Traveling waves in a mean field learning model
Preliminary draft

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Abstract
In this paper we prove existence of traveling wave solutions of mean field system for knowledge diffusion and economics growth. We solve the problem numerically and observe that the numerical results support and expend the theory.

1 Introduction

In this paper we study an extension a mean field game model for knowledge propagation and economic growth, proposed by Lucas and Moll [1]. Mean field models are used in optimal decision making based on stochastic games with a very large population of agents that are statistically the same [2–7]. In such models the overall effect of the other agents on a single one can be replaced by an averaged effect, hence the optimal behavior of a single agent can be determined as a solution of an optimal control problem that depends on the distribution of the other agents. The model is determined by two partial differential equations - a forward Kolmogorov equation that keeps track of the distribution of agents, and a backward Hamilton Jacobi Bellman (HJB) equation for the value function of the optimal stochastic control problem for each agent. An equilibrium solution corresponds to a solution of the coupled system of the two equations.

In the model proposed in [1] the propagation of knowledge is a Markovian processes that involves jumps, but no diffusion and the economic growth is modeled by a production function, that depend on the knowledge of each agent in the economy. The mean field model can be described by a system of coupled forward Kolmogorov equation for the propagation of knowledge, and backward Hamilton Jacobi Bellman equation for the value function of an individual in the economy [8], [1]. The two equations are coupled through a search function, representing the tradeoff between the time spent learning and the time spent producing.

A Balance Growth Path (BGP) solution of the mean field system of the forward - backward equations are special solutions that are valid for all time and admit an interpretation that corresponds to a constant growth rate in the economy in equilibrium. In their paper Lucas and Moll not only propose the interesting model for propagation of knowledge and economic growth, but also study numerically the BGP solutions of the system [1]. Such solutions are proved to exist in [9] and [10] using a fixed point method in function space.

In this paper we study a model, similar to the one introduced in [1], but in logarithmic variable and with diffusion added to the process modeling the propagation of knowledge, introducing an
additional level of uncertainty. Diffusion was also added in [10] in the original variables, where BGP solutions are constructed numerically. We choose to add diffusion after the change of variables, because the equation for propagation of knowledge looks like Fisher-KPP equation and traveling solutions of the system can be constructed in a similar way as traveling wave solution for the Fisher-KPP equation.

A Balance Growth Path (BGP) solutions of the mean field system with diffusion in logarithmic variables correspond to traveling wave solutions of (2.12-2.11). The main result of this paper is a proof of existence of such solution for for a specific choice of search function \( \alpha(s) = \alpha \sqrt{s} \). A traveling wave satisfies the following system:

\[
- cF_x - \kappa F_{xx} = \alpha F(x) \int_{-\infty}^{x} s^*(y)(-F_y(y))dy,
\]

\[
\rho Q(x) = cQ(x) - c \frac{\partial Q(x)}{\partial x} + \kappa \frac{\partial^2 Q(x)}{\partial x^2} + e^x H(R(x)),
\]

\[
R(x) = \frac{\alpha}{2} e^{-x} \int_{x}^{\infty} [Q(y) - Q(x)](-F_y(y))dy.
\]  

with \( s^*(x) = \min[1, R(x)] \), and with boundary conditions

\[
\lim_{x \to -\infty} F(x) = 1, \lim_{x \to \infty} F(x) = 0,
\]

\[
\lim_{x \to -\infty} Q(x) \quad \exists \text{ and is positive},
\]

\[
\lim_{x \to +\infty} (Q(x)e^{-x}) = \frac{1}{\rho - \kappa}.
\]  

The main result of this paper is the following theorem.

**Theorem 2.1.** There exists \( \rho_0 \) that depends only on \( \kappa \) and \( \alpha \) so that if \( \rho > \rho_0 \), then there exists \( 0 < c \leq 2\sqrt{\kappa \alpha} \) so that the system (2.31) has a solution such that \( F(x) \) is monotonically decreasing, \( Q(x) \) is monotonically increasing, and the boundary conditions (2.32) are satisfied. In addition, we have

\[
\int_{-\infty}^{\infty} |F_x|^2 dx < +\infty,
\]  

and there exist constants \( A_{1,2} > 0, B > 0 \) such that

\[
A_1 e^x \leq Q(x) \leq A_2 e^x + B,
\]  

and \( x_0 \in \mathbb{R} \) so that \( R(x) < 1 \) and \( s^*(x) < 1 \) for all \( x > x_0 \).

In Section 2 we review the mean field learning model, described in [1] and we formulate the mean filed system in logarithmic variable with diffusion added after the change of variables in (2.12-2.11). We study the existence of traveling wave solutions of this system for a specific choice of a search function \( \alpha(s) = \alpha \sqrt{s} \). Such solution correspond to a solution of the system defined in (1.1) with boundary condition (1.2).
In Section 3 we prove the theorem in two steps: first, we consider an approximate problem on a finite interval $[-a,a]$, for sufficiently big $a$ and with normalization $F^a(0) = 1/2$. We show that a solution $(F^a, Q^a, c^a)$ to the approximate problem exists by obtaining a priori bounds on the solutions and by using a degree argument. Next, using the a priori bounds on the solutions of the approximate problem on a finite intervals, we pass to the limit along a subsequence $a_n \to +\infty$ and show that $(F^{a_n}, Q^{a_n}, c^{a_n})$ converge uniformly on compact sets to a solution $(F, Q, c)$ to (2.31), and that the boundary conditions (2.32) are also satisfied by the functions $F$ and $Q$.

In Section 4 we give an iterative finite difference numerical algorithm solving the restricted problem in finite interval $[-a,a]$ and we present the numerical solution of the system for different values of $a$. As we expect exponential growth of the value function we introduce a rescaling in section 4.1. The numerical simulation show clearly the validity of the result and clarify the dependence of the solutions on the various parameters that enter the problem. The speed of the traveling wave is determined using the normalization $F(0) = \frac{1}{2}$. Different iterative numerical algorithms for BGP solutions of the mean field model are given in [1], [9] and [10]. The speed there is determined by using a relation between the speed and the solution, similar to the one given in Proposition 2.2.

An interesting open question is the existence of the traveling wave solution of the mean field learning model for general choice of search function. Another interesting question is the stability of the forward - backward mean field problem.

2 The mean field learning model

In this section we recall the basics of the mean field learning model introduced in [1]. In addition, we reformulate the model in the logarithmic variables and add diffusion in the knowledge space. We also define the notion of traveling wave solutions that correspond to what is known as the balanced growth paths in the original variables, and are an important class of solutions to the infinite time horizon problem.

2.1 The non-diffusive model

Consider a population of agents, such that each agent has a certain knowledge $z \geq 0$ at a given time $t \geq 0$. An agent can either produce or learn at each moment of time, and we denote by $s(t, z) \in [0, 1]$ the fraction of time an agent with knowledge $z \geq 0$ spends learning on a time interval $[t, t + \Delta t]$, so that $(1 - s(t, z))$ is the fraction of time he spends producing on this time interval. His total production between the times $t$ and $t + \Delta t$ is then

$$[1 - s(t, z)] z \Delta t. \quad (2.1)$$

Agents in the economy learn by meeting other agents, with a higher production knowledge. In order to describe the meetings, let $\Phi(t, z)$ be the fraction of the agents with knowledge less or equal to $z$ at time $t$, and let $\phi(t, z) = \Phi_z(t, z)$ be the corresponding density. The probability that the search by an agent $A$ with knowledge $z \geq 0$ is successful on a time interval $(t, t + \Delta t)$ is $\alpha(s(t, z)) \Delta t$, where $\alpha(s) : [0, 1] \to [0, 1]$ is a given concave function. Given that the search is successful, the probability that $A$ encounters an agent $B$ with knowledge in the interval $(z', z' + \Delta z')$ is proportional
to $\phi(t, z') \Delta z'$ – this is the mean field nature of the model. If the production knowledge of agent $A$ is lower than the production knowledge of agent $B$, then agent $A$ updates his production knowledge to that of agent $B$. The overall balance leads to the following nonlinear kinetic equation for the density $\phi(t, z)$:

$$
\frac{\partial \phi(t, z)}{\partial t} = -\alpha(s(t, z)) \phi(t, z) \int_0^\infty \phi(t, y) dy + \phi(t, z) \int_0^z \alpha(s(t, y)) \phi(t, y) dy.
$$

(2.2)

An agent with knowledge $z$ at time $t$ chooses the search time $s(t, z)$, so as to maximize the expected total production (value function) $V(t, z)$, discounted in time:

$$
V(t, z) = \max_{s \in \mathcal{A}} \mathbb{E} \left\{ \int_t^T e^{-\rho(\tau-t)} z(\tau)[1 - s(\tau, z(\tau))] + V_T(z)|z(t) = z \right\}.
$$

(2.3)

Here $\rho > 0$ is a discount parameter and $\mathcal{A}$ is the set of admissible control functions, $T > 0$ is a given terminal time, and $V_T(z)$ is a prescribed terminal value. The value function $V(t, z)$ satisfies a Hamilton-Jacobi-Bellman equation:

$$
\rho V(t, z) = \frac{\partial V(t, z)}{\partial t} + \sup_{s \in [0, 1]} \left\{ (1 - s) e^{x} + \alpha(s) \int_x^\infty [V(t, y) - V(t, z)] \psi(t, y) dy \right\},
$$

(2.4)

supplemented by the terminal condition $V(T, z) = V_T(z)$. We will denote by $s^*(t, x)$ the optimal control in (2.4). An informal derivation of (2.2) and (2.4) is given in [1].

In the economics context, especially since we are soon going to introduce the diffusion of knowledge, it is natural to consider an exponential change of variables $\phi(t, z) = \psi(t, \log z)/z$, so that the function $\psi(t, x)$ is also a density but in the logarithmic variables

$$
1 = \int_0^\infty \phi(t, z) dz = \int_0^\infty \psi(t, \log z) \frac{dz}{z} = \int_{-\infty}^\infty \psi(t, x) dx.
$$

(2.5)

This change of variables transforms (2.2), (2.4) into the following system:

$$
\frac{\partial \psi(t, x)}{\partial t} = \psi(t, x) \int_{-\infty}^x \alpha(s^*(t, y)) \psi(t, y) dy - \alpha(s^*(t, x)) \psi(t, x) \int_x^\infty \psi(t, y) dy.
$$

(2.6)

$$
\rho \psi(t, x) = \frac{\partial \psi(t, x)}{\partial t} + \max_{s \in [0, 1]} \left\{ (1 - s) e^{x} + \alpha(s) \int_x^\infty [V(t, y) - V(t, x)] \psi(t, y) dy \right\},
$$

(2.7)

with the initial condition $\psi(0, x) = \phi(0, e^x) e^x$, and the terminal condition $V(T, x) = V_T(e^x)$.

### 2.2 Adding diffusion

Equations (2.6)-(2.7) assume that the only changes in the productivity of the agents come from their interactions. It is reasonable from the economics point of view to assume that even in the absence of such interactions the productivity of each agent undergoes some diffusion, so that the agents learn not only from each other but also through experimenting, and it is natural to do that
in the logarithmic variables, as in (2.6)-(2.7). Adding diffusion to both equations transforms the system to

\[
\frac{\partial \psi(t,x)}{\partial t} = \kappa \frac{\partial^2 \psi}{\partial x^2} + \psi(t,x) \int_{-\infty}^{x} \alpha(s^*(t,y))\psi(t,y)dy - \alpha(s^*(t,x))\psi(t,x) \int_{x}^{\infty} \psi(t,y)dy \tag{2.8}
\]

and

\[
\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa \frac{\partial^2 V(t,x)}{\partial x^2} + \max_{s \in [0,1]} \left[(1-s)e^x + \alpha(s) \int_{x}^{\infty} [V(t,y) - V(t,x)]\psi(t,y)dy\right]. \tag{2.9}
\]

Note that we have added the diffusion after the exponential change of variables. In [10], diffusion was added before the change of variables and numerical solutions of the system are presented for a specific choice of search function. We choose to add the diffusion after the change of variables because from the economics point of view, the diffusion of the relative quantities is more natural.

We will also make use of the cumulative distribution function

\[
\Psi(t,x) = \int_{x}^{\infty} \psi(t,y)dy, \quad \Psi(-\infty) = 1, \quad \Psi(+\infty) = 0. \tag{2.10}
\]

A straightforward computation shows that \(\Psi(t,x)\) satisfies the following integro-differential equation:

\[
\frac{\partial \Psi}{\partial t} - \kappa \frac{\partial^2 \Psi}{\partial x^2} = -\Psi(t,x) \int_{-\infty}^{x} \alpha(s^*(t,y))\psi_y(t,y)dy. \tag{2.11}
\]

The equation for the value function \(V(t,x)\) in terms of \(\Psi(t,x)\) is

\[
\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa \frac{\partial^2 V(t,x)}{\partial x^2} + \max_{s \in [0,1]} \left[(1-s)e^x + \alpha(s) \int_{x}^{\infty} [V(t,y) - V(t,x)](-\Psi_y(t,y))dy\right]. \tag{2.12}
\]

Equation (2.11) should be supplemented by an initial condition for \(\Psi(0,x)\) and (2.12) should come with a terminal condition for \(V(T,x) = V_T(x)\) at some \(T > 0\). Existence of the solutions of the resulting forward-backward in time problem will be discussed elsewhere [11]. One natural terminal condition is \(V_T(x) = 0\), as there is no time left to produce at the end. This, however, is not the only possibility as one could also try to choose \(V_T(x)\) so as to approximate the solution to the infinite time horizon problem with \(T = +\infty\), so that \(V(t,x)\) in (2.3) is re-defined as

\[
V(t,z) = \max_{s \in \mathcal{A}} \mathbb{E}\left\{ \int_{t}^{\infty} e^{-\rho(\tau-t)}z(\tau)[1 - (\tau, z(\tau))]|z(t) = z \right\}. \tag{2.13}
\]

A very interesting question, to be addressed elsewhere, is whether the pair of solutions \(\Psi_T(t,x), V_T(t,x)\) defined for \(0 \leq t \leq T\), with some prescribed terminal conditions, have a well-defined limit \(\Psi(t,x), V(t,x)\) as \(T \to +\infty\). This would be a natural candidate for a "correct" solution to the infinite horizon problem, without an explicit terminal condition for \(V(t,x)\).

Let us also note that in the special case when \(\alpha(s)\) is a constant, the system (2.11)-(2.12) decouples, and (2.11) becomes the classical Fisher-KPP equation

\[
\frac{\partial \Psi}{\partial t} = \kappa \frac{\partial^2 \Psi}{\partial x^2} + \alpha \Psi(1 - \Psi). \tag{2.14}
\]
Its solutions in the long time limit converge to traveling waves moving with the speed \( c_\ast = 2\sqrt{\kappa\alpha} \). This direct analogy to the Fisher-KPP type problems works only in the special case when \( \alpha(s) \) is constant. However, in general, one still expects that, as in the FKPP case, the long time behavior of the solutions to (2.11) is governed to the leading order by the linearization as \( x \to +\infty \):

\[
\frac{\partial \tilde{\Psi}}{\partial t} - \kappa \frac{\partial^2 \tilde{\Psi}}{\partial x^2} = R(t) \tilde{\Psi}(t,x), \quad R(t) = \int_{-\infty}^{\infty} \alpha(s^\ast(t,y)) \Psi_y(t,y) dy, \tag{2.15}
\]

Note that, unlike in the true FKPP case, the linearized equation (2.15) is not closed in general as the rate \( R(t) \) depends on the function \( V(t,y) \) as well. Nevertheless, it is natural to conjecture that solutions to the full problem still belong to the so called class of pulled fronts [12], and significant intuition can be gained from the Fisher-KPP analogy.

### 2.3 The choice of the search function

The maximization problem in (2.12) is of the form

\[
\max_{s \in [0,1]} \left[ (1 - s) + B\alpha(s) \right], \tag{2.16}
\]

with

\[
B = e^{-x} \int_x^{\infty} [V(t,y) - V(t,x)](-\Psi_y(t,y)) dy, \tag{2.17}
\]

so that the optimal \( s \) is given by

\[
s^\ast = s^\ast(B) = \begin{cases} 0, & B \leq \frac{1}{\alpha'(0)}, \\ \beta \left(\frac{1}{B}\right), & \frac{1}{\alpha'(0)} < B \leq \frac{1}{\alpha'(1)}, \\ 1, & B > \frac{1}{\alpha'(1)}, \end{cases} \tag{2.18}
\]

where \( \beta = (\alpha')^{-1} \). In order to avoid the situation where agents of sufficiently advanced knowledge do not search at all, it is natural to assume that \( \alpha'(0) = +\infty \). To simplify some computations, we will make an assumption that \( \alpha(s) = \alpha \sqrt{s} \) with some \( \alpha > 0 \). Generalizations of our results to a general concave function \( \alpha(s) : [0,1] \to [0,1] \) with \( \alpha'(0) = +\infty \) are quite straightforward. Now, equations (2.12) and (2.11) become

\[
\rho V(t,x) = \kappa \frac{\partial V(t,x)}{\partial t} + \kappa \frac{\partial^2 V(t,x)}{\partial x^2} + e^x \max_{s \in [0,1]} \left[ (1 - s^2) + \alpha s e^{-x} \int_x^{\infty} [V(t,y) - V(t,x)](-\Psi_y(t,y)) dy \right], \tag{2.19}
\]

and

\[
\frac{\partial \Psi}{\partial t} - \kappa \frac{\partial^2 \Psi}{\partial x^2} = -\alpha \Psi(t,x) \int_{-\infty}^{x} s^\ast(t,y) \Psi_y(t,y) dy, \tag{2.20}
\]
To simplify (2.19), we introduce the auxiliary functions
\[
r(t,x) = \alpha^2 e^{-x} \int_x^\infty [V(t,y) - V(t,x)][(-\Psi_y(t,y))dy],
\]
(2.21)
\[
H(r(t,x)) = \max_{s \in [0,1]} [(1 - s^2) + \alpha s e^{-x} \int_x^\infty [V(t,y) - V(t,x)]\psi(t,y)dy]
\]
(2.22)
\[
= \max_{s \in [0,1]} [(1 - s^2) + 2sr(t,x)],
\]
so that \(H(r)\) and the maximizer \(S^*(r)\) are given by
\[
H(r) = \begin{cases} 2r, & r > 1, \\ 1 + r^2, & 0 < r < 1, \\ 1, & r < 0. \end{cases} \quad S^*(r) = \begin{cases} 1, & r > 1, \\ r, & 0 < r < 1, \\ 0, & r < 0. \end{cases}
\]
(2.23)
Now, we can write (2.19) - (2.20) as
\[
\partial_t \Psi - \kappa \partial_x^2 \Psi = \alpha \Psi(t,x) \int_{-\infty}^x S^*(r(t,y))(-\Psi_y(t,y))dy,
\]
(2.24)
and
\[
\rho V(t,x) = \frac{\partial V(t,x)}{\partial t} + \kappa \frac{\partial^2 V(t,x)}{\partial x^2} + e^x H(r(t,x)).
\]
(2.25)
Thus, the new formulation of the problem are equations (2.24)-(2.25) for \(\Psi(t,x)\) and \(V(t,x)\), with the function \(r(t,x)\) defined by (2.21), and \(H(r)\) and \(S^*(r)\) given by (2.23).

### 2.4 The traveling wave solutions

The infinite time horizon problem has special solutions that in the original variables are known as the balanced growths path (BGP). These are solutions to (2.2), (2.4) of the form
\[
\phi(t,z) = e^{-\gamma t} f(ze^{-\gamma t}), \quad V(t,z) = e^{\gamma t} v(ze^{-\gamma t}), \quad s(t,z) = \sigma(ze^{-\gamma t}),
\]
(2.26)
with some \(\gamma > 0\), and \(f(x), v(x) \in C^1(\mathbb{R})\) and \(\sigma(x) \in C(\mathbb{R})\). The BGP solutions are interesting from the economics point of view since they give a constant growth rate for the economy, but they also give a well-defined solution to the infinite time horizon problem, and it is natural to conjecture, from the numerical evidence, that they should be the long time limit of the finite horizon problems on a time interval \([0,T]\) as \(T \to +\infty\), with a proper terminal condition \(V_T(x)\). This is similar to the stability of the Fisher-KPP traveling waves.

It has been shown in [9] that there exists \(\mu_0 > 0\) so that the BGP solutions with the asymptotics
\[
\phi(z) \sim z^{-\mu}, \text{ as } z \to +\infty,
\]
(2.27)
exist for all \(0 < \mu < \mu_0\), with a corresponding growth rate \(\gamma(\mu) \in (0, \rho)\). After the exponential change of variables, a BGP solution defined for \(z \geq 0\) transforms to a traveling wave solution for \(x \in \mathbb{R}\) that moves with a constant speed equal to the growth rate \(\gamma\):
\[
\psi(t,x) = e^{\gamma t} \phi(t,e^x) = e^{x-\gamma t} f(e^x-\gamma t) = \Psi(x-\gamma t).
\]
(2.28)
Traveling waves are solutions to the system (2.24)-(2.25) of the form
\[ \Psi(t, x) = F(x - ct), \quad V(t, x) = e^{ct}Q(x - ct), \quad r(t, x) = R(x - ct). \] (2.29)

They correspond to the balanced growth paths before the logarithmic change of variables. Note that if \( F(x), Q(x) \) and \( R(x) \) form a traveling wave, with the corresponding search function \( s^*(x) \), then for any fixed shift \( y \in \mathbb{R} \), the functions
\[
F_y(x) := F(x - y), \quad s^*_y(x) := s^*(x - y), \quad Q_y(x) := e^{cy}Q(x - y), \quad R_y(x) := R(x - y)
\] (2.30)
also form a traveling wave solution, so that traveling waves form a one parameter family, which is a typical situation in the theory of traveling waves. The only difference is that the value function \( Q(y) \) is transformed slightly different under a shift by \( y \).

A traveling wave satisfies the following system:
\[
-cF_x - \kappa F_{xx} = \alpha F(x) \int_{-\infty}^{x} s^*(y)(-F_y(y))dy,
\rho Q(x) = cQ(x) - c\frac{\partial Q(x)}{\partial x} + \kappa \frac{\partial^2 Q(x)}{\partial x^2} + e^x H(R(x)),
R(x) = \frac{\alpha}{2} e^{-x} \int_{x}^{\infty} [Q(y) - Q(x)](-F_y(y))dy.
\] (2.31)

with \( s^*(x) = \min[1, R(x)] \), and with boundary conditions
\[
\lim_{x \to -\infty} F(x) = 1, \quad \lim_{x \to \infty} F(x) = 0,
\lim_{x \to -\infty} Q(x) \quad \text{exists and is positive},
\lim_{x \to +\infty} (Q(x)e^{-x}) = \frac{1}{\rho - \kappa}.
\] (2.32)

The main result of this paper is the following theorem.

**Theorem 2.1.** There exists \( \rho_0 \) that depends only on \( \kappa \) and \( \alpha \) so that if \( \rho > \rho_0 \), then there exists \( 0 < c \leq 2\sqrt{\kappa \alpha} \) so that the system (2.31) has a solution such that \( F(x) \) is monotonically decreasing, \( Q(x) \) is monotonically increasing, and the boundary conditions (2.32) are satisfied. In addition, we have
\[
\int_{-\infty}^{\infty} |F_x|^2 dx < +\infty,
\] (2.33)
and there exist constants \( A_{1,2} > 0, B > 0 \) such that
\[
A_1 e^x \leq Q(x) \leq A_2 e^x + B,
\] (2.34)
and \( x_0 \in \mathbb{R} \) so that \( R(x) < 1 \) and \( s^*(x) < 1 \) for all \( x > x_0 \).

As has been conjectured in [8], the speed and the wave profile are related as follows.
**Proposition 2.2.** The speed \( c \), the search function \( s^*(x) \) and \( F(x) \) are related as follows:

\[
c = 2 \sqrt{\kappa \alpha} \int_{-\infty}^{\infty} s^*(y)(-F_y(y))dy. \tag{2.35}
\]

This relation is similar to the minimal traveling front speed formula for the Fisher-KPP equation

\[
F_t = \kappa F_{xx} + \gamma F(1 - F),
\]

which is \( \kappa = 2 \sqrt{\kappa \gamma} \). In our case, the role of \( \gamma \) is played by the integral under the square root in (2.35). This proposition is proved at the end of Section 3.

The proof of Theorem 2.1 is presented in Section 3. It proceeds in two steps: first, we consider an approximate problem on a finite interval \([-a, a]\), with \( a \gg 1 \), and an additional normalization \( F^a(0) = 1/2 \). We obtain a priori bounds on the solutions and use a degree argument to show that there exists a solution \((F^a, Q^a, c^a)\) to the approximate problem. In the second step, using the a priori bounds on the finite intervals, we pass to the limit along a subsequence \( a_n \to +\infty \) and show that \((F^{a_n}, Q^{a_n}, c^{a_n})\) converge uniformly on compact sets to a solution \((F, Q, c)\) to (2.31), and that the boundary conditions (2.32) are also satisfied by the functions \( F \) and \( Q \).

The upper bound on the speed \( c \leq 2 \sqrt{\kappa \alpha} \) in Theorem 2.1 is very natural: as we have pointed out \( c^* = 2 \sqrt{\kappa \alpha} \) is the speed of the Fisher-KPP wave that would correspond to \( s^*(x) \equiv 1 \), so that agents only search and do not produce at all. This strategy leads to the fastest growth rate of the production knowledge. However, this community of learners also leads to no production whatsoever. The optimal speed \( c \) is chosen so that the growth rate of knowledge is smaller than the Fisher-KPP speed but ensures a maximal production, as a balance between learning and producing. The condition that the discount rate \( \rho \) is not too small is also natural as we are considering solutions to the infinite horizon problem – this prevents the blow-up of the value function.

### 3 Existence of traveling wave solution

In this section, we prove Theorem 2.1.

#### 3.1 The finite interval problem

In the first step, we restrict the system (2.31) to a finite interval \([-a, a]\), with \( a > 0 \) and consider the following approximate problem for the functions \( F^a(x), Q^a(x) \) and \( R^a(x) \), and a speed \( c^a \):

\[
-c^a F^a_x - \kappa F^a_{xx} = \alpha F^a(x) \int_{-a}^{x} s^a(y)(-F^a_y(y))dy, \tag{3.1}
\]

\[
\rho Q^a(x) = c^a Q^a_x - c^a \frac{\partial Q^a(x)}{\partial x} + \kappa \frac{\partial^2 Q^a(x)}{\partial x^2} + e^x H(R^a(x)), \tag{3.2}
\]

\[
R^a(x) = \frac{\alpha}{2} e^{-x} \int_{-a}^{x} [Q^a(y) - Q^a(x)](-F^a_y(y))dy, \tag{3.3}
\]

with \( s^a_s(x) = \min[1, R^a(x)] \) and with the boundary conditions

\[
F^a(-a) = 1, \quad F^a(a) = 0, \tag{3.4}
\]

\[
Q^a_x(-a) = 0, \quad Q^a_x(a) = Q^a(a). \tag{3.5}
\]
In addition, we impose a normalization for $F^a$:

$$F^a(0) = 1/2,$$  \hspace{1cm} (3.6)

that is needed to obtain uniform bounds on the speed $c^a$ that at the moment is assumed to be unknown. Let us define $x^a_0$ as

$$x^a_0 = \sup \{ x : s^a_*(x) = 1 \}.$$  \hspace{1cm} (3.7)

The main result of this step is the following proposition:

**Proposition 3.1.** There exists $a_0 > 0$ so that for all $a > a_0$ there exists a constant $c^a \in \mathbb{R}$ for which the system (3.1)-(3.3) has a solution such that $F^a(x)$ and $R^a(x)$ are monotonically decreasing, $Q^a(x)$ is increasing, and the boundary conditions (3.4)-(3.5), as well as the normalization (3.6), hold. Moreover, there exists a constant $C$ independent of $a$, and $a_0 > 0$ such that for all $a > a_0$ we have

$$|c^a| + \int_{-a}^{a} |F^a|^2 dx \leq C.$$  \hspace{1cm} (3.8)

There also exist constants $A_1, A_2, B, x^-_0$ and $x^+_0$ that do not depend on $a$, such that for all $a > a_0$ we have

$$A_1 e^{x^-} \leq Q(x) \leq A_2 e^{x^+} + B,$$  \hspace{1cm} (3.9)

and

$$x^-_0 < x^a_0 < x^+_0.$$  \hspace{1cm} (3.10)

We will drop the superscript $a$ below to simplify the notation, except where it will be important to emphasize that the a priori bounds do not depend on $a$. The proof of this proposition relies on a Leray-Schauder degree argument: we consider a family of systems of equations

$$-cF^\tau_x = \kappa F^\tau_{xx} + \alpha F^\tau(x) \int_{-a}^{x} [(1 - \tau) + \tau s^\tau_*(y)](-F^\tau_y(y)) dy$$  \hspace{1cm} (3.11)

$$(\rho - c)Q^\tau + cQ^\tau_x - \kappa Q^\tau_{xx} = \tau e^x H(R^\tau(x)),$$  \hspace{1cm} (3.12)

with

$$R^\tau(x) = (1 - \tau) + \frac{\alpha}{2} e^{-x} \int_{x}^{a} [Q^\tau(y) - Q^\tau(x)](-F^\tau_y(y)) dy, \quad s^\tau_*(x) = S^*(R^\tau(x)),$$  \hspace{1cm} (3.13)

and with the boundary conditions

$$F^\tau(-a) = 1, \quad F^\tau(a) = 0,$$  \hspace{1cm} (3.14)

$$Q^\tau_0(-a) = 0, \quad Q^\tau_0(a) = Q^\tau(a),$$  \hspace{1cm} (3.15)

together with the normalization

$$F^\tau(0) = \frac{1}{2}.$$  \hspace{1cm} (3.16)

This family is parametrized by $\tau \in [0, 1]$, so that at $\tau = 0$ it reduces to the classical Fisher-KPP equation

$$-cF^0_x = \kappa F^0_{xx} + F^0(1 - F^0),$$
and $Q^0(x) = 0$, $R^0(x) = s^*(x) = 1$ for all $x \in [-a,a]$, while at $\tau = 1$ the system (3.11)-(3.16) is exactly the problem (3.1)-(3.6) that we are interested in. We will show that the above system has a solution for all $\tau \in [0,1]$, and, in particular, for $\tau = 1$. The main difficulty in the proof of Proposition 3.1 is to obtain the uniform a priori bounds on the solutions to (3.11)-(3.16) that do not depend on $a$.

3.1.1 A priori bounds on a finite interval

We now prove the required a priori bounds for the solutions to (3.11)-(3.16) that are uniform in the parameter $\tau$ and do not depend on $a$ for $a > a_0$.

The monotonicity of $F^\tau$

We start by establishing monotonicity of $F^\tau$ for all $\tau \in [0,1]$.

**Lemma 3.2.** The function $F^\tau(x)$, satisfying (3.11) together with the boundary conditions (3.14) and normalization (3.16) is positive on $(-a,a)$ and decreasing in $x$ for all $\tau \in [0,1]$.

**Proof.** It is helpful to write

$$-cF^\tau_x - \kappa F^\tau_{xx} = \alpha F^\tau(x) \int_{-a}^{x} [(1 - \tau) + \tau s^*_\tau(y)](-F^\tau_y(y))dy$$

$$= \alpha(1 - \tau)F^\tau(x)(1 - F^\tau(x)) + \alpha \tau F^\tau(x) \int_{-a}^{x} s^*_\tau(y)(-F^\tau_y(y))dy. \tag{3.17}$$

Note that for $\tau = 0$ this is the Fisher-KPP equation

$$-cF^0_x - \kappa F^0_{xx} = \alpha F^0(x) \int_{-a}^{x} (-F^0_y(y))dy = \alpha F^0(x)(1 - F^0(x)), \tag{3.18}$$

with the boundary conditions (3.14), for which we know that the solution $F^0(x)$ is positive on $(-a,a)$ and is strictly decreasing in $x$, so that $F^0_x(x) < 0$ for all $x \in [-a,a]$. By continuity, we have $0 < F^\tau(x) < 1$ for all $x \in (-a,a)$ and $F^\tau_x(x) < 0$ for all $x \in [-a,a]$ for $\tau > 0$ sufficiently small. Furthermore, note that if $x_0$ is the local minimum or maximum of $F^\tau(x)$ that is closest to $(-a)$, then $F^\tau(x)$ does not change sign on $(-a,x_0)$, so that the integral term in the right side of (3.17) is either strictly positive if $x_0$ is a minimum, or strictly negative if $x_0$ is a maximum, which immediately gives a contradiction unless $F^\tau(x_0) < 0$. To rule out this possibility, let $\tau_1 > 0$ be the smallest $\tau \in [0,1]$ such that either there exists $x' \in (-a,a)$ such that $F^{\tau_1}(x') = 0$ or $F^{\tau_1}_x(a) = 0$. In the latter case, we have $F^{\tau_1}(x) \geq 0$ for all $x \in [-a,a]$, hence $F^{\tau_1}_x(a) = 0$ would contradict the Hopf lemma. On the other hand, the former situation would imply that the closest minimum of $F^{\tau_1}(x)$ to $(-a)$ is non-negative, which is also a contradiction. Thus, such $\tau_1$ can not exist, which means that $F^\tau(x) > 0$ for all $x \in (-a,a)$ and $F^\tau_x(a) < 0$ for all $\tau \in [0,1]$. As a consequence, by the same token, $F^\tau(x)$ can not attain a minimum on $[-a,a]$. The only possibility to rule out then is that $F^\tau(x)$ would attain a single local maximum on $[-a,a]$. On the other hand, that maximum would have to be larger than 1, and, as we have explained, this is impossible. Now, the conclusion of Lemma 3.2 follows.

$\square$
An a priori bound on the speed

Now, we obtain a uniform bound on the speed \( c_a \).

**Lemma 3.3.** For any \( \varepsilon > 0 \) there exists \( a_0 > 0 \) such that

\[
-\varepsilon < c_a < 2\sqrt{\kappa a} + \varepsilon \text{ for all } a > a_0 \text{ and for all } \tau \in [0, 1].
\]  

(3.19)

**Proof.** As \( s^*(y) \leq 1 \) for all \( y \), and \( F^\tau(y) \) is monotonically decreasing, the function \( F^\tau(y) \) satisfies

\[
-c_a F^\tau_x - \kappa F^\tau_{xx} \leq \alpha F^\tau(x)(1 - F^\tau(x)) \leq \alpha F^\tau(x),
\]

(3.20)

for all \( \tau \in [0, 1] \). On the other hand, the function \( \psi^A(x) = Ae^{-\beta(x+a)} \) satisfies

\[
-c_a \psi^A_x - \kappa \psi^A_{xx} \geq \alpha \psi^A(x),
\]

(3.21)

as long as

\[
c_a \beta \geq \kappa \beta^2 + \alpha.
\]

(3.22)

Note that if \( \beta > 0 \) and \( A \) is sufficiently large, then \( F^\tau(x) < \psi^A(x) \) for all \( x \in [-a, a] \). As we decrease \( A \), we see from (3.20) and (3.21) that \( F^a(x) \) and \( \psi^A(x) \) can not touch except at the boundary. Since \( F^a(a) = 0 \), this can only happen at \( x = -a \), which means that \( A = 1 \). It follows that

\[
F^a(x) \leq e^{-\beta(x+a)} \text{ for all } -a \leq x \leq a,
\]

and, in particular, we have \( F^a(0) \leq e^{-\beta a} \). This is a contradiction to (3.16) if \( \beta > \log 2/a \), and the upper bound for \( c^a \) in (3.19) follows.

For the lower bound we proceed in exactly the same way. Once again, monotonicity of \( F^a(x) \) implies that

\[
-c_a F^\tau_x - \kappa F^\tau_{xx} \geq 0.
\]

(3.23)

However, the function \( \psi(x) = 1 - Be^{\beta(x-a)} \) satisfies

\[
-c_a \psi_x(x) - \kappa \psi_{xx} \leq 0,
\]

(3.24)

provided that

\[
c_a \beta + \kappa \beta^2 \leq 0.
\]

(3.25)

Hence, if \( c^a < 0 \), we can find \( \beta > 0 \) such that (3.24) holds. As before, if \( B > 0 \) is sufficiently large, we automatically have \( F^\tau(x) > \psi(x) \). Decreasing \( B \), we see that (3.23) and (3.24) do not allow \( F^\tau(x) \) and \( \psi(x) \) to touch inside \([-a, a]\), and they can not intersect at \( x = -a \) either. Thus, they touch at \( x = a \) for the first time, with \( B = 1 \). It follows that

\[
F^\tau(x) \geq 1 - e^{\beta(x-a)} \text{ for all } x \in [-a, a],
\]

and, in particular, we have

\[
\frac{1}{2} = F^\tau(0) > 1 - e^{-\beta a},
\]

which is a contradiction if \( \beta > \log 2/a \), and the lower bound on \( c^a \) in (3.19) follows. \( \square \)
A lower bound for $Q^*_a(x)$

We now obtain a series of bounds for the function $Q^*_a(x)$. First, we establish a lower bound on $Q^*_a(x)$ and, in particular, show that it is positive. To this end, we need the following auxiliary lemma. Consider the eigenvalue problem

$$c\psi'(x) - \kappa \psi''(x) = \mu_a(c)\psi, \quad \psi(x) > 0 \text{ for all } -a < x < a,$$

with the boundary conditions

$$\psi'(-a) = 0, \quad \psi'(a) = \psi(a).$$

Existence of such principal eigenfunction and eigenvalue follows from the standard Sturm-Liouville theory – see, for instance, Theorem 4.1 in [13]. The next lemma gives a uniform bound on $\mu_a(c)$ as $a \to +\infty$.

**Lemma 3.4.** For any $K > 0$ there exists $C_K$ so that $|\mu_a(c)| \leq C_K$ for all $|c| < K$.

**Proof.** Writing

$$\psi(x) = \phi(x) \exp \left( \frac{c}{2\kappa} x \right)$$

turns (3.26)-(3.27) into

$$-\phi''(x) = -\gamma_a \phi, \quad \phi(x) > 0 \text{ for all } -a < x < a,$$

with

$$\gamma_a = -\frac{1}{\kappa} \left( \mu_a(c) - \frac{c^2}{4\kappa} \right)$$

and with the boundary conditions

$$\phi'(-a) = -\frac{c}{2\kappa} \phi(-a), \quad \phi'(a) = \left( 1 - \frac{c}{2\kappa} \right) \phi(a).$$

Note that if $\gamma_a < 0$ then the eigenfunction is of the form

$$\phi(x) = \cos(\sqrt{-\gamma_a}(x - z_a)),$$

with some $z_a \in \mathbb{R}$. As $\phi(x) > 0$ for all $x \in (-a, a)$, it follows that $\sqrt{-\gamma_a} \leq \pi/2a$ in this case. As $|c| \leq 2\sqrt{\kappa a}$, we conclude that there exists $a_0 > 0$ so that for all $a > a_0$ if $\gamma_a \leq 0$, then

$$|\mu_a(c)| \leq \alpha + 1.$$

Let us now assume that $\gamma_a > 0$ and set

$$r_1 = -\frac{c}{2\kappa}, \quad r_2 = 1 + r_1.$$

If $\gamma_a > 0$, then the positive eigenfunction has the form

$$\eta(x) = \exp(\sqrt{\gamma_a}x) + \beta \exp(-\sqrt{\gamma_a}x).$$
As we are only interested in bounds on $\gamma a$, we may assume without loss of generality that

$$|\sqrt{\gamma a} + r_1| > 1,$$  \hspace{1cm} (3.33)

for otherwise $\gamma$ is automatically bounded, and thus so is $\mu_a(c)$. The boundary condition at $x = -a$

$$\sqrt{\gamma a} \exp(-\sqrt{\gamma a} a) - \beta \sqrt{\gamma a} \exp(\sqrt{\gamma a} a) = r_1 \exp(-\sqrt{\gamma a} a) + r_1 \beta \exp(\sqrt{\gamma a} a),$$  \hspace{1cm} (3.34)

implies that

$$\beta = \frac{\sqrt{\gamma a} - r_1}{\sqrt{\gamma a} + r_1} \exp(-2\sqrt{\gamma a} a).$$  \hspace{1cm} (3.35)

Using this in the boundary condition at $x = a$

$$\sqrt{\gamma a} \exp(\sqrt{\gamma a} a) - \beta \sqrt{\gamma a} \exp(-\sqrt{\gamma a} a) = r_2 \exp(\sqrt{\gamma a} a) + r_2 \beta \exp(-\sqrt{\gamma a} a)$$

(3.36)

gives

$$\sqrt{\gamma a} \left(1 - \frac{\sqrt{\gamma a} - r_1}{\sqrt{\gamma a} + r_1} \exp(-4\sqrt{\gamma a} a)\right) = r_2 \left(1 + \frac{\sqrt{\gamma a} - r_1}{\sqrt{\gamma a} + r_1} \exp(-4\sqrt{\gamma a} a)\right),$$

so that

$$\frac{r_2}{\sqrt{\gamma a}} = \frac{\sqrt{\gamma a} + r_1 - (\sqrt{\gamma a} - r_1) \exp(-4\sqrt{\gamma a} a)}{\sqrt{\gamma a} + r_1 + (\sqrt{\gamma a} - r_1) \exp(-4\sqrt{\gamma a} a)} = 1 - \frac{2(\sqrt{\gamma a} - r_1) \exp(-4\sqrt{\gamma a} a)}{\sqrt{\gamma a} + r_1 + (\sqrt{\gamma a} - r_1) \exp(-4\sqrt{\gamma a} a)}. \hspace{1cm} (3.37)$$

Let us assume that there exists a sequence $a_k \rightarrow +\infty$ such that

$$\sqrt{\gamma a_k} \geq \frac{1}{a_k}. \hspace{1cm} (3.39)$$

Then we have

$$|\sqrt{\gamma a_k} - r_1| \exp(-4\sqrt{\gamma a_k} a_k) \leq (\sqrt{\gamma a_k} + |r_1|) \exp(-4\sqrt{\gamma a_k} a_k) \leq \frac{C}{a_k} + |r_1| \exp(-4\sqrt{a_k}). \hspace{1cm} (3.40)$$

Passing to the limit $a_k \rightarrow +\infty$ in (3.38) using (3.33) and (3.40) gives in that case

$$\gamma a_k \rightarrow r_2^2 \text{ as } k \rightarrow +\infty. \hspace{1cm} (3.41)$$

On the other hand, for any sequence $a_k \rightarrow +\infty$ for which (3.39) does not hold, we automatically have (3.31). This finishes the proof.

Now, we can prove the following lower bound on $Q^*_{a}(x)$.

**Lemma 3.5.** Let $\rho > C_K$, with $K = 2\sqrt{\kappa a}$, and $g(x)$ be the solution to

$$(\rho - c^a)g(x) + c^a g'(x) - \kappa g''(x) = \tau e^x,$$  \hspace{1cm} (3.42)

with the boundary conditions

$$g'(-a) = 0, \quad g'(a) = g(a),$$  \hspace{1cm} (3.43)

then $Q^*_{a}(x) \geq g(x)$ for all $x \in [-a, a]$. 

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\textbf{Proof.} Recall that \( Q^\tau(x) \) satisfies
\[(\rho - c^a)Q(x) + c^a Q'(x) - \kappa Q''(x) = \tau e^x H(R^a(x)) \geq \tau e^x.\]
Hence, the difference \( f(x) = Q(x) - g(x) \) satisfies
\[(\rho - c^a)f(x) + c^a f'(x) - \kappa f''(x) \geq 0, \quad (3.44)\]
with the boundary conditions \( g'(-a) = 0, g'(a) = g(a). \) Lemmas 3.3 and 3.4 imply that under the assumptions of the current lemma on the parameter \( \rho, \) the principal eigenvalue of the operator in the left side, with the boundary conditions (3.43), is positive, so that the comparison principle applies, thus \( f(x) \geq 0 \) for all \( x \in [-a,a]. \)

As a consequence of Lemma 3.5, we have the following more explicit lower bound.

\textbf{Lemma 3.6.} There exist \( \rho_0 > 0 \) and \( a_0 > 0 \) so that for \( \rho > \rho_0 \) and \( a > a_0 \) the function \( Q^\tau_a(x) \) satisfies \( Q^\tau_a(x) \geq A e^x \) for all \( x \in [-a,a], \) and \( \tau \in [0,1] \) and \( A < 1/(\rho - \kappa). \)

\textbf{Proof.} An explicit solution to (3.42)-(3.43) is
\[g(x) = \tau z_1 e^{\lambda_1 x} + \tau z_2 e^{-\lambda_2 x} + \frac{\tau}{\rho - \kappa} e^x, \quad (3.45)\]
where
\[\lambda_1 = \frac{c + \sqrt{c^2 + 4\kappa(\rho - c)}}{2\kappa} > 0, \quad \lambda_2 = \frac{-c + \sqrt{c^2 + 4\kappa(\rho - c)}}{2\kappa} > 0, \quad (3.46)\]
and the constants \( z_1 \) and \( z_2 \) are given by
\[z_1 = \frac{e^{-a}}{\rho - \kappa} \left( \frac{\lambda_2(\lambda_1 - 1)}{\lambda_2 + 1} e^{(\lambda_1 + 2\lambda_2)a} - \lambda_1 e^{-\lambda_2 a} \right)^{-1}, \quad z_2 = \frac{e^{-a}}{\rho - \kappa} \left( \frac{\lambda_2 e^{\lambda_2 a}}{\lambda_1(\lambda_2 + 1)} - \frac{\lambda_1}{\lambda_2(\lambda_1 - 1)} e^{-(\lambda_2 + 2\lambda_1)a} \right)^{-1}.\]
Note that \( \lambda_1 > 0 \) and \( \lambda_2 > 0 \) for \( \rho \) sufficiently large, and for \( a > a_0 \) sufficiently large we also have that both \( z_1 > 0 \) and \( z_2 > 0, \) and the conclusion of the present Lemma follows from Lemma 3.5.

\textbf{The monotonicity of} \( Q^\tau_a(x) \) \textbf{and} \( R^\tau_a(x) \)

Next, the uniform bound on the speed \( c^a \) and positivity of \( Q^\tau_a(x) \) allow us to show monotonicity of \( Q^\tau_a(x) \) and \( R^\tau_a(x). \)

\textbf{Lemma 3.7.} There exists \( a_0 > 0 \) so that function \( Q^\tau_a(x) \) is increasing in \( x \) and the functions \( R^\tau_a(x) \) and \( s^\tau_a(x) \) are decreasing in \( x \) for all \( a > a_0. \)

\textbf{Proof.} In order to simplify the notation we omit the \( \tau \) superscript. Note that if \( Q^a(x) \) is increasing in \( x, \) then, as
\[R_a(x) = 1 - \frac{\alpha}{2} e^{-x} \int_x^a \left[ Q^a(y) - Q^a(x) \right] (-F^a_y(y)) dy = 1 - \frac{\alpha}{2} e^{-x} \int_x^a Q^a_y(y) F^a(y) dy, \quad (3.47)\]
Lemma 3.2 implies that $R_a(x)$ is decreasing in $x$. In addition, monotonicity of $R_a(x)$ implies monotonicity of $s^*_a(x)$, hence we only need to study monotonicity of $Q^a(x)$. Differentiating (3.12) shows that
\[(\rho - c^a)Q' + c^aQ'_x - \kappa Q''_x = \tau e^x H(R^a(x)) + \tau e^x H'(R(x))R'(x),\] (3.48)
with $Q'(x) = \partial Q^a(x)/\partial x$, and from (3.13) we see that
\[R'(x) = -R(x) + 1 - \tau - \frac{\alpha}{2} e^{-x}Q'(x)F(x).\] (3.49)
Recalling that $H(R)$ is given explicitly by (2.23), we now write (3.48) as
\[(\rho - c^a)Q' + c^aQ'_x - \kappa Q''_x = \begin{cases} 2 - 2\tau - \tau \alpha e^{-x}Q'(x)F(x), & \text{if } R > 1, \\ 1 - R^2 + 2R(1 - \tau) - \tau \alpha e^{-x}Q'(x)F(x), & \text{if } 0 \leq R \leq 1, \\ 1, & \text{if } R < 0. \end{cases}\] (3.50)
It follows that
\[-\kappa Q''_x + c^aQ'_x + (\rho - c^a)Q'(x) + \tau \alpha F(x)Q'(x)S^a(R(x)) \geq 0.\]
Assumption $\rho > 2\sqrt{\kappa \alpha}$ in Theorem 2.1 together with Lemma 3.3 implies that $c^a < \rho$ for $a > a_0$.
It follows that $Q'(x)$ can not attain an interior negative minimum. We also have $Q'(-a) = 0$ and $Q'(a) = Q^a(a) > 0$, thus a negative minimum of $Q'(x)$ can not be attained at $x = \pm a$ either. Therefore, we have $Q'(x) \geq 0$ for all $x \in [-a, a]$ and $Q^\tau(x)$ is increasing in $x$. \hfill \Box

An upper bound for $Q^\tau(x)$

In this section we show that $Q^\tau_a(x)$ is bounded from above. To start, we show that the first derivative of $Q^\tau_a$ is bounded from above for all $\tau$.

**Lemma 3.8.** For all $A > 1/(\rho - \kappa)$ there exist $a_0 > 0$ and $C > 0$ so that $Q^\tau(x) = (Q^\tau_a)'(x)$ satisfies
\[Q'(x) \leq Ae^x + Ce^{-\lambda_2 a}\] (3.51)
for all $a > a_0$, $x \in [-a, a]$ and all $\tau \in [0, 1]$, with $\lambda_2 > 0$ as in (3.46).

**Proof.** As we have already shown that $Q'(x) \geq 0$, it follows from (3.50) that
\[-\kappa Q''_x + c^aQ'_x + (\rho - c^a)Q'(x) \leq 2e^x.\]
Therefore, we have we get that $Q'(x) \leq 2g(x)$ for all $a > a_0$ and $x \in [-a, a]$, with the function $g(x)$ that satisfies (3.42)-(3.43). An explicit calculation shows that
\[g(x) \leq Ae^x + Be^{-\lambda_2 a},\]
for all $A \geq 1/(\rho - \kappa)$, with a suitable $B > 0$. \hfill \Box

We will also need a bound on the integral that appears in the expression (3.47) for the function $R_a(x)$. 

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Lemma 3.9. There exists constants $B$ and $a_0$, such that

$$ \int_{-a}^{a} (Q^\tau_a)'(x) F^\tau_a(x) dx \leq B $$

for all $a > 0$ and $\tau$.

Proof. Once again, we drop the subscripts $a$ and $\tau$ in the notation. Since $0 < F(x) < 1$, Lemma 3.8 implies

$$ \int_{-a}^{0} Q'(x) F(x) dx \leq \int_{-a}^{0} e^{x} dx < A. \quad (3.53) $$

For the positive side of the integral, the same upper bound for $Q'(x)$ gives

$$ \int_{0}^{a} Q'(x) F(x) dx \leq \int_{0}^{a} e^{x} F(x) dx \quad (3.54) $$

Recall that $x_0$, as defined in (3.7) is $x_0 = \sup \{ x : R(x) > 1 \}$. If $x_0 \leq 0$ we have that

$$ \int_{0}^{a} Q'(x) F_a(x) dx \leq \int_{x_0}^{a} Q'(x) F_a(x) dx = \frac{2}{\alpha} e^{x_0} R(x_0) \leq \frac{2}{\alpha}. \quad (3.55) $$

The case $x_0 \geq 0$ is handled by the following upper bound on $F^\tau_a$.

Lemma 3.10. There exists $\alpha_0 > 0$ and a constant $C$, independent of $a$ and $\tau$, such that for all $\alpha > \alpha_0$ if $x_0 > 0$ then

$$ F^\tau_a(x) \leq C e^{-2x}. \quad (3.56) $$

Note that the conclusion of Lemma 3.9 is an immediate consequence of Lemma 3.10. We now prove the latter. Since $x_0 > 0$ we have $s^*(x) = 1$ for $x < 0$. Therefore, we have for all $x > 0$

$$ \int_{-a}^{x} s^*(y)(-F^\tau_y) dy \geq \int_{-a}^{0} s^*(y)(-F^\tau_y) dy = \frac{1}{2}. \quad (3.57) $$

It follows that for $x > 0$ we have

$$ \int_{-a}^{x} [(1 - \tau) + \tau s^*(y)](-F^\tau_y) dy = (1 - \tau)(1 - F^\tau(x)) + \tau \int_{-a}^{x} s^*(y)(-F^\tau_y) dy $$

$$ \geq \frac{(1 - \tau)}{2} + \frac{\tau}{2} = \frac{1}{2}. \quad (3.58) $$

Therefore, for $x > 0$ we have that

$$ -cF^\tau_x \geq \kappa F^\tau_{xx} + \frac{\alpha}{2} F^\tau. \quad (3.59) $$

We integrate from $x$ to $y$, for $0 < x < y$ to get

$$ cF^\tau(x) - cF^\tau(y) \geq \kappa F^\tau_x(y) - \kappa F^\tau_x(x) + \frac{\alpha}{2} \int_{x}^{y} F^\tau(\xi)d\xi. \quad (3.60) $$
Next, integrate in $y$ from $z$ to $z + 1$, with $z > x$, to get
\[ cF^\tau(x) - c \int_z^{z+1} F^\tau(y)dy \geq \kappa F^\tau(z + 1) - \kappa F^\tau(z) - \kappa F^\tau_x(x) + \frac{\alpha}{2} \int_z^{z+1} \int_x^y F^\tau d\xi dy. \] (3.61)

The left side we can estimated simply as
\[ cF^\tau(x) - c \int_z^{z+1} F^\tau(y)dy \leq cF^\tau(x). \] (3.62)

For the right side of (3.61) we have
\[ \kappa F^\tau(z + 1) - \kappa F^\tau(z) - \kappa F^\tau_x(x) + \frac{\alpha}{2} \int_z^{z+1} F^\tau(y)(y - x)dy \geq \kappa F^\tau(z + 1) - \kappa F^\tau(z) - \kappa F^\tau_x(x) + \frac{\alpha}{2} F^\tau(z + 1)(z - x). \] (3.63)

In particular, we get
\[ cF^\tau(x) \geq \kappa F^\tau(z + 1) - \kappa F^\tau(z) + \frac{\alpha}{2} F^\tau(z + 1)(z - x). \] (3.64)

Adding $\kappa F^\tau(x)$ to both sides gives, as $x < z$:
\[ (c + \kappa)F^\tau(x) \geq \kappa F^\tau(z + 1) + \kappa F^\tau(x) - \kappa F^\tau(z) + \frac{\alpha}{2} F^\tau(z + 1)(z - x) \geq \kappa F^\tau(z + 1) + \frac{\alpha}{2} F^\tau(z + 1)(z - x). \] (3.65)

Taking $z = x + 1$ leads to
\[ (c + \kappa)F^\tau(x) \geq F^\tau(x + 2) \left( \kappa + \frac{\alpha}{2} \right), \] (3.66)

thus
\[ F^\tau(x + 2) \leq \frac{c + \kappa}{\kappa + \alpha/2} F^\tau(x). \] (3.67)

Now, (3.56) follows if we take $\alpha$ sufficiently large, as $|c| \leq 2\sqrt{\kappa \alpha}$ by Lemma 3.3.

Lemma 3.11. There exists $a_0 > 0$ so that for any $A > 1(\rho - \kappa)$ there exists $B > 0$ so that
\[ Q^\tau_a(x) \leq Ae^x + B, \]
for all $a > a_0$, $\tau \in [0, 1]$ and $x \in [-a, a]$. 

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Proof. From Lemma 3.8 we already know that \( Q_x \leq Ae^x \), hence (dropping the subscripts \( a \) and \( \tau \)):

\[
Q(x) \leq Ae^x - Ae^{-a} + Q(-a). \tag{3.68}
\]

Let us now show that \( Q'(a) \) is bounded above by a constant independent of \( a \) and \( \tau \). Note that \( H(r) \leq 1 + 2r \) for all \( r \geq 0 \), hence

\[
\begin{align*}
(\rho - c)Q(x) + cQ'(x) - \kappa Q''(x) & = \tau e^x H(R(x)) \leq \tau e^x + 2\tau e^x R(x) \\
& \leq \tau e^x + 2\tau(1 - \tau) e^x + \tau^2 \alpha \int_x^a Q'(x) F(y) dy. \tag{3.69}
\end{align*}
\]

We used (3.47) above. Next, we integrate (3.69) from \(-a\) to \(-a + 1\) to get

\[
\begin{align*}
(\rho - c) \int_{-a}^{-a+1} Q(x) dx + c \int_{-a}^{-a+1} Q'(x) dx - \kappa Q'(-a + 1) + \kappa Q'(-a) & \leq 3\tau e^{-a+1} + \tau^2 \alpha \int_{-a}^{-a+1} \int_x^a Q'(x) F(y) dy dx. \tag{3.70}
\end{align*}
\]

Monotonicity of \( Q \) allows us to bound the left side from below as

\[
(\rho - c) \int_{-a}^{-a+1} Q(x) dx + c \int_{-a}^{-a+1} Q'(x) dx - \kappa Q'(-a + 1) + \kappa Q'(a) \\
\geq (\rho - c) Q(-a) - \kappa Q'(-a + 1) + (\rho - c) Q(-a) - \kappa Ae^{-a+1} - \kappa Ce^{-a}. \tag{3.71}
\]

In the last step we have used the upper bound for \( Q'(x) \) in (3.51). For the right side in (3.70) we have

\[
\alpha \int_{-a}^{-a+1} \int_x^a Q'(x) F(y) dy dx \leq \alpha \int_{-a}^{-a+1} \int_{-a}^a Q'(x) F(y) dy dx \leq \alpha B, \tag{3.72}
\]

where \( B \) is as in (3.52). Therefore we have that

\[
(\rho - c^a) Q(-a) < \kappa Ae^{-a+1} + \kappa Ce^{-a} + \alpha \tau B. \tag{3.73}
\]

Using this bound in (3.68) completes the proof. \( \square \)

**A uniform gradient bound for \( F^\tau_a(x) \)**

We first obtain a uniform bound for the derivative of \( F^\tau_a(x) \). To simplify the notation, we drop the subscripts \( a \) and \( \tau \).

**Lemma 3.12.** There exists \( a_0 > 0 \) and \( C > 0 \) such that

\[
\int_{-a}^{a} |F_x|^2 dx \leq C \text{ for all } a > a_0 \text{ and for all } \tau \in [0, 1]. \tag{3.74}
\]
Proof. Integrating (3.17) from \(-a\) to \(a\) gives
\[
c = \kappa F_x(a) - \kappa F_x(-a) + \alpha(1 - \tau) \int_{-a}^{a} F(1 - F) dx + \alpha \tau \int_{-a}^{a} F(x) \int_{-a}^{x} s^*(y)(-F_y(y)) dy dx. \tag{3.75}
\]
We also multiply both sides of (3.11) by \(F\), integrate from \(-a\) to \(a\), and use (3.75):
\[
\frac{c}{2} + \kappa \int_{-a}^{a} F_x^2 dx + \kappa F_x(-a) = \alpha(1 - \tau) \int_{-a}^{a} F^2(1 - F) dx + \alpha \tau \int_{-a}^{a} F^2(x) \int_{-a}^{x} s^*(y)(-F_y(y)) dy dx
\leq c - \kappa F_x(a) + \kappa F_x(-a), \tag{3.76}
\]
so that
\[
\kappa \int_{-a}^{a} F_x^2 dx \leq \frac{c}{2} - \kappa F_x(a). \tag{3.77}
\]
The uniform bounds on \(c\) in Lemma 3.3 allow to apply the standard elliptic regularity results to conclude that \(|F_x(a)| \leq C\), with \(C\) that does not depend on \(a\), and (3.74) follows. \(\square\)

**A uniform bound on the transition point \(x_0^a\)**

Next, we prove a uniform bound on the point \(x_0^a\).

**Lemma 3.13.** There exist constants \(x_0^+, x_0^-\) and \(a_0\) such that for all \(a > a_0\) we have
\[
x_0^- \leq x_0^a \leq x_0^+. \tag{3.78}
\]

**Proof.** Recall that the point \(x_0^a\) is determined by \(R(x_0^a) = 1\), so that
\[
1 = \frac{\alpha}{2} e^{-x_0^a} \int_{x_0^a}^{a} Q_y^a(y) F^a(y) dy.
\]
Using Lemma 3.9, we obtain
\[
\exp\{-x_0^a\} \geq \frac{2}{\alpha B},
\]
hence
\[
x_0^a \leq \log(\alpha B). \tag{3.79}
\]
For a lower bound on \(x_0^a\), let us assume that \(x_0^a < 0\), and write, for any \(z > 0\):
\[
\frac{2}{\alpha} e^{-x_0^a} \int_{x_0^a}^{a} [Q^a(y) - Q^a(x_0^a)](-F_y^a(y)) dy > e^{-x_0^a} \int_{0}^{a} [Q^a(y) - Q^a(x_0^a)](-F_y^a(y)) dy
\]
\[
> e^{-x_0^a} \int_{0}^{a} [Q^a(y) - Q^a(0)](-F_y^a(y)) dy > e^{-x_0^a} \int_{z}^{a} [Q^a(y) - Q^a(0)](-F_y^a(y)) dy
\]
\[
\geq e^{-x_0^a} \int_{z}^{a} [Ae^y - B](-F_y^a(y)) dy. \tag{3.79}
\]
We used Lemma 3.6 in the last step above to bound $Q^a(y)$ from below and Lemma 3.11 to bound $Q^a(0)$ from above. We may now choose $z > 0$ so that $Ae^y > 2B$ for all $y > z$, so that

$$\frac{2}{\alpha} \geq Be^{-x_0^a} \int_{z}^{a} (-F_y^a(y)) dy = Be^{-x_0^a} F^a(z).$$

(3.80)

As $z$ does not depend on $a$, and $F^a(0) = 1/2$, the Harnack inequality implies that there exists $s > 0$ that does not depend on $a$ so that $F^a(z) > s$, so that

$$e^{x_0^a} > \frac{\alpha Bs}{2},$$

finishing the proof of the lower bound for $x_0^a$.

3.1.2 The degree argument

We have by now proved the a priori bounds in Proposition 3.1. We now use these a priori bounds to finish the proof of the existence part of Proposition 3.1 using a Leray-Schauder degree argument. Let us define the map $L_\tau(c,F,Q) = (\theta,G,T)$ as the solution operator for the system

$$
\begin{align*}
-cG_x &= \kappa G_{xx} + \int_{-a}^{x} [(1 - \tau) + \tau s(t)(-F_y(t))] dy \\
(\rho - c)T + cT_x - \kappa T_{xx} &= \tau e^x H(R(x))
\end{align*}
$$

(3.81)

with the boundary conditions

$$G(-a) = 1, \ G(a) = 0, \ T_x(-a) = 0 \ \text{and} \ T_x(a) = T(a),$$

and with

$$R(x) = 1 - \tau + \tau \frac{\alpha}{2} e^x \int_{-a}^{a} [Q(y) - Q(x)](-F_y^\tau(y)) dy$$

(3.82)

and $s^*(x) = \min\{1, R(x)\}$. The constant $\theta$ is defined as

$$\theta = \frac{1}{2} - \max_{x \in [0,a]} F(x) + c.$$ 

(3.83)

This operator maps the Banach space $X = \mathbb{R} \times C^1([-a,a]) \times C^1([-a,a])$ with the norm

$$\|c,F,Q\|_X = \max\{|c|, \|F\|_{C^1}, \|Q\|_{C^1}\},$$

to itself, and its fixed points are solutions to (3.11). Therefore, it suffices to show that the operator $F_\tau = \text{Id} - L_\tau$ has a nontrivial kernel for all $\tau \in [0,1]$. Let $B_M$ be a ball of radius $M$ in $X$ centered at the origin. Using the a priori bounds obtained above we can choose $M$ sufficiently large to ensure that $F_\tau$ does not vanish on the boundary $\partial B_M$. As the Leray-Schauder degree is homotopy invariant, it is enough to show that $\text{deg}(F_0, B_M, 0) \neq 0$. Note that

$$F_0(c,F,Q) = \left( \max_{x>0} F_0^c(x) - \frac{1}{2}, F - F_0^c, Q \right),$$

(3.84)

where $F_0^c$ solves

$$-cF_0^c = \kappa F_0'' + \alpha F(1 - F), \ F_0(-\infty) = 1, \ F_0(+\infty) = 0.$$ 

(3.85)

Hence, $\text{deg}(F_0, B_M, 0) = -1$, thus $F_\tau$ has a nontrivial kernel. Therefore, a solution to (3.11) exists for all $\tau \in [0,1]$, which proves the existence part of Proposition 3.1.
3.2 Identification of the limit

The a priori bounds obtained in Proposition 3.1 allows us to extract a subsequence \( a_n \to +\infty \) such that the corresponding sequence \((c^{a_n}, F^{a_n}, Q^{a_n})\) converges to a limit \((c, F, Q)\), in \( C_{\text{loc}}^{2,\alpha}(\mathbb{R})\). Moreover, the functions \( F \) and \( Q \) are monotonic. The upper bound on \( Q_a'(x) \) in Lemma 3.8 and the upper bound on \( F_a(x) \) in Lemma 3.13 imply that

\[
\int_{x}^{a_n} Q_a'(y)F_a(y)dy \to \int_{x}^{\infty} Q'(y)F(y)dy,
\]

hence the corresponding sequences \( s_a^*(x) \) and \( R_a(x) \) converge as well to their respective limits \( s^*(x) \) and \( R(x) \) such that \( R(x) \geq 0 \) and \( s^*(x) = \min(1, R(x)) \), and

\[
R(x) = \frac{\alpha}{2} e^{-x} \int_{x}^{\infty} Q'(y)F(y)dy.
\]

In particular, as a consequence, the function \( Q \) satisfies the second equation in (2.31):

\[
\rho Q = cQ - c \frac{\partial Q}{\partial x} + \kappa \frac{\partial^2 Q}{\partial x^2} + e^x H(R).
\]

In order to see that \( F(x) \) satisfies the first equation in (2.31), let us take \( x_0^\pm \) as in Lemma 3.13 and write

\[
\int_{-a_n}^{x} s_a^*(y)(-F_y^{a_n}(y))dy = \int_{-a_n}^{x_0^-} (-F_y^{a_n}(y))dy + \int_{x_0^-}^{x_0^+} s_a^*(y)(-F_y^{a_n}(y))dy + \int_{x_0^+}^{a_n} s_a^*(y)(-F_y^{a_n}(y))dy = 1 - F^{a_n}(x_0^-) + I_n + II_n.
\]

The bounded convergence theorem implies that

\[
I_n \to \int_{x_0^-}^{x_0^+} s^*(y)(-F_y(y))dy,
\]

while the Lebesgue dominated convergence theorem and (3.56) imply that

\[
II_n \to \int_{x_0^+}^{\infty} s^*(y)(-F_y(y))dy.
\]

It follows that \( F \) satisfies

\[
-cF_x - \kappa F_{xx} = \alpha F^a(x) \int_{-a}^{x} s_a^*(y)(-F_y^a(y))dy.
\]

It remains to show that the limit \( F \) satisfies the correct boundary conditions and that \( Q(x) \) converges to a positive constant on the left and grows exponentially on the right, as in (2.32). Note that Lemma 3.10 implies that \( F(x) \to 0 \) as \( x \to +\infty \). The next lemma takes care of the limit on the left.
Lemma 3.14. The limiting function $F(x)$ converges to 1 as $x \to -\infty$.

Proof. Let $x^-_0$ be the lower bound on $x_0^a$ as in Lemma 3.13. Then, for $a > a_0$ the function $F^a$ satisfies
\[-c^a F^a_x - \kappa F^a_{xx} = \alpha F^a \int_{-a}^x (-F^a_y(y))dy = \alpha F^a (1 - F^a), \quad \text{for } x < x^-_0.\]

Integrating both sides from $(-a)$ to $x^-_0$ gives
\[c^a (1 - F^a(x^-_0)) - \kappa F^a_x (-a) + \kappa F^a_x (x^-_0) = \int_{-a}^{x^-_0} \alpha F^a (1 - F^a) dx.\]

Note that $c^a$ is bounded by Lemma 3.3, $F^a(x)$ is bounded for all $x$ and $F^a_x (-a)$ and $F^a_x (x^-_0)$ are also bounded by elliptic regularity. Therefore, the left side is bounded independently of $a$ for $a > a_0$, hence so is the integral in the right side. It follows that the integral
\[\int_{-\infty}^{x^-_0} F(1 - F) dx\]
is finite. As $F(x)$ is monotonically decreasing, and $F(x) \geq 1/2$ for $x \leq 0$, it follows that $F(x) \to 1$ as $x \to -\infty$. \square

Next, we look at the left limit of $Q(x)$.

Lemma 3.15. The limiting function $Q(x)$ converges to a positive constant $q_-$ as $x \to -\infty$.

Proof. Note that for $x < x^-_0$ we have $R(x) > 1$, so that $H(R(x)) = 2R(x)$, and (3.88) becomes
\[\rho Q(x) = cQ(x) - c \frac{\partial Q(x)}{\partial x} + \kappa \frac{\partial^2 Q(x)}{\partial x^2} + \alpha \int_x^{\infty} Q_y(y) F(y) dy. \quad (3.93)\]

As the function $Q(x)$ is monotonically increasing, and the derivatives $Q'(x)$ and $Q''(x)$ are uniformly bounded for $x < 0$, there exists a sequence $x_n \to -\infty$ such that both $Q'(x_n) \to 0$ and $Q''(x_n) \to 0$. Passing to the limit $n \to +\infty$ in (3.93) leads to
\[\left(\rho - c\right) q_- = \alpha \int_{-\infty}^{\infty} Q_y(y) F(y) dy, \quad (3.94)\]
where
\[q_- = \lim_{x \to -\infty} Q(x).\]

It follows that $q_- > 0$. \square

Finally, we look at the behavior of $Q(x)$ on the right.

Lemma 3.16. The limit
\[\lim_{x \to +\infty} Q(x)e^{-x} \quad (3.95)\]
exists and equals $1/(\rho - \kappa)$. 23
Proof. Lemmas 3.6 and 3.11 imply that there exist $0 < A_1 < A_2$ and $B > 0$ such that
\[
A_1 e^x \leq Q(x) \leq A_2 e^x + B.
\] (3.96)

Consider the function $Z(x) = Q(x)e^{-x}$, which satisfies
\[
\rho Z = -c \frac{\partial Z}{\partial x} + \kappa \frac{\partial^2 Z}{\partial x^2} + 2\kappa \frac{\partial Z}{\partial x} + \kappa Z + H(R).
\] (3.97)

Note that for $A_1 \leq Z(x) \leq A_2$ and for $x > x_0^+$ we have $R(x) < 1$, so that
\[
H(R) = 1 + R^2,
\] and (3.97) becomes
\[
(\rho - \kappa)Z = -(c - 2\kappa) \frac{\partial Z}{\partial x} + \kappa \frac{\partial^2 Z}{\partial x^2} + 1 + R^2, \text{ for } x > x_0^+.
\] (3.98)

Let us assume that $y_n \to +\infty$ such that $Z(y_n) \to \zeta$ as $n \to +\infty$. Since $Z(x)$ is uniformly bounded and positive, (3.97) implies that $\|Z\|_{C^{2,\alpha}} \leq C$, hence the functions $Z_n(x) = Z(x + y_n)$ converge, after extracting a subsequence to a function $\bar{Z}$. As $R(x) \to 0$ as $x \to +\infty$, the function $\bar{Z}(x)$ is a bounded solution to
\[
(\rho - \kappa) \bar{Z} = -(c - 2\kappa) \frac{\partial \bar{Z}}{\partial x} + \kappa \frac{\partial^2 \bar{Z}}{\partial x^2} + 1, \text{ for } x \in \mathbb{R}.
\] (3.99)

It follows that $\bar{Z}(x) \equiv \kappa/(\rho - \kappa)$, and, in particular, $\zeta = 1/(\rho - \kappa)$, finishing the proof.

This also completes the proof of Theorem 2.1.

The proof of Proposition 2.2

We prove the matching lower and upper bounds on $c$. First, exactly as in the proof of Lemma 3.3, using an exponential super-solution and the normalization at $x = 0$, we can show that for any $\varepsilon > 0$ there exists $a_0 > 0$ such that
\[
-\varepsilon < a^\varepsilon < 2\sqrt{\kappa \int_{-a}^{a} s_n^*(y)(-F_y^0(y)) dy} + \varepsilon \text{ for all } a > a_0.
\] (3.100)

Passing to the limit $a \to +\infty$, we get an upper bound
\[
c \leq 2\sqrt{\kappa \int_{-\infty}^{\infty} s^{*}(y)(-F_y(y)) dy}.
\] (3.101)

For the lower bound, let $F(x)$ be the solution to the traveling wave equation
\[
-cF_x - \kappa F_{xx} = \alpha F(x) \int_{-\infty}^{x} s^{*}(y)(-F_y(y)) dy,
\] (3.102)
with \( F(\infty) = 1 \) and \( F(-\infty) = 0 \) that we have just constructed, and set \( F_n(x) = F(x+n)/F(n) \).

The functions \( F_n \) satisfy

\[
-cF_n - \kappa F_n^{xx} = \alpha \gamma_n(x) F^n, \quad \gamma_n(x) = \int_{-\infty}^{x+n} s^*(y)(-F_y(y))dy, \tag{3.103}
\]

with \( F_n(0) = 1 \). The standard elliptic regularity estimates and the Harnack inequality imply that after extracting a subsequence, the functions \( F_n(x) \) converge locally uniformly to a limit \( G(x) \) that satisfies

\[
-cG - \kappa G^{xx} = \alpha \gamma, \quad \gamma = \int_{-\infty}^{\infty} s^*(y)(-F_y(y))dy > 0, \tag{3.104}
\]

and \( G(0) = 1 \). In addition, the function \( G(x) \) is positive and monotonically decreasing. As a consequence, since \( c > 0 \), we must have

\[
c \geq 2\sqrt{\kappa \gamma}, \tag{3.105}
\]

and the proof of Proposition 2.2 is complete. □

4 Numerical results

In this section, we describe the numerical results obtained via an iterative finite differences scheme for solving the traveling wave system on a finite interval \([-a,a]\), using the following algorithm:

- Start with an initial guess for \( F(x) \) and \( Q(x) \) and compute \( R(x) \) from (3.3), \( H(x) \) using (2.23), and set \( s^*(x) = \min(1, R(x)) \). A good initial guess for \( F(x) \) is the solution of the Fisher-KPP equation for the traveling wave on \([-a,a]\). We can take \( Q(x) = e^x \) as an initialization.

- Given \( H(x) \), we solve (3.2) for \( Q(x) \) on \([-a,a]\) with the boundary conditions (3.5).

- Given \( s^*(x) \) we solve (3.1) for \( F(x) \) and \( c \), with the boundary conditions (3.4) and normalization (3.6), using an iterative finite difference scheme.

Note that both \( Q(x) \) and \( R(x) \) grow exponentially, on the right and on the left, respectively. Accordingly, we rescale the problem, so that all functions involved are bounded. As the equation for \( F \) is not linear, we use another iterative finite difference scheme to solve it and we relax the boundary condition by using the solution of the linearized problem.

4.1 A rescaling for \( Q(x) \) and \( R(x) \)

As \( Q(x) \) goes to a positive constant on the left, rescaling by \( e^{-x} \) would not help, as the rescaled function would grow exponentially on the left, which is to be avoided. Define a function \( g(x) \) as

\[
g(x) = \begin{cases} 
1 + 2 \tan^{-1}(1) - 2 \tan^{-1}(x+1) & \text{for } x \leq 0, \\
e^{-x} & \text{for } x \geq 0.
\end{cases} \tag{4.1}
\]
Note that \( g(x) \) is continuous function with continuous first three derivatives, that converges to a constant on the left and decays exponentially on the right. The function \( \tilde{Q}(x) = g(x)Q(x) \) satisfies
\[
\phi_1 \tilde{Q} + \phi_2 \tilde{Q}_x + \phi_3 \tilde{Q}_{xx} = g e^x H(R(x)),
\]
with
\[
\phi_1 = \rho - c - \frac{g_1}{g} - \frac{\kappa(2g_1^2 - gg_2)}{g^2} = \rho - \kappa \quad \text{for } x > 0,
\]
\[
\phi_2 = c + 2 \frac{\kappa g_1}{g} = c - 2 \kappa \quad \text{for } x > 0,
\]
\[
\phi_3 = -\kappa.
\]
The boundary conditions \( Q'(-a) = 0 \) and \( Q'(a) = Q(a) \) become
\[
\tilde{Q}'(-a) = \frac{g_1(-a)}{g(-a)} \tilde{Q}(-a) \quad \text{and} \quad \tilde{Q}'(a) = 0.
\]
We solve numerically for \( \tilde{Q} \) using a finite difference scheme.

We use a similar strategy to rescale \( R(x) \). Note that for \( x < 0 \) we have that \( R(x) \sim e^{-x} \). The function \( \tilde{R}(x) = e^x R(x) \) satisfies
\[
\tilde{R}'(x) = -\frac{\alpha}{2} Q'(x)F(x)
\]
with terminal condition \( \tilde{R}(a) = 0 \). Note that when solving for \( R \) we don’t know \( F(x) \) and \( Q'(x) \) explicitly. We can use the numerical values of \( F \) and \( Q \) on the finite grid. We approximate \( Q'(x) \) as
\[
Q'(x_i) = \frac{1}{g} \tilde{Q}_x - \frac{g_1}{g^2} \tilde{Q} \approx \frac{1}{g} \frac{\tilde{Q}_{i+1} - \tilde{Q}_{i-1}}{2h} - \frac{g_1}{g^2} \tilde{Q}_i.
\]

### 4.2 Numerical Solution for the Fisher-KPP equation on the interval \([-a, a]\)

Let us first explain an iterative finite difference scheme and relaxed boundary conditions for the Fisher-KPP equation
\[
-cF_x = \kappa F_{xx} + \alpha F(1 - F),
\]
on a finite interval \([-a, a]\), with \( F(-a) = 1 \), \( F(a) = 0 \) and \( F(0) = 1/2 \). Near \( x = a \), the solution to the Fisher-KPP equation is well approximated by the linearized equation
\[
-cF_x - \kappa F_{xx} = \alpha F.
\]
A solution to (4.4) with the initial condition \( F(0) = 1/2 \) is
\[
F(x) = \frac{1}{2} \exp\{-\beta x\},
\]
where \( \beta \) is given by
\[
\kappa \beta^2 - c \beta + \alpha = 0, \quad \beta(c) = \begin{cases} 
\frac{c}{2\kappa} & \text{if } 0 < c < 2, \\
\frac{c - \sqrt{c^2 - 4\kappa \alpha}}{2\kappa} & \text{if } c \geq 2.
\end{cases}
\]
We use
\[
F(a) = \frac{1}{2} \exp\{-\beta(c)a\},
\]
as a new boundary condition for the Fisher-KPP equation on \([-a,a]\). It will change at each step of the iterative algorithm to reflect the change in \(c\).

The iterative algorithm solves
\[
-cF_{x}^{k+1} - \kappa F_{xx}^{k+1} = \alpha F^{k}(1 - F^{k}),
\]
on the interval \([-a,a]\), with the boundary condition \(F^{k+1}(-a) = 1, F^{k+1}(a) = (1/2) \exp\{-\beta(c)a\}\)
with the initialization \(F_{0}(x) = (a-x)/(2a)\). The first and the second derivatives are approximated using the central differences. The speed \(c\) is updated after each iteration, to enforce \(F^{k}(0) = 1/2\) for each iteration.

4.2.1 Solution for \(F(x)\) for general \(s^{*}(x)\)

We solve numerically the equation:
\[
-cF_{x} - \kappa F_{xx} = \alpha F \int_{-a}^{x} s^{*}(y)(-F_{y}(y))dy
\]
on the interval \([-a,a]\), with the boundary condition as in the scheme for the Fisher-KPP equation. Take a partition with step \(h\) and let \(n = 2a/h\). Let \(x_{i} = -a + ih\) and \(F_{i}^{k} = F_{i}^{k}(x_{i})\) be the value of the \(k\)-th approximation of the solution at \(x_{i}\). At each step of the iterative scheme the speed \(c_{k}\) is updated, so that \(F^{k}(0) = 1/2\). We take \(F^{0}\) to be the solution of the Fisher-KPP equation
\[
-c_{0}F_{x}^{0} - \kappa F_{xx}^{0} = \alpha F^{0}(1 - F^{0})
\]
on the interval \([-a,a]\), with \(c_{0}\) chosen, so that \(F^{0}(0) = 1/2\). Using the central differences, we approximate the left side of (4.7) as
\[
\frac{c_{k}^{F_{i+1}^{k} - F_{i-1}^{k}}}{2h} - \kappa \frac{F_{i+1}^{k} - 2F_{i}^{k} + F_{i-1}^{k}}{h^{2}}.
\]
To reduce the error in the approximation of the right side of (4.7), we integrate by parts
\[
\int_{-a}^{x} s^{*}(y)(-F_{y}(y))dy = \int_{-a}^{x} s^{*}(y)(-F_{y}(y))dy + \int_{x}^{x_{i}} s^{*}(y)(-F_{y}(y))dy
\]
\[
= 1 - F(x_{0}) + \int_{x_{0}}^{x} s^{*}(y)(-F_{y}(y))dy = 1 - F(x_{0}) + F(x_{0}) - F(x_{i})s^{*}(x_{i}) - \int_{x_{0}}^{x_{i}} s_{y}^{*}(y)(-F_{y}(y))dy
\]
\[
= 1 - F(x_{i})R(x_{i}) + \int_{x_{0}}^{x_{i}} R_{y}(y)F(y)dy \approx 1 - F_{i}^{k}R_{i} + \sum_{m=1}^{i-1} \frac{R_{y}(x_{j})F_{j}^{k}}{2}.
\]
In the last computation we take \(x_{0} = \max\{x|s^{*}(x) = 1\}\) and \(m\) such that \(x_{0} = -a + mh\). We also use that \(s^{*}(x) = \min\{1, R(x)\}\), so for \(x > x_{0}\) we have that \(s_{y}^{*}(x) = R_{x}(x)\). Thus, the discretized version of (4.7) is
\[
-c_{k} \frac{F_{i+1}^{k} - F_{i-1}^{k}}{2h} - \kappa \frac{2F_{i}^{k} + F_{i-1}^{k}}{h^{2}} = \alpha F_{i}^{k-1}\left(1 - F_{i}^{k}R_{i} + \sum_{m=1}^{i-1} \frac{R_{y}(x_{j})F_{j}^{k}}{2} \right).
\]
(4.8)
4.3 Discussion of the numerical results and comparison to the Fisher-KPP traveling wave

In the numerical simulations, we choose $\kappa = 1$, $h = 0.02$, $\alpha = 2$, $\rho = 10$. Recall that we need to take $\alpha > 1$ to ensure that $F(x)$ decays sufficiently fast so that $R(x)$ is finite, and $\rho > \kappa$ to control $Q(x)$. We implement the above algorithm for $a \in \{15, 20, 25, 30, 35, 40\}$ and observe that, as expected, the solutions for $F$ and $Q$ converge pointwise as $a$ grows.

As expected, the value function $Q(x)$ behaves as an exponential on the right and converges to a constant on the left. We observe that on the right the rescaled value function $\tilde{Q}(x) = g(x)Q(x)$, with $g(x)$ as in (4.1), converges to 0.1111 for all values of $a$, so that $Q(x)$ behaves as $0.1111e^x$. This limit is obtained at $a = 15$ and remains the same for larger values of $a$. On the left $Q(x)$, as well as $\tilde{Q}(x)$, converge to a constant. We get the following limits on the left for $Q(x)$ and for different values of $a$:

<table>
<thead>
<tr>
<th>$a$</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q$</td>
<td>0.2891</td>
<td>0.3425</td>
<td>0.3834</td>
<td>0.4004</td>
<td>0.4017</td>
<td>0.4041</td>
</tr>
</tbody>
</table>

Table 1: Left limit for $Q(x)$

As we expect that $R(x) \sim Ce^{-x}$, we solve for $\tilde{R} = e^xR(x)$, and get the following numerical solutions for $\tilde{R}$ for $a = 35$ and $a = 40$.

Figure 1: $\tilde{Q}(x)$ on $[-40, 40]$
Recall that $s^*(x) = \min\{R(x), 1\}$ and the transition point $x_0^a = \sup\{x|s^*(x) = 1\}$. We observe that $x_0^a$ converges numerically to 0.42. The values of the transition point for different values of $a$ are given in the following table:

<table>
<thead>
<tr>
<th>$a$</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>35</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0^a$</td>
<td>0.12</td>
<td>0.28</td>
<td>0.38</td>
<td>0.42</td>
<td>0.42</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 2: $x_0^a$ for $a \in \{15, 20, 25, 30, 35, 40\}$

The speed $c^a$ for different values of $a$ is given in the following table:
\begin{center}
\begin{tabular}{cccccc}
\hline
$a$ & $a = 15$ & $a = 20$ & $a = 25$ & $a = 30$ & $a = 35$ & $a = 40$
\hline
2.3150 & 2.3725 & 2.4025 & 2.4133 & 2.4136 & 2.4144 & \\
\hline
\end{tabular}
\end{center}

\textbf{Table 3: }\( c^a \) for \( a \in \{15, 20, 25, 30, 35, 40\} \)

Note that the Fisher-KPP speed with these parameters is \( c_{FKPP} = 2\sqrt{2} \), and, as expected, the full system speed is slower than for the Fisher-KPP speed. The solution for \( F \) is plotted vs the solution of the Fisher-KPP equation with \( \alpha = 2 \) on the plot below. As expected, we observe that \( F \) is above Fisher-KPP on the left and below on the right, although they are very close. Interestingly, the shape of the Fisher-KPP traveling wave gives a much better approximation of the traveling wave profile for the full system than the approximation of the speed of the full system by the Fisher-KPP speed.

\begin{center}
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{KPP vs \( F \) on \([-40, 40]\)}
\end{figure}
\end{center}

The difference between the two profiles is better seen when the level slope of the function \( F \) is plotted vs the level slope of the solution of the Fisher-KPP equation with \( \alpha = 2 \) in the plot below. We observe that \( F \) is steeper for all values of \( F(x) \).
Figure 4: Level slope comparison - $KPP$ vs $F$ on $[-35, 35]$

Figure 5: Level slope comparison - $KPP$ vs $F$ on $[-40, 40]$

References


