{k}| c_s \hat{s}^{(s)} = -\frac{4}{3}\bar{\nu}|\vec{k}|^2 \hat{s}^{(s)} \Rightarrow c_s = -\frac{4}{3}\bar{\nu}|\vec{k}| j$$

ALSO SUBSTITUTING c_s INTO THE ENTROPY MODE GIVES

$$\frac{s'^{(s)}}{c_p} = \hat{s}^{(s)} \underbrace{e^{-\frac{4}{3}\bar{\nu}|\vec{k}|^2 t}}_{\text{DECAYING}} \underbrace{e^{j|\vec{k}|(\vec{n}\cdot\vec{r})}}_{\text{PROPAGATING AT ZERO SPEED IN THE MOVING FRAME } (U \text{ IN THE LAB FRAME})}$$

THE CHARACTERISTIC DECAY TIME IS $t_D^{(s)} \sim \frac{3}{4}\frac{1}{\bar{\nu}|\vec{k}|^2} \overset{\lambda = 2\pi/|\vec{k}|}{=} \frac{3\lambda^2}{16\pi^2\bar{\nu}} = 1.2 \cdot 10^{-3}\ s$

AND THE DISTANCE TRAVELLED BEFORE DECAY IS $L_D^{(s)} \sim U \cdot t_D^{(s)} = 0.76\ m$

3. THE TEMPERATURE AND DENSITY FLUCTUATIONS CAN BE WRITTEN AS

$$\left.\begin{aligned} \frac{T'}{\bar{T}} &= \hat{T}^{(s)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c_T t)} + \hat{T}^{(p)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c t)} \\ \frac{\rho'}{\bar{\rho}} &= \underbrace{\hat{\rho}^{(s)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c_s t)}}_{\text{ENTROPY MODE}} + \underbrace{\hat{\rho}^{(p)} e^{j|\vec{k}|(\vec{n}\cdot\vec{r}' - c t)}}_{\text{ACOUSTIC MODE}} \end{aligned}\right\}$$

a) SUBSTITUTING THE ENTROPY MODE OF T' AND s' INTO (2) AND NOTICING THAT THE ENTROPIC PRESSURE DISTURBANCES ARE ZERO NECESSARILY LEADS TO $\hat{s}^{(s)} = \hat{T}^{(s)}$ AND $c_T = c_s$, SO THE CONCLUSIONS FROM 2b ARE APPLICABLE TO $T'^{(s)}$.

SUBSTITUTING THE ACOUSTIC MODE OF T' INTO (2) AND USING THE RESULT FROM 2a LEADS TO

$$\hat{s}^{(p)} = \hat{T}^{(p)} - \left(\frac{\gamma-1}{\gamma}\right) \hat{p}^{(p)} \Rightarrow \hat{T}^{(p)} = \left(\frac{-\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c} \left(\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c} + j \right)}{1 + \left(\frac{4}{3}\frac{\bar{\nu}|\vec{k}|}{c}\right)^2} + 1 \right) \left(\frac{\gamma-1}{\gamma}\right) \hat{p}^{(p)}$$

b) SUBSTITUTING THE ENTROPY MODE OF THE DENSITY FLUCTUATIONS INTO (1) AND NOTICING THAT THE ENTROPIC PRESSURE DISTURBANCES ARE ZERO NECESSARILY LEADS TO $c_{fs} = c_s$ AND $\hat{\rho}^{(s)} = -\hat{T}^{(s)} = -\hat{s}^{(s)}$. THEREFORE ALL RESULTS FROM 2b ARE APPLICABLE TO $\rho'^{(s)}$

THE SAME SUBSTITUTION FOR THE ACOUSTIC MODE YIELDS

$$\hat{\rho}^{(p)} = \hat{p}^{(p)} - \hat{T}^{(p)}, \quad \text{WHERE } \hat{T}^{(p)} \text{ IS GIVEN IN 3a.}$$

c) SUBSTITUTING THE ENTROPY MODE OF ρ' INTO (1) AND ASSUMING THAT $\frac{\vec{u}'^{(s)}}{U} = \hat{\vec{u}}^{(s)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}' - c_s t)}$, THEN

$$-i|\vec{k}|c_s \hat{\rho}^{(s)} = -i\vec{k}\cdot\hat{\vec{u}}^{(s)} U$$

AND SINCE $\hat{\rho}^{(s)} = -\hat{s}^{(s)} \Rightarrow \vec{k}\cdot\vec{u}^{(s)} U = -|\vec{k}| c_s \hat{s}^{(s)}$.

NOTE THAT THE ENTROPIC MODE OF THE VELOCITY FLUCTUATIONS IS IRROTATIONAL, AND AS A RESULT $\vec{u}^{(s)}$ DERIVES FROM A POTENTIAL, WHICH IN VIEW OF THE ABOVE EXPRESSION MUST BE GIVEN BY

$$\vec{k}\cdot\left(-i\frac{\vec{k}}{|\vec{k}|}\left(\frac{c_s \hat{s}^{(s)}}{i}\right)\right) = -|\vec{k}| c_s \hat{s}^{(s)} \Rightarrow \vec{u}^{(s)} = -iU c_s \hat{s}^{(s)} \frac{\vec{k}}{|\vec{k}|}$$

WITH $c_s = -\frac{4}{3}\nu|\vec{k}|i$

4. a) CONSIDER THE VORTICITY FLUCTUATION

$$\vec{\omega}' = \hat{\omega}^{(\omega)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}' - c_\omega t)}$$

AND SUBSTITUTE IT INTO (6), WHICH GIVES

$$-i|\vec{k}| c_\omega = -|\vec{k}|^2 \nu \Rightarrow c_\omega = -\nu|\vec{k}|i$$

SO THAT $\vec{\omega}' = \hat{\omega}^{(\omega)} \underbrace{e^{+\nu|\vec{k}|^2 t}}_{\text{DECAYING}} \underbrace{e^{i|\vec{k}|(\vec{m}\cdot\vec{r}')}}_{\text{PROPAGATING AT ZERO SPEED WITH RESPECT TO THE MEAN FLOW}}$.

CHARACTERISTIC DECAY TIME: $t_\nu^{(\omega)} \sim \frac{1}{\nu|\vec{k}|^2} = \frac{\lambda^2}{4\pi^2\nu} = 1.7 \cdot 10^{-3}\ s$

LENGTH TRAVELED BEFORE DECAY: $L_\nu^{(\omega)} \sim U t_\nu^{(\omega)} \sim 1\ m$

b) SINCE $p^{(\omega)} = T^{(\omega)} = \rho^{(\omega)} = 0$ IN THE VORTICAL MODE, THE ASSOCIATED FLUCTUATION VELOCITY FIELD MUST SATISFY THE SOLENOIDAL CONDITION $\nabla\cdot\vec{u}'^{(\omega)} = 0$, WHICH IS EQUIVALENT TO $\vec{k}\cdot\hat{\vec{u}}^{(\omega)} = 0 \Rightarrow \hat{\vec{u}}^{(\omega)} \perp \vec{k}$

5. VISCOSITY CAN BE NEGLECTED $X \ll \min\left(L_D^{(p)}, L_D^{(\omega)}, L_D^{(s)}\right) = 0.46\ m$.

SETTING $D=0$ IN THE RESULTS LEADS TO

$$q = \pm a, \quad \frac{p'^{(p)}}{\bar{p}} = \hat{p}^{(p)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}' \pm at)}, \quad \frac{\vec{u}'^{(p)}}{U} = \frac{\bar{a}_0^2}{\gamma U q} \frac{\vec{k}}{|\vec{k}|} \hat{p}^{(p)}$$

$$s'^{(p)} = 0, \quad q_s = 0, \quad \frac{s'^{(s)}}{c_p} = \hat{s}^{(s)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}')}, \quad \frac{T'}{\bar{T}} = \hat{T}^{(s)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}')} + \hat{T}^{(p)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}' \pm at)}$$

$$\frac{\rho'}{\bar{\rho}} = \hat{\rho}^{(s)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}')} + \hat{\rho}^{(p)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}' \pm at)}$$

$$\hat{T}^{(p)} = \left(\frac{\gamma-1}{\gamma}\right) \hat{p}^{(p)}, \quad \vec{u}'^{(s)} = 0, \quad \vec{\omega}'^{(\omega)} = \hat{\omega}^{(\omega)} e^{i|\vec{k}|(\vec{m}\cdot\vec{r}')}.$$

THUS ILLUSTRATING THAT THE ENTROPY, VORTICITY AND PRESSURE EVOLVE INDEPENDENTLY AND NONE OF THE MODES IS SUBJECT TO VISCOUS DECAY.

ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Homework 3: Compressible-turbulence energetics and shock-turbulence interactions

Due Thursday, June 1, in class.

Guidelines: Please turn in a *neat* and *clean* homework that gives all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions.

Student's Name:..... **Student's ID:**.....

1. Why does the streamwise vorticity component of the flow remain mostly constant across a shock wave? Justify your answer with appropriate equations.

2. Provide a brief explanation of what an eddy shocklet is and how are they typically detected in numerical simulations.

3. What is the critical incidence angle of a vorticity-entropy wave on a Mach-3.0 shockwave for which the acoustic wave becomes an evanescent wave behind the shock?

4. Starting from the momentum equation, show that in compressible homogeneous isotropic turbulent flows integrated in triply periodic domains the conservation equation for the mean turbulent kinetic energy is

$$\frac{dk}{dt} = \overline{P}\Delta_v - \epsilon \tag{1}$$

where $k = \widetilde{\overline{\rho} u_i'' u_i''}/2$.

ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Midterm Exam

Tuesday, May 9

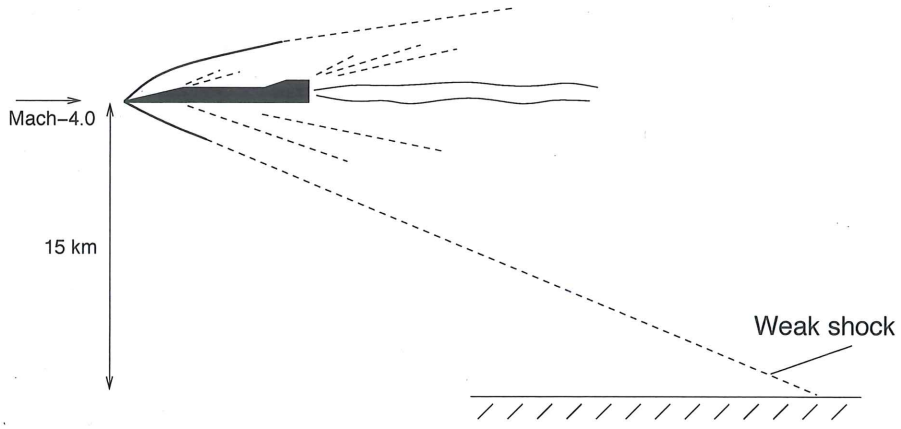
Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name: JAVIER URZAY **Student's ID:**.....

PART I: Closed books, closed notes, calculators allowed
Time: 30 mins

Questions (40 pts)

1. - Explain the basis of Lighthill's analogy for computing the sound radiated from a turbulent flow, including the associated conservation equations for the acoustic field.
   - If the integral size of the eddies is $\ell \sim 1$ cm and the corresponding turnover velocity is $u_\ell \sim 1$ m/s, estimate the characteristic wavelength L_a of the acoustic radiation (assume $T = 300$ K).
   - Estimate the turbulent dissipation, ϵ, and compare it with the amount of acoustic energy emitted per unit time and per unit mass by the turbulent flow, ϵ_s.
2. During flight at an altitude of $h = 15$ km in the stratosphere, where $T = 216.5$ K, a pilot of a supersonic airplane reads Mach-4.0 in the Mach-meter, which is an instrument that approximately provides the flight speed in units of the local speed of sound in the undisturbed air environment around the aircraft.
   - Estimate the Mach number based on ground conditions, $T = 288$ K.
   - Assuming that the nose shock wave around the aircraft has become a weak shock near the ground with a small streamline deflection angle $\delta \sim 10^{-4}$ rad, compute the characteristic angle formed by the shock and the horizontal ground surface, along with the corresponding sound-pressure level (SPL) caused by the sonic boom by taking $P_{ref} = 20$ μPa as the reference pressure.



ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Midterm Exam

Tuesday, May 9

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name:..... **Student's ID:**.....

PART II: Open books, open notes, calculators allowed
Time: 50 mins

Problem 1 (60 pts)

A turbulent compressible turbulent shear layer is formed by two parallel gaseous streams (denoted by the subindexes $_1$ and $_2$ as shown in the Fig. 1a) with free-stream velocities U_1 and $U_2 < U_1$ downstream from a splitter plate. The corresponding densities are ρ_1 and ρ_2. The molecular weights of the gases are W_1 and W_2, while their adiabatic coefficients are γ_1 and γ_2. The static pressure is the same on both sides of the mixing layer and equals P_∞. As a result of vortex pairing, the thickness of the mixing layer is observed to increase with increasing distances from the splitter plate, as sketched in Fig. 1a.

a) An important parameter that characterizes the relative importance of compressibility and its effects on the growth of the mixing layer is the convective Mach number, $\mathrm{Ma}_{C,1} = (U_1 - U_C)/a_1$, which represents a Mach number in a relative frame moving at the convective speed U_C of the disturbances in the mixing layer (see Fig. 1b) [1]. In this formulation, a_1 and a_2 are the speed of the sound waves in each stream. A method to compute the convective Mach number is outlined below.

- In the moving reference frame depicted in Fig. 1b, the point O is a saddle point between the structures and represents a common stagnation point for both streams. As a result, the stagnation pressures in that frame are the same for both streams. If, additionally, the deceleration at the stagnation point is assumed to be isentropic, find an expression that relates $\mathrm{Ma}_{C,1} = (U_1 - U_C)/a_1$, $\mathrm{Ma}_{C,2} = (U_C - U_2)/a_2$, γ_1 and γ_2.

- Prove that the expression found above leads to the convective velocity

$$U_C = \frac{a_2 U_1 + a_1 U_2}{a_2 + a_1} \tag{1}$$

when $\gamma_1 = \gamma_2$ and the convective Mach numbers are moderately small. Note that Eq. (1) implies that both convective Mach numbers are equal, $\mathrm{Ma}_{C,1} = \mathrm{Ma}_{C,2} = \mathrm{Ma}_C$. Using expression Eq. (1) for U_C, compute the characteristic convective Mach number when stream 1 is air at $\mathrm{Ma}_1 = 4$, $T_1 = 1300$ K, and stream 2 is molecular hydrogen at $\mathrm{Ma}_2 = 0.5$ and $T_2 = 500$ K, which correspond to conditions typical of fuel-air mixing layers in scramjets.

[1] Note that compressibility effects necessarily emerge from the amount of relative motion between both streams.

b) In order to reduce the computational cost of simulating the spatially development of the mixing layer, most approaches involve simulation of the problem in a reference frame moving at the mean velocity $U_m = (U_1 + U_2)/2$, with the inflow/outflow streamwise boundary conditions of the original problem being replaced by periodic boundary conditions. The boundary conditions at the top and bottom boundaries become free-stream conditions with $u_2 = u_3 = 0$. In this way, the problem reduces to two supersonic free streams flowing in opposite directions at the same velocity U (see Fig. 1c).

- Derive an expression for U as a function of U_1 and U_2, and show that the resulting convective Mach number is

$$\mathrm{Ma_C} = 2U/(a_1 + a_2). \tag{2}$$

- Given U and the speeds of sound a_1 and a_2 as the only input data provided in the problem, would it be possible to retrieve the values of the absolute velocities U_1, U_2, the mean velocity U_m and the convective speed U_C? If the answer is affirmative, provide details about how such retrieval would be done[2].

c) In the relative reference frame described in part b), the solution to the compressible shear layer involves a momentum thickness δ_θ that grows in time -rather than space as in the spatially-evolving mixing layer-, thereby leading to unsteady statistics. In particular, δ_θ is defined as

$$\delta_\theta(t) = \frac{1}{4\rho_m U^2} \int_{-\infty}^{+\infty} \overline{\rho} \left(U^2 - \widetilde{u}_1^2\right) dx_2, \tag{3}$$

where $\rho_m = (\rho_1 + \rho_2)/2$ is the mean density. In this formulation, $\overline{\rho}$ and $\widetilde{u}_1$ represent, respectively, the Reynolds-averaged density and the Favre-averaged streamwise velocity (refer to the coordinate system in Fig. 1c). All types of averaging in this problem are performed along planes x_2 =constant. The rate of growth of the mixing layer can be computed from

$$\dot{\delta}_\theta = \frac{d\delta_\theta}{dt}. \tag{4}$$

Figure 2a shows the ratio of $\dot{\delta}_\theta$ to the incompressible growth rate, as a function of the convective Mach number Ma_C for a number of simulations and experiments. The results suggest that δ_θ rapidly decreases with increasing convective Mach numbers as a result of compressibility effects, a phenomenon that is related to the decrease of the turbulent production, as shown in Fig. 2b. To see this, consider the Favre-averaged mass and momentum equations

$$\frac{\partial \overline{\rho}}{\partial t} + \frac{\partial}{\partial x_i}(\overline{\rho}\widetilde{u}_i) = 0, \tag{5}$$

$$\frac{\partial}{\partial t}(\overline{\rho}\widetilde{u}_i) + \frac{\partial}{\partial x_j}(\overline{\rho}\widetilde{u}_i\widetilde{u}_j) = -\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial \overline{\tau'}_{ij}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\widetilde{\overline{\rho}u_i''u_j''}\right). \tag{6}$$

The last term in Eq. (6) involves the unclosed Reynolds stresses, $\widetilde{\overline{\rho}u_i''u_j''}$. Note that all averaged quantities are, by definition of the averaging operators, functions of t and x_2.

- Substituting (3) into the right-hand side of (4) and using (5)-(6), find an expression for $\dot{\delta}_\theta$ as a function of the streamwise turbulent production $-\widetilde{\overline{\rho}u_1''u_2''}(\partial\widetilde{u}_1/\partial x_2)$.

[2]Note that retrieving U_m would allow one to compute the approximate streamwise distance $x \sim U_m t$ traveled by the moving frame and therefore link space in the absolute frame with the simulation time t in the relative frame.

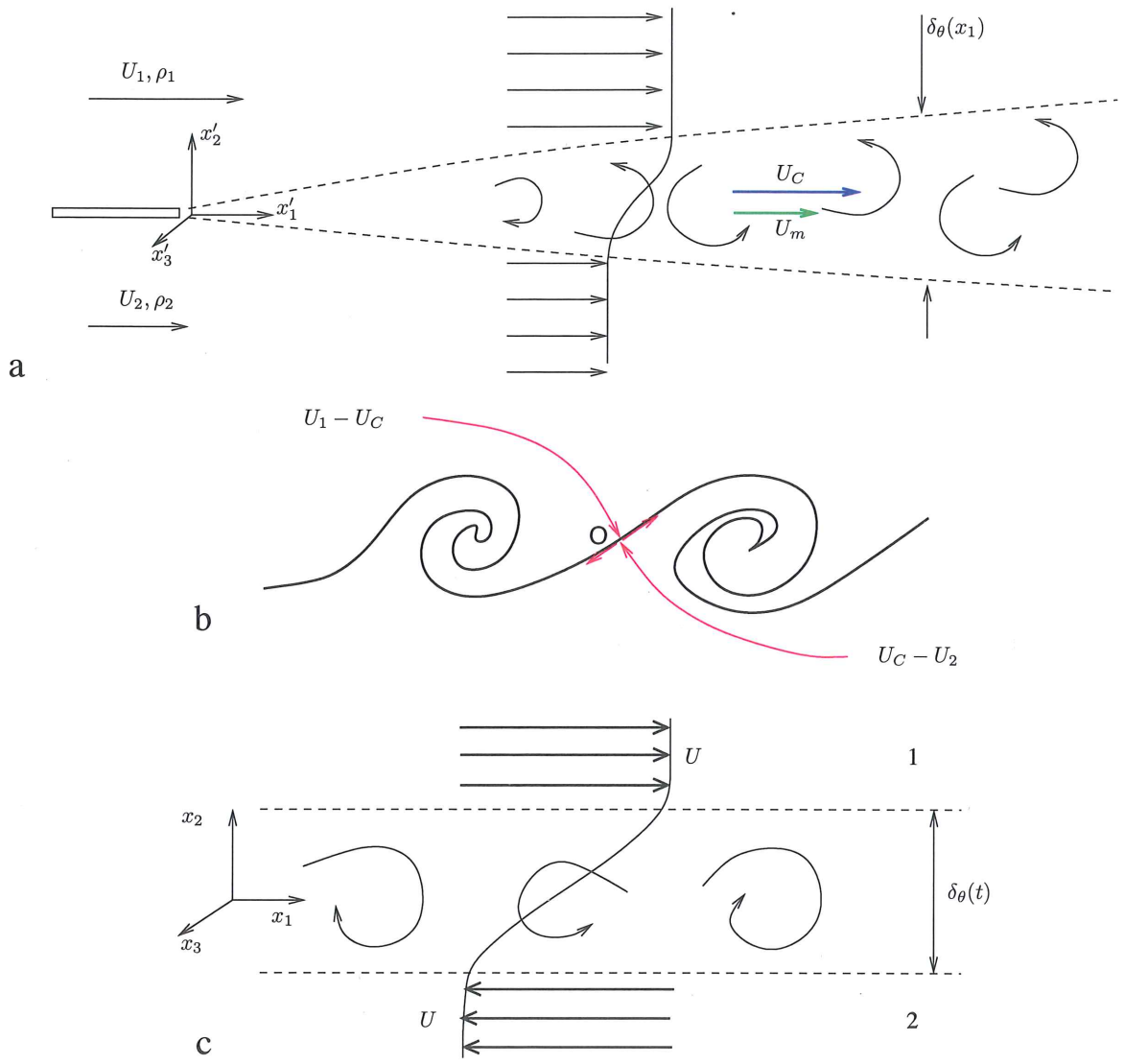


Figure 1: (a) Spatially developing mixing layer. (b) Sketch of streamlines in the convective frame of reference. (c) Description of the problem in a reference frame moving with the mean speed U_m.

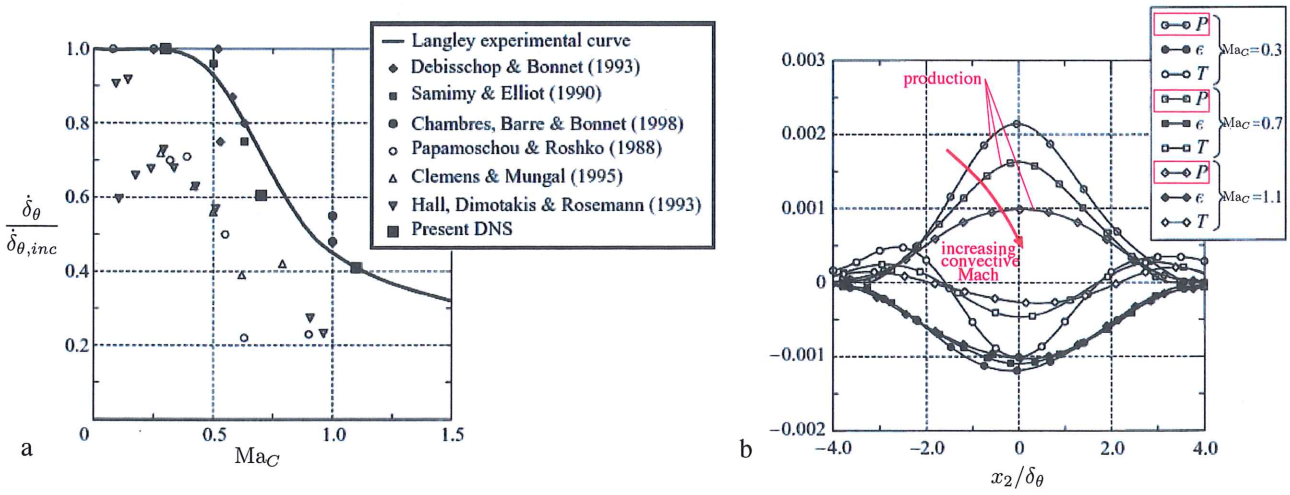


Figure 2: (a) Dependence of shear-layer growth on Ma_C. (b) Turbulent production as a function of the x_2 coordinate for three values of Ma_C. Figures adapted from Pantano & Sarkar, JFM (2002).

SOLUTIONS

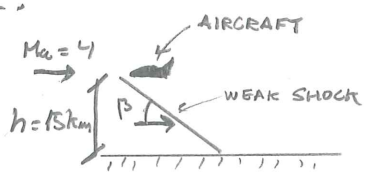
QUESTIONS

1. – SEE PAGES 35-37 FROM CLASS NOTES.

– $\left.\begin{array}{l} \ell \sim 1\,cm \\ U_\ell \sim 1\,m/s \\ T = 300\,K \end{array}\right\}$ $\quad La \sim \frac{\ell}{U_\ell}\, a = \frac{\ell}{U_\ell}\sqrt{\gamma R_g T} = \underline{\underline{3.4\,m}}$

– $\mathcal{E} \sim U_\ell^3/\ell \sim 100\ \frac{m^2}{s^3}$. $\quad \mathcal{E}_s \sim \mathcal{E}\left(\frac{U_\ell}{a}\right)^5 \sim \underline{\underline{2.2\cdot 10^{-11}\ m^2/s^3}}$

2.



– ON THE GROUND: $Ma_G \frac{U}{a_G} = \frac{a_h}{a_G}\frac{U}{a_h} = \sqrt{\frac{216.5}{288}}\cdot 4 = \underline{\underline{3.46}}$

– THE INCIDENCE ANGLE OF A WEAK SHOCK IS VERY CLOSE TO THE MACH ANGLE IN THE FIRST APPROXIMATION,

$$\beta \approx \mu' = \arctan\left(\frac{1}{\sqrt{Ma_G^2 - 1}}\right) = 16.8^\circ$$

FOR $\delta = 10^{-4}$ rad, THE CARACTERISTIC PRESSURE VARIATION ACROSS THE SHOCK IS

$$\frac{P_2 - P_1}{P_1} = \frac{\gamma Ma_G^2}{\sqrt{Ma_G^2 - 1}}\,\delta \simeq 5.05\cdot 10^{-4} \Rightarrow P' = P_2 - P_1 \approx 50.6\ Pa$$

THE CORRESPONDING SPL IS $20\log_{10}\left(\frac{P'}{P_{ref}}\right) \approx \underline{\underline{128\,dB}}$

PROBLEMS

4. a) – $P_{o_1} = \cancel{P_\infty}\left(1 + \left(\frac{\gamma_1 - 1}{2}\right)Ma_{C_1}^2\right)^{\frac{\gamma_1}{\gamma_1 - 1}} = \cancel{P_\infty}\left(1 + \left(\frac{\gamma_2 - 1}{2}\right)Ma_{C_2}^2\right)^{\frac{\gamma_2}{\gamma_2 - 1}} = P_{o_2}$

SO THAT $\left[1 + \left(\frac{\gamma_1 - 1}{2}\right)Ma_{C_1}^2\right]^{\frac{\gamma_1}{\gamma_1 - 1}} = \left[1 + \left(\frac{\gamma_2 - 1}{2}\right)Ma_{C_2}^2\right]^{\frac{\gamma_2}{\gamma_2 - 1}}$

– FOR $Ma_{C_1}, Ma_{C_2} \ll 1$, A SERIES EXPANSION OF THE ABOVE EXPRESSION LEADS TO

$$1 + \left(\frac{\gamma_1 - 1}{2}\right)\frac{\gamma_1}{\gamma_1 - 1}Ma_{C_1}^2 + O\left(Ma_{C_1}^2\right) \sim$$

$$\sim 1 + \left(\frac{\gamma_2 - 1}{2}\right)\frac{\gamma_2}{\gamma_2 - 1}Ma_{C_2}^2 + O\left(Ma_{C_2}^2\right)$$

AND CANCELLING TERMS; $Ma_{c_1} = \left(\frac{\gamma_2}{\gamma_1}\right)^{1/2} Ma_{c_2}$

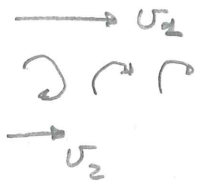
IF $\gamma_1 = \gamma_2$, THEN THIS EXPRESSION REDUCES TO $Ma_{c_1} = Ma_{c_2}$, OR EQUIVALENTLY;

$$\frac{U_1 - U_c}{a_1} = \frac{U_c - U_2}{a_2} \Rightarrow U_c = \frac{a_1 U_2 + a_2 U_1}{a_1 + a_2}$$

REWRITING U_c AS $\frac{U_c}{a_1} = \frac{Ma_2 + Ma_1}{1 + a_1/a_2} = \frac{Ma_2 + Ma_1}{1 + \sqrt{\frac{W_2 T_2}{W_1 T_2}}} = 3.15$

AND THEREFORE $Ma_c = \frac{U_1 - U_c}{a_1} = Ma_1 - \frac{U_c}{a_1} = 4 - 3.15 = \underline{0.85}$

b) – ABSOLUTE FRAME



FRAME MOVING AT MEAN SPEED U_m



$U_1 - U_m = U_1 - \frac{1}{2}(U_1 + U_2) = \frac{1}{2}(U_1 - U_2)$

$U_m - U_2 = \frac{1}{2}(U_1 + U_2) - U_2 = \frac{1}{2}(U_1 - U_2)$

$$\Rightarrow U = \frac{1}{2}(U_1 - U_2)$$

THE CONVECTIVE MACH NUMBER IS $Ma_c = \frac{U_1 - U_c}{a_1} = \frac{U_1}{a_1} - \frac{1}{a_1}\left(\frac{a_1 U_2 + U_1 a_2}{a_1 + a_2}\right) =$

$$= \frac{U_1}{a_1} - \frac{U_2 + U_1 a_2/a_1}{a_1 + a_2} = \frac{(a_1 + a_2)U_1 - a_1 U_2 - U_1 a_2}{a_1(a_1 + a_2)} = \frac{U_1 - U_2}{a_1 + a_2} = \frac{2U}{a_1 + a_2}$$

– GIVEN U, a_1 AND a_2, ONE CAN COMPUTE THE CONVECTIVE MACH NUMBER FROM $Ma_c = 2U/(a_1 + a_2)$ AND THE VELOCITY DIFFERENCE $U_1 - U_2 = 2U$, BUT IT IS NOT POSSIBLE TO RECOVER THE VALUES OF U_1, U_2, U_m NOR U_c.

c) $\dot{\delta}_\theta = \frac{d\delta_\theta}{dt} = \frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty}\left(\frac{\partial \bar{\rho}}{\partial t}U^2 - \frac{\partial}{\partial t}(\rho \tilde{U}_1^2)\right)dx_2 =$

$$= -\frac{1}{4\rho_m}\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(\rho \tilde{U}_2)\,dx_2 - \frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}(\rho \tilde{U}_1^2)\,dx_2 =$$

$= 0$ IF $\tilde{U}_2 = 0$ AT $x_2 \to \pm\infty$

IN ORDER TO SIMPLIFY THE LAST TERM, MULTIPLY THE x_1-COMPONENT OF THE MOMENTUM EQUATION (6) BY $\tilde{U}_1$:

$$\tilde{U}_1\frac{\partial}{\partial t}(\bar{\rho}\tilde{U}_1) + \tilde{U}_1\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{U}_1\tilde{U}_2) = -\tilde{U}_1\cancel{\frac{\partial \bar{p}}{\partial x_1}}^{\,0} + \tilde{U}_1\frac{\partial \bar{\tau}_{12}}{\partial x_2} - \tilde{U}_1\frac{\partial}{\partial x_2}\left(\bar{\rho}\widetilde{U_1'' U_2}\right)$$

ALSO NOTE THAT

$$\tilde{U}_1 \frac{\partial}{\partial t}(\bar{\rho}\tilde{U}_1) + \tilde{U}_1 \frac{\partial}{\partial x_2}(\bar{\rho}\tilde{U}_1\tilde{U}_2) = \bar{\rho}\tilde{U}_1 \frac{\partial \tilde{U}_1}{\partial t} + \bar{\rho}\tilde{U}_1\tilde{U}_2 \frac{\partial \tilde{U}_1}{\partial x_2} = \bar{\rho}\frac{\partial}{\partial t}\left(\frac{\tilde{U}_1\tilde{U}_1}{2}\right) + \bar{\rho}\tilde{U}_2 \frac{\partial}{\partial x_2}\left(\frac{\tilde{U}_1\tilde{U}_1}{2}\right)$$

$$= \frac{\partial}{\partial t}\left(\frac{\bar{\rho}\tilde{U}_1\tilde{U}_1}{2}\right) + \frac{\partial}{\partial x_2}\left(\bar{\rho}\tilde{U}_2 \frac{\tilde{U}_1\tilde{U}_1}{2}\right),$$

WHERE USE OF CONTINUITY $\left(\frac{\partial \bar{\rho}}{\partial t} + \frac{\partial}{\partial x_2}(\bar{\rho}\tilde{U}_2) = 0\right)$ HAS BEEN MADE. AS A RESULT,

$$\frac{\partial}{\partial t}(\bar{\rho}\tilde{U}_1\tilde{U}_1) = -\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{U}_2\tilde{U}_1\tilde{U}_1) + \frac{\partial}{\partial x_2}(2\tilde{U}_1\bar{\tau}'_{12}) - 2\bar{\tau}'_{12}\frac{\partial \tilde{U}_1}{\partial x_2}$$

$$- \frac{\partial}{\partial x_2}(2\bar{\rho}\tilde{U}_1\widetilde{U_1''U_2''}) + 2\bar{\rho}\widetilde{U_1''U_2''}\frac{\partial \tilde{U}_1}{\partial x_2}.$$

SUBSTITUTING THIS EXPRESSION INTO $\dot{\delta}_\theta$:

$$\dot{\delta}_\theta = -\frac{1}{4\rho_m U^2}\int_{-\infty}^{+\infty}\frac{\partial}{\partial t}(\bar{\rho}\tilde{U}_1\tilde{U}_1)\,dx_2 = -\frac{1}{4\rho_m U^2}\left\{-\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(\bar{\rho}\tilde{U}_2\tilde{U}_1\tilde{U}_1)\,dx_2 + \right.$$

(0 SINCE $\tilde{U}_2 \to 0$ AT $x_2 \to \pm\infty$)

$$+\int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(2\tilde{U}_1\bar{\tau}'_{12})\,dx_2 - \int_{-\infty}^{+\infty}\frac{\partial}{\partial x_2}(2\bar{\rho}\tilde{U}_1\widetilde{U_1''U_2''})\,dx_2$$

(0 SINCE $\bar{\tau}'_{12} = 0$ AT $x_2 \to \pm\infty$; 0 SINCE $\widetilde{U_1''U_2''} = 0$ AT $x_2 \to \pm\infty$)

$$\left. + \int_{-\infty}^{+\infty} 2\bar{\rho}\widetilde{U_1''U_2''}\frac{\partial \tilde{U}_1}{\partial x_2}\,dx_2 - \int_{-\infty}^{+\infty} 2\bar{\tau}'_{12}\frac{\partial \tilde{U}_1}{\partial x_2}\,dx_2 \right\}.$$

THE LAST TERM CORRESPONDS TO TWICE THE MEAN VISCOUS DISSIPATION,

$$\bar{\varepsilon} = +\bar{\tau}'_{12}\frac{\partial \tilde{U}_1}{\partial x_2} = +\bar{\tau}_{12}\tilde{S}_{12}$$

NOTE THIS INVOLVES THE PLANAR-AVERAGED GRADIENTS → NEGLIGIBLE

WHICH IS NEGLIGIBLE AT HIGH REYNOLDS NUMBERS.

AS A RESULT,

$$\dot{\delta}_\theta = -\frac{1}{2\rho_m U^2}\int_{-\infty}^{+\infty}\bar{\rho}\widetilde{U_1''U_2''}\frac{\partial \tilde{U}_1}{\partial x_2}\,dx_2$$

ME 451C: Compressible Turbulence, Spring 2017

Stanford University

Final Exam

Thursday, June 8 (Due Tuesday June 13)

Guidelines: Please turn in *neat* and *clean* exam solutions that give all the formulae that you have used as well as details that are required for the grader to understand your solution. Attach these sheets to your solutions. Assume $\gamma = 1.4$ and $c_p = 1.0$ KJ/KgK for all problems.

Student's Name:..... **Student's ID:**.....

PART I
Estimated Time: 30 mins

Questions (40 pts)

1. Why does the streamwise vorticity component of the flow remain mostly constant across a shock wave? Support your answer by making use of appropriate shock jump conditions.

2. Provide a brief explanation of what an eddy shocklet is and how are they typically detected in today's numerical simulations.

3. Simplify the conservation equations for the mean kinetic energy, turbulent kinetic energy and mean thermal energy for compressible turbulent flows integrated in triply periodic domains. Which ones are the most important terms at high Reynolds numbers? Based on the different terms in the equations you derived, sketch an energy diagram that shows the modes of conversion between the three types of energy.

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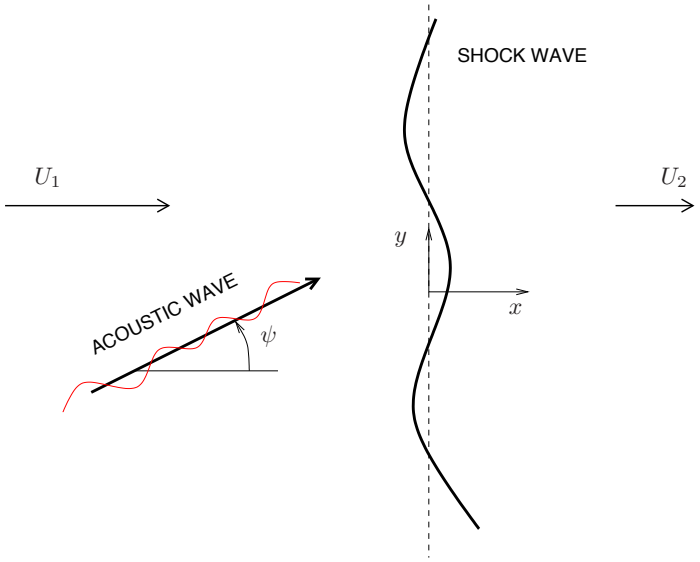
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Student's Name:..... **Student's ID:**.....

PART II

Estimated Time: 120 mins Pick just one of the two following problems.

Problem 1 (60 pts) Using the notation and linear-interaction analysis we introduced in class to solve the problem of the interaction of a vorticity-entropy wave with a shock wave, modify the methodology to address the problem of an acoustic wave impinging on a shock wave, as shown in the figure below. In particular, a) compute the interval of incident angles ψ over which the pressure wave downstream of the shock is an evanescent wave that decays exponentially, and b) determine whether vortical and entropy modes exist downstream of the wave and how would you compute them (i.e. just write the necessary equations and boundary conditions, but it is not required to solve them for the wave coefficients).



Problem 2 (60 pts) The equilibrium wall model for large-eddy simulations (LES) of compressible turbulent wall-bounded flows integrates the momentum and total-energy conservation equations

$$\frac{d}{dy}\left[(\mu+\mu_{t,wm})\,\frac{du}{dy}\right]=0 \tag{1}$$

$$\frac{d}{dy}\left[(\mu+\mu_{t,wm})\,u\frac{du}{dy}+c_p\left(\frac{\mu}{\mathrm{Pr}}+\frac{\mu_{t,wm}}{\mathrm{Pr}_{t,wm}}\right)\frac{dT}{dy}\right]=0 \tag{2}$$

within a layer spanning from the wall to a matching location, where appropriate boundary conditions are applied, as schematically shown in the figure below and described in this problem.

In this formulation, y is the wall-normal coordinate, u is the total wall-parallel velocity including both the streamwise and spanwise components, T is the static temperature, c_p is the specific heat at constant pressure, $\mathrm{Pr}=0.7$ is the molecular Prandtl number, μ is the molecular dynamic viscosity, and the subscript "wm" indicates variables in the wall model. The molecular viscosity μ is proportional to the temperature, $\mu/\mu_{ref}=(T/T_{ref})^{\sigma}$ with $\sigma=0.7$. Additionally, the eddy viscosity $\mu_{t,wm}$ is modeled using the mixing-length model

$$\mu_{t,wm}=\kappa\rho y\sqrt{\frac{\tau_w}{\rho}}D, \tag{3}$$

where $\kappa=0.4$ is the von Kármán constant, ρ is the density and $\tau_w=\mu_w du/dy|_{y=0}$ is the local wall shear stress, with μ_w the dynamic viscosity evaluated at the wall. The symbol D is a damping function given by

$$D=\left[1-\exp\left(-\frac{y^+}{A^+}\right)\right]^2, \tag{4}$$

where $A^+=17$ is a constant. The superscript "+" indicates length normalized with wall units, i.e. $y^+=\rho_w u_\tau y/\mu_w$, where $u_\tau=\sqrt{\tau_w/\rho_w}$ is the friction velocity and ρ_w is the density at the wall. The density and the temperature are related by the equation of state

$$P=\rho R_g T, \tag{5}$$

where R_g is the gas constant and P is the static pressure, the latter of which is modeled as a constant across the wall-modeled region and matches with the LES outside. Lastly, $\mathrm{Pr}_{t,wm}=0.9$ is the eddy Prandtl number.

Equations (1) and (2), along with (3)-(5) are integrated on a one-dimensional grid $0\le y\le h$ bounded by the wall at $y=0$ and by a matching location at $y=h$, where the wall-model solution matches with the LES solution on the coarse grid. The boundary conditions for the wall model at the wall $y=0$ are

$$u=0,\quad T=T_w \tag{6}$$

where T_w is the temperature imposed at the wall. The corresponding boundary conditions at the matching location $y=h$ are

$$u=\widetilde{U},\quad T=\widetilde{T},\quad P=\overline{P}, \tag{7}$$

where $\widetilde{U}$, $\widetilde{T}$ and $\overline{P}$ are the resolved LES values of wall-parallel velocity, static temperature and static pressure, the former also includes the streamwise and spanwise components as appropriate (i.e., these are all known values). Equations (1)-(7) determine the solution profiles $u(y)$ and $T(y)$.

a) Show that the problem described has the following two first integrals

$$u=\int_0^y\frac{\tau_w dy}{\mu+\mu_{wm}} \tag{8}$$

and

$$\int_0^y \frac{\tau_w^2 dy}{\mu + \mu_{wm}} + c_p \left(\frac{\mu}{\mathrm{Pr}} + \frac{\mu_{wm}}{\mathrm{Pr}_t} \right) \frac{dT}{dy} = -q_w \tag{9}$$

where $q_w = -(\mu c_p/\mathrm{Pr}) dT/dy|_{y=0}$ is the heat flux at the wall. The first term in Eq. (9) represents aerodynamic heating that is only important when the characteristic Mach number is sufficiently large. Note that τ_w and q_w are fundamental constants that are part of the solution of the problem and are determined by using the boundary conditions (6)-(7).

b) Your goal in this problem is to investigate the existence of closed or approximate relations of the type

$$\tau_w = \tau_w(\widetilde{U}, \widetilde{T}) \tag{10}$$

$$q_w = q_w(\widetilde{U}, \widetilde{T}), \tag{11}$$

in such a way that one could compute the wall shear stress and the wall heat flux as a function of the velocities and temperatures of the outer LES field at the matching location, without the need of having to solve a system of ordinary differential equations (for instance, note that a straightforward solution exists for the less interesting case of incompressible and constant-viscosity flows). This would be attractive since the quantities τ_w and q_w are the ones to be passed as a boundary condition for the outer LES. To investigate the existence of such simplified solutions, it may be expedient to nondimensionalize equations (8)-(9) and search for relevant asymptotic limits in which the solution may simplify to a form similar to (10)-(11), since in the most general case a closed-form solution may not exist.

In normalizing the equations, note that there are two external length scales in the problem, namely, h and $A^+\mu_w/(\rho_w u_\tau)$, with $\rho_w u_\tau h/\mu_w \gg 1$. In particular, D is vanishingly small near the wall at distances $y \ll A^+\mu_w/(\rho_w u_\tau)$ (i.e. in the viscous sublayer), in such a way that the eddy viscosity μ_{wm} is much smaller than the molecular viscosity μ there. Conversely, $D \sim 1$ at distances $y \sim h$ (i.e. outside the viscous sublayer), where $\mu_{wm} \gg \mu$.

Appropriate normalization variables could be $\theta = (T - T_w)/(\widetilde{T} - T_w)$, $\eta = y/h$, and $\hat{u} = u/\widetilde{U}$, with which you may be able to rewrite (1)-(2) or (8)-(9) in terms of characteristic dimensionless parameters and subject to simpler boundary conditions. Numerical investigations of these dimensionless equations are also encouraged since those may inspire simplifications of the problem within parameter ranges that are to be discovered.

