Delta function for a double root (corrected version)

In lecture I explained how to deal with $\delta(g(x))$ if $g'(x)$ is nonzero at the roots of $g(x)$. Here I show how to handle the case where $g'(x) = 0$ but $g''(x) \neq 0$ at a root. This case corresponds to the merging of two roots of $g(x)$, i.e. a double root. For simplicity, take $g(x) = ax^2$ where a is a constant.

To evaluate $I \equiv \int_{-\infty}^{\infty} f(x)\delta(ax^2)\,dx$, we must change variables from x to $y = ax^2$. However, we must first write two separate integrals for $x < 0$ and $x > 0$:

$$I = \int_{-\infty}^{\infty} f(x)\delta(ax^2)\,dx = \int_{-\infty}^{0} f(x)\delta(ax^2)\,dx + \int_{0}^{\infty} f(x)\delta(ax^2)\,dx \ .$$

Now change variables to $x = -\sqrt{y/|a|}$ for the first integral and $x = +\sqrt{y/|a|}$ for the second one. This gives

$$I = \frac{1}{2|a|} \int_{0}^{\infty} \sqrt{\frac{|a|}{y}} \left[f\left(-\sqrt{\frac{y}{|a|}}\right) + f\left(\sqrt{\frac{y}{|a|}}\right) \right] \delta(y)dy \ ,$$

which we may rewrite

$$I = \frac{1}{2|a|} \int_{0}^{\infty} \left|\frac{a}{y}\right|^{1/2} \left[f\left(-\left|\frac{y}{a}\right|^{1/2}\right) + f\left(\left|\frac{y}{a}\right|^{1/2}\right) \right] \delta(y)dy \ .$$

This is still difficult to evaluate because the zero of the delta function occurs on the boundary of the integration region. This can be fixed by noting that the integrand is an even function of y, so that $\int_{0}^{\infty} = \int_{-\infty}^{0}$ and therefore

$$I = \frac{1}{4|a|} \int_{-\infty}^{\infty} \left|\frac{a}{y}\right|^{1/2} \left[f\left(-\left|\frac{y}{a}\right|^{1/2}\right) + f\left(\left|\frac{y}{a}\right|^{1/2}\right) \right] \delta(y)dy \ .$$

Finally, using the delta function to evaluate the integral gives

$$I = \frac{1}{4|a|} \lim_{x \to 0} \frac{f(-x) + f(x)}{|x|} \ .$$

If $f(0) \neq 0$ the result is infinity. By comparing this result with the original integral, we deduce

$$\delta(ax^2) = \lim_{\epsilon \to 0} \frac{\delta(x+\epsilon) + \delta(x-\epsilon)}{4|ax|} \ .$$

This result is a factor of two smaller than what one gets for a pair of simple roots, $\delta(g(x)) = \sum_i |g'(x_i)|^{-1} \delta(x - x_i)$ where $g(x_i) = 0$. You might have guessed the factor of two because when roots merge there is no integration domain between them. However, it is always safest to go through the steps of changing variables and rearranging the integral as was done above.

Notice that we treated $\delta(x)$ as though it were an ordinary function until it was used to evaluate an integral. This approach always works (and is how I like to think of the delta function) because $\delta(x)$ can be defined as the limit of a sequence of functions, with the limit taken only at the very end of a calculation.

To demonstrate the last point, consider $\int_{-\infty}^{\infty} |x| \delta(x^2) \, dx$. Using

$$\delta(t) = \lim_{\sigma \to 0} \frac{e^{-t^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \ ,$$

we find

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x| \delta(x^2) \, dx &= \lim_{\sigma \to 0} \int_{-\infty}^{\infty} |x| \frac{e^{-x^4/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \, dx \\
&= 2 \lim_{\sigma \to 0} \int_0^{\infty} x \frac{e^{-x^4/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \, dx \ , \ \text{now change variables } x = (2y\sigma^2)^{1/4} \\
&= 2 \lim_{\sigma \to 0} \int_0^{\infty} \frac{(2y\sigma^2)^{1/4}}{\sqrt{2\pi\sigma^2}} e^{-y} \frac{1}{4} \left(\frac{2\sigma^2}{y^3} \right)^{1/4} dy \\
&= \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y} \, dy = \frac{1}{2} \ ,
\end{aligned}
$$

in agreement with our result

$$|x| \delta(x^2) = \frac{1}{2} \delta(x) \ .$$