Stochastic Integration for non-Martingales
Stationary Increment Processes
Multi-color noise approach

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Outline

1. Introduction
   - Motivation
   - Fractional Brownian Motion

2. Main Result
   - Stochastic Processes Induced by Operators
   - The $m$-Noise Space and the Process $B_m$
   - The $S_m$ Transform
   - Stochastic Integration with respect to $B_m$

3. Applications
   - Optimal Control
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3 Applications
   • Optimal Control
Stochastic Processes and Colored noises

- Stochastic stationary noises with dependent distinct time samples do exist in nature.
- We wish to model physical phenomena by stochastic differential equations of this form
  \[ \mathrm{d}X_t = F (X, dB_m). \]
- If \( B_m \) is a Brownian motion, the notion of Itô integral can be used so the differential \( dB_m \) is what we intuitively think of as white noise.
- Such notion does not exist in general if \( B_m \) is a stationary increment Gaussian process that is not a semi-martingale.
- The aim of this talk is to give meaning to this notation by extending Itô’s integration theory to these processes.
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Fractional Brownian Motion

The fractional Brownian motion with Hurst parameter $0 < H < 1$ is a zero mean Gaussian stochastic process with covariance function

$$COV(t, s) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} + |t - s|^{2H} \right), \quad t, s \in \mathbb{R}.$$ 

In particular, for $H \neq \frac{1}{2}$ it is not a semi-martingale.

Stochastic calculus for fractional Brownian (fBm) has attracted much attention in the last two decades, especially due to apparent application in economics.

The Itô-Wick integral for fBm seems to be the most natural extension of the Itô integral for this class of non-semi-martingale processes.
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The Itô-Wick integral for fBm seems to be the most natural extension of the Itô integral for this class of non-semi-martingale processes.
We have the following relation:

\[
\frac{1}{2} \left( |t|^{2H} + |s|^{2H} + |t - s|^{2H} \right) = \int_{-\infty}^{\infty} \hat{1}_{[0,t]}(0) \hat{1}_{[0,s]}(0)^* m(\xi) d\xi,
\]

where

- \( \hat{1}_{[0,t]} \) is the indicator function of the interval \([0, t]\)
- \( \hat{f} = \int_{-\infty}^{\infty} e^{-i\xi f(u)} du \)
- \( m(\xi) = M(H)|\xi|^{1-2H} \) and \( M(H) = \frac{H(1-H)}{\Gamma(2-2H) \cos(\pi H)} \)

According to the theory of Gelfand-Vilenkin on generalized stochastic processes, the time derivative of the fBm is a stationary stochastic distribution with spectral density \( m(\xi) \).
Fractional Brownian Motion
Spectral Properties

- We have the following relation:
  \[
  \frac{1}{2} \left( |t|^{2H} + |s|^{2H} + |t - s|^{2H} \right) = \int_{-\infty}^{\infty} 1_{[0, t]} 1_{[0, s]}^* m(\xi) d\xi,
  \]

  where
  - \( 1_{[0, t]} \) is the indicator function of the interval \([0, t]\)
  - \( \hat{f} = \int_{-\infty}^{\infty} e^{-iu\xi} f(u) du \)
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  - According to the theory of Gelfand-Vilenkin on generalized stochastic processes, the time derivative of the fBm is a stationary stochastic distribution with spectral density \( m(\xi) \).
Fractional Brownian Motion
Member of a Wide Family

- It suggests the fBm is a member of a wide family of stationary increments Gaussian processes whose covariance function is of the form

\[
\text{COV}_m(t, s) = \int_{-\infty}^{\infty} 1_{[0,t]} 1_{[0,s]}^* m(\xi) d\xi
\]  

for a function \( m(\xi) \) satisfies

\[
\int_{-\infty}^{\infty} \frac{m(\xi)}{1+\xi^2} d\xi < \infty.
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Main Goal of this Talk
Extend the Itô integral for Brownian motion to this family of non-martingales stationary increments processes.
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Stochastic Processes Induced by Operators

Definition

For a given spectral density function \( m(\xi) \) such that
\[
\int_{-\infty}^{\infty} \frac{m(\xi)}{1+\xi^2} d\xi < \infty,
\]
we associate an operator
\[
T_m : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad \hat{T}_m f(\xi) = \hat{f}(\xi) \sqrt{m(\xi)}, \quad f \in L^2(\mathbb{R}).
\]
or
\[
f \xrightarrow{\sqrt{m}} T_m f
\]

This operator is in general unbounded.

\( 1_{[0,t]} \in \text{dom}T_m \) for each \( t \geq 0 \).

The covariance function (1) can now be rewritten as
\[
COV_m(t, s) = \int_{-\infty}^{\infty} \hat{1}_{[0,t]} \hat{1}_{[0,s]}^* m(\xi) d\xi = (T_m 1_{[0,t]}, T_m 1_{[0,s]})_{L^2(\mathbb{R})}.
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Multi-color noise spaces
Stochastic Processes Induced by Operators

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  \]
To each operator $T_m$ we associate a Gaussian probability space $(\Omega, \mathcal{F}, P_m)$ which will be called the $m$-noise space.

Stochastic process with covariance function $\left( T_m 1_{[0,t]}, T_m 1_{[0,s]} \right)_{L^2(\mathbb{R})}$ is naturally defined on the $m$-noise space.

We use the analogue of the $S$-transform to define a Wick-Itô integral on this space.

Application to optimal control theory.
To each operator $T_m$ we associate a Gaussian probability space $(\Omega, \mathcal{F}, P_m)$ which will be called the $m$-noise space.

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The $m$-Noise Space

Notations

We use an analogue of Hida’s white noise space as our underlying probability space. Notations:

- $\mathcal{S}$ - Schwartz space of real rapidly decreasing functions.
- $\Omega$ is the dual of $\mathcal{S}$, the space of tempered distributions.
- $\mathcal{B}(\Omega)$ is the Borel $\sigma$-algebra.
- $\langle \omega, s \rangle = \langle \omega, s \rangle_{\Omega, \mathcal{S}}$, $s \in \mathcal{S}$ and $\omega \in \Omega$ will denote the bilinear pairing between $\mathcal{S}$ and $\Omega$.

Lemma

$T_m$ as an operator from $\mathcal{S} \subset L_2(\mathbb{R})$, endowed with the Fréchet topology, into $L_2(\mathbb{R})$ is continuous.
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Definition of the Probability Space

Bochner-Minlus Theorem

It follows that \( C_m(s) = e^{\frac{1}{2} \| T_m s \|_{L^2(\mathbb{R})}^2} \) is a characteristic functional on \( \mathcal{S} \).

By the Bochner-Minlos theorem there is a unique probability measure \( P_m \) on \( \Omega \) such that for all \( s \in \mathcal{S} \),

\[
C_m(s) = \exp \left\{ -\frac{1}{2} \| T_m s \|_{L^2(\mathbb{R})}^2 \right\} = \int_{\Omega} e^{i\langle \omega, s \rangle} dP_m(\omega) = \mathbb{E} \left[ e^{i\langle \cdot, s \rangle} \right]
\]

\( \langle \omega, s \rangle \) is viewed as a random variable on \( \Omega \).

The triplet \((\Omega, \mathcal{B}(\Omega), P_m)\) will be called the \( m \)-noise space.

The case \( T_m = \text{id}_{L^2(\mathbb{R})} \) (\( m \equiv 1 \)) will lead back to Hida’s white noise space.
It follows that $C_m(s) = e^{\frac{1}{2} \| T_m s \|^2_{L^2(\mathbb{R})}}$ is a characteristic functional on $\mathcal{S}$.

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The Process $B_m$

**Definition**

- $\langle \omega, s \rangle$, $s \in \mathcal{S}$, is a zero mean Gaussian random variable with variance

$$E \left[ \langle \cdot, s \rangle^2 \right] = \| T_m s \|_{L_2(\mathbb{R})}^2.$$

- The last isometry $L_2 (\Omega, \mathcal{B}(\mathcal{S}'), P_m) \mapsto T_m \mathcal{S}$ can be extended such that $\langle \omega, f \rangle$, $f \in \text{Dom}(T_m)$ is meaningful and

$$E \left[ \langle \cdot, f \rangle^2 \right] = \| T_m f \|_{L_2(\mathbb{R})}^2.$$

- In particular, for $t \geq 0$ we may define the stochastic process $B_m : \Omega \times [0, \infty] \mapsto \mathbb{R}$ by

$$B_m(t) := B_m(\omega, t) := \langle \omega, 1_{[0,t]} \rangle.$$
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The Process $B_m$

Properties

- The process $\{B_m\}_{t \geq 0}$ is a zero mean Gaussian process with covariance function
  \[
  \mathbb{E}[B_m(t)B_m(s)] = \left( T_m1_{[0,t]}, T_m1_{[0,s]} \right)_{L_2(\mathbb{R})}.
  \]

- $\frac{d}{dt} B_m$ (in the sense of distribution) has spectral density $m(\xi)$.

- In view of the previous isometry, it is natural to define for $f \in \text{Dom}(T_m)$,
  \[
  \int_0^t f(u) dB_m(u) = \langle \omega, 1_{[0,t]} f \rangle, \quad t \geq 0.
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The Process $B_m$

Examples

**Example (Standard Brownian Motion)**

Take $m \equiv 1$, then $T_m = id_{L_2(\mathbb{R})}$ and

$$\mathbb{E}[B_m(t)B_m(s)] = (T_m\mathbf{1}_{[0,t]}, T_m\mathbf{1}_{[0,s]}) = \int_{-\infty}^{\infty} \mathbf{1}_{[0,t]}\mathbf{1}_{[0,s]}^*du = t \wedge s.$$

**Example (Fractional Brownian Motion)**

Take $m(\xi) = M(H)|\xi|^{1-2H}$, then

$$\mathbb{E}[B_m(t)B_m(s)] = \int_{-\infty}^{\infty} \mathbf{1}_{[0,t]}\mathbf{1}_{[0,s]}^* \ m(\xi)d\xi = \frac{|t|^{2H} + |s|^{2H} - |t-s|^{2H}}{2}.$$
Outline

1. Introduction
   - Motivation
   - Fractional Brownian Motion

2. Main Result
   - Stochastic Processes Induced by Operators
   - The $m$-Noise Space and the Process $B_m$
   - The $S_m$ Transform
   - Stochastic Integration with respect to $B_m$

3. Applications
   - Optimal Control
We wish to define a Wick-Itô-Skorohod stochastic integral based on the process \( \{B_m\}_{t \geq 0} \).

A standard definition in Hida’s white noise space would be

\[
\int_0^\Delta X(t) dB(t) \triangleq \int_0^\Delta X(t) \diamond \frac{d}{dt} B_m(t) dt,
\]

where

\( \{X(t)\}_{0 \leq t \Delta} \) is a stochastic process

\( \frac{d}{dt} B_m(t) \) is the time derivative (in the sense of distributions) of the Brownian motion.

\( \diamond \) is the Wick product.

Those definitions make use of the Wiener-Itô Chaos decomposition of the white noise space.
Any $X \in L_2(\Omega, \mathcal{B}, P_m)$ can be represented as

$$X = \sum_{\alpha} f_{\alpha} H_{\alpha}(\omega).$$

Any such basis for $L_2(\Omega, \mathcal{B}(\mathcal{S}'), P_m)$ depends explicitly on $m(\xi)$.

In order to keep our construction as general as possible, we take an $S$-transform approach for the Wick-Itô-Skhorhod integral, which does not use chaos decomposition.
Definition of the $S_m$-Transform

- We reduce to the $\sigma$-field $\mathcal{G}$ generated by $\{\langle \omega, f \rangle \}_{f \in \text{Dom}(T_m)}$.

Definition

For a random variable $X \in L_2 (\Omega, \mathcal{G}, P_m)$ define

$$(S_m X)(s) \triangleq \mathbb{E} \left[ e^{\langle \cdot, s \rangle} X(\cdot) \right] e^{-\frac{1}{2} \| T_m s \|^2}, \quad s \in \mathcal{S}.$$  

- Any $X \in L_2 (\Omega, \mathcal{G}, P_m)$ is uniquely determined by $(S_m X)(s)$.

Lemma

$$(S_m B_m(t))(s) = (T_m s, T_m 1_{[0,t]})_{L_2(\mathbb{R})}$$

is everywhere differentiable with respect to $t$.  

D. Alpay and A. Kipnis  Multi-color noise spaces
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Multi-color noise spaces
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Multi-color noise spaces
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Definition of the Stochastic Integral

A stochastic process \( X(t) : [0, \Delta] \rightarrow L_2(\Omega, \mathcal{G}, P_m) \) will be called Wick-Itô integrable if there exists a random variable \( \Phi \in L_2(\Omega, \mathcal{G}, P_m) \) such that

\[
(S_m \Phi)(s) = \int_0^\Delta (S_m X(t))(s) \frac{d}{dt} (S_m B_m(t))(s) dt.
\]

In that case we define \( \Phi(\Delta) = \int_0^\Delta X(t) dB_m(t). \)

- For any polynomial \( p \in \mathbb{R}[X] \), \( p(B_m(t)) \) is integrable.
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- For any polynomial \( p \in \mathbb{R} [X] \), \( p (B_m(t)) \) is integrable.
The Wick product of $X, Y \in L_2(\Omega, \mathcal{G}, P_m)$ can be defined by

$$(S_m (X \diamond Y)) (s) = S_m X(s) S_m Y(s)$$

So

$$\int_0^\Delta X(t) dB_m(t) = \int_0^\Delta X(t) \diamond \frac{d}{dt} B_m(t)$$

where the integral on the right is a Pettis integral.

If $B_m$ is the Brownian motion ($m(\xi) \equiv 1$), our definition of the stochastic integral coincides with the Itô-Hitsuda integral [Hida1993].

If $B_m$ is the fractoinal Brownian motion ($m(\xi) = |\xi|^{1-2H}$), our definition of the stochastic integral reduces to the one given in [Bender2003] which coincides the Wick-Itô-Skorokhod integral defined in [Duncan,Hu 2000] and [Hu,Øksendal 2003].
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Itô’s Formula

We have the following version of Itô’s Formula:

- Let \( X(t) = \int_0^t f(u) dB_m(u) = \langle \omega, 1_{[0,t]} f \rangle \)
  where \( f \in domT_m \) and \( t \geq 0 \), such that \( \| T_m 1_{[0,t]} f \| ^2 \) is absolutely continuous in \( t \).
- \( F \in C^{1,2} ([0, t], \mathbb{R}) \) with \( \frac{\partial}{\partial t} F(X_t), \frac{\partial}{\partial x} F(X_t), \frac{\partial^2}{\partial x^2} F(X_t) \) all in \( L_1 (\Omega \times [0, t]) \).
- The following holds in \( L_2 (\Omega, \mathcal{G}, P_T) \):

\[
F(t, X_t) - F(0, 0) = \int_0^t f(u) \frac{\partial}{\partial x} F(u, X(u)) dB_m(u) \\
+ \int_0^t \frac{\partial}{\partial u} F(u, X(u)) du + \frac{1}{2} \int_0^t \frac{d}{du} \| T_m 1_{[0,u]} f \|^2 \frac{\partial^2}{\partial x^2} F(u, X(u)) du
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- $F \in C^{1,2}([0, t] \times \mathbb{R})$ with $\frac{\partial}{\partial t}F(X_t), \frac{\partial}{\partial x}F(X_t), \frac{\partial^2}{\partial x^2}F(X_t)$ all in $L_1(\Omega \times [0, t])$.

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Introduction

Motivation
Fractional Brownian Motion

Main Result

Stochastic Processes Induced by Operators
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Stochastic Integration with respect to $B_m$

Applications

Optimal Control
Consider the scalar system subject to

\[
\begin{aligned}
\frac{dx_t}{dt} &= (A_t dt + C_t dB_m(t)) x_t + F_t u_t dt \\
x_0 &\in \mathbb{R} \quad \text{(deterministic)}
\end{aligned}
\]

where \( A(\cdot), C(\cdot), F(\cdot) : [0, \Delta] \to \mathbb{R} \) are bounded deterministic functions.

Using Itô’s formula, one may verify that

\[
x_\Delta = x_0 \exp \left\{ \int_0^\Delta (A_t + F_t u_t) dt + \int_0^\Delta C_t dB_m(t) - \frac{1}{2} \| T_m^1_{[0,\Delta]} \|^2 \right\}
\]
Consider the scalar system subject to

\[
\begin{aligned}
\text{d}x_t &= (A_t \text{d}t + C_t \text{d}B_m(t)) \ x_t + F_t u_t \text{d}t \\
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Using Itô's formula, one may verify that

\[x_\Delta = x_0 \exp \left\{ \int_0^\Delta (A_t + F_t u_t) dt + \int_0^\Delta C_t dB_m(t) - \frac{1}{2} \| T_m 1_{[0,\Delta]} \|^2 \right\}\]
We present a quadratic cost functional

\[ J(x_0, u(\cdot)) := \mathbb{E} \left[ \int_0^\Delta \left( Q_t x_t^2 + R_t u_t^2 \right) dt + G x_\Delta^2 \right]. \]

where \( R(\cdot), Q(\cdot) : [0, \Delta] \to \mathbb{R}, R_t > 0, Q_t \geq 0 \forall t \geq 0 \) and \( G \geq 0 \).

We reduce ourselves to control signals of linear feedback type:

\[ u_t = K_t \cdot x_t. \]

so the control dynamics reduces to

\[
\begin{cases}
    dx_t = [(A_t + F_t K_t) dt + C_t dB_m(t)] x_t \\
    x_0 \in \mathbb{R} \quad \text{(deterministic)}
\end{cases}
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\[ J(x_0, u(.)) := \mathbb{E} \left[ \int_0^\Delta \left( Q_t x_t^2 + R_t u_t^2 \right) dt + G x_0^2 \right]. \]

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x_0 \in \mathbb{R} \quad \text{(deterministic)}
\end{cases}
\]
And the cost may be associated directly with the feedback gain $K_t : [0, \Delta] \rightarrow \mathbb{R}$:

$$J(x_0, K(\cdot)) := \mathbb{E} \left[ \int_0^\Delta \left( Q_t + K_t^2 R_t \right) x_t^2 \,dt + Gx_\Delta^2 \right], \quad (2)$$

The optimal stochastic control problem:

Minimize the cost functional (2), for each given $x_0$, over the set of all linear feedback controls $K(\cdot) : [0, \Delta] \rightarrow \mathbb{R}$.

This control problem was formulated and solved in the case of fractional Brownian motion by Hu and Yu Zhou in 2005, and appears in [Biagini, Hu, Øksendal, Zhang 2008].

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And the cost may be associated directly with the feedback gain $K_t : [0, \Delta] \rightarrow \mathbb{R}$:

$$J \left( x_0, K(\cdot) \right) := \mathbb{E} \left[ \int_0^\Delta \left( Q_t + K_t^2 R_t \right) x_t^2 dt + Gx^2_{\Delta} \right],$$  \quad (2)$$

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This control problem was formulated and solved in the case of fractional Brownian motion by Hu and Yu Zhou 2005, and appears in [Biagini,Hu,Øksendal,Zhang 2008].
If \( \frac{d}{dt} \left\| T_m1_{[0,t]} C(\cdot) \right\|^2 \) is bounded in \((0, \Delta)\), then the optimal linear feedback gain \( \tilde{K}_t \) is given by

\[
\tilde{K}_t = -\frac{F_t}{R_t} p_t. \tag{3}
\]

where \( \{p_t, t \in [0, \Delta]\} \) is the unique positive solution of the Riccati equation

\[
\begin{aligned}
\dot{p}_t + 2p_t \left[ A_t + \frac{d}{dt} \left\| T_m1_{[0,t]} C(\cdot) \right\|^2 \right] + Q_t - \frac{F_t^2}{R_t} p_t^2 &= 0 \\
p_\Delta &= G
\end{aligned} \tag{4}
\]
Proof.

Using Itô’s formula with:
\[ x_t = x_0 \exp \left[ \int_0^t c_u dBm(u) + \int_0^t (A_u + F_u K_u) \, du - \frac{1}{2} \left\| T_m (1_t C) \right\|^2 \right], \]
leads to

\[
p_{\Delta} x^2_{\Delta} = p_0 x^2_0 + 2 \int_0^\Delta x^2_t C_t p_t dBm(t)
+ \int_0^\Delta x^2_t \left[ \dot{p}_t + 2p_t (A_t + F_t K_t) + 2p_t \frac{d}{dt} \left\| T_m 1_t \right\|^2 \right] \, dt.
\]

Taking the expectation of both sides and substituting the Riccati equation (4) yields

\[
J(x_0, K(\cdot)) = p_0 x^2_0 + \mathbb{E} \int_0^\Delta \left( K_t + \frac{B_t}{R_t} p_t \right)^2 \, dt,
\]
of which the result follows.
We use the following specification
\[ \frac{A}{C} = \text{SNR}, \quad x_0 = 5, \quad F = 0.3 \]
in the state-space model which results in
\[
\begin{cases}
    dx_t = \left( A + \frac{1}{2} 0.3 K_t \right) x_t dt + x_t C dB_m(t), & (\text{SNR} = \frac{A}{C}) \\
    x_0 = 5.
\end{cases}
\]

We take \( B_m \) to have a spectral density:
\[
m(\xi) = \alpha |\xi|^{1-2H} + \beta \sin^2 \left( \Delta (\xi - 2\pi f_0) \right),
\]
with \( \Delta = 20, \quad f_0 = 2, \quad H = 0.6, \quad \alpha = 0.05 \) and \( \beta = 80. \)
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\end{align*} \]

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\[ m(\xi) = \alpha |\xi|^{1-2H} + \beta \sin^2 (\Delta(\xi - 2\pi f_0)), \]
with \( \Delta = 20, f_0 = 2, H = 0.6, \alpha = 0.05 \) and \( \beta = 80. \)
We design to different controllers:

- $K_{Opt}(\cdot)$ is the optimal controller from Theorem 7 for a system perturbated by $dB^m$.
- $K_{Nai}(\cdot)$ is the optimal controller designed for a system perturbated by the time derivative of a Brownian motion, so it corresponds to a naive design.

We compare the cost function

$$J_{(Opt,Nai)} = \mathbb{E} \left[ \int_0^\Delta \left( 1 + 2K_{(Opt,Nai)}(t)^2 \right) x_t^2 dt + 2x_\Delta^2 \right],$$

for the two controllers $K_{Opt}(\cdot)$ and $K_{Nai}(\cdot)$ and their corresponding state-space trajectories.
We design to different controllers:

- $K_{Opt}(\cdot)$ is the optimal controller from Theorem 7 for a system perturbated by $dB^m$.
- $K_{Nai}(\cdot)$ is the optimal controller designed for a system perturbated by the time derivative of a Brownian motion, so it corresponds to a naive design.

We compare the cost function

$$J_{(Opt,Nai)} = \mathbb{E} \left[ \int_0^\Delta \left( 1 + 2K_{Opt,Nai}(t)^2 \right) x_t^2 dt + 2x_{\Delta}^2 \right],$$

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Simulation
Over 10,000 independent sample paths

Average ratio \( \frac{J_{\text{Nai}}}{J_{\text{Opt}}} \) for different SNR values
We have used a variation on Hida’s white noise space and the $S$-transform to develop Wick-Itô stochastic calculus for non-martingales Gaussian processes with covariance function

$$\text{COV}(t, s) = \int_{-\infty}^{\infty} 1_{[0,t]}(\xi) 1_{[0,s]}^{*}(\xi) m(\xi) d\xi,$$

In particular, it extends many works on stochastic calculus for fractional Brownian motion from the past two decades.

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D. Alpay and A. Kipnis.
Stochastic integration for a wide class of non-martingale Gaussian processes
In preparation.

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Stochastic control for linear systems driven by fractional noises.

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