A Brief Introduction to Supersymmetric Quantum Mechanics *

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In class, we've learned about the supersymmetric, or factorization, method for solving some quantum mechanical potentials. It's difficult to see how this method relates to everything else you might have heard about supersymmetry and high energy particle physics. In these notes, we will attempt to build a bridge between these two applications, first by giving an introduction to supersymmetric quantum mechanics, then by connecting it to the methods learned in class. Finally, we will mention the essential elements that generalize to quantum field theory and those that do not. Most of the exposition in these notes will follow closely the very excellent notes of [2].

1 SUSY QM

Let us postulate a Hilbert space of quantum mechanical states. In addition to the Hamiltonian, H, we will assume that their exist two other operators on the space, Q and $Q^\dagger$. We require the operators to satisfy the *supersymmetry algebra*:

$$\begin{aligned} \{Q, Q\} &= 0 \\ \{Q^\dagger, Q^\dagger\} &= 0 \\ \{Q, Q^\dagger\} &= 2H, \end{aligned} \tag{1}$$

where $\{A, B\} \equiv AB + BA$ is known as the *anticommutator* of A and B. Notice, in particular, that (1) implies that $Q^2 = (Q^\dagger)^2 = 0$. It is also a straightforward exercise to show that $[Q, H] = [Q^\dagger, H] = 0$.

From the algebra alone, we can immediately derive one of the most important consequences of supersymmetry, the positivity of energy. In any state, the expectation value of the Hamiltonian is

$$\langle H \rangle = \frac{1}{2}\langle \Omega | \{Q^\dagger, Q\} | \Omega \rangle$$

*These notes have been prepared as a supplement for 8.05 students by David Guarrera.

$$
\begin{aligned}
&+ \quad \frac{1}{2}\left|Q|\Omega\rangle\right|^2 + \frac{1}{2}\left|Q^\dagger|\Omega\rangle\right|^2 \\
&\geq \quad 0. \qquad (2)
\end{aligned}
$$

In particular, for any energy eigenstate, the energy is always greater than or equal to zero. Also, a zero energy state is necessarily annihilated by both Q and $Q^\dagger$.

We can diagonalize the Hamiltonian and work with a particular subspace spanned by eigenstates of a particular energy, E_n. On this subspace, if $E_n > 0$, we can define

$$
a = \frac{Q}{\sqrt{2E_n}} \quad \text{and} \quad a^\dagger = \frac{Q^\dagger}{\sqrt{2E_n}}. \qquad (3)
$$

The supersymmetry algebra (1) becomes

$$
\begin{aligned}
\{a, a\} = \{a^\dagger, a^\dagger\} = 0 \\
\{a, a^\dagger\} = 1. \qquad (4)
\end{aligned}
$$

Since $[a, H] = [a^\dagger, H] = 0$, if $|n\rangle$ has energy eigenvalue E_n, so will $a^\dagger|n\rangle$ (if it is nonzero). In fact, since $(a)^2 = 0$, there will exist some $|-\rangle$ such that $a|-\rangle = 0$. $|+\rangle \equiv a^\dagger|-\rangle$ will then also have energy E_n. $(a^\dagger)^2|-\rangle$ is zero and the algebra (4) tells us that $aa^\dagger|-\rangle = |-\rangle$. In this way, for a particular energy that is greater than zero, a and $a^\dagger$ are raising and lowering operators. We see that, for any positive energy, states always come in pairs.

In the $|+\rangle, |-\rangle$ basis, the raising and lowering operators are

$$
a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad , \quad a^\dagger = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \qquad (5)
$$

We can define another operator, which anticommutes with all others. It is called the *fermion number operator* (the name will be explained later)

$$
(-)^F \equiv 2a^\dagger a - a = \frac{1}{2E}[Q^\dagger, Q] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (6)
$$

Note, in particular, its action on states

$$
\begin{aligned}
(-)^F|+\rangle &= |+\rangle \\
(-)^F|-\rangle &= -|-\rangle. \qquad (7)
\end{aligned}
$$

We must now deal with the possible $E = 0$ states. For these states $\{Q, Q^\dagger\} = 0$. We've also shown that for such a state, $|0\rangle$, $Q|0\rangle = Q^\dagger|0\rangle = 0$ (2) . Therefore, we cannot form raising and lowering operators for this state and, in general, there is no degeneracy. We still have not addressed the problem of whether or not a zero energy state exists; this will be discussed in the next section.

If such a state does exist, we'd still like to define how the fermion number operator acts on it. There is no way to do this in general, though in specific contexts, the correct action is usually clear (we will encounter this in the next section). Since we'd still like $\left((-)^F\right)^2 = 1$, $(-)^F|0\rangle = \pm|0\rangle$.

2 A One Dimensional Spinning Particle

Let's be more concrete and look in particular at a particle with spin moving in one dimension. The wavefunction is

$$\Psi(x) = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix}. \tag{8}$$

The first component represents spin up in the z direction, while the second component is spin down. We define the operators

$$\begin{aligned} Q &\equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} (p - iW'(x)) \\ Q^\dagger &\equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} (p + iW'(x)), \end{aligned} \tag{9}$$

with $p = -i\hbar\frac{d}{dx}$ and $W(x)$ is a yet unspecified function known as the *superpotential*. We are using units where the mass of the particle $m = 1$. For any W, we can satisfy the supersymmetry algebra (1) by requiring

$$H = \frac{1}{2}\{Q, Q^\dagger\} = \left(\frac{p^2}{2} + \frac{(W')^2}{2}\right)\mathbb{1} - \frac{\hbar}{2}\sigma^3 W''. \tag{10}$$

The first term is the usual kinetic energy, the second a potential energy for a spinless particle, and the third is the potential energy for a spin 1/2 particle in a (possibly position dependent) magnetic field. We can now define the fermion number operator for *all* states (not just the positive energy ones) as

$$(-)^F = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

On positive energy states, it is an easy check that this reduces to the definition (6), $(-)^F = \frac{1}{2E}[Q^\dagger, Q]$. Now, though, we have a perfectly good definition for all states.

As an example, let's take the particularly simple superpotential $W(x) = \frac{1}{2}\omega x^2$. The Hamiltonian (10) becomes

$$H = \frac{p^2}{2} + \frac{1}{2}\omega^2 x^2 - \frac{\hbar\omega\sigma^2}{2}. \tag{12}$$

This is the Hamiltonian for a spin 1/2 particle in a harmonic oscillator potential and a magnetic field ω (in $m = 1$ units). After 8.05, we can immediately write down the energies

$$E^n_\pm = \hbar\omega(n + \frac{1}{2} \mp \frac{1}{2}) \tag{13}$$

and eigenvectors

$$\begin{pmatrix} \psi_n(x) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \psi_n(x) \end{pmatrix}, \tag{14}$$

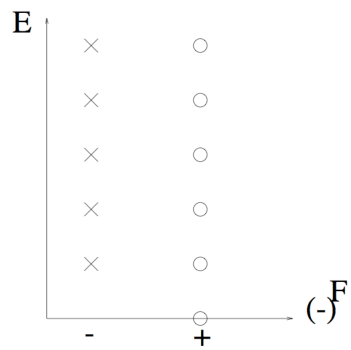

Figure 1: The energy diagram for the supersymmetric harmonic oscillator. This diagram was shamelessly stolen from [2].

with the ψ_n's the usual Hermite polynomials. The first solutions have fermion number $+1$, and the second have -1. This simple system exhibits many of the important qualities of supersymmetry discussed in the previoius section:

- All $E \geq 0$.
- All $E > 0$ states are paired. The paired states with degenerate energy have opposite fermion number.
- There is at most one $E = 0$ state. It's fermion number is, in general, not determined. In this case, there is one state with fermion number $(-)^F = +1$.

Going back to the case of general W, it is very easy to determine the zero energy states. Equation (2) says that for a state to satisfy $E = 0$, the first order equations

$$Q|\Psi\rangle = Q^\dagger|\Psi\rangle = 0 \tag{15}$$

are obeyed. For a state (8), this implies that

$$\begin{aligned} (p - iW')\psi_+ &= 0 \\ (p + iW')\psi_- &= 0. \end{aligned} \tag{16}$$

Integrating,

$$\begin{aligned} \psi_+ &\propto e^{-W/\hbar} \\ \psi_- &\propto e^{+W/\hbar}. \end{aligned} \tag{17}$$

If $W \to \infty$ as $x \to \pm\infty$, then ψ_+ is normalizable while ψ_- is not. Similarly if $W \to -\infty$ as $x \to \pm\infty$, then ψ_- is normalizable and ψ_+ is not. In all other cases, neither is normalizable and there is no zero energy state. In this case, we say (for reasons that will be explained later) that supersymmetry is broken.

3 Connection to the Supersymmetric Method

As you've read the previous sections, you may have noticed many parallels between some of the calculations in SUSY QM and the supersymmetric method. In this section, we try and make those connections more concrete. Recall that in the supersymmetric method $H^{(1)} = \mathcal{A}^\dagger\mathcal{A}$ and $H^{(2)} = \mathcal{A}\mathcal{A}^\dagger$. Here, there exists a symmetry between eigenstates of $H^{(1)}$ and $H^{(2)}$, while in SUSY QM there exists a symmetry between eigenstates of the *same* Hamiltonian, though the eigenstates differ by fermion number. By now, you might have noticed that the symmetry of the two systems is basically the same with replacements $Q \leftrightarrow \mathcal{A}$ and $Q^\dagger \leftrightarrow \mathcal{A}^\dagger$. The proofs of all the useful properties of the supersymmetric method map directly onto proofs of the interesting properties of the SUSY QM spectrum. We remind the reader of some of those proofs now:

1. In SUSY QM, the proof that the spectrum is nonnegative ($E \sim \langle\Omega|\{Q^\dagger, Q\}|\Omega\rangle \geq 0$) is exactly the same as the proof that the spectrum of $H^{(1)}$ and $H^{(2)}$ are nonnegative, e.g. $E^{(1)} \sim \langle\Omega|\mathcal{A}^\dagger\mathcal{A}|\Omega\rangle \geq 0$.

2. Both the supersymmetric method and SUSY QM have twofold energy degeneracies for positive energy states. In particular $\mathcal{A}$ ($or \mathcal{A}^\dagger$) maps eigenstates of $H^{(1)}$ ($H^{(2)}$) to eigenstates of $H^{(2)}$ ($H^{(1)}$) with the same energy. In an exactly parallel manner, in SUSY QM, Q and $Q^\dagger$ map eigenstates of the Hamiltonian to different eigenstates of the *same* Hamiltonian, but with different fermion numbers. This equivalence can be made more concrete. On states with $E_n > 0$, $QQ^\dagger|-\rangle = E_n|+\rangle$ and $Q^\dagger Q|+\rangle = E_n|-\rangle$. Therefore on "bosonic states" (states with $(-)^F = +1$), $H = H_b = Q^\dagger Q$, while on "fermionic states" ($(-)^F = +1$), $H = H_f = QQ^\dagger$. Therefore, in SUSY QM we can actually think of Q and $Q^\dagger$ as mapping eigenstates between *different* Hamiltonians that each act on states of opposite fermion number. This is the origin of equation (66) in reference [1].

3. In both the supersymmetric method and SUSY QM, the method for solving for zero energy solutions is exactly the same. In particular, one must solve $Q|\psi\rangle = 0$ ($\mathcal{A}|\psi\rangle = 0$) or $Q^\dagger|\psi\rangle = 0$ ($\mathcal{A}^\dagger|\psi\rangle = 0$). In both cases, at most one of these will have a solution. In SUSY QM, this one state can have either fermion number, while in the supersymmetric method the state can be a solution to either Hamiltonian. In both cases, if there is no zero energy solution, we say that SUSY is broken. One might notice that the exact definitions for Q, $Q^\dagger$ (9) and $\mathcal{A}$,$\mathcal{A}^\dagger$ are almost *exactly* the same, except for the fact that the SUSY QM definition involves W' while the factorization definition involves a superpotential I'll call $\widetilde{W}$. The definitions are the same provided $\widetilde{W} = W'$. This is only a matter of convention; the definition of the superpotential in these notes generalizes slightly more nicely to supersymmetric field theories.

4 Generalization to Quantum Field Theory and Particle Physics

In a very specific sense, quantum mechanics is a one dimensional quantum field theory (the one dimension is time). At high energies, the correct physics description is a four dimensional quantum field theory. In four dimensions, the correct SUSY algebra is not equation (1), but a generalization that accounts for, among other things, the fact that energy (and hence H) should be relativistically treated on equal footing with momentum (i.e. $\vec{v} = (E, \vec{p})$ transforms as a four vector). We will not write the exact generalization here, but it suffices to say that it is slightly more complicated. Many of the features we discussed for SUSY QM continue to hold for SUSY QFT, but others change.

1. If SUSY is not broken (i.e. there are zero energy states), the energy degeneracy amongst the $E > 0$ states continues to hold. Using the relativistic relation (in $c = 1$ units)

$$m^2 = E^2 - p^2, \tag{18}$$

   we can go to a particle's rest frame to find $E = m$. Therefore, the energy degeneracy means that there are two particles (superpartners) at each mass level.

2. We can still define a fermion number operator $(-)^F$. The particles with degenerate masses have opposite fermion numbers $(-)^F = \pm 1$, just as in SUSY QM. However, in QFT this fermion number operator genuinely corresponds to whether or not a particle is a boson or a fermion, meaning that $(-)^F = +1$ for bosons and -1 for fermions. We have learned that SUSY is a symmetry between bosons and fermions!

3. In QFT, it is still the case that the number of fermionic zero energy states need not equal the number of bosonic zero energy states (although in QFT, there will, in general, be more than one zero energy state). The quantity $Tr(-)^F$, known as the *Witten index* counts the number of bosonic zero energy states minus the number of fermionic zero energy states. To see this, note that the trace for $E > 0$ states cancels because the states enter in pairs, with different signs.

4. If there are no zero energy states, supersymmetry is broken. When this is the case, there are issues in defining Q and $Q^\dagger$ on states in the infinite spatial volume limit. Therefore, the degeneracy derived in section 2 need not hold. (This is unlike SUSY QM, where the QFT has no spatial dimensions, and so this limit need not be taken. The degeneracy holds even when supersymmetry is broken). Therefore, if supersymmetry is broken, the superpartners may be at a higher mass than currently observed particles (and we may observe some of them soon, at the LHC).

5. The concept of a superpotential remains very important for supersymmetric field theories. In particular, one can define an ($N = 1$) supersymmetric field theory by specifying only

   - The gauge group and matter content; this specifies how many and what types of particles will appear and the various (non-super) symmetries between them.
   - The superpotential.
   - The Kahler potential; this is a new ingredient which does not appear in SUSY QM.

   In fact, the superpotential satisfies some very beautiful physical properties that are beyond the scope of these notes.

References

[1] "8.05 Class Notes on Factorization/Supersymmetric Method." Available at http://web.mit.edu/8.05/handout.shtml."

[2] Argyres, Phillip. "Notes on N=1 d=4 Global Supersymmetry," 1996. Available at "http://www.physics.uc.edu/ argyres ."

[3] Witten, Edward. "Constraints on Supersymmetry Breaking," Nuclear Physics B202: 253, 1982.