18.100C Writing Assignment 4

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In this assignment we are to discuss either compactness or connectedness. I will give an alternative proof that compact sets are closed (Rudin 2.34); and solve Problem 6 in Rudin Ch. 4.

1 Compact subsets of metric spaces are closed.

Suppose not. Let K be a compact subset of a metric space X. We assume that K is not closed.

Since K is not closed, there exists a limit point x that lies in K^c. We will construct an open cover of K by the following. For each integer $n = 1, 2, 3, \ldots$, we define:

$$V_n = \left\{p \in K \mid p \notin N_{1/n}(x)\right\} \tag{1}$$

where $N_{1/n}(x)$ is as usual the neighborhood of radius $1/n$ about x.

Clearly $\bigcup_n V_n$ is an open cover of K. However it has no finite subcover. Since $V_{n+1} \supset V_n$, a finite union $V_{n_1} \cup V_{n_2} \cup \ldots \cup V_{n_k} = V_M$, where $M = max(n_1, n_2, \ldots, n_k)$. But since x is a limit point of K, the set $N_{1/M}(x) \cap K$ is nonempty; and hence the finite union does not cover K. Contradiction.

2 Rudin Ch 4, Problem 6.

2.1 If f is defined on E, the *graph* G of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, G is a subset of the plane. Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

We will first show that the continuity of f and the compactness of E imply that G is a bounded and closed subset of $\Re^2$; and hence that it is compact.

Since f is continuous and E is compact, the image $f(E)$ is also compact (Rudin 4.14). Furthermore, both E and $f(E)$ are subsets of $\Re$; hence by Heine-Borel they are closed and bounded. Let x_1, x_2, y_1, y_2 be such that $x_1 < x < x_2$ for every $x \in E$ and $y_1 < y < y_2$ for every $y \in f(E)$.

The graph G consists of points $(x, f(x))$ for $x \in E$. Hence, by the above inequalities, G is contained in the rectangular subset of the plane $\left\{(x, y) \in \Re^2 \mid x_1 < x < x_2;\ y_1 < y < y_2\right\}$. Clearly, G is bounded.

Now we show that G is closed. Suppose $p = (a, b)$ is a limit point of G. So for any neighborhood about p, we can find a point $(x, f(x))$ in G. In particular, this implies that a is a limit point of E. Since E is closed, a lies in E. We now show that it must be the case that $b = f(a)$.

Assume not. Suppose $D = d((a, b), (a, f(a))) > 0$. Since f is continuous, for $\epsilon = \frac{D}{2}$ there exists some corresponding δ such that whenever $d(x, a) < \delta$ for any $x \in E$, we have $d(f(x), f(a)) < \frac{D}{2}$. Choose $r = min(\frac{D}{2}, \delta)$; the neighbordhood $N_r((a, b))$ then has no points of G. To show this, we consider the interval $(a - r, a + r)$ only, since any point that lies in N has an x-coordinate within this segment. But since $r \leq \delta$, for every x in the interval we have $f(x) < f(a) + \frac{D}{2}$. On the other hand, any point that lies in N has a y-coordinate y that satisfies $y > f(a) - r \geq f(a) - \frac{D}{2}$. No point can satisfy both inequalities and hence $N \cap G$ is empty. Contradiction.

Thus G is a closed and bounded subset of $\Re^2$. Heine-Borel applies and G is compact.

To prove the converse, let G be compact but suppose that f is not continuous. Then there is some point $p \in E$ such that $\lim_{x \to p} f(x) \neq f(p)$.

Recall that for $\lim_{x \to p} f(x) \neq f(p)$, p must be a limit point of E. If p were instead an isolated point, we can find a δ such that $N_\delta(p) \cap E = \{p\}$. Then, for any $\epsilon > 0$, we have for every $x \in E$ for which $d(x, p) < \delta$ that $d(f(x), f(p)) < \epsilon$, simply because the only point in E that satisfies the criterion is $x = p$. So if p is not a limit point, then we trivially have $\lim_{x \to p} f(x) = p$, contrary to our condition.

For $\lim_{x \to p} f(x) \neq p$, there are two possibilities:

Suppose $q = \lim_{x \to p} f(x)$ exists but $q \neq f(p)$. Here, we will argue that (p, q) is a limit point of G but does not lie in it. So G is not closed and therefore cannot be compact.

Fix $\epsilon > 0$. Since $\lim_{x \to p} f(x) = q$, there exists a $\delta > 0$ such that for every $x \in E$ that satisfies $d(x, p) < \delta$, we have $d(f(x), f(p)) < \frac{\epsilon}{2}$. Put $r = min(\delta, \frac{\epsilon}{2})$. Since p is a limit point of E, we are able to select a point x from $N_r(p) \cap E$. Then, by the triangle inequality:

$$\begin{aligned} d((x, f(x)), (p, q)) &\leq d((x, f(x)), (p, f(x))) + d((p, f(x)), (p, q)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

Since $(x, f(x)) \in G$, (p, q) is a limit point of G. However (p, q) cannot lie in G since the only element in G with p as its x-coordinate is $(p, f(p)) \neq (p, q)$. So G is not closed, and hence it cannot be compact. Contradiction.

Now we consider the case when $\lim_{x \to p} f(x)$ does not exist. Again, we take advantage of the fact that p is a limit point in E.

First, we define the "lower" and "upper" sequences x_k^- and x_k^+ in E by the following.

For $n = 1, 2, 3, ...$, define x_n^- by choosing an element from the set $(p - \frac{1}{n}, p) \cap E$. Likewise, we define x_n^+ from $(p, p + \frac{1}{n}) \cap E$. Since the sequences are in E, $f(x_k^{-/+})$ is defined and we can consider the sequences in $\Re^2$:

$$\begin{aligned} G_k^- &= (x_k^-, f(x_k^-)) \\ G_k^+ &= (x_k^+, f(x_k^+)) \end{aligned}$$

By construction $\{G_k^-\}$ and $\{G_k^+\}$ are infinite subsets of G; and, by sequential compactness, must have limits in G. Furthermore, it is clear that the x-coordinate of both limits must be p. Let $G_k^- \to (p, y^-)$ and $G_k^+ \to (p, y^+)$. Since G is compact, it is closed and both points must lie in G.

However, (p, y^-) must be distinct from (p, y^-). If instead $q = y^- = y^+$, then for any $\epsilon > 0$ we can find δ^- such that $d(x, p) < \delta^-$ with $x \in (p-1, p) \cap E$ implies that $d(f(x), f(p)) < \epsilon$. Similarly, we can obtain a δ^+ for the upper seqeunce. If we let $\delta = min(\delta^-, \delta+)$, we see that $\lim_{x \to p} f(x) = (p, q)$, contrary to our premise that the limit does not exist.

So, (p, y^-) and (p, y^+) are in G and they are distinct. This contradicts the notion that any function assigns only one value to an element in its domain.

Therefore, the compactness of G implies the continuity of f.