# Price and Capacity Competition

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**Abstract**

We study the efficiency of oligopoly equilibria in a model where firms compete over capacities and prices. Our model economy corresponds to a two-stage game. First, firms choose their capacity levels. Second, after the capacity levels are observed, they set prices. Given the capacities and prices, consumers allocate their demands across the firms. We establish the existence of pure strategy oligopoly equilibria and characterize the set of equilibria. We then investigate the efficiency properties of these equilibria, where "efficiency" is defined as the ratio of surplus in equilibrium relative to the first best. We show that efficiency in the worst oligopoly equilibria can be arbitrarily low. However, if the best oligopoly equilibrium is selected (among multiple equilibria), the worst-case efficiency loss is $2(\sqrt{N}-1)/(N-1)$ with $N$ firms, and this bound is tight. We also suggest a simple way of implementing the best oligopoly equilibrium.

*Keywords:* Capacity; Competition; Efficiency Loss; Industry Structure; Investment Oligopoly

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## 1 Introduction

This paper studies oligopoly competition in the presence of capacity investments. Our motivation comes from large-scale communication networks, particularly the Internet, which has

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undergone major decentralization since the mid 1990s. These changes have spurred interest in new decentralized network protocols and architectures that take into account the noncooperative interactions between users and service providers. A key question in the analysis of these new network structures is the extent of efficiency losses in the decentralized equilibrium relative to the efficient allocation of resources. Most of the work in this literature investigates the efficiency losses resulting from the allocation of users and information flows across different paths or administrative domains in an already established network (see, for example, Roughgarden and Tardos (2002), Correa et al. (2002), Correa et al. (2005), Acemoglu and Ozdaglar (2007a), Hayrapetyan et al. (2005), Bimpikis and Ozdaglar (2007), Ozdaglar (2008)). Arguably, the more important economic decisions in large-scale communication networks concern the investments in the structure of the network and in bandwidth capacity. In fact, the last 20 years have witnessed significant investments in broadband, high-speed and optical networks. Our objective in this paper is to model price and capacity competition between service providers and investigate the efficiency properties of the resulting equilibria. Following previous research in this area, we provide explicit bounds on the efficiency losses by providing various worst-case performance results for equilibria.

Our model consists of $N$ firms (service providers) and a mass of consumers wishing to send a fixed amount of flow from a fixed source origin to a given destination using subnetworks operated by these firms. Each user has an inelastic demand with a reservation utility $R$. Firms face a linear and potentially different cost of investing to expand the capacity of their subnetwork. For simplicity, we assume that once capacity is installed, there is no additional cost of allowing consumers to use the subnetwork. In our baseline model, firms play a two-stage game. They first choose the level of capacity in their subnetwork, and then set prices for consumers to use their subnetwork. This game has an obvious similarity to Kreps and Scheinkman's well-known model of quantity precommitment and price competition for two firms (Kreps and Scheinkman (1983)), but it is simpler because demand is inelastic.

For expositional purposes, we start with the special case with two firms. For this case, we fully characterize the set of pure strategy subgame perfect equilibria and prove that a pure strategy equilibrium always exists. As in Kreps and Scheinkman (1983), subgame perfect equilibria in which firms use pure strategies along the equilibrium path are nonetheless supported by mixed strategies off the equilibrium path. As part of our equilibrium analysis, we also provide a complete characterization of the set of mixed strategy equilibria following any choices of capacities by firms.

We then investigate the efficiency properties of equilibria in the worst-case scenarios. We quantify efficiency as the ratio of social surplus in equilibrium relative to the maximum value of social surplus (in the hypothetical first best). Since the game typically has multiple pure strategy equilibria, there are two possible approaches to quantifying worst-case scenarios. The first, referred to as the "*Price of Anarchy*" in the computer science and previous network economics literature, looks at the worst-case scenario in terms of the possible values of the parameters and selects the worst equilibrium if there are multiple equilibria. The second, referred to as the "*Price of Stability*," selects the best equilibrium for any given set of parameters and then looks for the worst-case values of the parameters (see Koutsoupias and Papadimitriou (1999), Correa et al. (2002)).

Our first result is that even in the simplest structure with linear costs, the Price of Anarchy is equal to zero, meaning that the equilibrium can be arbitrarily inefficient. Our second major result is that once we focus on the Price of Stability there is a tight bound of $2\sqrt{2} - 2 \simeq 5/6$, meaning that if the (socially) best equilibrium is selected, the maximum inefficiency that may result from capacity competition is no more than approximately $1/6$ of the maximal social surplus. These results suggest that even in the simplest capacity games if the "incorrect" equilibria arise, there could be very large inefficiencies, but if the "appropriate" equilibrium is selected, capacity and price competition between two firms is sufficient to ensure a high degree of efficiency.

We also suggest a simple way of implementing the best equilibrium, by considering a game form in which firms make their capacity choices sequentially, in reverse order according to their costs of investing in capacity. In the special case with two firms, this corresponds to a situation in which the firm with the lower cost of capacity investment acts as the Stackelberg leader. This "Stackelberg" game may be implemented by some type of regulation, for example by giving a first-mover advantage to lower-cost firms, or it may arise as the focal point in the game. We show that this Stackelberg game has a unique (pure strategy) equilibrium and inefficiency in this equilibrium is bounded by $2\sqrt{2} - 2 \simeq 5/6$.

We also show that our main results generalize to the game with $N$ firms. For this case, we characterize the pure strategy equilibria using a slightly different argument, and then show that the Price of Anarchy (the combination of worst-case parameters and worst equilibrium) is again equal to zero. Moreover, there is again a bound on the Price of Stability (the combination of worst-case parameters with the selection of best equilibrium), equal to $2(\sqrt{N} - 1)/(N - 1)$, and we show that this bound is also tight.

The differences in the structure of equilibria and the extent of inefficiency between our baseline game and the Stackelberg game suggest that the timing of moves is an important determinant of the extent of inefficiency in this class of games. This raises the natural question of how the set of equilibria will be affected when pricing and capacity decisions are made simultaneously. We show that in this case there *never* exists a pure strategy equilibrium, which starkly contrasts with the result that a pure strategy equilibrium always exists in the sequential game. This nonexistence of equilibrium results from the ability of the firms to deviate simultaneously on their capacities and prices. In contrast, in the sequential game, a firm could only deviate by changing its capacity first, and then its rivals could also respond by adjusting their prices to this deviation. Since the sequence of events in which capacities are chosen first and then prices are set later is more reasonable (in the sense that it constitutes a better approximation to a situation in which prices can change at much higher frequencies than capacities), we do not view this result as negative. Nonetheless, it suggests that it is important for industries with major capacity investments to choose structures of regulation that do not allow simultaneous deviations on capacities and prices.

Two modeling assumptions that are important in our analysis deserve a brief discussion here. First, in the context of communication network applications, our model corresponds to a network where service providers operate parallel links (or subnetworks). This is a natural starting point for analysis of equilibria in such markets since many service providers offer end-to-end service. Moreover, as shown in Acemoglu and Ozdaglar (2007b), when the network involves serial providers, the potential double marginalization problem can lead to much greater inefficiencies even without capacity investments. Second, the equilibria we construct are supported by mixed strategy play off the equilibrium path (though along the equilibrium path actions are pure). Mixed strategy equilibria in this context can be interpreted as resulting from explicit randomization by the firms or they can be justified by Harsanyi type purification arguments (Harsanyi (1973)).

In addition to the newly-burgeoning literature on competition and cooperation between users and firms in communication networks, this paper is closely related to the industrial organization literature on capacity competition. Classic contributions here include Levitan and Shubik (1980), Kreps and Scheinkman (1983) and Davidson and Deneckere (1986). A key issue in these papers, especially in Davidson and Deneckere (1986), is the rationing rule when total demand exceeds capacity. Our simpler framework with inelastic demand avoids this issue and enables us to provide a complete characterization of the full set of subgame perfect Nash equilibria. In another related work, Fabra et al. (2006) study a similar model with two firms and inelastic demand, where the focus is on understanding the implications of demand uncertainty on equilibrium characterization. Unlike our work, efficiency properties of equilibria have not been considered in this work.

Most closely related to our paper is the recent work, Weintraub et al. (2006), who add investment decisions to the model of price competition with congestion externalities in Acemoglu and Ozdaglar (2007a) and study the efficiency properties of oligopoly equilibria. Weintraub, Johari, and Van Roy put very little restriction on how investments may affect congestion costs, but only focus on the case in which all firms are symmetric and there are no capacity constraints. In this case, an equilibrium, when it exists, is always efficient. The distinguishing feature of our work is to consider and fully characterize the equilibria in the general non-symmetric case (where inefficiencies are indeed important as shown by our unbounded Price of Anarchy result) and also to introduce capacity constraints, which are a realistic feature of most communication networks.

The rest of the paper is organized as follows. Section 2 describes the model. Section 3 defines the price-capacity competition game and the oligopoly equilibria in this game. Section 4 characterizes the continuation price equilibria and the profits in the capacity subgames. Section 5 focuses on the special case with two firms and characterizes pure strategy oligopoly equilibria of the game (as well as the mixed strategy off-the-equilibrium play). Section 6 contains our main results and provides various efficiency bounds for the set of pure strategy oligopoly equilibria for the case of two firms. Section 7 generalizes the existence and efficiency results to an arbitrary number of firms. Section 8 shows how the best oligopoly equilibria can be implemented by a multi-stage game, where the low-cost firm acts as the Stackelberg leader. Section 9 analyzes a related game with simultaneous capacity-price decisions and shows that this game never has a pure strategy equilibrium. Section 10 concludes.

# 2 Model

We start with the general model with $N$ firms. Each firm can be thought of as a service provider operating its own communication subnetwork. For this reason, we refer to the demands for the firms' services as "flows". We denote total flow for firm $i \in \{1, ..., N\}$ by $x_i \geq 0$, and use $x = (x_1, ..., x_N)$ to denote the vector of flows. We assume that firm $i$ has a capacity $c_i \geq 0$, and flow allocated to firm $i$ cannot exceed its capacity, i.e., $x_i \leq c_i$. We denote the vector of capacities by $c = (c_1, ..., c_N)$. Investing in capacity is costly. In particular, the cost of capacity $c_i$ for firm $i$ is $\gamma_i c_i$, where $\gamma_i > 0$ for $i \in \{1, ..., N\}$.[1] For simplicity (and without loss of generality), we ignore additional costs of servicing flows. We denote the price charged by firm $i$ (per unit flow) by $p_i$ and denote the vector of prices by $p = (p_1, ..., p_N)$.

We are interested in the problem of allocating $d$ units of aggregate flow between these $N$ firms and without loss of generality, we set $d = 1$. We assume that this is the aggregate flow of many "small" users.[2] We also assume that the users have a *reservation utility* $R$; they choose the lowest price firm whenever there is unused capacity with this firm and do not participate if the lowest available cost exceeds the reservation utility. Further, we assume throughout the paper that $\gamma_i \leq R$ for all $i \in \{1, ..., N\}$. This is without loss of generality, since any firm with $\gamma_i > R$ will have no incentive to be active and can be excluded from the set $i \in \{1, ..., N\}$.

We start with the definition of *flow equilibrium* given a vector of capacities $c$ and a vector of prices $p$.

**Definition 1** **[Flow Equilibrium]** For a given capacity vector $c \geq 0$ and price vector $p \geq 0$, a vector $x^*$ is a *flow equilibrium* if

$$x^* \in \arg \max_{\substack{0 \leq x_i \leq c_i \\ \sum_{i=1}^{N} x_i \leq 1}} \left\{ \sum_{i=1}^{N} (R - p_i)\, x_i \right\}. \tag{1}$$

We denote the set of flow equilibria at a given $p$ and $c$ by $W[p, c]$.

This definition captures the simple notion that users will allocate their demand to the lowest price firm up to the point where the capacity constraint of this firm is reached. After this,

[1] Alternatively, we could assume $\gamma_i \geq 0$, with essentially the same results, but in this case Propositions 7 and 13 below need to be modified slightly, since there could be excess capacity in some equilibria.

[2] In the presence of additional congestion costs, this small users assumption would lead to the Wardrop principle, commonly used in communication and transport networks (see Wardrop (1952)), where flows are routed along paths with minimum effective cost (see, for example, Larsson and Patriksson (1994), Acemoglu and Ozdaglar (2007a)). In our context, there is no need to introduce this concept and it suffices to observe that users will choose a lower cost provider whenever this is possible.

if there are any more users, they will allocate their capacity to the second lowest price firm (as long as its price does not exceed their reservation utility, $R$), and so on.

Using the optimality conditions for problem (1), it follows that a vector $x^* \geq 0$ is a flow equilibrium if and only if $\sum_{i=1}^N x_i^* \leq 1$ and there exists $\lambda \geq 0$ such that $\lambda \left(\sum_{i=1}^N x_i^* - 1\right) = 0$ and for all $i \in \{1, ..., N\}$,

$$
\begin{aligned}
R - p_i &\leq \lambda \text{ if } x_i^* = 0, \\
&= \lambda \text{ if } 0 < x_i^* < c_i, \\
&\geq \lambda \text{ if } x_i^* = c_i.
\end{aligned}
\tag{2}
$$

This is a convenient representation of the flow equilibrium, which will be used in the analysis below. The following result on the structure of flow equilibria is an immediate consequence of this characterization (proof omitted):

**Proposition 1** Let $c = (c_1, ..., c_N) \geq 0$ be a capacity vector and $p = (p_1, ..., p_N) \geq 0$ be a price vector. Suppose that for some $M \leq N$, we have $p_1 < p_2 < ... < p_M \leq R < p_{M+1}$ (with the convention that $p_{M+1} = +\infty$ if $M = N$). Then, at all flow equilibria $x \in W[p, c]$, we have

$$
\begin{aligned}
x_1 &= \min\{c_1, 1\}, \\
x_m &= \min\left\{c_m, \max\left\{0, 1 - \sum_{i=1}^{m-1} x_i\right\}\right\}, \qquad \forall\ 2 \leq m < M.
\end{aligned}
$$

Moreover, if $p_M < R$, then there exists a unique flow equilibrium $x \in W[p, c]$ given by

$$
\begin{aligned}
x_1 &= \min\{c_1, 1\}, \\
x_m &= \min\left\{c_m, \max\left\{0, 1 - \sum_{i=1}^{m-1} x_i\right\}\right\}, \qquad \forall\ 2 \leq m \leq M.
\end{aligned}
$$

**Remark 1** If instead of $p_1 < p_2 < ... < p_M \leq R$, we have $p_i = p_j$ for some $i \neq j$, the flow equilibrium is not necessarily unique, since users would be indifferent between allocating their flow across these two firms. Note also that in the special case with $N = 2$, this proposition simply states that when $p_1 < p_2 < R$, the unique flow equilibrium will involve $x_1 = \min\{c_1, 1\}$ and $x_2 = \min\{c_2, 1 - x_1\}$.

We next define the social optimum, which is the capacity and flow allocation that would be chosen by a planner that has full information and full control over the allocation of resources. Since there is no cost of servicing flows beyond the capacity costs, the following definition for a social optimum follows immediately.

**Definition 2** A capacity-flow vector $(c^S, x^S)$ is a *social optimum* if it is an optimal solution of the *social problem*

$$
\begin{aligned}
\text{maximize}_{x\geq 0,\ c\geq 0}\ & R\sum_{i=1}^{N} x_i - \sum_{i=1}^{N} \gamma_i c_i \qquad (3)\\
\text{subject to}\ & \sum_{i=1}^{N} x_i \leq 1,\\
& x_i \leq c_i, \quad i \in \{1, ..., N\}.
\end{aligned}
$$

The social problem has a continuous objective function and a compact constraint set, guaranteeing the existence of a social optimum $(c^S, x^S)$. It is also clear from the preceding that we have $c_i^S = x_i^S$, for $i \in \{1, ..., N\}$. We refer to $c^S$ as the *social capacity.* In view of the fact that $c_i^S = x_i^S$, for $i \in \{1, ..., N\}$, the social capacity is given as the solution to the following maximization problem:

$$
c^S \in \arg \max_{c\geq 0,\ \sum_{i=1}^{N} c_i \leq 1} \left\{ \sum_{i=1}^{N} (R - \gamma_i) c_i \right\}. \qquad (4)
$$

For future reference, for a given capacity vector $c \geq 0$, we define the social surplus as

$$
\mathbb{S}(c) = \sum_{i=1}^{N} (R - \gamma_i) c_i, \qquad (5)
$$

i.e., the difference between the users' utility and the total capacity cost.

## 3 Price and Capacity Competition Game

We next consider the two-stage competition game in which capacities are chosen first and then firms compete in prices as outlined in the previous section.

The price-capacity competition game is as follows. First, the $N$ firms simultaneously choose their capacities, i.e., firm $i$ chooses $c_i$ at cost $\gamma_i c_i$. At the second stage, firms, having observed the capacities set at the first stage, simultaneously choose prices, i.e., firm $i$ charges a price $p_i$. Given the price vector of other firms, denoted by $p_{-i}$, the profit of firm $i$ is

$$
\Pi_i[p_i, p_{-i}, x, c_i, c_{-i}] = p_i x_i - \gamma_i c_i,
$$

where $x \in W[p, c]$ is a flow equilibrium given the price vector $p$ and the capacity vector $c$. The objective of each firm is to maximize profits. We refer to the dynamic game between the two firms as the *price-capacity competition game*, and look for the subgame perfect equilibria (SPE) of this game. Since the capacities set in the first stage are observed by all firms, every capacity vector $c = (c_1, ..., c_N)$ defines a proper subgame, and subgame perfection requires that in each subgame, the continuation equilibrium strategies constitute a Nash equilibrium.[3] For each *capacity subgame*, we first define the price equilibrium between the

[3] A subgame is identified with the public history (of previous moves). Hence, the SPE notion

firms, which we will also refer to as the (continuation) *Price Equilibrium.* As we will see below, pure strategy equilibria will fail to exist in some capacity subgames. For this reason, we define both pure and mixed strategy price equilibria. Let $\mathcal{B}$ denote the space of all (Borel) probability measures on the interval $[0, R]$. Let $\mu_i \in \mathcal{B}$ be a probability measure, and denote by $\mu \in \mathcal{B}^N$ the product measure $\mu_1 \times ... \times \mu_N$, and by $\mu_{-i}$ the product measure $\mu$ excluding $\mu_i$ (i.e., $\mu_{-i} = \mu_1 \times ... \times \mu_{i-1} \times \mu_{i+1} \times ... \times \mu_N$). We use the notation $[p(c), x(c)]$ to denote a pure strategy Price Equilibrium of the capacity subgame $c$, where $p(c)$ is a price vector and $x(c)$ is a flow vector. Similarly, we use the notation $[\mu^c, x^c(p)]$ to denote a mixed strategy Price Equilibrium of the capacity subgame $c$, where $\mu^c \in \mathcal{B}^N$ and $x^c(p)$ is a subgame perfect selection from $W[p, c]$.

**Definition 3** **[Price Equilibrium]** Let $c \geq 0$ be a capacity vector. A vector $[p(c), x(c)]$ is a *pure strategy Price Equilibrium* in the capacity subgame if $x(c) \in W[p(c), c]$ and for all $i \in \{1, ..., N\}$,

$$\Pi_i[p_i(c), p_{-i}(c), x(c), c] \geq \Pi_i[p_i, p_{-i}(c), x, c], \qquad \forall\ p_i \geq 0,\ \forall\ x \in W[p_i, p_{-i}(c), c]. \tag{6}$$

We denote the set of pure strategy price equilibria at a given $c$ by $PE(c)$.

A vector $[\mu^c, x^c(p)]$ is a *mixed strategy Price Equilibrium* in the capacity subgame if $\mu^c \in \mathcal{B}^N$ and the function $x^c(p) \in W[p, c]$ for every $p$ and

$$\int_{[0,R]^N} \Pi_i[p_i, p_{-i}, x^c(p_i, p_{-i}), c] \quad d\Big(\mu_i^c(p_i) \times \mu_{-i}^c(p_{-i})\Big)$$
$$\geq \int_{[0,R]^N} \Pi_i[p_i, p_{-i}, x^c(p_i, p_{-i}), c] d\Big(\mu_i(p_i) \times \mu_{-i}^c(p_{-i})\Big),$$

for all $i \in \{1, ..., N\}$ and $\mu_i \in \mathcal{B}$. We denote the set of mixed strategy price equilibria at a given $c$ by $MPE(c)$.

In the following, with a slight abuse of notation, we will write $[\mu, x(\cdot)] \in MPE(c)$ for mixed strategy equilibria.[4] Note that here $x(\cdot)$ is not a vector, but a function of $p$, i.e., $x(p)$ is a selection from the correspondence $W(p, c)$. We denote the profits for firm $i$ in the mixed

requires that the action prescribed by each player's strategy is optimal given the other player's strategies, after every history; see, for example, Fudenberg and Tirole (1991), Osborne and Rubinstein (1994).

[4] Note also that the pure strategy Price Equilibrium notion here may appear slightly stronger than the standard subgame perfection, since it requires that a strategy profile yields higher profits for each player for all $x \in W[p_i, p_{-i}(c), c]$, rather than for some such $x$. Nevertheless, Acemoglu and Ozdaglar (2007a) shows, for a more general game, that this definition of equilibrium coincides with the standard pure strategy subgame perfect equilibrium (but is slightly more convenient to work with). Given this relation, we have $PE(c) \subset MPE(c)$, in the sense that for every $[p, x] \in PE(c)$, there exists $[\mu, x(\cdot)] \in MPE(c)$ such that $\mu$ is the degenerate measure with $\mu(\{p\}) = 1$, and $x(\cdot)$ is an arbitrary selection from $W[p, c]$ with $x(p) = x$.

strategy price equilibria in the capacity subgame by $\Pi_i[\mu, x(\cdot), c]$, i.e.,

$$\Pi_i[\mu, x(\cdot), c] = \int_{[0,R]^N} \Pi_i[p, x(p), c] d\mu(p). \tag{7}$$

We will also use the notation $\Pi_i[p_i, \mu_{-i}, x(\cdot), c]$ for some $p_i \in [0, R]$ to denote the expected profits when firm $i$ uses the degenerate mixed strategy $\mu_i$ with $\mu_i(p_i) = 1$, while the remaining firms use the mixed strategy $\mu_{-i}$.

Note that since the profit functions are discontinuous in prices, it is not obvious that each capacity subgame has a mixed strategy price equilibrium. In the next section, we will show that in each capacity subgame $c$, a pure or mixed strategy price equilibrium always exists.

We next define the subgame perfect equilibrium of the entire game. For notational convenience, we focus on the actions along the equilibrium path to represent the subgame perfect equilibrium.

**Definition 4 [Oligopoly Equilibrium]** A vector $[c^{OE}, p(c^{OE}), x(c^{OE})]$ is a *(pure strategy) Oligopoly Equilibrium (OE)* if $[p(c^{OE}), x(c^{OE})] \in PE(c^{OE})$ and for all $i \in \{1, ..., N\}$,

$$\Pi_i[p(c^{OE}), x(c^{OE}), (c_i^{OE}, c_{-i}^{OE})] \geq \Pi_i[\mu, x(\cdot), (c_i, c_{-i}^{OE})], \tag{8}$$

for all $c_i \geq 0$, and for all $[\mu, x(\cdot)] \in MPE(c_i, c_{-i}^{OE})$. We refer to $c^{OE}$ as the *OE capacity.*

Note that pure strategy OE may involve pure strategies along the equilibrium path, but mixed strategy continuation price equilibria in some off-the-equilibrium subgames. Throughout the paper, pure strategy OE refers to equilibria where pure strategies are used along the equilibrium path.

## 4 Price Equilibria in the Capacity Subgame With Two Firms

Our first task is to characterize the entire set of subgame perfect equilibria in this game. For expositional purposes, we start with the case where $N = 2$, which enables us to provide an explicit characterization of the equilibria and the extent of the efficiency losses. We generalize our main results to an arbitrary number of firms in Section 7 below.

We consider an arbitrary capacity subgame and prove the existence of pure or mixed strategy price equilibria and provide a characterization of these equilibria. We will then use this characterization to determine the form of oligopoly equilibria and analyze their efficiency properties. Since in this and in the next section we consider only two firms, we sometimes refer to these two firms using the indices $i$ and $-i$.

**Proposition 2** Let $c$ be a capacity vector such that $c_1 + c_2 \leq 1$ and $c_i > 0$ for $i = 1, 2$. Then there exists a unique Price Equilibrium in the capacity subgame $[p, x]$ such that $p_i = R$ and $x_i = c_i$ for $i = 1, 2$.

**Proof.** Since $c_1 + c_2 \leq 1$, it follows by the equivalent characterization of a flow equilibrium [cf. equation (2)] that for all $p \in [0, R]^2$, the flow allocation $(c_1, c_2) \in W(p, c)$. Therefore, by charging a price $p_i$, firm $i$ can make a profit of

$$\Pi_i[p_i, p_{-i}, x, c] = p_i c_i,$$

for all $p_{-i} \in [0, R]$. This shows that $p_i = R$ strictly dominates all other price strategies of firm $i$, so that $p_i = R$ and $x_i = c_i$, $i = 1, 2$, is the unique Price Equilibrium. **Q.E.D.**

**Proposition 3** Let $c$ be a capacity vector such that $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Then there exists no pure strategy Price Equilibrium in the capacity subgame.

**Proof.** Suppose there exists a pure strategy Price Equilibrium $(p, x)$. The following list considers all candidates for a Price Equilibrium and profitable unilateral deviations from each, thus establishing the nonexistence of a pure strategy Price Equilibrium:

- Suppose $p_1 < p_2$. Then the profit of firm 1 is $\Pi_1[p, x, c] = p_1 \min\{c_1, 1\}$. A small increase in $p_1$ will increase firm 1's profits, thus firm 1 has an incentive to deviate.
- Suppose $p_1 = p_2 > 0$. If $x_1 < \min\{c_1, 1\}$, then firm 1 has an incentive to decrease its price. If $x_1 = \min\{c_1, 1\}$, then, since $c_1 + c_2 > 1$, firm 2 has an incentive to decrease its price.
- Suppose $p_1 = p_2 = 0$. Since by assumption $c_i < 1$ for some $i$, firm $-i$ has an incentive to increase its price and make positive profits.

**Q.E.D.**

**Proposition 4** Let $c$ be a capacity vector such that $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Then there exists a mixed strategy Price Equilibrium in the capacity subgame.

**Proof.** The subgame following any capacity choice $c$ is a special case of the model in Acemoglu and Ozdaglar (2007a). Building on Dasgupta and Maskin (1986), Proposition 4.3 in Acemoglu and Ozdaglar (2007a) establishes that there always exists a mixed strategy equilibrium in any such subgame. We do not repeat this proof here to avoid repetition. **Q.E.D.**

When $c_1, c_2 \geq 1$, the capacity subgame is an uncapacitated Bertrand price competition between two firms. Thus, we immediately have the following result (proof omitted).

**Proposition 5** Let $c$ be a capacity vector such that $c_1, c_2 \geq 1$. Then, for all Price Equilibria $[p, x]$, we have $p_i = 0$ for $i = 1, 2$, i.e., both firms make zero profits.

### 4.2 Characterization of Mixed Strategy Price Equilibria

We next provide an explicit characterization of the mixed strategy price equilibria and the profits in each capacity subgame.

Let $c = (c_1, c_2)$ be a capacity vector. Throughout this section, we focus on the case where $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. By Proposition 4, there exists a mixed strategy Price Equilibrium $[\mu, x(\cdot)]$ in the capacity subgame. Let $u_i$ denote the *upper support of* $\mu_i$, and $l_i$ denote the *lower support of* $\mu_i$, i.e.,

$$u_i = \inf \Big\{ \bar{p} \ : \mu_i(\{p \leq \bar{p}\}) = 1 \Big\},$$

$$l_i = \sup \Big\{ \underline{p} \ : \mu_i(\{p \geq \underline{p}\}) = 1 \Big\}.$$

Let $(F_1, F_2)$ denote the corresponding cumulative distribution functions for the measure $(\mu_1, \mu_2)$, i.e., $F_i(\bar{p}) = \mu_i(\{p \leq \bar{p}\})$, for $i = 1, 2$.

Recall that $[\mu, x(\cdot)]$ is a mixed strategy Price Equilibrium if and only if for some $\Pi_i^E \geq 0$, we have

$$\Pi_i[p, \mu_{-i}, x(\cdot), c] \leq \Pi_i^E, \tag{9}$$

for all $p \in [0, R]$, and there exists a set $\bar{P}_i \subseteq [l_i, u_i]$ such that $\mu_i(\bar{P}_i) = 1$ and

$$\Pi_i \left[p, \mu_{-i}, x\left(\cdot\right), c\right] = \Pi_i^E \text{ for all } p \in \bar{P}_i, \tag{10}$$

(see, e.g., Osborne and Rubinstein (1994)). It can be seen that the flow equilibrium correspondence $W[p, c]$ is upper semicontinuous (see Acemoglu and Ozdaglar (2007a)). This implies that the selection $x(p)$ and therefore the profit function $\Pi_i$ is continuous in $p$, unless $F_{-i}$ has an atom at $p$. Hence, it follows that relation (10) holds also for $p = l_i$ (and $p = u_i$) for $i = 1, 2$, unless $F_{-i}$ has an atom at $l_i$ (or $u_i$). This follows by the definition of $l_i$ (and $u_i$) since there exists some $p \in \bar{P}_i$ which is arbitrarily close to $l_i$ (and $u_i$).

We will now use this property of mixed strategy equilibria to derive three lemmas that will allow us to explicitly characterize the unique mixed strategy Price Equilibrium in the capacity subgame. [5]

**Lemma 1** Assume that $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Then, for any $i = 1, 2$, the mixed strategy $\mu_i$ cannot have all its mass concentrated at a single point, i.e., $\mu_i$ cannot be degenerate.

[5] Uniqueness here implicitly ignores variations on measure zero sets.

**Proof.** By Proposition 3, both $\mu_i$'s cannot be degenerate. To obtain a contradiction, assume that $\mu_1$ is degenerate at some $p_1 \in [0, R]$ (i.e., $\mu_1(\{p = p_1\}) = 1$). We first show that $\mu_2(\{p < p_1\}) = 0$. Consider $p_1 > 0$. Charging the price $p_2 = p_1 - \epsilon$ for some $\epsilon > 0$ yields a profit of $(p_1 - \epsilon)\min\{c_2, 1\}$ for firm 2, which is strictly decreasing in $\epsilon$, showing that $\mu_2(\{p < p_1\}) = 0$. We next show that $\mu_2$ cannot have an atom at $p = p_1$. Suppose it does; then there is a positive probability of both firms charging the price $p_1$. If $x_2(p_1, p_1) < \min\{c_2, 1\}$, then charging a price of $p_1 - \delta$ for some small $\delta > 0$ generates higher profits for firm 2. If $x_2(p_1, p_1) = \min\{c_2, 1\}$, then, since $c_1 + c_2 > 1$, the same applies to player 1, showing that $\mu_2$ cannot have an atom at $p_2$. Finally, if firm 2 charges the price $p_2 = p_1 + \epsilon$ for $0 < \epsilon \leq R - p_1$, it yields a profit of $(p_1 + \epsilon)(1 - \min\{c_1, 1\})$, which is strictly increasing in $\epsilon$, thus $\mu_2$ should have all its mass concentrated at $p_2 = R$. However both $\mu_i$'s cannot be degenerate, thus we arrive at a contradiction. **Q.E.D.**

**Lemma 2** Assume that $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Then:

(i) $F_1$ and $F_2$ have the same lower support, i.e., $l_1 = l_2 = l$.
(ii) $F_1$ and $F_2$ have the same upper support, i.e., $u_1 = u_2 = u$.
(iii) $F_1$ and $F_2$ are strictly increasing over $[l, u]$.

**Proof.**

(i) Assume that $l_1 < l_2$. This implies the existence of two prices $p_1 \in \bar{P}_1$ and $p_1' \in (0, R)$ such that $p_1 < p_1' < l_2$. Then, by Proposition 1, it follows that for all $p_2 \in \bar{P}_2$ and $p = p_1$ or $p = p_1'$, any flow equilibrium $x \in W[(p, p_2), c]$ satisfies $x_1 = \min\{c_1, 1\}$. Thus the profits of firm 1 at prices $p_1,\ p_1'$ are given by

$$\Pi_1[p_1, \mu_2, x, c] = p_1 x_1 < \Pi_1[p_1', \mu_2, x, c] = p_1' x_1,$$

contradicting the equilibrium characterization in (9)-(10).

(ii) Assume that $u_1 > u_2$. We consider the following three cases.
- $(u_2, u_1] \cap \bar{P}_1 = \{u_1\}$ and $F_1$ does not have an atom at $u_2$. Then, for all flow equilibria $x(p) \in W[p, c]$, we have $x_2(p, u_2) = x_2(p, u_2 + \epsilon)$ for some sufficiently small $\epsilon > 0$ and all $p \in \bar{P}_1$, implying that firm 2 has a profitable deviation.
- $(u_2, u_1] \cap \bar{P}_1 = \{u_1\}$ and $F_1$ has an atom at $u_2$. Then, if $F_2$ does not have an atom at $u_2$, we have $x_1(u_2, p) = x_1(u_2 + \epsilon, p)$ for some sufficiently small $\epsilon > 0$ and all $p \in \bar{P}_2$, implying that firm 1 has a profitable deviation. On the other hand, both $F_1$ and $F_2$ cannot have an atom at $u_2$, since one of them would then have a profitable deviation by setting price $p = u_2 - \epsilon$ with probability 1 for sufficiently small $\epsilon$.
- $(u_2, u_1] \cap \bar{P}_1 \neq \{u_1\}$. This implies the existence of two prices $p_1 \in \bar{P}_1$ and $p_1' \in [0, R)$ such that $u_2 < p_1 < p_1'$. By Proposition 1 and the assumption $c_1 + c_2 > 1$, it follows that for all $p_2 \in \bar{P}_2$ and $p = p_1$ or $p = p_1'$, any flow equilibrium $x \in W[(p, p_2), c]$ satisfies $x_1 = 1 - \min\{c_2, 1\}$. Moreover, we have $x_1 > 0$ since otherwise, we would have $\Pi_1[p, \mu_2, x, c] = 0$ for all $p \in \bar{P}_1$ and a deviation to $p_1 = u_2 - \epsilon$ would yield positive profits and would be a profitable deviation. Thus the profits of firm 1 at prices $p_1,\ p_1'$ are given by
$$\Pi_1[p_1, \mu_2, x, c] = p_1 x_1 < \Pi_1[p_1', \mu_2, x, c] = p_1' x_1,$$

contradicting the equilibrium characterization in (9)-(10).

(iii) Assume to arrive at a contradiction that $F_1$ is constant over the interval $[p_1, p_1']$ for some $p_1, p_1' \in [l, u]$ with $p_1 < p_1'$. We assume without loss of generality that $F_2$ does not have an atom at $p_1'$ [otherwise, we can replace $p_1'$ by $p_1' - \epsilon$ for some sufficiently small $\epsilon > 0$]. We will first show that this implies $F_2$ is constant over the same interval. Suppose to obtain a contradiction that $F_2$ is not constant over this interval. Then there exist $p_2 \in \bar{P}_2$ and $p_2' \in (0, R)$ such that $p_1 < p_2 < p_2' < p_1'$ and $F_2(p_2) < F_2(p_2')$ (the assumption that $F_2$ does not have an atom at $p_1'$ ensures that $p_2 < p_1'$ by the right continuity of a distribution function). By Proposition 1 and the assumption $c_1 + c_2 > 1$, for all flow equilibria $x(p) \in W[p, c]$, we have

$$x_2(p, p_2) = x_2(p, p_2') = \begin{cases} 1 - \min\{c_1, 1\} & \text{if } p < p_1, \\ \min\{c_2, 1\} & \text{if } p > p_1'. \end{cases}$$

Since $F_1$ is constant over $[p_1, p_1']$, this implies that the profits of firm 2 at $p_2'$ are higher than those at $p_2$, yielding a contradiction.

Next we have to consider the following three cases.

- $F_2$ has an atom at $p_1$. Then, if $F_1$ does not have an atom at $p_1$, setting price $p_1 + \epsilon$ with probability 1 is a profitable deviation for firm 2, yielding a contradiction. On the other hand, both $F_1$, $F_2$ cannot have an atom at $p_1$, since then there would exist a profitable deviation for both of them.
- $p_1 \in \bar{P}_1$ and $F_2$ does not have an atom at $p_1$. Then, consider a deviation by firm 1 to $p_1 + \epsilon$ instead of $p_1$ for some sufficiently small $\epsilon > 0$. Since $F_2$ does not have an atom at $p_1$ and is constant between $p_1$ and $p_1 + \epsilon$, we have for all flow equilibria $x(p) \in W[p, c]$, $x_1(p_1, p) = x_1(p_1 + \epsilon, p)$ for all $p \in \bar{P}_2$, and therefore deviating to $p_1 + \epsilon$ is profitable for firm 1, yielding a contradiction.
- $p_1 \notin \bar{P}_1$ and $F_2$ does not have an atom at $p_1$. Note that for all sufficiently small $\epsilon > 0$, it holds that $p_1 - \epsilon \in \bar{P}_1$ [otherwise we would have considered a larger interval for which $F_1$ is constant] and for every $\epsilon_1 > 0$ we can find $\epsilon$ such that $F_2(p_1 - \epsilon) \geq F_2(p_1) - \epsilon_1$ [since $F_2$ does not have an atom at $p_1$]. Then, there exists a profitable deviation for firm 1 to $p_1 + \epsilon_2$ for appropriately chosen $\epsilon_2 > 0$, yielding once again a contradiction.

**Q.E.D.**

**Lemma 3** Assume that $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Then:

(i) The distribution $F_i$, $i = 1, 2$, does not have any atoms except possibly at the upper support $u$.

(ii) Both distributions $F_i$ cannot have an atom at the upper support $u$.

(iii) The upper support $u$ is equal to $R$.

**Proof.**

(i) Without loss of generality, we consider $F_1$. We first show that $F_1$ cannot have an atom at any $p \in (l, u)$. Assume to arrive at a contradiction that there exists an atom at some

$p \in (l, u)$, i.e., $F_1(p+) > F_1(p-)$. By Lemma 2(iii), $F_2$ is strictly increasing over the interval $[l, u]$ (which satisfies $l < u$ in view of Lemma 1). Thus, there exists some $\epsilon > 0$ sufficiently small such that the prices $p - \epsilon$ and $p + \epsilon$ belong to $\bar{P}_2$, and $p + \epsilon < R$. Using Proposition 1, the profits of firm 2 at these two prices can be written as

$$\Pi_2[p-\epsilon, \mu_1, x(\cdot), c] = F_1(p-\epsilon)(p-\epsilon)(1-\min\{c_1, 1\}) + (1 - F_1(p-\epsilon))(p-\epsilon)\min\{c_2, 1\},$$

and

$$\Pi_2[p+\epsilon, \mu_1, x(\cdot), c] = F_1(p+\epsilon)(p+\epsilon)(1-\min\{c_1, 1\}) + (1 - F_1(p+\epsilon))(p+\epsilon)\min\{c_2, 1\}.$$

Since $F_1(p+) > F_1(p-)$ and $\min\{c_2, 1\} > 1 - \min\{c_1, 1\}$, it follows that for small enough $\epsilon$, we have

$$\Pi_2[p-\epsilon, \mu_1, x(\cdot), c] > \Pi_2[p+\epsilon, \mu_1, x(\cdot), c],$$

yielding a contradiction.

We next show that $F_1$ cannot have an atom at $p = l$. We first prove that the common lower support must satisfy $l > 0$. If $l = 0$, then by relations (9), (10) we get that the profits of either firm at any price vector are equal to 0. Since by assumption $c_i < 1$ for some $i$, the profits of firm $-i$ at $p = u$ are strictly positive at any flow equilibrium. Hence, it follows that $l > 0$. Consider the profits of firm 2 at price $l - \epsilon$ for some sufficiently small $\epsilon$,

$$\Pi_2[l-\epsilon, \mu_1, x(\cdot), c] = (l-\epsilon)\min\{c_2, 1\}.$$

Consider next the profits of firm 2 at the price $l + \epsilon$, which belongs to $\bar{P}_2$:

$$\Pi_2[l+\epsilon, \mu_1, x(\cdot), c] = (1 - F_1(l+\epsilon))(l+\epsilon)\min\{c_2, 1\} + F_1(l+\epsilon)(l+\epsilon)(1 - \min\{c_1, 1\}).$$

If there is an atom at $l$, i.e., $F_1(l+) > 0$, then since $1 - \min\{c_1, 1\} < \min\{c_2, 1\}$, it follows from the preceding two relations that for sufficiently small $\epsilon$,

$$\Pi_2[l-\epsilon, \mu_1, x(\cdot), c] > \Pi_2[l+\epsilon, \mu_1, x(\cdot), c],$$

contradicting equation (9).

(ii) Assume that both distributions have an atom at $p = u$. Then, it follows that with probability $[F_1(u+) - F_1(u-)] \cdot [F_2(u+) - F_2(u-)] > 0$, both firms will be charging a price of $p = u$. Suppose $x_1(u, u) < \min\{c_1, 1\}$. Then charging a price of $p_1 = u - \epsilon$ generates higher profits for firm 1 than charging a price of $p_1 = u$. If $x_1(u, u) = \min\{c_1, 1\}$, then, since $c_1 + c_2 > 1$, the same applies to player 2, establishing this part of the lemma.

(iii) Assume that $u < R$. By part (ii), it follows that there is no atom at $u$ for one of the players, say player 2. Then

$$\Pi_1[u, \mu_2, x(\cdot), c] = u(1 - \min\{c_2, 1\}) < \Pi_1[R, \mu_2, x(\cdot), c] = R(1 - \min\{c_2, 1\}),$$

showing that the upper support $u$ cannot be strictly less than $R$.

**Q.E.D.**

The next proposition characterizes the expected profits of the two firms in capacity subgames with continuation mixed strategy price equilibrium. This result follows from Lemma 5 in Kreps and Scheinkman (1983) by specializing it to the inelastic demand case. We provide here an alternative proof for completeness.

**Proposition 6** Let $c = (c_1, c_2)$ be a capacity vector with $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Let $[\mu, x(\cdot)]$ be a mixed strategy Price Equilibrium in the capacity subgame $c$. The expected profits $\Pi_i[\mu, x(\cdot), c]$, for $i = 1, 2$ are given by

$$\Pi_i[\mu, x(\cdot), c] = \begin{cases} \frac{R(1-c_i)c_i}{\min\{c_{-i},1\}} - \gamma_i c_i, & \text{if } c_i \leq c_{-i}, \\ R(1 - c_{-i}) - \gamma_i c_i, & \text{otherwise.} \end{cases}$$

**Proof.** Assume that $c_1 \leq c_2$. We denote the equilibrium profits of player $i$ in the capacity subgame $c$ by $\Pi_i^E$. We will now use the characterization of mixed strategy equilibrium in equation (10) to explicitly characterize the equilibrium distributions $F_1$ and $F_2$. Recall that the relation (10) holds for $p = l_i$ and $p = u_i$, when $F_{-i}$ does not have an atom at $p = l_i$ and $p = u_i$.

By Lemma 3(i), the distributions $F_1$ and $F_2$ do not contain an atom except possibly at the upper support $R$. Using the Flow Equilibrium characterization given in Proposition 1, we can write the expected profits of firm 1 for any $p_1 \in \bar{P}_1 \cup \{l_1\}$, $p_1 \neq R$ as

$$\begin{aligned} \Pi_1[p_1, \mu_2, x(\cdot), c] &= p_1 c_1 (1 - F_2(p_1)) + p_1(1 - \min\{c_2, 1\}) F_2(p_1) - \gamma_1 c_1 \\ &= \Pi_1^E. \end{aligned}$$

Similarly, for all $p_2 \in \bar{P}_2 \cup \{l_2\}$, and $p_2 \neq R$, we have

$$\begin{aligned} \Pi_2[p_2, \mu_1, x(\cdot), c] &= p_2 \min\{c_2, 1\}(1 - F_1(p_2)) + p_2(1 - c_1) F_1(p_2) - \gamma_2 c_2 \\ &= \Pi_2^E. \end{aligned}$$

Let $\bar{\Pi}_1^E = \Pi_1^E + \gamma_1 c_1$ and $\bar{\Pi}_2^E = \Pi_2^E + \gamma_2 c_2$. Solving for $F_1(p)$ and $F_2(p)$ in the preceding relations, we obtain

$$F_1(p) = \frac{\min\{c_2, 1\} - \bar{\Pi}_2^E / p}{c_1 + \min\{c_2, 1\} - 1}, \qquad \forall\, p \in \bar{P}_2 \cup \{l_2\},\ p \neq R, \tag{11}$$

$$F_2(p) = \frac{c_1 - \bar{\Pi}_1^E / p}{c_1 + \min\{c_2, 1\} - 1}, \qquad \forall\, p \in \bar{P}_1 \cup \{l_1\},\ p \neq R.$$

Let $l$ denote the common lower support of $\mu_1$ and $\mu_2$, i.e., $l_1 = l_2 = l$ (cf. Lemma 2). Using the preceding relations for $p = l$ and the facts that $F_1(l) = F_2(l) = 0$, it follows that

$l = \bar{\Pi}_2^E / \min\{c_2, 1\}$, and

$$\bar{\Pi}_1^E = \bar{\Pi}_2^E \frac{c_1}{\min\{c_2, 1\}}.$$

By Lemma 2 and Lemma 3(iii), we have $u_1 = u_2 = R$. We next show that $F_1$ does not have an atom at $u = R$, and therefore the characterization in (11) is also valid for $p = R$. Assume to arrive at a contradiction that $F_1$ has an atom at $R$, i.e., $F_1(R-) < 1$. Then, using $c_1 \leq c_2$ and the preceding relation between $\bar{\Pi}_1^E$ and $\bar{\Pi}_2^E$, it follows that $F_2(R-) < 1$. But, by Lemma 3(ii), both distributions cannot have an atom at the upper support, yielding a contradiction. Hence, we can use the characterization in (11) for $p = R$ to write

$$F_1(R) = 1 = \frac{\min\{c_2, 1\} - \bar{\Pi}_2^E / R}{c_1 + \min\{c_2, 1\} - 1},$$

which shows that

$$\bar{\Pi}_2^E = R(1 - c_1),$$
$$\bar{\Pi}_1^E = \frac{R(1 - c_1)c_1}{\min\{c_2, 1\}}.$$

The argument for $c_1 \geq c_2$ is similar and completes the proof. **Q.E.D.**

## 5 Oligopoly Equilibria With Two Firms

In this section, we prove the existence and characterize the properties of pure strategy Oligopoly Equilibria. We first provide a characterization of pure strategy OE capacities. Using this characterization, we show that the price-capacity competition game always has a pure strategy Oligopoly Equilibrium. We then use this characterization to study the efficiency properties of pure strategy Oligopoly Equilibria in the next section.

**Proposition 7** Assume that $\gamma_i < R$ for some $i = 1, 2$. A capacity vector $c = (c_1, c_2)$ is an OE capacity if and only if $c_1 + c_2 = 1$ and

$$\frac{R - \gamma_i}{2R - \gamma_i} \leq c_i \leq c_{-i}, \tag{12}$$

for some $i = 1, 2$.

**Proof. (Sufficiency)** We first show that $c_1 + c_2 = 1$ together with (12) is an OE capacity. First, since $c_1 + c_2 \leq 1$, Proposition 2 implies that the profits of firm $i = 1, 2$ are

$$\Pi_i[p\,(c)\,, x, c] = (R - \gamma_i)c_i, \tag{13}$$

where $p\,(c)$ denotes the continuation equilibrium price vector, which in this case is $(R, R)$.

Consider a deviation $\hat{c}_i \neq c_i$ by firm $i$. If $\hat{c}_i < c_i$, Proposition 2 still applies and the resulting profit for firm $i$ is $\hat{\Pi}_i[p\,(\hat{c}_i, c_{-i})\,, x, (\hat{c}_i, c_{-i})] = (R - \gamma_i)\hat{c}_i \leq \Pi_i[p\,(c)\,, x, c]$, establishing that there are no profitable deviations with $\hat{c}_i < c_i$.

Next consider $\hat{c}_i > c_i$. Clearly, if $\hat{c}_i, c_{-i} \geq 1$, Proposition 5 applies and $\hat{\Pi}_i[p(\hat{c}_i, c_{-i}), x, (\hat{c}_i, c_{-i})] = 0$ so that the deviation is not profitable. Suppose that $\hat{c}_i + c_{-i} > 1$, $\hat{c}_i, c_{-i} > 0$ and either $\hat{c}_i < 1$ or $c_{-i} < 1$. Proposition 4 applies and the deviation will induce a mixed strategy continuation equilibrium $\mu$. There are two cases to consider: $\hat{c}_i > c_{-i}$ and $\hat{c}_i \leq c_{-i}$.

- Suppose that $\hat{c}_i > c_{-i}$, which by Proposition 6 implies that the deviation profits of firm $i$ are

$$\begin{aligned}\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] &= R(1 - c_{-i}) - \gamma_i \hat{c}_i \\ &= (R - \gamma_i)\, c_i - \gamma_i (\hat{c}_i - c_i) \\ &\leq \Pi_i[p(c), x, c],\end{aligned}$$

  where the second line exploits the fact that $c_1 + c_2 = 1$ and the third line uses the definition of equilibrium profits from (13) together with $\hat{c}_i \geq c_{-i}$, establishing that there are no profitable deviations with $\hat{c}_i > c_{-i}$.
- Suppose that $\hat{c}_i \leq c_{-i}$. Then by Proposition 6, we have

$$\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] = \frac{R(1 - \hat{c}_i)\hat{c}_i}{c_{-i}} - \gamma_i \hat{c}_i. \tag{14}$$

  Let $\hat{c}_i^{max}$ denote the capacity that maximizes (14), given by

$$\hat{c}_i^{max} \equiv \frac{1}{2} - \frac{c_{-i}\gamma_i}{2R}. \tag{15}$$

  Since $c_i < \hat{c}_i \leq c_{-i}$, we obtain from equation (12) that $\hat{c}_i^{max} \leq c_i$. Since $\hat{\Pi}_i$ is a strictly concave quadratic function of $\hat{c}_i$ and we have $\hat{c}_i^{max} \leq c_i$, it follows that $\hat{\Pi}_i$ is a non-increasing function of $\hat{c}_i$ in the interval $c_i < \hat{c}_i \leq c_{-i}$. Combined with the fact that

$$\hat{\Pi}_i[\mu, x(\cdot), (c_i, c_{-i})] \leq \Pi_i[p(c), x, c] = (R - \gamma_i)c_i,$$

  this implies that for all $c_i < \hat{c}_i \leq c_{-i}$, we have

$$\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] \leq \Pi_i[p(c), x, c],$$

  establishing that there are no profitable deviations with $\hat{c}_i \leq c_{-i}$.

This proves that any $c_1 + c_2 = 1$ together with (12) is an OE capacity.

**(Necessity)** Clearly, any $c_1 + c_2 < 1$ cannot be a pure strategy OE capacity, since the firm with $\gamma_i < R$ can increase profits by raising $c_i$. Similarly, any $c_1, c_2 \geq 1$ cannot be a pure strategy OE capacity, since the profits of both firms are equal to 0. Suppose, to obtain a contradiction, that there exists an OE capacity equilibrium with $c_1 + c_2 > 1$, $c_i > 0$ for $i = 1, 2$ and $c_i < 1$ for some $i$. Without loss of generality, we assume that $c_1 \geq c_2$. Then Proposition 6 implies that

$$\Pi_1[\mu, x(\cdot), c] = R(1 - c_2) - \gamma_1 c_1. \tag{16}$$

Consider the deviation to $\hat{c}_1 = 1 - c_2 < c_1$ by firm 1, which by Proposition 2 yields profits

$$\begin{aligned}\hat{\Pi}_1[\mu, x(\cdot), (\hat{c}_1, c_2)] &= R(1 - c_2) - \gamma_1 \hat{c}_1 \\ &> R(1 - c_2) - \gamma_1 c_1 \\ &= \Pi_1[\mu, x(\cdot), c],\end{aligned}$$

where the inequality exploits the fact that $\gamma_1 > 0$ and establishes that such an equilibrium cannot exist.

Next, to obtain a contradiction, suppose that there exists an equilibrium with $c_1 + c_2 = 1$, but (12) is violated. Without loss of generality, assume that $c_1 \leq c_2$, so that

$$c_1 < \frac{R - \gamma_1}{2R - \gamma_1}. \tag{17}$$

Now consider a deviation by firm 1 to $\hat{c}_1 = \hat{c}_1^{max}$ as given by (15). In view of (17), $\hat{c}_1^{max} > c_1$ and from Proposition 6, the deviation profits are given by

$$\begin{aligned}\hat{\Pi}_1[\mu, x(\cdot), (\hat{c}_1^{max}, c_2)] &= \frac{(R - (1 - c_1)\gamma_1)^2}{4R(1 - c_1)} \\ &> Rc_1 - \gamma_1 c_1 \\ &= \Pi_1[\mu, x(\cdot), c].\end{aligned}$$

To see that the inequality holds, we consider the function

$$f(c_1) = \frac{(R - (1 - c_1)\gamma_1)^2}{4R(1 - c_1)(Rc_1 - \gamma_1 c_1)},$$

for $c_1 \neq 0$ (for $c_1 = 0$, the inequality holds trivially). Note that the function $f(c_1)$ is strictly decreasing in $c_1$ for $c_1 \leq \frac{R - \gamma_1}{2R - \gamma_1}$. Therefore, for all $c_1 < \frac{R - \gamma_1}{2R - \gamma_1}$,

$$f(c_1) > f\left(\frac{R - \gamma_1}{2R - \gamma_1}\right) = 1.$$

This implies that

$$\frac{(R - (1 - c_1)\gamma_1)^2}{4R(1 - c_1)} > Rc_1 - \gamma_1 c_1.$$

The right hand side in the preceding relation is equal to $\Pi_1[\mu, x(\cdot), c]$ by (13). This establishes that there cannot be any equilibrium OE capacity with $c_1 + c_2 = 1$ that does not satisfy (12), completing the proof. **Q.E.D.**

Since we have $c_1 + c_2 = 1$ for all OE capacities, the relation in (12) can equivalently be written as

$$\frac{R - \gamma_1}{2R - \gamma_1} \leq c_1 \leq \frac{R}{2R - \gamma_2}, \tag{18}$$

and

$$c_2 = 1 - c_1. \tag{19}$$

Note that for all $0 < \gamma_i \leq R$, $i = 1, 2$, the capacity vector $c = (1/2, 1/2)$ satisfies equations (18) and (19). Thus, we immediately obtain the existence of a pure strategy Oligopoly equilibrium as a corollary:

**Theorem 1** The price-capacity competition game has a pure strategy Oligopoly Equilibrium.

## 6 Efficiency of Oligopoly Equilibria

In this section, we quantify the efficiency losses of Oligopoly Equilibria. We take the measure of efficiency to be the ratio of the social surplus of the equilibrium capacity $c^{OE}$ to the social surplus of the social capacity $c^S$, $\mathbb{S}(c^{OE})/\mathbb{S}(c^S)$ [cf. equation (5)]. We investigate the worst-case bound on this metric over all problem instances characterized by $\gamma_1$ and $\gamma_2$, either for the worst equilibrium among the set of oligopoly equilibria or for the best equilibrium among the set of equilibria.

Given capacity costs $\gamma_1$ and $\gamma_2$, let $C(\{\gamma_i\})$ denote the set of OE capacities. We define the efficiency metric at some $c^{OE} \in C(\{\gamma_i\})$ as

$$r(\{\gamma_i\}, c^{OE}) = \frac{\sum_{i=1}^{2} (R - \gamma_i) c_i^{OE}}{\sum_{i=1}^{2} (R - \gamma_i) c_i^{S}},$$

where $c^S$ is the social capacity given the capacity costs $\gamma_i$ and reservation utility $R$ [cf. (4)].

Following the literature on the efficiency losses of equilibria, we are interested in the performance of both the worst and the best OE capacity equilibria of price-capacity competition games. In particular, we first look for a lower bound on the worst performance in a capacity equilibrium,

$$\inf_{\{0<\gamma_i<R\}} \inf_{c^{OE} \in \ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}),$$

which is commonly referred to as *the Price of Anarchy* in the literature (see Koutsoupias and Papadimitriou (1999)). We then study the best performance in a capacity equilibrium given an arbitrary price-competition game, and thus provide a lower bound on

$$\inf_{\{0<\gamma_i<R\}} \sup_{c^{OE} \in \ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}), \tag{20}$$

which is commonly referred to as *the Price of Stability* in the literature (see Correa et al. (2002)).

**Example 1** Consider a price-capacity competition game with two firms, and $\gamma_1 = R - \epsilon$ for some $0 < \epsilon < \min\{1, R\}$, $\gamma_2 = R - \epsilon^2$. The unique social capacity is $(c_1^S, c_2^S) = (1, 0)$

with social surplus

$$\mathbb{S}(c^S) = \epsilon.$$

Using Proposition 7, it follows that the capacity vector

$$c^{OE} = (c_1^{OE}, c_2^{OE}) = \left( \frac{\epsilon}{R+\epsilon}, \frac{R}{R+\epsilon} \right),$$

is an OE capacity with social surplus

$$\mathbb{S}(c^{OE}) = \frac{\epsilon^2(1+R)}{R+\epsilon}.$$

Therefore, as $\epsilon \to 0$, the efficiency metric gives

$$\lim_{\epsilon \to 0} r(\{\gamma_i\}, c^{OE}) = \lim_{\epsilon \to 0} \frac{\epsilon(1+R)}{R+\epsilon} = 0.$$

Recall that when $\gamma_1 = \gamma_2$, $\mathbb{S}(c^{OE}) = \mathbb{S}(c^S)$. Instead in the preceding example we have that as $\gamma_1 \to \gamma_2$ (as $\epsilon \to 0$), the efficiency metric converges to 0.

The preceding example implies the following efficiency result:

**Theorem 2** Consider the price-competition game with two firms. Then

$$\inf_{\{0 \le \gamma_i \le R\}} \inf_{c^{OE} \in \ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}) = 0,$$

i.e., the Price of Anarchy of the price-capacity competition game is 0.

We next provide a non-zero lower bound on the Price of Stability of a price-capacity competition game.

**Theorem 3** Consider the price-competition game with two firms. Then, for all $0 \le \gamma_i \le R$, $i \in \{1, ..., N\}$, we have

$$\inf_{\{0 \le \gamma_i \le R\}} \sup_{c^{OE} \in \ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}) \ge 2\sqrt{2} - 2,$$

i.e., the Price of Stability of the price-capacity competition game is $2\sqrt{2} - 2$ and this bound is tight.

**Proof.** We assume without loss of generality that $\gamma_1 \le \gamma_2$. Then, the capacity vector $(c_1^S, c_2^S) = (1, 0)$ is a social capacity (unique social capacity if $\gamma_1 < \gamma_2$), with social surplus $\mathbb{S}(c^S) = R - \gamma_1$. Using the definition of the efficiency metric $r(\{\gamma_i\}, x^{OE})$, we consider the following optimization problem:

$$\sup_{c^{OE} \in \ C(\{\gamma_i\})} \frac{R - \gamma_1 c_1^{OE} - \gamma_2 c_2^{OE}}{R - \gamma_1}. \tag{21}$$

Since for all $c^{OE} \in C(\{\gamma_i\})$, we have $c_1^{OE} + c_2^{OE} = 1$, the supremum in the above expression is clearly attained at some $c^{OE} \in C(\{\gamma_i\})$ with the maximum value of $c_1^{OE}$. By Proposition 7 and equations (18)-(19), the maximum value of $c_1^{OE}$ at an OE capacity is given by

$$c_1^{OE} = \frac{R}{2R - \gamma_2}. \tag{22}$$

Substituting $c_1^{OE} = \frac{R}{2R-\gamma_2}$ and $c_2^{OE} = \frac{R-\gamma_2}{2R-\gamma_2}$ in the objective function in (21), we see that the optimal value is given by

$$\frac{R - \frac{R\gamma_1}{2R-\gamma_2} - \frac{(R-\gamma_2)\gamma_2}{2R-\gamma_2}}{R - \gamma_1}.$$

We are interested in finding a lower bound on the preceding over all $0 \leq \gamma_i \leq R$ with $\gamma_1 \leq \gamma_2$, i.e., we consider the following optimization problem:

$$\inf_{\substack{0 \leq \gamma_i \leq R \\ \gamma_1 \leq \gamma_2}} \frac{R - \frac{R\gamma_1}{2R-\gamma_2} - \frac{(R-\gamma_2)\gamma_2}{2R-\gamma_2}}{R - \gamma_1}.$$

This problem has a compact constraint set and a lower semicontinuous objective function [note that for $\gamma_1 = R$, the efficiency metric satisfies $r(\{\gamma_i\}, c^{OE}) = 1$]. Therefore it has an optimal solution $(\bar{\gamma}_1, \bar{\gamma}_2)$. For $\gamma_2 = R$, the objective function value is 1, showing that $\bar{\gamma}_2 < R$. For all $\gamma_2 \neq R$, the objective function is strictly increasing in $\gamma_1$, showing that $\bar{\gamma}_1 = 0$. It follows then that the unique stationary point given by $\bar{\gamma}_2 = \left(2 - \sqrt{2}\right)R$ attains the infimum, showing that the optimal solution of the preceding problem is given by $(\bar{\gamma}_1, \bar{\gamma}_2) = \left(0, (2 - \sqrt{2})R\right)$ with optimal value $2\sqrt{2} - 2$.

Finally, to see that the bound of $2\sqrt{2} - 2$ is tight, consider the best OE capacity of the game with $\gamma_1 = \delta > 0$ and $\gamma_2 = \left(2 - \sqrt{2}\right) R$. As $\delta \to 0$, the surplus in the best oligopoly equilibrium relative to social optimum limits to $2\sqrt{2} - 2$. **Q.E.D.**

## 7 Equilibria and Efficiency With $N$ Firms

We now generalize the results on the characterization and existence of pure strategy Oligopoly Equilibria (cf. Section 5) and the efficiency bounds (cf. Section 6) to $N$ firms. While all the results provided so far generalize, the argument is slightly different, and does not rely on explicitly characterizing the expected profits of the firms for all mixed strategy Price Equilibria.

### 7.1 Preliminaries

The next set of results generalize Propositions 2-5 of Section 4. Note that in our analysis of mixed strategy price equilibria, it is sufficient to focus on capacity subgames in which

$c_i > 0$ for all $i \in \{1, ..., N\}$ (since if $c_i = 0$, profits are equal to 0 for that firm).

**Proposition 8** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i \leq 1$ and $c_i > 0$ for $i \in \{1, ..., N\}$. Then there exists a unique Price Equilibrium in the capacity subgame $[p, x]$ such that $p_i = R$ and $x_i = c_i$ for $i \in \{1, ..., N\}$.

**Proof.** Since $\sum_{i=1}^{N} c_i \leq 1$, it follows by the equivalent characterization of a flow equilibrium that for all $p \in [0, R]^N$, the flow allocation $(c_1, c_2, ..., c_N) \in W(p, c)$. Therefore, by charging a price $p_i$, firm $i$ can make a profit of

$$\Pi_i[p_i, p_{-i}, x, c] = p_i c_i,$$

for all $p_{-i} \in [0, R]^{N-1}$. This shows that $p_i = R$ strictly dominates all other price strategies of firm $i$, so that $p_i = R$ and $x_i = c_i$, $i \in \{1, ..., N\}$, is the unique Price Equilibrium. **Q.E.D.**

**Proposition 9** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for all $i \in \{1, ..., N\}$ and assume that there exists some $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Then there exists no pure strategy Price Equilibrium in the capacity subgame.

**Proof.** Suppose there exists a pure strategy Price Equilibrium $(p, x)$. Without loss of generality suppose that $p_1 \leq p_h$, for all $h$. Let $P_1$ be the set of players whose price is equal to $p_1$, i.e., $P_1 = \{h \ : \ p_h = p_1\}$. The following list considers all candidates for a Price Equilibrium and provides profitable unilateral deviations from each, thus establishing the nonexistence of a pure strategy Price Equilibrium:

- $p_1 < \min_{h \neq 1} p_h$, i.e., $P_1 = \{1\}$: Then the profit of firm 1 is $\Pi_1[p, x, c] = p_1 \min\{c_1, 1\}$. A small increase in $p_1$ will increase firm 1's profits, thus firm 1 has an incentive to deviate.
- $p_1 = \min_{h \neq 1} p_h > 0$: Let $C_{P_1} = \sum_{i \in P_1} c_i$ be the sum of capacities of the firms that belong to set $P_1$. If $C_{P_1} \leq 1$, then we have to consider the following two cases:
  · $p_1 = \min_{h \neq 1} p_h = R$: Then, since by assumption $\sum_{i=1}^{N} c_i > 1$, there exists firm $s$, $s \notin P_1$, such that $p_s > R$ and firm $s$ is making zero profits, since its price is greater than the reservation utility $R$. Firm $s$ can change its price to $p_s = R - \epsilon$, for some $\epsilon$ with $0 < \epsilon < R$, and make positive profits.
  · $p_1 = \min_{h \neq 1} p_h < R$: Then firm 1 can increase slightly its price without affecting its flow allocation and thus increase its profits.

  If $C_{P_1} > 1$, we consider the following two cases:
  · $x_1 < \min\{c_1, 1\}$: Firm 1 can decrease its price slightly, and increase its flow and its profits.
  · $x_1 = \min\{c_1, 1\}$: Since $C_{P_1} > 1$, there exists firm $s \neq 1$, such that $s \in P_1$ and $x_s < \min\{c_s, 1\}$, which can decrease its price and increase its profits.
- $p_1 = \min_{h \neq 1} p_h = 0$. If $C_{P_1} \leq 1$, then firm 1 can increase its price and make positive profits. Let's consider next the case when $C_{P_1} > 1$. By assumption there exists some $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Note that $j \in P_1$, since otherwise $C_{P_1} \leq \sum_{i=1}^{N} c_i - c_j < 1$. Firm $j$ can increase its price and make positive profits.

**Q.E.D.**

Similar to Proposition 4, the next proposition establishes the existence of a mixed strategy Price Equilibrium in capacity subgames with no pure strategy price Equilibrium (proof follows from Proposition 4.3 in Acemoglu and Ozdaglar (2007a), and therefore is omitted).

**Proposition 10** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and suppose that there exists some $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Then there exists a mixed strategy Price Equilibrium in the capacity subgame.

**Proposition 11** Let $c$ be a capacity vector such that for each $j \in \{1, ..., N\}$, $\sum_{i=1}^{N} c_i - c_j \geq 1$ and $c_i > 0$ for $i \in \{1, ..., N\}$. Then, for all Price Equilibria $[p, x]$, we have $p_i = 0$ for $i \in \{1, ..., N\}$, i.e., all firms make zero profits.

**Proof.** The proof follows from a Bertrand price competition argument among the $N$ firms. **Q.E.D.**

In the remainder of this section, we consider a subgame defined by a capacity vector $c$, where $c$ is such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$, and there exists $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Proposition 9 implies that there does not exist a pure strategy Price Equilibrium in this subgame. However, Proposition 10 implies that a mixed strategy Price Equilibrium exists. Let $\mu_i$ denote the probability measure of prices used by firm $i$ in this equilibrium. We denote the (essential) support of $\mu_i$ by $[l_i, u_i]$ and the corresponding cumulative distributions by $F_i$. Next, we will provide a series of lemmas regarding the structure of the mixed strategy Price Equilibrium.

**Lemma 4** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and assume that there exists some $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Let $l$ denote the minimum of the lower supports of the mixed strategies, i.e., $l = \min_{i \in \{1,2,...,N\}} l_i$. Let $P_l$ denote the set of firms whose lower support is $l$, i.e., $P_l = \{i \in \{1, \dots, N\} \ : \ l_i = l\}$. Then:

(i) $\sum_{i \in P_l} c_i > 1$.

(ii) Let $P_{l,atom}$ denote all firms, such that $i \in P_l$ and distribution $F_i$ has an atom at $l$. Then,

$$\sum_{i \in P_{l,atom}} c_i + c_j \leq 1, \text{ for all } j \in P_l \text{ but } j \notin P_{l,atom}$$

Note that if there is no firm $j$ such that $j \in P_l$ but $j \notin P_{l,atom}$ we have

$$\sum_{i \in P_{l,atom}} c_i \leq 1$$

**Proof.**

(i) Suppose to obtain a contradiction that $\sum_{i \in P_l} c_i \leq 1$. Let $l' = \min_{i \notin P_l} l_i$. Then consider firm $j \in P_l$ deviating to $\hat{\mu}_j$ such that the new cumulative distribution $\hat{F}_j$ is given by:

$$\hat{F}_j(p) = \begin{cases} 0 & p < l' - \epsilon, \\ F_j(l' - \epsilon) & p = l' - \epsilon, \\ F_j(p) & p > l' - \epsilon, \end{cases}$$

where $F$ is the original cumulative distribution and $\epsilon > 0$ is sufficiently small, so that $l < l' - \epsilon$. Essentially all the mass between $l$ and $l' - \epsilon$ is shifted to $l' - \epsilon$. In such a deviation profile, the flow equilibrium remains unchanged since, $\sum_{i \in P_l} c_i \leq 1$, but the prices charged for positive flows by firm $j$ have increased, thus its profits increase, leading to a contradiction.

(ii) We first show that $j \in P_l$. Assume to arrive at a contradiction that $j \notin P_l$. Then, we have

$$\sum_{i \in P_l} c_i \leq \sum_{i \neq j} c_i < 1,$$

where the second inequality holds by the assumption that $\sum_{i=1}^{N} c_i - c_j < 1$. But this contradicts part (i), showing that $j \in P_l$.

We next show that $l > 0$. Assume to arrive at a contradiction that $l = 0$. This implies that the profits of firm $j$ at any price vector are equal to 0 [see the characterization of mixed strategy equilibria; cf. equations (9)-(10)]. However, by the assumption that $\sum_{i=1}^{N} c_i - c_j < 1$, there exists a price vector and a flow equilibrium at which the profits of firm $j$ are nonzero, thus showing that $l > 0$.

Note that the set $P_l$ cannot consist of only one firm, since then this firm has an incentive to increase its price. Suppose next that $P_l$ consists of two firms $m \neq n$. Assume to arrive at a contradiction that distribution $F_m$ has an atom at $l$. Consider the profits of firm $n$ at price $l - \epsilon$ for some sufficiently small $\epsilon > 0$,

$$\Pi_n[l - \epsilon, \mu_m, x(\cdot), c] = (l - \epsilon) \min\{c_n, 1\}.$$

Consider next the profits of firm $n$ at the price $l + \epsilon$ (we can assume without loss of generality that $l + \epsilon$ belongs to $\bar{P}_n$):

$$\Pi_n[l + \epsilon, \mu_m, x(\cdot), c] = (1 - F_m(l + \epsilon))(l + \epsilon) \min\{c_n, 1\} + F_m(l + \epsilon)(l + \epsilon)(1 - \min\{c_m, 1\}).$$

Since $F_m$ has an atom at $l$, i.e., $F_m(l+) > 0$ and $1 - \min\{c_m, 1\} < \min\{c_n, 1\}$, it follows from the preceding two relations that for sufficiently small $\epsilon$,

$$\Pi_n[l - \epsilon, \mu_m, x(\cdot), c] > \Pi_n[l + \epsilon, \mu_m, x(\cdot), c],$$

yielding a contradiction and showing that $F_m$ cannot have an atom at $l$. This shows the claim for two firms.

The proof for the case when $P_l$ contains more than two firms is now straightforward. In particular, note that if

$$\sum_{i \in P_{l,atom}} c_i + c_j > 1$$

for some $j \in P_l$ (but $j \notin P_{l,atom}$) we can show that firm $j$ has a profitable deviation to $l - \epsilon$. When there is no $j \in P_l$ but $j \notin P_{l,atom}$ we have to show that $\sum_{i \in P_{l,atom}} \leq 1$. If this is not the case, i.e. $\sum_{i \in P_{l,atom}} > 1$, then we can consider $j \in P_{l,atom}$ such that $\sum_{i \in P_{l,atom}, i \neq j} c_i + c_j > 1$, use the above arguments and obtain a contradiction.

**Q.E.D.**

**Lemma 5** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and assume that there exists $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Let $u$ denote the maximum of the upper supports of the mixed strategies, i.e., $u = \max_{i \in \{1,2,...,N\}} u_i$. Let $k$ be a firm with the maximum capacity, i.e., $c_k \geq c_i$, for all $i \in \{1, \ldots, N\}$. Then:

(i) At most one distribution $F_i$ can have an atom at the maximum upper support $u$.
(ii) The maximum upper support $u$ is equal to $R$.
(iii) If the distribution $F_i$ has an atom at $u$, then $c_i = c_k$.

**Proof.**

(i) We first show that at most one distribution can have an atom at $u$. We define the set

$$P_{atom} = \{i \in \{1, \ldots, N\} : F_i \text{ has an atom at } u\}.$$

Suppose to arrive at a contradiction that $P_{atom}$ has more than one element. It follows that with probability $\Pi_{i \in P_{atom}}(F_i(u+) - F_i(u-)) > 0$, all firms that belong to $P_{atom}$ will charge a price of $p = u$.

Let $C_{atom} = \sum_{i \in P_{atom}} c_i$ and $D_{res} = \max\{0, 1 - \sum_{i \notin P_{atom}} c_i\}$. We have

$$C_{atom} = \sum_{i=1}^{N} c_i - \sum_{i \notin P_{atom}} c_i > \max\left\{0, 1 - \sum_{i \notin P_{atom}} c_i\right\} = D_{res},$$

where the strict inequality follows by the assumption that $\sum_{i=1}^{N} c_i > 1$. This implies that there exists some $h \in P_{atom}$, such that $x_h(p^u) < \min\{c_h, 1\}$, where $p^u$ is the price vector for which all firms in $P_{atom}$ charge the price $u$. Then, firm $h$ can increase its profits by reducing its price to $u - \epsilon$ for some $\epsilon > 0$ (since firm $h$ is undercutting the rest of the firms in $P_{atom}$). This shows that there exists at most one distribution, which has an atom at $p = u$.

(ii) Suppose to arrive at a contradiction that $u < R$. Let $u_1 = u$, i.e., the upper support of the mixed strategy of firm 1 is equal to $u$. By part (i) at most one distribution can have an atom at $u$. If a firm has an atom at $u$, we assume without loss of generality that it is firm 1. Then, consider the following deviation by player 1 to $p_1 = u + \epsilon$ for some sufficiently small $\epsilon > 0$. Since no other distribution has an atom at $u$, for all flow equilibria $x(p) \in W[p, c]$, we can find an $\epsilon_1 > 0$ such that for every $\epsilon_2 > 0$, $u - \epsilon_1 \in \bar{P}_1$ and $x_1(u + \epsilon, p_{-1}) \geq x_1(u - \epsilon_1, p_{-1}) - \epsilon_2$ for all $p_{-1}$. Moreover, $x_1(u - \epsilon_1, p_{-1}) > 0$ for some $p_{-1}$ and all $\epsilon_1$. We can conclude that firm 1 has a profitable deviation yielding a contradiction. Note that we did not consider the profits of firm 1 at $u$, since $u$ may not necessarily belong to the support of 1's equilibrium profile.

(iii) Assume that the distribution $F_i$ has an atom at $u$. We will show that $c_i = c_k$.

Let $P_l$ denote the set of firms whose lower support is $l$, i.e., $P_l = \{i \in \{1, \ldots, N\} : l_i = l\}$. We first show that $i \in P_l$. Suppose that $i \notin P_l$. Then since $\sum_{h \in P_l} c_h > 1$ [cf. Lemma 4(i)], firm $i$'s profits when he charges the price $p = u$ are equal to 0. Using the assumption that $\sum_{i=1}^{N} c_i - c_k < 1$, and an argument similar to that in the proof of Lemma 4(ii), it can also be seen that $k \in P_l$.

Suppose to obtain a contradiction that for the only firm with atom at $u$, firm $i$, we have $c_i < c_k$. Let $\bar{\Pi}_i$ and $\bar{\Pi}_k$ denote the expected profits (plus the capacity costs) of firms $i$ and $k$ respectively at the mixed strategy Price Equilibrium (i.e., $\bar{\Pi}_i = \Pi_i + \gamma_i c_i$ and $\bar{\Pi}_k = \Pi_k + \gamma_k c_k$, where $\Pi_j$ denotes the equilibrium profits of firm $j$ in the mixed strategy Price Equilibirum). Using $i, k \in P_l$, and the fact that $F_i$, $F_k$ do not have an atom at the lower support $l$ [cf. Lemma 4(ii)], it can be seen that $\bar{\Pi}_i = c_i l$ and $\bar{\Pi}_k = c_k l$, which implies

$$\bar{\Pi}_i = \bar{\Pi}_k \frac{c_i}{c_k}. \tag{23}$$

Next note that since the upper support of $F_i$ is $u$ [which is equal to $R$ by part (ii)] and no other firm has an atom at $u$ [by part (i)], it is also the case that

$$\bar{\Pi}_i = R\left(1 - \sum_{j \neq i} c_j\right).$$

using equation (23), this implies

$$\bar{\Pi}_k = R\left(1 - \sum_{j \neq i} c_j\right)\frac{c_k}{c_i}.$$

Now consider a deviation for firm $k$ to charging a price $p = R$ with probability 1. The expected profits for firm $k$ following this deviation satisfy

$$\bar{\Pi}_k \geq R\left(1 - \sum_{j \neq k} c_j\right)$$

Since $c_k > c_i$ and $\sum_{j=1}^{N} c_j > 1$, we have that

$$\left(1 - \sum_{j \neq k} c_j\right) = \left(1 - \sum_{j=1}^{N} c_j + c_k\right) > \left(1 - \sum_{j=1}^{N} c_j + c_i\right)\frac{c_k}{c_i} = \left(1 - \sum_{j \neq i} c_j\right)\frac{c_k}{c_i}.$$

Therefore, the deviation for firm $k$ is profitable, yielding a contradiction and proving that $c_i = c_k$.

**Q.E.D.**

**Proposition 12** Let $c$ be a capacity vector such that $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and suppose that there exists $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Let $\bar{c} = \max_{i=1,\ldots,N} c_i$. Let $u$ denote the maximum of the upper supports of the mixed strategies, i.e., $u = \max_{i\in\{1,2,\ldots,N\}} u_i$. For firm $j$, the expected profits $\Pi_j[\mu, x(\cdot), c]$ are given by

$$\Pi_j[\mu, x(\cdot), c] = \begin{cases} R\left(1+\bar{c}-\sum_{i=1}^N c_i\right)\frac{c_j}{\bar{c}} - \gamma_j c_j, & \text{if } F_j \text{ has no atom at } u, \\ R\left(1+\bar{c}-\sum_{i=1}^N c_i\right) - \gamma_j c_j, & \text{if } F_j \text{ has an atom at } u. \end{cases}$$

**Proof.** Let $\bar{\Pi}_j = \Pi_j + \gamma_j c_j$ as in the proof of Lemma 5. If the distribution of firm $j$, $F_j$, has an atom at the maximum upper support, then Lemma 5 implies that $F_j$ is the only distribution having an atom at the maximum upper support $u = R$ and, moreover, $c_j = \bar{c}$. Firm $j$ is charging price $p = R$ with positive probability and

$$\begin{aligned}\Pi_j[R, \mu_{-j}, x(\cdot), c] &= R\left(1 - \sum_{i\neq j} c_i\right) \\ &= R\left(1 + \bar{c} - \sum_{i=1}^N c_i\right).\end{aligned}$$

Thus, $\bar{\Pi}_j = R\left(1+\bar{c}-\sum_{i=1}^N c_i\right)$ and the expected profits of firm $j$ are given by

$$\Pi_j[\mu, x(\cdot), c] = R\left(1+\bar{c}-\sum_{i=1}^N c_i\right) - \gamma_j c_j,$$

as claimed in the proposition.

Suppose next that $F_j$ does not have an atom at the maximum upper support. We consider the following two cases:

- None of the distributions has an atom at $u$. Then, let $k$ denote a firm with $c_k = \bar{c}$. We claim that $u_k = u = R$, i.e. the upper support of the equilibrium distribution for firm $k$ is the maximum upper support, $u = R$. If this is not the case, using an argument similar to that of Lemma 5[iii], we can show that firm $k$ has a profitable deviation to $p = R$. Using the Flow Equilibrium characterization given in Proposition 1, we have that the equilibrium profits for firm $k$ at price $p_k = R$ are given by

$$\bar{\Pi}_k = R\left(1 - \sum_{i\neq k} c_i\right).$$

  Then, as argued in Lemma 4, both firm $j$, which is such that $\sum_{i=1}^N c_i - c_j < 1$, and firm $k$ belong to set $P_l$ (recall that $P_l = \{i \ : l_i = l\}$, where $l$ is the minimum lower support). Therefore,

$$\bar{\Pi}_j = \bar{\Pi}_k \frac{c_j}{c_k} = R\left(1+\bar{c}-\sum_{i=1}^N c_i\right)\frac{c_j}{c_k}.$$

and

$$\Pi_j[\mu, x(\cdot), c] = R\left(1 + \bar{c} - \sum_{i=1}^{N} c_i\right)\frac{c_j}{\bar{c}} - \gamma_j c_j.$$

- The distribution $F_k$ has an atom at the maximum upper support, for some $k \neq j$. Then by the first part of the proof $c_k = \bar{c}$ and $\bar{\Pi}_k = R(1 + \bar{c} - \sum_{i=1}^{N} c_i)$. Moreover, $j, k \in P_l$, which implies that

$$\bar{\Pi}_j = \bar{\Pi}_k \frac{c_j}{c_k} = R\left(1 + \bar{c} - \sum_{i=1}^{N} c_i\right)\frac{c_j}{c_k}.$$

  We conclude that

$$\Pi_j[\mu, x(\cdot), c] = R\left(1 + \bar{c} - \sum_{i=1}^{N} c_i\right)\frac{c_j}{\bar{c}} - \gamma_j c_j.$$

**Q.E.D.**

### 7.2 Oligopoly Equilibria With $N$ Firms

In this section, we provide a characterization of Oligopoly Equilibria capacities with $N \geq 2$ firms. Similar to the analysis for two firms, we will use this characterization to establish the efficiency properties of Oligopoly Equilibria.

**Proposition 13** Assume that $\gamma_i < R$ for some $i$. Let $k$ be a firm with the maximum capacity, i.e., $c_k \geq c_i$ for all $k \in \{1, \ldots, N\}$. A capacity vector $c$ is an OE capacity if and only if $\sum_{i=1}^{N} c_i = 1$ and

$$\frac{R - \gamma_i}{2R - \gamma_i} \cdot (c_i + c_k) \leq c_i \leq c_k, \tag{24}$$

for all $i \neq k$.

**Proof. (Sufficiency)** We first show that $\sum_{i=1}^{N} c_i = 1$ together with (24) define an OE capacity. Note that since $\sum_{i=1}^{N} c_i \leq 1$, Proposition 8 implies that the profits of firm $i, i \in \{1, \ldots N\}$, are

$$\Pi_i[p(c), x, c] = (R - \gamma_i)c_i, \tag{25}$$

where $p(c)$ denotes the continuation equilibrium price vector, which in this case is $(R, \ldots, R)$.

Consider a deviation $\hat{c}_i \neq c_i$ by firm $i$. If $\hat{c}_i < c_i$, Proposition 8 still applies and the resulting profit for firm $i$ is $\hat{\Pi}_i[p(\hat{c}_i, c_{-i}), x, (\hat{c}_i, c_{-i})] = (R - \gamma_i)\hat{c}_i \leq \Pi_i[p(c), x, c]$, establishing that there are no profitable deviations with $\hat{c}_i < c_i$.

Next consider $\hat{c}_i > c_i$. Clearly, if $c_i = 0$, $\sum_{i=1}^{N} c_i = 1$ and $\hat{c}_i = 1$ (i.e., firm $i$ changed its capacity from 0 in the original vector to 1 in the new), Proposition 11 applies and $\hat{\Pi}_i[p(\hat{c}_i, c_{-i}), x, (\hat{c}_i, c_{-i})] = 0$ so that the deviation is not profitable. Therefore, we must have $\sum_{j \neq i} c_j + \hat{c}_i > 1$, $c_j > 0$ for $j \in \{1, \ldots, N\}$ and $\sum_{j \neq i} c_j < 1$. In this case, Proposition 10 applies and the deviation will induce a mixed strategy continuation equilibrium $\mu$. We consider the following two cases:

- Firm $i$ has the maximum capacity in the new subgame, i.e., $\hat{c}_i \geq c_j$, for all $j$. Then, Proposition 12 implies that the deviation profits of firm $i$ are

$$\begin{aligned}\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] &= R\left(1 - \sum_{j \neq i} c_j\right) - \gamma_i \hat{c}_i \\ &= (R - \gamma_i)\, c_i - \gamma_i\,(\hat{c}_i - c_i) \\ &\leq \Pi_i[p\,(c)\,, x, c].\end{aligned}$$

  Thus, in this case, there is no profitable deviation.

- Firm $i$ does not have the maximum capacity in the new subgame, i.e., there exists some $k$ such that $c_k$ is the maximum capacity and $c_k > \hat{c}_i$. Then by Proposition 12,

$$\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] = \frac{R\left(1 - \sum_{j \neq k} c_j\right)\hat{c}_i}{c_k} - \gamma_i \hat{c}_i. \tag{26}$$

  Let $\hat{c}_i^{max}$ denote the capacity that maximizes (26), given by

$$\hat{c}_i^{max} \equiv \frac{1}{2} - \frac{c_k \gamma_i}{2R} - \frac{\sum_{j \neq k,i} c_j}{2}. \tag{27}$$

  From equation (24) we have that $\frac{R-\gamma_i}{2R-\gamma_i} \cdot (c_i + c_k) \leq c_i$ , which implies that $\hat{c}_i^{max} \leq c_i$. Therefore, for all $c_i < \hat{c}_i \leq c_k$, we have

$$\hat{\Pi}_i[\mu, x(\cdot), (\hat{c}_i, c_{-i})] \leq \Pi_i[p\,(c)\,, x, c],$$

  establishing that there are no profitable deviations with $\hat{c}_i \leq c_k$.

This proves that any $\sum_{i=1}^{N} c_i = 1$ together with (24) is an OE capacity.

**(Necessity)** Any capacity vector $c$ such that $\sum_{i=1}^{N} c_i < 1$ cannot be a pure strategy OE capacity, since the firm with $\gamma_i < R$ can increase profits by raising $c_i$. Similarly, any capacity vector $c$ such that for all $j$, $\sum_{i=1}^{N} c_i - c_j \geq 1$ cannot be a pure strategy OE capacity, since the profits of all firms are equal to 0. Suppose, to obtain a contradiction, that there exists an OE capacity equilibrium with $\sum_{i=1}^{N} c_i > 1$, $c_i > 0$ for $i \in \{1, ..., N\}$ and suppose that there exists $j$ with $\sum_{i=1}^{N} c_i - c_j < 1$. Consider the profits of firm $k$ for which $c_k \geq c_j$ for all $j$. Then, we have

$$\Pi_k[\mu, x(\cdot), c] = R\left(1 - \sum_{i \neq k} c_i\right) - \gamma_k c_k. \tag{28}$$

Consider the deviation to $\hat{c}_k = 1 - \sum_{i \neq k} c_i < c_k$ by firm $k$, which yields profits

$$
\begin{aligned}
\hat{\Pi}_k[\mu, x(\cdot), (\hat{c}_k, c_{-k})] &= R\left(1 - \sum_{i \neq k} c_i\right) - \gamma_k \hat{c}_k \\
&> R\left(1 - \sum_{i \neq k} c_i\right) - \gamma_k c_k \\
&= \Pi_1[\mu, x(\cdot), c],
\end{aligned}
$$

establishing that such an equilibrium cannot exist.

Next, to obtain a contradiction, suppose that there exists an equilibrium with $\sum_{i=1}^{N} c_i = 1$, but (24) is violated. Without loss of generality, assume that firm 1 violates (24), i.e.,

$$
c_1 < \frac{R - \gamma_1}{2R - \gamma_1}(c_1 + c_k). \tag{29}
$$

Now consider a deviation by firm 1 to $\hat{c}_1 = \hat{c}_1^{max}$ as given by (27). In view of (29), $\hat{c}_1^{max} > c_1$ and from Proposition 12, the deviation profits are given by

$$
\begin{aligned}
\hat{\Pi}_1[\mu, x(\cdot), (\hat{c}_1^{max}, c_{-1})] &= \frac{\left(R\left(1 - \sum_{j \neq 1,k} c_j\right) - \left(1 - \sum_{j \neq 1,k} c_j - c_1\right)\gamma_1\right)^2}{4R(1 - \sum_{j \neq 1,k} c_j - c_1)} \\
&> Rc_1 - \gamma_1 c_1 \\
&= \Pi_1[\mu, x(\cdot), c].
\end{aligned}
$$

This establishes that there cannot be any equilibrium OE capacity with $\sum_{i=1}^{N} c_i = 1$ that does not satisfy (24), completing the proof. **Q.E.D.**

### 7.3 Efficiency of Oligopoly Equilibria With $N$ Firms

We next investigate the Price of Anarchy and Price of Stability for oligopoly equilibria with $N$ firms. The following example shows that the efficiency loss in the worst oligopoly equilibrium (Price of Anarchy) can again be arbitrarily high.

**Example 2** Consider a price-capacity competition game with $N$ firms, and $\gamma_1 = R - \epsilon$ for some $0 < \epsilon < \min\{1, R\}$, $\gamma_2 = ... = \gamma_N = R - \epsilon^2$. The unique social capacity is $(c_1^S, \ldots, c_N^S) = (1, 0, \ldots, 0)$ with social surplus

$$
\mathbb{S}(c^S) = \epsilon.
$$

Using Proposition 13, it follows that the capacity vector

$$
c^{OE} = (c_1^{OE}, c_2^{OE}, ..., c_N^{OE}) = \left(\frac{\epsilon}{R+\epsilon}, \frac{R}{(R+\epsilon)(N-1)}, ..., \frac{R}{(R+\epsilon)(N-1)}\right),
$$

is an OE capacity with social surplus

$$\mathbb{S}(c^{OE}) = \frac{\epsilon^2(1+R)}{R+\epsilon}.$$

Therefore, as $\epsilon \to 0$, the efficiency metric satisfies

$$\lim_{\epsilon\to 0} r(\{\gamma_i\}, c^{OE}) = \lim_{\epsilon\to 0} \frac{\epsilon(1+R)}{R+\epsilon} = 0.$$

The preceding example implies the following efficiency result:

**Theorem 4** Consider the price-competition game with $N$ firms, $N \geq 2$. Then

$$\inf_{\{0\leq\gamma_i\leq R\}} \inf_{c^{OE}\in\ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}) = 0,$$

i.e., the Price of Anarchy of the price-capacity competition game is 0.

Next we provide a non-zero lower bound on the Price of Stability of a price-capacity competition game.

**Theorem 5** Consider the price-competition game with $N$ firms, $N \geq 2$. Then, for all $0 \leq \gamma_i \leq R$, $i \in \{1, ..., N\}$, we have

$$\sup_{c^{OE}\in\ C(\{\gamma_i\})} r(\{\gamma_i\}, c^{OE}) \geq 2\frac{\sqrt{N}-1}{N-1},$$

i.e., the Price of Stability of the price-capacity competition game is $2(\sqrt{N}-1)/(N-1)$ and this bound is tight.

**Proof.** We assume without loss of generality that $\gamma_1 \leq \min_{i\in\{2,...,N\}} \gamma_i$ and that $\gamma_1 < R$ [if $\gamma_1 = R$, then by definition $\gamma_j = R$ for all $j$, so that the equilibrium and social surpluses coincide]. Then, the capacity vector $(c_1^S, c_2^S, ..., c_N^S) = (1, 0, ..., 0)$ is a social capacity, with social surplus $\mathbb{S}(c^S) = R - \gamma_1$. Using the definition of the efficiency metric $r(\{\gamma_i\}, x^{OE})$, we consider the following optimization problem:

$$\sup_{c^{OE}\in\ C(\{\gamma_i\})} \frac{R - \sum_{i=1}^N \gamma_i c_i^{OE}}{R-\gamma_1}. \tag{30}$$

Since for all $c^{OE} \in\ C(\{\gamma_i\})$, we have $\sum_{i=1}^N c_i^{OE} = 1$, the supremum in the above expression is clearly attained at some $c^{OE} \in C(\{\gamma_i\})$ with the maximum value of $c_1^{OE}$. By Proposition 13, the maximum value of $c_1^{OE}$ at an OE capacity is given by

$$c_1^{OE} = \frac{R}{R + \sum_{i=2}^N (R-\gamma_i)}. \tag{31}$$

Substituting $c_1^{OE} = R/(R + \sum_{i=2}^N (R - \gamma_i))$ and $c_i^{OE} = (R - \gamma_i)/(R + \sum_{i=2}^N (R - \gamma_i))$ for $i \in \{2, ..., N\}$ in the objective function in (30), we see that the optimal value is given by

$$\frac{R - \frac{R\gamma_1}{\left(R + \sum_{j=2}^N (R-\gamma_j)\right)} - \frac{\sum_{i=2}^N (R-\gamma_i)\gamma_i}{\left(R + \sum_{j=2}^N (R-\gamma_j)\right)}}{R - \gamma_1}.$$

We are interested in finding a lower bound on the preceding over all $0 \leq \gamma_i \leq R$ with $\gamma_1 \leq \min_{i \in \{2,...,N\}} \gamma_i$, i.e., we consider the following optimization problem:

$$\inf_{\substack{0 \leq \gamma_i \leq R \\ \gamma_1 \leq \gamma_i}} \frac{R - \frac{R\gamma_1}{\left(R + \sum_{j=2}^N (R-\gamma_j)\right)} - \frac{\sum_{i=2}^N (R-\gamma_i)\gamma_i}{\left(R + \sum_{j=2}^N (R-\gamma_j)\right)}}{R - \gamma_1}. \tag{32}$$

This problem has a compact constraint set and a lower semicontinuous objective function [note that for $\gamma_1 = R$, the efficiency metric satisfies $r(\{\gamma_i\}, c^{OE}) = 1$]. Therefore it has an optimal solution $(\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_N)$. For $\gamma_2 = ... = \gamma_N = R$, the objective function value is 1, showing that there should exist at least an i such that $\bar{\gamma}_i < R$. For all $(\gamma_2, ..., \gamma_N) \neq (R, ..., R)$, the objective function is strictly increasing in $\gamma_1$, showing that $\bar{\gamma}_1 = 0$. Moreover it is not hard to see that the optimal solution to (32) will satisfy $(\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_N) = (0, \bar{\gamma}, ..., \bar{\gamma})$, i.e., $\bar{\gamma}_2 = ... = \bar{\gamma}_N = \bar{\gamma}$. It follows then that the optimal solution is given by $(\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_N) = (0, \frac{N-\sqrt{N}}{N-1}R, ..., \frac{N-\sqrt{N}}{N-1}R)$ with optimal value $2\frac{\sqrt{N}-1}{N-1}$.

Finally, to see that this bound is tight, consider the best OE capacity of the game with $\gamma_1 = \delta > 0$ and

$$\gamma_2 = \cdots = \gamma_N = \frac{N - \sqrt{N}}{N-1} R.$$

In this case, as $\delta \to 0$ the ratio of the surplus in the equilibrium and the surplus in the social optimum is $2\frac{\sqrt{N}-1}{N-1}$. **Q.E.D.**

An interesting implication of this result is that as the number of players increases not only is the Price of Anarchy equal to zero, but the Price of Stability also goes to zero. Therefore, while coordination with a limited number of players can ensure that inefficiencies remain bounded when there are many competing firms even the best equilibrium has unbounded inefficiency. This result is interesting in part because it goes against a naive conjecture that increasing the number of oligopolistic competitors should increase efficiency (or even ensure that the equilibrium limits to a competitive allocation). The reason why this naive intuition does not apply in this case is that as the number of firms increases, investment incentives become potentially more distorted.

# 8 Stackelberg Leader Game

We have so far characterized the set of pure strategy equilibria in the baseline price-capacity competition game, where a set of competing firms choose capacity simultaneously first and then compete in prices (and users allocate their demands in the third stage). The analysis has shown that different equilibria within this set have widely differing efficiency features. In particular, the worst equilibrium from the set of pure strategy equilibria can have arbitrarily low efficiency, while if we select the best equilibrium from the set of equilibria, the worst efficiency performance will be $2(\sqrt{N}-1)/(N-1)$ (in particular, $2\sqrt{2}-2$ with two firms). This raises the question of how the equilibrium will be selected from the set of pure strategy equilibria and whether some type of regulation may be used to affect equilibrium selection.

While an analysis on equilibrium selection is beyond the scope of the current paper, there is a natural and simple multi-stage game that implements the best equilibrium. In this section, we discuss this multi-stage game, which involves the firms choosing their capacities sequentially, acting in reverse order of their capacity costs. In the special case with two firms, this is equivalent to the lower-cost firm acting as the Stackelberg leader and choosing its capacity first.

To simplify the exposition, in this section we suppose that $N$ =2 and again use $i$ and $-i$ to denote the two firms. In this case, the Stackelberg game works as follows: if $\gamma_i < \gamma_{-i}$, firm $i$ moves first and chooses $c_i$. Then firm $-i$, after observing $c_i$, chooses $c_{-i}$. After the capacity choices, the two firms simultaneously choose prices, and after capacities and prices are revealed, users allocate their demand. If $\gamma_i = \gamma_{-i}$, the two firms choose their capacities at the same time.

This game form may result as a focal point, giving the first-mover advantage to the low cost firm. Alternatively, if the low cost firm is an incumbent in the industry, we may think that this equilibrium will arise naturally, since the incumbent may have chosen its capacity before the new entrant. However, it is possible to imagine situations in which the lower cost firm is the entrant not the incumbent, in which case such a Stackelberg game will not arise naturally.

For the rest of this section, let us suppose that $\gamma_1 < \gamma_2$, and by a Stackelberg game, we refer to the multi-stage game where firm 1 chooses its capacity first, followed by firm 2, and then the two firms choose their prices simultaneously. A pure strategy Stackelberg equilibrium is defined as follows.

**Definition 5 [Stackelberg Equilibrium]** For a given $c_1 \geq 0$, let $BR_2(c_1)$ denote the set of best response capacities for firm 2, i.e.,

$$BR_2(c_1) = \arg \max_{\substack{c_2 \geq 0 \\ [\mu, x(\cdot)] \in MPE(c_1, c_2)}} \Pi_2[\mu, x(\cdot), c_1, c_2].$$

A vector $[c^{SE}, p(c^{SE}), x(c^{SE})]$ is a *(pure strategy) Stackelberg Equilibrium (SE)* if $[p(c^{SE}), x(c^{SE})] \in$

$PE(c^{SE})$, $c_2^{SE} \in BR_2\left(c_1^{SE}\right)$, and

$$\Pi_1[p(c^{SE}), x(c^{SE}), (c_1^{SE}, c_2^{SE})] \geq \Pi_1[\mu, x(\cdot), c_1, c_2], \tag{33}$$

for all $c_1 \geq 0$, $[\mu, x(\cdot)] \in MPE(c_1, c_2)$, and $c_2 \in BR_2\left(c_1\right)$.

**Proposition 14** Suppose that $\gamma_1 < \gamma_2 < R$. Then there exists a unique Stackelberg equilibrium in which

$$c_1^{SE} = 1 - \frac{R - \gamma_2}{2R - \gamma_2}$$
$$c_2^{SE} = \frac{R - \gamma_2}{2R - \gamma_2},$$

$p_1^{SE} = p_2^{SE} = R$ *and* $x_1^{SE} = c_1^{SE}, x_2^{SE} = c_2^{SE}$.

**Proof. (Existence)** It follows from the sufficiency part of proof of Proposition 7 that given $c_1^{SE}$, $c_2^{SE}$ is a best response for firm 2, i.e., $c_2^{SE} \in BR_2\left(c_1^{SE}\right)$. To see that there is no deviation for firm 1, first note that any $c_1 < c_1^{SE}$ gives lower profits. Next consider $c_1 > c_1^{SE}$. An argument identical to that in the proof of Proposition 7 shows that the best response of firm 2 to such $c_1$ will satisfy $c_1 + c_2 > 1$. Since

$$c_1 > 1 - \frac{R - \gamma_2}{2R - \gamma_2},$$

the analysis in the proof of Proposition 7 establishes that firm 1 will make lower profits.

**(Uniqueness)** From Proposition 7, this is the equilibrium with the highest level of $c_1$. Any other choice of $c_1$ can be improved upon by firm 1 deviating to $c_1^{SE}$. **Q.E.D.**

Denote the set of Stackelberg equilibria by $SE\left(\{\gamma_i\}\right)$. Combining this result with Theorem 2, we have the following result.

**Theorem 6** Consider the Stackelberg game described above with two firms. Then, for all $0 \leq \gamma_i \leq R$, $i = 1, 2$, we have

$$\inf_{c^{SE} \in \ SE(\{\gamma_i\})} r(\{\gamma_i\}, c^{SE}) = \sup_{c^{SE} \in \ SE(\{\gamma_i\})} r(\{\gamma_i\}, c^{SE}) = 2\sqrt{2} - 2,$$

i.e., both the Price of Anarchy and the Price of Stability of the Stackelberg game are $2\sqrt{2} - 2$ and this bound is tight.

## 9 Simultaneous Capacity-Price Selection Game

In this section, we consider the alternative one-stage competition between the two firms: firms simultaneously choose the capacity levels $c_i$ on their links and the price $p_i$ they will charge per unit bandwidth. Given the price and the capacity set by the other firm, $p_{-i},\ c_{-i}$, the profit of firm $i$ is

$$\Pi_i[(p_i, p_{-i}), x, (c_i, c_{-i})] = p_i x_i - \gamma_i c_i,$$

where $x \in W[p, c]$, i.e., $x$ is a flow equilibrium given the price vector $p$ and the capacity vector $c$. The objective of each firm is to maximize profits. We next define the *one-stage Oligopoly Equilibrium* for this competition model.

**Definition 6** A vector $[c^*, p^*, x^*]$ is a *(pure strategy) one-stage Oligopoly Equilibrium (OE)* if $x^* \in W[p^*, c^*]$ and for all $i \in \{1, ..., N\}$,

$$\Pi_i[(p_i^*, p_{-i}^*), x^*, (c_i^*, c_{-i}^*)] \geq \Pi_i[(p_i, p_{-i}^*), x, (c_i, c_{-i}^*)], \tag{34}$$

for all $p_i \geq 0,\ c_i \geq 0$, and for all $x \in W[(p_i, p_{-i}^*), (c_i, c_{-i}^*)]$.

**Proposition 15** Consider $N$ firms playing the one-stage game described above with $N \geq 2$. Given any $\gamma_i$, with $0 < \gamma_i < R,\ i \in \{1, ..., N\}$, there does not exist a one-stage Oligopoly Equilibrium.

**Proof.** Suppose, to obtain a contradiction, that there exists a one-stage Oligopoly Equilibrium $[c^*, p^*, x^*]$. We first show that in this equilibrium, we must have $\sum_{i=1}^N c_i^* = 1$ and $p_i^* = R$. If $\sum_{i=1}^N c_i^* < 1$, then since the flow allocation vector $x \in W[p, c]$ for all $p \in [0, R]^N$ and $\sum_{i=1}^N c_i^* < 1$, the profit of firm 1 is given by

$$\Pi_1[p^*, x^*, c^*] = (p_1 - \gamma_1)c_1^*. \tag{35}$$

Since $\gamma_1 < R$, by increasing $c_1^*$ slightly, firm 1 increases its profits, contradicting the claim that $[c^*, p^*, x^*]$ is a one-stage OE.

Consider next $\sum_{i=1}^N c_i^* > 1$. Then there exists $j \in \{2, 3, ..., N\}$ for which $x_j^* < c_j^*$. Clearly it is profitable for firm $j$ to deviate to $(c_j, p_j) = (x_j^*, p_j)$, since it reduces its capacity costs without affecting its price and flow allocation.

Hence, we must have $\sum_{i=1}^N c_i^* = 1$ and also $p_i^* = R$ by equation (35). If $c_1^* = 0$, then since $x_1^* = 0$, firm 1 can increase its capacity level and make positive profits. Assume next that $c_1^* = \epsilon$ for some $\epsilon > 0$. Then the profit of any firm $j \in \{2, ..., N\}$ is at most $(R - \gamma_j)(1 - \epsilon)$. But if firm $j$ changes its capacity and price to $(c_j, p_j) = (1, R - \delta)$ for some $\delta > 0$ and $\delta < (R - \gamma_j)\epsilon$, it will make a profit of $R - \delta - \gamma_j > (R - \gamma_j)(1 - \epsilon)$, showing that there does not exist a one-stage Oligopoly Equilibrium. **Q.E.D.**

# 10 Conclusions

In this paper, we studied the efficiency of oligopoly equilibria in a model where firms compete over capacities and prices. This problem is not only of theoretical interest, but it is relevant for understanding the extent of potential inefficiencies that may arise in the process of capacity extension in modern communication networks.

To isolate the main economic interactions, we considered the following simple game form. First, firms independently choose their capacity levels. Second, after the capacity levels are observed, they set prices. Finally, consumers allocate their demands across the firms. This game has an obvious similarity to Kreps and Scheinkman's model of quantity precommitment and price competition, Kreps and Scheinkman (1983), but it is simpler because demand is inelastic and because results do not have to rely on specific rationing rules.

Using similar ideas to the analysis in Kreps and Scheinkman (1983) and in Acemoglu and Ozdaglar (2007a), we characterized the entire set of pure strategy equilibria. A pure strategy oligopoly equilibrium always exists in this game but is supported by mixed strategies off-the-equilibrium path. The complete characterization of the equilibrium set enables us to investigate the worst-case efficiency properties of oligopoly equilibria.

Our first result here is that efficiency in the worst oligopoly equilibria (also referred to as the Price of Anarchy) of this game can be arbitrarily low. However, we also show that if the best oligopoly equilibrium is selected, the worst-case efficiency loss (also referred to as the Price of Stability) can be bounded. With two firms, this bound is tight and equal to $2\sqrt{2} - 2$. With an arbitrary number of firms, $N$, the bound is again tight and equal to $2(\sqrt{N} - 1)/(N - 1)$. Interestingly, this bound goes to zero as the number of firms, $N$, increases. This result contrasts with a naive intuition that the efficiency of oligopoly equilibrium should improve as the number of firms increases. The reason why this intuition does not apply in the current context is that with the greater number of competitors, ex ante investment incentives become potentially more distorted.

We also suggested a simple way of implementing the best oligopoly equilibrium, which involves the lower cost firms acting before higher cost firms as the "Stackelberg leaders" and choosing their capacities. With two firms, the Stackelberg game gives a unique equilibrium, with the efficiency loss bounded by $2\sqrt{2} - 2$.

Finally, we studied an alternative game form where capacities and prices are chosen simultaneously and showed that it always fails to have a pure strategy equilibrium. These results suggest that the timing of capacity and price choices in oligopolistic environments is important both for the existence of equilibrium and the extent of efficiency losses.

Many features of the model analyzed here were chosen to simplify the exposition. The analysis here can be easily generalized to arbitrary (convex) costs functions for investment in capacities, without changing the essence of the analysis or the results.

Another more important generalization is to include potential congestion costs, which are an important feature of many communication networks. Existence and efficiency of oligopoly equilibria with congestion costs (but without capacity investments) are analyzed in Acemoglu and Ozdaglar (2007a), and existence and efficiency of oligopoly equilibria with congestion costs and with capacity investments in the case with symmetric firms are studied in Weintraub et al. (2006). The problem is much more challenging when there are asymmetries, either in the costs of investing in capacity or in the extent of congestion costs within a subnetwork. We leave the analysis of this general model to future work.

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