Abstract—Achievable rate distortion region for the quadratic Gaussian multi-terminal source coding was recently solved by Wagner, Tavildar, and Viswanath [1]. Although it is a very important result, the proofs are somewhat ad-hoc and hard to be generalized. Recently, Courtade and Weissman [2] established the complete achievable rate region of two-encoder multi-terminal source coding problem under logarithmic loss for any finite alphabet sources. In this project, we show it is natural to extend logarithmic loss to any sources, and in general it provides a tighter outer bound for corresponding source coding problems in the quadratic Gaussian settings. Our conjecture is the logarithmic loss function. In this case, the reconstruction and its Gaussian Counterparts

I. INTRODUCTION

Consider the multi-terminal source coding problem described in [3, Chapter 12]. The general setup is shown in Fig. 1. In [2], Courtade and Weissman considered a special case of this problem in which the distortion is defined by the logarithmic loss function. In this case, the reconstruction alphabets \( \hat{X} \) and \( \hat{Y} \) are the set of probability mass functions on \( X \) and \( Y \), respectively, and the distortion between a symbol \( x \in X \) and its reconstruction \( \hat{x} \in \hat{X} \) is defined by

\[
d(x, \hat{x}) = \log \left( \frac{1}{\hat{x}(x)} \right).
\]

The main result in [2] is the following theorem, which gives a full characterization of the rate-distortion region under logarithmic loss for the case of two sources.

Theorem 1 (Courtade and Weissman 2012) A rate-distortion quadruple \( (R_1, R_2, D_1, D_2) \) is achievable for the two terminals source coding if and only if there exists a joint distribution of the form \( p(x, y)p(u|x, y)p(v|y, q)p(q) \), where \( |U| \leq |X| \), \( |V| \leq |Y| \), and \( |Q| \leq 5 \), which satisfies

\[
\begin{align*}
R_1 &\geq I(X; U|V, Q), \\
R_2 &\geq I(Y; V|U, Q), \\
R_1 + R_2 &\geq I(X, Y; U, V|Q), \\
D_1 &\geq H(X|U, V, Q), \\
D_2 &\geq H(Y|U, V, Q).
\end{align*}
\]

We note that the region above is merely the Burger-Tung inner bound specialized to the case of logarithmic distortion. Another important special case in which the Burger-Tung inner bound was proven to be tight, is the case of jointly Gaussian sources and quadratic distortion. This case was solved by Wagner, Tavildar, and Viswanath in [1]. In this work we study the relation between the these two special cases of the multi-terminal source coding problem.

As a first step, we extend Theorem 1 to the case of inputs which are not limited to discrete alphabets. We define

\[
I_1 \triangleq h(X) - D_1, \quad I_2 \triangleq h(Y) - D_2,
\]

and denote by \( \mathcal{RL}_L \) the set of quadruples \( (R_1, R_2, I_1, I_2) \) such that \( (R_1, R_2, h(X) - I_1, h(Y) - I_2) \in \mathcal{RD}_L \). A characterization of \( \mathcal{RL}_L \) is obtained from Theorem 1 by replacing the inequalities (1) with

\[
\begin{align*}
R_1 &\geq I(X; U|V, Q), \\
R_2 &\geq I(Y; V|U, Q), \\
R_1 + R_2 &\geq I(U, V; X, Y|Q), \\
I_1 &\leq I(X : U|V, Q), \\
I_2 &\leq I(Y : U|V, Q).
\end{align*}
\]

So \( \mathcal{RL}_L \) is described only in terms of mutual information, which is defined between any two distributions. Consider now any input distribution on arbitrary alphabet, via a discretization argument we get that the rate distortion trade-off for the logarithmic setting is described by \( \mathcal{RL}_L \).
A. An outer bound for the quadratic Gaussian multi-terminal source coding

Under the quadratic distortion, the Burger-Tung inner bound takes the form

\[
\begin{align*}
R_1 & \geq I(X; U|V, Q), \\
R_2 & \geq I(Y; V|U, Q), \\
R_1 + R_2 & \geq I(U, V; X, Y|Q), \\
D_1 & \geq MMSE(X|U, V, Q), \\
D_2 & \geq MMSE(Y|U, V, Q),
\end{align*}
\]

where for any two random variable \(W\) and \(V\),

\[
MMSE(V|W) \triangleq \mathbb{E}[(V - \mathbb{E}[V|W])^2].
\]

From now on we assume that \(X\) and \(Y\) are jointly Gaussian with covariance matrix

\[
K_{X,Y} = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}.
\]

Denote by \(\mathcal{RD}_Q\) the rate-distortion region in the quadratic Gaussian multi-terminal source coding setting (QG-MTSC). We have the following proposition which essentially shows that Theorem 1 implies an outer bound for \(\mathcal{RD}_Q\).

Proposition 1 If \((R_1, R_2, D_1, D_2) \in \mathcal{RD}_Q\) then \((\tilde{R}_1, \tilde{R}_2, \frac{1}{2} \log \frac{1}{D_1}, \frac{1}{2} \log \frac{1}{D_2}) \in \mathcal{RI}_L\). In other words, for any achievable rate-distortion quadruple \((R_1, R_2, D_1, D_2)\) for the QG-MTSC, there exist random variables \(U, V\) and \(Q\), such that given \(Q\), \(U - X - Y - V\) form a Markov chain, and

\[
\begin{align*}
R_1 & \geq I(X; U|V, Q), \\
R_2 & \geq I(Y; V|U, Q), \\
R_1 + R_2 & \geq I(U, V; X, Y|Q), \\
\frac{1}{2} \log \frac{1}{D_1} & \leq I(X; U, V|Q), \\
\frac{1}{2} \log \frac{1}{D_2} & \leq I(Y; U, V|Q).
\end{align*}
\]

Proof: If \((R_1, R_2, D_1, D_2)\) is achievable for the QG-MTSC, then there exist \(f_1 : X^n \rightarrow [1: 2^nR_1]\) and \(f_2 : Y^n \rightarrow [1: 2^nR_2]\) such that

\[
\begin{align*}
D_1 \geq & MMSE(X^n|f_1(X^n), f_2(Y^n)), \\
D_2 \geq & MMSE(Y^n|f_1(X^n), f_2(Y^n)).
\end{align*}
\]

Denote

\[
\tilde{I}_1 := \frac{1}{n} I(X^n; f_1(X^n), f_2(Y^n)), \\
\tilde{I}_2 := \frac{1}{n} I(Y^n; f_1(X^n), f_2(Y^n)),
\]

then \((R_1, R_2, \tilde{I}_1, \tilde{I}_2) \in \mathcal{RI}_L\). To see this, let \(\tilde{X}\) be a discrete approximation to \(X\). The expected distortion under logarithmic loss achieved by the reconstruction function

\[
\hat{x}^n = Pr \left\{ \tilde{X}^n = \tilde{x}^n | f_1(\tilde{X}^n), f_2(Y^n) \right\}
\]
is \(H(\tilde{x}^n|f_1(\tilde{x}^n), f_2(Y^n))\). So

\[
\frac{1}{n} I(\tilde{X}^n; f_1(\tilde{X}^n), f_2(Y^n)) = \frac{1}{n} H(\tilde{X}^n) - \frac{1}{n} H(\tilde{X}^n|f_1(\tilde{X}^n), f_2(Y^n))
\]
is achievable for the log-loss, and in the discretization limit we get that \(\tilde{I}_1\) is also achievable, and similarly \(\tilde{I}_2\). Now the key point is that Theorem 1 provides us a converse: there exist \(U, V\) and \(Q\) such that given \(Q\), \(U - X - Y - V\) form a Markov chain, and

\[
\begin{align*}
R_1 & \geq I(X; U|V, Q), \\
R_2 & \geq I(Y; V|U, Q), \\
R_1 + R_2 & \geq I(U, V; X, Y|Q), \\
\tilde{I}_1 & \leq I(X; U, V|Q), \\
\tilde{I}_2 & \leq I(Y; U, V|Q).
\end{align*}
\]

By the maximal differential entropy lemma [3, 2.7],

\[
I(X; U, V|Q) \geq \tilde{I}_1 = \frac{1}{n} I(X^n; f_1(X^n), f_2(Y^n)) = \frac{1}{2} \log (2\pi e) - \frac{1}{n} h(X^n|f_1(X^n), f_2(Y^n)) \geq \frac{1}{2} \log (2\pi e) - \frac{1}{2} \log (2\pi e MMSE(X^n|f_1(X^n), f_2(Y^n))) = \frac{1}{2} \log \frac{1}{MMSE(X^n|f_1(X^n), f_2(Y^n))} \geq \frac{1}{2} \log \frac{1}{D_1}.
\]

Similarly,

\[
I(Y; U, V|Q) \geq \frac{1}{2} \log \frac{1}{MMSE(Y^n|f_1(X^n), f_2(Y^n))} \Rightarrow \frac{1}{2} \log \frac{1}{D_2}.
\]

Remark 1 Note that if we can show the region \(\mathcal{RI}_L\) is exhausted by Gaussian test channels, i.e., we can always find auxiliaries \((U, V)\) such that \((U, X, Y, V)\) is jointly Gaussian with Markov relation \(U - X - Y - V\) and satisfy the inequalities in Proposition 1 then we have

\[
\begin{align*}
I(X; U, V) & = \frac{1}{2} \log \left( \frac{1}{MMSE(X|U, V)} \right), \\
I(Y; U, V) & = \frac{1}{2} \log \left( \frac{1}{MMSE(Y|U, V)} \right),
\end{align*}
\]

and from (3) and (4) we get

\[
\begin{align*}
MMSE(X^n|f_1(X^n), f_2(Y^n)) & \geq MMSE(X|U, V), \\
MMSE(Y^n|f_1(X^n), f_2(Y^n)) & \geq MMSE(Y|U, V).
\end{align*}
\]

This means that the Burger-Tung scheme achieves quadratic distortion which is not higher than any given coding scheme for the QG-MTSC. From this we conclude that if we can
show that the boundary of $\mathcal{R}_{Q,L}$ is achieved by Gaussian test channels $P_{U|X}$ and $P_{V|Y}$, then we would have achieved an elegant converse proof for the Quadratic Gaussian multi-terminal source coding problem.

In the rest of this work, we describe our results toward proving the following:

**Conjecture 1** The boundary of $\mathcal{R}_{Q,L}$ is achieved by Gaussian test channels $P_{U|X}$ and $P_{V|Y}$.

The rest of this work is organized as follows. First we describe the source coding with rate-constrained side information problem, which is a special case of the general multi-terminal source coding problem, and show the logarithmic loss easily implies the converse of its quadratic Gaussian version. Second, we deal with the sum rate constraint in QG-MTSC, and conjecture that it is also implied by logarithmic loss results by formulating a related optimization problem. In the end, we point out some future directions to pursue.

## II. Results

### A. Quadratic Gaussian Source Coding with Rate Constrained Side Information

In Fig[1] if we take $D_2 = D_{\text{max}}$, which indicates we are not interested in reconstructing $V''$, then the general QG-MTSC is reduced to the so-called quadratic Gaussian source coding with rate constrained side information problem. This problem was first solved by Oohama[4], and the solution is very elegant:

$$R_1 \geq \frac{1}{2} \log \frac{1 - \rho^2 + \rho^2 2^{-2D_1}}{D_1}.$$  (5)

Note that the solution to this problem subsumes the quadratic Gaussian Wyner–Ziv problem by taking $R_2 = \infty$ and the point to point quadratic Gaussian rate distortion problem by further taking $\rho = 0$.

Now we show that solution in Equation (5) can be implied by Proposition[1]. Before the proof, we show two lemmas to help decompose and simplify the arguments.

**Lemma 1** Let $Q$ be independent of $X, Y$, and suppose $U - X - Y - V$ form a Markov chain given $Q$. If

- $R_1 \geq I(X; U|V, Q)$
- $R_2 \geq I(Y; V|U, Q)$
- $R_1 + R_2 \geq I(X, Y; U, V|Q)$
- $\frac{1}{2} \log \frac{1}{D_1} \leq I(X; U, V|Q)$

then there exists $X - Y - \tilde{V}$ satisfying

$$R_1 \geq \frac{1}{2} \log \frac{1}{D_1} - I(X; \tilde{V})$$

$$R_1 \geq I(Y; \tilde{V}).$$

**Proof:** We will assume that $|Q| = 1$, as the time sharing extension follows easily. Note that if $R_2 \geq I(Y; V)$, then we are done since $R_1 \geq I(X; U|V) = I(X; U, V) - I(X; V) \geq \frac{1}{2} \log \frac{1}{D_1} - I(X; V)$. Therefore, we will assume that $R_2 < I(Y; V)$. Let $T$ be a Bernoulli random variable, independent of $X, Y, U, V$, defined by

$$T = \begin{cases} 0 & \text{with probability } \lambda \\ 1 & \text{with probability } 1 - \lambda. \end{cases}$$  (6)

Let $V''$ be the result of passing $X$ through the test channel $P_{V|X}$. Now define $U'$ and $V'$ as follows:

$$U' = \begin{cases} (U, V'') & \text{if } T = 0 \\ U & \text{if } T = 1 \end{cases}$$

$$V' = \begin{cases} \emptyset & \text{if } T = 0 \\ V & \text{if } T = 1. \end{cases}$$

Observe that $U' - X - Y - V'$ form a Markov chain given $T$, and that $I(X; U, V) = I(X; U', V'|T)$. Since $R_2 < I(Y; V)$, we can take $\lambda$ to satisfy $I(Y; V''|T) = R_2$. Then,

$$I(X, U'|V', T) + R_2 = \lambda I(X; U, V) + (1 - \lambda) I(X, Y; U, V) \leq I(X, Y; U, V) \leq R_1 + R_2.$$  

Therefore,

$$R_1 \geq I(X, U'|V', T) = I(X, U', V'|T) - I(X; V'|T) \geq \frac{1}{2} \log \frac{1}{D_1} - I(X; V'|T).$$

Taking $\tilde{V} = (V', T)$ proves the lemma.

**Lemma 2** $\sup \{ I(X; V) : I(Y; V) \leq \eta \text{ and } X - Y - V \} = -\frac{1}{2} \log (1 - \rho^2 + \rho^2 2^{-2\eta})$.

**Proof:** Consider any $V$ satisfying $X - Y - V$. Let $Z \sim \mathcal{N}(0, 1 - \rho^2)$, and let $Y(v), X(v)$ denote the random variables $X, Y$ conditioned on $V = v$. By Markovity and the definition of $X, Y$, we have that $X(v) = \rho Y(v) + Z$. Hence, the conditional entropy power inequality implies that

$$2^{2h(X|V)} \geq \rho^2 2^{2h(Y|V)} + 2\pi e(1 - \rho^2) \geq 2\pi e(1 - \rho^2) + 2\pi e(1 - \rho^2) = 2\pi e(1 - \rho^2 + \rho^2 2^{-2\eta}).$$

Therefore, $h(X|V) \geq \frac{1}{2} \log (2\pi e(1 - \rho^2 + \rho^2 2^{-2\eta}))$. As a consequence,

$$I(X; V) \leq -\frac{1}{2} \log (1 - \rho^2 + \rho^2 2^{-2\eta}).$$  (7)

Now we formally state the converse theorem for quadratic Gaussian source coding with rate constrained side information.

**Theorem 2** If rate distortion triple $(R_1, R_2, D_1)$ in the quadratic Gaussian source coding with rate constrained side information problem is achievable, then it must satisfy

$$R_1 \geq \frac{1}{2} \log \left(\frac{1 - \rho^2 + \rho^2 2^{-2D_1}}{D_1}\right).$$

**Proof:** By Proposition[1] we know there exists random variable $Q$ independent of $X, Y$ and conditioned on $Q, U -$
$X - Y - V$ forms a Markov chain, such that
\[
R_1 \geq I(X; U|V, Q) \\
R_2 \geq I(Y; V|U, Q) \\
R_1 + R_2 \geq I(X, Y; U, V|Q) \\
\frac{1}{2} \log \frac{1}{D_1} \leq I(X; U, V|Q).
\]
By Lemma 1 there exists $\tilde{V}$ such that $X - Y - \tilde{V}$ form a Markov chain and
\[
R_1 \geq \frac{1}{2} \log \frac{1}{D_1} - I(Y; \tilde{V}) \\
R_2 \geq I(Y; \tilde{V}).
\]
By Lemma 2 $I(Y; \tilde{V}) \leq -\frac{1}{2} \log (1 - \rho^2 + \rho^2 2^{-2R_2})$, which completes the proof.

**B. Sum Rate of Quadratic Gaussian Multi-terminal Source Coding**

Now we consider the general QT-MTSC. The only thing left is to show the sum rate constraint can be implied by logarithmic loss via Proposition 1. As shown in 1, the sum rate can be dealt with separately, i.e., we only need to show the following region
\[
R_1 + R_2 \geq I(X, Y; U, V) \\
\frac{1}{2} \log \frac{1}{D_1} \leq I(X; U, V) \\
\frac{1}{2} \log \frac{1}{D_2} \leq I(Y; U, V)
\]
is exhausted by Gaussian test channels $P_{U|X}$ and $P_{V|Y}$.

Suppose this region is strictly convex, then we just need to show that the optimum of the following optimization problem can be achieved by Gaussian test channels
\[
\min \ I(X, Y; U, V) - \lambda_1 I(X; U, V) - \lambda_2 I(Y; U, V) \quad (8) \\
\text{s.t.} \quad U - X - Y - V, (\lambda_1, \lambda_2) \in [0, 1]^2
\]

**Remark 2** We can show that if $\max{\lambda_1, \lambda_2} > 1$, then the resulted minimum is $-\infty$. Note that here we can exclude the time-sharing random variable $Q$.

Now we do some transformations on the optimization problem in Equation (8).
\[
I(X, Y; U, V) - \lambda_1 I(X; U, V) - \lambda_2 I(Y; U, V) \\
= h(X, Y) - h(X, Y|U, V) - \lambda_1(h(X) - h(X|U, V)) \\
- \lambda_2(h(Y) - h(Y|U, V)) \\
= \lambda_1 h(X|U, V) + \lambda_2 h(Y|U, V) - h(X, Y|U, V) + C \\
= \lambda_1 h(X|U, V) + (\lambda_2 - 1)h(Y|U, V) - h(X|U, V) + C,
\]
where $C$ has nothing to do with test channels $P_{U|X}$, $P_{V|Y}$, and the last equality is by Markov chain assumption $U - X - Y - V$.

Thus, the optimization problem has been transformed into the following form:
\[
\max \ (1 - \lambda_2)[h(Y|U, V) - \mu h(X|U, V)] + h(X|U, V) \\
\text{s.t.} \quad U - X - Y - V, (\lambda_1, \lambda_2) \in [0, 1]^2,
\]
where $\mu = \frac{\lambda_1}{1 - \lambda_2} \geq 0$.

Denote the best linear estimator of $X$ given $U, Y$ as $BLE(X|U, Y)$. Then we have
\[
h(X|U, Y) = h(X - BLE(X|U, Y)|U, Y) \\
\leq h(X - BLE(X|U, Y)) \\
\leq \frac{1}{2} \log 2\pi e MMSE(X|U, Y) \\
= h(X - BLE(X|U, G, Y)|U, G, Y) \\
= h(X|U, G, Y),
\]
where $(U_G, X, Y)$ and $(U, X, Y)$ have the same covariance, and $(U_G, X, Y)$ is jointly Gaussian.

However, it is still not clear how to show $h(Y|U, V) - \mu h(X|U, V)$ is maximize by Gaussian test channels $P_{U|X}$, $P_{V|Y}$. Our conjecture is that the optimum of the optimization problem in Equation (8) is achieved by Gaussian test channels, which directly implies the sum rate constraint of QG-MTSC.

**III. Summary**

In this report, we show the recent results on multi-terminal source coding under logarithmic loss provides a “better” outer bound for quadratic Gaussian source coding problems, and we conjecture that in general it gives a unified approach to establish converses of quadratic Gaussian source coding problems by showing Gaussian auxiliaries exhaust the region. We have shown that for the source coding with rate constrained side information problem, logarithmic loss indeed implied its converse, and we conjecture that the sum rate constraint can also be established via logarithmic loss, which is still under investigation. Some possible future directions include showing logarithmic loss could imply the converses of sum rate of quadratic Gaussian multi-terminal source coding, $m$-encoder quadratic Gaussian CEO problem, and the quadratic Gaussian multiple description coding.

**References**


