Ion trajectory simulation - THK

The objective is to perform a numerical simulation of the classical equation of motion:

$$M\ddot{z} = -QE(z,t) \tag{1}$$

where $M = 1.4 \times 10^{-25}$kg is the Strontium mass and $Q = 1.6 \times 10^{-19}$C is the unit charge. I will use the z-coordinate for the discussion, although the analysis will be equally valid for other coordinates. Note that I have assumed that the z-coordinate is decoupled from the others, which is strictly true only when z is one of the principal axes of the trap in the linear approximation.

One then separates the driving field into the rf and stray field components. The rf term can be decomposed into a product of a spatial profile and a sinusoidal modulation in time at frequency Ω. The stray field is assumed to be a constant dc field. Additionally, I'll also add a velocity-proportional damping force:

$$\begin{aligned} M\ddot{z} &= -\beta\dot{z} - Q\left(E_{\text{rf}}(z)\cos(\Omega t) + E_{\text{stray}}\right) &(2)\\ &= -\beta\dot{z} - QE_{\text{rf}}(z)\cos(\Omega t) - QE_{\text{stray}}. &(3)\end{aligned}$$

Here, note that we can then substitute a numerically-computed (e.g. CPO) field profile for $E_{\text{rf}}(z)$ and numerically integrate the equation of motion. For now, we will proceed with the linear approximation, which will yield the standard Mathieu equation.

We perform a linearization of the rf field $E_{\text{rf}}(z)$ at its node z_0. It then follows:

$$E_{\text{rf}}(z) \approx V_{\text{rf}} f \cdot (z - z_0) \tag{4}$$

where f is a constant of unit [length]$^{-2}$ dependent on the geometry of the trap; it can be thought of as a voltage-normalized coefficient for the quadrupole field. For example, for the symmetric point Paul trap with inner and outer radii a and b,

$$f(a,b) = \sqrt{\frac{9(b^{\frac{2}{3}} - a^{\frac{2}{3}})^2(b^{\frac{2}{3}} + a^{\frac{2}{3}})^6}{b^{\frac{4}{3}}a^{\frac{4}{3}}(b^{\frac{4}{3}} + b^{\frac{2}{3}}a^{\frac{2}{3}} + a^{\frac{4}{3}})^5}}. \tag{5}$$

Substituting the linear approximation into the EOM, we find:

$$M\ddot{z} = -\beta\dot{z} - QV_{\text{rf}} f(z - z_0)\cos(\Omega t) - QE_{\text{stray}}. \tag{6}$$

We then "recenter" the coordinate t the rf node z_0 and introduce the standard (Mathieu) change of variables $\tau = \Omega t/2$, yielding:

$$\frac{M\Omega^2}{4} \cdot \frac{d^2 z}{d\tau^2} = -\frac{\beta\Omega}{2} \cdot \frac{dz}{d\tau} - QV_{\text{rf}} f \cdot z\cos(\Omega t) - QE_{\text{stray}}. \tag{7}$$

Multiplying through by $\left(\frac{M\Omega^2}{4}\right)^{-1}$, we then find:

$$\frac{d^2 z}{d\tau^2} = -\left(\frac{2\beta}{M\Omega}\right) \cdot \frac{dz}{d\tau} - \left(\frac{4QV_{\text{rf}} f}{M\Omega^2}\right)\cos(\Omega t) \cdot z - \left(\frac{4Q}{M\Omega^2}\right) E_{\text{stray}}. \tag{8}$$

The result is a normalized equation of motion whose units are those of z, as shown below:

$$\frac{d^2 z}{d\tau^2} + b\frac{dz}{d\tau} + 2q\cos(2\tau) \cdot z = c \tag{9}$$

where we define the following nondimensional parameters:

- $q = \frac{2QV_{\mathrm{rf}}f}{M\Omega^2}$: Standard definition of the Mathieu q-parameter
- $b = \frac{2}{M\Omega} \cdot \beta$: Nondimensional damping parameter
- $c = -\frac{4Q}{M\Omega^2} \cdot E_{\mathrm{stray}}$: Nondimensional stray charge