1 Introduction

My dissertation research is on the mathematical analysis of systems of partial differential equations that arise in modeling the equilibrium of mean field games. I am interested in finding traveling wave solutions of such systems and studying the large time behavior of the associated time dependent problem.

Mean field models are used in optimal decision making based on stochastic games with a very large population of agents that are statistically the same [1–6]. The main advantage of such models is that the overall effect of the other agents on a single one can be replaced by an averaged effect. Hence, the dynamics of a single agent can be determined as a solution of an optimal control problem that depends on the overall distribution of the agents. These models are governed by two partial differential equations, a forward Kolmogorov equation that keeps track of the distribution of agents in the model, and a backward Hamilton Jacobi Bellman (HJB) equation for the value function of the optimal stochastic control problem for each agent. An equilibrium solution to this mean field problem is a solution of the coupled system of the two equations. Such models are what is called a "mean-field game" in [1,2].

During my PhD I have been focusing on an extension of a mean field model for knowledge propagation and economic growth, proposed by Lucas and Moll [7] and studied in [8,9]. Their mean field model is an extension of mean field game theory involving kinetic equations because the interaction between agents is represented by collisions. The equilibrium solution of such a mean field model is the solution of a Boltzmann equation for the agent distribution coupled to an HJB equation for the optimal strategy for the control problem. In their paper, Lucas and Moll study a special, invariant in time, class of solutions that generates constant growth rate for the overall economy, a Balanced Growth Path (BGP) solutions. They also explore numerically such solutions, but do not address questions about their existence, uniqueness and stability. A proof that BGP solutions of the mean field model without diffusion, under the condition that the initial distribution of knowledge has Pareto-tail, is presented in [9]. The core of the proof is a fixed point argument. The time dependent problem without diffusion is studied locally in time in [8].

The problem that I have studied has diffusion added to the original mean field system, in order to get a more realistic model for knowledge propagation. I have studied the existence of a BGP solution of the mean field learning model with diffusion in logarithmic variables. Such solution can be considered as a generalization of traveling wave solutions of a system of Fisher-KPP (Fisher- Kolmogorov, Petrovsky, Piskunov) type of equation and an HJB equation. There are similarities between the traveling wave solution of the Lucas-Model with diffusion and that of the Fisher-KPP equation, which makes the analysis interesting from a mathematical perspective. In [10] we prove that a traveling wave solution of the mean field learning model with diffusion exists and we have explored it numerically. The method that we use to construct traveling wave solutions is a variation of a well known method for the construction of traveling wave solutions of Fisher-KPP type equations [11]. The numerical results that we have obtained support and extend the theory as many features observed numerically can be proved analytically.

We have also studied analytically and numerically the forward-backward system, supplemented by an initial condition for the Fisher-KPP type of equation and a terminal condition for the HJB equation. We have proved existence of solutions for a fixed terminal time $T$. We have tested the stability of the traveling
wave numerically, by using perturbed traveling wave profiles as initial/terminal values and we found that as time increases the solution of the time dependent problem tends to the traveling wave profiles. This, however, remains to be proved mathematically.

2 Current research

2.1 The Time Dependent Model

I give first an overview of the mean field model for knowledge propagation, described in [7].

Consider a population of agents, in which each agent at time \( t \) has knowledge \( z(t) \geq 0 \). At any time \( t > 0 \) any agent can either produce or learn from the other agents in the economy. We denote by \( s(t, z) \in [0,1] \) the fraction of time that an agent with knowledge \( z(1) \geq 0 \) spends learning during a time interval \( [t, t + \Delta t] \), so that \((1 - s(t, z))\) is the fraction of time he/she spends producing during this time interval. The total production in the time interval \( t \) and \( t + \Delta t \) depends on both the time he/she spend producing and his production knowledge and is given by

\[
[1 - s(t, z)]z \Delta t. \tag{2.1}
\]

Agents in the economy can learn in two ways: by meeting other agents in the economy, with a higher production knowledge, or by experimenting.

In order to describe the first process, let us denote by \( \Phi(t, z) \) the fraction of agents that have knowledge less than or equal to \( z \) at time \( t > 0 \), and let \( \phi(t, z) = \Phi_z(t, z) \) be the corresponding density function. During the time interval \( [t, t + \Delta t] \) an agent \( A \) with production knowledge \( z(t) \) meets another agent from the economy with probability \( \alpha(s(t, z)) \Delta t \), where \( \alpha(s) : [0,1] \rightarrow [0,1] \) is a given concave function. Given that a meeting occurs, the probability that \( A \) meets an agent \( B \) with knowledge in the interval \((z', z' + \Delta z')\) is proportional to \( \phi(t, z') \Delta z' \). If the production knowledge of agent \( A \) is lower than the production knowledge of agent \( B \), then agent \( A \) updates his production knowledge to that of agent \( B \). Therefore the density \( \phi(t, z) \) satisfies the following nonlinear equation:

\[
\frac{\partial \phi(t, z)}{\partial t} = -\alpha(s(t, z))\phi(t, z) \int_{z}^{\infty} \phi(t, y)dy + \phi(t, z) \int_{0}^{z} \alpha(s(t, y))\phi(t, y)dy. \tag{2.2}
\]

An agent with knowledge \( z \) at time \( t \) chooses the search time \( s(t, z) \) so as to maximize the expected total production (value function) \( V(t, z) \), discounted in time:

\[
V(t, z) = \max_{s \in \mathcal{A}} \mathbb{E} \{ \int_{t}^{T} e^{-\rho(t-\tau)} z(\tau) [1 - s(\tau, z(\tau))] + V_T(z) \} \tag{2.3} \]

Here \( \rho > 0 \) is a discount parameter and \( \mathcal{A} \) is the set of admissible control functions, \( T > 0 \) is a given terminal time, and \( V_T(z) \) is a prescribed terminal value. The value function \( V(t, z) \) satisfies a Hamilton-Jacobi-Bellman equation:

\[
\rho V(t, z) = \frac{\partial V(t, z)}{\partial t} + \sup_{s \in [0,1]} \left( (1 - s)z + \alpha(s) \int_{z}^{\infty} [V(t, y) - V(t, z)] \phi(t, y)dy \right), \tag{2.4}
\]

with given terminal condition \( V(T, z) = V_T(z) \)

We transform the system (2.2-2.4) by first going to logarithmic variables in \( z \), and second by adding diffusion to both equations to model learning by experimenting. After a sequence of algebraic manipulations, described in [10], we obtain the following system of equations:

\[
\frac{\partial \Psi(t, x)}{\partial t} - \kappa \frac{\partial^2 \Psi(t, x)}{\partial x^2} = -\Psi(t, x) \int_{-\infty}^{x} \alpha(s^*(t, y)) \Psi_y(t, y)dy. \tag{2.5}
\]

Here \( \Psi \) is 1 minus the distribution of knowledge function in logarithmic variables, and \( s^*(t, x) \) denotes the optimal search time in (2.6), and

\[
\rho V(t, x) = \frac{\partial V(t, x)}{\partial t} + \kappa \frac{\partial^2 V(t, x)}{\partial x^2} + \max_{s \in [0,1]} \left( (1 - s)e^x + \alpha(s) \int_{x}^{\infty} [V(t, y) - V(t, x)](-\Psi_y(t, y))dy \right). \tag{2.6}
\]
In my research I study the system (2.5) - (2.6). I prove existence of traveling waves solutions of this system. I also study the finite horizon problem, supplemented by an initial condition for $\Psi(0, x) = \Psi_0(x)$ and terminal condition for $V(T, x) = V_T(x)$. In Section 2.2 I state a theorem for existence of traveling wave solutions of the system (2.5) - (2.6) and give a sketch of the proof. In Section 2.3 I state and motivate the choice of initial condition for $\Psi(0, x) = \Psi_0(x)$ and terminal condition for $V(T, x) = V_T(x)$ and I sketch the proof for existence of solution of the time dependent problem with this particular choice of initial/terminal conditions.

2.2 Existence of traveling wave solutions

In [10] we prove existence of a traveling wave solution for a specific choice of search function $\alpha(s) = \alpha\sqrt{s}$. Such a solution of the mean field system correspond to a solution of the following system over the whole real line:

\[-cF_x(x) - \kappa F_{xx}(x) = \alpha F(x) \int_{-\infty}^{x} s^*(y)(-F_y(y))dy,\]
\[\rho Q(x) = cQ(x) - c\frac{\partial Q(x)}{\partial x} + \kappa Q(x) + e^x H(R(x)),\]

\[R(x) = \frac{\alpha}{2} e^{-x} \int_{x}^{\infty} [Q(y) - Q(x)](-F_y(y))dy.\]

with $s^*(x) = \min[1, R(x)]$, and

\[H(r) = \begin{cases} 
1 + r^2 & \text{for } 0 \leq r \leq 1, \\
2r & \text{for } 1 \leq r, 
\end{cases}\]

and with boundary conditions

\[\lim_{x \to -\infty} F(x) = 1, \quad \lim_{x \to \infty} F(x) = 0,
\]
\[\lim_{x \to \infty} Q(x) \text{ exists and is positive,}
\]
\[\lim_{x \to +\infty} (Q(x)e^{-x}) = \frac{1}{\rho - \kappa}.\]

The main result proved in [10] is the following theorem

**Theorem 2.2.1.** There exists $\rho_0$ that depends only on $\kappa$ and $\alpha$ so that if $\rho > \rho_0$, then there exists $0 < c \leq 2\sqrt{\kappa \alpha}$ so that the system (2.7) has a solution such that $F(x)$ is monotonically decreasing, $Q(x)$ is monotonically increasing, and the boundary conditions (2.9) are satisfied. In addition, we have

\[\int_{-\infty}^{\infty} |F_x|^2 dx < +\infty,\]

and there exist constants $A_{1,2} > 0, B > 0$ such that

\[A_1 e^x \leq Q(x) \leq A_2 e^x + B,\]

and $x_0 \in \mathbb{R}$ so that for all $x > x_0$ $R(x) = s^*(x) < 1$ and for all $x < x_0$, $R(x) > 1$ and $s^*(x) = 1$.

**Sketch of the proof**

The proof of theorem 2.2.1 in [10] is in two steps. In the first step we construct a solution to an approximation of the problem on a finite interval $[-a, a]$ for $a$ sufficiently big, and with the additional normalization $F(0) = 1/2$. To show that a solution to the approximate problem exists we use a priori bounds on the solutions and then apply a degree argument. In the second step, we pass to the limit along a sequence $a_n \to \infty$ and, using the a priori bounds on the finite intervals, we show that the sequence of approximate solutions converge uniformly on compact sets to a traveling wave solution of the mean filed system (2.7) that satisfies the boundary conditions (2.9). The method is similar to the one used for constructing traveling wave solutions of Fisher-KPP type equations in [11].
2.3 Time dependant problem

The traveling wave solution of the mean field learning system represents a special type of solution that is essentially independent of time. An interesting question is if solutions exist for the finite time horizon problem, where the system defined in (2.5) - (2.6) is supplemented by an initial condition for \( \Psi(0, x) = \Psi_0(x) \) and a terminal condition for \( V(T, x) = V_T(x) \), and on how the existence of a solution depends on the choice of \( \Psi_0(x) \) and \( V_T(x) \). In [9] it is proved that solutions for the problem without diffusion exist locally in time using a fixed point argument.

We extend the idea used to prove existence of traveling wave solutions to prove existence of solutions for the finite time horizon. The structure of the argument repeats that of the proof in [10]. First, we consider an approximate problem on a domain \([0, T] \times [-a, a]\). We obtain a priori bounds on the solutions in order to be able to apply a Leray-Schauder degree argument to prove existence of a solution \((\Phi_a, V_a)\) to the approximate problem. In the second step we take the limit along a sequence \(a_n \to \infty\) and, using the a priori bounds on the finite intervals, we show that the sequence of approximate solutions converge uniformly on compact sets to a solution on \(\mathbb{R} \times [0, T]\).

The needed a priori bounds depend not only on the parameters in the problem, but also on the initial and terminal conditions. This introduces an extra level of complexity to the problem, as it is not clear what the “right” choice of initial/terminal conditions should be. In [9] it is assumed that \(V_T(x)\) is identically 0. We use a different approach and we chose initial/terminal conditions that are close to the traveling wave profiles. We assume that \(\Psi_0(x) : [-a, a] \rightarrow [0, 1]\) is decreasing and decays faster than \(e^{-\sqrt{\alpha}x}\), and that

\[
\Psi(0, x) = \Psi_0(x), \quad V^\alpha(T, x) = \min\{L_T, e^x\} \quad \text{for} \quad L_T > 0, \tag{2.12}
\]

where \(L_T\) is chosen such that \(\Psi(T, L_T) = 1/2\).

We also impose the following boundary conditions:

\[
\lim_{x \to -\infty} \Psi(t, x) = 1 \quad \text{and} \quad \lim_{x \to \infty} \Psi(t, x) = 0 \quad \text{for all} \quad t \in [0, T] \tag{2.13}
\]

and

\[
\lim_{x \to -\infty} V(t, x) \quad \text{exists and it is positive, and} \tag{2.14}
\]

\[
\lim_{x \to \infty} (V(t, x)e^{-x}) = \frac{1}{\rho - 1} \quad \text{for all} \quad t \in [0, T]. \tag{2.15}
\]

The main result, in a paper that is still in preparation, is the following theorem

**Theorem 2.3.1.** There exists \(\rho_0\) that depends only on \(\kappa\) and \(\alpha\) so that if \(\rho > \rho_0\), then there exists \(0 < L_T \leq 2\sqrt{\kappa\alpha}T\) so that the system (2.5) - (2.6) with initial/terminal condition as in (2.12), has a solution such that \(\Psi(t, x)\) is monotonically decreasing in \(x\), \(V(t, x)\) is monotonically increasing in \(x\), and the boundary conditions (2.13-2.15) are satisfied. In addition, the solution satisfy the following bounds

\[
\Psi(t, x) \leq e^{2\alpha t - \sqrt{\alpha}x} \tag{2.16}
\]

and there exist constants \(A_{1,2} > 0, B(t) > 0\) such that

\[
A_1e^x \leq V(t, x) \leq A_2e^x + B(t). \tag{2.17}
\]

2.4 Numerical Results

In [10] we explore numerically the solution of the approximation of the problem defined in (2.7) on a finite interval \([-a, a]\) with boundary conditions

\[
F^a(-a) = 1, \quad F^a(a) = 0 \tag{2.18}
\]
and

\[ V_x^a(-a) = 0 \quad \text{and} \quad V_x^a(a) = V^a(a). \]  

(2.19)

and with additional normalization \( F^a(0) = 1/2 \). We solve a rescaled problem numerically using an iterative finite difference algorithm. We start with an initial guess for \( F^a(x) \) and \( V^a(x) \) and compute \( R^a(x) \) using the equation in (2.7). Given \( R^a(x) \), we compute \( s^a(x) = \min[1, R^a(x)] \) and we update the values of \( F^a(x) \) and \( c^a \), the numerical speed. Then we update the value of \( Q^a(x) \). We repeat until the numerical speed \( c^a \) converges.

The boundary condition (2.19) suggests that \( Q^a(x) \) increases exponentially on the right and will tend to a constant on the left, and the way \( R^a(x) \) is defined in (2.7) suggests that \( R^a(x) \) will increase exponentially on the left. In order to ensure that all functions stay bounded in the numerical calculations, we rescale both \( Q^a(x) \) and \( R^a(x) \). The details of the rescaling are given in [10].

As the equation for \( F^a \) is nonlinear, we use an additional iterative finite difference scheme to solve it. For better accuracy, we also relax the boundary condition by using the solution of the linearized problem

\[ -c^a F_x^a - \kappa F_x^a = \alpha F^a. \]  

(2.20)

The boundary conditions change at each step of the iterative algorithm to reflect the change in \( c^a \). The additional normalization for \( F^a(0) = 1/2 \) is used to determine uniquely the numerical speed on the finite domain.

We find that the numerical results support quite well the theoretical results described in Section 2.2. We also observe that \( c^a, F^a, Q^a \), and \( R^a \) tend to converge as we increase \( a \). Moreover, they satisfy the bounds in Theorem 2.2.1.

We have also explored numerically the solution of the approximation of the problem defined in (2.5-2.6), supplemented by an initial condition for \( \Psi^a(0, x) = F^a(x) \), where \( F^a(x) \) is the numerical solution for the traveling wave problem on \([-a,a]\) and terminal condition for \( V(T, x) = V_T(x) \) is as in (2.12). As before, we use an iterative finite difference scheme and rescaling of \( V(t, x) \) and \( R(t, x) \) similar to the one used in [10]. We again use a relaxed boundary condition for \( \Psi(t, x) \), similar to the one in 2.20.

Finally, we have tested numerically the stability of the traveling wave. We have designed an iterative finite difference scheme in a moving frame, similar to the one used to solve the time dependent problem where the speed is modeled as a function of time that satisfies an ordinary differential equation obtained from \( \Psi(t, 0) = 1/2 \) for all \( t \in [0,T] \). We use perturbed traveling wave profiles as initial/terminal values for the distribution function and for the value function. We find that with increasing time the solution of the time dependent problem converges to the traveling wave profiles.

### 3 Summary and Future work

During my PhD I have studied an extension of a mean field model for knowledge propagation, proposed by [7] and described in Section 2.1. An equilibrium solution of the mean field model is a solution of the coupled system (2.5) - (2.6). I have studied the existence of traveling wave solutions of the system. Such solutions can be expressed as solutions to the system (2.7) with boundary condition as in (2.9). In [10] and as explained in Section 2.2 we prove that traveling wave solution of the mean field system exists by first solving an approximation of the problem on a finite domain and then constructing the solution on \( \mathbb{R} \) as the limit of a sequence of the approximate solutions. In a paper still in preparation and as explained in Section 2.3 we prove the existence of a solution of the time dependent problem (2.5) - (2.6) on a time interval \([0,T]\), supplemented by initial/terminal conditions as in (2.12) and boundary conditions (2.13-2.15), using an argument similar to the one for the proof of existence of traveling waves. Finally, we have explored numerically the traveling wave problem as well as the stability of the traveling wave.

In the future I will be interested in extending my studies of mean field models in the following directions.

**Asymptotic behavior of the time dependence problem and stability of the traveling wave**

I am interested in studying the long time behavior of the time dependent system defined in (2.5 - 2.6). I would like to address the question of convergence of the solution of the time dependent problem to the traveling wave solution as time goes to infinity, for different choices of initial/terminal conditions. Other interesting questions to consider are how the convergence depends on the initial/terminal conditions and if there is a limit for the level set locations for \( \Psi \).
Generalizations of the mean field learning model

In my research I have made a specific choice of search function $\alpha(s) = \alpha \sqrt{s}$. I am interested to see if the argument used to prove existence of traveling wave solutions can be applied for general $\alpha(s)$ and to study how the behavior of the traveling wave profiles depends on the choice of such a search function. This last issue is addressed numerically for the model without diffusion in [7] for $\alpha(s) = \alpha s^\eta$, for $\eta \in [0, 1]$. I am interested in first exploring numerically the traveling wave solutions for the mean field learning model with diffusion and general search function and then studying it analytically.

Traveling wave solutions in an Optimally Planned Economy

In [7] Lucas and Moll propose another very interesting extension of the mean field learning model. They introduce the total production of the economy at time $t \geq 0$ as the integral of the production of each agent. The total production of the economy can be computed as follows:

$$\int_0^\infty (1 - s(t, z))z\phi(t, z)dz$$

(3.1)

where $s(t, z)$ and $\phi(t, z)$ are as in Section 2.1. They observe that in equilibrium each agent will construct their strategy so that they maximize their own value function, but this will not necessary maximize the total, discounted in time, value of the economy, so they ask the question: what search function $\sigma(t, z)$ would be optimal for the economy as a whole. Such a question gives raise to a dynamic programming problem in which the state variable is a distribution. The problem is to find $\sigma : \mathbb{R}^2_+ \to [0, 1]$, such that the expected total production of the economy (the value function of the economy) $W[\phi(t, \cdot)]$ discounted in time is maximized:

$$W[\phi(t, \cdot)] = \max_{\sigma(\cdot, \cdot)} \int_t^\infty e^{-\rho(t-t)} \int_0^\infty (1 - \sigma(\tau, z))z\phi(\tau, z)dzd\tau.$$  

(3.2)

subject to a law of motion for $\phi$ as in (2.2), and with $\phi(t, \cdot)$ given.

In [7] they show how such a dynamic programming problem, in which the state variable is a distribution, can be brought into a manageable form - a coupled system of an HJB equation for the value function of the economy and a Boltzmann type equation for the distribution of the agents. This system is similar to that of a mean field problem, except that the HJB equation for the optimal planned economy is different. Lucas and Moll study numerically the BGP solutions of the optimally planned economy and compare them with the BGP solutions of the mean field learning model. I am interested in studying this model mathematically and in addressing the existence of traveling wave solutions for the optimally planned economy, with and without diffusion. It would be interesting to compare analytically the optimal strategy for the economy $\sigma(t, z)$ with $s(t, z)$ - the strategy that comes out from the solution of the control problem as described in (2.3).

The objective of a central planner in the economy is to induce each agent to follow the optimal strategy $\sigma(t, z)$. As none of the agents will deviate from their optimal strategies - $s(t, z)$, the question for the central planner is how to change the value functions of each of the agents so that $s(t, z) = \sigma(t, z)$. Lucas and Moll propose to do this by redistributing "value" - imposing a flat tax $\tau_0$ on agents with high productivity knowledge and giving a subsidy $\tau(t, z)$ to people with low productivity knowledge, under the budget constrain that the total tax collected from the first group of agents is equal to the total subsidy given to the second. The problem is to find $\tau_0$ and $\tau(t, z)$. In [7] Lucas and Moll do this for the BGP systems numerically. This is much harder problem both analytically and numerically that I want to explore.

Last, but not least, I am interested in expanding my knowledge and learning other analytical tools. Directions in which I will be happy to work include analysis of general mean field models, the study of classical problems in fluid dynamics, analysis of reaction-diffusion equations, and so on.

References


