Recreational mathematics in Leonardo of Pisa's *Liber abbaci*

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Leonardo of Pisa's classic, medieval text *Liber abbaci* was long believed to have been the major work that introduced Hindu-Arabic arithmetic into Europe and thereby gave rise to the computational, financial, and commercial revolutions that began in thirteenth century Tuscany and from there spread throughout Europe. But it was not until 2003 that a key piece of evidence was uncovered that convinced historians that such was indeed the case.

I tell that fascinating story in my book *The Man of Numbers: Fibonacci's Arithmetic Revolution*.[1] In this article, I will say something about Leonardo's use of what we now call recreational mathematics in order to help his readers master the arithmetical techniques his book described. One example is very well known: his famous rabbit problem, the solution to which gives rise to the Fibonacci sequence. But there are many others.

Liber abbaci manuscripts

Leonardo lived from around 1170 to 1250; the exact dates are not know, nor is it known where he was born or where he died, but it does appear he spent much of his life in Pisa. As a teenager, he traveled to Bugia, in North Africa, to join his father who had moved there to represent Pisa's trading interests with the Arabic speaking world. It was there that he observed Muslim traders using what was to him a novel notation to represent numbers and a remarkable new method for performing calculations. That system had been developed in India in the first seven centuries of the first Millennium, and had been adopted and thence carried

[1] Devlin, 2011.

northwards by the Arabic-Speaking traders who taveled the Silk Route. He wrote *Liber abbaci* (the title translates as “Book of Calculation”) in 1202, soon after he returned to Pisa.

Leonardo began ths book with the words “Here begins the Book of Calculation compsed by Leonardo Pisano, Family Bonacci, in the year 1202.” The Latin phrase that I have translated as “family Bonacci” was *filius Bonacci*. In 1838, the historian Gillaume Libri took that phrase as the basis for a surname he gave him, “Fibonacci.” (Surnames were not common in medeival times.)

No copies of the 1202 manuscript have survived, but much later in Leonardo’s life, in 1228, by which time he had become famous throughout Italy, he wrote a much enlarged, second edition, of which fourteen copies exist today in various degrees of completeness. Seven are mere fragments, consisting of between one-and-a-half and three of the book’s fifteen chapters. Of the remaining seven, more substantial manuscripts, three are complete or almost so, and are generally regarded as the most significant. Those three are all in Italy.

One is housed in the Biblioteca Communale di Siena (Siena Public Library), where it has the reference number L.IV.20. It is generally believed to date from the thirteenth century, and according to some scholars is possibly the oldest, with a possible date of 1275, perhaps not long after Leonardo died — though others have suggested it may have been written as much as a century later.

Another, also believed to date from the late thirteenth, or perhaps the early fourteenth century, is in the Biblioteca Nazionale Centrale di Firenze (BNCF — Florence National Central Library), where it is listed in the catalogue as Conventi Sopressi C.1.2616. This manuscript is complete, which probably explains why the publisher Baldassarre Boncompagni used it as the basis for the first printed edition, which he brought out in the mid nineteenth century, even though it is not the best preserved, and perhaps not the oldest.

The third is in the Vatican Library in Rome, where it bears the reference mark Vatican Palatino #1343. This manuscript, from which Chapter 10 is missing, is also believed to date back to the late thirteenth century.

Of the remaining, more fragmented manuscripts, four are housed in the BNCF, along with the one mentioned above, one is in the Biblioteca Laurentiana Gadd in Florence (Gadd. Reliqui 36, dated to the 14^{th} century), one in the Biblioteca Riccardiana in Florence, one in the Biblioteca Ambrosiana in Milan, one in the Biblioteca Nazionale Centrale in Naples, and three in Paris (one in the Bibliothèque Mazarine, two in the Bibliothèque National de France).

Contents of *Liber abbaci*

Liber abbaci was a huge work. For instance, the Siena manuscript has 448 sides. The first, and only, printed edition of Leonardo's Latin text, published in Rome in 1857 by Baron Baldassare Boncompagni, an Italian bibliophile and medieval mathematical historian, has 387 densely packed pages.[2] The printed English language translation of *Liber abbaci*, by Laurence Sigler, which is based on Boncompagni's edition and published in 2002, runs to 672 pages.[3] It is the only translation of Leonardo’s text into a modern language.

Following a dedication and a short prologue, Leonardo divided *Liber abbaci* into fifteen chapters. Their titles vary from manuscript to manuscript, suggesting that

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[2] *Liber abbaci* was the first volume of a two-volume, printed collection of all of Leonardo’s works that Boncompagni published in Rome under the title *Scritti di Leonardo Pisano*. The second volume, containing all of Leonardo’s other works, appeared in 1862.
[3] Sigler, 2002.

the scribes who made copies felt free to make what they felt were clarifying improvements.

Chapter 1. On the recognition of the nine Indian figures and how all numbers are written with them; and how the numbers must be held in the hands, and on the introduction to calculations

Chapter 1 occupies 6 pages in Sigler's translation, and describes how to write — and read — whole numbers in the Hindus' decimal system. For large numbers, the numerals are grouped in threes to facilitate reading.

Chapter 2. On the multiplication of whole numbers

This is a 16-pages-long "how to" manual. [4] The approach differs little from the one used today to teach children how to multiply two whole numbers together. Leonardo begins with the multiplication of pairs of two-digit numbers and of multi-digit numbers by a one-place number, and then works up to more complicated examples.

Chapter 3. On the addition of them, one to the other

Chapter 3 is short, with just 5 pages of instructions. Giving a hit of things to come, he describes a procedure for keeping expenses in a table with columns for librae, soldi, and denari, the coinage used at the time (pounds, shillings, and pence in old English terms).

Chapter 4. On the subtraction of lesser numbers from greater numbers

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[4] The page counts are for Sigler's English language translation.

This, the shortest chapter in the book, occupies a mere 3 pages of Sigler's translation. The title explains the contents.

Chapter 5. On the divisions of integral numbers

Chapter 5 focuses on divisions by small numbers and on simple fractions. It has 28 pages of instructions. It describes the familiar "long-division algorithm," still taught today in many schools.

Chapter 6. On the multiplication of integral numbers with fractions

The topic is what are today called mixed numbers, numbers that comprise both a whole number and a fractional part. Leonardo explains that you calculate with them by first changing them to fractional form (what we would today call "improper fractions"), computing with them, and then converting the answer back to mixed form. This chapter takes up 22 pages.

Chapter 7. On the addition and subtraction and division of numbers with fractions and the reduction of several parts to a single part

Leonardo fills 28 pages showing how to combine everything that has been learned so far.

Chapter 8. On finding the value of merchandise by the Principal Method

Chapter 8 prvides the first real examples of practical mathematics, in the form of 51 pages of worked examples on the value of merchandise, using what we would today call reasoning by proportions. For example, Leonardo asks: if 2 pounds of barley cost 5 soldi, how much do 7 pounds cost? He then proceeds to show how to work out the answer. He makes use of simple diagrams of proportion, which he calls the "method of negotiation." Examples include monetary exchange, the sale of goods by weight, and the sale of cloth, pepper, cheese, canes, and bales.

Chapter 9. On the barter of merchandise and similar things

Leonardo presents another 33 pages (in the Sigler translation) of worked practical examples extending the discussion from the previous chapter. There are problems on the barter of common things, on the sale and purchase of money, on horses that eat barley in a certain number of days, on men who plant trees, and men who eat corn.

Chapter 10. On companies and their members

These 14 pages provide worked examples on investments and profits of companies and their members, showing how to decide who should be paid what.

Chapter 11. On the alloying of monies

Chapter 11 occupies 31 pages in Sigler's English language translation. The need for the methods Leonardo describes in this chapter was considerable. At that time, Italy had the highest concentration of different currencies in the world, with 28 different cites issuing coins during the course of the Middle Ages, seven in Tuscany alone. Their relative value and the metallic composition of their coinage varied considerably, both from one city to the next and over time. This state of affairs not only meant good business for money changers — and *Liber abbaci* provided plenty of examples on problems of that nature — but with governments regularly re-valuing their currencies, gold and silver coins provided a more stable base, and since most silver coinage of the time were alloyed with copper, problems of minting and alloying of money were important.

Chapter 12. On the solutions to many posed problems

This enormous chapter filled a staggering 186 pages with miscellaneous worked examples. Its primary focus is algebra. Not the symbolic reasoning we associate with the word today, rather "algebraic reasoning," expressed in ordinary language (and often referred to as "rhetorical algebra"). Much of Leonardo's focus is on applications of what is generally known as the "method of false position," which he refers to as the "tree method." This is a procedure used to solve problems equivalent (in modern terms) to a simple linear equation of the type $Ax = B$. The solver first picks an approximate answer and then reasons to adjust it to give the correct solution. Many of the problems Leonardo looks at he called "tree problems," which is why he speaks of solving them by the "method of trees." This is a class of problems he named after a particular puzzle he introduces in the chapter, where you want to know the total length of a tree when you are given the proportion that lies beneath the ground.

He also shows how to solve the same kinds of problems using what he called the "direct method" (*regula recta*), where you begin by calling the sought-after quantity a "thing" (*res*), and then forming an equation (expressed in words), which is then solved step-by-step to give the answer. Expressed symbolically, this is precisely the modern algebraic method. It was known to the Arab scholars, and was described by the Persian mathematician al-Khwārizmī around 830 in the book from whose title the modern word "algebra" stems.

Many of the problems Leonardo presents are of a financial nature, providing the businessman of the thirteenth century and subsequent with some extremely powerful tools that helped to revolutionize European trade and commerce.

Chapter 13. On the method elchataym and how with it nearly all problems of mathematics are solved

In modern terminology, *elchataym* is a rule, known also as “double false position,” used to solve one or more linear equations. The word “elchataym” is Leonardo’s Latin transliteration of the Arabic *al-khata’ayn*, which means “the two errors”. The name reflects the fact that you start with two approximations to the sought-after answer, one too low, the other too high, and then reason to adjust both until the correct answer is arrived at. It can be used to solve linear equations not only of the form $Ax = B$, for which single false position can be used, but also the more general form $Ax + B = C$. This chapter provides 41 pages of worked examples. Leonardo formulates the problems in several ingenious ways, in terms of snakes, four-legged animals, eggs, business ventures, ships, vats full of liquid which empty through holes, how a group of men should share out the proceeds when they find a purse or purses, subject to various conditions, how a group of men should each contribute to the cost of buying a horse, again under various conditions, as well as some in pure number terms.

Chapter 14. On finding square and cubic roots, and on the multiplication, division, and subtraction of them, and on the treatment of binomials and apotomes and their roots

Leonardo’s penultimate chapter offers 42 pages of worked examples. His main focus is on methods for handling roots. He uses the classifications given by Euclid in Book X of *Elements* for the sums and differences of unlike roots, namely binomials and apotomones. (The discovery that $\sqrt{2}$ is irrational led the ancient Greeks to a study of what they called “incommensurable magnitudes.” Euclid’s term for a sum of two incommensurables, such as $\sqrt{2} + 1$, was *binomial* (a "two-name" magnitude), and a difference, such as $\sqrt{2} - 1$, he called an *apotome*. Handling incommensurables by means of what we would now regard as algebraic expressions was a common feature of Greek and medieval mathgematics.)

In terms of mathematical content, Chapter 14 is little more than a collection of known methods and results, and Leonardo presents nothing significant not already found in *Elements*.

Chapter 15. On pertinent geometric rules and on problems of algebra and al-muchabala

This final chapter occupies 85 pages, again filled with worked examples. In modern terms, *al-muchabala* corresponds to manipulating the two sides of an equation while keeping it balanced. Once you know that, the chapter title says it all. Leonardo's approach differs little from that found in al-Khwārizmī's earlier book on algebra.

Word problems and recreational mathematics in *Liber abbaci*

It is with Chapters 8 and 9 that the reader first encounters real-world examples. Many of these involve items called "Pisan rolls"; the Pisan roll was a unit of weight, equal to 12 Pisan ounces.

Units of weight differed from one city to another. One worked problem in Chapter 8 is titled:[5]

[5] Page numbers refer to Sigler [2002].

> *On finding the worth of Florentine rolls when the worth of those of Genoa is known.* [p.148]

A typical worked problem in this section of the book starts like this:

> *If one hundredweight of linen or some other merchandise is sold near Syria or Alexandria for 4 Saracen bezants, and you will wish to know how much 37 rolls are worth, then …* [p.142]

Chapter 10, on companies and their members, demonstrates obviously valuable methods for solving problems such as determining the payouts in the following scenario:

> *Three men made a company in which the first man put 17 pounds, the second 29 pounds, the third 42 pounds, and the profit was 100 pounds.* [p.220]

Toward the end of Chapter 11, we find a curious problem that became quite well known to mathematicians (though nothing like as famous as his "rabbit problem"). It is called "Fibonacci's Problem of the Birds". Here is what Leonardo asks: [p.256]

> *On a Man Who Buys Thirty Birds of Three Kinds for 30 Denari*

> *A certain man buys 30 birds which are partridges, pigeons, and sparrows, for 30 denari. A partridge he buys for 3 denari, a pigeon for 2 denari, and 2 sparrows for 1 denaro, namely 1 sparrow for ½ denaro. It is sought how many birds he buys of each kind.*

What makes this problem particularly intriguing is that, on the face of it you don't have enough information to solve it. You arrive at that conclusion as soon as you try to solve it using modern symbolic algebra. If you let x be the number of partridges, y the number of pigeons, and z the number of sparrows, then the information you are given leads to two equations:

$x + y + z = 30$ (the number of birds bought equals 30)

$3x + 2y + \frac{1}{2}z = 30$ (the total price paid equals 30)

But as everyone learns in the high-school algebra class, you need three equations to find three unknowns. Well, in general that is true, but in this case you have one crucial additional piece of information that enables you to solve the problem. I'll give the solution to the problem at the end of the article. (Leonardo, as usual, presents the solution in words, not symbols, but apart from that, the solution I will give is his.)

Some of Leonardo's problems are presented in more whimsical terms. For instance, many are like this one in Chapter 12:

> A certain lion is in a certain pit, the depth of which is 50 palms, and he ascends daily 1/7 of a palm, and descends 1/9. It is sought in how many days will he leave the pit. [p.273]

But for the most part, the examples of *Liber abbaci* are couched in self-evidently practical terms. In fact, Chapter 12 is a mammoth piece of work that presents 259 worked examples, each of which Leonardo works through in great detail. Some examples require only a few lines to solve, others spread over several densely packed pages. In Sigler's English language translation, the entire chapter takes up 187 printed pages.

Like mathematics teachers and authors before him and since, Leonardo clearly knew that many of the people who sought to learn from him would have little interest for theoretical, abstract problems. And so, in order to explain how to use the new methods he learned during his visit to North Africa, Leonardo looked for ways to dress up the abstract ideas in familiar, everyday clothing. The result is that many of the problems he gave, while given with obvious practical applications in mind, would be classifed today as "recreational mathematics."

For instance, he presents a series of "purse problems" to try to put into everyday terms the mathematical problem associated with dividing up an amount of money — or anything else that people may want to divide up — according to certain rules. The first one goes like this:

> Two men who had denari found a purse with denari in it; thus found, the first man said to the second, If I take these denari of the purse, then with the denari I have I shall have three times as many as you have. Alternately the other man responded, And if I shall have the denari of the purse with my denari, then I shall have four times as many denari as you have. It is sought how many denari each has, and how many denari they found in the purse. [p.317]

Students today would be expected to solve this problem using elementary algebra (equations), and it takes at most a few lines. But modern symbolic algebra is a much later invention. Leonardo filled almost half a parchment page with his solution. (At heart, it is the same solution as today's algebra student will — or at least should — come up with, but without the simplicity given by symbolic equations it takes a lot more effort, and a great deal more space on the page, to work through to the answer.)

More complicated variations follow, including a purse found by three men, a purse found by four men, and finally a purse found by five men. Each problem took a full parchment sheet to solve. Adding still more complexity, he presented a particularly challenging problem in which four men with denari find four purses of denari, the solution of which fills four entire pages of Sigler's English language translation. In all, Leonardo presented eighteen different purse problems, which occupy nineteen-and-a-half pages of the English language translation.

Although many of the variants to the purse problem that Leonardo presents seem to be of his own devising, the original problem predated him by at least four hundred years. In his book *Ganita Sara Sangraha*, the ninth-century Jain mathematician Mahavira (ca. 800 – 870) presented his readers with this problem:

> Three merchants find a purse lying in the road. The first asserts that the discovery would make him twice as wealthy as the other two combined. The second claims his wealth would triple if he kept the purse, and the third claims his wealth would increase five fold.

The reader has to determine how much each merchant has and how much is in the purse. This is precisely Leonardo's first purse problem in Chapter 12 of *Liber abbaci*. Presumably Leonardo came across the puzzle by way of an Arab text.

Leonardo's purse problems involve divisions that require only whole numbers. To explain how to proceed when fractions are involved, he used a different scenario his readers could relate to: buying horses. On page 337 of Sigler's translation, we read:

> Here Begins the Fifth Part on the Purchase of Horses
> among Partners According to Some Given Proportion.

The first horse problem reads:

> *Two men having bezants found a horse for sale; as they wished to buy him, the first said to the second, If you will give me 1/3 of your bezants, then I shall have the price of the horse. And the other man proposed to have similarly the price of the horse if he takes 1/4 of the first's bezants. The price of the horse and the bezants of each man are sought.* [p.337]

Again, a mathematics student today would solve this problem using (symbolic) algebraic equations. Leonardo solved the problem using arithmetic:

> *You put 1/4 1/3 in order, and you subtract the 1 which is over the 3 from the 3 itself; there remains 2 that you multiply by the 4; there will be 8 bezants, and the first has this many. Also the 1 which is over the 4 is subtracted from the 4; there remains 3 that you multiply by the 3; there remains 9 bezants, and the other man has this many. Again you multiply the 3 by the 4; there will be 12 from which you take the 1 that comes out of the multiplication of the 1 which is over the 3 by the 1 which is over the 4; there remain 11 bezants for the price of the horse; this method proceeds from the rule of proportion, namely from the finding of the proportion of the bezants of one man to the bezants of the other; the proportion is found thus.* [p.337]

This definitely does not read like a modern day arithmetic textbook. What Leonardo is doing is explain, step-by-step, what digits you must write where, and what you do to them. A modern textbook would supply an algebraic formula into which you can simply plug in numbers, but algebraic notation was still several centuries away. Instead, Leonardo had to convey the method by giving many concrete examples, each with its unique twist and each using slightly different numbers.

Thirty-six pages and twenty-nine horse-type problems later, Leonardo evidently decided he had provided enough variations that his readers will have mastered the general technique. Along the way, he worked out a problem in which five men buy five horses [p.350], another, particularly tricky puzzle that he titles

> *A Problem Proposed to Us by a Most Earned Master of a Constantinople Mosque* [p.362]

in which five men buy not a horse but a ship, and another problem where seven men buy a horse, which, despite its seemingly greater complexity, turns out to be less intricate to solve [p.366].

With many of his fellow citizens frequent travelers, Leonardo knew that money problems about traveling were sure to arouse wide interest, so these provide his next set of examples. For his first traveler problem, he wrote:

> *A certain man proceeding to Lucca on business to make a profit doubled his money, and he spent there 12 denari. He then left and went through Florence; he there doubled his money, and he spent 12 denari. Then he returned to Pisa, doubled his money, and spent 12 denari, and it is proposed that he had nothing left. It is sought how much he had at the beginning.* [p.372]

While the ending of his little scenario might strike a familiar chord to many a vacation traveler to Tuscany today, this particular problem has a relatively easy solution. So too do some, though not all, of the many variants of the problem that Leonardo solves in the ensuing pages. He also illustrates the same arithmetical principles and solution methods with some other problems, including several about calculating interest on house purchases. [pp.384–392].

One problem leads to the particularly nasty answer that a certain businessman walks away from a partnership in Constantinople with a profit of

$$\frac{1}{2}\frac{7}{8}\frac{1}{8}\frac{4}{8}\frac{0}{8}\frac{21169}{24767}206 \text{ bezants}$$

To read this, you need to understand that, when Europeans in Leonardo's time learned the Hindu-Arabic number system, they wrote fractions before the whole number part, built up from right to left, with each new fraction representing that part of what is to the right. For example,

$$\frac{1}{2}\frac{2}{3}\frac{4}{5} \text{ means } \frac{1}{2\times3\times5}+\frac{2}{3\times5}+\frac{4}{5}\text{ , i.e. }\frac{29}{30}$$

The right-to-left ordering may simply be a carry-over from the writing of Arabic, although it is of interest to note that, for the most part, Arabic texts expressed Hindu-Arabic numbers rhetorically, using words instead of symbols. Leonardo would have articulated the above fraction as the Arabic mathematicians would both write and speak it: "For fifths, and two thirds of a fifth, and one half of a third of a fifth."

Decimal expansions are a special case of this notation when the denominators are all 10. For example, Leonardo would have written today's decimal number 3.14159 as

$$\frac{9}{10}\frac{5}{10}\frac{1}{10}\frac{4}{10}\frac{1}{10}3$$

Though decimal representation seems far simpler to us today, there was little need for it in Leonardo's time, as no one counted anything special in tenths. In fact, the method used to represent fractions was particularly well-suited for calculations involving money. In the monetary system used in medieval Pisa, 12 denari equaled 1 soldus and 20 soldi equaled 1 libra, so 2 librae, 7 soldi, and 3 denari would be written

$$\frac{3}{12}\frac{7}{20}2$$

Units of weight and measure could be even more complex. According to Leonardo, Pisan hundredweights:[6]

> ... have in themselves one hundred parts each of which is called a roll, and each roll contains 12 ounces, and each of which weighs ½ 39 pennyweights; and each pennyweight contains 6 carobs and a carob is 4 grains of corn.

Imagine having to calculate with those units.

Interestingly, an Arabic arithmetic text written by al-Uqlidisi in Damascus in 952 did in fact use place-value decimals to the right of a decimal point, but no one saw any particular reason to adopt it, and so the idea died, not to reappear again for five hundred years, when Arab scholars picked up the idea once more. Decimal fractions were not used in Europe until the sixteenth century.

[6] Sigler, 2002, p.128.

Fractions written after the whole number part in Leonardo's time denote multiplication. For example, ½ of 3.14159 could be written

$$\frac{9}{10}\frac{5}{10}\frac{1}{10}\frac{4}{10}\frac{1}{10}3\frac{1}{2}$$

The Fibonacci sequence

Leonardo's most well known connection to present day recreational mathematics comes toward the latter part of his Chapter 12. Nestled between problems involving the division of food and money, Leonardo throws in a whimsical problem about a growing rabbit population. He did not invent the problem; it dates back at least to the Indian mathematicians in the early centuries of the Current Era who developed the number system *Liber abbaci* describes. He clearly realized, however, as did his Hindu predecessors, that it is an excellent, easy problem for practicing how to use the new number system. And so he included it. What he obviously could not foresee was that, although later generations of historians of mathematics would consider *Liber abbaci* one of the most influential books of all times, it would be in associaion with that one little problem that he would be most widely known.

In what was to become his most famous passage, Leonardo wrote his way into twentieth century popular culture with these words [p.404]:

How Many Pairs of Rabbits Are Created by One Pair in One Year.

A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

As usual, Leonardo explained the solution in full detail, but the modern reader can rapidly discern the solution method by glancing at the table Leonardo also presented, giving the rabbit population each month:

Beginning	1
first	2
second	3
third	5
fourth	8
fifth	13
sixth	21
seventh	34
eighth	55
nintth	89
tenth	144
eleventh	233
twelfth	377

The general rule is that each successive number is the result of adding together the previous two; thus, 1 + 2 = 3, 2 + 3 = 5, 3 + 5 = 8, etc. As Leonardo observes at the end of his solution, although he has calculated the population at the end of one year, namely 377, this simple rule gives you the population after any number of months.

The numbers generated by the addition process Leonardo described to solve the rabbit problem are known today as the Fibonacci numbers. They first appeared, it seems, in the *Chandahshastra* (The Art of Prosody) written by the Sanskrit grammarian Pingala some time between 450 and 200 BCE. Prosody was important in ancient Indian ritual. In the sixth century, the Indian mathematician Virahanka showed how the sequence arises in the analysis of metres with long and short syllables. Subsequently, the Jain philosopher Hemachandra (c.1150) composed a text on them.

The Fibonacci numbers were given their name by the French mathematician Edouard Lucas in the 1870s, after the French historian Guillaume Libri gave Leonardo the nickname Fibonacci in 1838. A lot of the initial — and subsequent — fascination with them is due to the surprising frequency with which they seem to arise when you go out into the garden and count things. Many of those occurrences arise because of a well known rconnection to the Golden Ratio. But that is a well known story, repeated often — though in many cases with false claims alongside the valid ones, perhaps the most frequent false claim being that the Fibonacci sequence arises naturally in the shell of the Chambered Nautilus.[7]

APPENDIX: The solution to Leonardo's Problem of the Birds

The problem of the birds says:

> A certain man buys 30 birds which are partridges, pigeons, and sparrows, for 30 denari. A partridge he buys for 3 denari, a pigeon for 2 denari, and 2 sparrows for 1 denaro, namely 1 sparrow for ½ denaro. It is sought how many birds he buys of each kind.

Here is a modern solution. You let x be the number of partridges, y the number of pigeons, and z the number of sparrows. The information you are given then leads to two equations:

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[7] The Chambered Nautlus shell has the shape of a logarithmic spiral, but its angle of rotation is not the goden ratio, so you don't get the Fibonacci numbers.

$$x + y + z = 30$$ (the number of birds bought equals 30)

$$3x + 2y + \tfrac{1}{2}z = 30$$ (the total price paid equals 30)

This looks like an impossible task, since you have three unknowns but only two equations. But the problem provides you with a crucial additional piece of information that enables you to solve it: The values of the three unknowns must all be positive whole numbers. (You are told he buys three kinds of birds, so none of the unknowns can be zero, and he surely does not buy fractions of birds.)

Start by doubling every term in the second equation to get rid of that fraction:

$$x + y + z = 30$$

$$6x + 4y + z = 60$$

Subtract the first equation from the second to eliminate z:

$$5x + 3y = 30$$

Notice that 5 divides the first term and the third, so it must also divide y. So y is one of 5, 10, 15, etc. But y cannot be 10 or anything bigger, since then it could not satisfy that last equation! Thus y = 5. It follows that x = 3 and z = 22. Neat, eh?

Bibliograpy

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