Data Tracking under Competition

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We explore the welfare implications of *data-tracking* technologies that enable firms to collect consumer data and use it for price discrimination. The model we develop centers around two features: competition between firms and consumers' level of sophistication. Our baseline environment features a firm that can collect information about the consumers it transacts with in a duopoly market, which it can then use in a second, monopoly market. We characterize and compare the equilibrium outcomes in three settings: (i) an economy with *myopic* consumers, who, when making purchase decisions, do not internalize the fact that firms track their behavior and use this information in future transactions, (ii) an economy with *forward-looking* consumers, who take into account the implications of data tracking when determining their actions, and (iii) an economy where no data-tracking technologies are used due to technological or regulatory constraints. We find that the absence of data tracking may lead to a decrease in consumer surplus, even when consumers are myopic. Importantly, this result relies critically on *competition:* consumer surplus may be higher when data-tracking technologies are used only when multiple firms offer substitutable products.

Key words: competition; privacy; consumer data; game theory.

1. Introduction

As firms adopt and employ increasingly sophisticated information technology tools, economic transactions not only involve the exchange of goods and services, but also generate potentially valuable information about consumers' preferences. The availability of such data provides firms with the opportunity to personalize their interactions with their customers, for example, by offering personalized prices, promotions, or product recommendations.

Motivated by these observations, we study an economy with multi-product firms, which can leverage the information generated by early transactions to learn consumer-specific attributes and use them to personalize subsequent interactions. This dynamic arises in several settings of interest. For example, in the process of applying for a loan, customers disclose detailed information about their net worth, sources of income, and financial goals. Such data is often used by financial services firms to personalize subsequent product offerings (Kim and Wagman 2015). As another example, the automotive industry is increasingly adopting data-tracking technologies to better understand the driving behavior of its customers. Such information may be useful in several application domains ranging from predictive maintenance to insurance products tailored to individual drivers.[1] Finally, online retailers that operate in several product markets track and may use consumer information from one market to tailor their offerings in another.[2]

While the personalization enabled by data tracking may benefit consumers, e.g., by allowing firms to make better product recommendations, there are also potential downsides, e.g., using customer-specific data for price discrimination. Not surprisingly, this has sparked a debate among firms, consumer advocates, and regulatory authorities centered around consumers' *privacy* (Goldfarb and Tucker 2019). In fact, several firms offer "privacy," i.e., a commitment not to share or exploit the data they collect on consumers, as an appealing feature of their products.[3] Our primary goal in this paper is to better understand the implications of data-tracking technologies primarily for consumer welfare but also for other market-wide outcomes. In particular, we focus on two potentially first-order features affecting the interaction between firms and consumers in data-rich environments: (i) the structure of competition in the market, and (ii) the level of consumer sophistication, i.e., the extent to which consumers understand how information pertaining to their interactions with firms may be used in the future.

In order to study these questions, we consider a stylized model of an economy with two firms and two products. The first product market is a duopoly, whereas one of the firms (firm B) is a monopoly in the second product market. The interaction between firms and consumers takes place over two time periods. In the first period, firms compete in prices for product 1. After consumers make their purchase decisions for product 1, in the second period, firm B sets its price for product 2, which consumers choose to buy or not. We introduce *data tracking* in the economy by assuming that firm B collects information about the consumers it transacts with in the first product market that it can then use to offer personalized prices to those same consumers when selling the second product. On one hand, we consider *forward-looking* consumers who anticipate that buying the first product from firm B may lead to price discrimination in the second product market. Thus, consumers may have an incentive to avoid transacting with firm B in order to preserve their "privacy," whose value in the context of our model is not exogenously given but can be endogenously quantified based on the purchasing decisions consumers make and the prices they are quoted. Furthermore, as a way to better understand the relationship between consumers' understanding of how their data may be

[1] As an example of a firm moving in this direction, General Motors recently announced plans to sell car insurance and use individual driving data to set personalized prices (`https://www.wsj.com/articles/gm-wants-to-not-only-sell-cars-but-insure-them-too-11605708000`).

[2] When Amazon acquired Whole Foods in 2017, the Wall Street Journal reported that part of the acquisition's value would result from enabling Amazon to gather customer-specific information about grocery shopping habits (`https://www.wsj.com/articles/big-prize-in-amazon-whole-foods-deal-data-1497951004`).

[3] For example, Apple recently launched ads that emphasize the iPhone's privacy protection features (`https://www.theverge.com/2020/9/3/21420108/apple-new-over-sharing-ad-privacy-security-iphone`).

used and market-wide outcomes, we also consider a setting with consumers who act myopically; i.e., they do not internalize the future consequences of them revealing information to firm B when choosing to transact with it in the first product market.

Finally, motivated by the broader debate on privacy, we also consider whether consumers may benefit from privacy-protecting regulations, i.e., imposing constraints on firms in using the consumer data they obtain through tracking technologies. To this end, we study a *restricted* setting, where firm B has no access to tracking technologies and therefore its operations in the two product markets are decoupled as far as information on individual consumers is concerned.

Our results can be briefly summarized as follows. First, we establish that as long as the value of consumer data is not too high, consumers are worse off in an economy in which firms do not employ data tracking, due to technological or regulatory constraints, even if they act myopically and do not internalize how their information may be used in future transactions. The intuition for this result is that data tracking provides an incentive for firm B to subsidize its product in the first market as a way of increasing its share of transactions and obtaining information about a larger fraction of the consumer population. In turn, competition drives lower prices for both firms in the first product market relative to a setting without tracking. Finally, we show that the benefit of lower prices in the first market offsets the potentially adverse effects of price discrimination in the second one.

Competition is a key driver of this result. Indeed, we also consider an alternative economy in which firm B is a monopolist in both product markets. Then, data tracking always leads to a decrease in consumer surplus when consumers are myopic and, typically, i.e., for a wide range of parameters, it makes forward-looking consumers worse off as well. To some extent, this result contributes to the debate on whether regulating such technologies benefits consumers, by illustrating that their effect *depends not only on the value of consumer information or the consumers' level of sophistication, but also on how competitive the relevant market(s) are.*

Furthermore, we establish that firms with data-tracking ability have no incentive to self-regulate; i.e., in the context of our model, data tracking makes firm B better off even when consumers fully internalize the effect of their actions on firms' pricing strategies. This result is in stark contrast to recent literature, which suggests that firms may find it optimal to self-regulate their use of data tracking when consumers develop privacy concerns, i.e., when they are forward-looking in the context of our model (refer to Acquisti, Taylor, and Wagman 2016 for an extensive discussion). Moreover, we show that the gains resulting from data tracking and the ensuing competition between the firms are unevenly distributed among consumers. In particular, we establish that consumers at the lower end of the willingness-to-pay spectrum benefit the most.

Finally, we study a number of extensions that showcase that our main finding—that consumer surplus can be higher in the presence of data tracking even if consumers act myopically—is robust to some of the key assumptions of our baseline model, e.g., the competitive structure of the market, the accuracy of firm B's tracking ability, and the choice of the prior distribution of consumer types.

1.1. Related Literature

Our work is related to the literature on the economics of privacy and behavior-based price discrimination (refer to Acquisti et al. 2016, Fudenberg and Villas-Boas 2006, and Armstrong 2006 for excellent surveys). A series of papers in this stream study the implications of firms having the ability to set personalized prices based on consumers' past purchase information. Taylor (2004) considers an economy with two products that are sold by two monopolists, where one of the firms can sell the information it collects about consumers to the other firm. Villas-Boas (2004) studies an infinite-horizon model with a monopolist that may set different prices for new and old customers, respectively, while Acquisti and Varian (2005) analyze a model where a monopolist can offer personalized prices based on consumers' purchase histories. Conitzer, Taylor, and Wagman (2012) consider consumers that have access to "anonymizing technologies," while Bonatti and Cisternas (2020) study a setting where consumers face a sequence of monopolists that have access to a score aggregating purchase histories that can be used for price discrimination. Ichihashi (2020) studies a setting where consumers may disclose personal information to a firm that may use it both for product recommendations and for price discrimination. Finally, a series of related papers, e.g., Hart and Tirole (1988), Schmidt (1993), Devanur, Peres, and Sivan (2019), Immorlica, Lucier, Pountourakis, and Taggart (2017), consider models where a seller repeatedly interacts with buyers and can choose to use purchase histories to set prices.[4] All the aforementioned papers analyze settings that feature a monopolistic firm. By contrast, our model is designed to study the interplay between competition and the degree of consumer awareness of firms' data-tracking practices, which results in considerably different insights.

A series of related papers in this stream of literature have considered competition, albeit in quite different settings. Thisse and Vives (1988) analyze a model where firms know the types of all consumers and may choose to engage in price discrimination or price uniformly. By contrast, consumer types are private in our setting and firms may observe information about them only after transacting with consumers. Villas-Boas (1999), Fudenberg and Tirole (2000), and Esteves

[4] Another notable difference of our work from the literature on repeated sales is that the latter focuses on a single-product setting; i.e., the consumer has a fixed utility for a product and considers whether to buy a fresh copy in every period, whereas the seller's beliefs on the consumer's valuation evolve according to the history of prices and the consumer's actions. By contrast, our model features two different products that are sold by the same firm, which has a tracking mechanism that enables it to learn the consumer's valuation for the second product after the consumer has purchased the first one.

(2014) focus on symmetric settings where firms repeatedly compete over time and have the ability to price discriminate based on consumers' past purchase behavior. Our setting features asymmetric competition, where only one of the firms employs data tracking. As a result of these differences, our model allows us to study the role of data collection as a differentiator between firms, and produces different insights; notably, we find that the firm with access to data tracking is always better off using this technology to inform pricing, i.e., it has no incentive to self-regulate.[5]

Another related stream of work considers settings where, rather than having information readily available, firms may buy consumer information from a data broker to set personalized prices. Montes, Sand-Zantman, and Valletti (2019) analyze the implications of allowing consumers to have their information deleted from the dataset by paying a fee as well as the data broker's optimal selling strategy. Interestingly, the optimal strategy is to sell the data to a single firm, resulting in asymmetric competition, which is what we consider. By contrast, Bounie, Dubus, and Waelbroeck (2021) show that this strategy is suboptimal if the data broker can sell different subsets of data to each firm, while Clavorà Braulin (2023) finds that both exclusive and non-exclusive data allocations can be optimal, depending on the data available to the broker. A related line of work (e.g., Bimpikis, Crapis, and Tahbaz-Salehi 2019 and Drakopoulos and Makhdoumi 2022) studies the interaction between data brokers and firms that purchase information to estimate payoff-relevant parameters that need not determine pricing decisions. We refer the reader to Bergemann and Bonatti (2019) for a survey on the broader literature that studies markets for information. Our work differs from this branch of the literature in several aspects. In particular, the central feature of our modeling framework is that firms learn consumer information from previous transactions rather than by acquiring data from a third party, which results in different incentives for firms and consumers.

Moreover, recent work has focused on competition dynamics in the context of endogenous data acquisition and privacy. In particular, Kim and Wagman (2015) study competition between financial services providers that may sell data collected from loan applications. They provide empirical evidence to illustrate that privacy-oriented regulation may result in under-collection of information and subsequent efficiency losses. Closer to our work, Ali, Lewis, and Vasserman (2022) study a setting where consumers have the option to voluntarily disclose their data to firms, which in turn use it to price discriminate. They find that whether voluntary disclosure improves welfare depends on the flexibility of the consumers' ability to communicate, i.e., how much information consumers can share with firms, and market competitiveness. While related, our setting differs from theirs in a number of ways, primarily because we consider a firm that may infer consumer information solely based on their actions, e.g., prior transactions, rather than by direct disclosure from consumers.

[5] Prior work (including the aforementioned papers) argued that firms may have the incentive to self-regulate their use of consumer data, which does not necessarily conform with real-world practice.

A key feature of our model is that one firm may initially acquire consumer data that it later exploits, which creates an incentive to subsidize the transactions of one product to increase the profits associated with another. Similar tradeoffs are present in other settings, although they arise from different mechanisms. For instance, in the presence of switching costs or network effects, firms may compete aggressively for early purchases in exchange for future gains (e.g., Farrell and Klemperer 2007), and in the context of two-sided markets, platforms may choose to subsidize one side of the market to create additional value for the other side (e.g., Rochet and Tirole 2006).

Another strand of the literature has focused on the relationship between privacy protection policies and the equilibrium outcomes they induce in different settings, such as targeted advertising (Cummings, Ligett, Pai, and Roth 2016b, Anderson, Baik, and Larson 2022), service systems (Hu, Momot, and Wang 2022), and where data collection may determine firms' value proposition or enhance its operations (e.g., Casadesus-Masanell and Hervas-Drane 2015, Fainmesser, Galeotti, and Momot 2022). Our results contribute to this line of work as we establish that limiting firms' data-tracking abilities may lead to a decrease in consumer surplus, even when consumers act myopically, i.e., they do not internalize how firms may use the data they generate. In addition, concurrent work by Argenziano and Bonatti (2021) evaluates alternative privacy-oriented policies in a setting where a consumer sequentially transacts with two firms that may agree to share data. Although our high-level motivation is similar, our work differs in several ways from theirs. Mainly, they analyze firms' incentives to establish data linkages in a setting without competition, while we study asymmetric competition between a firm that collects consumer data and one that does not.

Furthermore, there is a large literature that focuses on decision making in the presence of privacy concerns. In particular, the notion of *differential privacy* has been studied in a variety of settings (see Dwork and Roth 2014 for an introduction to the topic), focusing on how to perturb datasets to ensure that decision makers can use them without being able to identify information about specific individuals. Recent contributions in this area develop algorithms that satisfy privacy constraints for personalized pricing (Lei, Miao, and Momot 2023) and network analytics (Hastings, Falk, and Tsoukalas 2022). Also related, Cummings, Echenique, and Wierman (2016a) study choice theory with consumers that have intrinsic preferences for privacy, whereas Acemoglu, Makhdoumi, Malekian, and Ozdaglar (2017) explore a network formation game where agents face a trade-off between the benefits of adding friends to their social network and the associated privacy loss resulting from sharing personal informtion. By and large, in this literature, the value for privacy is an exogenous modeling primitive that is taken as fixed and given. By contrast, a key driving force behind our results is that consumers develop privacy concerns endogenously, i.e., due to the potential of their data being used for personalized pricing.

A series of recent papers, such as Acemoglu, Makhdoumi, Malekian, and Ozdaglar (2022), Bergemann, Bonatti, and Gan (2022), Liang and Madsen (2020), and Ichihashi (2021), study externalities in a context where data has a social dimension, i.e., data pertaining to an individual is informative about her peers. While we do not explicitly consider such correlations, externalities arise in our model since the actions of some consumers (rather than the data itself) may provide information about others.[6]

Lastly, the literature has studied widely the importance of taking into account the degree of consumer sophistication in a variety of pricing and revenue management settings, including the design of markdown policies (Gallego, Phillips, and Şahin 2008, Cachon and Swinney 2009), optimal pricing strategies (Su 2010, Besbes and Lobel 2015, Cui, Duenyas, and Şahin 2018, Chen and Hu 2020), and the timing of new product launches (Lobel, Patel, Vulcano, and Zhang 2016). Perhaps surprisingly, we find that in the presence of competition, data tracking may increase consumer surplus *irrespective* of the degree of consumer sophistication.[7]

The rest of the paper is organized as follows. We introduce our baseline model and discuss the implications of the main assumptions in Section 2. The characterization of equilibria in the three settings we consider follows in Section 3. Then, we discuss the welfare implications of equilibrium behavior in Section 4. We then summarize several extensions that relax some assumptions of the baseline model and discuss the robustness of our main findings in Section 5. Finally, we conclude and discuss directions for further research in Section 6. The proofs of our results and the details of the extensions we considered are presented in the Online Appendix.

2. Model

Our model economy consists of a consumer, two firms, A and B, and two products, 1 and 2. Both firms sell differentiated variants of product 1, while only firm B sells product 2.[8] The interaction between the consumer and the firms proceeds as follows. First, the consumer decides from which firm to buy product 1 based on the prices chosen by the firms and her *type*, which captures both her preference for firm A's product 1 relative to firm B's (modeled as the consumer's location on a Hotelling line) and her valuation for product 2. Then, firm B sets a price for product 2, which may depend on the consumer's purchase decision for product 1, and the consumer chooses whether to buy the product. Figure 1 illustrates the timing of events in the model economy we consider.

[6] Recently, Aridor, Che, and Salz (2022) provide empirical evidence to support such a dynamic in the context of the European Union's GDPR. They show that as some consumers opted out of data collection, it became easier for firms to track and interpret the data associated with consumers that did not opt out.

[7] We refer the reader to Lobel (2021) for an overview of the recent work on revenue management as it relates to the algorithmic economy.

[8] To ease exposition, we assume that firms face no production costs. Our results extend to the case where both firms have equal and constant marginal costs for product 1.

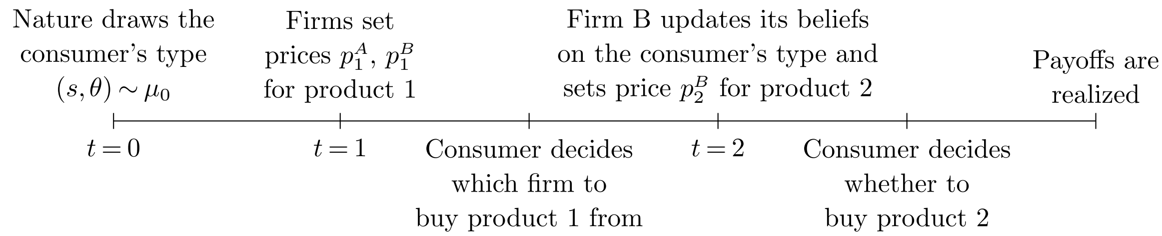

Figure 1 Timeline of the game

A key feature in our framework is that firm B has the ability to observe the type of the consumer after she buys product 1 from it. In addition, firm B can use this information to set its (personalized) price for product 2. We refer to this as *data tracking*. We consider two different settings in this environment: one in which the consumer is *forward-looking*, i.e., she internalizes the implications of her actions (purchase decisions) on firms' future pricing, and another one where the consumer is *myopic*, i.e., she makes purchase decisions solely to maximize her per-period utilities. Moreover, as a benchmark, we also consider a *restricted* setting, where data tracking is not possible due to technological or regulatory constraints.

In what follows, we describe our *baseline framework* and the sequence of events in more detail in the presence of data tracking and forward-looking consumers, and briefly describe the additional settings we consider. We discuss several extensions to the baseline model in Section 5.

Consumer. The consumer's type is two-dimensional and consists of (i) her location in the Hotelling line for product 1, which we denote by $s \in [0,1]$ with the endpoints of the interval representing the locations of firms A and B, respectively, and (ii) her valuation for product 2, which we denote by $m\theta$ with $\theta \in [0,1]$, where $m > 0$ is a parameter meant to capture the *value of consumer data* to firms as will become evident in what follows.[9] Thus, the consumer can be represented by her *type* $\tau = (s,\theta) \in \mathcal{T} = [0,1] \times [0,1]$, which we assume is ex ante known to her but not to the firms, whereas the value of m is commonly known.[10] Throughout the paper, we assume that the consumer's type (s,θ) is drawn from the uniform distribution in the unit square, which, in turn, implies that s and θ are independent. Moreover, we assume that the prior distribution over types is common knowledge.

First, the consumer interacts with the firms in product market 1. In particular, she observes their prices, p_1^A and p_1^B, and decides whether to buy product 1 from either firm. When buying product 1, the consumer derives a baseline utility of $\bar{u} > 0$ and faces linear transporation costs, where the unit

[9] Intuitively, as we assume that θ is uniformly distributed in $[0,1]$, the prior distribution of the consumer's valuation for product 2 is uniform in $[0,m]$. Therefore, the additional expected profit for firm B associated with data tracking (i.e., due to observing the consumer's type before setting the price of product 2) is larger as m increases.

[10] Although we develop and present the model with a single consumer for simplicity, our results can be interpreted similarly for a continuum of consumers.

transportation cost is normalized to 1/2. To simplify exposition, we assume that the consumer always buys product 1 from one of the firms; i.e., we explicitly restrict the consumer's actions in period 1 to be whether she buys product 1 from firm A or B.[11] We denote the consumer's action in the first period by $a_1 \in \{0,1\}$, where $a_1 = 1$ denotes buying from firm B. Thus, the consumer's utility associated with product 1 given her action, her type, and the prices is equal to

$$u_1\left(a_1; s, \theta, p_1^A, p_1^B\right) = (1-a_1)\left(\bar{u} - s/2 - p_1^A\right) + a_1\left(\bar{u} - (1-s)/2 - p_1^B\right). \tag{2.1}$$

Then, in period 2, the consumer observes firm B's price p_2^B and decides whether to buy product 2 ($a_2 = 1$) or not ($a_2 = 0$). Thus, the consumer's utility associated with product 2 can be written succinctly as

$$u_2\left(a_2; s, \theta, p_2^B\right) = a_2\left(m\theta - p_2^B\right), \tag{2.2}$$

where recall that $m\theta$ is the valuation for product 2 for a consumer with type (s,θ). Finally, the consumer's aggregate utility is then the sum of the utilities she derives from each product.

Firms. At the beginning of period 1, both firms set prices p_1^A, $p_1^B \in \mathbb{R}$ simultaneously. When making these pricing decisions, firms have no information about the consumer's type other than its prior distribution. Given product 1 prices, the consumer's action a_1, and assuming that production costs are normalized to zero, the profits associated with product 1 for firms A and B are, respectively,

$$\pi_1^A\left(p_1^A, p_1^B, a_1\right) = (1-a_1)\,p_1^A \quad \text{and} \quad \pi_1^B\left(p_1^A, p_1^B, a_1\right) = a_1 p_1^B. \tag{2.3}$$

Firm A participates only in product market 1, and so its total payoff is π_1^A. At the beginning of period 2, firm B sets a price p_2^B for product 2. When firm B makes this decision, its available information depends on the events of the first period as we describe below. Then, given the consumer's action a_2, firm B's profit associated with product 2 is

$$\pi_2^B\left(p_2^B, a_2\right) = a_2 p_2^B. \tag{2.4}$$

Much of our analysis centers on the implications of *data tracking*, i.e., the ability of firms to collect/infer consumer information in a transaction and use it in subsequent transactions (e.g., to set personalized prices). We incorporate data tracking into the model by making the information available to firm B when setting the price of product 2 dependent on the consumer's purchase decision for product 1. Concretely, we assume that if the consumer bought product 1 from firm B (i.e., chose $a_1 = 1$), the firm observes the consumer's type before setting the price of product 2. On

[11] The characterization of equilibria in our model extends to the setting where consumers are allowed to forgo buying product 1 (for a payoff of zero) if we assume, as is common in the literature, that $\bar{u}$ in (2.1) is large enough so that all consumers buy product 1 in equilibrium. In particular, it can be shown that $\bar{u} \geq 4/5$ suffices to achieve this.

the other hand, if the consumer did not buy product 1 from firm B (i.e., chose $a_1 = 0$), the firm does not observe the consumer's type and recalls the history of product 1 prices only.[12] We provide a formal description of the collection of histories of the game and the information structures they induce in Appendix A. Finally, note that as firm B participates in both product markets, its total payoff is $\pi_1^B + \pi_2^B$.

Strategies, beliefs and equilibrium. Next, we provide a brief description of the notions of strategies and beliefs for the consumer and the firms and we also discuss the equilibrium concept we adopt. We relegate the formal definitions to Appendix A.

A *strategy for the consumer* consists of a pair of functions $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$, where γ_i maps the information available to the consumer before making her purchase decision for product $i \in \{1,2\}$ to a probability distribution over the available actions $a_i \in \{0,1\}$.

A *pricing strategy for firm A* is a function of the information available at the beginning of period 1 to a non-negative real number that represents the firm's price for product 1. Similarly, a *pricing strategy for firm B* consists of a pair of functions $\boldsymbol{\sigma}^B = (\sigma_1^B, \sigma_2^B)$, where σ_i^B maps the information available to firm B when setting the price of product $i \in \{1,2\}$ to a real number representing the price it sets for the product. For simplicity, we focus on the case where both firms set prices according to pure strategies. Moreover, we allow firm B to set negative prices for product 1, which we interpret as pricing below cost. Restricting firm A to non-negative prices is without loss of generality as it interacts only once with the consumer.

A *belief system* consists of a pair of functions $\boldsymbol{\mu} = (\mu_1, \mu_2)$ that map firms' available information in each period to probability distributions over consumer types. In particular, μ_1 represents the firms' common beliefs on the consumer's type when choosing their product 1 prices. Note that μ_1 is defined as the prior distribution of consumer types, since this is the only information available to both firms at the beginning of the first period. For the second period, μ_2 maps firm B's available information when setting the price of product 2 to a distribution over the types (s, θ).

We adopt Perfect Bayesian Equilibrium (PBE) as our equilibrium notion. In what follows, we briefly describe this equilibrium concept in the context of our model, while we provide its formal definition in Appendix A. A PBE consists of a strategy profile $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$ for all agents (the firms and the consumer), and a belief system $\boldsymbol{\mu}$ such that the strategy profile is sequentially rational for all agents and the belief system $\boldsymbol{\mu}$ is consistent with the strategy profile. In more detail, sequential rationality for the consumer requires that in period $i = 1, 2$, and given the firms' pricing strategies, the consumer maximize her continuation utility by making a purchase decision according to γ_i.

[12] In our baseline model firm B can infer that if the consumer's type is not observed, it must be that the consumer did not buy product 1 from it (i.e., chose $a_1 = 0$). However, in some of the extensions we consider, the firm may not be able to infer the consumer's product 1 purchase decision when her type is not observed.

Similarly, sequential rationality for firms requires that at any point in time, each firm's pricing choice maximize its expected continuation profit given the rest of the agents' strategies and the belief system $\boldsymbol{\mu}$. Finally, the belief system $\boldsymbol{\mu}$ is said to be consistent with the strategy profile $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$ if, when firm B observes the consumer's type, μ_2 assigns probability 1 to the observed type whereas when firm B does not observe the consumer's type, μ_2 is determined by Bayes' rule given μ_1 and the strategy profile at histories that are reached with positive probability.

Additional settings. The framework described above represents the setting where firm B employs data tracking and the consumer is forward-looking; i.e., she takes into account the firm's data tracking ability when making her purchase decisions. This is captured by requiring sequential rationality from the consumer, as she makes her purchasing decision in period 1 to maximize her total expected utility over *both* time periods. To best illustrate the implications of data tracking, we also consider the following two settings:

Myopic. Firm B employs data tracking but the consumer acts myopically. In particular, when deciding from which firm to buy product 1, she does not internalize the implications of her action on firm B's pricing for product 2. In other words, a *myopic* consumer determines her action a_1 to maximize her utility in period 1 (given by Expression (2.1)), whereas a *forward-looking* consumer determines a_1 to maximize her aggregate utility over both periods. Thus, in this setting, the consumer's sequential rationality condition in the equilibrium definition is modified to require that the consumer's period 1 strategy, γ_1, maximize her expected utility associated with product 1 only.

Restricted. In the restricted setting, firm B does not have any data-tracking capabilities, which in turn implies that the two markets in which firm B operates are entirely decoupled. In other words, the only information available to firm B when setting the price of product 2 is the history of product 1 prices (p_1^A, p_1^B), irrespective of the consumer's action in period 1.

Model discussion. As a preface to the discussion below, we note that our objective is to illustrate the implications of data tracking, the endogenous privacy concerns it may generate, and how competition affects the associated equilibrium outcomes. To this end, we develop a two-period model that captures the basic elements of the setting of our interest. We now discuss the main assumptions of the baseline model and their implications. We mention a number of extensions that relax some of these assumptions, which we discuss in more detail in Section 5.

Data tracking and personalization. In the model, if the consumer buys product 1 from firm B, she reveals her type and firm B can use this information to offer a personalized price for product 2. With this modeling choice, we are implicitly assuming that (i) firm B has access to data-tracking and personalization technologies, and (ii) it is able to infer the consumer's type after transacting with her.

Assumption (i) is reasonable in the context of e-commerce applications. Tracking technologies are widespread in practice: customers have accounts for online stores that keep track of various aspects of their behavior, such as purchases and browsing history. In addition, personalization is increasingly used and can take the form of personalized pricing, targeted coupons or promotions, or tailored product assortments.[13] In line with such practices, our model entails leveraging information about customers for personalization.

We make assumption (ii) to capture the fact that firms can learn information about their customers from previous interactions beyond just observing that the customer chose to buy some products at certain prices. We incorporate this feature into our model by positing that firms infer customers' types in the event of a transaction. This assumption may well capture certain contexts, e.g., when the population of customers can be clustered in a small number of types that can be easily identified by historical data (e.g., Moon, Bimpikis, and Mendelson 2018). In addition, it provides the intuition we aim to capture in a clean and transparent way, i.e., that firms may learn substantial information about their consumers based on data from previous transactions. In Section 5.2, we discuss how our main findings apply when we relax this assumption by considering a setting where data tracking is imperfect.

Endogenous privacy concerns. We study a setting where the consumer does not have an intrinsic value for privacy, but would like to preserve it only to the extent that doing so prevents a future economic loss. Therefore, in the context of our model "privacy concerns" arise solely due to the possibility that revealing one's data may be associated with economic consequences and not due to the consumer deriving a direct utility from maintaining her privacy.[14]

Timeline of consumer purchases. We assume that the order in which the consumer makes her purchase decisions is fixed, i.e., first for product 1 and then for product 2. Indeed, one could formulate a model with a more flexible timeline, where the order of purchases may be stochastic, or where the consumer may consider buying only one of the products, or even where she strategically chooses the sequence of her transactions. In addition to aiming for simplicity, we avoid a more flexible timeline for three main reasons. First, our model setup provides firm B with a clear path to obtain and, then, potentially exploit consumer data. Also, we are interested in isolating the effect of data tracking in the model and introducing a less stringent timeline would complicate this considerably. Finally, several papers that study related questions assume a similar timeline and maintaining it allows us to contrast our results with theirs (e.g., Taylor 2004, Acquisti and Varian 2005, and Conitzer et al. 2012).

[13] For a detailed discussion, refer to Acquisti et al. (2016) and Dubé and Misra (2023).

[14] We refer the reader to Lin (2022), who differentiates between consumers' intrinsic versus instrumental motives for protecting their privacy. In their terminology, our setting involves instrumental motives.

Competitive structure. We assume that both firms compete in the market for product 1, but firm B is a monopolist in the market for product 2. In so doing, we look to capture two features. First, we want to highlight that learning consumer information is valuable for firm B. Giving firm B monopoly power in the second market and the possibility of increasing her profits by leveraging its knowledge about the consumer achieves this. Second, we want to understand how the consumer's choice depends on her option to buy product 1 while preserving her privacy, which is the role that firm A plays in the first period. A natural extension is to incorporate competition into the second market in order to reduce the value of information for firm B; we discuss this setting in Section 5.1.

Consumer types distribution. We assume that the components of the consumer's type associated with the two products, s and θ, are independent and drawn from the uniform distribution. We make this assumption both for tractability, and to clearly isolate the effects of data tracking on market outcomes from other effects (e.g., possibly introduced by correlation between s and θ). This version of our model best captures settings where firms learn not only from the consumer's choices, but also from data that is generated through the transaction itself. Some of the motivating examples we provide in the introduction represent situations where such a dynamic is present, e.g., the ability to personalize car insurance prices based on driving history results from data that is generated by driving the car, rather than by inferences made on the consumer's purchase decision of a particular car model. Similarly, loan application data that enable banks to offer tailored financial products is generated by the application process itself rather than by the consumer's choice of a bank.

In Section 5.3 we consider two different extensions that relax this assumption, by introducing correlation between s and θ, and by considering more general type distributions.

3. Equilibrium Outcomes of the Baseline Model

As a first step in our analysis, we provide a characterization of the equilibrium outcomes in the model economy of Section 2. First, we describe the equilibrium in the environment where firm B employs data tracking and consumers are forward-looking (Theorem 1). Then, we also describe the equilbrium outcomes associated with (i) the setting with no data tracking, and (ii) with consumers who are myopic in their decision making (in Propositions 1 and 2, respectively). Armed with this characterization, we proceed in Section 4 to discuss the implications of adopting data tracking both for the firms and for consumers. We provide the proof of Theorem 1 in Online Appendix B, while the proofs of Propositions 1 and 2 can be found in Online Appendix E.

Equilibrium characterization. In the presence of data tracking, a consumer anticipates that buying product 1 from firm B (the firm that operates in both markets) would result in zero net utility in period 2 as firm B would use the information generated in the transaction to price product 2 at the consumer's valuation. Thus, all else equal, the consumer views firm A as a more attractive option than B given that firm A allows the consumer to maintain her "privacy," i.e., not to disclose her type to firm B. In turn, the firms determine their prices for product 1 accordingly. Theorem 1 below formalizes this intuition, states conditions that ensure existence, and provides a characterization of the equilibrium outcome.

THEOREM 1. *There exist constants m_L, m_H with $0 < m_L < m_H$ such that if $m < m_L$ or $m > m_H$, there exists an equilibrium in the forward-looking setting. Moreover, the equilibrium path can be characterized as follows:*

(a) If $0 < m < m_L$, there exist unique prices $p_1^{A*} = p_1^{A*}(m) > 0$, $p_1^{B*} = p_1^{B*}(m)$, with $p_1^{A*} > p_1^{B*} > p_1^{A*} - 1/2$ such that, in any equilibrium, the firms' prices for product 1 are equal to p_1^{A*} and p_1^{B*}, respectively. In addition, the expected product 1 demand for both firms is positive. Moreover, there exists a constant $\bar{\theta}^* = \bar{\theta}^*(m) \in (1/2, 1)$, and a function $g^* : [0,1] \to \mathbb{R}$ defined by

$$g^*(t) = p_1^{B*} - p_1^{A*} + 1/2 + m\left(t - \bar{\theta}^*\right)^+,$$

such that

(i) If the consumer's type (s, θ) is such that $s > g^*(\theta)$, then, with probability one, the consumer buys product 1 from firm B; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(ii) If the consumer's type (s, θ) is such that $s < g^*(\theta)$, then, with probability one, the consumer buys product 1 from firm A; firm B sets a price equal to $p_2^B = m\bar{\theta}^*$ for product 2, which the consumer buys if $\theta > \bar{\theta}^*$.

(b) If $m > m_H$, in any equilibrium, the firms' prices for product 1 are equal to $p_1^{A*} = 0$ and $p_1^{B*} = -1/2$, respectively, and the consumer buys product 1 from firm B with probability one. In addition, firm B perfectly observes the consumer's type and, in the second period, sets a price equal to her valuation for product 2, which the consumer buys.

The proof of Theorem 1 involves a backwards induction argument and proceeds in three steps (see Lemmas 1, 2, and 3 in Online Appendix B): first, for any pair of product 1 prices, p_1^A and p_1^B, we characterize the equilibria that arise in the subgame that follows the firms' choices of such prices. Then, we leverage this characterization to define a simultaneous-move pricing game for firms A and B, where their corresponding profit functions represent the "on-the-equilibrium-path" profits that each firm obtains when prices are set to be equal to p_1^A and p_1^B, respectively. We establish a formal mapping between the equilibrium strategies of the original game and those in the simultaneous-move pricing game. Finally, in the third step, we identify the range of values of m for which an equilibrium exists in the simultaneous-move pricing game and characterize its general structure. On the technical level, the proof, and in particular its third step, presents challenges

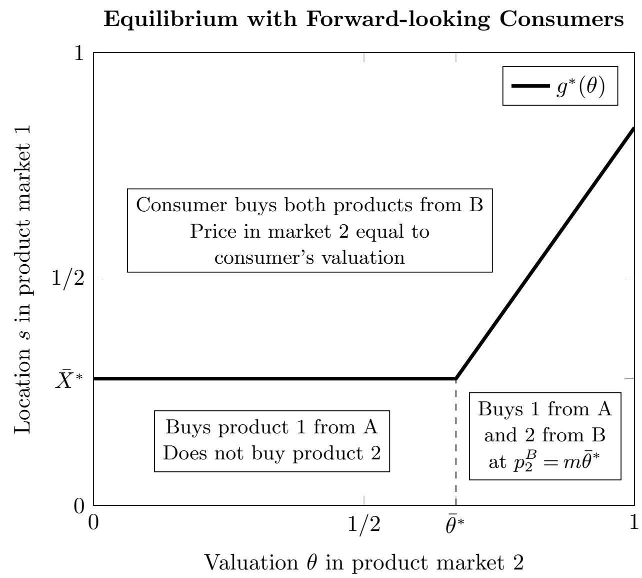

Figure 2 **Equilibrium outcome for a forward-looking consumer as a function of her type (s,θ) (we depict only interior equilibria; i.e., we assume that $m < m_L$). We let $\bar{X}^*$ denote the quantity $\bar{X}^* = 1/2 + p_1^{B*} - p_1^{A*}$**

given that essentially the two firms engage in an asymmetric pricing game, where firm B's profit function is not quasi-concave in its price. This prevents us from using well-known methods to establish equilibrium existence. Instead, we directly obtain the firms' best response correspondences and establish conditions that ensure the existence of a pure-strategy Nash equilibrium as well as its uniqueness (when it exists).[15]

As we already mentioned above, firm B has an incentive to offer a discount relative to firm A's product 1 price in order to induce consumers to buy from it and generate information about their valuations for product 2. This results in firm A's price being higher in equilibrium than firm B's, i.e., $p_1^{A^*} > p_1^{B^*}$. Moreover, Theorem 1 implies that two types of equilibria may arise: (i) *interior*, in which firm A sets a positive price for product 1 and transacts with the consumer with strictly positive probability, and (ii) *corner*, in which firm A sets its price for product 1 to zero and does not transact with the consumer. Corner equilibria arise only when the potential profits that firm B can generate in product market 2 are relatively large (i.e., when the parameter m takes relatively high values). Then, firm B finds it optimal to set a sufficiently low price to capture the entire market for product 1. Figure 2 provides an illustration of the equilibrium outcome as a function of the consumer's type.

Next, we describe the equilibria that arise in two alternative settings: (i) with no data tracking, and (ii) when firm B uses data tracking but consumers act myopically (Figure 3 provides an

[15] It is important to emphasize that Theorem 1 ensures that an equilibrium exists when $m < m_L$ or $m > m_H$. In Online Appendix B.3, we establish that $m_L \approx 3.98$ and $m_H \approx 4.02$. This implies that an equilibrium may not exist only in a small interval of the parameter space (i.e., $m \in (3.98, 4.02)$). In fact, we indeed can show that no equilibrium exists for specific values within the aforementioned interval.

illustration). In particular, in the absence of data tracking, the firms do not collect any consumer information that they can use in product market 2. Thus, the two product markets are entirely decoupled both for the firms and for the consumer. Proposition 1 summarizes the equilibrium outcome in this setting.

Proposition 1. *For any $m > 0$, there exists an equilibrium in the restricted setting. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) Both firms set a price of $1/2$ for product 1.

(ii) The consumer buys product 1 from firm A if her type satisfies $s < 1/2$, and she buys from firm B if $s > 1/2$. In particular, the expected demand for product 1 is $1/2$ for both firms.

(iii) Firm B sets a price of $p_2^{B,R} = m/2$ for product 2, which the consumer buys if $\theta > 1/2$.

In a nutshell, given that firm B does not collect any consumer information, it sets its price for product 2 based on the prior distribution for the consumer's type. Then, in product market 1, firms are symmetric since there is no information flowing from the first to the second period. Thus, firms compete in a standard Hotelling spatial competition model with fixed locations, which results in both firms setting the same price and splitting the market evenly (see e.g., d'Aspremont, Gabszewicz, and Thisse 1979 and Osborne and Pitchik 1987).

Finally, we consider the setting where firm B employs data tracking but consumers act myopically; i.e., they do not take into account the implications of their actions in the first period for firm B's pricing in the second. Proposition 2 summarizes the equilibrium outcome in this setting.

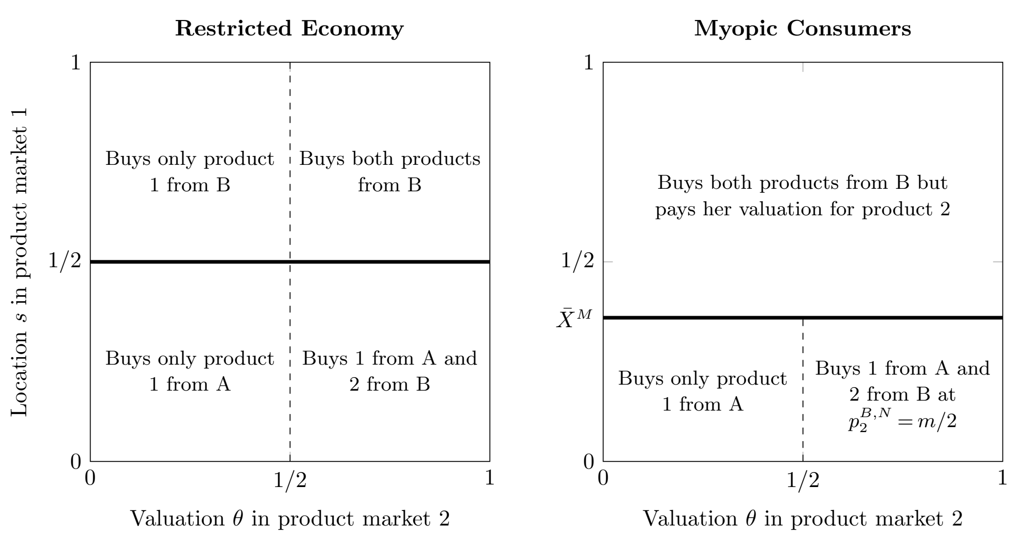

Figure 3 Equilibrium outcome in a restricted economy (left) and when the consumer behaves myopically (right) as a function of her type (s,θ). $\bar{X}^M$ denotes the quantity $\bar{X}^M = 1/2 + p_1^{B,M} - p_1^{A,M}$

PROPOSITION 2. *For any $m > 0$, there exists an equilibrium in the myopic setting. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) Firms' prices for product 1 are, respectively,

$$p_1^{A,M} = \max\left\{\frac{1}{2} - \frac{m}{12}, 0\right\}, \quad p_1^{B,M} = \max\left\{\frac{1}{2} - \frac{m}{6}, -\frac{1}{2}\right\}. \tag{3.1}$$

(ii) The consumer buys product 1 from firm B if her type (s,θ) satisfies $s > p_1^{B,M} - p_1^{A,M} + 1/2$. In that case, firm B perfectly observes the type of the consumer and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(iii) The consumer buys product 1 from firm A if her type (s,θ) satisfies $s < p_1^{B,M} - p_1^{A,M} + 1/2$. In that case, firm B sets a price of $p_2^{B,M} = m/2$ for product 2, which the consumer buys if $\theta > 1/2$.

(iv) Firm B faces a higher expected demand for product 1 than firm A. In particular, firm A's expected demand for product 1 is equal to $p_1^{A,M} < 1/2$.

The equilibrium outcome when consumers act myopically shares a number of features with the corresponding outcome in the forward-looking setting. In both cases, firm B finds it optimal to offer a discount for product 1 relative to firm A. In addition, as in the forward-looking setting, depending on the value of m, two types of equilibria may arise: (i) interior, in which both firms transact with the consumer with strictly positive probability, and (ii) corner, in which firm B's incentive to generate information for product 2 is sufficiently high for it to capture the entire product 1 market by pricing low.

However, there are also key differences between the structure of the equilibrium paths in the forward-looking and myopic settings. First, note that when the equilibrium is interior for the forward-looking setting (i.e., $0 < m < m_L$), the consumer strictly prefers to buy product 1 from firm A if her type satisfies $s < g^*(\theta)$, where g^* is a (weakly) increasing function. In other words, consumers with high values of θ are relatively more likely to buy product 1 from firm A (see Figure 2). This captures the fact that the consumers with high valuations for product 2 have relatively more to gain by hiding their type from firm B and buy product 1 from firm A. In contrast to the myopic setting, a forward-looking consumer considers this tradeoff when making her purchase decision for product 1.

A second and related key difference is that the price firm B sets for product 2 if the consumer bought product 1 from firm A is higher in the forward-looking setting than the corresponding price in the myopic setting, as $p_2^B = m\bar{\theta}^*(m) > m/2$. In particular, a myopic consumer's purchase decision for product 1 is independent of her valuation for product 2, i.e., the value of θ. Thus, when the consumer buys product 1 from firm A, firm B sets its price for product 2 based on the prior distribution for θ. By contrast, for a forward-looking consumer, her decision to purchase product 1 from firm A is informative about her valuation for product 2, as consumers with high valuations are relatively more likely to avoid buying product 1 from firm B (for an illustration refer to Figure 2). In turn, this is internalized by firm B when it updates its beliefs about the consumer's type after period 1, resulting in a distribution that is skewed toward high values of θ (when the consumer buys product 1 from firm A). This leads firm B to set a higher price for product 2 than it would do if such beliefs were uniform as in the myopic setting.

This dynamic highlights an important feature captured by our model: in the forward-looking setting, the actions of consumers that have a low opportunity cost for revealing their type (low θ) impact the equilibrium outcome for consumers that face a higher cost (high θ). The fact that low-valuation consumers are more likely to buy product 1 from firm B induces an externality on high-valuation consumers, since their choices result in firm B's equilibrium beliefs on θ to be skewed toward higher values, which drives firm B to set a higher price for product 2 than it would do if consumers acted myopically.

4. Implications of Data Tracking for Consumers and Firms

In this section, we build on the equilibrium characterization results of Section 3 to study the implications of data tracking on consumers and firms in our model economy. We begin our analysis by discussing consumer surplus (Theorem 2). Then, we consider firms' pricing decisions and profits (Propositions 4, 5, and 6). Throughout this section, we focus on the case where the forward-looking setting admits an interior equilibrium; i.e., we assume that $m \in (0, m_L)$. In our view, this regime represents the most plausible and interesting setting to study, as otherwise, i.e., when $m > m_H$, firm A does not engage in any transactions and makes zero expected profits. For the sake of completeness, we discuss the implications of data tracking for corner equilibria in Online Appendix D. The proof of Theorem 2 is provided in Online Appendix C and the proofs of Propositions 4, 5, and 6 can be found in Online Appendix E.

Consumer Surplus. Our first result establishes that data tracking leads to *higher* aggregate consumer surplus. Interestingly, this finding holds even if consumers act myopically, i.e., if they do not take into account the implications of their actions in product market 1 for pricing in product market 2.

THEOREM 2. *Suppose that $m \in (0, m_L)$. Then, aggregate consumer surplus is higher in the presence of data tracking for both myopic and forward-looking consumers.*

Theorem 2 highlights one of the main qualitative insights of our analysis: on aggregate, consumers are better off when firm B employs data tracking. The intuition for this result can be best explained as follows: the presence of data tracking creates the incentive for firm B to lower its price for product 1 as a way of increasing its share of transactions and generating consumer information. In turn, this also results in firm A setting a lower price due to competition. On the other hand, firm B can use the data it generates to extract additional surplus from consumers in product market 2. Our analysis establishes that the first effect, i.e., lower prices in product market 1, dominates the potentially adverse effects of price discrimination in product market 2. The left panel of Figure 4 depicts graphically the findings of Theorem 2, i.e., the increase of consumer surplus relative to the restricted economy, as a function of the value of consumer data (captured by parameter m).

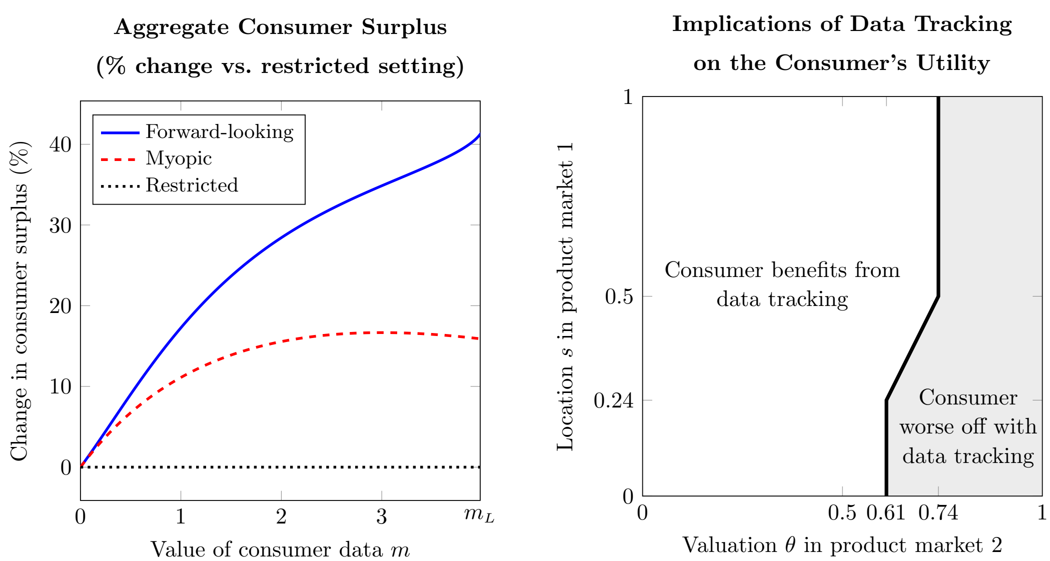

Figure 4 Percentage change in the equilibrium aggregate consumer surplus relative to the restricted setting, as a function of the value of consumer data m, for $\bar{u} = 1$ (left), and comparison of the consumer's utility as a function of her type (s, θ) in the forward-looking and restricted settings (right) when $m = 2$. Forward-looking consumers with relatively high values of θ, i.e., whose types fall into the shaded region, are better off in the absence of data tracking

Although consumers are on aggregate better off in the presence of data tracking, this does not hold across all consumer types. In particular, consumers with low valuations for product 2 (low θ) tend to gain the most from data tracking as they benefit from lower prices for product 1, while not experiencing a significant loss in net utility in product market 2 relative to a restricted economy. By contrast, high-valuation consumers would be better off if the two product markets were decoupled for firm B, i.e., if there was no information flow between them. The right panel of Figure 4 illustrates this point and highlights that the implications of data tracking on individual consumers may differ considerably depending on their idiosyncratic features. In particular, the shape of the boundary that divides the two regions in Figure 4 highlights that this comparison depends both on the tradeoff between the lower price the consumer pays for product 1 versus the higher price for product 2 as well as on whether the consumer buys from one firm in the restricted setting but chooses the other firm in the presence of data tracking, which increases the consumer's transportation cost. Moreover, Figure 4 also illustrates the fact that lower product 1 prices benefit all consumers, while only those with relatively high values of θ bear the surplus loss associated with higher product 2 prices, which is precisely the main mechanism behind Theorem 2.

Importantly, Theorem 2 is largely driven by competition: firm B's incentive to lower its prices for product 1 results in its competitor, firm A, doing the same, which in turn benefits (a fraction of) consumers. To best illustrate the role of competition, we consider an alternative formulation where firm B is a monopolist in both markets. Online Appendix G outlines the monopoly variant of our model economy and provides a characterization of the equilibria for the settings we consider.[16] In turn, this allows us to obtain the following result for consumer surplus.

PROPOSITION 3. *Consider an economy with $\bar{u} \geq 1/2$ and $m > 0$. Then, when firm B is a monopoly in both markets, we obtain that*

(i) Aggregate consumer surplus is lower in the presence of data tracking when consumers behave myopically.

(ii) In addition, if $m \geq 4(1-\bar{u})$, aggregate consumer surplus is also lower in the presence of data tracking when consumers are forward-looking.

The juxtaposition of Proposition 3 with Theorem 2 establishes that competition plays a fundamental role in determining whether data-tracking technologies may lead to an increase in consumer surplus. In particular, when consumers fail to internalize the impact of their actions on firm B's pricing decisions in market 2, they are on aggregate worse off if firm B faces no competition and can employ data tracking. Thus, the findings of Theorem 2 are driven by the combination of the incentive that data tracking provides to firm B to lower its price for product 1 and the competitive response from firm A, which increases the size of the price discount. As statement (ii) of the proposition highlights, a similar insight holds for an economy with forward-looking consumers: the effects of data tracking on aggregate consumer surplus are largely dependent on competition. In fact, when consumers' valuations for product 1 are sufficiently high, i.e., $\bar{u} \geq 1$, and firm B is a monopoly in both markets, they are better off if firm B has *no* access to data tracking, regardless of the value of m.

The above findings about consumer surplus may help inform the discussion on policy design relative to data tracking. Although prior work has emphasized the adverse effects of tracking

[16] In particular, the consumer decides sequentially whether to buy product 1 and then product 2 from firm B. Then, her utility associated with product 1 is simply given as $a_1\left(\bar{u}-(1-s)/2-p_1\right)$.

for myopic consumers (Taylor 2004, Acquisti and Varian 2005, Bonatti and Cisternas 2020), it has mainly focused on a monopoly context. We illustrate that the implications may be more nuanced and largely dependent not only on the consumers' level of sophistication, i.e., whether they internalize the effect of their purchase decisions on (future) firms' actions, but also on the structure of competition in the market.

Prices. As we discussed above, data tracking creates the incentive for firm B to set a lower price in the first product market as a way to increase its market share and obtain additional consumer information. The effect of firm B's data-tracking capabilities on firm A's price is driven by the following tradeoff. On the one hand, firm A faces pressure to decrease its price for product 1 relative to an economy where no such capabilities exist given firm B's lower price. On the other hand, when consumers are forward-looking, they anticipate the implications of revealing their type to firm B and, as a result, a fraction of consumers find firm A's product relatively more attractive as transacting with firm A allows them to preserve their *privacy* regarding their type.

Data tracking results in price dispersion, i.e., firm A charging more for its product than firm B, relative to the economy where data-tracking capabilities are not present and both firms set the same price. What is more, the extent of price dispersion is higher when consumers are forward-looking, since in this case the aforementioned privacy effect is also present in the economy. In other words, the incentive to collect consumer data drives firm B to offer a discount relative to firm A's product 1 price. Moreover, the offered discount is larger with forward-looking than with myopic consumers, as the former factor in the opportunity cost of disclosing their data in making their purchase decision, resulting in higher data acquisition costs for firm B. These observations are formalized in Proposition 4 below and illustrated in Figure 5.

PROPOSITION 4. *Suppose that $m \in (0, m_L)$. Then, there exists dispersion in product 1 prices in equilibrium when firm B uses data tracking. Furthermore, the level of price dispersion is highest when consumers are forward-looking. In other words,*

$$\Delta^* = p_1^{A*} - p_1^{B*} > \Delta^M = p_1^{A,M} - p_1^{B,M} > 0,$$

where Δ^* and Δ^M denote the level of product 1 price dispersion in equilibrium in an economy with forward-looking and myopic consumers, respectively.

Profits. Next, we turn our attention to the profits firms generate. When firm B employs data tracking, it finds it optimal to "subsidize" product 1 (set a lower price than in an economy without tracking) and generate larger profits from product 2. Clearly, when consumers behave myopically, tracking yields the highest benefits for firm B. Proposition 5 below establishes that firm B is better off with data tracking even when consumers are forward-looking and internalize the implications of

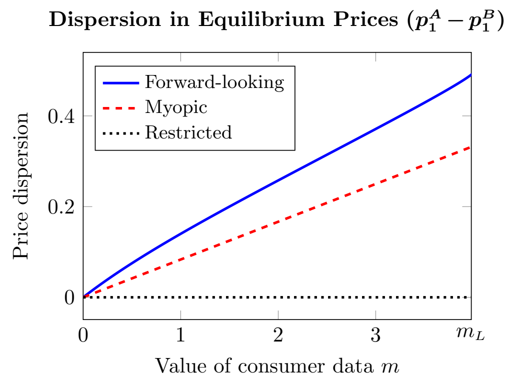

Figure 5 **Dispersion in the equilibrium prices for product 1 as a function of the value of consumer data m**

the firm's pricing decisions. Interestingly, this finding is somewhat at odds with prior work, which argues that firms may find it optimal to commit not to use consumer information for personalized pricing when facing forward-looking consumers, albeit in models that differ from ours in a number of ways, e.g., by featuring two customer types and different tracking and personalization technologies (Taylor 2004, Villas-Boas 2004, Acquisti and Varian 2005).

Proposition 5. *Suppose that $m \in (0, m_L)$. Then, firm B always benefits from using data-tracking technologies. In addition, its equilibrium profits are higher when consumers behave myopically than when they engage in forward-looking behavior.*

As one would perhaps expect, the implications of data tracking for firm A are considerably different and are summarized in Proposition 6 below.

Proposition 6. *Suppose that $m \in (0, m_L)$. Then, firm A's expected equilibrium profits are highest in a restricted economy. On the other hand, suppose that firm B uses data tracking. Then, there exists $\bar{m} \in (0, m_L)$ such that firm A's equilibrium profits are higher with forward-looking than with myopic consumers if and only if $m < \bar{m}$.*

The first part of the proposition establishes that firm A is better off in an economy where its competitor does *not* use data tracking. In other words, even though consumers attach a "privacy" premium to firm A's product, this does not compensate for the potential loss in its profits due to firm B's lower prices. The findings are more nuanced when we focus on consumer behavior. Assuming that firm B tracks consumer data, firm A is better off in an economy with forward-looking consumers only when the value of their data as captured by m is relatively low. In other words, even though forward-looking consumers attach additional value to firm A's product, the firms' endogenous pricing decisions may result in lower profits for A than in an economy where consumers act myopically, as firm B's incentive to lower the price of product 1 is intensified when consumers are forward-looking. Figure 6 provides a graphical illustration of our findings related to the firms' equilibrium profits.

As a final remark, note that the latter findings may point to a strategic opportunity for firm A. In particular, Proposition 6 provides conditions under which the firm would be better off in an economy with forward-looking consumers relative to one where they behave myopically. To the extent that highlighting the implications of data tracking, e.g., through firm A advertising its "privacy" features, would raise awareness among consumers and, in the context of our model, induce forward-looking behavior, the comparison of equilibrium profits in the proposition establishes when such an endeavor would benefit firm A. Interestingly, this depends on the value of consumer data as firm A's efforts to educate consumers may be countered by firm B's endogenous pricing response.

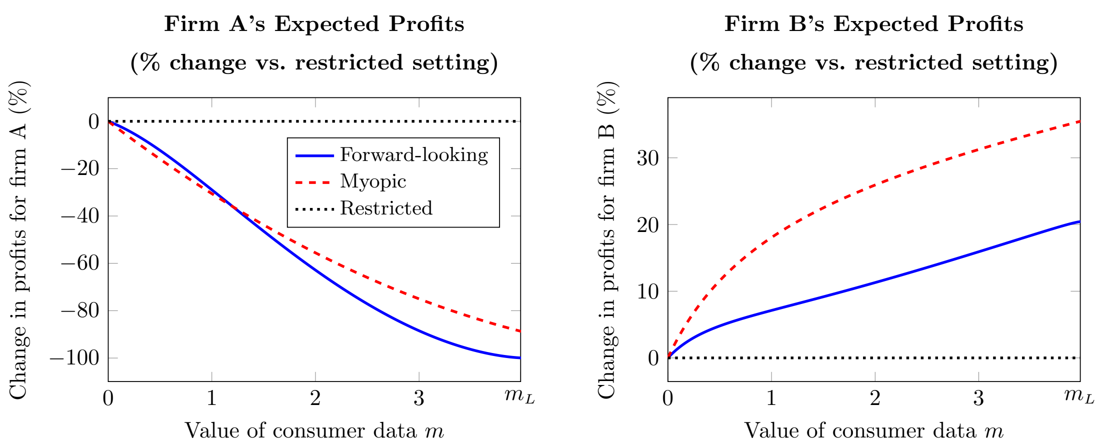

Figure 6 Percentage change in total expected profits in equilibrium for firm A (left) and B (right), relative to the restricted setting, as a function of the value of consumer data m

Social Welfare. To conclude this section, we compare the social welfare in equilibrium, i.e., the sum of firms' profits and aggregate consumer surplus, in the three settings we consider. As Figure 7 illustrates, social welfare is higher if firm B has access to data tracking regardless of the degree of consumer sophistication. Thus, data tracking increases consumer surplus and firm B's profits (as described in Theorem 2 and Proposition 5) in a way that more than compensates for the decrease in firm A's profits (Proposition 6).

To understand this tradeoff in more detail, note that since prices affect only monetary transfers between the consumer and the firms, the total social welfare is fully characterized by the expected transportation cost borne by the consumer in product market 1 and the expected value of the transactions realized in product markets 1 and 2. Data tracking affects these two terms in opposing directions: on the one hand, the expected value of product 2 transactions is higher in the presence of data tracking, as it enables firm B to set personalized prices based on the information collected from product market 1, which results in a higher volume of transactions in product market 2. On the other hand, the expected transportation cost borne by the consumer in product market 1 increases in the presence of data tracking, as it induces price dispersion in equilibrium. The comparison in Figure 7 shows that the increase in value generated in product market 2 outweighs the higher transportation cost in product market 1, regardless of whether the consumer is forward-looking or acts myopically.

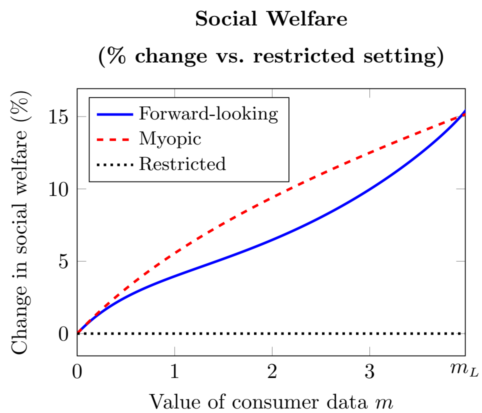

Figure 7 Percentage change in social welfare in equilibrium relative to the restricted setting, as a function of the value of consumer data m, for $\bar{u} = 1$

5. Robustness of Main Findings

To assess the extent to which our main findings apply in broader settings, in this section we discuss several alternative formulations of the baseline model, which relax or modify some of its assumptions. Specifically, we extend the model in three directions: introducing competition in the market for product 2, relaxing the strength of firm B's data-tracking ability, and allowing more general distributions of consumer types. In what follows, we summarize the analysis of these scenarios. Importantly, we find that in all these settings, data tracking may lead to higher consumer surplus even if consumers act myopically (as described in Theorem 2).

5.1. Duopoly in Both Markets

In our baseline model, firm B is a monopoly in the market for product 2 but faces competition to sell product 1. This market structure gives firm B a clear incentive to sell product 1 when data tracking is available since, after observing the consumer's type, it can extract full surplus by selling product 2 at the consumer's valuation. However, if competition was present in both product markets, firm B's incentive to decrease its price for product 1 in order to collect consumer information would be weaker, since its ability to extract surplus from product 2 transactions would be more limited. To understand the implications of data tracking in that scenario, we consider a setting in which the market for product 2 is also a duopoly. Specifically, we introduce a third firm to the model (firm C) that competes with firm B to sell product 2 in the second market.[17]

[17] This formulation also covers the case where firm A participates in both markets but has no data-tracking ability.

We analyze this extension and discuss its implications in Online Appendix H. To summarize, we consider myopic consumers and characterize the equilibrium outcomes for the settings with and without data tracking, and find that they follow a similar pattern to those of the baseline model (Propositions 1 and 2). In the presence of data tracking, firms A and B set lower prices for product 1 than in the restricted setting, with firm B pricing at a discount relative to firm A, which again reflects firm B's incentive to collect consumer information that it can subsequently use to inform pricing in the second period. However, in contrast to our baseline model, data tracking not only results in lower prices for product 1, but also intensifies competition in the market for product 2. Intuitively, data tracking enables firm B to capture a fraction of the demand that would otherwise flow to firm C as, when firm B observes the consumer's type, it can price product 2 at just the right level to induce the consumer to buy from it instead of from firm C. In turn, this drives firm C to lower its price in order to preserve demand from consumers that have a strong preference for buying from it. By way of this mechanism, data tracking creates competition at the individual consumer level instead of only "on aggregate," which leads to lower product 2 prices in expectation.

Thus, the presence of data tracking does not introduce a tradeoff between lower product 1 prices and firm B extracting more surplus associated with product 2 as it does in the baseline model but, instead, intensifies competition in both product markets. As a result, we find that a comparison in line with the one established in Theorem 2 holds: with myopic consumers and competition in both markets, the presence of data tracking increases consumer surplus.

5.2. Imperfect Data Tracking

In Online Appendix I, we consider an extension that relaxes the strength of firm B's data-tracking ability. Concretely, instead of assuming that firm B learns the consumer's valuation for product 2 when the consumer buys product 1 from it, we assume that the valuation is revealed only with some probability that we denote by $\beta \in (0,1)$ and interpret as firm B's tracking accuracy. Thus, the value of β influences firm B's incentive to price product 1 at a discount in order to attract consumers to buy from it, as it determines the likelihood that the firm will be able to price product 2 at the consumer's valuation. Moreover, forward-looking consumers also take into account firm B's tracking accuracy when making their product 1 purchase decision.

We find that the equilibrium outcomes in this context display a similar structure to those of the baseline model, with appropriate modifications that quantify the value of consumer data not only as a function of the range of consumer valuations for product 2, as given by m, but also as a function of the level of firm B's tracking accuracy β. Moreover, we show that in line with Theorem 2, consumer surplus may be higher in the presence of data tracking regardless of the level of consumer sophistication, provided that the value of consumer data (which now depends on both m and β) is not exceedingly high.

5.3. More General Type Distributions

Our baseline formulation assumes that the prior distribution of the consumer's type (s, θ) is uniform in the unit square, which implies that s and θ are independent. We consider two different extensions that relax this assumption. First, in Online Appendix J, we relax independence by focusing on a class of distributions that induces correlation between s and θ. Specifically, we assume that s is uniformly distributed in $[0,1]$ and that, conditional on s, θ is uniformly distributed in the interval $[rs, 1]$, where the parameter $r \in [0,1)$ modulates the correlation between s and θ. We find that our main findings extend to this context. In particular, the equilibrium outcomes of the model follow a similar pattern to those of the setting without correlation and, in line with Theorem 2, the presence of data tracking may increase consumer surplus.

Then, in Online Appendix K, we extend our analysis of the model with myopic consumers to the case where θ follows an arbitrary distribution F that takes values in $[0, \infty)$, and maintain that s is uniformly distributed in $[0,1]$ and independent of θ. We find that the comparison established in Theorem 2 also extends to this setting, provided that the distribution F satisfies two mild assumptions. Namely, if consumers act myopically, consumer surplus is higher in the presence of data tracking as long as the value of m is below a threshold that depends on the distribution F. In particular, this analysis shows that the observation that data tracking may benefit consumers even if they act myopically is by no means limited to the case where F is a uniform distribution.

6. Concluding Remarks

In this paper we study the implications of data-tracking technologies on firms' pricing decisions, their profits, and, most importantly, consumer surplus. The economy we consider, which features two markets (a duopoly and a monopoly), allows us to focus on the role of competition and the consumers' level of sophistication. In particular, our results establish that the adoption of data tracking may benefit consumers in the presence of competition. This result stems from the fact that the potential gains from acquiring consumer information through transactions provide the incentive for a firm to lower its prices. In turn, this increases the intensity of competition and results in lower prices for all consumers in the first market, the duopoly. Interestingly, given that the firm can use the consumer data it collects for price discrimination in the second market, the gains in consumer surplus are not uniform among different consumer types: those with high valuations for the product in the second market are adversely affected by data tracking in contrast to those with low valuations. Thus, our results may contribute to the ongoing discussion on regulation aimed at protecting consumers' data privacy, by illustrating that the debate on whether to regulate data-tracking practices should take into account not only the degree of consumers' understanding of how their data may be used (captured in our model by whether consumers are forward-looking or myopic), but also the competitive structure of the market.

Our baseline formulation makes a number of simplifying assumptions to allow for a tractable and transparent analysis. For example, we assume that firm B has monopoly power in the market for product 2, that data-tracking technologies enable firm B to perfectly infer the type of a consumer after a transaction, and that consumer types are drawn from the uniform distribution. However, we analyze various extensions that relax each of these assumptions and find that our main insights continue to hold beyond our baseline formulation. Specifically, the observation that consumer surplus may be higher in the presence of data tracking even if consumers act myopically also applies in the alternative formulations we considered.

In addition, we have assumed that consumers are either all taking into account the implications of their actions on firms' pricing decisions or they are all acting myopically. Extending our model to the setting where only a fraction of consumers are forward-looking is an interesting avenue for further research. Finally, in a context with imperfect tracking technologies, it would be instructive to study the incentives of firms to invest in adopting and improving the accuracy of such technologies.

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Appendix A: Model Description

In this Appendix we formalize the description of the model discussed in Section 2 and of the equilibrium concept we adopt. We start by formally defining the histories of the model and the information available to the firms and the consumer in each period in Appendix A.1. Then, we provide formal definitions for each agent's strategies and beliefs, and define their payoffs as a function of these objects in Appendix A.2. Next, in Appendix A.3, we provide a formal definition for the notion of perfect Bayesian equilibrium we consider. Finally, in Appendix A.4, we describe how these definitions change for the settings with no data tracking (restricted) and with data tracking and myopic consumers. The notation and concepts defined in what follows are used throughout the proofs of our results in the subsequent online appendices.

A.1. Timeline, histories, and information structure

In what follows we formally define the histories and information available to players at each stage of the game introduced in Section 2. Recall that the game begins with period $t = 0$ when the consumer's type is drawn, followed by periods $t = 1, 2$, during which the firms and the consumer transact. Each of these periods begins with firms setting prices (with only firm B in period 2), followed by the consumer's purchasing decision; see Figure 1 for an illustration of this timeline.

First, we introduce some notation. We denote a generic history by h. Given that the consumer knows her type and observes the firms' actions, i.e., prices, the information she has at her disposal when making purchase decisions is the full history h. Note that this is generally not true for the firms, which may set prices based on incomplete information. Thus, given a history h where one or both firms move, we denote the information available to them by $\mathcal{J}(h)$. In other words, $\mathcal{J}(h)$ represents the components of history h that are observed by the firms when setting their prices. We now describe the histories at each stage of the game. Note from Figure 1 that, within each period $t = 1, 2$, the consumer's purchasing decision follows the firms' pricing decisions. Thus, in what follows, we denote the collection of possible histories before the consumer moves in period t by H_t^c. Similarly, we use H_t^f for the collection of histories that immediately precede the firms' move in period t.

Period 0: The game begins with the empty history $h = \emptyset$. At time $t = 0$, Nature draws the consumer's type $(s, \theta) \in \mathcal{T} = [0,1] \times [0,1]$ according to the uniform distribution over the unit square, and the consumer observes her type.

Period 1: The history at the beginning of period 1 consists only of the realization of the consumer's type. Thus, we can write the history at this stage as $h = (s, \theta)$. We denote the collection of all such histories by H_1^f. Next, we describe the firms' and the consumer's sequential actions in period 1:

(i) *Firms.* Both firms set their product 1 prices p_1^A and p_1^B simultaneously at the beginning of period 1. We assume that the firms do not know the consumer's type when making this decision: given history $h \in H_1^f$, the firms observe $\mathcal{J}(h) = \emptyset$ (i.e., no information) and choose product 1 prices.[18]

[18] However, we assume that firms know the prior distribution of consumer types when setting prices. We do not explicitly include this as part of the information available when setting prices, since the prior distribution is later incorporated when we define the firms' beliefs over the consumer types.

(ii) *Consumer.* The consumer then makes her purchasing decision for product 1 after observing the firms' prices for product 1, i.e., after histories of the form $h = (s, \theta, p_1^A, p_1^B)$. We denote the collection of all such histories by H_1^c.

Period 2: The history at the beginning of period 2 consists of the consumer's type, the firms' prices for product 1, and the consumer's purchasing decision for product 1; i.e., we can write the history at the beginning of period 2 as $h = (s, \theta, p_1^A, p_1^B, a_1)$. We let H_2^f denote the collection of all such histories. Then, firm B and the consumer move sequentially as follows:

(i) *Firm B.* At this stage, firm B sets the price of product 2, but the information available to make this decision depends on the consumer's purchasing decision for product 1. In particular, firm B's data-tracking ability comes into play at this point. If the consumer did not buy product 1 from firm B (i.e., chose $a_1 = 0$), firm B observes the prices for product 1 but does not observe the consumer's type. On the other hand, if the consumer bought product 1 from firm B (i.e., chose $a_1 = 1$), the firm observes the complete history of the game and, in particular, the consumer's type.[19] Formally, given a history $h = (s, \theta, p_1^A, p_1^B, a_1) \in H_2^f$, the information available to firm B at the beginning of period 2 is

$$\mathcal{J}(h) = \begin{cases} h & \text{if } a_1 = 1, \\ (p_1^A, p_1^B) & \text{if } a_1 = 0. \end{cases} \tag{A.1}$$

Then, given a history $h \in H_2^f$, firm B observes $\mathcal{J}(h)$ and sets the price of product 2.

(ii) *Consumer.* Finally, the consumer decides whether to buy product 2 after firm B sets its price p_2^B, i.e., after a history of the form $h = (s, \theta, p_1^A, p_1^B, a_1, p_2^B)$; we denote the collection of all such histories by H_2^c. After the consumer's purchasing decision for product 2, payoffs are realized and the game ends.

Before moving on, we establish some additional notation. Given history $h = (s_0, \theta_0, \bar{p}_1^A, \bar{p}_1^B, \bar{a}_1)$, we let $s(h) = s_0$, $p_1^A(h) = \bar{p}_1^A$, $a_1(h) = \bar{a}_1$, etc. Similarly, if $I = \mathcal{J}(h)$ for some history h, we define $p_1^A(I)$ and $p_1^B(I)$ as the corresponding product 1 prices that are common for all histories that result in firms observing information I. In addition, given a history h and an action a, we denote the concatenation of h and a by $\langle h, a \rangle$. Finally, we denote the set of possible information vectors observed by firms when setting prices for product t by $\mathcal{I}_t$. It follows that $\mathcal{I}_t$ can be written as the image of H_t^f under $\mathcal{J}$; more precisely, we define $\mathcal{I}_t = \left\{\mathcal{J}(h) : h \in H_t^f\right\}$, for $t = 1, 2$. Note that for the first period, we simply have $\mathcal{I}_1 = \{\emptyset\}$. Moreover, we can write $\mathcal{I}_2$ as the union of two sets, which differ on whether the consumer's type is known by firm B when setting the price of product 2. In turn, this depends on the consumer's decision in the first period. That is, we write $\mathcal{I}_2 = \mathcal{I}_2^0 \cup \mathcal{I}_2^1$, where

$$\mathcal{I}_2^j = \left\{\mathcal{J}(h) : h \in H_2^f,\, a_1(h) = j\right\}, \text{ for } j = 0, 1. \tag{A.2}$$

A.2. Strategies, beliefs, and payoffs

We define an *assessment* as a tuple $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ of strategies for the consumer, firms A and B, and a belief system $\boldsymbol{\mu}$. In more detail, a *strategy for the consumer* consists of a pair of functions $\boldsymbol{\gamma} = (\gamma_1, \gamma_2)$ with $\gamma_t : H_t^c \to \Delta(\{0, 1\})$ for $t = 1, 2$, where $\gamma_t(h)$ denotes the probability of buying product $t = 1, 2$ from firm B after history h.

[19] One could assume that instead of observing the consumer's type (s, θ), firm B observes only the value of θ. In that case, all our results continue to hold as θ contains all the payoff-relevant information after the first period.

A *pricing strategy for firm A* is then a function of the information available at the beginning of period 1 to firm A's action set,[20] $\sigma_1^A : \mathcal{I}_1 \to S^A$. A *pricing strategy for firm B* consists of a pair of functions (one for each product) $\boldsymbol{\sigma}^B = (\sigma_1^B, \sigma_2^B)$, where $\sigma_1^B : \mathcal{I}_1 \to S^B$ and $\sigma_2^B : \mathcal{I}_2 \to \mathbb{R}$. Note that, for simplicity, we focus on the case where both firms set prices according to pure strategies.[21]

A *belief system* consists of a pair of functions $\boldsymbol{\mu} = (\mu_1, \mu_2)$ that map firms' available information at a given time to probability distributions on the consumer type space, i.e., $\mu_t : \mathcal{I}_t \to \Delta(\mathcal{T})$ for $t = 1, 2$. In particular, $\mu_1(\emptyset)$ represents the firms' common beliefs on the consumer types when choosing their product 1 prices. Since the only information available to both firms when setting product 1 prices is the prior distribution of consumer types, we define these beliefs to be equal to this prior distribution; i.e., we define $\mu_1(\emptyset) = \mu_0$, where μ_0 is the uniform distribution on the unit square. For the second period, given firm B's available information, $I \in \mathcal{I}_2$, $\mu_2(I)$ represents firm B's information on the type (s, θ) when setting product 2's price.

Payoffs First, we define the continuation utility for the consumer in time periods $t = 1, 2$. Given a history $h \in H_2^c$, we slightly abuse notation to write the consumer's utility as a function of her action as $u_2(a_2; h) = u_2(a_2; s(h), \theta(h), p_2^B(h))$, where u_2 is as given in (2.2). Note that we omit the explicit dependence on the histories whenever it is clear from the context. We let U_2 denote the consumer's continuation utility in period 2 given an assessment $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ and a history $h \in H_2^c$. Then, we have that

$$U_2\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right) = \mathbb{E}\left[u_2\left(a_2; h\right) \mid h\right], \tag{A.3}$$

where the expectation is taken with respect to the consumer's purchasing decision in period 2, i.e., $a_2 \sim \gamma_2(h)$. Similarly, for every history where the consumer moves in period 1, $h \in H_1^c$, we write $u_1(a_1; h) = u_1(a_1; s(h), \theta(h), p_1^A(h), p_1^B(h))$, where u_1 is as given in (2.1), and define the total expected utility given the assessment and h as

$$U_1\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right) = \mathbb{E}\left[u_1(a_1; h) + u_2\left(a_2; \langle h, a_1, p_2^B \rangle\right)\right], \tag{A.4}$$

where the expectation is taken with respect to consumer purchase decisions a_1, a_2, and firm B's product 2 price p_2^B. Similarly, we define firm A's total payoff as

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \mathbb{E}\left[\pi_1^A\left(p_1^A, p_1^B, a_1\right)\right],$$

where $p_1^j = \sigma_1^j(\emptyset)$ for $j = A, B$, and the expectation is taken with respect to the firms' beliefs on the consumer types for the first period, $\mu_1(\emptyset)$, and the consumer's decision a_1. Proceeding similarly for firm B, given an information vector $I \in \mathcal{I}_2$ and an assessment, we define firm B's continuation payoff at time 2 as

$$\Pi_2^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid I\right) = \mathbb{E}\left[\pi_2^B\left(p_2^B, a_2\right) \mid I\right],$$

[20] For the rest of the paper, we assume that the firms' action spaces in period 1 are $S^A = [0, 1]$ and $S^B = [-1/2, 1]$, interpreting negative prices as pricing below cost or paying consumers in exchange for their data. This choice of action spaces does not have a material effect on our findings; i.e., it allows us to study all equilibria in which firms play undominated strategies (see Claim 19 in Online Appendix B.3).

[21] In proving our results, we allow firm B to use a mixed strategy when pricing product 2. As we establish, in equilibrium, firm B's pricing strategy for product 2 is indeed pure.

where the expectation is taken assuming that the consumer types are distributed according to $\mu_2(I)$, taking firm B's product 2 price as $p_2^B = \sigma_2^B(I)$, and with respect to the consumer's decision a_2. Finally, firm B's total expected profit in the game is

$$\Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \mathbb{E}\left[\pi_1^B\left(p_1^A, p_1^B, a_1\right) + \pi_2^B\left(p_2^B, a_2\right)\right],$$

where $p_1^j = \sigma_1^j(\emptyset)$ for $j = A, B$. The expectation is taken with respect to the consumer's decisions a_1, a_2 and firm B's price for product 2, p_2^B, assuming that this price is set according to σ_2^B and that the consumer types are distributed according to $\mu_1(\emptyset)$.

A.3. Equilibrium definition

We consider Perfect Bayesian Equilibria (PBE) in the game we described above. An assessment $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is a PBE if and only if the strategy profile $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$ is sequentially rational for all players given the belief system $\boldsymbol{\mu}$, and $\boldsymbol{\mu}$ is consistent with the strategy profile. In particular, a PBE $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ satisfies the following conditions:

(i) For any period $t = 1, 2$, any history $h \in H_t^c$, and any consumer strategy $\boldsymbol{\gamma'}$ we have that

$$U_t\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right) \geq U_t\left(\boldsymbol{\gamma'}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right).$$

(ii) For any period $t = 1, 2$, any information vector $I \in \mathcal{I}_t$, and any firm B's pricing strategy $\boldsymbol{\sigma}^{B'}$ we have that

$$\Pi_t^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid I\right) \geq \Pi_t^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^{B'}, \boldsymbol{\mu} \mid I\right).$$

(iii) For any firm A's pricing strategy $\sigma_1^{A'}$, we have that

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) \geq \Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^{A'}, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right).$$

(iv) The belief system $\boldsymbol{\mu}$ is consistent with the strategy profile $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$. That is, given any information vector $I \in \mathcal{I}_2$ we have that

  (a) If $I \in \mathcal{I}_2^1$ (as defined in (A.2)), then $\mu_2(\cdot \mid I)$ assigns probability 1 to the true consumer type $(s(I), \theta(I))$. That is, for any Borel set $A \subseteq \mathcal{T}$,

$$\mu_2(A \mid I) = \mathbf{1}_A(s(I), \theta(I)), \tag{A.5}$$

  where $\mathbf{1}_A$ denotes the indicator function of set A.

  (b) If $I \in \mathcal{I}_2^0$ (as defined in (A.2)), then for any Borel set $A \subseteq \mathcal{T}$,

$$\mu_2(A \mid I) = \mathbb{P}_{(s,\theta)\sim\mu_1(\emptyset)}\left[(s, \theta) \in A \middle| I, \boldsymbol{\gamma}, \boldsymbol{\sigma}\right],$$

  and moreover, this expression satisfies Bayes' rule whenever the event of reaching a history that results in firm B observing information vector I has positive probability (according to $\mu_1(\emptyset)$), if players follow the strategy profile $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$.

A.4. Modifications for restricted and myopic settings

The definitions of histories, payoffs and equilibrium described above consider the setting with data tracking and forward-looking consumers. For the settings with no data tracking (restricted) and with data tracking and myopic consumers, these definitions remain the same except for the following modifications:

Restricted. We assume that firm B's information in period 2 given history h is given by $\mathcal{J}(h) = (p_1^A, p_1^B)$, irrespective of the consumer's action in period 1 (contrast this with Expression (A.1)). Thus, there is no consumer-related information flowing from the first to the second period.

Myopic. Recall that a *myopic* consumer determines her action a_1 to maximize her utility in period 1, as given by Expression (2.1), whereas a *forward-looking* consumer determines a_1 to maximize her aggregate utility over both product markets, as given by Expression (A.4). Thus, in this setting, condition (i) in the equilibrium definition for period 1 changes to

$$\gamma_1(h) \in \underset{\gamma_1'(h) \in \Delta(\{0,1\})}{\arg\max} \mathbb{E}\left[u_1(a_1; h) \mid h\right],$$

for all $h \in H_1^c$, where the expectation above is taken with respect to $a_1 \sim \gamma_1'(h)$.

Online Appendix
Data Tracking under Competition

The online appendix is organized as follows. First, we provide the proofs of the results for the baseline model in Appendices B–F. In particular, Appendix B provides the proof of Theorem 1, which characterizes the equilibrium outcomes in the setting with data tracking and forward-looking consumers. Appendix C provides the proof of Theorem 2, which results from comparing the aggregate consumer surplus for the three settings we consider. Appendix D compares the equilibrium outcomes across these three settings in the corner equilibrium regime (i.e, when $m > m_H$), complementing the analysis of Section 4. Appendix E contains the proofs of Propositions 1 and 2 which, respectively, characterize the equilibrium in the restricted and myopic settings, as well as of Propositions 4–6, which compare the equilibrium outcomes for the three settings we consider in the baseline model. Finally, Appendix F provides the proofs of several auxiliary results that are used in proving Theorem 1.

Then, in Appendices G–K we extend the model and our main results in various directions. In Appendix G, we consider the case where firm B is a monopolist in both product markets, and provide the proof of Proposition 3. In Appendices H–K, we develop a series of extensions that result in the conclusions summarized in Section 5. Specifically, in Appendix H, we consider a setting with competition in both product markets. Subsequently, in Appendix I, we study a model of imperfect data tracking. Then, in Appendix J we analyze the setting where the consumer types s and θ are correlated, and finally, in Appendix K, we extend the analysis of the model with myopic consumers to more general distributions for the consumer types.

Online Appendices

Appendix B: Proof of Theorem 1

Overview of the proof As discussed in Section 3, the proof of Theorem 1 proceeds in three steps, each of which we will state as an independent lemma. Informally, Lemma 1 characterizes the equilibria that arise in the continuation game that follows the firms' choices for product 1 prices, p_1^A and p_1^B, i.e., the subgame that follows histories of the form $h=(s,\theta,p_1^A,p_1^B)$. Then, based on the characterization established in Lemma 1, we define a simultaneous-move pricing game for firms A and B, where their corresponding profit functions in terms of product 1 prices represent the "on-the-equilibrium-path" profits that each firm will obtain when the prices are p_1^A and p_1^B, respectively. Lemma 2 shows that any equilibrium in the original game is associated with a Nash equilibrium in this simultaneous-move game and vice versa. Finally, in Lemma 3, we identify the range of values of m for which an equilibrium exists in this simultaneous-move game and characterize its general structure (i.e., the interior and corner regimes) and establish the uniqueness of such an equilibrium.

To formally introduce each of these lemmas, we first define some auxiliary notation that we use to characterize the equilibrium strategies in the continuation game that follows the firms' choices for product 1 prices. First, given product 1 prices p_1^A and p_1^B, let us define

$$\bar{X}(p_1^A,p_1^B)=\max\left\{0,\min\left\{p_1^B-p_1^A+1/2,\,1\right\}\right\}. \tag{B.1}$$

Note that $\bar{X}(p_1^A,p_1^B)$ is a measure of the price dispersion for product 1. Let us also define the following auxiliary function that will allow us to characterize firm B's pricing strategy for product 2, as we will see in Lemma 1. We define $\bar{\theta}:[0,1]\times\mathbb{R}^{++}\to\mathbb{R}$ as

$$\bar{\theta}(x,m)=\begin{cases}\frac{1}{m}\left(2x+m-\sqrt{2x\,(2x+m)}\right), & \text{if } x\le\tilde{x}(m),\\ \frac{1}{1+x}\left[1-\frac{1}{2m}(1-x)^2\right], & \text{if } x>\tilde{x}(m),\end{cases} \tag{B.2}$$

where the cutoff value for the above two cases, $\tilde{x}:\mathbb{R}^+\to\mathbb{R}$, is given by

$$\tilde{x}(m)=\frac{1}{3}\left(\sqrt{(m-1)^2+3}-(m-1)\right). \tag{B.3}$$

The function $\bar{\theta}$ represents the type θ of a consumer that is indifferent between whether to buy product 2 from firm B or not, conditional on firm B not having perfectly observed the consumer's type after a transaction for product 1. Finally, given any product 1 prices p_1^A and p_1^B, and $t\in[0,1]$, we define

$$g(t\mid p_1^A,p_1^B)=p_1^B-p_1^A+1/2+m\left(t-\bar{\theta}\left(\bar{X}\left(p_1^A,p_1^B\right),m\right)\right)^+. \tag{B.4}$$

As we will see, this quantity will help us characterize the type of the consumer that is indifferent between buying product 1 from either firm. We show that this type satisfies $s=g(\theta\mid p_1^A,p_1^B)$, which defines a piecewise linear boundary that splits the consumer type space in two regions as in Figure 2.

Now that we have defined these functions, we can characterize the equilibrium strategies after firms set product 1 prices as follows.

Lemma 1. *Fix an assessment* $(\boldsymbol{\gamma},\sigma_1^A,\boldsymbol{\sigma}^B,\boldsymbol{\mu})$ *and let* $p_1^A=\sigma_1^A(\emptyset)$ *and* $p_1^B=\sigma_1^B(\emptyset)$*. Then, in any equilibrium, we have that*[22]

[22] We do not explicitly characterize the equilibrium outcomes when the consumer types satisfy $s=g(\theta\mid p_1^A,p_1^B)$ since the consumer is indifferent between buying product 1 from either firm. However, this occurs with probability zero and has no impact on subsequent results. The same reasoning applies for the case where $\theta=\bar{\theta}(\bar{X}\left(p_1^A,p_1^B\right),m)$.

(a) If $p_1^B - p_1^A \leq -1/2$, the consumer buys product 1 from firm B with probability one; the firm perfectly observes the type of the consumer and, in the second period, sets a price equal to the consumer's valuation for product 2, which the consumer buys.

(b) If $p_1^B - p_1^A > -1/2$, we have that

(i) If the consumer's type (s,θ) is such that $s > g(\theta \mid p_1^A, p_1^B)$, then the consumer buys product 1 from firm B with probability one; the firm perfectly observes the consumer's type and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer buys.

(ii) If the consumer's type (s,θ) is such that $s < g(\theta \mid p_1^A, p_1^B)$, then the consumer buys product 1 from firm A with probability one; firm B sets the price for product 2 equal to $p_2^B = m\bar{\theta}(\bar{X}(p_1^A, p_1^B), m)$, which the consumer buys if $\theta > \bar{\theta}(\bar{X}(p_1^A, p_1^B), m)$,

where $\bar{X}$, $\bar{\theta}$, and g are defined in (B.1), (B.2), and (B.4), respectively.

The analysis of the continuation game yields interesting managerial insights. First, note that a (forward-looking) consumer is indifferent between buying product 1 from either firm if and only if her type satisfies $s = g(\theta \mid p_1^A, p_1^B)$. This condition contrasts with the corresponding one for the restricted and myopic settings, which is given by $s = p_1^B - p_1^A + 1/2$. In particular, if $p_1^B > p_1^A - 1/2$, the indifference condition in the forward-looking setting satisfies $s = g(\theta \mid p_1^A, p_1^B) \geq p_1^B - p_1^A + 1/2$, with the inequality being strict with positive probability. This inequality captures the fact that given product 1 prices that satisfy $p_1^B > p_1^A - 1/2$, there are consumers that would choose to buy product 2 from firm B in the myopic setting but prefer to buy from firm A instead in the forward-looking setting, since, when making their purchase decision for product 1, they consider the cost of revealing their type to firm B by buying from it.[23]

In addition, we establish that, in equilibrium, firm A captures a positive share of the product 1 market if and only if $p_1^B - p_1^A > -1/2$. That is, by offering a price that is lower than firm A's by $1/2$, firm B has the ability to capture all the market of product 1 transactions and, as a result, learn the consumer's type with probability 1. Observe that this condition is exactly the same as in the setting with myopic consumers (see Proposition 2). Given that in the forward-looking setting firm B has data-tracking ability and consumers are aware of it, one might expect that firm B would need to offer a larger discount to ensure that all consumers buy from it; Lemma 1 shows that this is not the case.

The force driving this result is that as the price gap between p_1^A and p_1^B increases (with price p_1^B being the lowest), more consumers are "persuaded" to buy product 1 from firm B even though they reveal their type. Moreover, the consumers that still prefer to buy from firm A (and avoid revealing their type) are those with relatively high valuations for product 2 (i.e., high θ). However, this information is captured by firm B when updating its beliefs: as the price gap increases, firm B's beliefs assign relatively more probability to high values of θ, which leads to firm B setting a higher price for product 2 when the consumer buys from firm A in the first period. This can be seen by noting that this price is $p_2^B = m\bar{\theta}(\bar{X}(p_1^A, p_1^B), m)$, which is decreasing

[23] This observation does not rely on the assumption that types are uniformly distributed. For a general distribution, the indifference condition can also be written as $s = p_1^B - p_1^A + 1/2 + m\left(t - \bar{\theta}\left(\bar{X}\left(p_1^A, p_1^B\right), m\right)\right)^+$, but with a different expression for $\bar{\theta}$ that will depend on the distribution of types.

in p_1^B. In fact, when $p_1^B \leq p_1^A - 1/2$, we have that $\bar{\theta}(0,m) = 1$; i.e., firm B sets the price of product 2 for consumers that bought product 1 from firm A as the largest possible valuation, which induces all consumers to buy product 1 from firm B given the discount it offers relative to firm A.[24]

Finally, observe that the equilibrium in the continuation game depends only on product 1 prices p_1^A and p_1^B through the quantity $\bar{X}\left(p_1^A, p_1^B\right)$, which essentially measures the gap between these prices.[25] This property is helpful to characterize the equilibria of the game, since the problem of finding product 1 equilibrium prices can be reduced to finding the equilibrium value of $\bar{X}$. Indeed, this property allows us to reduce a two-dimensional problem to a single dimension, which simplifies our analysis considerably (see Appendix F.2).

Having characterized the equilibrium in the continuation game, we now focus on the second step of the proof, which consists of deriving the firms' profit functions in terms of product 1 prices, incorporating the subsequent equlibrium path that we characterized in Lemma 1, and defining a simultaneous-move pricing game with these functions. We define this game as follows.

DEFINITION 1. For $m > 0$, let $\mathbf{G}(m)$ be the two-player normal-form game with action spaces[26] $S^A = [0,1]$ and $S^B = [-1/2, 1]$, and profit functions π^A and π^B that we define as[27]

$$\pi^A\left(p_1^A, p_1^B, m\right) = p_1^A \psi\left(\bar{X}(p_1^A, p_1^B), m\right), \tag{B.5}$$

$$\pi^B\left(p_1^A, p_1^B, m\right) = \left(1 - \psi(\bar{X}(p_1^A, p_1^B), m)\right) p_1^B + m\phi\left(\bar{X}(p_1^A, p_1^B), m\right), \tag{B.6}$$

where $\psi, \xi, \phi : [0,1] \times \mathbb{R}^{++} \to \mathbb{R}$ are defined by

$$\psi(x,m) = 2x\bar{\theta}(x,m), \tag{B.7}$$

$$\xi(x,m) = \mathbb{E}_{\mu_0}\left[\theta; s \geq x + m\left(\theta - \bar{\theta}(x,m)\right)^+\right], \tag{B.8}$$

$$\phi(x,m) = \xi(x,m) + \frac{1}{2}\psi(x,m)\bar{\theta}(x,m). \tag{B.9}$$

While these expressions are formally derived in the proof of Claim 8, it is useful to (informally) preview their interpretation. Based on the equilibrium path described in Lemma 1, $\psi\left(\bar{X}(p_1^A, p_1^B), m\right)$ represents the expected demand for buying product 1 from firm A, given prices p_1^A and p_1^B. Since firm A participates in the market for product 1 only, its profit function is simply the product of the price it sets, p_1^A, and its expected demand, $\psi\left(\bar{X}(p_1^A, p_1^B), m\right)$.

[24] This observation also extends beyond the setting with uniformly distributed types. In particular, as long as θ is a continuous random variable that admits a density, the same intuition described here continues to hold.

[25] Indeed, the price firm B sets for product 2 if the consumer bought product 1 from firm A is $p_2^B = m\bar{\theta}(\bar{X}\left(p_1^A, p_1^B\right), m)$. In addition, when $p_1^B - p_1^A > -1/2$, we can write the condition that defines the indifferent consumer's type as $s = \bar{X}\left(p_1^A, p_1^B\right) + m\left(\theta - \bar{\theta}\left(\bar{X}\left(p_1^A, p_1^B\right), m\right)\right)^+$. The condition that induces the corner equilibrium, $p_1^B - p_1^A \leq -1/2$, can be rewritten as $\bar{X}\left(p_1^A, p_1^B\right) = 0$.

[26] Our choice of action spaces requires some justification. Note that setting a negative price is a dominated strategy for firm A (as it is better off setting a price of zero), and so no equilibria in undominated strategies are lost by requiring firm A prices to be non-negative. Once we take this into account, any price below $-1/2$ is strictly dominated for firm B (as it is better off with $p_1^B = -1/2$), and so no equilibria are lost by bounding firm B's action space below by $-1/2$. Claim 19 in Appendix B.3 shows that restricting firms' action spaces to S^A and S^B results in no loss of equilibria in undominated strategies for $\mathbf{G}(m)$.

[27] We include the dependency of profit functions on the parameter m explicitly.

Similarly, firm B's expected profit associated with product 1 is $p_1^B\left(1-\psi(\bar{X}(p_1^A,p_1^B),m)\right)$. However, firm B also sells product 2 in the second period, which results in a profit that is represented by the term $m\phi\left(\bar{X}(p_1^A,p_1^B),m\right)$ in (B.6). This last term is further split into two components: first, the term $m\xi\left(\bar{X}(p_1^A,p_1^B),m\right)$ represents the profit firm B makes by selling product 2 to a consumer that previously bought product 1 from it, and therefore revealed her valuation for product 2,[28] while the remaining term in (B.9) represents the expected product 2 profit in the event that the consumer buys product 1 from firm A instead.

The second step of our proof concludes by establishing the following relationship between the equilibria of our original game and those in game $\mathbf{G}(m)$. Formally,

Lemma 2. *If $\mathbf{G}(m)$ admits a pure-strategy Nash equilibrium (p_1^{A*},p_1^{B*}), then there exists an equilibrium in the forward-looking setting with $(\sigma_1^A(\emptyset),\sigma_1^B(\emptyset))=(p_1^{A*},p_1^{B*})$.*

Conversely, if $(\boldsymbol{\gamma},\sigma_1^A,\boldsymbol{\sigma}^B,\mu)$ is an equilibrium in the forward-looking setting, then $(p_1^{A*},p_1^{B*})=(\sigma_1^A(\emptyset),\sigma_1^B(\emptyset))$ is a pure-strategy Nash equilibrium in $\mathbf{G}(m)$.

In the third and final step, we characterize the pure-strategy Nash equilibria of the game $\mathbf{G}(m)$. Formally, we show that

Lemma 3. *There exist constants m_L,m_H with $0<m_L\le m_H$ such that*

1. *If $m<m_L$ (Interior equilibrium regime): $\mathbf{G}(m)$ admits a unique pure-strategy Nash equilibrium (p_1^A,p_1^B). Moreover, these prices satisfy $p_1^A>0$ and $p_1^B>p_1^A-1/2$.*
2. *If $m>m_H$ (Corner equilibrium regime): $(p_1^A,p_1^B)=(0,-1/2)$ is the unique pure-strategy Nash equilibrium in $\mathbf{G}(m)$.*

Note that $\mathbf{G}(m)$ is an asymmetric pricing game where, crucially, firm B's profit function is not quasi-concave in p_1^B in general (see Appendix B.3). In particular, firm B's best-response correspondence is not convex-valued everywhere, as illustrated in Figure 8. Indeed, due to this fact, the game $\mathbf{G}(m)$ admits no pure-strategy Nash equilibria for some values of m. Therefore, to prove Lemma 3, we directly characterize the firms' best-response correspondences for $\mathbf{G}(m)$ and establish conditions that ensure existence of a pure-strategy Nash equilibrium, as well as its uniqueness (when it exists). We provide an overview of the proof of Lemma 3 in Appendix B.3, which consists of a series of claims. In the interest of clarity, we have relegated the proofs of these claims to Appendix F.1 as they are, for the most part, algebraic exercises.

Moreover, in Appendix F.2, we establish additional results to fully characterize the set of values of m for which $\mathbf{G}(m)$ admits a pure-strategy Nash equilibrium, and leverage these results to approximately compute the values of m_L and m_H, and find that $m_L\approx 3.98$ and $m_H\approx 4.02$. This implies that an equilibrium may not exist only in a small interval of the parameter space (i.e., $m\in(3.98,4.02)$). In fact, for some of these values, we indeed can show that no equilibrium exists[29] (for instance, for $m=4$).

In what follows, Appendices B.1, B.2, and B.3 provide the proofs of Lemmas 1, 2, and 3, respectively.

[28] To see this, note that $\xi\left(\bar{X}(p_1^A,p_1^B),m\right)$ is defined as the expected value of θ, conditional on the consumer preferring to buy product 1 from firm B, multiplied by the probability that this event occurs.

[29] We require the firms' pricing strategies to be pure strategies. However, Nash equilibria would exist if we allowed for mixed strategies. However, this may not be a particularly illuminating exercise given that pure-strategy equilibria exist for most of the parameter space.

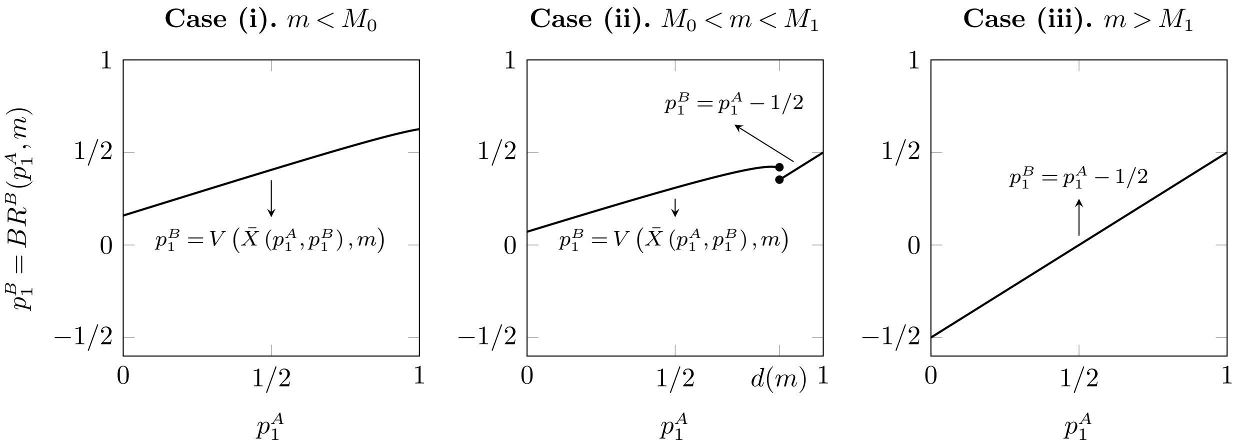

Figure 8 Illustration of the three possible shapes of firm B's best-response correspondence in G(m), BR^B, as we establish in Claim 16. The three charts above display the shape of $BR^B(p_1^A, m)$ for $m \in \{1/2, 1, 5\}$, from left to right. As can be seen in the second panel, $BR^B(p_1^A, m)$ need not be convex-valued for all $p_1^A \in S^A$ when $M_0 \leq m \leq M_1$

B.1. Proof of Lemma 1

To prove Lemma 1, we establish some necessary conditions that any equilibrium $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ must satisfy. Later on, when proving Lemma 2, we show that these conditions are indeed sufficient.

To characterize the equilibrium conditions, we proceed by backwards induction in a series of claims as follows. Claim 1 characterizes the consumer's purchasing strategy for product 2. Claim 2 then pins down firm B's pricing strategy for product 2, in the subset of histories where the consumer buys product 1 from firm B, and therefore the firm perfectly observes the consumer's type. Then, Claim 3 provides the form of the consumer's purchasing strategy for product 1. Next, in Claim 4, we show that the probability of the consumer being indifferent between buying product 1 from either firm is zero, which simplifies the argument later on. Claim 5 then characterizes firm B's beliefs on the consumer's type, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability. Then, given these beliefs, we obtain the closed-form expression for firm B's pricing strategy for product 2 following such histories in Claim 6. Finally Claim 7 provides a closed-form expression for the probability that the consumer buys product 1 from firm A as a function of product 1 prices and, in particular, establishes that this probability is zero when product 1 prices satisfy $p_1^B \leq p_1^A - 1/2$. In what follows, we state and prove these claims, and explain the relationships between them in more detail. Then, we prove Lemma 1.

We start by characterizing the consumer's purchasing strategy in the second period, which follows a simple form: buying when the price is lower than her valuation for product 2, not buying in the opposite case, and mixing between both actions when indifferent.

Claim 1. *In any equilibrium, given any history $h \in H_2^c$, γ_2 takes the following form:*

$$\gamma_2(h) = \begin{cases} 1, & \text{if } m\theta(h) > p_2^B(h) \\ q(h) \in [0,1], & \text{if } m\theta(h) = p_2^B(h) \\ 0, & \text{if } m\theta(h) < p_2^B(h). \end{cases} \tag{B.10}$$

Proof of Claim 1. Given a history $h \in H_2^c$, by sequential rationality in period 2, the consumer chooses a_2 to maximize $a_2\left(m\theta(h) - p_2^B(h)\right)$. It follows that she is only indifferent when $m\theta(h) = p_2^B(h)$. □

Next, we show that when firm B observes the consumer's type, it sets the price for product 2 equal to the consumer's valuation and the consumer buys the product with probability 1.

Claim 2. *Suppose that strategy $\boldsymbol{\gamma}$ satisfies the condition in Claim 1. Fix any history $h \in H_2^f$ where the consumer bought product 1 from firm B (i.e., $a_1(h) = 1$). Then, in any equilibrium, we have that $\sigma_2^B(h) = m\theta(h)$ with probability 1. Moreover, we have that the mixing probability for γ_2 at the resulting subsequent history of h is $q(\langle h, m\theta(h)\rangle) = 1$ (see (B.10)).*

Proof of Claim 2. Since $a_1(h) = 1$, we have that $\mathcal{J}(h) = h$. Then, by consistency of beliefs, we have that, in equilibrium, $\mu_2(\cdot \mid h)$ assigns probability 1 to the true type $(s(h), \theta(h))$. Moreover, notice that it is always suboptimal for firm B to choose p_2^B outside of $[0, m]$ (since this is the support of consumer valuations for product 2). Thus, sequential rationality for firm B implies that

$$\sigma_2^B(h) \in \underset{p_2^B \in [0,m]}{\arg\max}\, p_2^B \left(\mathbf{1}_{\{x:\, x < m\theta(h)\}}(p_2^B) + q\left(\langle h, p_2^B\rangle\right) \mathbf{1}_{\{x:\, x = m\theta(h)\}}(p_2^B)\right).$$

Notice that an equilibrium can exist only if $q(\langle h, p_2^B\rangle) = 1$, in which case $\sigma_2^B(h) = m\theta(h)$. Otherwise, if $q(\langle h, p_2^B\rangle) < 1$, then all prices of the form $m\theta(h) - \varepsilon$ for $\varepsilon > 0$ small enough result in higher profit than $m\theta(h)$. □

Next, we derive the consumer's equilibrium strategy for period 1, given any strategies γ_2 and σ_2^B that satisfy the conditions given in the previous two claims.

Claim 3. *Suppose that strategies $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma^B}$ satisfy the conditions in Claims 1 and 2, respectively. Given any product 1 prices p_1^A, p_1^B, and any $t \in [0, 1]$, define*[30]

$$\tilde{g}(t \mid p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 + \mathbb{E}\left[\left(mt - p_2^B\right)^+\right], \tag{B.11}$$

where the expectation is taken w.r.t. p_2^B, and $p_2^B \sim \sigma_2^B(p_1^A, p_1^B)$. Then, in any equilibrium, given a consumer history $h \in H_1^c$, γ_1 takes the following form:

$$\gamma_1(h) = \begin{cases} 1, & \text{if } s(h) > \tilde{g}(\theta(h) \mid p_1^A(h), p_1^B(h)) \\ \beta(h) \in [0,1], & \text{if } s(h) = \tilde{g}(\theta(h) \mid p_1^A(h), p_1^B(h)) \\ 0, & \text{if } s(h) < \tilde{g}(\theta(h) \mid p_1^A(h), p_1^B(h)). \end{cases} \tag{B.12}$$

Proof of Claim 3. Given history $h = (s, \theta, p_1^A, p_1^B) \in H_1^c$, the consumer's total utility when buying product 1 from firm B ($a_1 = 1$) is

$$U_1 = u_1(1; h) = \left(\bar{u} - (1-s)/2 - p_1^B\right),$$

as, by assumption, σ_2^B and γ_2 satisfy the conditions in Claims 1 and 2, respectively. That is, firm B learns the consumer's type and sets a price equal to the consumer's valuation for product 2. On the other hand, choosing not to buy from firm B ($a_1 = 0$) yields

$$U_1 \;=\; u_1(0; h) + \mathbb{E}\left[u_2\left(a_2; s, \theta, p_2^B\right) \mid s, \theta\right] \;=\; \left(\bar{u} - s/2 - p_1^A\right) + \mathbb{E}\left[\left(m\theta - p_2^B\right)^+ \,\middle|\, \theta\right],$$

[30] $\tilde{g}$ is defined given a strategy σ_2^B that satisfies the condition of Claim 2. By contrast, g, as defined in (B.4), results from plugging in the form of $\sigma_2^B\left(p_1^A, p_1^B\right)$ that we derive in Claim 6. Thus, in equilibrium we will have that $g = \tilde{g}$.

where the equality follows since p_2^B and a_2 are chosen according to σ_2^B and γ_2, respectively. In addition, the last expectation is taken w.r.t. $p_2^B \sim \sigma_2^B\,(p_1^A, p_1^B)$.

The consumer is indifferent between both actions only if the previous two expressions are equal, which is equivalent to $s = \tilde{g}(\theta \mid p_1^A, p_1^B)$. □

Given any choice for product 1 prices p_1^A, p_1^B, and any pair of strategies $\boldsymbol{\gamma}$, σ_2^B that satisfy the previous claims, we split the consumer type space according to the behavior induced by γ_1. We define $\mathcal{A}_1(p_1^A, p_1^B)$ as the set of consumer types that strictly prefer buying from firm B given those prices. Analogously, let $\mathcal{A}_0(p_1^A, p_1^B)$ and $\mathcal{A}_I(p_1^A, p_1^B)$ be the set of consumer types that strictly prefer to buy from firm A and are indifferent between both firms, respectively. By Claim 3, we have that

$$\begin{aligned} \mathcal{A}_1(p_1^A, p_1^B) &= \left\{(s,\theta) \in [0,1]\times[0,1] : s > \tilde{g}(\theta \mid p_1^A, p_1^B)\right\}, \\ \mathcal{A}_0(p_1^A, p_1^B) &= \left\{(s,\theta) \in [0,1]\times[0,1] : s < \tilde{g}(\theta \mid p_1^A, p_1^B)\right\}, \\ \mathcal{A}_I(p_1^A, p_1^B) &= \left\{(s,\theta) \in [0,1]\times[0,1] : s = \tilde{g}(\theta \mid p_1^A, p_1^B)\right\}. \end{aligned} \tag{B.13}$$

We now show that any history in which the consumer is indifferent between buying product 1 from either firm occurs with probability zero in an equilibrium.

CLAIM 4. *Suppose that strategies $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma^B}$ satisfy the conditions in Claim 3. Then, given product 1 prices p_1^A and p_1^B, in any equilibrium we have that $\mu_0\left(\mathcal{A}_I(p_1^A, p_1^B)\right) = 0$.*

Proof of Claim 4. The set of consumer types (s,θ) that satisfy $s = \tilde{g}(\theta \mid p_1^A, p_1^B)$ has Lebesgue measure zero, and therefore has measure zero (relative to μ_0). □

The next result characterizes the equilibrium beliefs following a history where the consumer buys product 1 from A, assuming that this event occurs with positive probability, i.e., assuming that the event $\mathcal{A}_0(p_1^A, p_1^B)$ occurs with positive probability if players follow $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$. In what follows, we denote by $\mu_2^\theta(\cdot \mid I)$ the CDF of the marginal distribution of θ induced by μ_2, given $I \in \mathcal{I}_2$, i.e., for $t \in [0,1]$,

$$\mu_2^\theta(t \mid I) = \mu_2\left([0,1]\times[0,t] \mid I\right) = \int_{[0,1]\times[0,t]} d\mu_2(s,\theta \mid I).$$

CLAIM 5. *Fix prices p_1^A, p_1^B and suppose that strategies $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma^B}$ satisfy the conditions in Claim 3. Let $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$, and suppose that $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) > 0$. Then, $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$ at I if and only if $\mu_2(\cdot \mid I)$ is the uniform probability measure on $\mathcal{A}_0(p_1^A, p_1^B)$. In addition, $\mu_2^\theta(\cdot \mid I)$ admits a density that is proportional to*

$$\bar{g}(z \mid p_1^A, p_1^B) = \max\left\{0, \min\left\{\tilde{g}(z \mid p_1^A, p_1^B), 1\right\}\right\}. \tag{B.14}$$

Proof of Claim 5. Let B be a Borel set in the unit square. By Bayes' rule, and since $\mu_1(\emptyset) = \mu_0$ by definition, we have that

$$\mu_2(B \mid I) = \mathbb{P}_{\mu_0}\left[(s,\theta) \in B \middle| I, \boldsymbol{\gamma}, \boldsymbol{\sigma}\right] = \frac{\mu_0\left(B \cap \mathcal{A}_0(p_1^A, p_1^B)\right)}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)} = \frac{\int_B I_{\mathcal{A}_0(p_1^A, p_1^B)}(s,\theta)\, d(s,\theta)}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)}.$$

By Fubini's theorem and the definition of $\mathcal{A}_0(p_1^A, p_1^B)$, the marginal CDF of θ induced by $\mu_2(\cdot \mid I)$ satisfies

$$\mu_2^\theta(t \mid I) = \frac{1}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)} \int_0^t \int_0^{\bar{g}(\theta \mid p_1^A, p_1^B)} ds\, d\theta = \frac{1}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)} \int_0^t \bar{g}(\theta \mid p_1^A, p_1^B)\, d\theta,$$

as desired. □

The next claim characterizes the form of firm B's pricing strategy for product 2 in an equilibrium, following histories in which the consumer does not buy product 1 from firm B, assuming that this event occurs with positive probability.

Claim 6. *Fix prices p_1^A, p_1^B and suppose that strategies $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma^B}$ satisfy the conditions in Claim 3. Let $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$, and suppose that $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) > 0$. Then, in any equilibrium, we have that $\sigma_2^B(I) = m\bar{\theta}\left(\bar{X}(p_1^A, p_1^B), m\right)$ with probability 1, where $\bar{\theta}$ is defined as in (B.2).*

Proof of Claim 6. Fix a history $h = (s, \theta, p_1^A, p_1^B, a_1) \in H_2^f$ such that the consumer does not buy product 1 from firm B, i.e., such that $a_1 = 0$. Let $I = \mathcal{J}(h) = (p_1^A, p_1^B)$ be the information vector observed by firm B following such a history. The proof consists of two parts. First, we show that in any equilibrium, there exists $t^*(I) \in [0,1]$ such that $\sigma_2^B(I) = mt^*(I)$ with probability 1. In the second part, we show that $t^*(I) = \bar{\theta}\left(\bar{X}(p_1^A, p_1^B), m\right)$.

Consider firm B's pricing decision for product 2 given information I. Since γ_2 satisfies Claim 1 and the consumer's valuation for product 2 is in $[0,m]$, it is suboptimal for firm B to choose p_2^B outside of $[0,m]$. Then, by sequential rationality, $\sigma_2^B(I)$ takes values in $[0,m]$, in any equilibrium. We will now show that there exists $t^*(I) \in [0,1]$ such that $\sigma_2^B(I) = mt^*(I)$ with probability 1.

To do so, we first write firm B's continuation profit when setting the price of product 2, given I. First, given the beliefs that $(s,\theta) \sim \mu_2(\cdot \mid I)$, Claims 4 and 5 imply that in any equilibrium we have that $\mu_2\left(\mathcal{A}_I(p_1^A, p_1^B) \mid I\right) = 0$ and, moreover, by Claim 5 we have that $\mu_2^\theta(\cdot \mid I)$ is atomless. Then, given γ_2 and information I, firm B's expected continuation profit at price $p_2^B \in [0,m]$ is

$$p_2^B\left(1 - \mu_2^\theta(p_2^B/m \mid I)\right).$$

We change variables by letting $p_2^B = mt$ with $t \in [0,1]$ and use the definition of μ_2^θ to rewrite the previous expression as $mF(t)$, with

$$F(t) = t\left(1 - \mu_2^\theta(t \mid I)\right) = \frac{t}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)} \int_t^1 \bar{g}(z \mid p_1^A, p_1^B)dz. \tag{B.15}$$

Consider the problem of maximizing $F(t)$ for $t \in [0,1]$. A solution to this problem exists as F is a continuous function and $[0,1]$ is compact. In addition, F is differentiable in $(0,1)$ and

$$F'(t) = 1 - \mu_2^\theta(t \mid I) - \frac{t}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)}\bar{g}(t \mid p_1^A, p_1^B). \tag{B.16}$$

Note that $F(0) = F(1) = 0$, and also that $F'(0) = 1$. Thus, there exist values of $t \in (0,1)$ such that $F(t) > 0$. Hence, any solution $t^* \in \arg\max_{t \in [0,1]} F(t)$ lies in $(0,1)$, and satisfies $F'(t^*) = 0$.

We claim that this optimization problem has a unique solution. To see this, let $t_0 = \inf\{t \in [0,1] : \mu_2^\theta(t \mid I) > 0\}$ and observe that $\bar{g}(t \mid p_1^A, p_1^B)$ is non-decreasing in t (by (B.11) and (B.14)). Then, by (B.16), we have that $F'(t) = 1$ for $t < t_0$, and that $F'(t)$ is strictly decreasing for $t_0 \le t \le 1$. Therefore, there exists a unique $t^* \in [0,1]$ that maximizes $F(t)$. By sequential rationality, we have that $\sigma_2^B(I) = mt^*$ with probability 1, which concludes the first step of the proof.

We now show that $t^* = \bar{\theta}(\bar{X}(p_1^A, p_1^B), m)$, where $\bar{\theta}$ is defined as in (B.2). To derive this expression, first notice that as $\sigma_2^B(I) = mt^*$, it follows from (B.14) that

$$\bar{g}(t \mid p_1^A, p_1^B) = \max\left\{0, \min\left\{p_1^B - p_1^A + 1/2 + m\,(t - t^*)^+\,,\,1\right\}\right\},$$

and that $\bar{g}(t^* \mid p_1^A, p_1^B) = \bar{X}(p_1^A, p_1^B)$. In addition, notice that

$$\mu_2^\theta(t^* \mid I) = \frac{1}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)} \int_0^{t^*} \bar{g}(t \mid p_1^A, p_1^B)dt = \frac{t^* \bar{X}(p_1^A, p_1^B)}{\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right)}.$$

Moreover, t^* satisfies $F'(t^*) = 0$. Plugging the previous two equations back into (B.16) yields

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = 2\bar{X}(p_1^A, p_1^B)t^*. \tag{B.17}$$

Moreover, we have assumed that $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) > 0$, which implies that $\bar{X}(p_1^A, p_1^B) > 0$, and therefore $p_1^B - p_1^A + 1/2 > 0$. In particular, we can write

$$\bar{g}(t \mid p_1^A, p_1^B) = \min\left\{\bar{X}(p_1^A, p_1^B) + m\,(t - t^*)^+\,,\,1\right\}.$$

On the other hand, by Claim 5 we have that

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = \mu_0\left(\{s < \tilde{g}(\theta \mid p_1^A, p_1^B)\}\right) = \int_0^1 \bar{g}(t \mid p_1^A, p_1^B)dt = \int_0^1 \min\left\{\bar{X} + m\,(t - t^*)^+\,,\,1\right\}dt, \tag{B.18}$$

where we abbreviate $\bar{X} = \bar{X}(p_1^A, p_1^B)$. We then solve the system given by equations (B.17) and (B.18) to obtain the expression for t^*, which, as we argued above, exists and is uniquely defined in $[0,1]$. By computing the integral on (B.18), we obtain that

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = \begin{cases} \bar{X} + \frac{1}{2}m\,(1 - t^*)^2\,, & \text{if } \bar{X} + m(1 - t^*) \le 1. \\ \bar{X} + (1 - \bar{X})(1 - t^*) - \frac{1}{2m}\left(1 - \bar{X}\right)^2, & \text{if } \bar{X} + m(1 - t^*) > 1. \end{cases} \tag{B.19}$$

Plugging (B.19) into (B.17) and some simple algebra yields $t^* = \bar{\theta}(\bar{X}, m)$, as desired. □

The next claim provides an expression for the probability that the consumer buys product 1 from firm A, given product 1 prices p_1^A and p_1^B, i.e., the probability (under μ_0) that the event $\mathcal{A}_0(p_1^A, p_1^B)$ occurs.

Claim 7. *Fix strategies $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}^{\boldsymbol{B}}$ that satisfy the conditions in Claims 3 and 6, respectively. Then, for any prices $p_1^A \in S^A$ and $p_1^B \in S^B$, we have that*

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = \psi\left(\bar{X}(p_1^A, p_1^B), m\right),$$

where ψ is defined as in (B.7). In particular, $\mathcal{A}_0(p_1^A, p_1^B)$ occurs with positive probability under μ_0 if and only if $p_1^B - p_1^A > -1/2$.

Proof of Claim 7. First assume that $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) > 0$. It follows from (B.17) in the proof of Claim 6 that we can write this probability as

$$\psi\left(\bar{X}(p_1^A, p_1^B), m\right) = 2\bar{X}(p_1^A, p_1^B)\bar{\theta}\left(\bar{X}(p_1^A, p_1^B), m\right) > 0,$$

which implies that $\bar{X}(p_1^A, p_1^B) > 0$, and therefore $p_1^B - p_1^A > -1/2$.

Now assume that $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = 0$. We claim that $\bar{X}(p_1^A, p_1^B) = 0$ (i.e., that $p_1^B - p_1^A \le -1/2$). To show this, assume by way of contradiction that $p_1^B - p_1^A > -1/2$. Then, there exists $\varepsilon > 0$ such that $p_1^B - p_1^A + 1/2 > \varepsilon$, which implies that $\bar{g}(t \mid p_1^A, p_1^B) > \varepsilon$ for all $t \in [0,1]$. It follows that

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) = \mu_0\left(\{s < \tilde{g}(\theta \mid p_1^A, p_1^B)\}\right) > \mu_0\left(\{s < \varepsilon\}\right) = \min\{\varepsilon, 1\} > 0,$$

a contradiction. Therefore, $\bar{X}(p_1^A, p_1^B) = 0$. The result follows since $\psi(0, m) = 0$ for all $m > 0$. □

Finally, we leverage the previous claims to prove Lemma 1.

Proof of Lemma 1. First suppose that $p_1^B - p_1^A \leq -1/2$. By Claims 4 and 7, we have that $\mu_0\left(\mathcal{A}_1(p_1^A, p_1^B)\right) = 1$; i.e., the consumer buys product 1 from firm B with probability 1. Since firm B observes the consumer's type if she buys product 1 from it, we have that firm B indeed observes the type with probability 1, and, by Claim 2, sets the price of product 2 equal to the consumer's valuation for that product, which the consumer buys with probability 1.

Now suppose that $p_1^B - p_1^A > -1/2$. By Claim 6, we have that $\tilde{g}(t \mid p_1^A, p_1^B) = g(t \mid p_1^A, p_1^B)$, for all $t \in [0,1]$, where these expressions are given in (B.11) and (B.4), respectively.

Then, by Claim 3, we have that if the consumer's type satisfies $s < g(\theta \mid p_1^A, p_1^B)$, she buys product 1 from firm A with probability 1. If that is the case, then, by Claim 6, firm B sets the price of product 2 equal to $p_2^B = m\bar{\theta}(\bar{X}\left(p_1^A, p_1^B\right), m)$. By Claim 1, if the consumer's type is such that $\theta > \bar{\theta}(\bar{X}\left(p_1^A, p_1^B\right), m)$, she buys product 2 with probability 1.

Similarly, Claim 3 implies that if the consumer's type satisfies $s > g(\theta \mid p_1^A, p_1^B)$, she buys product 1 from firm B with probability 1. By Claim 2, firm B then sets the price of product 2 equal to the consumer's valuation (i.e., $m\theta$), and the consumer buys it with probability 1. $\square$

B.2. Proof of Lemma 2

The proof of Lemma 2 consists of two steps. First, in Claim 8, we formally derive firms' profit functions in terms of product 1 prices, assuming that the consumer and firm B play equilibrium strategies following the firms' choices for product 1 prices. We show that these profit functions are indeed the profit functions of the auxiliary game $\mathbf{G}(m)$. In the second step (Claim 9), we show that given a pure-strategy Nash equilibrium for $\mathbf{G}(m)$, we can construct an equilibrium for the original game.

Claim 8. *Fix an assessment $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ that satisfies the conditions of Lemma 1, and let $p_1^A = \sigma_1^A(\emptyset)$ and $p_1^B = \sigma_1^B(\emptyset)$. Then,*

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \pi^A\left(p_1^A, p_1^B, m\right), \quad \textit{and} \quad \Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \pi^B\left(p_1^A, p_1^B, m\right),$$

where π^A and π^B are defined as in (B.5) and (B.6), respectively.

Proof of Claim 8. The proof consists of computing the firms' profit functions while taking into account the necessary conditions for equilibrium given in Lemma 1. Note that these necessary conditions fully characterize the equilibrium after product 1 prices are set, except for events that occur with probability zero.[31] First, consider firm A. By the definition of A's total payoff and as $\mu_1(\emptyset) = \mu_0$, we have that

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \mathbb{E}_{\mu_0}\left[\pi_1^A\left(p_1^A, p_1^B, a_1\right)\right],$$

where the expectation is taken with respect to the consumer's type and a_1, which is chosen according to strategy γ_1, which, by assumption, satisfies Claim 3. Recall that firm A participates in period 1 only, and so its total payoff is $\pi_1^A\left(p_1^A, p_1^B, a_1\right) = p_1^A\left(1 - a_1\right)$. Thus,

$$\begin{aligned}\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) &= \mathbb{E}_{\mu_0}\left[\pi_1^A\left(p_1^A, p_1^B, a_1\right)\right] = \mathbb{E}_{\mu_0}\left[p_1^A\left(1 - \gamma_1(s, \theta, p_1^A, p_1^B)\right)\right] \\ &= p_1^A \mu_0\left(\{(s,\theta) \in \mathcal{A}_0(p_1^A, p_1^B)\}\right) = p_1^A \psi\left(\bar{X}(p_1^A, p_1^B), m\right) = \pi^A\left(p_1^A, p_1^B, m\right),\end{aligned}$$

[31] Namely, the event that the consumer's type satisfies either $s = g(\theta \mid p_1^A, p_1^B)$ or $\theta = \bar{\theta}\left(p_1^A, p_1^B, m\right)$.

where the third and fourth equalities follow from Claims 4 and 7, respectively, and $\mathcal{A}_0(p_1^A, p_1^B)$ and ψ are defined as in (B.13) and (B.7), respectively.

Now consider firm B. Again, since $\mu_1(\emptyset) = \mu_0$, we have that

$$\Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \mathbb{E}_{\mu_0}\left[\pi_1^B\left(p_1^A, p_1^B, a_1\right) + \pi_2^B\left(p_2^B, a_2\right)\right],$$

where the expectation is taken with respect to the consumer's type and the consumer's actions a_1, a_2, which are chosen according to γ_1 and γ_2, respectively, and p_2^B is chosen according to σ_2^B. Following the same argument as with firm A, we have that

$$\mathbb{E}_{\mu_0}\left[\pi_1^B\left(p_1^A, p_1^B, a_1\right)\right] = p_1^A\left(1 - \psi\left(\bar{X}(p_1^A, p_1^B), m\right)\right). \tag{B.20}$$

Moreover, since the probability that the consumer is indifferent between buying product 1 from either firm is zero (by Claim 4), we have that

$$\mathbb{E}_{\mu_0}\left[\pi_2^B\left(p_2^B, a_2\right)\right] = \mathbb{E}_{\mu_0}\left[p_2^B a_2 \mathbf{1}_{\mathcal{A}_0(p_1^A, p_1^B)}(s,\theta)\right] + \mathbb{E}_{\mu_0}\left[p_2^B a_2 \mathbf{1}_{\mathcal{A}_1(p_1^A, p_1^B)}(s,\theta)\right]. \tag{B.21}$$

We now analyze these two components separately. For the event where the consumer buys product 1 from firm B, we have (by Claim 7 and abbreviating $\bar{X} = \bar{X}(p_1^A, p_1^B)$) that

$$\begin{aligned}\mathbb{E}_{\mu_0}\left[p_2^B a_2 \mathbf{1}_{\mathcal{A}_1(p_1^A, p_1^B)}(s,\theta)\right] &= \mu_0\left(\mathcal{A}_1(p_1^A, p_1^B)\right)\mathbb{E}_{\mu_0}\left[m\theta \mid (s,\theta) \in \mathcal{A}_1(p_1^A, p_1^B)\right] \\ &= m\left(1 - \psi(\bar{X}, m)\right)\mathbb{E}_{\mu_0}\left[\theta \middle| s > \bar{X} + m\left(\theta - \bar{\theta}\left(\bar{X}, m\right)\right)^+\right] \\ &= m\xi\left(\bar{X}, m\right),\end{aligned} \tag{B.22}$$

where the last step simply uses the definition of ξ in (B.8). On the other hand, for the event where the consumer buys product 1 from firm A, we have (by Claims 1 and 6) that

$$\begin{aligned}\mathbb{E}_{\mu_0}\left[p_2^B a_2 \mathbf{1}_{\mathcal{A}_0(p_1^A, p_1^B)}(s,\theta)\right] &= \mathbb{E}_{\mu_0}\left[m\bar{\theta}(\bar{X}, m)\gamma_2(s, \theta, p_1^A, p_1^B, 0)\mathbf{1}_{\mathcal{A}_0(p_1^A, p_1^B)}(s,\theta)\right] \\ &= \mathbb{E}_{\mu_0}\left[m\bar{\theta}(\bar{X}, m)\mathbf{1}_{\mathcal{A}_0(p_1^A, p_1^B)\cup\{\theta > \bar{\theta}(\bar{X}, m)\}}(s,\theta)\right] \\ &= m\bar{\theta}(\bar{X}, m)\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B) \cap \{\theta > \bar{\theta}(\bar{X}, m)\}\right).\end{aligned} \tag{B.23}$$

We now compute the probability $\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B) \cap \{\theta > \bar{\theta}(\bar{X}, m)\}\right)$. We do this in two steps; first notice that

$$\begin{aligned}\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B) \cap \{\theta > \bar{\theta}(\bar{X}, m)\}\right) &= \mu_0\left(\mathcal{A}_0(p_1^A, p_1^B)\right) - \mu_0\left(\mathcal{A}_0(p_1^A, p_1^B) \cap \{\theta \le \bar{\theta}(\bar{X}, m)\}\right) \\ &= \psi(\bar{X}, m) - \mu_0\left(\{\theta \le \bar{\theta}(\bar{X}, m), s < g(\theta \mid p_1^A, p_1^B)\}\right),\end{aligned} \tag{B.24}$$

where we use the definition of $\mathcal{A}_0(p_1^A, p_1^B)$, given in (B.13), and Claim 7 to obtain the last equality.

We then compute $\mu_0\left(\{\theta \le \bar{\theta}(\bar{X}, m), s < g(\theta \mid p_1^A, p_1^B)\}\right)$:

$$\begin{aligned}\mu_0\left(\{\theta \le \bar{\theta}(\bar{X}, m), s < g(\theta \mid p_1^A, p_1^B)\}\right) &= \int_0^{\bar{\theta}(\bar{X}, m)}\int_0^{g(\theta \mid p_1^A, p_1^B)} ds d\theta = \int_0^{\bar{\theta}(\bar{X}, m)} g(\theta \mid p_1^A, p_1^B) d\theta \\ &= \int_0^{\bar{\theta}(\bar{X}, m)} \bar{X} d\theta = \bar{X}\bar{\theta}(\bar{X}, m),\end{aligned} \tag{B.25}$$

where the final step follows since $g(\theta \mid p_1^A, p_1^B) = \bar{X}$ for $\theta \le \bar{\theta}(\bar{X}, m)$. Plugging this expression back into (B.24) and using the fact that $\psi(\bar{X}, m) = 2\bar{X}\bar{\theta}(\bar{X}, m)$ results in

$$\mu_0\left(\mathcal{A}_0(p_1^A, p_1^B) \cap \{\theta > \bar{\theta}(\bar{X}, m)\}\right) = \psi(\bar{X}, m) - \bar{X}\bar{\theta}(\bar{X}, m) = \psi(\bar{X}, m)/2. \tag{B.26}$$

Thus, we can rewrite (B.23) as

$$\mathbb{E}_{\mu_0}\left[p_2^B a_2 \mathbf{1}_{\mathcal{A}_0(p_1^A,p_1^B)}(s,\theta)\right] = m\psi(\bar{X},m)\bar{\theta}(\bar{X},m)/2.$$

Plugging this expression and (B.22) back into (B.21) results in

$$\mathbb{E}_{\mu_0}\left[\pi_2^B\left(p_2^B,a_2\right)\right] = m\left(\xi\left(\bar{X},m\right)+\psi(\bar{X},m)\bar{\theta}(\bar{X},m)/2\right) = m\phi\left(\bar{X},m\right),$$

where ϕ is defined as in (B.9). Finally, it follows from (B.20) that

$$\Pi_1^B\left(\boldsymbol{\gamma},\sigma_1^A,\boldsymbol{\sigma}^B,\boldsymbol{\mu}\,|\,\emptyset\right) = p_1^A\left(1-\psi\left(\bar{X}(p_1^A,p_1^B),m\right)\right)+m\phi\left(\bar{X}(p_1^A,p_1^B),m\right) = \pi^B\left(p_1^A,p_1^B,m\right),$$

as desired. □

Next, we establish that given a pure-strategy Nash equilibrium in $\mathbf{G}(m)$, we can construct an equilibrium for the forward-looking setting.

CLAIM 9. *If $\mathbf{G}(m)$ admits a pure strategy Nash equilibrium (p_1^{A*},p_1^{B*}), then there exists an equilibrium in the forward-looking setting with $(\sigma_1^A(\emptyset),\sigma_1^B(\emptyset))=(p_1^{A*},p_1^{B*})$.*

Proof of Claim 9. Suppose that there exists a PSNE (p_1^{A*},p_1^{B*}) in $\mathbf{G}(m)$. We will construct an equilibrium $(\boldsymbol{\gamma},\sigma_1^A,\boldsymbol{\sigma}^B,\boldsymbol{\mu})$ for the forward-looking setting. To do so, we rely on the necessary conditions for equilibrium established in Lemma 1, adding some refinements to fully define the strategies and beliefs at histories that occur with probability zero.

Let us first define consumer strategies. For period 1, given a history $h\in H_1^c$, define γ_1 by

$$\gamma_1(h)=\begin{cases}1, & \text{if } s(h)\geq g\left(\theta(h)\,|\,p_1^A(h),p_1^B(h)\right),\\ 0, & \text{if } s(h)< g\left(\theta(h)\,|\,p_1^A(h),p_1^B(h)\right),\end{cases}$$

where g is defined as in (B.4). For the second period, given a history $h\in H_2^c$, define γ_2 by

$$\gamma_2(h)=\begin{cases}1, & \text{if } m\theta(h)\geq p_2^B(h),\\ 0, & \text{if } m\theta(h)< p_2^B(h).\end{cases}$$

For firm A, simply take $\sigma_1^A(\emptyset)=p_1^{A*}$. For firm B, let the pricing strategy for product 1 be $\sigma_1^B(\emptyset)=p_1^{B*}$. For product 2, given $I\in\mathcal{I}_2$, define

$$\sigma_2^B(I)=\begin{cases}m\theta(I), & \text{if } I\in\mathcal{I}_2^1,\\ m\bar{\theta}\left(\bar{X}\left(p_1^A(I),p_1^B(I)\right),m\right), & \text{if } I\in\mathcal{I}_2^0,\end{cases}$$

where $\mathcal{I}_2^0$ and $\mathcal{I}_2^1$ are defined as in (A.2). Finally, we complete the definition of our proposed assessment by defining a belief system as follows. For period 1, simply let $\mu_1(\emptyset)=\mu_0$. For the second period, given $I\in\mathcal{I}_2$, define $\mu_2(\cdot\,|\,I)$ as follows:

1. If $I\in\mathcal{I}_2^1$, let $\mu(\cdot\,|\,I)$ be the probability distribution that assigns probability 1 to the true consumer type $(s(I),\theta(I))$. That is, for any Borel set $B\subseteq\mathcal{T}$, let $\mu(B\,|\,I)=\mathbf{1}_B(s(I),\theta(I))$.

2. If $I\in\mathcal{I}_2^0$, we have two cases.

(a) If $p_1^B(I) - p_1^A(I) + 1/2 > 0$, let $\mu_2(\cdot \mid I)$ be the uniform probability distribution in the set $\tilde{\mathcal{A}}_0(I)$, which we define as

$$\tilde{\mathcal{A}}_0(I) = \mathcal{A}_0(p_1^A(I), p_1^B(I)) = \left\{(s,\theta) \in \mathcal{T} : s < g\left(\theta \mid p_1^A(I), p_1^B(I)\right)\right\};$$

that is, for any Borel set $B \subseteq \mathcal{T}$, we have[32]

$$\mu_2(B \mid I) = \frac{1}{\mu_0\left(\tilde{\mathcal{A}}_0(I)\right)} \int_B \mathbf{1}_{\tilde{\mathcal{A}}_0(I)}(s,\theta)\, d(s,\theta). \tag{B.27}$$

(b) If $p_1^B(I) - p_1^A(I) + 1/2 = 0$, let $\mu_2(\cdot \mid I)$ be the uniform probability distribution in $[0,1] \times \{1\}$.

We claim that $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is an equilibrium. To show this, we first show that this assessment satisfies sequential rationality for each player, and then proceed to show that the belief system $\boldsymbol{\mu}$ is consistent given the strategies.

Consumer. Consider period 2 and let $h \in H_2^c$. It follows that for any strategy $\boldsymbol{\gamma}'$ we have that

$$U_2\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \mu \mid h\right) = \left(m\theta(h) - p_2^B(h)\right)^+ \geq \left(m\theta(h) - p_2^B(h)\right)\gamma_2'(h) = U_2\left(\boldsymbol{\gamma}', \sigma_1^A, \boldsymbol{\sigma}^B, \mu \mid h\right).$$

For period 1 and $h \in H_1^c$, with some algebra we can write

$$\begin{aligned} U_1\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right) &= \bar{u} - (1 - s(h))/2 - p_1^B(h) + \left(g\left(\theta(h) \mid p_1^A(h), p_1^B(h)\right) - s(h)\right)^+ \\ &\geq \bar{u} - (1 - s(h))/2 - p_1^B(h) + (1 - \gamma_1'(h))\left(g\left(\theta(h) \mid p_1^A(h), p_1^B(h)\right) - s(h)\right) \\ &= U_1\left(\boldsymbol{\gamma}', \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid h\right). \end{aligned}$$

Thus, $\boldsymbol{\gamma}$ is sequentially rational for the consumer given $\boldsymbol{\sigma}$ and $\boldsymbol{\mu}$.

Firm B. Let $\boldsymbol{\sigma}^{B'}$ be a strategy for firm B and take $I \in \mathcal{I}_2$. Given the belief system $\boldsymbol{\mu}$ described above, we have three cases. First, consider the case where $I \in \mathcal{I}_2^1$. It follows that for any $\sigma_2^{B'}(I) \in \mathbb{R}$,

$$\begin{aligned} \Pi_2^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid I\right) = m\theta(I) &\geq \sigma_2^{B'}(I)\mathbf{1}_{\{\sigma_2^{B'}(I) \leq m\theta(I)\}}(\sigma_2^{B'}(I)) \\ &= \sigma_2^{B'}(I)\gamma_2(\langle I, \sigma_2^{B'}(I)\rangle) = \Pi_2^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^{B'}, \boldsymbol{\mu} \mid I\right). \end{aligned}$$

Now consider the case with $I \in \mathcal{I}_2^0$ and $p_1^B(I) - p_1^A(I) + 1/2 > 0$. Given $\boldsymbol{\gamma}$ and $\boldsymbol{\mu}$, we can write firm B's continuation profit as a function of the price it sets for product 2 as

$$p_2^B \mu_2\left([0,1] \times [p_2^B/m, 1] \mid I\right).$$

Following the same argument as in the proof of Claim 6, we can rewrite this expression as

$$\frac{p_2^B}{\mu_0\left(\tilde{\mathcal{A}}_0(h)\right)} \int_{\min\{p_2^B/m,1\}^+}^1 \min\{g\left(t \mid p_1^A(I), p_1^B(I)\right), 1\} dt.$$

In the proof of Claim 6, we have shown that this expression has a unique maximizer, which is $p_2^B = \sigma_2^B(I) = m\bar{\theta}\left(\bar{X}(p_1^A(I), p_1^B(I)), m\right)$. Thus, $\boldsymbol{\sigma}^B$ satisfies sequential rationality in the second period for firm B given information vector I, $\boldsymbol{\gamma}$, σ_1^A, and $\boldsymbol{\mu}$.

[32] It is easy to see, following the same argument as in the proof of Claim 7, that $\mu_0\left(\tilde{\mathcal{A}}_0(I)\right) > 0$.

Finally, if $I \in \mathcal{I}_2^0$ and $p_1^B(I) - p_1^A(I) + 1/2 \le 0$, then $\mu_2(\cdot \mid I)$ assigns probability 1 to the event that $\{\theta = 1\}$ and we can therefore write B's continuation of profit as a function of the price it sets for product 2 (given $\boldsymbol{\gamma}$ and $\boldsymbol{\mu}$) as

$$p_2^B \mathbf{1}_{\{p_2^B \le m\}}(p_2^B),$$

which is clearly maximized with $p_2^B = \sigma_2^B(I) = m\bar{\theta}(0,m) = m$. It follows that $\boldsymbol{\sigma}^B$ satisfies sequential rationality for firm B in the second period, given any information vector $I \in \mathcal{I}_2$, and given $\boldsymbol{\gamma}$, σ_1^A, and $\boldsymbol{\mu}$.

To show that $\boldsymbol{\sigma}^B$ satisfies sequential rationality for firm B in period 1, notice that $\boldsymbol{\gamma}$ and σ_2^B satisfy the conditions in Claims 1, 2, 3, and 6. Then, by Claim 8, and since prices (p_1^{A*}, p_1^{B*}) form a Nash equilibrium in $\mathbf{G}(m)$, it follows that, for any strategy $\boldsymbol{\sigma}^{B'}$,

$$\Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \pi^B\left(p_1^{A*}, p_1^{B*}, m\right) \ge \pi^B\left(p_1^A, \sigma_1^{B'}(\emptyset), m\right) = \Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^{B'}, \boldsymbol{\mu} \mid \emptyset\right).$$

Firm A. Following the same argument as for firm B, we have that σ_1^A satisfies sequential rationality for firm A given $\boldsymbol{\sigma}^B$, $\boldsymbol{\gamma}$, and $\boldsymbol{\mu}$.

Consistency of beliefs. It remains to show that the belief system $\boldsymbol{\mu}$ is consistent given the strategy profile $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$. First, notice that if $I \in \mathcal{I}_2^1$, then $\mu_2(\cdot \mid I)$ satisfies our definition of consistency as it assigns probability 1 to the event associated with the true consumer type observed in information vector I.

Now consider $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$. By the definition of γ_1, it follows that I is reached given prices p_1^A and p_1^B if and only if the event $\tilde{\mathcal{A}}_0(I)$ occurs. By following the same argument as in the proof of Claim 7, we have that

$$\mu_0\left(\tilde{\mathcal{A}}_0(I)\right) = \mu_0\left(\mathcal{A}_0\left(p_1^A(I), p_1^B(I)\right)\right) = \psi\left(\bar{X}(p_1^A(I), p_1^B(I), m\right).$$

Therefore, if $\bar{X}\left(p_1^A(I), p_1^B(I)\right) > 0$, we have that $\mu_0\left(\tilde{\mathcal{A}}_0(I)\right) > 0$. In such cases, notice from (B.27) that we have defined $\mu_2(\cdot \mid I)$ by applying Bayes' rule to $\mu_0 = \mu_1(\emptyset)$, and so $\boldsymbol{\mu}$ satisfies consistency with the strategy profile at such information vectors.

Finally, if $\bar{X}\left(p_1^A(I), p_1^B(I)\right) = 0$, we have that $\mu_0\left(\tilde{\mathcal{A}}_0(I)\right) = 0$. Thus, we can define beliefs arbitrarily at these information sets while maintaining consistency. We conclude that $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$. Thus, $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is an equilibrium in the forward-looking setting, as desired. □

Finally, given that we can construct an equilibrium in the forward-looking setting given a PSNE for $\mathbf{G}(m)$, we can establish the relationship between the equilibria in both games as stated in Lemma 2.

Proof of Lemma 2. First, suppose that $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is an equilibrium in the forward-looking setting, and define $(p_1^{A*}, p_1^{B*}) = (\sigma_1^A(\emptyset), \sigma_1^B(\emptyset))$. By sequential rationality for firms in period 1, and Claim 8, it follows that (p_1^{A*}, p_1^{B*}) is a PSNE in $\mathbf{G}(m)$.

Conversely, if (p_1^{A*}, p_1^{B*}) is a PSNE in $\mathbf{G}(m)$, it follows from Claim 9 that there exists an equilibrium in the forward-looking setting with $(\sigma_1^A(\emptyset), \sigma_1^B(\emptyset)) = (p_1^{A*}, p_1^{B*})$. □

B.3. Proof of Lemma 3

In what follows, we prove Lemma 3, which states that $\mathbf{G}(m)$ admits pure-strategy Nash equilibria (henceforth, PSNE) when the parameter m satisfies $m < m_L$ or $m > m_H$, where m_L and m_H are constants that we will characterize. In addition, we show that when $\mathbf{G}(m)$ admits a PSNE, it is the unique PSNE of the game.

To do so, we establish a series of claims that characterize the firms' best-response correspondences in $\mathbf{G}(m)$. We start by providing some technical conditions about the auxiliary functions $\bar{\theta}$, ψ, ξ, and ϕ (defined in (B.2), (B.7), (B.8), and (B.9), respectively) in Claim 10, and then show that the firms' profit functions π^A and π^B are continuous in prices in Claim 11. Then, we characterize firm A's best-response correspondence in Claim 12. In particular, we show that it is always convex-valued.

We then turn our focus to firm B's best-response correspondence, which need not be convex-valued in general, as illustrated in Figure 8. Claims 13 and 14 provide some monotonicity properties for firm B's best-response correspondence. In particular, they provide conditions under which firm B has an incentive to choose $p_1^B \le p_1^A - 1/2$, which induces the consumer to buy product 1 from it regardless of her type, as established in Lemma 1. These two claims allow us to show that when the value of m is large enough, the only PSNE of $\mathbf{G}(m)$ is for firm A to set a price of zero, and for firm B to set a price of $-1/2$, which we do in Claim 15. We refer to this equilibrium as the *corner equilibrium*.[33]

Claim 16 then provides a characterization of firm B's best-response correspondence and, in particular, it shows that it is single-valued everywhere for all small enough values of m. Combined with the fact that firm A's best-response correspondence is always convex-valued, this allows us to establish in Claim 17 that $\mathbf{G}(m)$ admits a PSNE for all small enough values of m. Moreover, we show that is this case, the equilibrium prices satisfy $p_1^A > 0$ and $p_1^B > p_1^A - 1/2$; we refer to such equilibria as *interior equilibria*. Finally, Claim 18 establishes that $\mathbf{G}(m)$ admits at most one interior PSNE. We then prove Lemma 3 based on all these claims, and we formally prove Theorem 1.

To conclude this section, we establish in Claim 19 that restricting the firms' action spaces in $\mathbf{G}(m)$ to be $S^A = [0,1]$ and $S^B = [-1/2, 1]$ (as given in Definition 1) is without loss, in the sense that the set of PSNE in undominated strategies of the game remains the same if we make no restrictions on the firms' action spaces.

In order to keep our present exposition as brief as possible, and since the proofs of all these claims are primarily algebraic exercises, we have relegated the proofs of Claims 10–19 to Appendix F.1.

The following claim provides various properties of the auxiliary functions we have previously defined. These properties will be useful for proving our results later on.

CLAIM 10. *Let $\bar{\theta}$, ψ, ξ, and ϕ be as defined in* (B.2), (B.7), (B.8), *and* (B.9), *respectively. Then, for every fixed $m > 0$, the functions $\bar{\theta}(x,m)$, $\psi(x,m)$, $\xi(x,m)$, and $\phi(x,m)$ are all continuous and bounded functions of $x \in [0,1]$, and continuously differentable for $x \in (0,1]$. Moreover, we have that*

1. *$\bar{\theta}(x,m)$ is strictly decreasing in x.*

2. *$\psi(x,m)$ is strictly increasing and strictly concave in x.*

3. *$\xi(x,m)$ is strictly decreasing in x.*

4. *$\phi(x,m)$ is strictly decreasing and strictly convex in x.*

The following claim establishes that the firms' profit functions in $\mathbf{G}(m)$ are continuous in prices. This implies that the firms' best-response correspondences are upper hemicontinuous in the competitor's price, which enables us to use fixed point theorems later on. The proof follows directly from the properties given in Claim 10.

[33] To make this definition complete, we say that a PSNE (p_1^A, p_1^B) of $\mathbf{G}(m)$ is a corner equilibrium if $p_1^B \le p_1^A - 1/2$.

Claim 11. *For fixed $m > 0$, π^A and π^B are continuous functions of (p_1^A, p_1^B).*

Next, we characterize the form of firm A's best-response correspondence. In particular, note that this correspondence is always convex-valued.

Claim 12. *For $m > 0$, let $BR^A(p_1^B, m)$ be firm A's best-response correspondence in $\mathbf{G}(m)$, given firm B's price $p_1^B \in S^B$. Then, $BR^A(-1/2, m) = S^A$. Moreover, $BR^A(p_1^B, m)$ is single-valued for any $p_1^B > -1/2$ and, in addition, $p_1^A = BR^A(p_1^B, m)$ is the unique solution to*

$$p_1^A = Z\left(\bar{X}\left(p_1^A, p_1^B\right), m\right), \tag{B.28}$$

where we define $Z : [0,1] \times \mathbb{R}^{++} \to \mathbb{R}$ as[34]

$$Z(x, m) = \frac{\psi(x, m)}{\psi_x(x, m)}. \tag{B.29}$$

We now study firm B's best-response correspondence. We denote firm B's best-response correspondence in $\mathbf{G}(m)$ given firm A's price p_1^A by $BR^B(p_1^A, m)$. In contrast to the case for firm A, a challenge that arises is that firm B's profit function is not always quasiconcave in its own price. To see why quasiconcavity may fail, recall from equation (B.6) that firm B's profit function can be written as $\pi^B(p_1^A, p_1^B) = \left(1 - \psi(\bar{X}(p_1^A, p_1^B), m)\right) p_1^B + m\phi\left(\bar{X}(p_1^A, p_1^B), m\right)$. The first term is quasiconcave in p_1^B, whereas the second term is in fact convex in p_1^B; adding these two terms produces instances where π^B is not quasiconcave in its own price. This results in cases where firm B's best-response correspondence is not convex-valued (see Figure 8). Moreover, this leads to the game $\mathbf{G}(m)$ having no PSNE for some values of m (see Remark 1 at the end of this section). The following two claims establish a series of properties that help us characterize firm B's best-response correspondence.

The next result establishes two properties. First, given firm A's price p_1^A, firm B will have an incentive to capture the entire market for product 1 transactions by setting a price of $p_1^A - 1/2$ provided that m is large enough. Second, if for a fixed value of m and some firm A's price $p_1^A \in S^A$, firm B has an incentive to capture all the product 1 market by setting a price of $p_1^A - 1/2$, then this is also the case for any $m' > m$.

Claim 13. *Fix $p_1^A \in S^A$. Then, the following two properties hold:*

1. *There exists $m_0 = m_0(p_1^A)$ such that for all $m > m_0$ we have that $BR^B(p_1^A, m) = p_1^A - 1/2$.*
2. *If $BR^B(p_1^A, m) = p_1^A - 1/2$ for some $m > 0$, then $BR^B(p_1^A, m') = p_1^A - 1/2$ for any $m' > m$.*

Next, we establish that if firm B has an incentive to capture the entire market for product 1 when firm A chooses some price p_1^A, that will also be the case if firm A sets any higher price.

Claim 14. *Fix $m > 0$, and let $p_1^A \in S^A$ be such that $p_1^A - 1/2 \in BR^B(p_1^A, m)$. Then, for any $y > p_1^A$, we have that $BR^B(y, m) = y - 1/2$.*

The previous two claims allow us to show that $(p_1^A, p_1^B) = (0, -1/2)$ is the only PSNE in $\mathbf{G}(m)$ for all m large enough. Recall that we refer to this type of equilibrium as the *corner case*.

[34] We denote the partial derivative of ψ w.r.t. x by ψ_x and similarly for other functions. Note that, by Claim 10, $\psi_x(x, m)$ is formally defined only for $x \in (0, 1]$. Slightly abusing notation, we define $\psi_x(0, m) = \lim_{t \to 0^+} \psi_x(t, m) = 2$, so that $Z(x, m)$ is well defined for all $x \in [0, 1]$.

Claim 15. *There exists $m_H > 0$ such that for every $m > m_H$, $(p_1^{A*}, p_1^{B*}) = (0, -1/2)$ is the only pure-strategy Nash equilibrium of $\boldsymbol{G}(m)$. In addition, there are no corner equilibria when $m < m_H$.*

Claim 15 characterizes the unique PSNE of $\mathbf{G}(m)$ for $m > m_H$. We now proceed to do so for small values of m, and to prove that these equilibria are *interior*, i.e., that prices satisfy $p_1^A > 0$, $p_1^B > -1/2$ and $p_1^B > p_1^A - 1/2$. The first step toward this goal is to characterize the shape of firm B's best-response correspondence, which is addressed by the following result. As we can see in the second panel of Figure 8, there is a range of values of m for which firm B's best-response correspondence need not be convex-valued everywhere (i.e., when $M_0 \le m \le M_1$). However, Claim 16 shows that $BR^B(p_1^A, m)$ will indeed be single-valued everywhere except for at most one price p_1^A and, in particular, that it is single-valued for all $p_1^A \in S^A$, provided that m is small enough.

Claim 16. *There exist constants $0 < M_0 < M_1$ that define the shape of BR^B as follows:*

(i) If $0 < m < M_0$, $BR^B(p_1^A, m)$ is single-valued for all $p_1^A \in S^A$. Moreover, $p_1^B = BR^B(p_1^A, m)$ satisfies $p_1^B > p_1^A - 1/2$ and

$$p_1^B = V\left(\bar{X}\left(p_1^A, p_1^B\right), m\right), \tag{B.30}$$

where we define $V : [0,1] \times \mathbb{R}^{++} \to \mathbb{R}$ as[35]

$$V(x, m) = \frac{1 - \psi(x, m) + m\phi_x(x, m)}{\psi_x(x, m)}. \tag{B.31}$$

(ii) If $M_0 \le m \le M_1$, there exists $d(m) \in S^A$ such that

(a) *If $p_1^A < d(m)$, $BR^B(p_1^A, m)$ is single-valued and satisfies equation (B.30).*

(b) *If $p_1^A > d(m)$, $BR^B(p_1^A, m) = p_1^A - 1/2$.*

(iii) If $m > M_1$, we have that $BR^B(p_1^A, m) = p_1^A - 1/2$ for all $p_1^A \in S^A$.

This last result shows that even though π^B is not quasiconcave in p_1^B in general, firm B's best-response correspondence $BR^B(p_1^A, m)$ is single-valued for all $p_1^A \in S^A$ and all small enough m. This allows us to use standard results from Game Theory to prove the existence of a PSNE in $\mathbf{G}(m)$ for all small enough m.

Claim 17. *There exists $M_0 > 0$ such that if $m < M_0$, $\mathbf{G}(m)$ admits a pure-strategy Nash equilibrium. Moreover, any such equilibrium is interior.*

The next claim establishes that $\mathbf{G}(m)$ admits at most one interior PSNE.

Claim 18. *If $\mathbf{G}(m)$ admits an interior PSNE, it is the unique interior PSNE of the game.*

Finally, we prove Lemma 3 based on the previous results.

Proof of Lemma 3. Let m_H be as in Claim 15, so that $(p_1^{A*}, p_1^{B*}) = (0, -1/2)$ is the unique PSNE of $\mathbf{G}(m)$ when $m > m_H$. Now define $\mathcal{E}$ as the set of positive values of m for which $\mathbf{G}(m)$ admits an interior PSNE, and let

$$m_L = \sup\{m' > 0 : [0, m'] \subseteq \mathcal{E}\}. \tag{B.32}$$

Take M_0 as in Claim 17, so that $m_L \ge M_0 > 0$. In addition, by Claim 15, we have that $m_L \le m_H$. By the definition of m_L, $\mathbf{G}(m)$ admits an interior PSNE for $m \in (0, m_L)$. Moreover, in this case, such an equilibrium is the only PSNE of $\mathbf{G}(m)$ by Claims 15 and 18. □

[35] $V(x, m)$ is well defined since $\psi_x(x, m) > 0$ for $x \in [0, 1]$ and $m > 0$.

We now prove Theorem 1 based on Lemmas 1, 2, and 3.

Proof of Theorem 1. Let $m \in (0, m_L) \cup (m_H, \infty)$, where m_L and m_H are as in Lemma 3. By Lemma 3, $\mathbf{G}(m)$ admits a unique PSNE (p_1^{A*}, p_1^{B*}). By Lemma 2, there exists an equilibrium $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ in the forward-looking setting with $(\sigma_1^A(\emptyset), \sigma_1^B(\emptyset)) = (p_1^{A*}, p_1^{B*})$. We now consider the two cases for the value of m separately.

If $m > m_H$, we have that $(p_1^{A*}, p_1^{B*}) = (0, -1/2)$ by Lemma 3. The description of the resulting equilibrium path follows from Lemma 1.

Now consider $m < m_L$. By Lemma 3, (p_1^{A*}, p_1^{B*}) is an interior PSNE; i.e., we have that $p_1^{A*} > 0$ and $p_1^{B*} > p_1^{A*} - 1/2$. In particular, this implies that $\bar{X}(p_1^{A*}, p_1^{B*}) > 0$ and thus, by Claim 7, the expected product 1 demand for both firms is positive. Moreover, let us define $\bar{\theta}^*(m) = \bar{\theta}\left(\bar{X}(p_1^{A*}, p_1^{B*}), m\right)$. The description of the resulting equilibrium path follows from Lemma 1. Finally, the fact that $p_1^{A*} > p_1^{B*}$ follows from Claim 22 in Appendix C. □

Note that Theorem 1 only ensures that an equilibrium exists when either $m < m_L$ or $m > m_H$. Although the constant m_H is fully characterized in the proof of Claim 15 as the smallest value of m such that $BR^B(0, m) = -1/2$, we have not yet provided any tool to compute the value of m_L. However, we do so in Appendix F.2 (see Proposition 8), where we characterize the set of values of m for which $\mathbf{G}(m)$ admits an interior PSNE. In particular, this characterization allows us to show that

REMARK 1. The constants m_L and m_H satisfiy $m_L < 4 < m_H$. In addition, $\mathbf{G}(4)$ admits no PSNE. Moreover, we can numerically approximate $m_L \approx 3.98$ and $m_H \approx 4.02$.

We refer the reader to Appendix F.2 for the proof of Remark 1.

To conclude this section, we establish that the restriction of the action spaces we have imposed (i.e., choosing $S^A = [0, 1]$ and $S^B = [-1/2, 1]$) leads to no loss of PSNE in undominated strategies. Formally, the following claim shows that the PSNE of $\mathbf{G}(m)$ remain unchanged, if we define action spaces as $\tilde{S}^A = \tilde{S}^B = \mathbb{R}$ instead and consider only equilibria in which firm A plays no dominated strategies, i.e., such that firm A sets non-negative prices.[36]

CLAIM 19. *Let $\tilde{\mathbf{G}}(m)$ be the two-player normal-form game with action spaces $\tilde{S}^A = \tilde{S}^B = \mathbb{R}$, and profit functions π^A and π^B as defined in (B.5) and (B.6). Then, for any $m > 0$, $\tilde{\mathbf{G}}(m)$ and $\mathbf{G}(m)$ have the same pure-strategy Nash equilibria in undominated strategies.*

Appendix C: Proof of Theorem 2

In this appendix, we establish a series of claims that allow us to compare the equilibrium expected consumer surplus for our three settings. In what follows, we refer to the equilibrium expected consumer surplus for each setting simply as "consumer surplus" (CS). These comparisons result in the proof of Theorem 2, which states that in the interior equilibrium regime ($m < m_L$), consumer surplus is higher with data tracking in the economy, and that this holds both with myopic and with forward-looking consumers.

[36] Setting a negative price need not be a dominated strategy for firm B, as it has an incentive to learn the type of the consumer by setting low prices for product 1.

The results we prove are as follows. Claim 20 provides the expressions for consumer surplus in the myopic and restricted settings and establishes that consumer surplus is larger in the myopic setting when $0 < m < m_L$. Then, Claim 21 provides the expression for consumer surplus in the forward-looking setting, and Claim 22 provides some auxiliary properties about the product 1 price dispersion in the forward-looking setting. Finally, Claim 23 shows that consumer surplus is higher in the forward-looking than in the myopic setting, for $0 < m < m_L$. We conclude by proving Theorem 2, which follows directly from Claims 20 and 23.

CLAIM 20. *For any $m > 0$, the equilibrium consumer surplus in the restricted and myopic settings is given, respectively, by*

$$CS^R(m) = \bar{u} - \frac{5}{8} + \frac{m}{8}, \quad \text{and} \quad CS^M(m) = \begin{cases} \bar{u} - \frac{5}{8} + \frac{m}{144}(27 - m), & \text{if } m \le 6, \\ \bar{u} + \frac{1}{4}, & \text{otherwise.} \end{cases} \tag{C.1}$$

In particular, for $m \in (0, m_L)$, consumer surplus is higher in the myopic than in the restricted setting.

Proof of Claim 20. First consider the restricted setting, and assume that the firms set prices p_1^A, p_1^B for product 1, and let $CS_1(p_1^A, p_1^B)$ be the expected utility associated with product 1 that the consumer obtains if the firms set such prices. It is easy to show that, in equilibrium, the consumer strictly prefers to buy product 1 from firm A if and only if her type (s, θ) satisfies $s < \bar{X}(p_1^A, p_1^B)$ (see Claim 25 in Appendix E.1). We then compute $CS_1(p_1^A, p_1^B)$ (abbreviating $\bar{X}(p_1^A, p_1^B)$ as $\bar{X}$) as

$$\begin{aligned} CS_1(p_1^A, p_1^B) &= \mathbb{E}_{\mu_0}\left[u_1\left(\mathbf{1}_{\{s > \bar{X}\}}(s,\theta); s, \theta, p_1^A, p_1^B\right) \middle| p_1^A, p_1^B\right] \\ &= \bar{u} - \left(\bar{X} p_1^A + (1 - \bar{X}) p_1^B\right) - \frac{1}{4}\left(\bar{X}^2 + (1 - \bar{X})^2\right) \\ &= \bar{u} - p_1^B - \frac{1}{4} + \frac{1}{2}\bar{X}^2, \end{aligned} \tag{C.2}$$

where, as before, μ_0 is the uniform distribution on the unit square.

By Proposition 1, both firms set a price of 1/2 for product 1 in equilibrium, so we have that the equilibrium consumer surplus associated with product 1 is $CS_1^R(m) \equiv CS_1(1/2, 1/2) = \bar{u} - 5/8$.

Now consider product 2. When firm B sets a price of p_2^B for product 2 (regardless of the consumer's type, since we are considering the restricted setting), the consumer surplus associated with product 2 is

$$CS_2(p_2^B) = \mathbb{E}_{\mu_0}\left[u_2\left(\mathbf{1}_{\{p_2^B/m < \theta\}}(\theta); s, \theta, p_2^B\right) \middle| p_2^B\right] = \mathbb{E}_{\mu_0}\left[\left(m\theta - p_2^B\right)^+\right].$$

In equilibrium, firm B sets a price of $p_2^{B,R} = m/2$ for product 2, and so we have that the consumer surplus associated with product 2 is $CS_2^R(m) \equiv CS_2(m/2) = m/8$. By adding the surplus across both products, we have that the total consumer surplus is $CS^R(m) = \bar{u} - 5/8 + m/8$.

Now consider the myopic setting, and recall from Proposition 2 that in this case, the equilibrium prices for product 1 are

$$p_1^{A,M}(m) = \max\{0, 1/2 - m/12\}, \quad p_1^{B,M}(m) = \max\{-1/2, 1/2 - m/6\}.$$

The surplus derived from product 1 can be obtained by plugging these prices into equation (C.2) (and taking $\bar{X} = \bar{X}^M(m) \equiv p_1^{B,M}(m) - p_1^{A,M}(m) + 1/2$), since in equilibrium, the consumer makes product 1 purchase decisions as in the restricted setting (by Claim 29 in Appendix E.2). By computation we obtain

$$CS_1^M(m) \equiv CS_1\left(p_1^{A,M}(m), p_1^{B,M}(m)\right) = \begin{cases} \bar{u} - \frac{5}{8} + \frac{m}{288}(m + 36), & \text{if } m \le 6, \\ \bar{u} + \frac{1}{4}, & \text{if } m > 6. \end{cases}$$

Regarding product 2, by Proposition 2, the consumer receives zero surplus with probability $1-\bar{X}^M(m)$ (since, if $s>\bar{X}^M(m)$, she buys product 1 from firm B and is subsequently offered a personalized price equal to her valuation for product 2). With the remaining probability, $\bar{X}^M(m)$, the consumer is offered a price of $m/2$, as in the restricted setting. It follows that the consumer surplus associated with product 2 in the myopic setting is (by independence of s and θ):

$$CS_2^M(m)=CS_2^R(m)\bar{X}^M(m)=\frac{m}{8}\max\left\{0,\frac{1}{2}-\frac{m}{12}\right\}=\max\left\{0,\frac{m(6-m)}{96}\right\}.$$

By computing $CS^M(m)\equiv CS_1^M(m)+CS_2^M(m)$, we get the expression given in (C.1).

Finally, comparing the expressions for $CS^R(m)$ and $CS^M(m)$ given in (C.1), we have that $CS^R(m)<CS^M(m)$ if and only if $0<m<7$. In particular, since $m_L<4$, this holds for all $0<m<m_L$. □

We now consider the consumer surplus comparisons involving the forward-looking setting. The following claim provides an expression for consumer surplus in this setting.

Claim 21. *For $0<m<m_L$, let $p_1^{A*}(m)$ and $p_1^{B*}(m)$ be the unique product 1 equilibrium prices in the forward-looking setting. Let $\bar{X}^*(m)=p_1^{B*}(m)-p_1^{A*}(m)+1/2$, and $\bar{\theta}^*(m)=\bar{\theta}(\bar{X}^*(m),m)$, where $\bar{\theta}$ is defined as in (B.2). Then, consumer surplus in the forward-looking setting is given by*

$$CS^{FL}(m)=\bar{u}-\frac{1}{4}-p_1^{B*}(m)+\frac{1}{2}(\bar{X}^*(m))^2+\frac{1}{2}m\bar{X}^*(m)\left(1-\bar{\theta}^*(m)\right)^2+\frac{1}{6}m^2\left(1-\bar{\theta}^*(m)\right)^3. \tag{C.3}$$

Proof of Claim 21. Let $0<m<m_L$, so that by Theorem 1, there exists an equilibrium in the forward-looking setting, say $(\boldsymbol{\gamma}^*,\sigma_1^{A*},\boldsymbol{\sigma}^{B*},\boldsymbol{\mu}^*)$. Without loss of generality, we take the equilibrium in which the consumer buys product 1 from firm B if indifferent.[37] Let $U^*(s,\theta)$ be the total utility if the consumer's type is (s,θ) in that equilibrium, i.e.,

$$U^*(s,\theta)=U_1\left(\boldsymbol{\gamma}^*,\sigma_1^{A*},\boldsymbol{\sigma}^{B*},\boldsymbol{\mu}^*\mid(s,\theta,p_1^{A*}(m),p_1^{B*}(m))\right),$$

where $(p_1^{A*}(m),p_1^{B*}(m))$ are the unique product 1 equilibrium prices from Theorem 1. By Theorem 1, the consumer strictly prefers to buy product 1 from firm B if her type (s,θ) satisfies $s>g^*(\theta)$, where $g^*(\theta)=\bar{X}^*(m)+m\left(\theta-\bar{\theta}^*(m)\right)^+$. It follows that the equilibrium utility for the consumer as a function of her type (s,θ) is

$$U^*(s,\theta)=\begin{cases}\bar{u}-\frac{1}{2}s-p_1^{A*}(m)+m\left(\theta-\bar{\theta}^*(m)\right)^+, & \text{if } s<g^*(\theta),\\ \bar{u}-\frac{1}{2}(1-s)-p_1^{B*}(m), & \text{if } s\geq g^*(\theta).\end{cases}$$

We now compute $CS^{FL}(m)=\mathbb{E}_{\mu_0}[U^*(s,\theta)]$, where μ_0 is the uniform distribution on the unit square, as before. For a fixed θ, let $h(\theta)=\int_0^1 U^*(s,\theta)ds$, so that $CS^{FL}(m)=\int_0^1 h(\theta)d\theta$. First consider $\theta\leq\bar{\theta}^*(m)$ so that $g^*(\theta)=\bar{X}^*(m)\in(0,1)$ in this case. Thus, we have that (omitting the dependency on m in the notation)

$$\begin{aligned}h(\theta)&=\mathbb{E}[U^*(s,\theta);s<g^*(\theta)]+\mathbb{E}[U^*(s,\theta);s\geq g^*(\theta)]\\&=\bar{u}-\int_0^{\bar{X}^*}\left(\frac{1}{2}s+p_1^{A*}\right)ds-\int_{\bar{X}^*}^1\left(\frac{1}{2}(1-s)+p_1^{B*}\right)ds\\&=\bar{u}-\bar{X}^*p_1^{A*}-(1-\bar{X}^*)p_1^{B*}-\frac{1}{4}(\bar{X}^*)^2-\frac{1}{4}(1-\bar{X}^*)^2\\&=\bar{u}-p_1^{B*}+\bar{X}^*\left(p_1^{B*}-p_1^{A*}\right)-\frac{1}{4}\left(2(\bar{X}^*)^2-2\bar{X}^*+1\right)\\&=\bar{u}-p_1^{B*}+\frac{1}{2}(\bar{X}^*)^2-\frac{1}{4}.\end{aligned}$$

[37] Recall that the equilibrium outcome is unique, except for the zero probability event in which the consumer is indifferent between her available actions, which does not impact the expected utility computation once we take the expectation over consumer types.

Similarly, for $\theta > \bar{\theta}^*(m)$ we have that[38]

$$\begin{aligned}
h(\theta) &= \bar{u} - \int_0^{g^*(\theta)} \left(\frac{1}{2}s + p_1^{A*} - m\left(\theta - \bar{\theta}^*\right)\right) ds - \int_{g^*(\theta)}^1 \left(\frac{1}{2}(1-s) + p_1^{B*}\right) ds \\
&= \bar{u} + mg^*(\theta)\left(\theta - \bar{\theta}^*\right) - g^*(\theta)p_1^{A*} - (1-g^*(\theta))p_1^{B*} - \frac{1}{4}(g^*(\theta))^2 - \frac{1}{4}(1-g^*(\theta))^2 \\
&= \bar{u} + mg^*(\theta)\left(\theta - \bar{\theta}^*\right) - p_1^{B*} + g^*(\theta)\left(p_1^{B*} - p_1^{A*}\right) - \frac{1}{4}\left(2(g^*(\theta))^2 - 2g^*(\theta) + 1\right) \\
&= \bar{u} + mg^*(\theta)\left(\theta - \bar{\theta}^*\right) - p_1^{B*} + g^*(\theta)\bar{X}^* - \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4} \\
&= \bar{u} - p_1^{B*} + g^*(\theta)\left(\bar{X}^* + m\left(\theta - \bar{\theta}^*\right)\right) - \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4} \\
&= \bar{u} - p_1^{B*} + \frac{1}{2}(g^*(\theta))^2 - \frac{1}{4}.
\end{aligned}$$

Finally, we compute consumer surplus by integrating $h(\theta)$:

$$\begin{aligned}
CS^{FL}(m) &= \int_0^1 h(\theta)d\theta \\
&= \bar{u} - p_1^{B*} + \frac{1}{2}(\bar{X}^*)^2 - \frac{1}{4} + \frac{1}{2}\int_{\bar{\theta}^*}^1 \left(2m\bar{X}^*\left(\theta - \bar{\theta}^*\right) + m^2\left(\theta - \bar{\theta}^*\right)^2\right) d\theta \\
&= \bar{u} - p_1^{B*} + \frac{1}{2}(\bar{X}^*)^2 - \frac{1}{4} + \frac{1}{2}m\bar{X}^*\left(1 - \bar{\theta}^*\right)^2 + \frac{1}{6}m^2\left(1 - \bar{\theta}^*\right)^3.
\end{aligned}$$

□

Before comparing consumer surplus in the forward-looking and myopic settings, we establish the following technical conditions on the quantity $\bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2$.

Claim 22. *For $0 < m < m_L$, $\bar{X}^*(m)$ is continuous in m and $0 < \bar{X}^*(m) < 1/2$.*

Proof of Claim 22. By Step 3 in the proof of Claim 44 (see Appendix F.1.1), we have that $x = \bar{X}^*(m)$ solves $N(x,m) = 0$ given fixed m for $x \in [0,1]$, where

$$N(x,m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x+m)}\,(1-2x). \tag{C.4}$$

Note that $N(0,m) = m(m-4) < 0$ for $m \in (0,4)$. In particular, this holds for $m \in (0, m_L)$ since $m_L < 4$ by Remark 1. In addition, $N(1/2, m) = m^2 + 2m > 0$.

Moreover, since $N(x,m)$ is strictly convex in $x \in [0,1]$ for fixed $m > 0$, and since $N(0,m) < 0 < N(1/2,m)$, $x = \bar{X}^*(m)$ is the unique solution to $N(x,m) = 0$ with $x \in [0,1]$ and, in particular, we have that $0 < \bar{X}^*(m) < 1/2$ for all $m \in (0, m_L)$.

Finally, we have that $N_x(\bar{X}^*(m), m) > 0$ by strict convexity of $N(x,m)$ and, therefore, $\bar{X}^*(m)$ is a continuous function of $m \in (0, m_L)$ by the implicit function theorem. □

Next, we show that consumer surplus is higher in the forward-looking than in the myopic setting when $0 < m < m_L$.

Claim 23. *For $0 < m < m_L$, we have that $CS^{FL}(m) > CS^M(m)$.*

Let us first describe the approach we follow to prove the result, which consists of the following steps.

[38] Claim 44 establishes that $\bar{X}^*(m) < \tilde{x}(m)$, which implies that $g^*(1) < 1$, and so integrating from 0 to $\min\{1, g^*(\theta)\}$ gives the same result as integrating from 0 to $g^*(\theta)$.

1. Find a function $F:(0,m_L)\to\mathbb{R}$ such that $CS^{FL}(m)-CS^{M}(m)\geq F(m)$ for $0<m<m_L$.
2. Show that (i) $F(m)$ is continuous in m, (ii) it has no roots in $(0,m_L)$, and (iii) $F(m)>0$ for some $m\in(0,m_L)$. We do this as follows.
   (i) Show that we can write $F(m)=\Gamma(\bar{X}^*(m),m)$, for $m\in(0,m_L)$, where $\Gamma:[0,1]\times\mathbb{R}^{++}\to\mathbb{R}$ is a continuous function. The continuity of F follows from the fact that $\bar{X}^*(m)$ is continuous in m.
   (ii) To show that $F(m)=\Gamma(\bar{X}^*(m),m)$ has no roots in $(0,m_L)$, we first note that given $0<m<m_L$, $x=\bar{X}^*(m)$ is the unique solution to $N(x,m)=0$ with $x\in(0,1/2)$, where $N(x,m)$ is defined as in (C.4). Therefore, showing that F has no roots in $(0,m_L)$ is equivalent to showing that the system of equations given by $\{\Gamma(x,m)=0,N(x,m)=0\}$ has no solutions with $x\in(0,1/2)$ and $m>0$.

   While obtaining the solutions of this system of equations is complicated, we achieve this by finding a change of variables that allows us to transform the above system into one with two polynomial equations with integer coefficients in two unknowns. We then use a well-established method to obtain the solutions of the transformed system, and find that none of them are feasible in the original system. Formally, the method relies on computing the Gröbner basis of the polynomial system, and obtaining an equivalent system in triangular form, for which the solutions can be approximated with high precision. We refer the reader to Cox, Little, and O'Shea (2015) and Sturmfels (2002) for an introduction to these tools.[39]

   (iii) Directly show that $F(m_0)>0$ for some $m_0\in(0,m_L)$.

We now prove Claim 23 by following these steps.

Proof of Claim 23. Let $m\in(0,m_L)$. We proceed according to the steps described above.

Step 1. By equations (C.1) and (C.3) we have that (omitting the dependence of $\bar{X}^*(m)$, $\bar{\theta}^*(m)$, $p_1^{A*}(m)$ and $p_1^{B*}(m)$ on m)

$$\begin{aligned}CS^{FL}(m)-CS^{M}(m)&=\frac{3}{8}-\frac{m}{144}(27-m)-p_1^{B*}+\frac{1}{2}(\bar{X}^*)^2+\frac{1}{2}m\bar{X}^*\left(1-\bar{\theta}^*\right)^2+\frac{1}{6}m^2\left(1-\bar{\theta}^*\right)^3\\&>\frac{3}{8}-\frac{m}{144}(27-m)-p_1^{B*}+\frac{1}{2}(\bar{X}^*)^2+\frac{1}{2}m\bar{X}^*\left(1-\bar{\theta}^*\right)^2\\&=\underbrace{\frac{7}{8}-\frac{m}{144}(27-m)-p_1^{A*}-\bar{X}^*+\frac{1}{2}(\bar{X}^*)^2+\frac{1}{2}m\bar{X}^*\left(1-\bar{\theta}^*\right)^2}_{F(m)},\end{aligned}$$

where the inequality follows since $\frac{1}{6}m^2\left(1-\bar{\theta}^*\right)^3>0$ and the final equality follows by plugging in $\bar{X}^*(m)=p_1^{B*}(m)-p_1^{A*}(m)+1/2$.

Step 2. Recall that by Claim 12, $p_1^{A*}(m)=Z(\bar{X}^*(m),m)$, where Z is defined as in (B.29). Therefore, we can write $F(m)=\Gamma(\bar{X}^*(m),m)$, where

$$\Gamma(x,m)=\frac{7}{8}-\frac{m}{144}(27-m)-Z(x,m)-x+\frac{1}{2}x^2+\frac{1}{2}mx\left(1-\bar{\theta}(x,m)\right)^2. \tag{C.5}$$

[39] Intuitively, this method performs a similar procedure to Gaussian elimination but with a system of polynomial rather than linear equations. This method is intractable by hand, but it is implemented in various software packages, such as Mathematica and Sage.

Since both $Z(x,m)$ and $\bar{\theta}(x,m)$ are continuous functions of $(x,m) \in [0,1] \times \mathbb{R}^{++}$, $\Gamma(x,m)$ is also continuous.[40] By Claim 22, we know that $\bar{X}^*(m)$ is continuous for $0 < m < m_L$. Thus $F(m) = \Gamma(\bar{X}^*(m), m)$ is continuous for $0 < m < m_L$.

Moreover, by the proof of Claim 22, we know that $\bar{X}^*(m)$ is the unique solution to $N(x,m) = 0$ with $0 < x < 1$, where $N(x,m)$ is defined as in (C.4), and, by Claim 22, we know that $0 < \bar{X}^*(m) < 1/2$ for all $0 < m < m_L$.

Then, by the preceding argument, showing that $F(m) = \Gamma(\bar{X}^*(m), m)$ has no roots in $(0, m_L)$ is equivalent to showing that the following system of equations admits no solution with $0 < x < 1/2$ and $0 < m < m_L$.

$$\begin{aligned} \Gamma(x,m) &= 0, \\ N(x,m) &= 0. \end{aligned} \tag{C.6}$$

However, we know that $\bar{X}^*(m) < \tilde{x}(m)$ (by Claim 44 in Appendix F.1), and so it suffices to show that this system of equations has no solution with $0 < x < \min\{1/2, \tilde{x}(m)\}$ and $0 < m < m_L$.

By plugging the expressions for $Z(x,m) = \psi(x,m)/\psi_x(x,m)$ and $\bar{\theta}(x,m)$ for $x < \tilde{x}(m)$ into (C.5) (from equations (B.2), (B.7), and (F.4)) and simplifying the resulting expression, we can write $\Gamma(x,m) = J_1(x,m)/J_2(x,m)$ for $0 < x < \tilde{x}(m)$, where

$$\begin{aligned} J_1(x,m) &= 2J_3(x,m) + \sqrt{2x(2x+m)}\,J_4(x,m), \\ J_2(x,m) &= 144m\left(2m + 6x - \sqrt{2x(2x+m)}\right), \\ J_3(x,m) &= m^4 + 3m^3(x-9) + 2304x^4 + 9m^2(24x^2 - 41x + 14) + 18mx(84x^2 - 40x + 21), \\ J_4(x,m) &= 27m^2 - m^3 - 2304x^3 - 18m(44x^2 - 8x + 7). \end{aligned}$$

Notice that $J_2(x,m) > 0$ for all $x, m > 0$. Therefore, for $0 < x < \tilde{x}(m)$ and $m > 0$, system (C.6) can be written equivalently as $\{J_1(x,m) = 0,\ N(x,m) = 0\}$. Thus, it suffices to show that this system has no solutions with $0 < x < 1/2$ and $0 < m < m_L$.

To do so, we now change variables to $w = \sqrt{2x}$, $z = \sqrt{2x+m}$, so that $\sqrt{2x(2x+m)} = wz$. By plugging the change of variables into J_1 and N we can write

$$\begin{aligned} J_1(w^2/2, z^2 - w^2) &= (w-z)Q_1(w,z), \\ N(w^2/2, z^2 - w^2) &= Q_2(w,z), \end{aligned} \tag{C.7}$$

where

$$\begin{aligned} Q_1(w,z) &= 17w^7 - 72w^6 z - 110w^4 z^3 - z^2(z^2 - 21)(z^2 - 6)(w + 2z) + w^5(91z^2 + 45) \\ &\quad + w^3(z^4 + 216z^2 - 126) + w^2 z^3(4z^2 + 234), \\ Q_2(w,z) &= -w^2 - wz + w^3 z - 4z^2 + 4w^2 z^2 + z^4. \end{aligned}$$

Showing that the system $\{J_1(x,m) = 0,\ N(x,m) = 0\}$ has no solutions with $0 < x < 1/2$ and $0 < m < m_L$ is then equivalent to showing that the transformed system given by $\{J_1(w^2/2, z^2 - w^2) = 0,\ N(w^2/2, z^2 - w^2) = 0\}$ has no solutions with $0 < w < 1$ and $0 < z^2 - w^2 < m_L$. In particular, it suffices to show that the system admits no solutions with $w > 0$ and $w > z$.

[40] It is easy to establish the continuity of Z and $\bar{\theta}$ by following a similar argument as in the proof of Claim 10.

It follows from (C.7) that it suffices to analyze the solutions of the following system:

$$Q_1(w,z)=0,$$
$$Q_2(w,z)=0. \tag{C.8}$$

Since Q_1 and Q_2 are polynomials with integer coefficients, this system can be solved by computing the the Gröbner basis of Q_1, Q_2, and solving the resulting triangular system (see Cox et al. 2015 and Sturmfels 2002 for a detailed introduction to these tools, or Sturmfels 2005 for a short overview). We find that system (C.8) has five real solutions,[41] but none of them satisfy that $0<w<1$ and $z>0$. It follows that system (C.6) has no solution with $0<x<1/2$ and $0<m<m_L$ and, therefore, $F(m)$ has no roots in $(0,m_L)$.

To conclude, we want to show that $F(m_0)>0$ for some $m_0\in(0,m_L)$. To do so, we take $m_0=2$ (by Remark 1, we have that $m_L>2$). Direct computation shows that $\bar{X}^*(2)\approx 0.2423$, $\bar{\theta}^*(2)\approx 0.6937$, $p_1^{A*}(2)\approx 0.2763$, and thus $F(2)\approx 0.06126>0$. □

Based on the previous results, we can now prove Theorem 2.

Proof of Theorem 2. Let $m\in(0,m_L)$. By Claims 20 and 23, we have that $CS^{FL}(m)>CS^{M}(m)>CS^{R}(m)$, as desired. □

Appendix D: Implications of Data Tracking in the Corner Equilibrium Regime

In this appendix, we consider the regime where $m>m_H$. In this case, the value of consumer data is high enough so that the forward-looking setting admits a corner equilibrium; i.e., firm B captures the entire market of product 1 transactions in equilibrium (see Theorem 1). In parallel with the comparisons presented in Section 4, Proposition 7 compares the equilibrium aggregate consumer surplus, firms' profits, and dispersion in product 1 prices when $m>m_H$.

It is important to remark that in contrast to the comparisons established in Theorem 2, Proposition 7 finds that consumers may be worse off when firm B uses data tracking if the value of their data (i.e, m) is high enough. However, this only occurs in the corner equilibrium regime, in which firm B has a strong enough incentive to set a low enough price for product 1 to induce the consumer to buy product 1 from it, regardless of her type. As we have argued, we find this to be the less interesting and plausible case of our model, as it implies that firm A receives no purchases.

The intuition for this result is the following: in the corner equilibrium regime, i.e., when $m>m_H$, we obtain from Theorem 1 that by setting the price of product 1 to be equal to $-1/2$, firm B ensures that the consumer will buy product 1 from it, and therefore that it will observe the value of θ with probability 1. In this equilibrium, the consumer derives no surplus from product 2 while receiving a fixed discount of $1/2$ when buying product 1. By contrast, with no data tracking, the aggregate consumer surplus associated with product 2 increases linearly with m. In turn, this implies that the restricted setting favors consumers for all large values of m. Thus, the result is driven by the fact that firm B can ensure perfect data collection for a fixed cost, while the corresponding value of such data increases linearly in m.

[41] We obtain the solutions by using the Solve routine in Mathematica. The real solutions to system (C.8) are $(w,z)=(0,0)$, $\pm(1,1)$, $\pm(1.32385, 0.589199)$.

Proposition 7. *Let m_H be as defined in Theorem 1. Then, for $m \in (m_H, \infty)$ we have that*

(i) The comparisons regarding consumer surplus depend on the value of m as follows:

(a) *If $m_H < m < 7$, the aggregate consumer surplus is highest in the forward-looking setting and lowest in the restricted setting. That is, the comparisons established in Theorem 2 hold in this region.*[42]

(b) *If $m > 7$, the aggregate consumer surplus is highest in the restricted setting, while the forward-looking and myopic settings result in the same consumer surplus. Thus, in contrast to Theorem 2, aggregate consumer surplus is lower in the presence of data tracking if the value of consumer data is high enough.*

(ii) As in Proposition 4, the dispersion in product 1 equilibrium prices is higher with forward-looking than with myopic consumers if $m < 6$. For $m \geq 6$, both the forward-looking and the myopic setting result in the same equilibrium prices.

(iii) As in Proposition 5, firm B benefits from using data-tracking technologies, even if consumers are forward-looking.

(iv) As in Proposition 6, firm A's expected profits are highest in the restricted setting, followed by the myopic and forward-looking settings. In particular, firm A's expected profit is zero in the forward-looking setting for all $m > m_H$, while this occurs in the myopic setting when $m \geq 6$.

(v) Social welfare is lowest in the restricted setting. If $m < 6$, social welfare is strictly higher in the forward-looking than in the myopic setting, whereas both of these settings yield the same outcome if $m \geq 6$.

Proof of Proposition 7. Let $m > m_H$, and recall from Theorem 1 that in equilibrium, the firms' prices for product 1 are $p_1^{A*}(m) = 0$, $p_1^{B*}(m) = -1/2$, with the consumer buying product 1 from firm B, and firm B observing her type with probability 1.

Part (i). For $m \geq 6$, the myopic and forward-looking settings result in the same equilibrium outcomes (by Theorem 1 and Proposition 2), and therefore the same level of aggregate consumer surplus, which is $\bar{u} + 1/4$ (by equation (C.1)). It follows from equation (C.1) that consumer surplus is larger in the restricted setting than in the myopic and forward-looking settings if and only if

$$\bar{u} + \frac{1}{4} < \bar{u} - \frac{5}{8} + \frac{m}{8}.$$

This inequality reduces to $m > 7$, which proves point (b), and point (a) for $6 \leq m \leq 7$. Now consider the case with $m < 6$, for which $CS^M(m) = \bar{u} - 5/8 + m(27 - m)/144$. Straightforward calculus shows that this expression is increasing for $m \in [0, 27/2]$ and so, in particular, $CS^M(m) < CS^M(6)$ for $m \in (m_H, 6)$. But since we have a corner equilibrium in the forward-looking setting for such values of m, it follows that $CS^M(6) = \bar{u} + 1/4 = CS^{FL}(m)$. Thus, $CS^{FL}(m) > CS^M(m)$. It remains to show that $CS^M(m) > CS^R(m)$ for $m_H < m < 6$, which is equivalent to

$$\bar{u} - \frac{5}{8} + \frac{m}{144}(27 - m) > \bar{u} - \frac{5}{8} + \frac{m}{8}.$$

[42] However, for $m \geq 6$, both the forward-looking and the myopic setting result in the same corner equilibrium, which results in the same level of consumer surplus.

This inequality holds for $m<9$ and, in particular, for $m\in(m_H,6)$, which concludes the proof of (i).

Part (ii). For $m>m_H$, we have that

$$p_1^{A*}(m)-p_1^{B*}(m)=1/2\geq \min\left\{\frac{1}{2},\frac{m}{12}\right\}=p_1^{A,M}(m)-p_1^{B,M}(m).$$

Part (iii). For $m>m_H$, it is easy to compute firm B's equilibrium profit in the restricted and forward-looking settings, which are $\pi_R^B(m)=(m+1)/4$ and $\pi_{FL}^B(m)=(m-1)/2$, respectively. Since $m>m_H>4$ (by Remark 1), we have that $(m-1)/2>(m+1)/4$, and so the profit is higher in the forward-looking than in the restricted setting. For $m\geq 6$, the myopic and the forward-looking settings result in the same outcome. For $m<6$, firm B's equilibrium profit in the myopic setting is $\pi_M^B(m)=m/4+(1/2+m/12)^2$ (see equation (E.8) in Appendix E.4). Then, for $m_H<m<6$, profits are highest in the myopic setting since the following inequality holds for $m_H<m<6$:

$$\pi_M^B(m)=\frac{m}{4}+\left(\frac{1}{2}+\frac{m}{12}\right)^2>\frac{m-1}{2}=\pi_{FL}^B(m).$$

Part (iv). In the forward-looking setting, firm A's profit is zero while this is also the case in the myopic setting when $m\geq 6$. By contrast, firm A's profit in the restricted setting is $1/4$ regardless of the value of m. It remains to see that firm A's profit in the restricted setting is higher than in the myopic one for $m_H<m<6$, which indeed holds since $\pi_R^A(m)=\frac{1}{4}>\max\left\{\frac{1}{2}-\frac{m}{12},0\right\}^2=\pi_M^A(m)$.

Part (v). Follows from a similar straightforward algebraic comparison as in part (ii). □

Appendix E: Proofs of Propositions 1, 2, 4, 5, and 6

In this appendix we provide the proofs of a series of propositions given in Sections 3 and 4. Specifically, Propositions 1, 2, 4, 5, and 6 are proved, respectively, in Appendices E.1, E.2, E.3, E.4, and E.5.

E.1. Proof of Proposition 1

To prove Proposition 1, we proceed by backwards induction to establish necessary conditions that any equilibrium $(\boldsymbol{\gamma},\sigma_1^A,\boldsymbol{\sigma}^B,\boldsymbol{\mu})$ must satisfy in the restricted setting, and then show that these conditions are sufficient. In what follows, we present a series of claims that state these conditions and conclude with the proof of Proposition 1.

First, Claim 24 characterizes the beliefs in the second period and firm B's pricing strategy for product 2 in any equilibrium. Then, Claim 25 provides the form of the consumer's purchasing strategy for product 1. This is complemented with Claim 1, which is easily seen to hold in the restricted setting as well, and determines the consumer's purchasing strategy for product 2. Then, Claim 26 provides the form of firms' expected profit functions in terms of product 1 prices, and Claim 27 leverages these functions to show that both firms' set a price of 1/2 for product 1 in equilibrum. Finally, we argue that an equilibrium for the restricted setting exists for any value of $m>0$ in Claim 28, and conclude with the proof of Proposition 1.

The following claim shows that the beliefs for firm B in the second period are equal to the prior distribution of consumer types, since the consumer's action in the first period result in no information for firm B in the restricted setting. In addition, since the prior distribution of consumer types is the uniform distribution on the unit square, firm B sets a price of $m/2$ for product 2, regardless of the actions played in the first period.

Claim 24. *In any equilibrium in the restricted setting, we have that $\mu_2(p_1^A, p_1^B) = \mu_0$ for any product 1 prices p_1^A and p_1^B. In addition, firm B sets the price of product 2 as $\sigma_2^B(p_1^A, p_1^B) = m/2$.*

Proof of Claim 24. Fix any product 1 prices p_1^A and p_1^B. Then, the information available to firm B after any history of the form $h = (s, \theta, p_1^A, p_1^B, a_1) \in H_2^f$ is $\mathcal{J}(h) = (p_1^A, p_1^B)$. Since these prices are set independently of the consumer's type, the only consistent belief system is such that $\mu_2(p_1^A, p_1^B)$ is equal to the prior distribution of consumer types, μ_0.

Now consider's firm B's pricing decision for product 2. First note that Claim 1 also holds in the restricted setting. Moreover, since $\mu_2(p_1^A, p_1^B) = \mu_0$, the firm's profit maximization problem when setting the price of product 2 is

$$\max_{p_2^B \in \mathbb{R}} p_2^B \mu_2 \left([0,1] \times [p_2^B/m, 1] \mid p_1^A, p_1^B\right) = \max_{p_2^B \in \mathbb{R}} p_2^B \min\left\{\left(1 - p_2^B/m\right)^+, 1\right\}, \tag{E.1}$$

where the equality follows since $\mu_2(p_1^A, p_1^B) = \mu_0$. The unique solution to problem (E.1) is $p_2^{B,R} = m/2$. Thus, in any equilibrium, $\sigma_2^B(p_1^A, p_1^B) = m/2$. □

Next, we show that since the consumer's actions have no implications for pricing in the second period, the consumer decides which firm to buy product 1 from by maximizing her utility associated with product 1.

Claim 25. *In any equilibrium for the restricted setting, given a consumer history $h \in H_1^c$, γ_1 takes the following form:*

$$\gamma_1(h) = \begin{cases} 1, & \text{if } s(h) > p_1^B(h) - p_1^A(h) + 1/2, \\ \beta(h) \in [0,1], & \text{if } s(h) = p_1^B(h) - p_1^A(h) + 1/2, \\ 0, & \text{if } s(h) < p_1^B(h) - p_1^A(h) + 1/2. \end{cases} \tag{E.2}$$

Proof of Claim 25. Fix a history $h = (s, \theta, p_1^A, p_1^B) \in H_1^c$ and note that Claim 1 also holds in the restricted setting. Moreover, the price firm B sets for product 2, is independent of the consumer's action in the first period (by definition of the restricted setting). Therefore, regardless of her action in the first period, the consumer's utility associated with product 2 in any equilibrium is $(m\theta - \sigma_2^B(p_1^A, p_1^B))^+$. Thus, given the history, the consumer's total utility as a function of her action in the first period is

$$u_1(a_1; h) + \left(m\theta - \sigma_2^B\left(p_1^A, p_1^B\right)\right)^+.$$

Therefore, in equilibrium, the consumer chooses a_1 to maximize

$$u_1(a_1; h) = \bar{u} + a_1\left(s - p_1^B + p_1^A - 1/2\right).$$

The consumer is only indifferent between either action if $s = p_1^B - p_1^A + 1/2$, therefore any sequentially rational strategy for the consumer takes the form of γ_1 given in (E.2). □

The next claim provides expressions for the the firms' expected profit functions in terms of product 1 prices, assuming that the consumer and firm B play equilibrium strategies following the firms' choices for product 1 prices.

Claim 26. *Fix an assessment $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ that satisfies the conditions of Claims 1, 24 and 25, and let $p_1^A = \sigma_1^A(\emptyset)$ and $p_1^B = \sigma_1^B(\emptyset)$. Then,*

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = p_1^A \bar{X}\left(p_1^A, p_1^B\right), \quad \textit{and} \quad \Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = p_1^B\left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right) + m/4,$$

where, as before, $\bar{X}(p_1^A, p_1^B) = \max\{0, \min\{p_1^B - p_1^A + 1/2, 1\}\}$.

Proof of Claim 26. Fix product 1 prices p_1^A, p_1^B. By Claim 25, we have that the expected demand to buy product 1 from firm A is

$$\mu_0\left(\{\gamma_1(s,\theta,p_1^A,p_1^B)=1\}\right)=\mu_0\left(\{s<p_1^B-p_1^A+1/2\}\right)=\bar{X}(p_1^A,p_1^B),$$

where the last equality follows since μ_0 is the uniform distributon on the unit square, and since the event $\{s=p_1^B-p_1^A+1/2\}$ occurs with probability zero under μ_0. Then, it follows that firm A's expected profit in terms of product 1 prices is $p_1^A\bar{X}(p_1^A,p_1^B)$.

Moreover, by the previous argument, the expected demand to buy product 1 from firm B is $1-\bar{X}(p_1^A,p_1^B)$, and therefore its expected profit associated with product 1 is $p_1^B\left(1-\bar{X}\left(p_1^A,p_1^B\right)\right)$. In addition, by Claims 1 and 24, firm B sets the price of product 2 as $m/2$ which the consumer buys with probability 1/2. Thus, firm B's expected profit associated with product 2 is $m/4$ regardless of the prices set for product 1, and its total expected profit is $p_1^B\left(1-\bar{X}\left(p_1^A,p_1^B\right)\right)+m/4$. □

Next, based on the profit functions given in Claim 26, we show that both firms set a price of 1/2 for product 1 in any equilibrium.

Claim 27. *In any equilibrium in the restricted setting, both firms set their product 1 price as* $1/2$, *i.e.,* $\sigma_1^A(\emptyset)=\sigma_1^B(\emptyset)=1/2$.

Proof of Claim 27. By the previous three claims, the firms' expected profit functions in terms of product 1 prices are given by

$$\pi_R^A\left(p_1^A,p_1^B\right)=p_1^A\bar{X}\left(p_1^A,p_1^B\right),\quad \pi_R^B\left(p_1^A,p_1^B\right)=p_1^B\left(1-\bar{X}\left(p_1^A,p_1^B\right)\right)+m/4. \tag{E.3}$$

Thus, in any equilibrium the prices $p_1^j=\sigma_1^j(\emptyset)$, for $j=A,B$ must form a Nash equilibrium in the two-player normal form game with profit functions π_R^A, π_R^B, and where we take firms' action spaces as[43] $S^A=[0,1]$ and $S^B=[-1/2,1]$. Note that this game is equivalent to the classic Hotelling model with linear transportation costs with firms being located at the extemes of the unit interval (e.g., d'Aspremont et al. 1979, Osborne and Pitchik 1987), and therefore the unique equilibrium is for both firms to set prices equal to the unit transportation cost, i.e., $p_1^A=p_1^B=1/2$. □

The following claim constructs an equilibrium for the restricted setting that follows the same structure as established by the necessary conditions in Claims 24–27.

Claim 28. *For any* $m>0$, *there exists an equilibrium in the restricted setting,*

Proof of Claim 28. Following a similar argument as in Claim 9, we will construct an assessment with the structure given in Claims 24–27 and show that it is indeed an equilibrium in the restricted setting.

Let $\sigma_1^A(\emptyset)=\sigma_1^B(\emptyset)=1/2$, and for any product 1 prices p_1^A and p_1^B, let $\sigma_2^B\left(p_1^A,p_1^B\right)=m/2$ and $\mu_2(p_1^A,p_1^B)=\mu_0$. In addition, define the beliefs in the first period as $\mu_1(\emptyset)=\mu_0$. Moreover, define γ_2 as in (B.10) with $q(h)=1$ for any $h\in H_2^c$ such that $m\theta(h)=p_2^B(h)$, and define γ_1 as in (E.2), with $\beta(h)=1$ for any $h\in H_1^c$ such that $s(h)=p_1^B(h)-p_1^A(h)+1/2$.

[43] As in the forward-looking setting, the argument still works even if we take firms' action spaces to be $\mathbb{R}$.

We claim that $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is an equilibrium in the restricted setting. To see this, observe that by the same argument as in the proof of Claim 24, $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$. In addition, by the proof of Claim 24, σ_2^B is sequentially rational at time $t=2$ for firm B given the other players' strategies and the belief system $\boldsymbol{\mu}$. By Claims 26 and the proof of Claim 27, σ_1^A and σ_1^B are sequentially rational for firms at time $t=1$. Finally, it follows from the same arguments as in the proofs of Claims 1 and 25 that $\boldsymbol{\gamma}$ is sequentially rational for the consumer at times $t=1,2$. □

Finally, given that we know that the restricted setting admits an equilibrium, and we have established necessary conditions for equilibrium, we can show that these are also sufficent and prove Proposition 1.

Proof of Proposition 1. By Claim 28, there exists an equilibrium $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ for the restricted setting. By Claim 27, we have that in any equilibrium, both firms set the price of product 1 as $1/2$. By Claim 25, in the equilibrium path, the consumer strictly prefers to buy product 1 from firm A if $s<1/2$, and strictly prefers to buy from firm B if $s>1/2$. Finally, by Claim 24, firm B sets a price of $p_2^{B,R}=m/2$ for product 2, which, by Claim 1, the consumer strictly prefers to buy if $\theta>1/2$. □

E.2. Proof of Proposition 2

To prove Proposition 2, we follow a similar argument as in Appendix E.1. We consider the myopic setting and derive necessary conditions that any equilibrium $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ must satisfy, and then show that these conditions are sufficient. In what follows, we state a series of claims that derive these conditions and conclude with the proof of Proposition 2.

Briefly, our claims are as follows. Claim 29 establishes the form of the consumer's strategy in any equilibrium. Then, Claim 30 pins down firm B's beliefs in the second period, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability. Then, given these beliefs, Claim 31 provides firm B's pricing strategy for product 2 following such histories. This is complemented with Claim 2, which also holds in the myopic setting. Then, Claim 32 provides the form of firms' expected profit functions in terms of product 1 prices, assuming that an equilibrium is played following such prices, and Claim 33 derives the Nash equilibrium in a game with these functions to obtain the equilibrium product 1 prices. Then, we show that an equilibrium for the myopic setting exists for any value of $m>0$ in Claim 34, and finally conclude with the proof of Proposition 2.

First, we show that the consumer's strategy is as in the restricted setting, in any equilibrium. This follows from noting that in the myopic setting, the consumer maximizes her current period payoff, rather than her total payoff.

Claim 29. *In any equilibrium for the myopic setting, γ_1 and γ_2 take the forms given in* (E.2) *and* (B.10)*, respectively.*

Proof of Claim 29. By the same logic as in Claim 1, γ_2 take the form given in (B.10) in equilibrium. In addition, recall that we define equilibrium in the myopic setting with the consumer deciding which firm to buy product 1 from in order to maximize her utility associated with product 1 (instead of her total utility). Thus, by the proof of Claim 25, γ_1 has the form given in (E.2). □

The next claim characterizes the beliefs for firm B in the second period, following a history where the consumer does not buy product 1 from firm B, assuming that this occurs with positive probability (which, as we will see, is equivalent to the condition $p_1^B - p_1^A + 1/2 > 0$).

CLAIM 30. *Fix prices p_1^A, p_1^B and suppose that γ_1 satisfies (E.2). Let $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$, and suppose that $p_1^B - p_1^A + 1/2 > 0$. Then, $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$ at I if and only if $\mu_2(I)$ is the uniform probability measure on $[0, \bar{X}(p_1^A, p_1^B)] \times [0,1]$.*

Proof of Claim 30. Since γ_1 satisfies (E.2), by the same argument as in the proof of Claim 26, the expected demand to buy product 1 from firm A if prices p_1^A and p_1^B are set is $\bar{X}(p_1^A, p_1^B) = \max\{0, \min\{p_1^B - p_1^A + 1/2, 1\}\}$. In particular, this quantity is positive if and only if $p_1^B - p_1^A + 1/2 > 0$.

In that case, the consumer buys product 1 from firm A if her type satisfies $s < \bar{X}(p_1^A, p_1^B)$, which occurs with positive probability.[44] Therefore, we can pin down the beliefs in the second period given information vector $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$ by Bayes' rule as follows. For any Borel set B in the unit square we have

$$\mu_2(B \mid I) = \mathbb{P}_{\mu_0}\left[(s,\theta) \in B \middle| I, \boldsymbol{\gamma}, \boldsymbol{\sigma}\right] = \frac{\mu_0\left(B \cap \{s < \bar{X}(p_1^A, p_1^B)\}\right)}{\mu_0\left(\{s < \bar{X}(p_1^A, p_1^B)\}\right)} = \frac{\int_B \mathbf{1}_{\{s < \bar{X}(p_1^A, p_1^B)\}}(s,\theta)\, d(s,\theta)}{\bar{X}(p_1^A, p_1^B)}.$$

Thus, $\mu_2(I)$ is the uniform probability measure on $[0, \bar{X}(p_1^A, p_1^B)] \times [0,1]$, as desired. □

Next, taking the beliefs characterized in the previous claim, we derive firm B's pricing strategy for histories where it does not perfectly observe the consumer's type, when this occurs with positive probability.

CLAIM 31. *Fix prices p_1^A, p_1^B and suppose that $\boldsymbol{\gamma}$ and $\boldsymbol{\sigma}^{\boldsymbol{B}}$ satisfy the conditions in Claims 2, 29, and 30. Let $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$, and suppose that $p_1^B - p_1^A + 1/2 > 0$. Then, in any equilibrium for the myopic setting, we must that $\sigma_2^B(I) = m/2$ with probability 1.*

Proof of Claim 31. Consider firm B's pricing decision for product 2 given information vector $I = (p_1^A, p_1^B) \in \mathcal{I}_2^0$. By Claim 30, in any equilibrium, the marginal distribution for θ induced by $\mu_2(I)$ is the uniform distribution in $[0,1]$. It follows that we can write firm B's profit maximization problem when choosing the price of product 2 as in (E.1), which is maximized by choosing $p_2^B = m/2$. Thus, in any equilibrium, we have $\sigma_2^B(I) = m/2$ by sequential rationality for firm B. □

The next claim provides expressions for the firms' expected profit functions in terms of product 1 prices, assuming that agents play an equilibrium after these prices are set.

CLAIM 32. *Fix an assessment $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ that satisfies the conditions of Claims 2, 29, 30, and 31. Let $p_1^A = \sigma_1^A(\emptyset)$ and $p_1^B = \sigma_1^B(\emptyset)$. Then, firms' expected profit functions in terms of product 1 prices are, respectively,*

$$\Pi_1^A\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = p_1^A \bar{X}\left(p_1^A, p_1^B\right)$$
$$\Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) = \left(p_1^B + m/4\right)\left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right) + m/4,$$

where, as before, $\bar{X}(p_1^A, p_1^B) = \max\{0, \min\{p_1^B - p_1^A + 1/2, 1\}\}$.

[44] We ignore the event in which the consumer is indifferent between buying from either firm, $\{s = \bar{X}(p_1^A, p_1^B)\}$, which occurs with probability zero under μ_0.

Proof of Claim 32. By the same argument as in the proof of Claim 26, firm A's expected profit in terms of product 1 prices is $p_1^A \bar{X}(p_1^A, p_1^B)$. To obtain firm B's profit function, note that since γ_1 satisfies (E.2) (by Claim 29), we have two cases (ignoring the event $\{s = \bar{X}(p_1^A, p_1^B)\}$ which occurs with probability zero):

1. If this consumer's type is such that $s < \bar{X}(p_1^A, p_1^B)$, she buys product 1 from firm A. By Claim 31, firm B then offers a price of $m/2$ for product 2, which the consumer buys with probability $1/2$ (only if $\theta > 1/2$, by Claim 29). Thus, if $s < \bar{X}(p_1^A, p_1^B)$, firm B's expected profit is $m/4$.

2. On the other hand, if $s > \bar{X}(p_1^A, p_1^B)$, the consumer buys product 1 from firm B, and, by Claim 2, buys product 2 at a price of $p_2^B = m\theta$. Therefore, if $s > \bar{X}(p_1^A, p_1^B)$, firm B's total expected profit is

$$p_1^B + \mathbb{E}_{\mu_0}\left[m\theta \mid s > \bar{X}(p_1^A, p_1^B)\right] = p_1^B + m/2,$$

where the equality follows since μ_0 is the uniform distribution in the unit square.

By considering these two cases, we have that firm B's expected profit in terms of product 1 prices is

$$\begin{aligned}\Pi_1^B\left(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu} \mid \emptyset\right) &= \mu_0\left(\{s > \bar{X}(p_1^A, p_1^B)\}\right)\left(p_1^B + m/2\right) + \mu_0\left(\{s < \bar{X}(p_1^A, p_1^B)\}\right) \cdot m/4 \\ &= \left(1 - \bar{X}(p_1^A, p_1^B)\right)\left(p_1^B + m/2\right) + \bar{X}(p_1^A, p_1^B) \cdot m/4 \\ &= \left(1 - \bar{X}(p_1^A, p_1^B)\right)\left(p_1^B + m/4\right) + m/4,\end{aligned}$$

as desired. □

Given these profit functions, the next claim derives the pure-strategy Nash equilibrium of the simulatenous-move game where firms set the price of product 1. As in the forward-looking setting, we restrict our attention to equilibria in undominated strategies (which implies that firm A chooses non-negative prices).

Claim 33. *In any equilibrium in the myopic setting such that firms play according to undominated strategies, firms' set their product 1 prices as $\sigma_1^A(\emptyset) = p_1^{A,M}(m)$, and $\sigma_1^B(\emptyset) = p_1^{B,M}(m)$, respectively, where*

$$p_1^{A,M}(m) = \max\left\{\frac{1}{2} - \frac{m}{12}, 0\right\}, \quad p_1^{B,M}(m) = \max\left\{\frac{1}{2} - \frac{m}{6}, -\frac{1}{2}\right\}.$$

Proof of Claim 33. Consider the two-player normal-form game with action spaces S^A, S^B (which we define below) and profit functions given by

$$\pi_M^A\left(p_1^A, p_1^B\right) = p_1^A \bar{X}\left(p_1^A, p_1^B\right), \quad \pi_M^B\left(p_1^A, p_1^B\right) = \left(p_1^B + m/4\right)\left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right) + m/4, \tag{E.4}$$

respectively. As these are the expected profit functions given in Claim 32, we have that in any equilibrium for the myopic setting, the firms' prices for product 1 must form a Nash equilibrium in the game that we just described.

Note that in this game, setting any negative price is a stricly dominated strategy for firm A (as it is better off setting a price of zero and receiving zero profits). Thus, to focus in undominated strategies, we define the game's action spaces as $S^A = [0, \infty)$ and $S^B = \mathbb{R}$. Considering this choice for action spaces, it is straightforward to show that firms' best-response correspondences are, respectively,

$$BR_M^A(p_1^B) = \begin{cases} p_1^B/2 + 1/4 & \text{if } p_1^B > -1/2, \\ [0, \infty) & \text{if } p_1^B \leq -1/2. \end{cases} \quad \text{and} \quad BR_M^B(p_1^A) = \max\left\{-1/2, \left(p_1^A - m/4 + 1/2\right)/2\right\}.$$

One can verify that the only pure-strategy Nash equilibrium induced by these correspondences is $\left(p_1^{A,M}(m), p_1^{B,M}(m)\right)$. □

Finally, we show that the myopic setting admits an equilibrium. To prove this, we construct an equilibrium that follows the structure established in the previous claims.

CLAIM 34. *For any $m > 0$, there exists an equilibrium in the myopic setting,*

Proof of Claim 34. We follow a similar argument as in the proof of Claim 9, i.e., we will construct an assessment that follows the same structure given in Claims 29–33 and show that it is an equilibrium in the myopic setting.

Firms' strategies. Let $\sigma_1^A(\emptyset) = p_1^{A,M}(m)$ and $\sigma_1^B(\emptyset) = p_1^{B,M}(m)$, where these prices are given in Claim 33. Moreover, define firm B's pricing strategy for product 2, given $I \in \mathcal{I}_2$ as

$$\sigma_2^B(I) = \begin{cases} m\theta(I), & \text{if } I \in \mathcal{I}_2^1, \\ m/2, & \text{if } I \in \mathcal{I}_2^0. \end{cases}$$

Consumer's strategy. As in the proof of Claim 28, define γ_2 as in (B.10) with $q(h) = 1$ for any $h \in H_2^c$ such that $m\theta(h) = p_2^B(h)$, and define γ_1 as in (E.2), with $\beta(h) = 1$ for any $h \in H_1^c$ such that $s(h) = p_1^B(h) - p_1^A(h) + 1/2$.

Belief system. Define the beliefs in the first period as $\mu_1(\emptyset) = \mu_0$. For the second period, given $I \in \mathcal{I}_2$, define $\mu_2(\cdot \mid I)$ as follows:

1. If $I \in \mathcal{I}_2^1$, let $\mu(\cdot \mid I)$ be the probability distribution that assigns probability 1 to the true consumer type $(s(I), \theta(I))$. That is, for any Borel set $B \subseteq \mathcal{T}$, let $\mu(B \mid I) = \mathbf{1}_B(s(I), \theta(I))$.
2. If $I \in \mathcal{I}_2^0$, let $\mu(\cdot \mid I)$ be the uniform probability distribution on $[0, \bar{X}(p_1^A, p_1^B)] \times [0, 1]$.

We claim that $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ is an equilibrium in the myopic setting. To see this, first observe that $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B)$; indeed, if $I \in \mathcal{I}_2^1$, $\boldsymbol{\mu}$ is consistent by definition, and if $I \in \mathcal{I}_2^0$, $\boldsymbol{\mu}$ is consistent by the same argument[45] as in the proof of Claim 30.

In addition, by the proofs of Claims 2 and 31, σ_2^B is sequentially rational at time $t = 2$ for firm B given the other players' strategies and the belief system $\boldsymbol{\mu}$. By Claims 32 and 33, σ_1^A and σ_1^B are sequentially rational for firms at time $t = 1$. Finally, it follows from the same arguments as in the proofs of Claims 1 and 25 that $\boldsymbol{\gamma}$ is sequentially rational for the consumer at times $t = 1, 2$. □

To conclude, we complete this appendix by proving Proposition 2, leveraging the claims we have established.

Proof of Proposition 2. By Claim 34, there exists an equilibrium $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ for the myopic setting. By Claim 33, we have that in any equilibrium, firms' prices for product 1 are $p_1^{A,M}(m)$ and $p_1^{B,M}(m)$, respectively. Let us denote $\bar{X}^M(m) = p_1^{B,M}(m) - p_1^{A,M}(m) + 1/2$.

Then, by Claim 29, in the equilibrium path, the consumer buys product 1 from firm A if $s < \bar{X}^M(m)$, in which case we have from Claim 31 that firm B sets a price of $p_2^{B,N} = m/2$ for product 2, which, by Claim 29, the consumer strictly prefers to buy if $\theta > 1/2$.

On the other hand, if the consumer's type satisfies $s > \bar{X}^M(m)$, by Claim 29, in the equilibrium path, the consumer buys product 1 from firm B. Due to data tracking, firm B observes the consumer's type and (by Claim 2) sets a price of $p_2^B = m\theta$ for product 2, which the consumer buys with probability 1.

[45] If $\bar{X}\left(p_1^A(I), p_1^B(I)\right) = 0$, the beliefs cannot be obtained by Bayes' rule since the consumer buys product 1 from firm B with probability 1 (see the proof of Claim 30). Thus, we can define beliefs arbitrarily in this case.

Finally, observe that the expected demand to buy product 1 from firm A is $\bar{X}^M(m)$, which can easily shown to be equal to $p_1^{A,M}(m)$ by straightforward computation. □

E.3. Proof of Proposition 4

In order to prove Proposition 4, we first establish the following claim.

CLAIM 35. *Fix $0 < m < m_L$, and let W and $\tilde{x}$ be defined as in (F.28) and (B.3), respectively. Suppose that $x \in (0, \tilde{x}(m))$ is such that $W(x,m) < 0$, then $x > \bar{X}^*(m)$.*

Proof of Claim 35. From Equation (F.37) in the proof of Claim 44, we can write for $0 < x < \tilde{x}(m)$,

$$W(x,m) = -\frac{N(x,m)\left(\sqrt{2x(2x+m)} - 2x\right)}{D(x,m)},$$

where

$$\begin{aligned} N(x,m) &= m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x+m)}\,(1-2x)\,, \\ D(x,m) &= 2\left((8x+2m)\sqrt{2x(2x+m)} - 2x(8x+3m)\right). \end{aligned} \tag{E.5}$$

We have shown that $D(x,m) > 0$ for $x, m > 0$ in (F.34) in the proof of Claim 43. Moreover, we know by Claim 44 that $W(\bar{X}^*(m), m) = 0$ and $\bar{X}^*(m) < \tilde{x}(m)$. In particular, it follows that $N(\bar{X}^*(m), m) = 0$. Then, it sufices to show that if $N(x,m) > 0$, then $x > \bar{X}^*(m)$. To do so, first notice that $N(0,m) = m^2 - 4m$, and since $m_L < 4$ (by Remark 1), we have that $N(0,m) < 0$ for all $m \in (0, m_L)$.

By step 4 in the proof of Claim 44, $N(x,m)$ is strictly convex in x for $m > 0$ fixed. Therefore, $x = \bar{X}^*(m)$ is the unique root of $N(x,m)$, which implies that $N(x,m) > 0$ for $x > \bar{X}^*(m)$. □

Given this result, we now prove Proposition 4.

Proof of Proposition 4. We want to show that for $0 < m < m_L$, we have

$$0 < p_1^{A,M}(m) - p_1^{B,M}(m) < p_1^{A*}(m) - p_1^{B*}(m),$$

where $p_1^{A,M}(m)$ and $p_1^{B,M}(m)$ are firms' equilibrium prices for product 1 in the myopic setting, and $p_1^{A*}(m)$ and $p_1^{B*}(m)$ are the corresponding equilibrium prices in the forward-looking setting. From Proposition 2, it is easy to see that $p_1^{A,M}(m) > p_1^{B,M}(m)$. To show that the second inequality holds, it is equivalent to prove that $\bar{X}^M(m) > \bar{X}^*(m)$, where, by Proposition 2,

$$\bar{X}^M(m) = p_1^{B,M}(m) - p_1^{A,M}(m) + \frac{1}{2} = \max\left\{\frac{1}{2} - \frac{m}{12}, 0\right\},$$

and $\bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2$.

First, recall that, by Claim 44, $\bar{X}^*(m)$ satisfies $W(\bar{X}^*(m), m) = 0$, where W is defined in (F.28), and that $\bar{X}^*(m) < \tilde{x}(m)$. Thus, if $\bar{X}^M(m) \geq \tilde{x}(m)$, the result follows.

Consider then the case where $\bar{X}^M(m) < \tilde{x}(m)$. By Claim 35, it suffices to show that $W(\bar{X}^M(m), m) < 0$, which we do now.[46] By the proof of Claim 35, we just need to show that $N(\bar{X}^M(m), m) > 0$. By plugging in $\bar{X}^M(m) = 1/2 - m/12$ and simplifying the resulting expression we have that

$$N(\bar{X}^M(m), m) = \frac{m}{36}\left(42 + 5m - \sqrt{(6-m)(6+5m)}\right).$$

Straightforward algebra shows that this expression is positive for $m \in (0, 6]$. In particular, since $m_L < 4$ (by Remark 1), it follows that $N(\bar{X}^M(m), m) > 0$ for $m \in (0, m_L)$. Therefore, $\bar{X}^M(m) > \bar{X}^*(m)$ for any $m \in (0, m_L)$, as desired. □

[46] We have that $\bar{X}^M(m) > 0$ for $0 < m < m_L$ since $m_L < 4$ (by Remark 1).

E.4. Proof of Proposition 5

In this appendix, we compare the equilibrium expected profit for firm B across our three settings, for the range of parameters in which the forward-looking setting admits an interior equilibrium (i.e., $0 < m < m_L$). First, we show that firm B's equilibrium expected profit is lower in the restricted than in the forward-looking setting (Claim 36), and then that this expected profit is lower than in the myopic setting (Claim 37). Both propositions result in Proposition 5. In what follows, we refer to expected profit simply as "profit", and denote firm B's equilibrium expected profit on each setting by $\Pi_R^B(m)$, $\Pi_N^B(m)$, and $\Pi_{FL}^B(m)$ respectively.

CLAIM 36. *For all $m \in (0, m_L)$, we have that $\Pi_{FL}^B(m) > \Pi_R^B(m)$.*

Proof of Claim 36. Firm B's equilibrium profit in the restricted setting is $\Pi_R^B(m) = (m+1)/4$, since it sells product 1 at a price of 1/2 with probability 1/2 and, independently, sells product 2 for $m/2$, also with probability 1/2. In the forward-looking setting, for $0 < m < m_L$, we have that $\Pi_{FL}^B(m) = \pi^B\left(p_1^{A*}(m), p_1^{B*}(m), m\right)$ by Claim 8, where $(p_1^{A*}(m), p_1^{B*}(m))$ are the unique equilibrium prices for $\mathbf{G}(m)$. We now compare $\Pi_R^B(m)$ and $\Pi_{FL}^B(m)$, considering two cases.

Case 1. $3 \le m < m_L$. Since $p_1^{B*}(m)$ is firm B's best response to $p_1^{A*}(m)$ in $\mathbf{G}(m)$ and $p_1^{A*}(m) > 0$ (since, by Theorem 1, $\mathbf{G}(m)$ admits an interior equilibrium), we have that

$$\begin{aligned}\pi_{FL}^B(m) = \pi^B\left(p_1^{A*}(m), p_1^{B*}(m), m\right) &\ge \pi^B\left(p_1^{A*}(m), p_1^{A*}(m) - 1/2, m\right)\\ &= p_1^{A*}(m) + \frac{m-1}{2} > \frac{m-1}{2} \ge \frac{m+1}{4} = \pi_R^B(m),\end{aligned}$$

where the last inequality follows since $m \ge 3$.

Case 2. $0 < m < 3$. Since $p_1^{B*}(m)$ is firm B's best response to $p_1^{A*}(m)$ in $\mathbf{G}(m)$, firm B is (weakly) better off choosing $p_1^{B*}(m)$ instead of $p_1^{A*}(m) - m/6$, if firm A sets $p_1^{A*}(m)$. Therefore,

$$\begin{aligned}\pi_{FL}^B(m) &= \pi^B\left(p_1^{A*}(m), p_1^{B*}(m), m\right)\\ &\ge \pi^B\left(p_1^{A*}(m), p_1^{A*}(m) - m/6, m\right) \qquad \text{(E.6)}\\ &= \left(p_1^{A*}(m) - m/6\right)\left(1 - \psi\left(1/2 - m/6, m\right)\right) + m\phi\left(1/2 - m/6, m\right).\end{aligned}$$

In order to give a lower bound for the RHS of the last inequality, we will establish a bound for $p_1^{A*}(m)$. We claim that for $0 < m < 3$, we have that $p_1^{A*}(m) > Z\left(1/2 - m/6, m\right)$, where Z is defined as in (B.29).

To prove this, recall that by Claim 12, $p_1^{A*}(m)$ satisfies firm A's first order condition for profit maximization, $p_1^{A*}(m) = Z(\bar{X}^*(m), m)$. Since Z is strictly increasing in its first argument,[47] it suffices to show that $\bar{X}^*(m) > 1/2 - m/6$. The proof of Claim 35 implies that for $m \in (0, 3)$, $N(x, m) < 0$ if and only if $x < \bar{X}^*(m)$, where N is defined as in (E.5). Evaluating $N(x, m)$ for $x = 1/2 - m/6$ results in

$$N\left(1/2 - m/6, m\right) = \frac{m}{9}\left(3 - 4m - \sqrt{(3-m)(2m+3)}\right) < 0.$$

Therefore, $p_1^{A*}(m) > Z\left(1/2 - m/6, m\right)$ for $0 < m < 3$. Combining this with (E.6) results in $\pi_{FL}^B(m) > b(m)$, where we define

$$b(m) = \left(Z\left(1/2 - m/6, m\right) - m/6\right)\left(1 - \psi\left(1/2 - m/6, m\right)\right) + m\phi\left(1/2 - m/6, m\right).$$

[47] To see this, note that $Z(x, m) = \psi(x, m)/\psi_x(x, m)$. Since $\psi(x, m)$ is non-negative, strictly increasing and strictly concave in x (by Claim 10), it follows that $Z(x, m)$ is strictly increasing in x.

Therefore, it suffices to show that $b(m) > \pi_R^B(m) = (m+1)/4$ to complete the proof. To do so, define $\bar{b}(m) = b(m) - (m+1)/4$. After plugging in the functional forms for Z, ψ, ϕ (see equations (B.7), (F.4), and (F.12)) into $b(m)$ and simplifying the resulting expression we have:[48]

$$\bar{b}(m) = \frac{(m-3)^2\left(324m + 153m^2 - 25m^3 - 216 + 2(m+6)(6-11m)\sqrt{(3-m)(3+2m)}\right)}{324m\left((m-3)(12+5m) + (12+2m)\sqrt{(3-m)(3+2m)}\right)}. \tag{E.7}$$

We want to show that this expression is positive for $m \in (0,3)$. First, we show that the denominator is positive. Note that we can write the term inside parentheses as

$$\sqrt{3-m}\left[(12+2m)\sqrt{(3+2m)} - (12+5m)\sqrt{3-m}\right],$$

which is positive for $m \in (0,3)$ since

$$(12+2m)^2(3+2m) - (12+5m)^2(3-m) = 216m + 153m^2 + 33m^3 > 0.$$

Now consider the numerator of (E.7), and note that it suffices to show that for $0 < m < 3$,

$$h(m) \equiv \underbrace{324m + 153m^2 - 25m^3 - 216}_{P_1(m)} + \underbrace{2(m+6)(6-11m)}_{P_2(m)}\sqrt{(3-m)(3+2m)} > 0.$$

We have that $h(0) = 0$ and $h(3) = 1458$, and so we just need to show that $h(m)$ has no roots in $(0,3)$. If h had such a root, it would satisfy the following equation:

$$\left(324m + 153m^2 - 25m^3 - 216\right)^2 = 4(m+6)^2(6-11m)^2(3-m)(3+2m),$$

which, after expanding terms, simplifies to

$$27m^3(59m^3 + 54m^2 + 351m - 216) = 0.$$

Since $m > 0$, the equation is equivalent to $\hat{h}(m) \equiv 59m^3 + 54m^2 + 351m - 216 = 0$. Computation shows that $\hat{h}(1/2) < 0$ and $\hat{h}(6/11) > 0$, and so $\hat{h}$ admits a root in $(1/2, 6/11)$. Denote this root by m_*. Furthermore, since $\hat{h}$ is strictly increasing for positive m, m_* is the only positive root of $\hat{h}$. However, we claim that $h(m_*) > 0$. To prove this, we show that $P_1(m_*)$, $P_2(m_*) > 0$. First, we have that

$$P_1(m) - \hat{h}(m) = 3m(4m-3)(3-7m),$$

and so $P_1(m) - \hat{h}(m) > 0$ for $m \in (3/7, 3/4)$. In particular, $m_* \in (3/7, 3/4)$, thus $P_1(m_*) > 0$.

For $P_2(m) = 2(m+6)(6-11m)$, we have that $P_2(m) > 0$ for all $m \in (0, 6/11)$, and therefore $P_2(m_*) > 0$. It follows that $h(m_*) > 0$, and therefore h has no roots in $(0,3)$.

Therefore, the numerator of (E.7) is positive for $0 < m < 3$, and it follows that $\bar{b}(m) > 0$, as desired. ☐

The next proposition compares firm B's profits in the myopic and forward-looking settings, when the forward-looking setting admits an interior equilibrium ($0 < m < m_L$).

Claim 37. *For all $m \in (0, m_L)$, we have that $\Pi_M^B(m) > \Pi_{FL}^B(m)$.*

[48] Note that as $1/2 - m/6 < \bar{X}^*(m) < \tilde{x}(m)$, we use the functional forms for Z, ψ, and ϕ for the case when $x < \tilde{x}(m)$.

Proof of Claim 37. We first derive the equilibrium profit for firm B in the myopic setting. Recall from (E.4) thaf firm B's expected profit function in terms of product 1 prices is

$$\pi_M^B\left(p_1^A, p_1^B\right) = \left(1 - \bar{X}(p_1^A, p_1^B)\right)\left(p_1^B + m/4\right) + m/4.$$

By Proposition 2, the unique equilibrium product 1 prices in the myopic setting are $p_1^{A,M}(m) = \max\{1/2 - m/12, 0\}$ and $p_1^{B,M}(m) = \max\{1/2 - m/6, -1/2\}$. By plugging in these prices, we have that firm B's equilibrium expected profit is

$$\Pi_M^B(m) = \begin{cases} \frac{m}{4} + \left(\frac{1}{2} + \frac{m}{12}\right)^2, & \text{if } m < 6, \\ \frac{m-1}{2}, & \text{if } m \geq 6. \end{cases} \tag{E.8}$$

Now fix $m \in (0, m_L)$. We will show that $\Pi_M^B(m) > \Pi_{FL}^B(m)$.

By Remark 1, we know that $m_L < 4$ and therefore $\Pi_M^B(m) = \frac{1}{4} + \frac{m}{3} + \frac{m^2}{144}$. Thus, it suffices to show that $\frac{1}{4} + \frac{m}{3} > \pi_2^{FL}(m)$. Let us define

$$F(m) = \frac{1}{4} + \frac{m}{3} - \Pi_{FL}^B(m) = \frac{1}{4} + \frac{m}{3} - p_1^{B*}(m)\left(1 - \psi\left(\bar{X}^*(m), m\right)\right) - m\phi\left(\bar{X}^*(m), m\right).$$

We want to show that $F(m) > 0$ for all $m \in (0, m_L)$. As in the proof of Claim 23, we will show that $F(m)$ (i) is continuous for $m \in (0, m_L)$, (ii) has no roots in $(0, m_L)$, and (iii) $F(m) > 0$ for some $m \in (0, m_L)$.

To prove (i), recall that by Claim 16, $p_1^{B*}(m) = V(\bar{X}^*(m), m)$, where V is defined as in (B.31). Therefore, we can write $F(m) = f(\bar{X}^*(m), m)$ where

$$f(x, m) = \frac{1}{4} + \frac{m}{3} - V(x, m)\left(1 - \psi(x, m)\right) - m\phi(x, m). \tag{E.9}$$

Then, $f(x, m)$ is continuous since both $V(x, m)$ and $\phi(x, m)$ are continuous functions of $(x, m) \in [0, 1] \times \mathbb{R}^{++}$ (see the proof of Claim 10). By the proof of Claim 23, we know that $\bar{X}^*(m)$ is continuous for $m \in (0, m_L)$. Thus, $F(m) = f(\bar{X}^*(m), m)$ is continuous in $(0, m_L)$.

To show (ii), we follow the same approach as in the proof of Claim 23. That is, we show that $F(m) = f(\bar{X}^*(m), m)$ has no roots in $(0, m_L)$ by showing that the following system admits no solution with $0 < x < \min\{1/2, \tilde{x}(m)\}$ and $0 < m < m_L$.

$$\begin{aligned} f(x, m) &= 0, \\ N(x, m) &= 0, \end{aligned} \tag{E.10}$$

where $N(x, m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x + m)}\,(1 - 2x)$.

By plugging in the expressions for $V(x, m) = (1 - \psi(x, m) + m\phi_x(x, m))/\psi_x(x, m)$ and $\psi(x, m)$ for $x < \tilde{x}(m)$ into (E.9) (from equations (B.7), (F.4), and (F.13)) and simplifying the resulting expression, we can write $f(x, m) = K_1(x, m)/K_2(x, m)$, where

$$\begin{aligned} K_1(x, m) &= xK_3(x, m) + \sqrt{2x(2x + m)}K_4(x, m), \\ K_2(x, m) &= 12m\left(\sqrt{2x(2x + m)}\,(4x + m) - x(8x + 3m)\right), \\ K_3(x, m) &= 128x^4 - 2m^3x + 8mx(16x^2 - 3) + m^2(32x^2 - 8x - 9) \\ K_4(x, m) &= m^3 - 64x^4 - m^2(12x^2 - 16x + 3) + 12mx(1 - 4x^2). \end{aligned}$$

It is straightforward to show that $K_2(x,m)>0$ for all $x,m>0$. Therefore, system (E.10) can be written equivalently as

$$\begin{aligned} K_1(x,m) &= 0, \\ N(x,m) &= 0. \end{aligned} \tag{E.11}$$

We now show that this system has no solution with $0<x<1/2$ and $0<m<m_L$. As in the proof of Claim 23, we change variables to $w=\sqrt{2x}$, $z=\sqrt{2x+m}$, so that $\sqrt{2x(2x+m)}=wz$. By plugging the change of variables into (E.11) we can write

$$K_1(w^2/2, z^2-w^2) = w(w-z)^2 Q_0(w,z)/2,$$
$$N(w^2/2, z^2-w^2) = Q_2(w,z),$$

where,

$$\begin{aligned} Q_0(w,z) &= 3w^3-4w^5+w^7-12w^2z+6w^4z-21wz^2+24w^3z^2-4w^5z^2-6z^3 \\ &\quad +16w^2z^3-8w^4z^3+4wz^4-w^3z^4+2z^5, \\ Q_2(w,z) &= -w^2-wz+w^3z-4z^2+4w^2z^2+z^4. \end{aligned}$$

Showing that system (E.11) has no solutions with $0<x<1/2$ and $0<m<m_L$ is then equivalent to showing that the transformed system has no solutions with $0<w<1$ and $0<z^2-w^2<m_L$. Moreover, since we look for solutions where $w>0$ and $w>z$, it suffices to analyze the solutions of the following system:

$$Q_0(w,z)=0,$$
$$Q_2(w,z)=0.$$

This system consists of two polynomial equations with integer coefficients in two variables, and it can be solved by computing the the Gröbner basis of Q_0, Q_2, and solving the resulting triangular system (see, e.g., Cox et al. (2015), Sturmfels (2002), Sturmfels (2005)). We find that the system has seven real solutions,[49] but none of them satisfy $0<w<1$ and $z>0$. It follows that system (E.11) has no solutions with $0<x<1/2$ and $0<m<m_L$ and, thus, F has no roots in $(0,m_L)$.

Finally, to show (iii), we evaluate F at $m=2$. We have that $\Pi^B_{FL}(2)\approx 0.834716$. Therefore, $F(2)\approx 1/4+2/3-0.834716\approx 0.08195>0$. It follows that $F(m)>0$ for $m\in(0,m_L)$. □

We conclude this section with the proof of Proposition 5, which results from the previous two claims.

Proof of Proposition 5. Let $m\in(0,m_L)$. By Claims 36–37, we have that $\Pi^B_M(m)>\Pi^B_{FL}(m)>\Pi^B_R(m)$. □

E.5. Proof of Proposition 6

In this appendix we compare firm A's equilibrium expected profit in our three settings, focusing on the values of m for which the forward-looking setting admits an interior equilibrium ($0<m<m_L$). These comparisons derive in the proof of Proposition 6.

We proceed in two steps. First, Claim 38 establishes that firm A's equilibrium profit is highest in the restricted setting than in any of the other two settings. Then, in Claim 39, we show that firm A's profit is higher in the forward-looking than in the myopic setting for small values of m, but the opposite occurs

[49] We obtain the solutions by using the Solve routine in Mathematica. The real solutions to system are $(0,0)$, $\pm(1,1)$, $\pm(0.884,-1.14201)$, $\pm(1.27263,0.626246)$.

for large values. These comparisons result in Proposition 6. At the end of this section, we state and prove Claim 40, an auxiliary result that we use in the proof of Claim 38.

In what follows, we denote by $\Pi_R^A(m)$, $\Pi_M^A(m)$, and $\Pi_{FL}^A(m)$ the corresponding equilibrium expected profit for firm A in the three settings.

CLAIM 38. *For any $m \in (0, m_L)$, firm A's equilibrium expected profit is highest in the restricted setting.*

Proof of Claim 38. First, we claim that the corresponding expressions for firm A's equilibrium profit on each setting are as follows:

$$\Pi_R^A(m) = \frac{1}{4}, \quad \Pi_M^A(m) = \left(\max\left\{0, \frac{1}{2} - \frac{m}{12}\right\}\right)^2, \quad \Pi_{FL}^A(m) = p_1^{A*}(m)\psi^*(m). \tag{E.12}$$

In the restricted setting, by Proposition 1, we know that in equilibrium, firm A sets a price of $1/2$ for product 1 and the consumer buys with probability $1/2$. Therefore, $\Pi_R^A(m) = 1/4$.

For the myopic setting, by Proposition 2, the corresponding equilibrium outcome is such that firm A sets a price of $p_1^{A,M}(m) = \max\{1/2 - m/12, 0\}$ and the consumer buys from it with probability $\bar{X}^M(m) = p_1^{A,M}(m)$. Thus, $\Pi_M^A(m) = \left(p_1^{A,M}(m)\right)^2 = (\max\{1/2 - m/12, 0\})^2$.

Finally, for the forward-looking setting we have that, by Claim 8,

$$\Pi_{FL}^B(m) = \pi^B\left(p_1^{A*}(m), p_1^{B*}(m), m\right) = p_1^{A*}(m)\psi^*(m),$$

where $(p_1^{A*}(m), p_1^{B*}(m))$ are the unique equilibrium prices for $\mathbf{G}(m)$, and we denote $\psi^*(m) = \psi\left(\bar{X}^*(m), m\right)$.

The comparison between the restricted and myopic settings is immediate as for any $m > 0$ we have $\max\left\{0, \frac{1}{2} - \frac{m}{12}\right\} < 1/2$, and taking squares on both sides of this inequality yields that $\Pi_M^A(m) < \Pi_R^A(m)$.

We now compare the restricted with the forward-looking setting. When $m \to 0^+$, we have that $p_1^A(m), \psi^*(m) \to 1/2$ as formally established in Claim 40. Therefore, $\lim_{m\to 0^+} \Pi_{FL}^A(m) = \Pi_R^A(0) = 1/4$. Moreover, note that $\Pi_{FL}^A(m)$ is strictly decreasing for $m \in (0, m_L)$ since both $p_1^{A*}(m)$ and $\psi^*(m)$ are positive and strictly decreasing in m (by Claim 40). Thus, for $0 < m < m_L$, $\Pi_{FL}^A(m) < 1/4 = \Pi_R^A(m)$. □

Now that we have established that firm A's is better off in the restricted setting than when firm B employs data tracking, we compare firm A's equilibrium profit in the forward-looking and myopic settings.

CLAIM 39. *There exists a cutoff $\bar{m} \in (0, m_L)$ such that $\Pi_{FL}^A(m) > \Pi_M^A(m)$ when $m \in (0, \bar{m})$ and vice versa when $m \in (\bar{m}, m_L)$.*

Proof of Claim 39. We proceed in a similar fashion to the proofs of Claims 23 and 37, but rather than showing that the function defined as the difference in firm A profit between the forward-looking and myopic scenarios has no roots in $(0, m_L)$, we will show that only one such root exists.

Let $F(m) = \Pi_{FL}^A(m) - \Pi_M^A(m)$. Since $0 < m < m_L$, we know that $(p_1^{A*}(m), p_1^{B*}(m))$ is an interior PSNE in $\mathbf{G}(m)$. By Claim 12, we have that $p_1^{A*}(m) = Z(\bar{X}^*(m), m)$, where Z is defined as in (B.29). Thus, since $\Pi_{FL}^A(m) = p_1^{A*}(m)\psi^*(m)$ and $\Pi_M^A(m) = (1/2 - m/12)^2$ (as $m < m_L < 6$, by Remark 1), we can write $F(m) = \Omega(\bar{X}^*(m), m)$ where

$$\Omega(x, m) = Z(x, m)\psi(x, m) - \left(\frac{1}{2} - \frac{m}{12}\right)^2.$$

By a similar argument to the proofs of Claims 23 and 37, we have that $F(m) = \Omega(\bar{X}^*(m), m)$ is continuous in $(0, m_L)$.

We will now show that F has a unique root $\bar{m} \in (0, m_L)$, and then that F changes sign at $\bar{m}$. Following the same argument as in proof of Claim 23, recall that $x = \bar{X}^*(m)$ is the unique solution in $(0, 1/2)$ to $N(x, m) = 0$ given fixed $m > 0$, where N is defined in (F.38). Therefore, $m \in (0, m_L)$ solves $F(m) = 0$ if and only if the following system has a solution with with $0 < x < \min\{1/2, \tilde{x}(m)\}$ and $0 < m < m_L$.

$$\begin{aligned} \Omega(x, m) &= 0, \\ N(x, m) &= 0. \end{aligned} \tag{E.13}$$

By plugging in the functional forms for Z and ψ (from equations (B.7) and (F.4)) and simplifying the resulting expression, we have that for all $x < \tilde{x}(m)$,

$$\Omega(x, m) = \frac{4x^2(2x+m)^2}{2(2x+m)^2 + (4x+m)\sqrt{2x(2x+m)}} - \frac{1}{144}(m-6)^2.$$

We claim that System (E.13) has a unique solution such that $x \in (0, 1/2)$ and $m \in (0, m_L)$. To do so, we change of variables by letting $w = \sqrt{2x}$, $z = \sqrt{2x+m}$, so that $\sqrt{2x(2x+m)} = wz$. Then, we look for solutions such that $0 < w < 1$, $z > 0$ and $0 < z^2 - w^2 < m_L$ to the following transformed system:

$$\begin{aligned} \Omega(w^2/2, z^2 - w^2) &= 0, \\ N(w^2/2, z^2 - w^2) &= 0. \end{aligned}$$

By plugging in the change of variables, we have that $N(w^2/2, z^2 - w^2) = Q_2(w, z)$ and $\Omega(w^2/2, z^2 - w^2) = Q_3(w, z)$ where

$$\begin{aligned} Q_2(w, z) &= -w^2 - wz + w^3 z - 4z^2 + 4w^2 z^2 + z^4, \\ Q_3(w, z) &= \frac{w^4 z^3}{w^3 + wz^2 + 2z^3} - \frac{1}{144}\left(z^2 - w^2 - 6\right)^2. \end{aligned}$$

Note that even though Q_3 is not a polynomial, the denominator of the first term is always positive for $0 < w < 1$ and $z > 0$. Therefore the roots of $Q_3(w, z)$ are the same of $\tilde{Q}_3(w, z)$ where:

$$\tilde{Q}_3(w, z) = 144\left(w^3 + wz^2 + 2z^3\right) Q_3(w, z) = 144 w^4 z^3 - \left(w^3 + wz^2 + 2z^3\right)\left(z^2 - w^2 - 6\right)^2.$$

We then look for solutions to the following system of equations:

$$\begin{aligned} Q_2(w, z) &= 0, \\ \tilde{Q}_3(w, z) &= 0. \end{aligned} \tag{E.14}$$

System (E.14) is a system of two polynomial equations with integer coefficients in two variables. We can compute all the solutions to the system by computing the Gröbner basis of the system and solving the resulting triangular system (as in the proofs of Claims 23 and 37). We find that there exists a unique solution to (E.14) such that[50] $0 < w < 1$ and $z > 0$, which is approximately $(w_{sol}, z_{sol}) \approx (0.818686, 1.3689)$. By reversing the change of variables, we let $\bar{m} = z_{sol}^2 - w_{sol}^2 \approx 1.20365 < m_L$. It follows that F has a unique root $\bar{m} \in (0, m_L)$ which is close to 1.2.

Finally, it remains to show that F changes its sign for some $m \in (0, m_L)$. To see this, we simply compute $F(1/2) \approx 0.0092 > 0$ and $F(3) \approx -0.0337 < 0$. This implies that $F(m) > 0$ for $0 < m < \bar{m}$ and $F(m) < 0$ for $\bar{m} < m < m_L$, as desired. □

[50] There are 7 real solutions to system (E.14): $(0, 0)$, $\pm(1, 1)$, $\pm(0.651494, -1.49704)$, $\pm(0.818686, 1.3689)$.

Proposition 6 now follows directly from the previous two claims.

Proof of Proposition 6. Let $m \in (0, m_L)$. By Claim 38, firm A's expected equilibrium profit is higher in the restricted setting than in the two scenarios with data tracking. Among the two other settings, by taking $\bar{m}$ as in Claim 38, we have that firm A's profits are higher when consumers are forward-looking if $m < \bar{m}$, while profits are higher with myopic consumers if $m > \bar{m}$. □

To end this section, we state and prove the following result, which was used in the proof of Claim 38.

CLAIM 40. *Consider $0 < m < m_L$, let $p_1^{A*}(m)$ and $p_1^{B*}(m)$ the unique product 1 equilibrium prices in the forward-looking setting, and let $\bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2$. In addition, let $\psi^*(m) = \psi(\bar{X}^*(m), m)$ be the expected product 1 demand for firm A in equilibrium. Then, the following properties hold:*

(i) $\bar{X}^*(m)$, $p_1^{A^*}(m)$, *and* $\psi^*(m)$ *are strictly decreasing in* m.

(ii) $\lim_{m\to 0^+} \bar{X}^*(m) = \lim_{m\to 0^+} p_1^{A*}(m) = \lim_{m\to 0^+} \psi^*(m) = 1/2$.

Proof of Claim 40. We first prove (i), which involves a series of algebraic arguments. We will prove that $\bar{X}^*(m)$ is strictly decreasing in m. The same can be shown for $p_1^{A^*}(m)$, and $\psi^*(m)$ using similar arguments, which we omit for brevity.

By the proof of Claim 18, it follows that $W(\bar{X}^*(m), m) = 0$, where W is defined in (F.28). In addition, by Claim 18, we know that $\bar{X}^*(m) < \tilde{x}(m)$, which implies that $W(x, m)$ is differentiable at $(\bar{X}^*(m), m)$, and so we can compute the derivative of $\bar{X}^*(m)$ as follows by the implicit function theorem:

$$\bar{X}^{*'}(m) = -\frac{W_m(\bar{X}^*(m), m)}{W_x(\bar{X}^*(m), m)}. \tag{E.15}$$

We claim that $\bar{X}^{*'}(m) < 0$. To prove this, we will show in two steps that $W_x(\bar{X}^*(m), m) < 0$ and $W_m(\bar{X}^*(m), m) < 0$.

Step 1. $W_x(\bar{X}^*(m), m) < 0$.

Write $W(x, m) = \varphi(x, m)/\psi_x(x, m)$, where φ is defined by

$$\varphi(x, m) = 1 - 2\psi(x, m) + m\phi_x(x, m) - \left(x - \frac{1}{2}\right)\psi_x(x, m).$$

By Claim 10, both $\varphi(x, m)$ and $\psi_x(x, m)$ are differentiable in x when $x < \tilde{x}(m)$ and, by taking the partial derivative w.r.t. x, we have

$$W_x(x, m) = \frac{\varphi_x(x, m)\psi_x(x, m) - \varphi(x, m)\psi_{xx}(x, m)}{(\psi_x(x, m))^2}.$$

In particular, $W_x(\bar{X}^*(m), m)$ is well-defined since $\bar{X}^*(m) < \tilde{x}(m)$, by Claim 44. In addition, since $W(\bar{X}^*(m), m) = 0$, it follows that $\varphi(\bar{X}^*(m), m) = 0$. Plugging back into the previous equation yields

$$W_x(\bar{X}^*(m), m) = \frac{\varphi_x(\bar{X}^*(m), m)}{\psi_x(\bar{X}^*(m), m)}.$$

The denominator is positive since ψ is strictly increasing in x (by Claim 10), and, by following the same argument as in the proof of Claim 18, we conclude that the numerator is negative. Thus, $W_x(\bar{X}^*(m), m) < 0$.

Step 2. $W_m(\bar{X}^*(m), m) < 0$.

We will show that $W_m(x, m) < 0$ for all $m > 0$, $x \in (0, \tilde{x}(m))$. Taking the derivative of W w.r.t. m yields

$$W_m(x, m) = \frac{\varphi_m(x, m)\psi_x(x, m) - \varphi(x, m)\psi_{xm}(x, m)}{(\psi_x(x, m))^2}. \tag{E.16}$$

By computing the derivatives involved in this expression (based on equations (B.7), (F.4), (F.12), and (F.13)) and simplifying the resulting expression, we can write

$$W_m(x,m) = x\frac{C_1(x,m)}{C_2(x,m)},$$

where

$$\begin{aligned}
C_1(x,m) &= C_3(x,m) - C_4(x,m) + \sqrt{2x(2x+m)}\left(C_6(x,m) - C_5(x,m)\right),\\
C_2(x,m) &= \sqrt{2x(2x+m)}\left(\sqrt{2x(2x+m)}(8x+2m) - 2x(8x+3m)\right)^2,\\
C_3(x,m) &= x\left(11m^3 + 100m^2x + 352mx^2 + 384x^3\right),\\
C_4(x,m) &= 2x\left(3m^2 + 24mx + 32x^2\right),\\
C_5(x,m) &= 2m^3 + 26m^2x + 128mx^2 + 192x^3,\\
C_6(x,m) &= 16x(2x+m).
\end{aligned} \tag{E.17}$$

From (F.34) in the proof of Claim 43, we know that $C_2(x,m) > 0$, and so it remains to show that $C_1(x,m) < 0$. We show this by proving the following two inequalities:

$$C_3(x,m) - \sqrt{2x(2x+m)}C_5(x,m) < 0,$$
$$-C_4(x,m) + \sqrt{2x(2x+m)}C_6(x,m) < 0.$$

By inspection, we note that $C_i(x,m) \geq 0$ for $x > 0$ and $i = 3,4,5,6$. The first inequality is then equivalent to $C_7(x,m) < 0$ where

$$C_7(x,m) = C_3(x,m)^2 - 2x(2x+m)C_5(x,m)^2.$$

Indeed, by plugging in from (E.17), we have that

$$C_7(x,m) = -m^2x\left(8m^5 + 103m^4x + 592m^3x^2 + 1856m^2x^3 + 3584mx^4 + 3072x^5\right) < 0.$$

Similarly, for the second inequality we show that $C_8(x,m) < 0$ where

$$C_8(x,m) = 2x(2x+m)C_6(x,m)^2 - C_4(x,m)^2.$$

By plugging in C_4 and C_6 from (E.17), we have that $C_8(x,m) = -4m^3x^2(9m+16x) < 0$. Adding the two inequalities that we just proved results in $C_1(x,m) < 0$, as desired. This implies that $W_m(x,m) < 0$ for all $0 < x < \tilde{x}(m)$. In particular, since $\bar{X}^*(m) < \tilde{x}(m)$, we have that $W_m(\bar{X}^*(m),m) < 0$.

Finally, by (E.15) and steps 1 and 2, it follows that $\bar{X}^{*\prime}(m) < 0$, i.e., $\bar{X}^*(m)$ is strictly decreasing in m.

Now, we prove property (ii). We start by showing that $\lim_{m\to 0^+}\bar{X}^*(m) = 1/2$.

From Step 3 in the proof of Claim 44, we have that $x = \bar{X}^*(m)$ solves $N(x,m) = 0$ given fixed m, where

$$N(x,m) = m^2 - 4m + 20x^2 + 12mx - 10x - \sqrt{2x(2x+m)}\left(1-2x\right).$$

Since $N(x,0) = 12x(2x-1)$, we have that $N(2/5,m) < 0$ for all small $m > 0$ by continuity. Moreover $N(1/2,m) = m^2 + 2m > 0$, for all $m > 0$. In addition, we have that $N(0,m) = m^2 - 4m < 0$ for all small $m > 0$. Since $N(x,m)$ is convex in x, it follows that given small enough $m > 0$, there is a unique $\hat{x} \in (0,1/2)$ such that $N(\hat{x},m) = 0$, and in addition, $\hat{x} \in (2/5,1/2)$. Moreover, by step 3 in the proof of Claim 44, we know that $\hat{x} = \bar{X}^*(m)$. Thus, for all small $m > 0$, it holds that $2/5 < \bar{X}^*(m) \leq 1/2$, and $N(\bar{X}^*(m),m) = 0$.

Let $\bar{X}^*(0) = \lim_{m\to 0^+} \bar{X}^*(m)$. By continuity of $N(x,m)$, we have that $N(\bar{X}^*(0), 0) = 0$, and $2/5 \le \bar{X}^*(0)$. Therefore, $\bar{X}^*(0) = 1/2$.

Next, we show that $\lim_{m\to 0^+} \psi^*(m) = 1/2$. Note that

$$\lim_{m\to 0^+} \psi\left(\bar{X}^*(m), m\right) = \lim_{m\to 0^+} 2\bar{X}^*(m)\bar{\theta}\left(\bar{X}^*(m), m\right) = \bar{X}^*(0) = 1/2,$$

where the second to last equality follows since $\lim_{m\to 0^+} \bar{\theta}\,(x,m) = 1/2$ for all $x \in (0,1)$, by the proof of Claim 42, and the fact that $\bar{X}^*(0) = 1/2$.

Finally, we show that $\lim_{m\to 0^+} p_1^{A^*}(m) = 1/2$. By Claim 12, we have that $p_1^{A^*}(m) = Z\left(\bar{X}^*(m), m\right)$ for $0 < m < m_L$, where $Z(x,m) = \psi(x,m)/\psi_x(x,m)$. Then, since $\psi^*(m) \to 1/2$ as $m \to 0^+$, it suffices to show that $\lim_{m\to 0^+} \psi_x\left(\bar{X}^*(m), m\right) = 1$. To prove this, we will first show that for all $x \in (0,1)$, $\lim_{m\to 0^+} \psi_x\,(x,m) = 1$. Indeed, for fixed $x \in (0,1)$ and small enough $m > 0$, it follows from (F.4) and (F.1) that

$$\psi_x(x,m) = 1 - \left(1 - \bar{\theta}(x,m)\right)\left[2 - \frac{4x+m}{\sqrt{4x^2+2mx}}\right].$$

By taking the limit as $m \to 0+$, we have that $\psi_x(x,m) \to 1$. Moreover, by Claim 10, we know that $\psi_x(x,m)$ is decreasing in x for fixed $m > 0$. Thus, for all small enough $m > 0$ we have that

$$\psi_x(2/5, m) \ge \psi_x\left(\bar{X}^*(m), m\right) \ge \psi_x\,(3/5, m)\,.$$

Taking $m \to 0^+$ shows that $\lim_{m\to 0^+} \psi_x\left(\bar{X}^*(m), m\right) = 1$, which implies that $\lim_{m\to 0^+} p_1^{A^*}(m) = 1/2$. □

Appendix F: Auxiliary Proofs for Appendix B.3

In this appendix, we prove several auxiliary results. Specifically, Appendix F.1 contains the proofs of Claims 10–19, which are intermediate steps in the proof of Lemma 3. Then, Appendix F.2 provides an additional result that characterize the set of values of m for which the game $\mathbf{G}(m)$ admits an interior pure-strategy Nash equilibrium (Proposition 8). This result allows us to provide numerical approximations for the constants m_L and m_H.

F.1. Proofs of Claims 10–19

In this appendix we prove Claims 10–19, which complete the proof of Lemma 3 as described in Appendix B.3. In the process of doing so, we state some additional results whose proofs we defer to the end of this appendix (see Appendix F.1.1).

Proof of Claim 10. Fix $m > 0$. The proof is organized as follows: first, we show all the stated properties for $\bar{\theta}$. Then, based on these results we prove the corresponding properties for ψ, ξ and ϕ respectively.

Properties of $\bar{\theta}$. We first show that $\bar{\theta}(x,m)$ is continuous in x, at any $x \in [0,1]$. Recall the definition of $\bar{\theta}$ from (B.2):

$$\bar{\theta}(x,m) = \begin{cases} \frac{1}{m}\left(2x + m - \sqrt{2x\,(2x+m)}\right), & \text{if } x \le \tilde{x}(m), \\ \frac{1}{1+x}\left[1 - \frac{1}{2m}(1-x)^2\right], & \text{if } x > \tilde{x}(m), \end{cases}$$

where $\tilde{x}(m) = \frac{1}{3}\left(\sqrt{(m-1)^2+3} - (m-1)\right)$. It is easy to verify continuity by inspection when $x \ne \tilde{x}(m)$, so we only need to prove continuity at $x = \tilde{x}(m)$. That is, we want to verify that the expression

$$\frac{1}{m}\left(2x + m - \sqrt{2x\,(2x+m)}\right) = \frac{1}{1+x}\left[1 - \frac{1}{2m}(1-x)^2\right]$$

is satisfied for $x = \tilde{x}(m)$. A few algebraic steps are necessary to verify this. First, one can verify that $x = \tilde{x}(m)$ is a solution for the quadratic equation given by $(1+x)^2 = 2x(2x+m)$, which can also be written as $(1-x)^2 = 4x^2 + 2(m-2)x$. Then, by plugging these two conditions above and simplifying the resulting expression, we have that continuity will hold if and only if

$$3\tilde{x}(m)^2 + 2(m-1)\tilde{x}(m) - 1 = 0.$$

This expression holds, since it can be rewritten as $(1+\tilde{x}(m))^2 = 2\tilde{x}(m)(2\tilde{x}(m)+m)$. Thus, $\bar{\theta}(x,m)$ is continuous in x for fixed $m > 0$.

To show that $\bar{\theta}(x,m)$ is differentiable in x, notice that this clearly holds by inspection at any $x \neq \tilde{x}(m)$, and the partial derivative of $\bar{\theta}$ w.r.t. x is[51]

$$\bar{\theta}_x(x,m) = \begin{cases} \frac{1}{m}\left[2 - \frac{4x+m}{\sqrt{4x^2+2mx}}\right], & \text{if } x \leq \tilde{x}(m), \\ \left[\frac{2}{m} - 1\right]\left[\frac{1}{(1+x)^2}\right] - \frac{1}{2m}, & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.1}$$

As before, we want to show that the expression for the two cases coincide when $x = \tilde{x}(m)$. Simple algebra shows that equating these two expressions of reduces to

$$(1+x)^2 = 2mx + 4x^2, \tag{F.2}$$

which holds for $x = \tilde{x}(m)$. Thus, $\bar{\theta}_x(x,m)$ is a continuous function of $x \in (0,1]$; i.e, $\bar{\theta}(x,m)$ is continuously differentiable in this region.[52] Moreover, simple algebra shows that $\bar{\theta}_x(x,m) < 0$ for all $x \in (0,1]$, which implies that $\bar{\theta}(x,m)$ is decreasing in x. Finally, by evaluating $\bar{\theta}(1,m) = 1/2$ and $\bar{\theta}(0,m) = 1$, we conclude that $\bar{\theta}(x,m) \in [1/2, 1]$ for all $x \in [0,1]$.

Properties of ψ. Recall from (B.7) that ψ is defined by $\psi(x,m) = 2x\bar{\theta}(x,m)$, for $x \in [0,1]$. Since $\bar{\theta}(x,m)$ is continuously differentiable in x and bounded, so is $\psi(x,m)$. Thus, we only need to show that ψ is strictly concave and strictly increasing in x. We will first prove strict concavity. Notice from (F.1) that $\bar{\theta}(x,m)$ is twice differentiable in x for all $x \in (0,1)$ with $x \neq \tilde{x}(m)$, and its second partial derivative w.r.t. x is

$$\bar{\theta}_{xx}(x,m) = \begin{cases} \frac{m}{(4x^2+2mx)^{3/2}}, & \text{if } x < \tilde{x}(m), \\ \frac{2(m-2)}{m(1+x)^3}, & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.3}$$

It follows that $\psi(x,m)$ is also twice differentiable in x for all $x \in (0,1)$, $x \neq \tilde{x}(m)$. Since $\psi_x(x,m)$ is continuous in x, to show strict concavity of ψ in x, it suffices to show that $\psi_{xx}(x,m) < 0$ for all $x \neq \tilde{x}(m)$.

To show this, we first obtain the expression for this second derivative. To derive a convenient expression, first note that by Claim 7 and equation (B.19) we have that

$$\psi(x,m) = \begin{cases} x + \frac{1}{2}m\left(1 - \bar{\theta}(x,m)\right)^2, & \text{if } x \leq \tilde{x}(m), \\ x + (1-x)(1-\bar{\theta}(x,m)) - \frac{1}{2m}(1-x)^2, & \text{if } x > \tilde{x}(m). \end{cases}$$

Thus, we can write the partial derivative of ψ w.r.t. x as

$$\psi_x(x,m) = \begin{cases} 1 - m\left(1 - \bar{\theta}(x,m)\right)\bar{\theta}_x(x,m), & \text{if } x \leq \tilde{x}(m), \\ \bar{\theta}(x,m) + \frac{1}{m}(1-x) - (1-x)\bar{\theta}_x(x,m), & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.4}$$

[51] Once we show $\bar{\theta}_x(x,m)$ is continuous at $x = \tilde{x}(m)$, we can define $\bar{\theta}_x(\tilde{x}(m),m)$ as any of the two cases by continuity.

[52] Note that $\lim_{t\to 0^+} \bar{\theta}_x(t,m) = -\infty$, but $\bar{\theta}_x(x,m)$ is finite for $x \in (0,1]$.

And therefore,

$$\psi_{xx}(x,m) = \begin{cases} m\left[\left(\bar{\theta}_x(x,m)\right)^2 - (1-\bar{\theta}(x,m))\bar{\theta}_{xx}(x,m)\right], & \text{if } x < \tilde{x}(m), \\ 2\bar{\theta}_x(x,m) - \frac{1}{m} - (1-x)\bar{\theta}_{xx}(x,m), & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.5}$$

We now show that this expression is negative for the two cases we have. First consider $0 < x < \tilde{x}(m)$. After plugging (F.3) into (F.5) and algebraic manipulation we have

$$\psi_{xx}(x,m) = \frac{m}{4x^2+2mx}\left[\frac{16x(2x+m) - 4(4x+m)\sqrt{4x^2+2mx}}{m^2} + \sqrt{\frac{2x}{2x+m}}\right].$$

Then, we want to show that for all $x \in (0, \tilde{x}(m))$,

$$\frac{16x(2x+m) - 4(4x+m)\sqrt{4x^2+2mx}}{m^2} + \sqrt{\frac{2x}{2x+m}} < 0.$$

By multiplying both sides by $m^2\sqrt{\frac{2x+m}{2x}}$, the inequality reduces to

$$(2x+m)\left[8\sqrt{2x(2x+m)} - 4(4x+m)\right] < -m^2. \tag{F.6}$$

Notice that the LHS of this inequality is strictly increasing in x. Therefore, it suffices to show that (F.6) holds, even if weakly, for $x = \tilde{x}(m)$. Plugging equation (F.2) into the LHS above results in (abbreviating $\tilde{x} = \tilde{x}(m)$)

$$\begin{aligned}(2\tilde{x}+m)\left[8\sqrt{2\tilde{x}(2\tilde{x}+m)} - 4(4\tilde{x}+m)\right] &= (2\tilde{x}+m)\left[8(1+\tilde{x}) - 4(4\tilde{x}+m)\right] \\ &= 4(2\tilde{x}+m)\left[2-m-2\tilde{x}\right] \\ &= -16\tilde{x}^2 - 16(m-1)\tilde{x} - 4m(m-2) \\ &= 8\tilde{x}^2 - 8 - 4m(m-2),\end{aligned}$$

where the last step follows since, by equation (F.2), $2(m-1)\tilde{x} = 1 - 3\tilde{x}^2$. Plugging back into inequality (F.6), we want to show that $8(1-\tilde{x}^2) + 4m(m-2) \geq m^2$, which is equivalent to

$$\tilde{x}(m)^2 \leq 1 - m + \frac{3}{8}m^2. \tag{F.7}$$

This inequality holds, in fact strictly,[53] for all $m > 0$. Thus, $\psi_{xx}(x,m) < 0$ for $0 < x < \tilde{x}(m)$.

Now consider $\tilde{x}(m) < x < 1$. Plugging in (F.3) to (F.5) results in

$$\begin{aligned}\psi_{xx}(x,m) &= 2\left[\frac{2-m}{m(1+x)^2} - \frac{1}{2m}\right] - \frac{1}{m} + 2(1-x)\frac{(2-m)}{m(1+x)^3} \\ &= 2\left(\frac{2-m}{m(1+x)^2}\right)\left(\frac{2}{1+x}\right) - \frac{2}{m} \\ &= \frac{2}{m}\left(\frac{4-2m}{(1+x)^3} - 1\right).\end{aligned}$$

Thus, it suffices to show that $4 - 2m < (1+x)^3$ for $x > \tilde{x}(m)$. In particular, for this to hold it suffices to verify that (abbreviating $\tilde{x} = \tilde{x}(m)$)

$$4 - 2m \leq (1+\tilde{x})^2 = 2m\tilde{x} + 4\tilde{x}^2. \tag{F.8}$$

[53] To see this, let $f(m) = 1 - m + \frac{3}{8}m^2 - \tilde{x}(m)^2$, and notice that $f(0) = 0$ and that $f(m)$ is increasing in m.

Since $\tilde{x} \in [0,1]$, this inequality is equivalent to

$$\tilde{x} = \frac{1}{3}\left(\sqrt{(m-1)^2+3}-(m-1)\right) \geq 1-\frac{m}{2},$$

which can be easily verified. Therefore, $\psi_{xx}(x,m) < 0$ for $\tilde{x}(m) < x < 1$.

We have shown that $\psi_{xx}(x,m) < 0$ for all $x \in (0,1)$ with $x \neq \tilde{x}(m)$. Since $\psi_x(x,m)$ is continuous in x, it follows that $\psi_x(x,m)$ is strictly decreasing in x, i.e., that $\psi(x,m)$ is strictly concave in x.

Finally, by strict concavity we have that $\psi_x(x,m) > \psi_x(1,m) = 1/2$ for all $x \in [0,1)$, which implies that $\psi(x,m)$ is increasing in x.

Properties of ξ. Recall from (B.8) that $\xi(x,m)$ is defined as

$$\xi(x,m) = (1-\psi(x,m))\,\mathbb{E}_{\mu_0}\left[\theta \middle| s \geq x + m\left(\theta - \bar{\theta}(x,m)\right)^+\right].$$

By conditioning the expectation of θ on whether the consumer buys from firm A or B in the first period, and by Claim 7, we have that

$$\frac{1}{2} = \mathbb{E}_{\mu_0}[\theta] = \psi(x,m)\mathbb{E}_{\mu_0}\left[\theta \middle| s < x + m\left(\theta - \bar{\theta}(x,m)\right)^+\right] + \xi(x,m).$$

Observe that $\xi(0,m) = 1/2$. Moreover, for $x > 0$, by Claim 5, we know that the conditional distribution of θ on $\{s < x + m\left(\theta - \bar{\theta}(x,m)\right)^+\}$ admits a density function given by $\hat{g}(t \mid x,m)/\psi(x,m)$, where, for $t \in [0,1]$,

$$\hat{g}(t \mid x,m) = \min\left\{x + m\left(t - \bar{\theta}(x,m)\right)^+, 1\right\}.$$

Then, for $x > 0$ we have that

$$\xi(x,m) = \frac{1}{2} - \int_0^1 t\hat{g}(t \mid x,m)dt = \int_0^1 t\left(1 - \min\left\{x + m\left(t - \bar{\theta}(x,m)\right)^+, 1\right\}\right)dt. \tag{F.9}$$

Since $\bar{\theta}(x,m)$ is continuous in x, so is the above expression. Thus $\xi(x,m)$ is continuous in x for all $x > 0$. Moreover, to verify that $\xi(x,m)$ is also continuous at $x = 0$, notice by plugging into the previous expression that $\lim_{x\to 0^+}\xi(x,m) = 1/2 = \xi(0,m)$.

In addition, note that $\xi(1,m) = 0$, and so by continuity, $\xi(x,m)$ is a bounded function of $x \in [0,1]$. Finally, to verify that $\xi(x,m)$ is continuously differentiable in x, for $x \in (0,1]$, we compute the integral in (F.9):

$$\xi(x,m) = \begin{cases} \frac{1-x}{2} - \frac{m}{6}(1-\bar{\theta}(x,m))^2(\bar{\theta}(x,m)+2), & \text{if } x \leq \tilde{x}(m), \\ \frac{(1-x)}{2}\bar{\theta}(x,m)^2 + \frac{(1-x)^2}{2m}\bar{\theta}(x,m) + \frac{(1-x)^3}{6m^2}, & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.10}$$

Since $\bar{\theta}(x,m)$ is differentiable in x (for $x \in (0,1]$), it can be verified by inspection that $\xi(x,m)$ is also differentiable when $x \neq \tilde{x}(m)$. To verify that ξ is also differentiable at $x = \tilde{x}(m)$, we compute the derivative of $\xi(x,m)$ w.r.t. x from (F.10) to obtain that[54]

$$\xi_x(x,m) = \begin{cases} -\frac{1}{2} + \frac{m}{2}\bar{\theta}_x(x,m)\left(1-\bar{\theta}(x,m)^2\right) & \text{if } x \leq \tilde{x}(m), \\ -\frac{1}{2}\bar{\theta}(x,m)^2 + (1-x)\left(\bar{\theta}_x(x,m) - \frac{1}{m}\right)\left(\bar{\theta}(x,m) + \frac{1-x}{2m}\right) & \text{if } x > \tilde{x}(m). \end{cases} \tag{F.11}$$

By continuity of $\bar{\theta}(x,m)$ and $\bar{\theta}_x(x,m)$, we have that $\xi_x(x,m)$ is continuous when $x \neq \tilde{x}(m)$. A similar algebraic argument as in the proof for $\bar{\theta}_x$ shows that this also holds when $x = \tilde{x}(m)$, and therefore $\xi_x(x,m)$ is continuous

[54] Formally, we define $\xi_x(\tilde{x}(m),m) = \lim_{x\to\tilde{x}(m)^-}\xi_x(x,m)$, which is equal to the limit from the right by continuity.

in x. In addition, since $\bar{\theta}_x(x,m)$ is finite for $x\in(0,1]$, so is $\bar{\xi}_x(x,m)$. Finally, notice that since $\bar{\theta}_x(x,m)<0$, it follows from (F.11) that $\xi_x(x,m)<0$ and, therefore, $\xi(x,m)$ is strictly decreasing in x.

Properties of ϕ. Recall from (B.9) that ϕ is defined as $\phi(x,m)=\xi(x,m)+\frac{1}{2}\psi(x,m)\bar{\theta}(x,m)$. By the previous arguments, $\xi(x,m)$, $\psi(x,m)$ and $\bar{\theta}(x,m)$ are continuously differentiable functions of x. Therefore, so is $\phi(x,m)$.

We now show that $\phi(x,m)$ is strictly convex in x. To do so, by plugging the closed-form expressions for $\bar{\theta}(x,m)$ and $\xi(x,m)$ given in (B.2) and (F.10), respectively, into $\phi(x,m)=\xi(x,m)+\frac{1}{2}\psi(x,m)\bar{\theta}(x,m)$, and simplifying the resulting expression, we have that

$$\phi(x,m)=\begin{cases}\frac{3m^2(1-x)+16x^3-2x\sqrt{2x(2x+m)}(4x-m)}{6m^2}, & \text{if } x\le\tilde{x}(m),\\ \frac{12m^2+(1-x)^3(1+7x)}{24m^2(1+x)}, & \text{if } x>\tilde{x}(m).\end{cases} \tag{F.12}$$

By taking the derivative w.r.t x, we have that

$$\phi_x(x,m)=\begin{cases}\frac{2x(m^2-4mx-16x^2)+\sqrt{2x(2x+m)}(16x^2-m^2)}{2m^2\sqrt{2x(2x+m)}}, & \text{if } x\le\tilde{x}(m),\\ -\frac{\left[4m^2+(1-x)^2\left(7x^2+10x-1\right)\right]}{8m^2(1+x)^2}, & \text{if } x>\tilde{x}(m).\end{cases} \tag{F.13}$$

We know that this expression is continuous in x by the previous argument. Moreover, note by inspection that $\phi(x,m)$ is twice differentiable at any $x\ne\tilde{x}(m)$ with $0<x<1$, and the second partial derivative is

$$\phi_{xx}(x,m)=\begin{cases}\frac{x(4x+m)(m^2-16mx-32x^2)+32x^2(2x+m)\sqrt{2x(2x+m)}}{m^2[2x(2x+m)]^{3/2}}, & \text{if } x<\tilde{x}(m)\\ \frac{(9-7x)(1+x)^3+4(m^2-4)}{4m^2(1+x)^3}, & \text{if } x>\tilde{x}(m).\end{cases} \tag{F.14}$$

One can show that $\phi_{xx}(x,m)>0$ for all $x\in(0,1)$ with $x\ne\tilde{x}(m)$ based on similar algebraic arguments to the ones we used before. Thus,[55] $\phi_x(x,m)$ is strictly increasing in x, which implies that $\phi(x,m)$ is strictly convex in x.

Finally, by strict convexity we have that for all $x\in(0,1)$, $\phi_x(x,m)\le\phi_x(1,m)=-1/8$, which implies that $\phi(x,m)$ is strictly decreasing in x. □

Proof of Claim 11. Recall from (B.5) and (B.6) that we can write the firms' profit functions as

$$\pi^A\left(p_1^A,p_1^B,m\right)=p_1^A\psi\left(\bar{X}(p_1^A,p_1^B),m\right),$$
$$\pi^B\left(p_1^A,p_1^B,m\right)=\left(1-\psi(\bar{X}(p_1^A,p_1^B),m)\right)p_1^B+m\phi\left(\bar{X}(p_1^A,p_1^B),m\right).$$

The result follows since $\bar{X}(p_1^A,p_1^B)=\max\{0,\min\{p_1^B-p_1^A+1/2,1\}\}$ is continuous in prices and, by Claim 10, both $\psi(x,m)$ and $\phi(x,m)$ are continuous functions of x. □

Next, we prove Claim 12, which characterizes firm A's best-response correspondence in $\mathbf{G}(m)$.

Proof of Claim 12. First note that if firm B sets $p_1^B=-1/2$, firm A obtains a profit of zero for any choice of $p_1^A\in S^A$, therefore any price in S^A is a best response to $p_1^B=-1/2$.

Now fix $p_1^B\in S^B$ such that $p_1^B>-1/2$. If firm A sets a price such that $p_1^A\ge p_1^B+1/2$, we have $\bar{X}\left(p_1^A,p_1^B\right)=0$, which implies that $\pi^A\left(p_1^A,p_1^B,m\right)=0$. Moreover, for any $p_1^A\in(0,p_1^B+1/2)$, we have that $\bar{X}\left(p_1^A,p_1^B\right)>0$, and therefore $\pi^A\left(p_1^A,p_1^B,m\right)>0$. Thus, any best response price lies in $[0,p_1^B+1/2)\cap S^A$.

[55] As $\psi_x(x,m)$ is continuous in x for all $x\in[0,1]$ and locally increasing at all $x\ne\tilde{x}(m)$, it is also increasing at $x=\tilde{x}(m)$.

Now consider $p_1^A < p_1^B - 1/2$. For any such price, we have $\bar{X}\left(p_1^A, p_1^B\right) = 1$, and thus $\pi^A\left(p_1^A, p_1^B, m\right) = p_1^A$. Thus, any price below $p_1^B - 1/2$ is dominated by taking $p_1^A = p_1^B - 1/2$, and it follows that any best-response price lies in $[\max\{0, p_1^B - 1/2\}, p_1^B + 1/2) \cap S^A$.

Since $S^A = [0,1]$, we can write firm A's profit maximization problem given firm B's price p_1^B as

$$\max\left\{\pi^A\left(p_1^A, p_1^B, m\right) : p_1^A \in [\max\{0, p_1^B - 1/2\}, \min\{p_1^B + 1/2, 1\}]\right\}. \tag{F.15}$$

We claim that this problem has a unique solution given any choice of $p_1^B \in S^B$ and, moreover, that this solution lies in the interior of the interval $[\max\{0, p_1^B - 1/2\}, \min\{p_1^B + 1/2, 1\}]$. In order to show this, first notice that for any p_1^A in this interval, we can write

$$\pi^A\left(p_1^A, p_1^B, m\right) = p_1^A \psi\left(p_1^B - p_1^A + 1/2, m\right).$$

By Claim 10, we have that $\psi(x,m)$ is differentiable, strictly increasing and strictly concave in x. Therefore, $\pi^A\left(p_1^A, p_1^B, m\right)$ is strictly concave and differentiable in p_1^A, for $p_1^A \in [\max\{0, p_1^B - 1/2\}, \min\{p_1^A + 1/2, 1\}]$. By strict concavity, problem (F.15) has a unique solution, and so it only remains to show that this solution is interior. We consider two cases depending on the value of p_1^A to show this.

Case 1. If $p_1^B \le 1/2$, it follows that the domain of problem (F.15) is $[0, p_1^B + 1/2]$. Setting either $p_1^A = 0$ or $p_1^A = p_1^B + 1/2$ results in a profit of zero, and therefore is dominated by small enough positive prices. Thus, the solution to (F.15) is interior.

Case 2. If $1/2 < p_1^B \le 1$, the interval of interest is $[p_1^B - 1/2, 1]$. To show that $p_1^A = p_1^B - 1/2$ is suboptimal, observe that

$$\frac{\partial}{\partial p_1^A}\pi^A\left(p_1^A, p_1^B, m\right) = \psi\left(p_1^B - p_1^A + 1/2, m\right) - p_1^A \psi_x\left(p_1^B - p_1^A + 1/2, m\right). \tag{F.16}$$

Plugging in $p_1^A = p_1^B - 1/2$ results in

$$\frac{\partial}{\partial p_1^A}\pi^A\left(p_1^B - 1/2, p_1^B, m\right) = \psi(1,m) - \left(p_1^B - 1/2\right)\psi_x(1,m) = 1 - \frac{1}{2}\left(p_1^B - \frac{1}{2}\right) = \frac{5}{4} - \frac{1}{2}p_1^B > 0,$$

since $p_1^B < 5/2$. Thus, $\pi^A\left(p_1^A, p_1^B, m\right)$ is increasing at $p_1^A = p_1^B - 1/2$. In particular, $p_1^A = p_1^B - 1/2$ is not a best-response price for firm A. It remains to show that $p_1^A = 1$ does not solve problem (F.15).

To do so, we will show that for any $m > 0$ and $1/2 < p_1^B \le 1$, we have that $\frac{\partial}{\partial p_1^A}\pi^A\left(1, p_1^B, m\right) < 0$. In order to show this, first note by taking the derivative of (F.16) w.r.t. to p_1^B that

$$\frac{\partial}{\partial p_1^B\, \partial p_1^A}\pi^A\left(p_1^A, p_1^B, m\right) = \psi_x\left(p_1^B - p_1^A + 1/2, m\right) - p_1^A \psi_{xx}\left(p_1^B - p_1^A + 1/2, m\right) > 0,$$

where the inequality follows since ψ is strictly concave and increasing in x. Thus, it suffices to show that $\frac{\partial}{\partial p_1^A}\pi^A(1,1,m) < 0$, i.e., that $\psi(1/2, m) - \psi_x(1/2, m) < 0$ for all $m > 0$. Computing this expression yields

$$\psi(1/2, m) - \psi_x(1/2, m) = \begin{cases} -\frac{(11+8m)}{36m} & \text{if } m > 5/4, \\ \frac{\sqrt{m+1} - m - 2}{\sqrt{m+1} + m + 1} & \text{if } m \le 5/4, \end{cases}$$

which is negative for all $m > 0$. Thus, $\pi^A\left(p_1^A, p_1^B, m\right)$ is decreasing at $p_1^A = 1$, given a fixed $p_1^B \in [1/2, 1]$.

In any case, the unique solution to (F.15) is interior, and therefore is such that that the partial derivative of π^A w.r.t. p_1^A is zero. From (F.16), we can rewrite this condition as $p_1^A = Z(p_1^B - p_1^A + 1/2, m)$, where Z is defined in (B.29).

That is, for any $p_1^B \in S^B$, there exists a unique $p_1^A \in [\max\{0, p_1^B - 1/2\}, \min\{p_1^B + 1/2, 1\}] \subset S^A$ that solves (F.15). In addition such p_1^A is the unique solution in S^A to $p_1^A = Z(p_1^B - p_1^A + 1/2, m)$. □

We next prove the series of claims that characterize firm B's best-response correspondence.

Proof of Claim 13. We start by proving the first property, i.e., we want to show that $\pi^B(p_1^A, p_1^B, m)$ is (strictly) maximized by choosing $p_1^B = p_1^A - 1/2$ for all m large enough, i.e., that for any $p_1^A \in S^A$ there exists $m_0 = m_0(p_1^A)$ such that, for all $m > m_0$ and any $p_1^B \in S^B$, we have

$$\pi^B\left(p_1^A, p_1^A - 1/2, m\right) > \pi^B\left(p_1^A, p_1^B, m\right). \tag{F.17}$$

First notice that choosing any price below $p_1^B < p_1^A - 1/2$ is suboptimal since for any such price, we have $\bar{X}(p_1^A, p_1^B) = \bar{X}(p_1^A, p_1^A - 1/2) = 0$, and therefore $\pi^B(p_1^A, p_1^B, m) = p_1^B + m/2$. Thus, any price such that $p_1^B < p_1^A - 1/2$ is dominated by setting a price of $p_1^A - 1/2$. Thus, (F.17) holds for any $p_1^B < p_1^A - 1/2$.

In addition, notice that setting a price $p_1^B > p_1^A + 1/2$ results in the same profits as choosing $p_1^A + 1/2$ since $\bar{X}(p_1^A, p_1^B) = \bar{X}(p_1^A, p_1^A + 1/2) = 1$. This implies that

$$\max_{p_1^B \in S^B} \pi^B\left(p_1^A, p_1^B, m\right) = \max_{p_1^B \in [p_1^A - 1/2, p_1^A + 1/2] \cap S^B} \pi^B\left(p_1^A, p_1^B, m\right). \tag{F.18}$$

To establish the result, we now show that given $p_1^A \in S^A$, $\pi^B(p_1^A, p_1^B, m)$ is strictly decrasing in p_1^B, for $p_1^B \in [p_1^A - 1/2, p_1^A + 1/2] \cap S^B$, for all m large enough. To see this, notice that in this region we can write

$$\pi^B\left(p_1^A, p_1^B, m\right) = p_1^B\left(1 - \psi\left(p_1^B - p_1^A + 1/2, m\right)\right) p_1^B + m\phi\left(p_1^B - p_1^A + 1/2, m\right).$$

Since, by Claim 10, $\psi(x,m)$ and $\phi(x,m)$ are differentiable for $x \in (0,1)$, we take the derivative of π^B w.r.t. p_1^B to obtain (changing variables to $x = p_1^B - p_1^A + 1/2 \in [0,1]$):

$$\frac{\partial}{\partial p_1^B}\pi^B\left(p_1^A, x + p_1^A - 1/2, m\right) = 1 - \psi(x,m) - \left(x + p_1^A - 1/2\right)\psi_x(x,m) + m\phi_x(x,m). \tag{F.19}$$

We will show that this expression is negative for all $x \in (0,1)$ and all m large enough. To do so, observe that since ϕ is strictly convex in x (by Claim 10), we have that for $0 < x < 1$, $\phi_x(x,m) < \phi_x(1,m) = -1/8$. Combining this with (F.19) results in

$$\frac{\partial}{\partial p_1^B}\pi^B\left(p_1^A, x + p_1^A - 1/2, m\right) < 1 - \left(x + p_1^A - \frac{1}{2}\right)\psi_x(x,m) - \frac{m}{8}. \tag{F.20}$$

Since $\psi_x > 0$ (by Claim 10), the expression in the RHS above is negative when $x \geq 1/2 - p_1^A$ for all m large enough. Consider then $0 < x < 1/2 - p_1^A$. By concavity of ψ in x, we have that $\psi_x(x,m) < \lim_{t \to 0^+} \psi_x(t,m) = 2$. Therefore, for $0 < x < 1/2 - p_1^A$ we have

$$\frac{\partial}{\partial p_1^B}\pi^B\left(p_1^A, x + p_1^A - 1/2, m\right) < 2\left(1 - p_1^A\right) - \frac{m}{8}, \tag{F.21}$$

which implies that (F.19) is strictly negative for all $x \in (0,1)$, provided that $m > 16(1 - p_1^A)$. Thus, we define $m_0(p_1^A) = 16(1 - p_1^A)$. It follows that for any $m > m_0(p_1^A)$ and any $p_1^B \in S^B$, (F.17) holds, as desired.

We now prove the second property. Suppose that $BR^B(p_1^A, m) = p_1^A - 1/2$ for some $m > 0$. Then, $\pi^B(p_1^A, p_1^A - 1/2, m) > \pi^B(p_1^A, p_1^B, m)$ for all $p_1^B \in S^B$. In particular, this implies that $\pi^B(p_1^A, p_1^B, m)$ is strictly decreasing at $p_1^B = p_1^A - 1/2$, i.e., that $\frac{\partial}{\partial p_1^B}\pi^B(p_1^A, p_1^A - 1/2, m) < 0$. Plugging in the expression for this derivative from (F.19), we have[56]

$$1 - \psi(0,m) - \left(p_1^A - 1/2\right)\psi_x(0,m) + m\phi_x(0,m) = 1 - 2\left(p_1^A - 1/2\right) - m/2 < 0. \tag{F.22}$$

By rearranging terms, it follows that $m > 4(1 - p_1^A)$. Given this condition, we rely on the following claim, which we prove in Appendix F.1.1, to complete the proof.

[56] We define $\psi_x(0,m) = \lim_{t \to 0^+} \psi_x(t,m) = 2$, and $\phi_x(0,m) = \lim_{t \to 0^+} \phi_x(t,m) = -1/2$.

CLAIM 41. *Let $f : [0,1] \times S^A \times \mathbb{R}^+ \to \mathbb{R}$ be given by*

$$f(x, p_1^A, m) = \pi^B\left(p_1^A, p_1^A - 1/2, m\right) - \pi^B\left(p_1^A, x + p_1^A - 1/2, m\right).$$

Fix $p_1^A \in S^A$ and $x \in (0,1]$. Then, $f(x, p_1^A, m)$ is strictly increasing in m for $m > 4(1 - p_1^A)$.

Since $BR^B(p_1^A, m) = p_1^A - 1/2$, it follows that $f(x, p_1^A, m) > 0$ for all $0 < x \leq 1$. By Claim 41, and since $m > 4(1 - p_1^A)$, this also holds for any $m' > m$, i.e.,

$$\pi^B\left(p_1^A, p_1^A - 1/2, m'\right) > \pi^B\left(p_1^A, p_1^B, m'\right),$$

for all $p_1^B \in S^B$ with $p_1^A - 1/2 < p_1^B \leq p_1^A + 1/2$. Thus, by (F.18), we have that $BR^B(p_1^A, m') = p_1^A - 1/2$. □

Proof of Claim 14. Fix $m > 0$, we omit the dependency on m in the rest of the proof to simplify the notation. By assumption, $\pi^B(p_1^A, p_1^A - 1/2) \geq \pi^B(p_1^A, p_1^B)$ for all $p_1^B \in S^B$, i,.e.,

$$p_1^A - 1/2 + \frac{m}{2} \geq \left(1 - \psi(\bar{X}(p_1^A, p_1^B))\right) p_1^B + m\phi\left(\bar{X}(p_1^A, p_1^B)\right).$$

In particular, it follows from (F.18) that for all[57] $x \in (0, \min\{1, 3/2 - p_1^A\}]$,

$$p_1^A - 1/2 + \frac{m}{2} \geq (p_1^A - 1/2 + x)(1 - \psi(x)) + m\phi(x). \tag{F.23}$$

Let $y > p_1^A$ such that $y \in S^A$, and assume towards a contradiction that there exists $p_1^{B'} \in S^B$ with $p_1^{B'} > y - 1/2$ such that $\pi^B(y, y - 1/2) \leq \pi^B(y, p_1^{B'})$. Without loss of generality, suppose that $p_1^{B'} \leq y + 1/2$, as otherwise setting a price of $y + 1/2$ yields the same profits as $p_1^{B'}$.

Define $x' = p_1^{B'} - y + 1/2$ and notice that $0 < x' \leq \min\{1, 3/2 - y\} < \min\{1, 3/2 - p_1^A\}$. We have that

$$\begin{aligned}
\pi^B(y, y - 1/2) = y - 1/2 + \frac{m}{2} &\leq (y - 1/2 + x')(1 - \psi(x')) + m\phi(x') \\
&= (p_1^A - 1/2 + x')(1 - \psi(x')) + m\phi(x') + (y - p_1^A)(1 - \psi(x')) \\
&\leq p_1^A - 1/2 + \frac{m}{2} + (y - p_1^A)(1 - \psi(x')),
\end{aligned}$$

where the last step follows from (F.23). The last inequality implies that $y - p_1^A \leq (y - p_1^A)(1 - \psi(x'))$, from where we have that $\psi(x') \leq 0$, and therefore $x' = 0$, a contradiction.

Finally, by (F.18), any possible best response to y lies in $[y - 1/2, \min\{1, y + 1/2\}]$, from where we have that $BR^B(y, m) = y - 1/2$. □

Proof of Claim 15. Define $m_H = inf\{m > 0 : -1/2 \in BR^B(0, m)\}$. By Claim 13, we have that $m_H < \infty$ and that $BR^B(0, m) = -1/2$ for all $m > m_H$.

Fix $m > m_H$. For firm A, we have that $BR^A(-1/2, m) = S^A$ for all $m > 0$ by Claim 12. It follows that $(p_1^{A*}, p_2^{B*}) = (0, -1/2)$ is a PSNE of $\mathbf{G}(m)$.

We now show that it is the unique PSNE. Suppose that there is another PSNE (p_1^A, p_1^B) in $\mathbf{G}(m)$, given $m > m_H$. If $p_1^A > 0$, then, by Claim 14, firm B's best response to p_1^A is $p_1^B = p_1^A - 1/2 > -1/2$. Thus, we have that $\bar{X}(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 = 0$, and therefore $\pi^A(p_1^A, p_1^B, m) = 0$. Then, for all $\varepsilon > 0$ small enough, we have $\bar{X}(p_1^A - \varepsilon, p_1^B) > 0$, and $\pi^A(p_1^A - \varepsilon, p_1^B, m) > 0$. In particular, $p_1^A \notin BR^A(p_1^B, m)$, contradicting that (p_1^A, p_1^B) is a Nash equilibrium. Thus, $(0, -1/2)$ is the unique PSNE of $\mathbf{G}(m)$.

Finally, by definition of m_H, if $m < m_H$, then $(0, -1/2)$ is not a PSNE of $\mathbf{G}(m)$. Moreover, by the preceding argument, any pair of prices (p_1^A, p_1^B) with $p_1^B \leq p_1^A - 1/2$ cannot be a PSNE if $m < m_H$. □

[57] We have this condition instead of $x \in (0,1]$ to ensure that $p_1^B \in S^B$.

To prove Claim 16, we rely on the following result, which establishes that as we take m to zero, firm B's profit function in $\mathbf{G}(m)$ converges to the corresponding profit function associated with product 1 in the restricted setting. We provide the proof of Claim 42 in Appendix F.1.1.

Claim 42. *Fix any $p_1^A \in S^A$, $p_1^B \in S^B$. Then,*

$$\lim_{m \to 0^+} \pi^B \left(p_1^A, p_1^B, m\right) = p_1^B \left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right).$$

Proof of Claim 16. Fix $m > 0$ and consider firm B's profit maximization problem when firm A sets a price of $p_1^A \in S^A$, which by (F.18) in the proof of Claim 13 can be written as:

$$\max\left\{\pi^B\left(p_1^A, p_1^B, m\right) : p_1^B \in [p_1^A - 1/2, \min\{p_1^A + 1/2, 1\}]\right\}. \tag{F.24}$$

Notice that this problem always has a solution for any $p_1^A \in S^A$ since π^B is a continuous function, and $[p_1^A - \frac{1}{2}, \min\{1, p_1^A + \frac{1}{2}\}]$ is a compact set, i.e., $BR^B(p_1^A, m)$ is non-empty for all $p_1^A \in S^A$.

Given $m > 0$, define $d(m)$ as the smallest value of p_1^A such that this problem has a corner solution, i.e., such that $p_1^B = p_1^A - 1/2$ solves the problem above:

$$d(m) = \inf\left\{p_1^A \in S^A : p_1^A - 1/2 \in BR^B(p_1^A, m)\right\},$$

where $d(m) = \infty$ if the set on which the infimum is taken is empty. The rest of the proof consists of studying the properties of $d(m)$, and in particular, showing that $BR^B(p_1^A, m)$ is single-valued when $p_1^A < d(m)$. We proceed in several steps.

Step 1. We show that for any $m > 0$, and any $p_1^A \in S^A$, $\min\{p_1^A + 1/2, 1\} \notin BR^B(p_1^A, m)$. Therefore, $BR^B(p_1^A, m)$ consists only of interior solutions to (F.24), or the corner solution $p_1^B = p_1^A - 1/2$.

If $p_1^A \leq 1/2$, we have that $\min\{p_1^A + 1/2, 1\} = p_1^A + 1/2$. Observe that $p_1^B = p_1^A + 1/2$ is suboptimal in (F.24) as it results in a profit of zero from product 1 and minimizes the profit from product 2. Thus, $p_1^A + 1/2 \notin BR^B(p_1^A, m)$.

Now suppose that $p_1^A > 1/2$. We will show that $p_1^B = 1$ is suboptimal in (F.24), by showing that choosing $p_1^B = p_1^A - 1/2$ results in higher profits for firm B. To see this, observe that given $p_1^A > 1/2$,

$$\pi^B\left(p_1^A, 1, m\right) = 1 - \psi(3/2 - p_1^A, m) + m\phi\left(3/2 - p_1^A, m\right).$$

On the other hand, $\pi^B\left(p_1^A, p_1^A - 1/2, m\right) = p_1^A - 1/2 + m/2$. We will now show that $\pi^B\left(p_1^A, 1, m\right) < \pi^B\left(p_1^A, p_1^A - 1/2, m\right)$.

Since $\phi(x, m) < \phi(0, m) = 1/2$ for any $x \in (0, 1]$ (by Claim 10), we have that $m\phi\left(3/2 - p_1^A, m\right) < m/2$, and so it suffices to show that $1 - \psi(3/2 - p_1^A, m) \leq p_1^A - 1/2$. This inequality can be rewritten as $3/2 - p_1^A \leq \psi(3/2 - p_1^A, m)$, which indeed holds.[58] Thus, $1 \notin BR^B(p_1^A, m)$.

Step 2. We show that if $p_1^A < d(m)$, there is only one interior solution to problem (F.24). In particular, $BR^B(p_1^A, m)$ is single-valued and defined by (B.30).

To show this, first notice that by the definition of $d(m)$, and by Step 1, $BR^B(p_1^A, m)$ consists only of interior solutions to problem (F.24) when $p_1^A < d(m)$. That is, $BR^B(p_1^A, m)$ contains only prices in $(p_1^A - 1/2, \min\{p_1^A + 1/2, 1\})$. We now show that in this case, there is only one interior solution to problem (F.24).

[58] Since $\psi(0, m) = 0$, $\psi(1, m) = 1$, and ψ is strictly concave in x, we have that $\psi(x, m) > x$ for all $x \in (0, 1)$, $m > 0$.

Fix $p_1^A \geq 0$ and assume that $p_1^A < d(m)$. By changing variables to $x = p_1^B - p_1^A + 1/2 \in [0,1]$, we can rewrite the first-order condition for profit maximization of problem (F.24) as

$$H(x,m) - p_1^A + \frac{1}{2} = 0, \tag{F.25}$$

where we define $H : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ as

$$H(x,m) = V(x,m) - x. \tag{F.26}$$

We next use the following result, which we prove in Appendix F.1.1.

Claim 43. *For any fixed $m > 0$, $H(x,m)$ is a strictly quasiconcave function of x.*

By Claim 43, there are at most two values of x that solve equation (F.25) given p_1^A. Note that if there exist two solutions, one of them must correspond to a local minimum and the other one to a local maximum of (F.24), and only the one corresponding to the local maximum can be a best-response. If there is only one solution, it must correspond to a maximizer (since it is the only candiate for an interior solution). Therefore, there exists at most one interior solution to (F.24). In particular, if $p_1^A < d(m)$, we know that an interior solution to (F.24) exists, and therefore it must be the unique solution to problem (F.24). By reversing the change of variables from p_1^B to x, we have that the best-response price p_1^B satisfies $p_1^B = V(p_1^B - p_1^A + 1/2, m)$, where V is defined in (B.31). In addition, such p_1^B lies in S^B since the domain of problem (F.24) is a subset of S^B.

Step 3. If $p_1^A > d(m)$, then $BR^B(p_1^A, m) = p_1^A - 1/2$.

This follows directly from Claim 14, and the definition of $d(m)$.

Step 4. For all small enough $m > 0$, we have that $d(m) = \infty$, i.e., $p_1^A - 1/2 \notin BR^B(p_1^A, m)$.

Recall that $\pi^B(p_1^A, p_1^B, m)$ is only defined for $m > 0$. We now augment this defintion by letting, for $m = 0$,

$$\pi^B\left(p_1^A, p_1^B, 0\right) = p_1^B\left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right).$$

By evaluating $\pi^B(p_1^A, p_1^B, 0)$ at $p_1^B = p_1^A/2 + 1/4$ and $p_1^B = p_1^A - 1/2$, we have that for any $p_1^A \in S^A$,

$$\pi^B\left(p_1^A, p_1^A/2 + 1/4, 0\right) - \pi^B\left(p_1^A, p_1^A - 1/2, 0\right) = \left(p_1^A/2 + 1/4\right)^2 - \left(p_1^A - 1/2\right).$$

Given that $p_1^A \in S^A = [0,1]$, it is easy to verify that this expression is minimized when $p_1^A = 1$, and therefore we have that $\pi^B(p_1^A, p_1^A/2 + 1/4, 0) - \pi^B(p_1^A, p_1^A - 1/2, 0) \geq 1/16$ for all $p_1^A \in S^A$.

It is easy to verify that for any fixed prices $p_1^A \in S^A$ and $p_1^B \in S^B$, $\pi^B(p_1^A, p_1^B, m)$ is a continuous function[59] of m. Therefore, by Claim 42, we have that for all small enough $m > 0$,

$$\pi^B\left(p_1^A, p_1^A/2 + 1/4, m\right) - \pi^B\left(p_1^A, p_1^A - 1/2, m\right) > 0,$$

which implies that $p_1^B = p_1^A - 1/2$ is suboptimal in problem (F.18) for all small enough $m > 0$. Thus, $d(m) = \infty$ for all small enough $m > 0$.

[59] By the same reasoning as in Claim 10, we can show that $\bar{\theta}$, ψ, ξ and ϕ, are continuous functions of m, when $m > 0$.

Let us then define $M_0 = \sup\{m > 0 : d(m) = \infty\}$. By the preceding argument, it follows that $M_0 > 0$. Moreover, if $m < M_0$, we have that $p_1^A < d(m)$ for all $p_1^A \in S^A$. Part (i) of the proposition then follows by Step 2.

Step 5. There exists $M_1 > 0$ such that if $m > M_1$, $BR^B(p_1^A, m) = p_1^A - 1/2$ for all $p_1^A \in S^A$.

To verify this, let $M_1 = m_H = inf\{m > 0 : -1/2 \in BR^B(0, m)\}$, as in the proof of Claim 15. By the proof of Claim 15, we have that if $m > M_1$, $BR^B(p_1^A, m) = p_1^A - 1/2$ for all $p_1^A \in S^A$. This proves (iii).

Step 6. $d(m)$ is non-increasing in m.

If $d(m) < \infty$, then for any $m' > m$, we have that $d(m') < d(m)$ by Claim 13. The property trivially holds if $d(m) = \infty$. Then, by steps 4 and 5, it follows that $d(m) \in (0,1)$ if and only if $M_0 < m < M_1$. Part (ii) of the claim then follows from Steps 2 and 3. □

Proof of Claim 17. Take M_0 as in Claim 16 and fix $m < M_0$. Then, both firms' best-response correspondences are single-valued (and therefore compact and convex-valued). Moreover, these correspondences are upper hemicontinuous by continuity of π^A and π^B (established in Claim 11). Then, by Kakutani's fixed point theorem, $\mathbf{G}(m)$ admits a PSNE. In addition, any such equilibrium is interior by Claims 12 and 16. □

Proof of Claim 18. Fix $m > 0$ and let (p_1^A, p_1^B) be an interior PSNE of $\mathbf{G}(m)$. Then, since (p_1^A, p_1^B) is not a corner equilibrium, it follows that $\bar{X}(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 \in (0,1)$. Moreover, by Claims 12 and 16, the following two conditions must hold:

$$\begin{aligned} p_1^A &= Z(p_1^B - p_1^A + 1/2, m), \\ p_1^B &= V(p_1^B - p_1^A + 1/2, m). \end{aligned} \tag{F.27}$$

By combining these two equations, we obtain that (p_1^A, p_1^B) satisfies $W(p_1^B - p_1^A + 1/2, m) = 0$, where we define $W : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ by

$$W(x,m) = V(x,m) - Z(x,m) + \frac{1}{2} - x. \tag{F.28}$$

Now suppose that there exist two interior PSNE for $\mathbf{G}(m)$, say (p_1^A, p_1^B) and $(p_1^{A'}, p_1^{B'})$. It follows that $\bar{X}(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 \in (0,1)$, and $\bar{X}(p_1^{A'}, p_1^{B'}) = p_1^{B'} - p_1^{A'} + 1/2 \in (0,1)$. For the remainder of the proof, let us abbreviate these two quantities by $\bar{X}$ and $\bar{X}'$, respectively. By the preceding argument, we have that $W(\bar{X}, m) = 0$ and $W(\bar{X}', m) = 0$. We will now prove that $\bar{X} = \bar{X}'$ by leveraging the following result, which we prove in Appendix F.1.1.

Claim 44. *Given $m > 0$, there are at most two values of $x \in (0,1)$ that solve $W(x,m) = 0$. Moreover, any such solution satisfies $x < \tilde{x}(m)$.*

By Claim 44, we have two cases to consider. If there exists only one solution to $W(x,m) = 0$ (with $0 < x < 1$), then it must be that $\bar{X} = \bar{X}'$. Otherwise, if there are two solutions, we will show that one of the previous price pairs is not a Nash equilibrium. To do so, notice that by plugging the expressions for $Z(x,m)$ and $V(x,m)$ given in (B.29) and (B.31) into (F.28), we can write

$$W(x,m) = \frac{\varphi(x,m)}{\psi_x(x,m)}, \tag{F.29}$$

where

$$\varphi(x,m) = 1 - 2\psi(x,m) + m\phi_x(x,m) - \left(x - \frac{1}{2}\right)\psi_x(x,m).$$

Since both $\bar{X}$ and $\bar{X}'$ solve $W(x,m)=0$, they also solve $\varphi(x,m)=0$. Since φ is continuous and differentiable[60] in x for $x<\tilde{x}(m)$, and $\bar{X}'$, $\bar{X}$ are the only two solutions of $\varphi(x,m)=0$, it must be that $\varphi_x(\bar{X},m)\varphi_x(\bar{X}',m)\le 0$. Therefore, it holds that $\varphi_x(\bar{X},m)\ge 0$ or $\varphi_x(\bar{X}',m)\ge 0$. Assume w.l.o.g. that $\varphi_x(\bar{X}',m)\ge 0$, so that

$$-3\psi_x(\bar{X}',m)+m\phi_{xx}(\bar{X}',m)-\left(\bar{X}'-\frac{1}{2}\right)\psi_{xx}(\bar{X}',m)\ge 0.$$

Since $p_1^{A'}=Z(\bar{X}',m)$ and $p_1^{B'}=\bar{X}'+p_1^{A'}-1/2$, we can rewrite the previous expression as

$$\left[-2\psi_x(\bar{X}',m)+m\phi_{xx}(\bar{X}',m)-p_1^{B'}\psi_{xx}(\bar{X}',m)\right]+\left[p_1^{A'}\psi_{xx}(\bar{X}',m)-\psi_x(\bar{X}',m)\right]\ge 0.$$

The second term is always negative since ψ is strictly increasing and strictly concave in x, and so the first term must be positive. However, the second-order condition from Firm 2's profit maximization is equivalent to having the first term to be non-positive, which implies that $p_1^{B'}$ does not maximize firm B's profit given $p_1^{A'}$, contradicting that $(p_1^{A'},p_1^{B'})$ is a Nash equilibrium.

We have shown that $\bar{X}=\bar{X}'$. It follows from (F.27) that $p_1^A=Z(\bar{X},m)=Z(\bar{X}',m)=p_1^{A'}$. Similarly, we conclude that $p_1^B=p_1^{B'}$, and therefore there exists a unique interior PSNE for $\mathbf{G}(m)$. □

Proof of Claim 19. Note that setting a negative price is a strictly dominated strategy for firm A, as it may result in negative profits while setting a price of zero results in a profit of zero. Thus, we restrict our attention to the case where $p_1^A\ge 0$.

Given this constraint, note that setting a price below $-1/2$ is a dominated strategy for firm B, since for any choice of $p_1^A\ge 0$, setting a price equal to $-1/2$ makes firm B better off than any price $p_1^B<-1/2$. Indeed, we have that for $p_1^B<-1/2$

$$\pi^B\left(p_1^A,p_1^B,m\right)=p_1^B+m/2<(m-1)/2=\pi^B\left(p_1^A,-1/2,m\right).$$

Thus, $\tilde{\mathbf{G}}(m)$ has no PSNE in undominated strategies such that firm B sets a price below $-1/2$. The rest of the proof consists of two steps. First, we show that any PSNE of $\mathbf{G}(m)$ is also a PSNE of $\tilde{\mathbf{G}}(m)$. The second step consists of showing the converse proposition; i.e., that any PSNE of $\tilde{\mathbf{G}}(m)$ (with $p_1^A\ge 0$) is a PSNE of $\mathbf{G}(m)$. In what follows, we denote the firms' best-response correspondences in $\tilde{\mathbf{G}}(m)$ by $\tilde{BR}^A$ and $\tilde{BR}^B$.

Step 1. Fix $m>0$ and suppose that (p_1^A,p_1^B) is a PSNE of $\mathbf{G}(m)$. We have two cases to cover. First, if (p_1^A,p_1^B) is a corner equilibrium, then, by Claim 15, $(p_1^A,p_1^B)=(0,-1/2)$. Since given $p_1^B=-1/2$, firm A cannot make a profit larger than zero, we have that $\tilde{BR}^A(-1/2,m)=[0,\infty)$. In addition, by following the same argument as in the proofs of Claims 13 and 14, we have that $-1/2\in\tilde{BR}^B(0,m)$. Thus, (p_1^A,p_1^B) is a PSNE in $\tilde{\mathbf{G}}(m)$.

The second case corresponds to when (p_1^A,p_1^B) is an interior equilibrium, i.e., $p_1^A>0$ and $p_1^B>p_1^A-1/2$. By following the same argument as in the proof of Claim 12, we have that

$$\begin{aligned}&\max\left\{\pi^A\left(y,p_1^B,m\right):y\in[\max\{0,p_1^B-1/2\},\min\{p_1^B+1/2,1\}]\right\}\\=&\max\left\{\pi^A\left(y,p_1^B,m\right):y\in[0,p_1^B+1/2]\right\}=\max\left\{\pi^A\left(y,p_1^B,m\right):y\ge 0\right\},\end{aligned}$$

[60] Note that φ is differentiable everywhere except at $x=\tilde{x}(m)$, since that is the case for ϕ_x and ψ_x, as shown in the proof of Claim 10. Then, by Claim 44, we are evaluating φ_x at a point where it is defined.

where the last step follows since firm A makes a profit of zero when choosing any price above $p_1^B+1/2$, and $p_1^B>-1/2$ by assumption. Moreover, $BR^A(p_1^B,m)$ is single-valued and $p_1^A=BR^A(p_1^B,m)$ is the unique solution to firm A's profit maximization problem above, by Claim 12. Hence, $BR^A(p_1^B,m)=\tilde{BR}^A(p_1^B,m)$ and, since (p_1^A,p_1^B) is a PSNE in $\mathbf{G}(m)$, $p_1^A\in\tilde{BR}^A(p_1^B,m)$.

We proceed similarly for firm B. By the same argument as in step 1 of the proof of Claim 16, we have

$$\max\left\{\pi^B\left(p_1^A,y,m\right):y\in[p_1^A-1/2,\min\{p_1^A+1/2,1\}]\right\}=\max\left\{\pi^B\left(p_1^A,y,m\right):y\in[p_1^A-1/2,p_1^A+1/2]\right\}$$
$$=\max\left\{\pi^B\left(p_1^A,y,m\right):y\geq-1/2\right\},$$

where the last equality follows since it is always suboptimal for firm B to set a price below $p_1^A-1/2$, and setting a price of $p_1^A+1/2$ or higher is also always suboptimal (by the same argument as in the proof of Claim 13). Then, since (p_1^A,p_1^B) is a PSNE in $\mathbf{G}(m)$, we have that $p_1^B\in BR^B(p_1^A,m)=\tilde{BR}^B(p_1^A,m)$. We have shown above that $p_1^A\in\tilde{BR}^A(p_1^B,m)$. Therefore, (p_1^A,p_1^B) is a PSNE of $\tilde{\mathbf{G}}(m)$.

Step 2. Suppose now that (p_1^A,p_1^B) is a PSNE in undominated strategies for $\tilde{\mathbf{G}}(m)$. Then, $p_1^A\geq 0$. Moreover, note that we have that $p_1^A>p_1^B-1/2$ (since otherwise firm B receives no demand in the market for product 1 and is better off matching firm A's price), and we also have $p_1^B\geq p_1^A-1/2$ (since otherwise firm B increases is better off choosing $p_1^A-1/2$). Moreover, these two conditions can be written as $0\leq\bar{X}(p_1^A,p_1^B)<1$. We now consider two cases: $\bar{X}(p_1^A,p_1^B)=0$, and $\bar{X}(p_1^A,p_1^B)>0$ (i.e., the interior and corner cases).

First, suppose that $\bar{X}(p_1^A,p_1^B)=0$. Then, $p_1^B=p_1^A-1/2$. By the same argument as in the proof of Claim 15, we have that $(p_1^A,p_1^B)=(0,-1/2)$. Since $BR^A(-1/2,m)=S^A$ (by Claim 12), it follows that $0\in BR^A(-1/2,m)$. In addition, by the same argument as in step 1, we have that $BR^B(0,m)=\tilde{BR}^B(0,m)$. Since $(p_1^A,p_1^B)=(0,-1/2)$ is a PSNE in $\tilde{\mathbf{G}}(m)$, it follows that $-1/2\in BR^B(0,m)$. Thus, $(p_1^A,p_1^B)=(0,-1/2)$ is a PSNE in $\mathbf{G}(m)$.

Now consider the case with $0<\bar{X}(p_1^A,p_1^B)<1$, i.e., $0<p_1^B-p_1^A+1/2<1$. Since $p_1^A\geq 0$, this implies that $p_1^B>-1/2$. By following the same argument as in the proofs of Claims 12 and 16, we have that

$$p_1^A\in\arg\max\left\{\pi^A\left(y,p_1^B,m\right):y\in[0,p_1^B+1/2]\right\},$$
$$p_1^B\in\arg\max\left\{\pi^B\left(p_1^A,y,m\right):y\in[p_1^A-1/2,p_1^A+1/2]\right\}.$$

In particular, we have that $p_1^A\in(0,p_1^B+1/2)$, as choosing a price of zero leads to zero profits for firm A given that $p_1^B>-1/2$. Thus, p_1^A satisfies the first-order condition for profit maximization for firm A. By the proof of Claim 12, we can write this condition as $p_1^A=Z\left(p_1^B-p_1^A+1/2,m\right)$, where Z is defined as in (B.29).

For firm B, by the same argument as in the proof of Claim 16, we have that $p_1^B\in(p_1^A-1/2,p_1^A+1/2)$. Thus, p_1^B satisfies firm B's first-order condition for profit maximization, which, by the proof of Claim 16, can be written as $p_1^B=V\left(p_1^B-p_1^A+1/2,m\right)$, where V is defined as in (B.31). Then, by following the same argument as in the proof of Claim 18, it follows that $W(p_1^B-p_1^A+1/2,m)=0$, where W is defined as in (F.28).

In addition, since $p_1^B\in\tilde{BR}^B(p_1^A,m)$, we have that $\pi^B\left(p_1^A,p_1^B,m\right)\geq\pi^B\left(p_1^A,p_1^A-1/2,m\right)=(m-1)/2+p_1^A$. Notice that we can write this condition as $T(p_1^B-p_1^A+1/2,m)\geq 0$, where we denote[61]

$$T(x,m)=x-V(x,m)\psi(x,m)+m\left(\phi(x,m)-1/2\right).$$

To summarize, we have that $0<p_1^B-p_1^A+1/2<1$, $T(p_1^B-p_1^A+1/2,m)\geq 0$, and $W(p_1^B-p_1^A+1/2,m)=0$. By Claim 45 and Proposition 8 in Appendix F.2, it follows that (p_1^A,p_1^B) is a PSNE of $\mathbf{G}(m)$, as desired. □

[61] See Appendix F.2 for a detailed derivation of $T(x,m)$.

F.1.1. Proofs of the Auxiliary Results of Appendix F.1 (Claims 41–44)

Proof of Claim 41. We want to show that for fixed $x \in (0,1]$ and $p_1^A \in S^A$, $f(x, p_1^A, m)$ is strictly increasing in m, provided that $m > 4(1-p_1^A)$, where

$$\begin{aligned} f(x,p_1^A,m) &= \pi^B\left(p_1^A, p_1^A - 1/2, m\right) - \pi^B\left(p_1^A, x + p_1^A - 1/2, m\right) \\ &= \frac{m-1}{2} + p_1^A - \left(p_1^A + x - \frac{1}{2}\right)(1-\psi(x,m)) - m\phi(x,m). \end{aligned}$$

From equations (B.2) and (B.7), we have that ψ and ϕ are differentiable in m, except when $x = \tilde{x}(m)$, it suffices to show that the derivative of this expression w.r.t. m is positive when $x \neq \tilde{x}(m)$ (since the expression is continuous in m). Thus, we want to show that for all $p_1^A \in S^A$, $m > 4(1-p_1^A)$, and $x \neq \tilde{x}(m)$, it holds that $f_m(x, p_1^A, m) > 0$, where

$$f_m(x,p_1^A,m) = 1/2 + \psi_m(x,m)\left(p_1^A + x - 1/2\right) - \phi(x,m) - m\phi_m(x,m). \tag{F.30}$$

The rest of the proof consists of showing that $f_m(x, p_1^A, m) > 0$ in the aforementioned region by plugging in the functional forms of the terms in the previous expression. We consider two cases.

Case 1. $\tilde{x}(m) < x \leq 1$. First, notice that $\psi_m > 0$. To see this, simply note from (B.2) that $\bar{\theta}_m > 0$, and recall that $\psi(x,m) = 2x\bar{\theta}(x,m)$. Thus, we have from (F.30) that $f_m(x, p_1^A, m)$ is increasing in p_1^A. This implies that

$$f_m(x,p_1^A,m) \geq f_m(x,0,m) = \frac{1 + 4(3m^2-2)x + 30x^2 - 40x^3 + 17x^4}{24m^2(1+x)},$$

where the last equality follows by computing the partial derivatives of ψ and ϕ w.r.t. m and plugging into (F.30). We want to show that the numerator of the previous expression is positive for all $m > 0$, i.e., that

$$1 + 4(3m^2-2)x + 30x^2 - 40x^3 + 17x^4 > 0.$$

Observe that this expression is strictly increasing in m, and so it suffices to show that the inequality holds (even if weakly) for $m = 0$, i.e., that

$$1 - 8x + 30x^2 - 40x^3 + 17x^4 = (x-1)^2(17x^2 - 6x + 1) \geq 0,$$

which holds, since both terms in the factorization are non-negative for all $x \in \mathbb{R}$.

Case 2. $0 < x < \tilde{x}(m)$. Notice that we can rewrite the constraint $m > 4(1-p_1^A)$ as $p_1^A > \max\{1-m/4, 0\}$. As in the previous case, since $f_m(x, p_1^A, m)$ is increasing in p_1^A, we have that[62]

$$f_m(x,p_1^A,m) > f_m(x, \max\{1-m/4,0\}, m) \geq f_m(x, 1-m/4, m).$$

We will now show that $f_m(x, 1-m/4, m) > 0$ for $m > 0$. To do so, we obtain the following expression by computing the partial derivatives of ψ and ϕ w.r.t. m and plugging into (F.30):

$$f_m(x,1-m/4,m) = \frac{x\left[(6m - 5m^2 + 24x - 8mx + 16x^2)\,x - \sqrt{2x(2x+m)}\,(8x^2 + 12x - 6mx - 3m^2)\right]}{6m^2\sqrt{2x(2x+m)}}.$$

[62] Formally, f_m is only defined when $p_1^A \in S^A = [0,1]$. For this proof, we simply extend the definition by plugging in $p_1^A = 1 - m/4$ to the original expression of f_m given in (F.30).

Since the denominator of this expression is positive, it suffices to show that the numerator is positive. Indeed, we have that

$$\begin{aligned}
&\left(6m-5m^2+24x-8mx+16x^2\right)x-\sqrt{2x(2x+m)}\left(8x^2+12x-6mx-3m^2\right)\\
=&\left(6m+24x-8mx+16x^2\right)x-\sqrt{2x(2x+m)}\left(8x^2+12x-6mx\right)+m^2\underbrace{\left(3\sqrt{2x(2x+m)}-5x\right)}_{>0}\\
>&\left(6m+24x+16x^2\right)x-\sqrt{2x(2x+m)}\left(8x^2+12x\right)+\underbrace{mx\left(6\sqrt{2x(2x+m)}-8x\right)}_{>4mx^2}\\
>&\left(6m+4mx+24x+16x^2\right)x-\sqrt{2x(2x+m)}\left(8x^2+12x\right)\\
=&2x\left(3+2x\right)\left(4x+m-2\sqrt{2x(2x+m)}\right)>0.
\end{aligned}$$

Thus, $f_m(x,p_1^A,m)>0$, as desired. □

Proof of Claim 42. Recall from (B.6) that firm B's profit function in $\mathbf{G}(m)$ is

$$\pi^B\left(p_1^A,p_1^B,m\right)=\left(1-\psi(\bar{X}(p_1^A,p_1^B),m)\right)p_1^B+m\phi\left(\bar{X}(p_1^A,p_1^B),m\right).$$

By Claim 10, we have that for all $m>0$, $1/4\le\phi(x,m)\le 1/2$ for all $x\in[0,1]$. Thus, we have that $m\phi\left(\bar{X}(p_1^A,p_1^B),m\right)\to 0$ as $m\to 0^+$. It remains to show that

$$\lim_{m\to 0^+}\left(1-\psi(\bar{X}(p_1^A,p_1^B),m)\right)p_1^B=p_1^B\left(1-\bar{X}\left(p_1^A,p_1^B\right)\right).$$

In particular, we just need to show that $\lim_{m\to 0^+}\psi(x,m)=x$ for any $x\in[0,1]$. To do so, recall that $\psi(x,m)=2x\bar{\theta}(x,m)$, where $\bar{\theta}(x,m)$ is given by

$$\bar{\theta}(x,m)=\begin{cases}\frac{1}{m}\left(2x+m-\sqrt{2x\left(2x+m\right)}\right), & \text{if } x\le\tilde{x}(m),\\ \frac{1}{1+x}\left[1-\frac{1}{2m}(1-x)^2\right], & \text{if } x>\tilde{x}(m),\end{cases}$$

where $\tilde{x}(m)=\frac{1}{3}\left(\sqrt{(m-1)^2+3}-(m-1)\right)$. We now show that $\lim_{m\to 0^+}\psi(x,m)=x$.

First, if $x=0$, we have that $\psi(0,m)=0$ for all $m>0$. Thus, $\lim_{m\to 0^+}\psi(0,m)=0$.

Next, we claim that for any $x\in(0,1]$, $\lim_{m\to 0^+}\bar{\theta}(x,m)=1/2$. To verify this, note that $\tilde{x}(m)$ is strictly decreasing in m, and that $\tilde{x}(0)=1$. Thus, for all $m>0$, we have $\bar{\theta}(1,m)=1/2$, which implies that $\lim_{m\to 0^+}\bar{\theta}(1,m)=1/2$.

Now consider $0<x<1$, and notice that $x<\tilde{x}(m)$, for all small enough $m>0$. Therefore

$$\lim_{m\to 0^+}\bar{\theta}(x,m)=\lim_{m\to 0^+}\sqrt{2x+m}\left[\frac{\sqrt{2x+m}-\sqrt{2x}}{m}\right]=\sqrt{2x}\frac{1}{2\sqrt{2x}}=\frac{1}{2}.$$

It follows that $\lim_{m\to 0^+}\psi(x,m)=x$ for any $x\in[0,1]$, as desired. □

Proof of Claim 43. The proof consists of three parts: first, we show that $H(x,m)$ is decreasing in x, for $\tilde{x}(m)<x<1$. Then, we show that this is also the case for $0<x<\tilde{x}(m)$ when $m\ge 6$. Finally, we show that when $m<6$, $H(x,m)$ is strictly concave in x, in the region $0<x<\tilde{x}(m)$.

Fix $m>0$. To simplify the notation, we omit the dependency of H and other functions on m, unless necessary. That is, we denote $H(x)=H(x,m)$, $\psi(x)=\psi(x,m)$, $\psi'(x)=\psi_x(x,m)$, and so on.

Part 1. $H(x)$ is decreasing in x for all $\tilde{x}(m)<x<1$.

Notice that we can write $H(x) = H_1(x) + H_2(x)$, where

$$H_1(x) = \frac{1-\psi(x)}{\psi'(x)}, \quad H_2(x) = \frac{m\phi'(x)}{\psi'(x)} - x.$$

We will show that both H_1 and H_2 are decreasing in x, for $x \in (\tilde{x}(m), 1)$. We start by computing the derivative of H_1,

$$H_1'(x) = -\frac{(\psi'(x)^2 + \psi''(x)(1-\psi(x)))}{\psi'(x)^2}.$$

We want to show that $H_1(x) < 0$ for $\tilde{x}(m) < x < 1$, which is equivalent to

$$\psi'(x)^2 + \psi''(x)(1-\psi(x)) > 0. \tag{F.31}$$

By plugging in the expressions for the derivatives of ψ (from equations (F.4) and (F.5) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of the previous inequality as

$$\frac{4m^2x + (1-x)^2(1-x^2+6x^3+2x^4) + 2m(1+2x^2-4x^3+x^4)}{m^2(1+x)^4}.$$

Since the denominator is positive, we just need to show that the numerator is positive for $\tilde{x}(m) < x < 1$. In fact, it is easy to show that it is positive for $x \in (0,1)$. Indeed, note by inspection that the first two terms in the denominator are positive. Simple calculus shows that the third term is positive as well.[63] It follows that (F.31) holds for $\tilde{x}(m) < x < 1$.

We now show that $H_2(x)$ is also decreasing for $\tilde{x}(m) < x < 1$. The derivative of H_2 is

$$H_2'(x) = \frac{m(\phi''(x)\psi'(x) - \phi'(x)\psi''(x))}{(\psi'(x))^2} - 1.$$

We want to show that $H_2'(x) < 0$ for $x \in (\tilde{x}(m), 1)$; which is equivalent to

$$\frac{1}{m}\psi'(x)^2 - (\phi''(x)\psi'(x) - \phi'(x)\psi''(x)) > 0.$$

By plugging in the derivatives of ψ and ϕ (from equations (F.4), (F.5), (F.13), and (F.14) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of the previous inequality as $f(x)/(4m^3(1+x)^4)$ where

$$f(x) = 4m^2(2+x+3x^2) + 4m(1-x)(x^2+8x-1) + x(1-x)^2(9x^3+27x^2+15x-19).$$

Therefore, it suffices to show that $f(x) > 0$ for $x \in (\tilde{x}(m), 1)$. To do so, we first obtain a lower bound for $f(x)$ by removing some of the positive terms in the previous expression, so that

$$f(x) > 8m^2 - 4m(1-x) - 19x(1-x)^2 \geq 8m^2 - 4m(1-x) - 19(1-x)^2.$$

We now show that $8m^2 - 4m(1-x) - 19(1-x)^2 > 0$ for all $x \in (\tilde{x}(m), 1)$. By changing variables to $y = 1-x$, we show that $8m^2 - 4my - 19y^2 \geq 0$ for $y \in [0, 1-\tilde{x}(m)]$. This quadratic expression has one positive root, namely $y(m) = 2m(\sqrt{39}-1)/19$; and so we just need to show that $1 - \tilde{x}(m) \leq y(m)$, where, as defined in (B.3), $\tilde{x}(m) = \frac{1}{3}\left(\sqrt{(m-1)^2+3} - (m-1)\right)$.

[63] To see this, let $f(x) = 1 + 2x^2 - 4x^3 + x^4$, and observe that $f(0) = 1$ and $f(1) = 0$. By differentiation we have $f'(x) = 4x(1-3x+x^2)$, which has two roots in $(0,1)$, namely $x_1 = 0$ and $x_2 = (3-\sqrt{5})/2$. Then, $f(x)$ is increasing in $(0, x_2)$ and decreasing in $(x_2, 1)$. Therefore, $\min_{x\in[0,1]} f(x) = f(1) = 0$, and $f(x)$ is positive for $x \in (0,1)$.

Equivalently, we show that $1 \le \tilde{x}(m) + y(m)$, for all $m \ge 0$. First note that both sides of this inequality are equal at $m=0$. Moreover, since $\tilde{x}(m)$ is convex and $y(m)$ is linear in m, their sum is convex and we just need to check that the derivative of that sum is positive at $m=0$. Indeed, we have that $y'(0)+\tilde{x}'(0) = 2\left(\sqrt{39}-1\right)/19 - 1/2 > 0$.

Therefore, $f(x)>0$ for $x \in (\tilde{x}(m),1)$, which implies that $H_2'(x)<0$ for $x\in(\tilde{x}(m),1)$.

Part 2. Fix $m \ge 6$, then $H(x,m)$ is strictly decreasing in x for all $0<x<\tilde{x}(m)$.

We show that the derivative of H w.r.t. x is negative under these conditions. Note that the derivative of H w.r.t. x is

$$H_x(x,m) = \frac{[m\left(\psi_x(x,m)\phi_{xx}(x,m) - \psi_{xx}(x,m)\phi_x(x,m)\right) - \psi_{xx}(x,m)\left(1-\psi(x,m)\right) - 2(\psi_x(x,m))^2]}{(\psi_x(x,m))^2}.$$

The denominator is always positive since $\psi(x,m)$ is strictly increasing in x, by Claim 10. Then, we want to show that the numerator is negative for all $m\ge 6$ and $0<x<\tilde{x}(m)$, i.e., that

$$2(\psi_x(x,m))^2 - m\left(\psi_x(x,m)\phi_{xx}(x,m) - \psi_{xx}(x,m)\phi_x(x,m)\right) + \psi_{xx}(x,m)\left(1-\psi(x,m)\right) > 0.$$

By plugging in the expressions for the derivatives of ψ and ϕ for the region $0<x<\tilde{x}(m)$ (from equations (F.4), (F.5), (F.13), and (F.14) in the proof of Claim 10) and simplifying the resulting expression, we can write the LHS of this inequality as $xf(x,m)/m^2(2x(2x+m))^{3/2}$, where

$$\begin{aligned} f(x,m) &= \left(m^4 - 6m^3 - 48m^2x - 28m^3x - 64mx^2 - 216m^2x^2 - 576mx^3 - 512x^4\right) \\ &\quad + 4\sqrt{2x(2x+m)}\left(2m^3 + 64x^3 + 8mx(1+7x) + m^2(4+17x)\right). \end{aligned} \tag{F.32}$$

Our claim reduces to proving that $f(x,m)>0$ for $x\in(0,\tilde{x}(m))$ and $m\ge 6$. To do so, we will first show that $f(x,m)$ is increasing in m for all $x>0$, and then show that $f(x,6)>0$ for all $0<x<\tilde{x}(m)$. With some algebra, we can write the partial derivative of f w.r.t. m as $A(x,m)/\sqrt{2x(2x+m)}$, where $A(x,m) = A_1(x,m) - A_2(x,m) + A_3(x,m)$, and

$$\begin{aligned} A_1(x,m) &= 4x\left(14m^3 + 32x^2(1+9x) + 8mx(7+38x) + m^2(20+109x)\right), \\ A_2(x,m) &= 4x\sqrt{2x(2x+m)}\left(16x(1+9x) + 12m(2+9x) + 21m^2\right), \\ A_3(x,m) &= 2m^2\sqrt{2x(2x+m)}\,(2m-9). \end{aligned}$$

Note that $A_3(x,m)$ is positive when $m>9/2$, and in particular when $m\ge 6$. Therefore, we just need to show that $A_1(x,m)-A_2(x,m)>0$. Since both $A_1(x,m)$ and $A_2(x,m)$ are non-negative, it suffices to show that $(A_1(x,m))^2 - (A_2(x,m))^2 > 0$. By plugging in the expressions for A_1 and A_2 and simplifying the resulting expression, we have that

$$(A_1(x,m))^2 - (A_2(x,m))^2 = 16mx^2A_4(x,m),$$

where $A_4(x,m)$ is a polynomial in x and m with strictly positive coefficients:

$$\begin{aligned} A_4(x,m) &= 196m^5 + 1024x^4(1+9x) + 70m^4(8+31x) + m^3(4+19x)(100+503x) \\ &\quad + 64mx^2(9+x(131+346x)) + 16m^2x(68+x(595+1298x)) > 0. \end{aligned}$$

It follows that $f(x,m)$ is strictly increasing in m when $m\ge 6$, for all $x>0$. It remains to show that show that $f(x,6)>0$ for all $0<x<\tilde{x}(m)$. By plugging in $m=6$ into (F.32) and rearranging terms we have

$$f(x,6) = 32\left[\sqrt{x(3+x)}\left(16x^3 + 84x^2 + 165x + 144\right) - x\left(16x^3 + 108x^2 + 255x + 243\right)\right].$$

From where it follows that

$$\begin{aligned}\frac{f(x,6)}{x} &= 32\left[\sqrt{1+\frac{3}{x}}\left(16x^3+84x^2+165x+144\right)-\left(16x^3+108x^2+255x+243\right)\right]\\ &> 32\left[2\left(16x^3+84x^2+165x+144\right)-\left(16x^3+108x^2+255x+243\right)\right]\\ &= 32\left[16x^3+60x^2+75x+45\right] > 0.\end{aligned}$$

Thus, $f(x,m) > 0$ for all $x > 0$ and $m \geq 6$. In particular, this holds for $x \in (0, \tilde{x}(m))$. Therefore, $H_x(x,m) < 0$ for all $x \in (0, \tilde{x}(m))$ and all $m \geq 6$, as desired.

Part 3. Fix $m < 6$, then $H(x,m)$ is strictly concave in x when $0 < x < \tilde{x}(m)$.

Recall that $H(x,m) = \frac{1-\psi(x,m)+m\phi_x(x,m)}{\psi_x(x,m)} - x$. By plugging in the expressions of ψ, ψ_x and ϕ_x given in the proof of Claim 10 for the case when $0 < x < \tilde{x}(m)$, we have that

$$H(x,m) = \frac{2x(m+2x)(m+4x)-\sqrt{2x(2x+m)}\,(m^2-2m+8mx+8x^2)}{4\sqrt{2x(2x+m)}(4x+m)-4x(8x+3m)}. \tag{F.33}$$

We want to show that this expression is concave in x, i.e., thet $H_{xx}(x,m) < 0$ for $0 < x < \tilde{x}(m)$. By taking the second derivative w.r.t. x of H and simplyfing the resulting expression, we can write $H_{xx}(x,m) = mxB_1(x,m)/B_2(x,m)$, where

$$\begin{aligned}B_1(x,m) &= B_3(x,m) - \sqrt{2x(2x+m)}B_4(x,m),\\ B_2(x,m) &= \sqrt{2}(x(2x+m))^{3/2}\left(\sqrt{2x(2x+m)}\,(8x+2m)-2x(8x+3m)\right)^3,\\ B_3(x,m) &= 9m^4x(6-m)+1136m^3x^2+52m^4x^2+6144m^2x^3+840m^3x^3\\ &\quad +12288mx^4+2304m^2x^4+8192x^5+1792mx^5,\\ B_4(x,m) &= m^4(6-m)+216m^3x-8m^4x+1920m^2x^2+216m^3x^2+5120mx^3\\ &\quad +928m^2x^3+4096x^4+896mx^4.\end{aligned}$$

We will show that $B_1(x,m) < 0$ and $B_2(x,m) > 0$ for all $0 < x < 1$ and $0 < m \leq 6$. In particular, this implies the result for $0 < x < \tilde{x}(m)$. First, notice that $B_2(x,m) > 0$ if and only if

$$\sqrt{2x(2x+m)}\,(8x+2m) > 2x(8x+3m). \tag{F.34}$$

By taking squares on both sides and rearranging terms, this inequality reduces to $4mx(2m^2+11mx+16x^2) > 0$, which holds for all $x, m > 0$. Thus, $B_2(x,m) > 0$.

To show that $B_1(x,m) < 0$, first note (by inspection) that both $B_3(x,m)$ and $B_4(x,m)$ are positive when $x > 0$ and $m \in (0,6)$. Then $B_1(x,m)$ is negative if and only if $B_5(x,m) > 0$, where

$$B_5(x,m) = (2x(2x+m))\,B_4(x,m)^2 - B_3(x,m)^2. \tag{F.35}$$

By plugging in the expressions for B_3 and B_4 above and simplifying the resulting expression we can write $B_5(x,m) = m^4xB_6(x,m)$, where

$$\begin{aligned}B_6(x,m) &= 2m^7 - 32768x^5(2x-3) - 3m^6(8+15x) - 1024mx^4\left(81x+14x^2-168\right)\\ &\quad + 12m^5\left(6-11x+22x^2\right) - 96m^3x^2\left(74x^2-103x-282\right)\\ &\quad + 4m^4x\left(603+828x+80x^2\right) - 64m^2x^3\left(300x+309x^2-1664\right).\end{aligned}$$

Notice that $B_6(0,m) = 2m^5(m-6)^2 > 0$, so it suffices to show that $B_6(x,m)$ is increasing in x for all fixed $0 < m < 6$. By taking the partial derivative of B_6 w.r.t. x and simplifying the resulting expression we have

$$\frac{\partial B_6}{\partial x}(x,m) = -3\left[B_7(x,m) + 32x^2 B_8(x,m)\right],$$

where

$$\begin{aligned}
B_7(x,m) &= (m-6)m^4(134+15m) - 16m^3 x\left(1128+138m+11m^2\right),\\
B_8(x,m) &= 1024x^2(4x-5) + m^3(296x-309) - 10m^4\\
&\quad + 32mx\left(28x^2+135x-224\right) + 2m^2\left(515x^2+400x-1664\right).
\end{aligned}$$

By inspection, we note that both $B_7(x,m)$ and $B_8(x,m)$ are strictly negative for $0 < m < 6$ and $0 \le x \le 1$, therefore $B_6(x,m)$ is increasing in x and it follows that $B_5(x,m) > B_5(0,m) = 0$. Thus, $B_1(x,m) < 0$, which implies that $H_{xx}(x,m) < 0$ for $m < 6$ and $0 < x < \tilde{x}(m)$.

Finally, Claim 43 follows from the previous three parts. From part 1, we have that $H(x,m)$ is strictly decreasing in x for $\tilde{x}(m) < x < 1$, and from parts 1 and 2 we know that $H(x,m)$ is either strictly concave of strictly decreasing, for $0 < x < \tilde{x}(m)$. Thus, $H(x,m)$ is either strictly decreasing for all x, or strictly concave in $(0,\tilde{x}(m))$ and strictly decreasing afterwards. Since H is continuous in x, strict quasiconcavity follows. ☐

Proof of Claim 44. As in the proof of Claim 43, we treat the cases where $0 \le x < \tilde{x}(m)$, and $\tilde{x}(m) < x \le 1$ separately. We prove the result in four steps.

Step 1. $W(x,m)$ is decreasing in x for $x > \tilde{x}(m)$.

Fix $m > 0$. From equations (F.26) and (F.28), we can write

$$W(x,m) = H(x,m) - Z(x,m) + \frac{1}{2}. \tag{F.36}$$

By Part 1 in the proof of Claim 43, we have that $H(x,m)$ is decreasing in x for $x \in (\tilde{x}(m),1)$. In addition, $Z(x,m) = \psi(x,m)/\psi_x(x,m)$ is increasing in x since, by Claim 10, $\psi(x,m)$ is strictly increasing and strictly concave in x. Thus, $W(x,m)$ is decreasing in x for $x \in (\tilde{x}(m),1)$.

Step 2. $W(\tilde{x}(m),m) < 0$ for all $m > 0$.

Let $0 < x \le \tilde{x}(m)$. By plugging in the expression for $H(x,m)$, $\psi(x,m)$ and $\psi_x(x,m)$ (see equations (F.1), (F.4), (F.33)) and simplifying the resulting expression, we can write

$$W(x,m) = -\frac{N(x,m)\left(\sqrt{2x(2x+m)} - 2x\right)}{D(x,m)}, \tag{F.37}$$

where $N, D : [0,1] \times \mathbb{R}^+ \to \mathbb{R}$ are defined as

$$\begin{aligned}
N(x,m) &= m^2 - 4m + 20x^2 + 12mx - 10x + \sqrt{2x(2x+m)}\,(2x-1),\\
D(x,m) &= 2\left((8x+2m)\sqrt{2x(2x+m)} - 2x(8x+3m)\right).
\end{aligned} \tag{F.38}$$

From (F.34) in the proof of Claim 43, we have that $D(x,m) > 0$ for $x, m > 0$, and in particular for $x = \tilde{x}(m)$. Since $\sqrt{2x(2x+m)} > 2x$, it suffices to show that $N(\tilde{x}(m),m) > 0$.

Recall that $\tilde{x}(m) = \frac{1}{3}\left(\sqrt{(m-1)^2+3} - (m-1)\right)$ and, from (F.2), that $2\tilde{x}(2\tilde{x}+m) = (1+\tilde{x})^2$ (abbreviating $\tilde{x} = \tilde{x}(m)$). Plugging this into $N(x,m)$ yields

$$\begin{aligned} N(\tilde{x},m) &= m^2 - 4m + 20\tilde{x}^2 + 12m\tilde{x} - 10\tilde{x} - (1+\tilde{x})(1-2\tilde{x}) \\ &= \frac{1}{9}\left(17m^2 - 61m + 74 + (17-8m)\sqrt{(m-1)^2+3}\right). \end{aligned}$$

We claim that this expression is positive. To see this, first notice that $17m^2 - 61m + 74$ is positive for all m, and so $N(\tilde{x},m)$ is clearly positive when $0 \le m \le 17/8$). If $m > 17/8$, then we can bound $N(\tilde{x},m)$ as follows:

$$\begin{aligned} 9N(\tilde{x},m) &= 17m^2 - 61m + 74 + (17-8m)\sqrt{(m-1)^2+3} \\ &\ge 17m^2 - 61m + 74 + (17-8m)\left(m - 1 + \sqrt{3}\right) \\ &> 17m^2 - 61m + 74 + (17-8m)(m+1) \\ &= 9m^2 - 52m + 91 > 0. \end{aligned}$$

Therefore, $N(\tilde{x}(m),m) > 0$, which implies that $W(\tilde{x}(m),m) > 0$ for all $m > 0$.

Step 3. If $x \in (0,1)$ solves $W(x,m) = 0$, then $x < \tilde{x}(m)$. In addition, x solves $W(x,m) = 0$ if and only if it solves $N(x,m) = 0$.

By steps 1 and 2, it follows that $W(x,m) < 0$ for all $\tilde{x}(m) \le x \le 1$. Thus, if $W(x,m) = 0$, it must be that $x < \tilde{x}(m)$. Moreover, it follows from equation (F.37) that the roots of $W(x,m)$ with $0 < x < \tilde{x}(m)$ for fixed m are the same as for $N(x,m)$.

Step 4. Given $m > 0$ fixed, $N(x,m)$ is strictly convex in x. Therefore, given $m > 0$ fixed, $N(x,m)$ has at most two roots.

Fix $m > 0$. Notice that the term without square root in $N(x,m)$, $m^2 - 4m + 20x^2 + 12mx - 10x$, is strictly convex in x. In addition, the remaining term $\sqrt{2x(2x+m)}\,(2x-1)$ is easily verified to be convex in x for $x > 0$, from where strict convexity of $N(x,m)$ follows. This implies that there are at most two values of $x \in [0, \tilde{x}(m))$ such that $N(x,m) = 0$.

To conclude the proof, note that by steps 1 and 2, we have that $W(x,m) < 0$ for all $\tilde{x}(m) \le x < 1$. This implies that if $W(x,m) = 0$, we have that $x \in (0, \tilde{x}(m))$. By steps 3 and 4, there are at most two values of $x \in (0, \tilde{x}(m))$ such that $W(x,m) = 0$. □

F.2. Characterization of Interior Equilibria in $\mathbf{G}(m)$

In this appendix, we provide a characterization of the set of values of m for which an interior PSNE exists in $\mathbf{G}(m)$. The main result in this appendix is Proposition 8, which provides a system of inequalities that allows to efficiently compute the quantity $\bar{X}^*(m) = p_1^{B*}(m) - p_1^{A*}(m) + 1/2$, for values of m such that $\mathbf{G}(m)$ admits an interior equilibrium $(p_1^{A*}(m), p_1^{B*}(m))$. This result allows us to efficiently compute the equilibrium outcomes of the baseline model numerically since, once we know the value of $\bar{X}^*(m)$, it is easy to compute the firms' equilibrium prices for product 1 from the first-order conditions for profit maximization, and the price firm B sets for "untracked" consumers in the second period as $m\bar{\theta}\left(\bar{X}^*(m), m\right)$. In addition, Proposition 8 allows us to prove Remark 1 (see Appendix B.3), which provides numerical approximations for the constants m_L and m_H, and shows that for some values of m, $\mathbf{G}(m)$ admits no PSNE.

The rest of this appendix is organized as follows. We first establish some preliminary notation that we use in subsequent proofs, then state and prove two intermediate results (Claims 45 and 46) that then allow us to prove Proposition 8. We conclude with the proof of Remark 1.

Preliminary definitions As we discussed in Appendix B.3, firm B's profit function is in general not quasiconcave in its own price, so solving for the first order conditions for profit maximization is not sufficient to characterize its best-response correspondence. However, Claim 16 shows that firm B's profit maximization problem admits at most one interior local maximum. Therefore, to verify that a price p_1^B is a best-response to some firm A's price p_1^A, it suffices to verify the second-order condition for profit maximization, and to check that such price results in a higher profit than choosing the "corner" price $p_1^A - 1/2$ (which results in firm B capturing the entire demand for product 1). We define two auxiliary functions for this purpose.

Let $U:[0,1]\times\mathbb{R}^{++}\to\mathbb{R}$ be defined by

$$U(x,m) = -2\psi_x(x,m) - V(x,m)\psi_{xx}(x,m) + m\phi_{xx}(x,m), \tag{F.39}$$

where ψ and V are defined in (B.7) and (B.31), respectively.

Recall from Claim 16 that if p_1^B solves firm B's first-order condition for profit maximization, given $p_1^A \in S^A$, we have that $p_1^B = V(p_1^B - p_1^A + 1/2, m)$. If that is the case, $U(p_1^B - p_1^A + 1/2, m)$ is equal to the second-derivative of firm 2's profit function with respect to its own price evaluated at p_1^B, which must be negative if p_1^B indeed maximizes firm B's profit.

Now, we define a function that allows us to check if a price that solves firm B's first-order condition for profit maximization yields a higher profit than the price that captures all the market, i.e., if $\pi^B\left(p_1^A, p_1^B, m\right) \geq \pi^B\left(p_1^A, p_1^A - 1/2, m\right)$. We want to define a function that is positive iff this condition holds (for a price p_1^B that satisfies firm B's first-order condition for profit maximization). Let us then define $T:[0,1]\times\mathbb{R}^{++}\to\mathbb{R}$ by

$$T(x,m) = x - V(x,m)\psi(x,m) + m\left(\phi(x,m) - 1/2\right), \tag{F.40}$$

where ψ, ϕ, and V are defined in (B.7), (B.9), and (B.31), respectively. To see why T serves the purpose we have described, notice that if $p_1^B = V(p_1^B - p_1^A + 1/2, m)$, we have that (abbreviating $\bar{X} = \bar{X}(p_1^A, p_1^B)$):

$$\begin{aligned}\pi^B\left(p_1^A, p_1^B, m\right) - \pi^B\left(p_1^A, p_1^A - 1/2, m\right) &= p_1^B\left(1 - \psi(\bar{X}, m)\right) + m\phi(\bar{X}, m) - m/2 + 1/2 - p_1^A \\ &= p_1^B - p_1^A + 1/2 - p_1^B\psi(\bar{X}, m) + m\left(\phi(\bar{X}, m) - 1/2\right) \\ &= \bar{X} - V(\bar{X}, m)\psi(\bar{X}, m) + m\left(\phi(\bar{X}, m) - 1/2\right) \\ &= T(\bar{X}, m).\end{aligned}$$

The next claim provides a system of equations and inequalities that characterize interior PSNE in $\mathbf{G}(m)$, based on the functions we have defined.

Claim 45. *Let $p_1^A \geq 0$ and $p_1^B \geq -1/2$. Then, (p_1^A, p_1^B) is an interior PSNE for $\mathbf{G}(m)$ if and only if $X^* = p_1^B - p_1^A + 1/2 \in (0,1)$, and the following conditions hold*

$$Z(X^*, m) = p_1^A, \tag{F.41}$$

$$V(X^*,m) = p_1^B, \tag{F.42}$$

$$T(X^*,m) \geq 0, \tag{F.43}$$

$$U(X^*,m) < 0. \tag{F.44}$$

Proof of Claim 45. If (p_1^A, p_1^B) is an interior PSNE in $\mathbf{G}(m)$, then $p_1^A \in S^A$ and $p_1^B \in S^B$ by definition of $\mathbf{G}(m)$. In addition, both firms' first order conditions for profit maximization must be satisfied (equations (F.41) and (F.42)). Since p_1^B is a best response to p_1^A for firm B, conditions (F.43) and (F.44) must also hold by the preceding discussion. Finally, to see that $X^* \in (0,1)$ holds, notice that, since the equilibrium is interior, $p_1^B > p_1^A - 1/2$. In addition, we have that $p_1^A > p_1^B - 1/2$ since otherwise firm B's price is not a best-response to p_1^A by Claim 16.

Now suppose that X^*, p_1^A, p_1^B solve the system above and that $X^* = p_1^B - p_1^A + 1/2 \in (0,1)$. We will show that $p_1^A \in S^A = [0,1]$ and $p_1^B \in S^B = [-1/2, 1]$, and that $p_1^A = BR^A(p_1^B, m)$ and $p_1^B \in BR^B(p_1^A, m)$. To prove this, note that by conditions (F.41) and (F.42), we have that

$$p_1^A = \frac{\psi(X^*,m)}{\psi_x(X^*,m)}, \quad p_1^B = \frac{1-\psi(X^*,m)+m\phi_x(X^*,m)}{\psi_x(X^*,m)}.$$

By adding these two conditions, we have that

$$p_1^A + p_1^B = \frac{1+m\phi_x(X^*,m)}{\psi_x(X^*,m)} < \frac{1}{\psi_x(X^*,m)} < 2,$$

where the last two inequalities follow since, by Claim 10, $\phi(x,m)$ is strictly decreasing in x, $\psi(x,m)$ is strictly concave in x and $\psi_x(1,m) = 1/2$. Therefore, at least one of the inequalities $p_1^A < 1$ or $p_1^B < 1$ holds. We now cover these two cases.

First suppose that $p_1^B < 1$. Since $\bar{X}^* = p_1^B - p_1^A + 1/2 > 0$, and $p_1^A > 0$ (as $Z(\bar{X}^*, m) > 0$), we have that $p_1^B > -1/2$. Since $p_1^A = Z(p_1^B - p_1^A + 1/2, m)$ (by (F.41)), it follows from the proof of Claim 12 that $p_1^A = BR^A(p_1^B, m)$, and that $p_1^A \in S^A$.

Now suppose that $p_1^A < 1$. Then, by following the same argument as in the proof of Claim 16, and by conditions (F.42) and (F.44), it follows that p_1^B is a local maximizer of the problem

$$\max\left\{\pi^B\left(p_1^A, y, m\right) : y \in [p_1^A - 1/2, p_1^A + 1/2]\right\} = \max\left\{\pi^B\left(p_1^A, y, m\right) : y \in S^B\right\},$$

where the equality follows from the proof of Claim 16 and by (F.18).

But we know, by the proof of Claim 16, that this problem has at most one local maximum, which must be p_1^B. In fact, we claim that p_1^B is a global maximizer of the problem. To verify this, note that since conditions (F.42) and (F.44) hold, p_1^B is a global maximizer of firm B's profit maximization problem (given p_1^A) if and only if it results in higher profits than choosing $\hat{p}_1^B = p_1^A - 1/2$, which holds by (F.43). Therefore, $p_1^B \in BR^B(p_1^A, m)$ and, by Claim 16, $p_1^B \in S^B$.

By the previous two cases, it follows that $p_1^A \in S^A$ and $p_1^B \in S^B$. In addition, $p_1^A = BR^A(p_1^B, m)$ and $p_1^B \in BR^B(p_1^A, m)$. Thus (p_1^A, p_1^B) is a PSNE of $\mathbf{G}(m)$. □

Next, we show that the system above can be simplified.

CLAIM 46. *Conditions (F.42) and (F.43) imply condition (F.44).*

Proof of Claim 46. Equation (F.25) in the proof of Claim 16 implies that firm B's first-order condition for profit maximiation (i.e., condition (F.42)) can be written as $H(p_1^B - p_1^A + 1/2, m) = p_1^A - 1/2$. By Claim 43, we know that $H(x,m)$ is a strictly quasiconcave function of x for fixed $m > 0$. Therefore, given $p_1^A \in S^A$, there can be at most two values of $p_1^B \in S^B$ that satisfy

$$H(p_1^B - p_1^A + 1/2, m) = p_1^A - 1/2.$$

If there is only one such value, it must be a local maximizer of firm B's profit maximization problem, since choosing $\hat{p}_1^B = p_1^A - 1/2$ is suboptimal by (F.43). Thus, we have that $\frac{\partial^2}{\partial (p_1^B)^2}\pi^B(p_1^A, p_1^B, m) < 0$, which implies that condition (F.44) holds.

Now assume that there are two values of $x \in [0,1]$ that satisfy $H(x,m) = p_1^A - 1/2$, say $x_1 < x_2$. Note that $\pi^B(p_1^A, p_1^B, m)$ is decreasing at p_1^B iff $H(p_1^B - p_1^A + 1/2, m) < p_1^A - 1/2$. Since $H(x,m)$ is strictly quasiconcave in x, it follows that $H(x,m) > p_1^A - 1/2$ for $x_1 < x < x_2$, and $H(x,m) < p_1^A - 1/2$ otherwise. This implies that $\pi^B(p_1^A, p_1^A - 1/2 + x, m)$ is increasing for $x \in (x_1, x_2)$ and decreasing otherwise. In particular, we have that

$$\pi^B(p_1^A, p_1^A - 1/2, m) > \pi^B(p_1^A, p_1^A - 1/2 + x_1, m),$$

which implies that $T(x_1, m) < 0$. By (F.43), It follows that $x_2 = p_1^B - p_1^A + 1/2$. In addition, $x = x_2$ is a local maximizer of $\pi^B(p_1^A, p_1^A - 1/2 + x, m)$, which implies that the second-order condition for firm B's profit maximization holds and therefore $U(p_1^B - p_1^A + 1/2, m) < 0$. □

With the previous two claims, we now characterize the set of values of m for which $\mathbf{G}(m)$ has an interior equilibrium.

PROPOSITION 8. *Fix $m > 0$ and consider the following system for $x \in [0,1]$*

$$\begin{aligned} W(x,m) &= 0, \\ T(x,m) &\geq 0, \end{aligned} \tag{F.45}$$

where W and T are defined in (F.28) and (F.40), respectively. Then $\mathbf{G}(m)$ has an interior PSNE if and only if some $x \in (0,1)$ satisfies system (F.45), and in that case, $\bar{X}^*(m) = x$.

Proof of Proposition 8. First assume $\mathbf{G}(m)$ has an PSNE (p_1^A, p_1^B) and take $x = \bar{X}(p_1^A, p_1^B) = p_1^B - p_1^A + 1/2 > 0$. By Claim 45, x satisfies equations (F.41) and (F.42), which implies that $W(x,m) = 0$, by (F.28). In addition, x satifies (F.43), and therefore system (F.45) is satisfied.

Now assume that some $x \in (0,1)$ satisfies system (F.45). Define $p_1^A = Z(x,m)$ and $p_1^B = V(x,m)$. Since $W(x,m) = 0$ and $W(x,m) = V(x,m) - Z(x,m) + 1/2 - x$, we have that $x = p_1^B - p_1^A + 1/2$. Since $T(x,m) \geq 0$, it follows from Claims 45 and 46 that (p_1^A, p_1^B) is a Nash Equilibrium for $\mathbf{G}(m)$. Since $0 < x < 1$, the equilibrium is interior, and by Claims 15 and 18, it is the unique PSNE for $\mathbf{G}(m)$. Thus, $\bar{X}^*(m) = x$. □

Finally, we leverage the characterization that we have established to provide bounds for m_L and m_H, given in Lemma 3, and in particular we show that $\mathbf{G}(m)$ admits no PSNE when $m = 4$.

Proof of Remark 1. To show that $m_L < 4$, we first notice by computation that $\lim_{x \to 0^+} W(x,4) = 0$. By Claim 44, there is at most one value of $x \in (0,1)$ such that $W(x,4) = 0$. We solve this equation numerically and obtain that this solution is $x_{sol} \approx 0.0054$. We compute that $T(x_{sol}, 4) \approx -6.01 \times 10^{-6} < 0$. Therefore, system (F.45) has no solution for $m = 4$, which implies, by Proposition 8, that $\mathbf{G}(4)$ has no interior PSNE. By the definition of m_L given in (B.32), it follows that $m_L < 4$.

To show that that $\mathbf{G}(4)$ has no corner PSNE, by Claim 15, it suffices to show that $-1/2 \notin BR^B(0,4)$; i.e., that there exists some price $p_1^B > -1/2$ such that $\pi^B(0,-1/2,4) < \pi^B(0,p_1^B,4)$. Numerical computation shows that this holds for $p_1^B = -0.48$, which implies that $-1/2 \notin BR^B(0,4)$. Therefore, $\mathbf{G}(4)$ has no PSNE and $m_H > 4$.

We now provide an approximation for m_L. By following a similar argument as in the proofs of Claims 22 and 44, we have that since $m_L < 4$, given a fixed $0 < m < 4$, there exists a unique $x = x(m) \in (0,1)$ that satisfies $W(x,m) = 0$. In particular, we have that $x(m) = \bar{X}^*(m)$ for $m < m_L$. Then, by Proposition 8 and the definition of m_L given in (B.32), we have that

$$m_L = \sup_{0 \le m \le 4} \{m : T(x(m'), m') \ge 0, \text{ for all } 0 < m' \le m\}.$$

By following a similar argument as in the proof of Claim 23, one can show that $T(x(m), m) > 0$ for all $m \in (0, 3.98)$. By computation, we obtain that $T(x(3.995), 3.995) < 0$, and so we have that $3.98 < m_L < 3.995$. We then use $m_L \approx 3.98$ as an approximation.

To approximate the value of m_H, computation shows that $-1/2 \notin BR^B(0,4.01)$, and, similarly, that $BR^B(0,4.03) = -1/2$. By Claim 13, we have that $4.01 < m_H \le 4.03$, and so we approximate $m_H \approx 4.02$. □

Appendix G: The Model with a Monopoly in Both Markets

In this appendix, we modify our baseline model to consider the case where firm B is a monopoly in both markets. We study this setting using a similar approach to that used in the baseline model: we characterize the equilibrium in each of the forward-looking, restricted and myopic settings, and compare their corresponding equilibrium consumer surplus. This analysis results in Proposition 3, which states that, unlike the model with competition, consumer surplus is always lower in the myopic than in the restricted setting. Therefore, *the presence of competition in product market 1 is essential to argue that myopic consumers can be better off with data tracking than without it.*

This appendix is organized as follows: we modify the baseline model to the monopoly context, and characterize the equilibrium outcomes in our three settings in Appendix G.1 (see Propositions 9, 10, and 11). Then, based on these characterizations, we compare the equilibrium levels of consumer surplus of the settings with data tracking with the restricted setting and contrast the implications of these comparisons with the ones in the baseline model in Appendix G.2. These comparisons result in the proof of Proposition 3.

G.1. Characterization of Equilibria in the Monopoly Model

G.1.1. The Monopoly Model We consider the baseline model, described in section 2, but removing firm A from the first product market. That is, with firm B as a monopoly for both products. Throughout this appendix, we refer to firm B simply as "the firm" or "the monopolist". In this context, we reinterpret the consumer's action in the first period $a_1 \in \{0,1\}$ to denote whether it buys product 1 from the monopolist ($a_1 = 1$) or not ($a_1 = 0$). Thus, the consumer's utility associated with product 1 given her action, her type, and the price for product 1 is equal to

$$u_1^{mon}(a_1; s, \theta, p_1) = a_1 (\bar{u} - (1-s)/2 - p_1). \tag{G.1}$$

Notice that we could rewrite this utility function by defining the consumer's valuation for product 1 as $v_1 = \bar{u} - (1-s)/2$, and simply write the payoff as $a_1(v_1 - p_1)$. With this parametrization, the prior distribution of the consumer's valuation for product 1 is uniform in the interval $[\bar{u} - 1/2, \bar{u}]$. In what follows, we assume that the consumer's valuation for product 1 is always positive.

Assumption 1. *$\bar{u} \geq 1/2$.*

The consumer's utility associated with product 2 remains as in the baseline model, i.e., we have

$$u_2^{mon}(a_2; s, \theta, p_2) = a_2 (m\theta - p_2), \tag{G.2}$$

where $a_2 = 1$ ($a_2 = 0$) denotes the decision to (not) buy product 2 from the monopolist. As in the baseline model, the consumer's total utility is defined as the sum of the utilities for each product.

The monopolist profit function is as firm B's profit function in the baseline model. That is, the profit associated with product $i = 1, 2$ is

$$\pi_i^{mon}(p_i, a_i) = a_i p_i,$$

and the monopolist's total profit is the sum of the profit associated with each product. The timeline, histories, strategies, expected payoffs and the equilibrium in this model are defined similarly to the baseline model, but removing firm A. In what follows, we characterize the equilibria in the forward-looking, restricted, and myopic settings of the monopoly model.

G.1.2. Forward-looking Setting We now characterize the equilibrium of the setting with data tracking and forward-looking consumers in the monopoly model. Proposition 9 establishes existence of an equilibrium in this setting and characterizes the resulting equilibrium path, which has a similar structure to the corresponding one in the baseline model (as given in Theorem 1).

Proposition 9. *For any $m > 0$ and $\bar{u} \geq 1/2$, there exists an equilibrium in the forward-looking setting of the monopoly model. In addition, for a fixed $\bar{u} \geq 1/2$, there exists a constant $K(\bar{u})$ such that the equilibrium path can be characterized as follows:*

(a) If $0 < m < K(\bar{u})$, there exists a unique price $p_1^{FL,mon} = p_1^{FL,mon}(m, \bar{u}) > \bar{u} - 1/2$ such that the monopolist sets $p_1^{FL,mon}$ as the price of product 1. Moreover, there exists a constant $\bar{\theta}^{mon} = \bar{\theta}^{mon}(m, \bar{u}) \in (1/2, 1)$, and a function $g_{mon}^* : [0,1] \to \mathbb{R}$ defined by

$$g_{mon}^*(t) = 1 - 2\left(\bar{u} - p_1^{FL,mon}\right) + 2m\left(t - \bar{\theta}^{mon}\right)^+,$$

such that

(i) If the consumer's type (s, θ) is such that $s > g_{mon}^*(\theta)$, then, with probability one, the consumer buys product 1; the firm perfectly observes the consumer's type and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(ii) If the consumer's type (s, θ) is such that $s < g_{mon}^*(\theta)$, then, with probability one, the consumer chooses not to buy product 1. In the second period, the firm sets a price equal to $p_2 = m\bar{\theta}^{mon}$ for product 2, which the consumer buys if $\theta > \bar{\theta}^{mon}$.

(b) If $m > K(\bar{u})$, in any equilibrium, the firm's price for product 1 is $p_1^{FL,mon} = \bar{u} - 1/2$. The consumer buys product 1 with probability one. In addition, the firm perfectly observes the consumer's type and, in the second period, sets a price equal to her valuation for product 2, which the consumer buys.

The proof of this result has a very similar structure to the proof of Theorem 1. We first characterize the equilibrium in the subgame that follows the monopolist's choice for product 1's price in Claim 47. Then, Claim 48 provides the form of the monopolist's total expected profit function in terms ot the price of product 1. Finally, Claim 49 establishes that the firm's profit maximization problem in the first period has a unique solution (except for at most one value of m), which fully determines the equilibrium path.

We start by characterizing the equilibrium in the subgame that follows the firm's choice of price of product 1. To do so, let us define some auxiliary notation. Let us define

$$\bar{X}^{mon}(p_1) = \min\{(1 - 2(\bar{u} - p_1))^+, 1\}. \tag{G.3}$$

This quantity plays the role of $\bar{X}$ in the baseline model (see (B.1)). To see this, note that if $\bar{X}^{mon}(p_1) \in (0,1)$, it holds that $p_1 = \bar{u} - (1 - \bar{X}^{mon}(p_1))/2$. Thus, $\bar{X}^{mon}(p_1)$ denote the location of the consumer that is indifferent between whether to buy product 1 or not without taking into account the utility derived from product 2. In addition, for a fixed p_1 and $t \in [0,1]$, we define

$$g_{mon}(t \mid p_1) = 1 - 2(\bar{u} - p_1) + 2m\left(t - \bar{\theta}\left(\bar{X}^{mon}(p_1), 2m\right)\right)^+, \tag{G.4}$$

where $\bar{\theta}$ is defined in (B.2). The function g_{mon} plays the same role as that of the function g in the baseline model (see (B.4)). With these auxiliary functions, we can now characterize the equilibrium in the subgame that follows the firm's choice of p_1 as follows.

Claim 47. *In any equilibrium where the monopolist chooses p_1 as the price of product 1, we have that*[64]

(a) If $p_1 \le \bar{u} - 1/2$, the consumer buys product 1 from the firm with probability one; the firm perfectly observes the consumer's type and, in the second period, sets a price equal to the consumer's valuation for product 2, that the consumer buys.

(b) If $p_1 > \bar{u} - 1/2$, we have that:

(i) If the consumer's type (s,θ) is such that $s > g_{mon}(\theta \mid p_1)$, then, the consumer buys product 1 from the firm with probability one; the firm perfectly observes the consumer's type and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer buys.

(ii) If the consumer's type (s,θ) is such that $s < g_{mon}(\theta \mid p_1)$, then, the consumer chooses not to buy product 1 with probability one. In the second period, the firm sets the price for product 2 equal to $p_2 = m\bar{\theta}\left(\bar{X}^{mon}(p_1), 2m\right)$, which the consumer buys if $\theta > \bar{\theta}\left(\bar{X}^{mon}(p_1), 2m\right)$,

where $\bar{\theta}$, $\bar{X}^{mon}$, and g_{mon} are defined in (B.2), (G.3), and (G.4), respectively

Proof of Claim 47. Follows the same argument as the proof of Lemma 1. □

[64] As in Lemma 3, we do not explicitly characterize the equilibrium outcomes when the consumer is indifferent between purchasing a product or not. This event occurs with probability zero and has no impact in subsequent results.

Next, we derive the monopolist's profit function in terms of the product 1 price p_1, incorporating the subsequent equilibrium play established in Claim 47. This step is analogous to Claim 8 in the baseline model.

Claim 48. *Suppose that the monpolists sets the price of product 1 equal to p_1, and the subsequent play follows the path given in Claim 47. Then, the monopolist's total expected profit is*

$$\pi^{FL}_{mon}(p_1,m,\bar{u}) = \left(1-\psi(\bar{X}^{mon}(p_1),2m)\right)p_1 + m\phi\left(\bar{X}^{mon}(p_1),2m\right), \tag{G.5}$$

where ψ, ϕ, and $\bar{X}^{mon}$ are given in (B.7), (B.9), and (G.3), respectively.

Proof of Claim 48. Follows the same argument as in the proof of Claim 8. □

Now, we show that the monopolist's profit maximization problem when choosing the price of product 1 always has a solution, which is unique except for at most one value of m (given fixed $\bar{u}$).

Claim 49. *For fixed $m>0$ and $\bar{u}\geq 1/2$, consider the firm's profit maximization problem when setting the price of product 1:*

$$\max_{p_1\in\mathbb{R}} \pi^{FL}_{mon}(p_1,m,\bar{u}). \tag{G.6}$$

Then, this problem has a solution. In addition, there exists a constant $K(\bar{u})$ such that if $m\neq K(\bar{u})$, the solution is unique.

Proof of Claim 49. First note that pricing at $\bar{u}-1/2$ dominates any price $p_1<\bar{u}-1/2$ as $\bar{X}^{mon}(p_1)=0$ for any such price. In addition any price larger than $\bar{u}$ yields the same profit as $p_1=\bar{u}$, which implies that

$$\max_{p_1\in\mathbb{R}} \pi^{FL}_{mon}(p_1,m,\bar{u}) = \max_{\bar{u}-1/2\leq p_1\leq \bar{u}} \pi^{FL}_{mon}(p_1,m,\bar{u}). \tag{G.7}$$

Since $[\bar{u}-1/2,\bar{u}]$ is compact and $\pi^{FL}_{mon}(p_1,m,\bar{u})$ is continuous in p_1 (by the same argument as in Claim 11), the problem above always has a solution. It remains to show that this solution is unique for all values of $m>0$, except perhaps for a single value.

Next note that $p_1=\bar{u}$ is suboptimal as it results in a profit of zero from product 1 and minimizes the profit associated with product 2. It follows that the solution of (G.6) is either $p_1=\bar{u}-1/2$, or some $p_1\in(\bar{u}-1/2,1)$ such that

$$\frac{\partial}{\partial p_1}\pi^{FL}_{mon}(p_1,m,\bar{u}) = 0. \tag{G.8}$$

Let us define $K(\bar{u}) = \inf\{m\geq 0 : p_1=\bar{u}-1/2 \text{ solves problem (G.6)}\}$. By the same reasoning as in the proof of Claim 13, it follows that if $m>K(\bar{u})$, $p_1=\bar{u}-1/2$ is the unique solution of (G.6).

Now consider $m<K(\bar{u})$. By defintion of $K(\bar{u})$, any solution to the optimization problem (G.6) lies in $(\bar{u}-1/2,1)$ and satisfies (G.8). As in the proof of Claim 16, by changing variables to $x=1-2(\bar{u}-p_1)$, we can write this condition as $V(\bar{x},2m)-\bar{x}+(1-2u)=0$ where, $V(x,m)$ is defined in (B.31). It follows from Claim 43 that for any fixed $m>0$, $V(x,2m)-x$ is a strictly quasiconcave function of x. Therefore, equation (G.8) has at most two solutions. Furthermore, if it has two solutions, one of them most correspond to a local minimum since $\pi^{FL}_{mon}(p_1,m,\bar{u})$ is continuously differentiable in p_1 (by Claim 10). Therefore, at only one solution to (G.8) corresponds to a local maximum, which must be a global maximum since the corner points are not solutions to problem (G.6), by the assumption that $m<K(\bar{u})$. Therefore, problem (G.6) admits a single solution when $m<K(\bar{u})$. □

Finally, we provide the proof of the equilibrium characterization given in Proposition 9.

Proof of Proposition 9. Fix $m > 0$, $\bar{u} \geq 1/2$, and let $K(\bar{u})$ be as in Claim 49. Suppose that $m > K(\bar{u})$. Then, by the proof of Claim 49, the unique solution to (G.6) is $p_1 = \bar{u} - 1/2$. By Claim 48 and by sequential rationality for the firm, this is the price set for product 1 in any equilibrium. The characterization of the equilibrium path following such price choice follows from Claim 47.

Suppose instead that $m < K(\bar{u})$. Let $p_1^{FL,mon}(m,\bar{u})$ be the solution to (G.6), which we know is unique by Claim 49. By Claim 48 and by sequential rationality for the firm, this is the price set for product 1 in any equilibrium. Let $g^*_{mon}(t) = g\left(t \,|\, p_1^{FL,mon}(m,\bar{u})\right)$ and $\bar{\theta}^{mon}(m,\bar{u}) = \bar{\theta}\left(\bar{X}^{mon}\left(p_1^{FL,mon}(m,\bar{u})\right), 2m\right)$. The form of the equilibrium path that follows the choice of the price of product 1 follows from Claim 47. ☐

G.1.3. Restricted Setting In this section we characterize the equilibrium in the restricted setting of the monopoly model. Recall that in this setting, the consumer's purchase decision for product 1 does not reveal anything about her valuation for product 2. Therefore, as in the baseline model, both markets are entirely decoupled and we can obtain the equilibrium by looking at two separate games (one for each market). Proposition 10 summarizes the equilibrium outcome in this setting.

PROPOSITION 10. *For any $m > 0$ and $\bar{u} \geq 1/2$, there exists an equilibrium in the restricted setting of the monopoly model. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) The monopolist sets $p_1^{R,mon} = \max\{\bar{u}/2, \bar{u} - 1/2\}$ as the price of product 1.

(ii) The consumer buys product 1 if her type satisfies $s > \max\{1 - \bar{u}, 0\}$.

(iii) The monopolist sets a price of $p_2^{R,mon} = m/2$ for product 2, which the consumer buys if $\theta > 1/2$.

Moreover, in equilibrium, the expected consumer surplus is

$$CS^{R,mon}(m,\bar{u}) = \begin{cases} 1/4 + m/8, & \text{if } \bar{u} \geq 1, \\ \bar{u}^2/4 + m/8, & \text{if } \bar{u} < 1. \end{cases} \tag{G.9}$$

Proof of Proposition 10. We proceed by backwards induction, following the same approach as in the restricted setting of the baseline model. As in the proof of Proposition 1, in any equilibrium, the monopolist's beliefs over the consumer's type when setting the price of product 2 are equal to the prior distribution μ_0. Thus, in equilibrium, the monopolist sets the price of product 2 assuming that θ is uniformly distributed in $[0,1]$, which results in a price of $p_2^{R,mon} = m/2$ for product 2. As in Proposition 1, the consumer prefers to buy product 2 if and only if $\theta \geq 1/2$, with indifference if $\theta = 1/2$. Finally, note that this outcome results in an expected consumer surplus of $m/8$ associated with product 2.

For product 1, denote $v_1(s) = u - (1-s)/2$, so that given a price p_1, the consumer strictly prefers to buy product 1 if and only if $p_1 < v_1(s)$ (since the utility derived from product 2 is independent of the purchase decision in the first period). Then, the monopolist's profit maximization problem is

$$\max_{p_1 \in \mathbb{R}} p_1 \mathbb{P}_{\mu_0}\left[v_1(s) \geq p_1\right] = \max_{p_1 \in [\bar{u}-1/2, \bar{u}]} p_1 \max\left\{2(\bar{u} - p_1), 1\right\}. \tag{G.10}$$

It follows that the profit maximizing price for product 1 is $p_1^{R,mon} = \max\{\bar{u}/2, \bar{u} - 1/2\}$, and the consumer is indifferent between buying product 1 or not if and only if her type satisfies $s = \max\{1 - \bar{u}, 0\}$.

Finally, to obtain the expressions for the equilibrium consumer surplus given in (G.9), we add the surplus associated with product 2 (which is $m/8$) and the corresponding one for product 1, which is:

$$CS_1^{mon} = \mathbb{E}_{\mu_0}\left[\left(\bar{u} - (1-s)/2 - p_1^{R,mon}\right)^+\right] = \begin{cases} 1/4, & \text{if } \bar{u} \geq 1, \\ u^2/4, & \text{if } \bar{u} < 1. \end{cases}$$

□

G.1.4. Myopic Setting We now consider the setting with myopic consumers in the monopoly model. Recall that in this setting, the firm has data tracking ability but consumers act as in the restricted setting, i.e., they decide whether to buy product 1 by maximizing the utility associated with that product only. Proposition 11 summarizes the equilibrium outcome in this setting.

PROPOSITION 11. *For any $m > 0$ and $\bar{u} \geq 1/2$, there exists an equilibrium in the myopic setting of the monopoly model. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) The monopolist sets $p_1^{M,mon}(m,\bar{u}) = \max\{\bar{u}/2 - m/8, \bar{u} - 1/2\}$ as the price of product 1.

(ii) The consumer buys product 1 if her type (s,θ) satisfies $s > \max\{1-\bar{u}-m/4, 0\}$. In that case, the firm perfectly observes the consumer's type and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys.

(iii) The consumer does not buy product 1 if her type (s,θ) satisfies $s < \max\{1-\bar{u}-m/4, 0\}$. In that case, the firm sets a price of $p_2^{M,mon} = m/2$ for product 2, which the consumer buys if $\theta > 1/2$.

Moreover, in equilibrium, the expected consumer surplus is

$$CS^{M,mon}(m,\bar{u}) = \begin{cases} 1/4, & \text{if } \bar{u} \geq 1 - m/4, \\ \frac{1}{4}\left(\bar{u} + m/4\right)^2 + \frac{m}{32}\left(4(1-\bar{u}) - m\right), & \text{if } \bar{u} < 1 - m/4. \end{cases} \tag{G.11}$$

Proof of Proposition 11. We proceed by backwards induction, following a similar approach to the proof of Proposition 2. As in the proof of Proposition 10, denote $v_1(s) = u - (1-s)/2$, so that given a price p_1, the (myopic) consumer strictly prefers to buy product 1 if and only if her type satisfies $p_1 < v_1(s)$. In addition, note that the game is identical to the baseline model after the consumer has decided whether to buy product 1 or not. Therefore, by following the same argument as in the proof of Proposition 2, we have that the equilibrium path after the price of product 1 has been set is as follows:

1. If the consumer's type (s,θ) satisfies $p_1 < v_1(s)$, the consumer buys product 1 and the monopolist observes the value of θ. The monopolist then sets the price of product 2 to be equal to the consumer's valuation, i.e., $p_2 = m\theta$, and the consumer buys the product with probability 1. Therefore, if $p_1 < v_1(s)$, the monopolist's total profit is $p_1 + m\theta$.

2. If $p_1 > v_1(s)$, the consumer does not buy product 1. Since the purchase decision for product 1 is independent of θ, the monopolist's beliefs on the consumer's type are such that the marginal distribution of θ is the uniform distribution in $[0,1]$. Given these beliefs, the monopolist sets $p_2 = m/2$ and receives a total expected profit of $m/4$.

Taking these two cases into account, we can write the monopolist's expected profit function in terms of p_1 as follows:

$$\begin{aligned}\pi^M_{mon}(p_1, m, \bar{u}) &= \mathbb{P}_{\mu_0}\left[p_1 < v_1(s)\right]\left(p_1 + \mathbb{E}\left[m\theta \mid p_1 < v_1(s)\right]\right) + \mathbb{P}_{\mu_0}\left[p_1 > v_1(s)\right] \cdot m/4 \\ &= \mathbb{P}_{\mu_0}\left[p_1 < v_1(s)\right]\left(p_1 + m/2\right) + \mathbb{P}_{\mu_0}\left[p_1 > v_1(s)\right] \cdot m/4 \\ &= \mathbb{P}_{\mu_0}\left[p_1 < v_1(s)\right]\left(p_1 + m/4\right) + m/4 \\ &= \left(1 - \bar{X}^{mon}(p_1)\right)\left(p_1 + m/4\right) + m/4,\end{aligned} \tag{G.12}$$

where $\bar{X}^{mon} = \min\{(1 - 2(\bar{u} - p_1))^+, 1\}$. The monopolist's profit maximization problem is given by $\max_{p_1 \in \mathbb{R}} \pi^M_{mon}(p_1, m, \bar{u})$. Simple calculus shows that there exists a unique optimal solution given by

$$p_1^{M,mon}(m, \bar{u}) = \max\{\bar{u}/2 - m/8, \bar{u} - 1/2\}. \tag{G.13}$$

To complete the characterization of the equilibrium path, note that the consumer is indifferent between buying product 1 or not if and only if her type satisfies $v_1(s) = p_1^{M,mon}(m, \bar{u})$. By plugging in (G.13), we can rewrite this condition as $s = \bar{X}^{M,mon}$, where

$$\bar{X}^{M,mon} = \bar{X}^{mon}\left(p_1^{M,mon}(m, \bar{u})\right) = \max\left\{1 - u - m/4, 0\right\}. \tag{G.14}$$

Finally, to compute consumer surplus, notice that if the the monopolist chooses $p_1 = \bar{u} - (1 - \bar{x})/2$ (so that the location of the indifferent consumer is $s = \bar{x}$), the expected consumer surplus associated with product 1 is $\frac{1}{4}(1 - \bar{x})^2$ as in the restricted setting. Moreover, since the consumer's purchase decisions for products 1 and 2 are independent in the myopic setting, the consumer surplus associated with product 2 is $\bar{x}m/8$, since a fraction $\bar{x}$ of consumers get an expected surplus of $m/8$ (as in the restricted setting), and the rest get zero (as they pay their valuation for product 2). Therefore, the total consumer surplus is $(1 - \bar{x})^2/4 + \bar{x}m/8$. Plugging in $\bar{x} = \bar{X}^{M,mon}$ results in (G.11). □

G.2. Consumer Surplus Comparisons in the Monopoly Model

In this section, we compare the expected consumer surplus in the three settings of the monopoly model, to obtain the proof of Proposition 3. To do so, we proceed as follows. Claim 50 establishes that unlike the baseline model, consumer surplus is higher in the restricted than in the myopic setting. Next, Claim 51 compares the firm's expected profit as a function of the price of product 1 for the myopic and forward-looking settings. This result is an auxiliary step to later show, in Claim 52, that if the paramters of the model satisfy $m \geq 4(1 - \bar{u})$, consumer surplus is higher in the restricted than in the forward-looking setting. These claims then allow us to prove Proposition 3.

Finally, we complement this discussion by comparing consumer surplus in the restricted and forward-looking setting numerically when the parameters satisfy $m < 4(1 - \bar{u})$. We find that neither setting dominates in this region; i.e., there are parameter choices such that forward-looking consumers are on average better off if the monopolist has access to data tracking (see Figure 9).

Claim 50. *In the monopoly model, consumer surplus is larger in the restricted than in the myopic setting for all $m > 0$ and $\bar{u} \geq 1/2$.*

Proof of Claim 50. We consider three cases separately, depdending on the parameters of the model.

Case 1. $\bar{u} \geq 1$. By equations (G.9) and (G.11), we have that $CS^{M,mon}(m,\bar{u}) = 1/4 < 1/4 + m/8 = CS^{R,mon}(m,\bar{u})$.

Case 2. $1 - m/4 \leq \bar{u} < 1$. In this case we have $m \geq 4(1-\bar{u})$, therefore

$$CS^{R,mon}(m,\bar{u}) = \frac{\bar{u}^2}{4} + \frac{m}{8} \geq \frac{\bar{u}^2}{4} + \frac{1-\bar{u}}{2} > \frac{1}{4} = CS^{M,mon}(m,\bar{u}),$$

where the second inequality follows since the expression $\bar{u}^2/4 + (1-\bar{u})/2$ is strictly decreasing for $\bar{u} \in [1/2, 1]$.

Case 3. $\bar{u} < 1 - m/4$. By equations (G.9) and (G.11), we have

$$CS^{M,mon}(m,\bar{u}) = \frac{1}{4}\left(\bar{u} + \frac{m}{4}\right)^2 + \frac{m}{32}(4(1-\bar{u}) - m) = \frac{\bar{u}^2}{4} + \frac{m}{8} - \frac{m^2}{64} = CS^{R,mon}(m,\bar{u}) - \frac{m^2}{64},$$

which implies that $CS^{M,mon}(m,\bar{u}) < CS^{R,mon}(m,\bar{u})$, as desired. □

Our next claim shows that, as one would expect, the monopolist's expected profit in terms of the price of product 1 is larger when facing myopic than forward-looking consumers.[65]

Claim 51. *Fix $m > 0$ and $\bar{u} \geq 1/2$. Then, for any $p_1 \in [\bar{u} - 1/2, \bar{u}]$, we have that $\pi^M_{mon}(p_1, m, \bar{u}) \geq \pi^{FL}_{mon}(p_1, m, \bar{u})$, where π^{FL}_{mon} and π^M_{mon} are given in (G.5) and (G.12), respectively.*

Proof of Claim 51. From equations (G.5) and (G.12), we have that

$$\begin{aligned}\pi^M_{mon}(p_1, m, \bar{u}) - \pi^{FL}_{mon}(p_1, m, \bar{u}) &= \left(\psi\left(\bar{X}^{mon}(p_1), 2m\right) - \bar{X}^{mon}(p_1)\right)p_1 \\ &\quad + m\left(\frac{1}{2} - \frac{1}{4}\bar{X}^{mon}(p_1) - \phi\left(\bar{X}^{mon}(p_1), 2m\right)\right)\end{aligned}$$

Since $\psi(x, 2m) \geq x$ for all $x \in [0,1]$ (see equation (B.19)) and $p_1 \geq \bar{u} - 1/2 \geq 0$, it follows that $\left(\psi\left(\bar{X}^{mon}(p_1), 2m\right) - \bar{X}^{mon}(p_1)\right)p_1 \geq 0$. To see that the second term is also positive let $f(x) = 1/2 - x/4$. Notice that $f(0) = 1/2 = \phi(0, 2m)$ and $f(1) = 1/4 = \phi(1, 2m)$. We know from Claim 10 that $\phi(x, 2m)$ is strictly convex in x; and since f is linear it follows that $f(x) > \phi(x, 2m)$ for all $x \in (0,1)$. In particular, we have that $1/2 - \bar{X}^{mon}(p_1)/4 \geq \phi\left(\bar{X}^{mon}(p_1), 2m\right)$. Thus, $\pi^M_{mon}(p_1, m, \bar{u}) \geq \pi^{FL}_{mon}(p_1, m, \bar{u})$, as desired. □

We now show that if the parameters of the model satisfy $m \geq 4(1-\bar{u})$, then consumer surplus is higher in restricted than in the forward-looking setting. As we show in the proofs, in this region of the parameter space, the equilibrium price for product 1 in the forward-looking setting is $p_1 = \bar{u} - 1/2$, which induces the consumer to buy product 1 from the firm with probability 1. The resulting equilibrium path is identical to the corresponding one in the myopic setting, which, by Claim 50 leads to an inferior consumer surplus compared to the restricted setting.

Claim 52. *Suppose that $m \geq 4(1-\bar{u})$. Then, in equilibrium, consumer surplus is larger in the restricted than in the forward-looking setting.*

Proof of Claim 52. Let $K(\bar{u})$ be as in Proposition 9. We claim that $K(\bar{u}) < 4(1-\bar{u})$. To show this, note that for any $p_1 \in (\bar{u} - 1/2, \bar{u}]$, we have from Proposition 11 and Claim 51 that

$$\pi^M_{mon}(\bar{u} - 1/2, m, \bar{u}) > \pi^M_{mon}(p_1, m, \bar{u}) \geq \pi^{FL}_{mon}(p_1, m, \bar{u}).$$

[65] Note that we require that $\bar{u} - 1/2 \leq p_1 \leq \bar{u}$. However, we know from (G.7) and (G.13) that any equilibrium price for product 1 lies in this region.

However, if the firm sets a price of $p_1 = \bar{u} - 1/2$ for product 1, the resulting expected profit is the same for the myopic and forward-looking settings, i.e., $\pi_{mon}^{FL}(\bar{u}-1/2, m, \bar{u}) = \pi_{mon}^{M}(\bar{u}-1/2, m, \bar{u}) = \bar{u} - 1/2 + m/2$. Plugging into the previous inequality yields that for any $p_1 \in (\bar{u}-1/2, \bar{u}]$,

$$\pi_{mon}^{FL}(\bar{u}-1/2, m, \bar{u}) > \pi_{mon}^{FL}(p_1, m, \bar{u}).$$

Then, by (G.7), we have that $p_1^{FL,mon}(m,\bar{u}) = \bar{u} - 1/2$ is the unique equilibrium price for product 1 in the forward-looking setting; i.e., we have that $m > K(\bar{u})$. By Proposition 9, the resulting equilibrium path is identical to the corresponding one for the myopic setting, which implies, by equation (G.11), that the expected consumer surplus is $1/4$. By the proof of Claim 50, the restricted setting results in higher consumer surplus. □

We now provide the proof of Proposition 3.

Proof of Proposition 3. The comparison between the myopic and restricted setting follows from Claim 50, and the one involving the forward-looking setting follows from Claim 52. □

Finally, we consider the case with with $m < 4(1-\bar{u})$, which, taking into account Assumption 1, is a subset of the rectangle given by $\mathcal{Z} = \{(m,\bar{u}) : 0 \le m \le 2,\ 1/2 \le \bar{u} \le 1\}$. Working with the consumer surplus expression analytically turns out to be complicated.[66] Therefore, we compute the consumer surplus values numerically for a grid of values in this rectangle,[67] and compare them with the corresponding expression for the restricted setting given in (G.9). This comparison is illustrated on Figure 9, which splits the region $\mathcal{Z}$ in two subsets defined by whether the forward-looking or the restricted setting dominates in terms of consumer surplus. We find that consumers can be better off (on average) in the forward-looking than in the restricted setting when both m and $\bar{u}$ are not large relative to the rectangle $[0,2] \times [1/2,1]$. Nonetheless, the subset of the parameter space where this holds is smaller than the corresponding one for the baseline model, which is defined by $m \in (0, m_L) \cup (m_H, 7)$, regardless of the value of $\bar{u}$. Indeed, this region is larger region than the subset of $0 < m < 2$ that is shaded in grey in Figure 9. In particular, this shows that there are parameter values for which forward-looking consumers benefit from data tracking in a competitive environment, but not when firm B acts as a monopoly in both markets.

Appendix H: Duopoly in Both Markets

In this appendix, we extend our model to a setting with competition in both product markets, as described in Section 5.1. That is, instead of assuming that firm B is a monopoly in the market for product 2, we introduce a third firm (firm C) that competes with firm B in the second period. In this context, we characterize and compare the equilibrium outcomes in the settings with and without data tracking under the assumption that

[66] Similar computations to the corresponding ones to obtain (C.3) show that the expression for consumer surplus in the monopoly model is (abbreviating $\bar{x}^* = \bar{X}^{mon}(p_1^{FL,mon}(m,\bar{u}))$ and $\bar{\theta}^* = \bar{\theta}^{mon}(m,\bar{u})$):

$$CS^{FL,mon}(m,\bar{u}) = \begin{cases} \frac{1}{4}\bar{\theta}\,(1-\bar{x}^*)^2 + \frac{m}{2}\left(1-\bar{\theta}^*\right)^2 + \frac{1}{24m}\,(1-\bar{x}^*)^3, & \text{if } \bar{x}^* \ge \tilde{x}(2m), \\ \frac{1}{4}\,(1-\bar{x}^*)^2 + \frac{1}{2}m\bar{x}^*\left(1-\bar{\theta}\right)^2 + \frac{1}{3}m^2\left(1-\bar{\theta}\right)^3, & \text{if } \bar{x}^* < \tilde{x}(2m). \end{cases}$$

[67] We compute the equilibrium and the resulting consumer surplus for parameters $(m,\bar{u})$ with $m \in \{0, 0.001, 0.002, ..., 2\}$ and $\bar{u} \in \{0.5, 0.501, 0.502, ..., 1\}$.

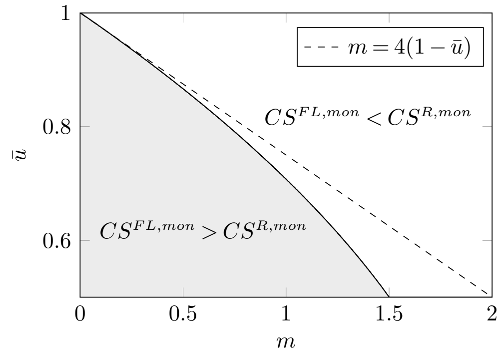

Figure 9 Numerical computations show that, in the monopoly model, consumer surplus is higher in the forward-looking than in the restricted setting only when $(m, \bar{u})$ belongs to the region shaded in grey.

consumers act myopically. As discussed in Section 5.1, these comparisons reveal that the main insights of the baseline model continue to hold when firm B faces competition in the second product market.

This appendix is organized as follows: we describe the extension of the model in Appendix H.1. Then, we characterize the equilibria of the restricted and myopic settings of this model (Propositions 12 and 13), and compare them in terms of consumer surplus (Proposition 14) in Appendix H.2. We omit the proofs of these results as they follow from similar arguments to the corresponding ones of the baseline model.

H.1. Model Extension

We incorporate competition to the second product market by introducing a third firm to the model, which we call firm C, that competes with firm B in the market for product 2. Thus, we modify the game we defined in Section 2 to have firms B and C set prices for product 2 in the second period,[68] with the consumer deciding which firm to buy product 2 from after observing these prices.

Mathematically, we modify the model as follows: the consumer's type now consists of a pair $(s_1, s_2) \in [0,1] \times [0,1]$, where s_i denotes the consumer's location in the Hotelling line for product i. The first component, s_1, takes the role of s in the baseline model while s_2 now represents the consumer's relative preferences for firms B and C in the market for product 2. That is, given firms' product 2 prices p_2^B and p_2^C, we modify the consumer's utility in period 2 to be

$$(1-a_2)\left(m(\bar{v}-s_2)-p_2^C\right)+a_2\left(m(\bar{v}-(1-s_2))-p_2^B\right), \tag{H.1}$$

where $a_2 = 1$ denotes the action of buying from firm B and $a_2 = 0$ denotes purchasing from firm C. As in product market 1, we do not allow the consumer to refrain from purchasing product 2. This can be relaxed by assuming that the baseline utility $\bar{v}$ is large enough so that, in equilibrium, the consumer prefers to buy product 2 than an outside option with utility normalized to zero.[69] In line with our baseline formulation, the

[68] In line with related literature (e.g., Montes et al. 2019) and to simlify the characterization of equilibrium outcomes, we assume that firm B sets personalized prices after setting uniform prices, which both firms do simultaneously. That is, if firm B does not observe the consumer's type, firms B and C set prices simultaneously, while if firm B observes the consumer's type, it sets its (personalized) price after firm C.

[69] For a direct comparison with the baseline model, one can set $\bar{v} = 1$ and allow the consumer to refrain from purchasing product 2 (for a utility of zero). The main results of the restricted and myopic settings of the baseline model extend

value of m captures the relative value of consumer data since, as m increases, so do the potential gains that firm B can capture by gathering consumer data to inform its pricing decision for product 2. The consumer's utility in the first period remains as in (2.1) in Section 2.

Moreover, we assume that when making its pricing decision, firm C's available information consists only of the prior distribution over consumer types and the product 1 prices set by firms in the first period, but that firm C does not observe the consumer's product 1 purchase decision nor her type. That is, we assume that firms' product 1 prices are public information that firm C observes, but that firm C has no data-tracking ability.[70] Firm C's profit function is then

$$\pi_2^C\left(p_2^C, a_2\right) = (1-a_2)p_2^C.$$

The rest of the model remains as described in Section 2 (with minor modifications to the definitions of total payoffs and histories). We conclude this section by noting that, while we consider a model where firm B faces different competitors in each period, this formulation also covers the case where firm A participates in both product markets but has no data-tracking ability.

H.2. Equilibrium Outcomes and Implications of Data Tracking on Consumers

To analyze the implications of data tracking when firm B faces competition in both product markets, we first characterize the equilibrium outcomes of the *restricted* setting, and the setting with data tracking and myopic consumers (i.e., the *myopic* setting). Proposition 12 characterizes the equilibrium outcomes for the restricted setting, which follow a similar pattern to the corresponding ones of the baseline model (see Proposition 1).

Proposition 12. *For any $m > 0$, there exists an equilibrium in the restricted setting of the model with competition in both product markets. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) Firms A and B set a price of $1/2$ for product 1. Then, the consumer buys product 1 from firm A if her type satisfies $s_1 < 1/2$, and she buys from firm B if $s_1 > 1/2$.

(ii) Firms B and C set a price of m for product 2. The consumer buys product 2 from firm C if her type satisfies $s_2 < 1/2$, and she buys from firm B if $s_2 > 1/2$.

The proof of Proposition 12 as it follows a similar argument to that of Proposition 1, but with firms B and C's product 2 prices forming an equilibrium in the pricing game that takes place in the second period, instead of with firm B pricing as a monopolist. The interpretation of the equilibrium outcomes is also similar to that of the baseline model: in the restricted setting, decisions are decoupled across time periods as no information flows from the first to the second period. Thus, the equilibrium outcomes correspond to firms competing according to a standard Hotelling model with fixed locations (d'Aspremont et al. 1979) in each product market. The left panel of Figure 10 illustrates the equilibrium outcomes as a function of the consumer's type in the restricted setting.

Next, we consider the setting where firm B employs data tracking while consumers act myopically. The equilibrium outcomes for this setting are summarized in Proposition 13 below, and are illustrated in the right panel of Figure 10.

to that formulation: the equilibrium outcomes display very similar structures, and consumer surplus is higher with data tracking if the value of m is below a threshold. We opted to present the model with no option to refrain from

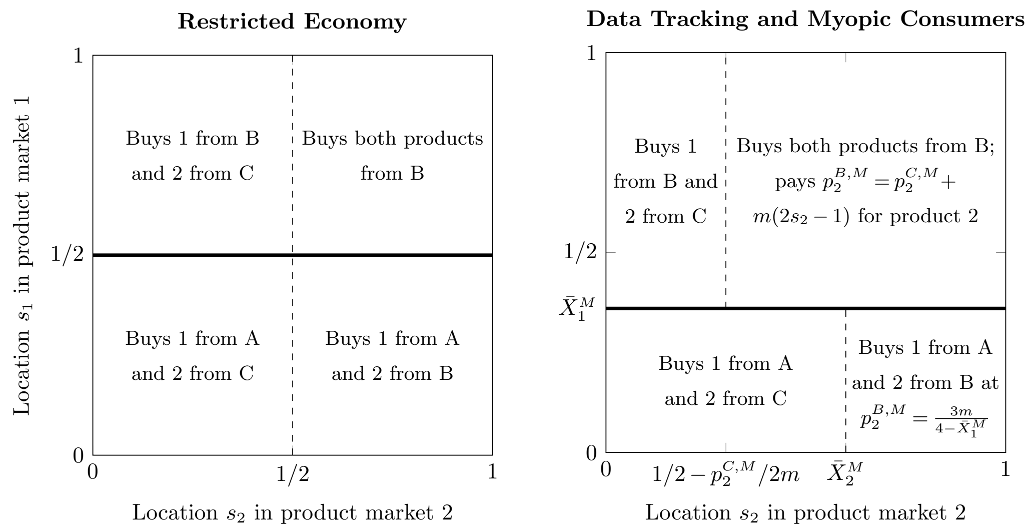

Figure 10 **Equilibrium outcome in a restricted economy (left), and when firm B has data tracking and the consumer behaves myopically (right), as a function of her type (s_1, s_2) in the model with competition in both product markets. $\bar{X}_2^M$ denotes the quantity $\bar{X}_2^M = (m + p_2^{B,M} - p_2^{C,M})/2m$**

PROPOSITION 13. *For any $m > 0$, there exists an equilibrium in the myopic setting of the model with competition in both product markets. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) There exist unique prices $p_1^{A,M} = p_1^{A,M}(m) \in (0, 1/2)$, $p_1^{B,M} = p_1^{B,M}(m)$, with $p_1^{A,M} > p_1^{B,M} > p_1^{A,M} - 1/2$ such that firms' prices for product 1 are equal to $p_1^{A,M}$ and $p_1^{B,M}$, respectively. In addition, the expected product 1 demand for firms A and B is positive. Then, given these prices, the consumer buys product 1 from firm A if her type satisfies $s_1 < \bar{X}_1^M$, and buys from firm B if $s_1 > \bar{X}_1^M$, where

$$\bar{X}_1^M = \frac{1}{2} + p_1^{B,M} - p_1^{A,M}.$$

(ii) If the consumer's type is such that $s_1 < \bar{X}_1^M$, then firms B and C prices for product 2 are, respectively:

$$p_2^{B,M} = \left[\frac{3}{4 - \bar{X}_1^M}\right] m, \quad p_2^{C,M} = \left[\frac{2 + \bar{X}_1^M}{4 - \bar{X}_1^M}\right] m. \tag{H.2}$$

Then, the consumer buys product 2 from firm C if her type satisfies $s_2 < (m + p_2^{B,M} - p_2^{C,M})/2m$, and she buys from firm B if $s_2 > (m + p_2^{B,M} - p_2^{C,M})/2m$.

buying product 2 as it results in more interesting competitive dynamics in the second market, and provides somewhat different findings relative to the baseline model.

[70] In particular, firm B remains as the only firm with data tracking. This allows us to analyze the effect of increased price competition in the product 2 market in isolation, as firm C's pricing may not depend on the consumer's product 1 purchase decision.

(iii) If the consumer's type sastisfies $s_1 > \bar{X}_1^M$, then firm C's price for product 2 is $p_2^{C,M}$ as given in (H.2). Moreover, firm B observes the consumer's type and sets a price that makes her indifferent between purchasing product 2 from firms B and C, as long as this price is positive,[71] that is:

$$p_2^{B,M}(s_2) = \left(p_2^{C,M} + m(2s_2 - 1)\right)^+. \tag{H.3}$$

Finally, the consumer buys product 2 from firm C if her type satisfies $s_2 < 1/2 - p_2^{C,M}/2m$, while she buys from firm B if $s_2 > 1/2 - p_2^{C,M}/2m$.

The characterization described in Proposition 13 also shares a number of features with the corresponding characterization for the baseline model (Proposition 2) and can be established by following a similar backwards induction argument (see Appendix E.2). First note that, as in the baseline model, firms' product 1 equilibrium prices are lower in the presence of data tracking. In addition, firm B prices product 1 at a discount relative to firm A (point (i) of Proposition 13), which captures firm B's incentive to observe the consumer's type by attracting her to buy product 1 from it. However, the participation of firm C in the second product market generates some notable differences.

In particular, the presence of firm C as a competitor in the second period limits the profit that firm B can generate after observing the consumer's type, as it no longer has the power to sell product 2 at the consumer's valuation. Instead, as point (iii) of Proposition 13 and the right panel of Figure 10 describe, there are two possible outcomes when firm B observes the consumer's type following a product 1 purchase: if the consumer's type s_2 is high enough, firm B prices product 2 at the highest value that makes the consumer indifferent between purchasing from firms B and C, which drives the consumer to buy product 2 from firm B in equilibrium. On the other hand, if s_2 is relatively low, the consumer prefers to buy from firm C even though firm B sets a price of zero.

With this mechanism, data tracking not only intensifies competition in the first period, but in the second period as well. If the consumer buys product 1 from firm B, data tracking enables it to capture the consumer's product 2 transaction as long as she is willing to buy from firm B at a non-negative price, given a fixed firm C price (as firm B can price product 2 at the exact level that leaves the consumer indifferent between buying from either firm). Intuitively, firm B can compete for "tracked" consumers at the individual level (i.e., by setting a type-dependent price) instead of only on aggregate.[72] On the other hand, firm C, which is restricted to uniform pricing, internalizes that a fraction of consumers that have been "tracked" in the first period will be lost to firm B unless they are offered a low enough price that drives them to buy from firm C, even if offered to buy from firm B at a price of zero. Therefore, firm C makes its pricing decision by balancing the individual-level competition it faces for the fraction of consumers that have been "tracked" in the first period, with the aggregate-level competition to attract "untracked" consumers.

[71] One can construct equilibria where firm B's price takes any non-negative value when $p_2^{C,M}(m) + m(2s_2 - 1) < 0$. However, refining the equilibrium concept to focus on pricing stratgies that are monotone in the consumer's type s_2 results in the price given in (H.3).

[72] A similar competitition dynamic appears in Thisse and Vives (1988), who study the setting where both firms have information about the consumer's type. In their model, firms' ability to set prices based on the consumer's type also leads to lower (average) prices and higher consumer surplus.

This dynamic also influences the product 2 prices that the consumer is offered after buying product 1 from firm A. In this case, firm C sets a lower price than firm B and captures a larger share of transactions (see point (ii) of Proposition 13 and the right panel of Figure 10). This asymmetry results from the competition mechanism described above: firm C sets a lower price than firm B as it is constrained to uniform pricing and aims to capture a fraction of demand from "tracked" consumers, for which it faces more intense competition by firm B. On the other hand, firm B best-responds to firm C's low product 2 price by lowering its product 2 price as well (in comparison to the restricted setting), but to a lesser extent than firm C since, as firm B recognizes that the consumer did not buy product 1 from it (due to data tracking), it realizes that there is no individual-level competition with firm C at this stage.

Moreover, an important difference relative to the baseline model is that the prices that consumers pay for product 2 can be lower with data tracking than in the restricted setting. Indeed, if the consumer does not buy product 1 from firm B, she faces product 2 prices that are unambiguously lower than in the restricted setting (as the prices given in (H.2) are lower than m), and even if the consumer purchases product 1 from firm B, it may buy product 2 from firm C at a lower price than it would without data tracking.

Thus, perhaps surprisingly, the presence of data tracking in the economy not only increases the intensity of competition in the market where consumer information is collected (product 1), but it also in the market where this information is exploited (product 2). A key implication of this fact is that, with competition in both product markets and consumers that act myopically, the presence of data tracking increases consumer surplus *for all values of* $m > 0$.

PROPOSITION 14. *Consider the model with competition in both product markets and myopic consumers. Then, for any* $m > 0$*, aggregate consumer surplus is higher in the presence of data tracking.*

The result described in Proposition 14 contrasts with the corresponding comparison for the baseline model (see Theorem 2 in Section 4 and Proposition 7 in Appendix D), in which data tracking increases consumer surplus only if the value of m is below a given threshold since, without competition in the market for product 2, the implications of data tracking on consumer surplus depend on the tradeoff between lower product 1 prices driven by firm B's incentive to gather consumer data vs. the utility loss associated with personalized pricing for product 2. By contrast, with firm C in the model, the presence of data tracking generates higher consumer surplus in both product markets, which results in the unambiguous comparison given in Proposition 14.

Appendix I: Imperfect Data Tracking

In this appendix, we extend our model by relaxing the assumptions around firm B's data-tracking ability. In particular, so far we have assumed that whenever the consumer buys product 1 from firm B, the firm learns the consumer's valuation for product 2. We relax this assumption by assuming instead that the type is revealed only with some probability, which we denote by β. More formally, we assume that given a history $h \in H_2^f$, the information available to firm B at the beginning of period 2 is given by the expression below.[73]

[73] Equivalently, one may introduce a move by Nature to the timeline after the first period, with Nature drawing a Bernoulli random variable with success probability $\beta a_1(h)$ to determine whether the consumer's type is revealed to firm B or not.

$$\mathcal{J}^{\beta}(h)=\begin{cases}h & \text{if } a_1(h)=1, \text{ with prob. } \beta,\\ (p_1^A,p_1^B) & \text{if } a_1(h)=1, \text{ with prob. } 1-\beta,\\ (p_1^A,p_1^B) & \text{if } a_1(h)=0, \text{ with prob. } 1.\end{cases}$$

One may compare this expression with $\mathcal{J}(h)$ in (A.1) to see that we essentially recover the baseline model when $\beta=1$. Also, $\beta=0$ recovers the restricted setting, in which firm B has no data-tracking ability. We interpret the probability β as firm B's *tracking accuracy.*[74]

In what follows, we characterize the equilibrium of this model in our three settings (Appendices I.1 and I.2), and compare the resulting levels of consumer surplus (Appendix I.3). We observe that, in line with the comparison established in Theorem 2, consumer surplus may be lower in the scenario where firm B has no data-tracking ability, as long as the value of consumer data is not exceedingly high.

I.1. Equilibrium with Forward-looking Consumers

To characterize the equilibrium with forward-looking consumers, we use backwards induction, similar to Appendix B.1. First, we characterize the equilibrium strategies after product 1 prices are set in Lemma 4. Then, we establish that any equilibrium in the model is associated with a Nash equilibrium in a simultaneous-move pricing game in Lemma 5, and finally discuss the properties of such equilibria. In what follows, we consider the case with $\beta\in(0,1)$ as the cases with $\beta\in\{0,1\}$ correspond to the settings with and without data-tracking that we have covered before. In particular, we assume that the value of β is common knowledge, but the consumer does not know in advance whether she will be tracked or not (i.e., does not know the realization of $\mathcal{J}^{\beta}(h)$ when making her product 1 purchase decision).[75]

We first introduce some auxiliary notation in similar fashion to Appendix B.1. To characterize firm B's pricing strategy for product 2, we modify the definition of $\bar\theta$ to depend on the tracking accuracy parameter β. We define $\bar\theta:\mathbb{R}\times\mathbb{R}^{++}\times(0,1)\to\mathbb{R}$ as

$$\bar\theta(x,m,\beta)=\begin{cases}1/2, & \text{if } x\geq 1 \text{ or } x\leq -m\beta/2,\\ \frac{1}{m\beta^2}\left(2(1-\beta)+m\beta^2+2\beta x-\sqrt{2(1-\beta(1-x))(m\beta^2+2(1-\beta(1-x)))}\right), & \text{if } 0\leq x\leq \tilde{x}^{+}(m,\beta),\\ \frac{1}{2-\beta(1-x)}\left[1-\frac{1}{2m}(1-x)^2\right], & \text{if } \tilde{x}^{+}(m,\beta)<x<1,\\ \frac{1}{m\beta^2}\left(2(1-\beta)+m\beta^2+\beta x-\sqrt{2(1-\beta)\left(m\beta^2+2(1-\beta(1-x))\right)}\right), & \text{if } -m\beta/2<x\leq\tilde{x}^{-}(m,\beta),\\ \frac{1}{2m(2-\beta)}\left[2m+2x-1\right], & \text{if } \tilde{x}^{-}(m,\beta)<x<0,\end{cases} \tag{I.1}$$

where the cutoff values for the various cases presented above, $\tilde{x}^{+},\tilde{x}^{-}:\mathbb{R}^{+}\times(0,1)\to\mathbb{R}$, are given by

$$\tilde{x}^{+}(m,\beta)=\max\left\{0,1-\frac{1}{3\beta}\left(2+m\beta^2+\sqrt{\left(1-\beta^2 m\right)^2+3}\right)\right\}\quad\text{and}\quad\tilde{x}^{-}(m,\beta)=\min\left\{0,\frac{4-3\beta}{4-4\beta}-\frac{\beta m}{2}\right\}.$$

In similar fashion to (B.4), given any product 1 prices p_1^A and p_1^B, and $t\in[0,1]$, we define

$$g_\beta(t\mid p_1^A,p_1^B)=p_1^B-p_1^A+1/2+m\beta\left(t-\bar\theta\left(p_1^B-p_1^A+1/2,m,\beta\right)\right)^{+}. \tag{I.2}$$

[74] An alternative modeling choice is to have firm B observe a noisy signal of the form $\hat\theta=\theta+\epsilon$ after selling product 1, where the distribution of ϵ captures firm B's tracking accuracy. With that formulation, we have obtained similar insights to the ones presented in this appendix with myopic consumers and assuming that ϵ follows a uniform distribution centered at zero. However, this alternative model is quite challenging to work with for the setting with forward-looking consumers, while the model presented here remains tractable.

[75] This assumption captures settings where the consumer is aware of the accuracy of firm B's tracking capabilities at the population level, but does not know whether the firm will be able to infer her specific preferences.

The following result characterizes the equilibrium strategies after the firms set product 1 prices.

LEMMA 4. *Fix an assessment* $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \boldsymbol{\mu})$ *and let* $p_1^A = \sigma_1^A(\emptyset)$ *and* $p_1^B = \sigma_1^B(\emptyset)$*. Then, in any equilibrium for the model with imperfect tracking, we have that*

(a) If $p_1^B - p_1^A \le -(1+m\beta)/2$*, the consumer buys product 1 from firm B with probability one. Then, with probability* β*, the firm observes the consumer's type and, in the second period, sets a price equal to the consumer's valuation for product 2, which the consumer buys. With probability* $1-\beta$*, the type is not observed and firm B sets a price of* $m/2$ *for product 2, which the consumer buys if* $\theta > 1/2$*.*

(b) If $p_1^B - p_1^A > -(1+m\beta)/2$*, we have that*

(i) If the consumer's type (s,θ) *is such that* $s > g_\beta(\theta \mid p_1^A, p_1^B)$*, then, the consumer buys product 1 from firm B with probability one. Then, with probability* β*, the firm observes the consumer's type and, in the second period, sets a price equal to* $m\theta$ *for product 2, which the consumer buys. With probability* $1-\beta$*, the type is not observed and the firm sets the price for product 2 equal to* $p_2^B = m\bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$*, which the consumer buys if* $\theta > \bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$*.*

(ii) If the consumer's type (s,θ) *is such that* $s < g_\beta(\theta \mid p_1^A, p_1^B)$*, then the consumer buys product 1 from firm A with probability one; firm B sets the price for product 2 equal to* $p_2^B = m\bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$*, which the consumer buys if* $\theta > \bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$*,*

where $\bar{\theta}$ *and* g_β *are defined as in (I.1) and (I.2), respectively.*

Proof of Lemma 4. The proof is similar to the one for Lemma 1 (see Appendix B.1). Analogous results to Claims 1–4 hold with minor changes to notation. The steps of Claims 5 and 6 are modified as follows.

To derive the equilibrium beliefs when firm B does not observe the consumer's type, we define, in similar fashion to (B.13),

$$\mathcal{A}_0^\beta(p_1^A, p_1^B) = \left\{(s,\theta) \in [0,1]\times[0,1] : s < g_\beta(\theta \mid p_1^A, p_1^B)\right\},$$

which corresponds to the subset of types for which the consumer strictly prefers to buy from firm A in period 1 given prices p_1^A, p_1^B. By Bayesian updating, it can be easily established that if $\boldsymbol{\mu}$ is consistent with $(\boldsymbol{\gamma}, \boldsymbol{\sigma})$, then the marginal CDF of θ induced by μ_2 at an information set of the form $I = \mathcal{J}^\beta(h) = (p_1^A, p_1^B)$, which we denote by $\mu_2^\theta(\cdot|I)$, is given by

$$\mu_2^\theta(t|I) = \int_0^t \frac{\bar{g}_\beta(z \mid p_1^A, p_1^B) + (1-\beta)(1-\bar{g}_\beta(z \mid p_1^A, p_1^B))}{\mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right) + (1-\beta)\left(1-\mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right)\right)} dz, \tag{I.3}$$

where $\bar{g}_\beta$ is defined similarly as in (B.14): $\bar{g}_\beta(z \mid p_1^A, p_1^B) = \max\{0, \min\{g_\beta(z \mid p_1^A, p_1^B), 1\}\}$.

Observe from (I.3) that instead of having a density proportional to $\bar{g}_\beta$ as in the baseline model, the density associated with μ_2^θ is now proportional to $\bar{g}_\beta + (1-\beta)(1-\bar{g}_\beta)$, where the second term factors in the probability of firm B not observing the consumer's type after selling product 1. Using these beliefs, we can characterize firm B's equilibrium product 2 pricing strategy when it does not observes the consumer's type.

CLAIM 53. *Fix prices* p_1^A, p_1^B *and suppose that strategies* $\boldsymbol{\gamma}$ *and* $\boldsymbol{\sigma}^{\boldsymbol{B}}$ *satisfy the conditions in Claim 3. Let* $I = (p_1^A, p_1^B)$*, and suppose that firm B observes* I *at the beginning of period 2. Then, in any equilibrium, we have that* $\sigma_2^B(I) = m\bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$ *with probability 1, where* $\bar{\theta}$ *is defined as in (I.1).*

This claim can be established following similar steps as in the proof of Claim 6. First, showing that $\sigma_2^B(I) = mt^*$ for some $t^* \in [0,1]$ follows from the same argument used in the proof of Claim 6. Then, to show that $t^* = \bar{\theta}_\beta(p_1^B - p_1^A + 1/2, m)$, we use the form of μ_2^θ given in (I.3) and the same argument used to derive condition (B.17) in the proof of Claim 6 to obtain that

$$\mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right) + (1-\beta)\left(1 - \mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right)\right) = 2t^*\left[\beta \bar{X}(p_1^A, p_1^B) + 1 - \beta\right], \tag{I.4}$$

where, as in (B.1), we denote $\bar{X}(p_1^A, p_1^B) = \max\{0, \min\{p_1^B - p_1^A + 1/2,\, 1\}\}$.

Note that if $\beta = 1$, we recover equation (B.17) as in the baseline model. Moreover, by computing $\mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right) = \mu_0\left(\{s < g_\beta(\theta \mid p_1^A, p_1^B)\}\right)$ we have that (denoting $X = p_1^B - p_1^A + 1/2$)

$$\mu_0\left(\mathcal{A}_0^\beta(p_1^A, p_1^B)\right) = \begin{cases} 1, & \text{if } X \geq 1, \\ X + \frac{1}{2}m\beta\left(1-t^*\right)^2, & \text{if } 0 \leq X < 1 \text{ and } X + m\beta(1-t^*) \leq 1, \\ X + (1-X)(1-t^*) - \frac{1}{2m\beta}\left(1-X\right)^2, & \text{if } 0 \leq X < 1 \text{ and } X + m\beta(1-t^*) > 1, \\ \frac{1}{2m\beta}\left(X + m\beta(1-t^*)\right)^2, & \text{if } X < 0 \text{ and } 0 < X + m\beta(1-t^*) \leq 1, \\ 1 - t^* + \frac{X}{m\beta} - \frac{1}{2m\beta}, & \text{if } X < 0 \text{ and } X + m\beta(1-t^*) > 1, \\ 0, & \text{if } X < 0 \text{ and } X + m\beta(1-t^*) \leq 0. \end{cases} \tag{I.5}$$

By plugging (I.5) into (I.4) and some algebra, we obtain that $t^* = \bar{\theta}(p_1^B - p_1^A + 1/2, m, \beta)$, as desired. Finally, the description of the equilibrium path follows by an analogous argument to that of Lemma 1. □

The next step consists of deriving the firms' profit functions in terms of product 1 prices, considering the equilibrium path that we have described in Lemma 4, and defining a simultaneous-move pricing game with these functions in similar fashion to Definition 1.

Definition 2. For $m > 0$ and $0 < \beta < 1$, let $\mathbf{G}(m, \beta)$ be the two-player normal-form game with action spaces $S^A = [0,1]$ and $S^B = [-(1+m\beta)/2, 1]$, and profit functions π^A and π^B that we define as

$$\pi^A\left(p_1^A, p_1^B, m, \beta\right) = p_1^A \psi\left(\min\{p_1^B - p_1^A + 1/2, 1\}, m, \beta\right),$$
$$\pi^B\left(p_1^A, p_1^B, m, \beta\right) = \left(1 - \psi(\min\{p_1^B - p_1^A + 1/2, 1\}, m, \beta)\right) p_1^B + m\phi\left(\min\{p_1^B - p_1^A + 1/2, 1\}, m, \beta\right),$$

where, slightly abusing notation, we define $\psi, \xi, \phi : [0,1] \times \mathbb{R}^{++} \times (0,1) \to \mathbb{R}$ by

$$\psi(x, m, \beta) = 2x\bar{\theta}(x, m, \beta) - \frac{1-\beta}{\beta}\left(1 - 2\bar{\theta}(x, m, \beta)\right),$$
$$\xi(x, m, \beta) = \mathbb{E}_{\mu_0}\left[\theta; s \geq x + m\beta\left(\theta - \bar{\theta}(x, m, \beta)\right)^+\right],$$
$$\phi(x, m, \beta) = \beta\xi(x, m, \beta) + \frac{1}{2}\bar{\theta}(x, m, \beta)\left[1 - \beta(1 - \psi(x, m, \beta))\right].$$

With these definitions, we can characterize the equilibrium of the model by following a similar procedure to that used for the baseline model in Lemma 2. Formally, we have that

Lemma 5. *If $\mathbf{G}(m, \beta)$ admits a pure strategy Nash equilibrium (p_1^{A*}, p_1^{B*}), then there exists an equilibrium in the imperfect tracking model with forward-looking consumers such that $(\sigma_1^A(\emptyset), \sigma_1^B(\emptyset)) = (p_1^{A*}, p_1^{B*})$.*

Conversely, if $(\boldsymbol{\gamma}, \sigma_1^A, \boldsymbol{\sigma}^B, \mu)$ is an equilibrium in the imperfect tracking model with forward-looking consumers, then $(p_1^{A*}, p_1^{B*}) = (\sigma_1^A(\emptyset), \sigma_1^B(\emptyset))$ is a pure-strategy Nash equilibrium in $\mathbf{G}(m, \beta)$.

Proof of Lemma 5. Follows from analogous arguments to Claims 8 and 9. □

As in the baseline model, we can derive the equilibria of the model by computing the equilibria of the simultaneous-move pricing game $\mathbf{G}(m, \beta)$. Thus, it only remains to establish a result that addresses existence of pure-strategy Nash equilibria for $\mathbf{G}(m, \beta)$ to fully determine when the model with imperfect tracking admits an equilibrium. Unfortunately, dealing with the profit functions defined above is significantly more challenging than in the baseline model, which has prevented us from establishing a result analogous to Lemma 3. The main challenge that complicates this task is that firm A's profit function does not preserve some of the appealing properties that it had in the baseline model, while the challenges associated with firm B's profit function remain (see Claims 12 and 16 and the related discussion in Appendix B). In particular, firm A's profit function is not single-peaked in general, which implies that its best-response correspondence is non-convex in some instances (the left panel of Figure 11 displays some instances of firm A's profit function). However, as we explain below, we can still numerically compute the equilibria of the game and confirm that similar trends to the ones established in the baseline model continue to hold in this model as well.

What determines the shape of firm A's profit function is that, in contrast to the baseline model, it is no longer sufficient for firm B to discount firm A's product 1 price by 1/2 to capture the entire product 1 demand, which creates two demand regimes. When the price difference between product 1 prices satisfies $-1/2 < p_1^B - p_1^A$, we observe a similar structure as in the baseline model (i.e., as in Figure 2), where if the consumer's location type for product 1 is close enough to firm A, she will buy from it. However, when $-1/2 > p_1^B - p_1^A > -1/2 - m\beta/2$, the expected demand for firm A is not zero as the consumer anticipates a lower price for product 2 than she would in the baseline model. To provide some intuition, note that if β is relatively low, firm B's beliefs on θ in period 2 are close to a uniform distribution (see (I.3)) and, therefore, firm B's price for product 2 (when it does not observe the consumer's type) is close to $m/2$, which results in the consumer preferring to buy product 1 from firm A only if her type θ is relatively large. However, firm B still captures the entire product 1 demand if the consumer's type is $\theta < 1/2$. Thus, for low values of β, we observe that the consumer prefers to buy from firm A even if the price discount would have sufficed to capture the entire product 1 demand in the baseline model.

While establishing an existence result like Lemma 3 is challenging, it is possible to establish some useful properties of the PSNE of $\mathbf{G}(m, \beta)$:

(i) There is no corner equilibrium when $\beta \in (0, 1)$, i.e., both firms have positive expected product 1 demand in any PSNE. It can be shown that it is never profitable for firm B to offer a large enough discount to induce all consumers to buy from it.

(ii) If $\mathbf{G}(m, \beta)$ admits a PSNE, it is unique. This can be established following a similar procedure as in Claim 18.

(iii) When it exists, the PSNE of $\mathbf{G}(m, \beta)$ can be computed by solving a system of equations, similar to the one we provide for the baseline model in Appendix F.2.

Using this machinery, we compute the equilibria of $\mathbf{G}(m, \beta)$ for a grid of values of parameters and observe that these equilibria display very similar features to those of the interior equilibria in the baseline model.[76]

[76] We numerically approximate the equilibrium (when it exists) for all combinations of $m \in \{0, 0.05, \ldots, 13\}$ and $\beta \in \{0, 0.01, \ldots, 0.99\}$.

In all our computations, given a fixed value of $\beta \in (0,1)$, we were able to approximate the equilibrium of $\mathbf{G}(m,\beta)$ for all values of m that are lower than some threshold (which depends on β). In addition, we observe that product 1 equilibrium prices satisfy $p_1^A - 1/2 < p_1^B < p_1^A$ (see the right panel of Figure 11). These observations closely resemble the properties we established in Theorem 1 for the interior equilibrium regime of the baseline model, suggesting that a similar relationship holds with imperfect tracking as well.

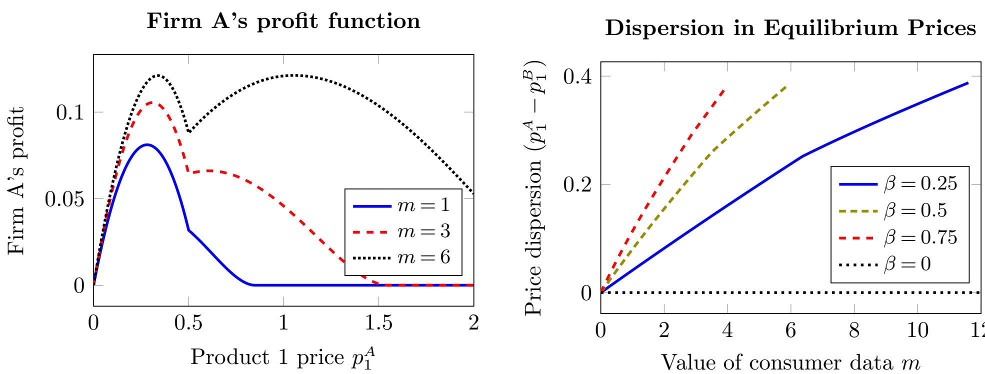

Figure 11 Firm A's profit function for select values of m, keeping $\beta = 0.7$ and $p_1^B = 0$ fixed (left), and dispersion in the equilibrium prices for product 1 (when the equilibrium exists) as a function of the value of consumer data m for some values of β (right)

I.2. Equilibrium in the Restricted and Myopic Settings

To make similar comparisons to those established in the baseline model, we now consider the restricted and myopic settings in the model with imperfect tracking. Recall that in the restricted setting, firm B has no data-tracking ability and therefore no information flows from the first to the second period. This scenario is effectively equivalent to one where firm B's tracking accuracy is zero and, therefore, the characterization of equilibrium provided in Proposition 1 continues to hold for the restricted setting in this context.

We now consider the setting with data tracking and myopic consumers for which the equilibrium can be characterized as follows.

PROPOSITION 15. *For any $m > 0$ and $\beta \in (0,1)$, there exists an equilibrium in the myopic setting for the model with imperfect tracking. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) The firms' prices for product 1 are, respectively,

$$p_1^{A,M} = \max\left\{\frac{1}{2} - \frac{m\beta}{12}, 0\right\}, \quad p_1^{B,M} = \max\left\{\frac{1}{2} - \frac{m\beta}{6}, -\frac{1}{2}\right\}.$$

(ii) The consumer buys product 1 from firm B if her type (s,θ) satisfies $s > p_1^{B,M} - p_1^{A,M} + 1/2$. In that case, with probability β, firm B observes the consumer's type and, in the second period, sets a price equal to $m\theta$ for product 2, which the consumer also buys. With probability $1-\beta$, firm B does not observe the consumer's type and sets a price of $p_2^{B,M} = m/2$ for product 2, which the consumer buys if $\theta > 1/2$.

(iii) The consumer buys product 1 from firm A if her type (s, θ) *satisfies* $s < p_1^{B,M} - p_1^{A,M} + 1/2$*. In that case, firm B sets a price of* $p_2^{B,M} = m/2$ *for product 2, which the consumer buys if* $\theta > 1/2$*.*

Proof of Proposition 15. The result is established by following a similar reasoning as in the proof of Proposition 2 (see Appendix E.2, with the only change being that the tracking accuracy parameter β appears in firm B's profit function in terms of product 1 prices. It is straightforward to show following the same steps as in the proof of Proposition 2 that the firms' profit functions in terms of product 1 prices considering subsequent equilibrium play are given by (see equation (E.4))

$$\pi_M^A\left(p_1^A, p_1^B\right) = p_1^A \bar{X}\left(p_1^A, p_1^B\right), \quad \pi_M^B\left(p_1^A, p_1^B\right) = \left(p_1^B + m\beta/4\right)\left(1 - \bar{X}\left(p_1^A, p_1^B\right)\right) + m/4,$$

where, as before, $\bar{X}(p_1^A, p_1^B) = \max\{0, \min\{p_1^B - p_1^A + 1/2,\, 1\}\}$. By computing the pure-strategy Nash equilibrium of the game with these profit functions we obtain the prices given above. Indeed, note that we recover the outcome of the baseline model when $\beta = 1$ (Proposition 2). □

I.3. Implications of Data Tracking on Consumer Surplus

We now compare the equilibrium expected consumer surplus for our three settings, and find that similar insights to those established in the baseline model (i.e., Theorem 2) continue to hold in the model with imperfect tracking. We first provide the comparison between the myopic and restricted settings in Proposition 16. Then, we consider the corresponding comparisons with forward-looking consumers numerically and provide evidence indicating that, in line with Theorem 2, consumer surplus is higher with data tracking as long as the value of m is not exceedingly high, confirming that the intuition provided by the baseline model holds in this more general setting.

Proposition 16. *In the model with imperfect tracking and myopic consumers, aggregate consumer surplus is higher with data tracking if and only if* $m\beta < 7$*.*

Proof of Proposition 16. By performing similar computations as in the proof of Claim 20 (see Appendix C), we obtain that the expected consumer surplus in equilibrium for the restricted and myopic settings are, respectively,

$$CS^R(m,\beta) = \bar{u} - \frac{5}{8} + \frac{m}{8}, \quad \text{and} \quad CS^M(m,\beta) = \begin{cases} \bar{u} - \frac{5}{8} + \frac{m}{144}\left(9(2+\beta) - m\beta^2\right), & \text{if } m\beta \le 6, \\ \bar{u} + \frac{1}{4} + \frac{m(1-\beta)}{8}, & \text{otherwise.} \end{cases} \tag{I.6}$$

By simple algebra, one can show that $CS^M(m,\beta) > CS^R(m,\beta)$ if and only if $m\beta < 7$. □

Proposition 16 shows that one of our main findings continues to hold even if firm B's data-tracking ability is imperfect, i.e., that in the presence of competition, data tracking may benefit even myopic consumers, as long as the value of m is lower than some threshold (given by $7/\beta$). In addition, we find that myopic consumers are always better off with some level of data tracking (i.e., allowing some degree of predictive power to firm B), than in the scenario with no data tracking.

Proposition 17. *For any* $m > 0$*, there exists some* $\beta > 0$ *such that with myopic consumers, aggregate consumer surplus is higher with data tracking.*

Proof of Proposition 17. Taking the expression for consumer surplus given in (I.6), we have that $\frac{\partial}{\partial \beta} CS^M(m,0) = \frac{9m}{144} > 0$, which implies that there exists $\beta > 0$ such that $CS^M(m,0) < CS^M(m,\beta)$. □

Finally, we turn our attention to the equilibrium consumer surplus in the setting with data tracking and forward-looking consumers and how it compares with the rest of the settings we consider. Deriving analytical comparisons involving the forward-looking setting turns out to be significantly more challenging than in the baseline model,[77] so we numerically compute the equilibrium of the model for a grid of parameter values and compare the resulting levels of consumer surplus for the three settings. These comparisons are illustrated in Figure 12, where we plot the equilibrium consumer surplus of the myopic and forward-looking settings relative to the restricted setting for two different values of β.

In line with the comparison established in Theorem 2, we observe that when consumers are forward-looking, consumer surplus is higher with data tracking as long as the value of m is lower than some threshold that depends on β (provided that there exists an equilibrium, which we observe numerically for all small enough values of m as discussed in Appendix I.1). In addition, Figure 12 reveals another interesting feature of the model with imperfect tracking: consumer surplus may be higher with myopic than with forward-looking consumers. We find this observation to be counterintuitive as one may expect that since forward-looking consumers are aware of the value of their information for firm B, they demand lower prices to reveal their type, which ends up making them better off. In fact, we showed that this is the case in the baseline model, i.e., when $\beta = 1$ (see Claim 23 in Appendix C); however, our numerical results indicate that this need not be the case when firm B's tracking ability is imperfect.

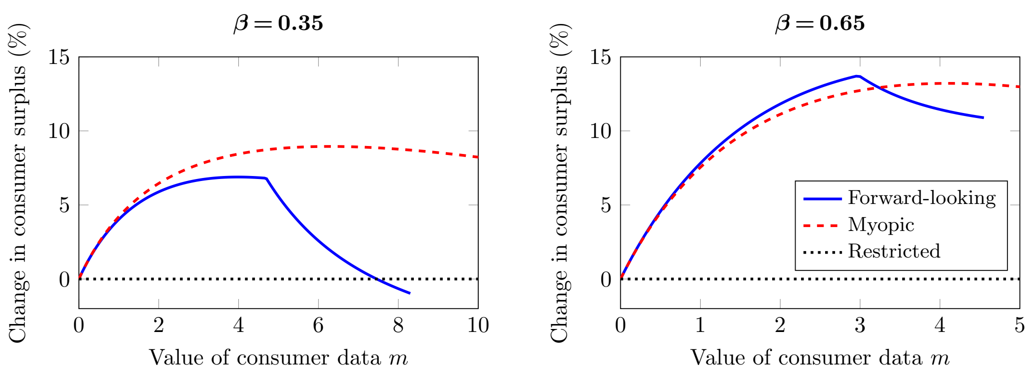

Figure 12 Percentage change in the equilibrium aggregate consumer surplus relative to the restricted setting for $\beta = 0.35$ (left) and $\beta = 0.65$ (right), as a function of the value of consumer data m, for $\bar{u} = 1$. The series corresponding to the forward-looking setting appears only for the values of m for which our numerical exercise finds an equilibrium, which corresponds to all values of m lower than some threshold

This observation is driven by a similar tradeoff as the one described in Proposition 6. On the one hand, the increase in demand to buy product 1 from firm A when consumers are forward-looking (due to the value

[77] The complication arises due to the fact that we now have two parameters (m and β) instead of only one. Indeed, when attempting to perform a similar argument as in the proof of Claim 23, we end up with a system of two polynomial equations and three unknowns, which complicates the argument. However, as we explain in Appendix I.1, it is possible to compute the equilibrium of the model numerically and obtain reliable comparisons, since the equilibrium can be shown to be unique when it exists.

that consumers associate with the option of not revealing their type to firm B by buying from firm A) pushes firm A's price for product 1 upwards, and may result in a higher price when consumers are forward-looking. On the other hand, firm B must offer a larger discount to forward-looking than to myopic consumers in order to induce product 1 transactions (as the former internalize the implications of data tracking), which results in firm B setting a lower price for product 1. When comparing the corresponding levels of consumer surplus with forward-looking and myopic consumers, we observe that the higher price set by firm A in the former setting is not always compensated by the lower price offered by firm B, resulting in the effect illustrated in Figure 12.

Appendix J: Correlation between Consumer Types

In this appendix, we consider an alternative distribution of consumer types to allow for correlation between the consumer's location in the Hotelling line for product 1 and her valuation for product 2 (i.e., we introduce correlation between s and θ). Specifically, we assume that the consumer's type $\tau = (s, \theta)$ is generated as follows: first, s is drawn from the uniform distribution in $[0,1]$ and then, conditional on the value of s, θ is drawn from the uniform distribution in $[rs, 1]$, where $r \in [0,1)$ is a parameter that modulates the correlation between s and θ. Equivalently, we assume that the type (s, θ) is drawn from the unit square according to the probability measure characterized by the following density function:

$$f(s, \theta; r) = \frac{1}{1 - rs} \mathbf{1}_{\{rs \leq \theta \leq 1, 0 \leq s \leq 1\}}(s, \theta).$$

This distribution captures a setting where the consumer's relative preference for firm B is somewhat persistent across both products. That is, if the consumer is relatively more likely to buy product 1 from firm B (i.e., her location s is relatively close to 1), then her valuation for product 2 (which is $m\theta$) will tend to be higher. Note that the strength of this relationship is determined by the value of the parameter r and, as a result, the correlation between s and θ is strictly increasing in r. Considering this distribution of consumer types, we analyze the outcomes of our model (i.e, all the components of the model except the prior distribution of consumer types remain as in Section 2). Indeed, the baseline formulation with a uniform prior distribution corresponds to setting $r = 0$. However, it is important to note that this model generates different distributions for θ depending on the value of r and that, in particular, the average valuation for product 2 is increasing in r.

In what follows, we describe how the equilibria of this model can be characterized for our three settings and we numerically compare the equilibrium outcomes across them, in order to assess the extent to which the results from our baseline model are robust to the introduction of correlation. We observe that the equilibria of the game have very similar structures to the corresponding equilibrium in the baseline model and that, in line with Theorem 2, consumer surplus is higher with data tracking for both myopic and forward-looking consumers, as long as the value of m is not relatively large.

The equilibrium of this extended version of the model can be characterized for the three settings that we study by following a similar procedure to that used for the baseline model. For the restricted and myopic settings, analogous results to Propositions 1 and 2 can be established. For the setting with forward-looking consumers, we follow an analogous reasoning to the one described in Appendix B, where we first characterize the equilibrium strategies in the continuation game that follows the firms' choices for product 1 prices by backwards induction (as in Appendix B.1), and then derive the firms' profit functions in terms of product 1 prices, considering the subsequent equilibrium path induced by the previously characterized strategies (as in Appendix B.2). The equilibria of the model are then fully characterized by the Nash equilibria of the simultaneous-move game associated with these profit functions. For brevity, we omit these derivations for the model with correlation as they follow very similar arguments to those for the baseline model. The main additional complication that arises is that establishing an existence and uniqueness result analogous to Lemma 3 turns out to be quite challenging, as the functions that define firm B's equilibrium strategies and profits (i.e, $\bar{\theta}$, ψ, ξ) have no closed-form expressions due to the fact that the prior distribution of consumer types is no longer uniform. Nonetheless, we can compute the equilibria of the game numerically to assess the extent to which the results from our baseline model are robust to the introduction of correlation.

To compare the equilibrium outcomes across the three settings, we numerically compute their corresponding equilibria for a grid of values in the parameter space.[78] In line with the baseline model, we observe the following structures in our computations:

(i) As in Theorem 1, we observe that given a fixed value of r, there are two equilibrium regimes for the forward-looking setting:

  (a) When m is lower than some threshold $m_L(r)$, we have an interior equilibrium, with product 1 prices satisfying $p_1^A - 1/2 < p_1^B < p_1^A < 1/2$.

  (b) When m is higher than some threshold $m_H(r)$, we have a corner equilibrium, with product 1 prices equal to $p_1^A = 0$, $p_1^B = -1/2$, and firm B capturing the entirety of product market 1.

  (c) We find that for all values of r, $m_L(r) \approx 3.98$ and $m_H(r) \approx 4.02$, as in the baseline model. The numerical method to compute equilibria fails to converge for some values of $m \in (3.98, 4.02)$, suggesting that the game might admit no equilibria in this region, as in the baseline model.

(ii) In line with Theorem 2, we observe that the expected consumer surplus is larger with data tracking for both forward-looking and myopic consumers as long as the value of m is not exceedingly high, suggesting that the intuition for Theorem 2 continues to hold even in the presence of correlation. We illustrate this comparison in Figure 13, where we compare the equilibrium levels of consumer surplus as a function of m for two fixed values of r.

(iii) As in Proposition 6, firm A's profits are highest when firm B has no data-tracking ability. With data tracking, the comparison of firm A's profits in the forward-looking and myopic settings depends on the values of the parameters.

Finally, we observe one departure from the results of the baseline model: in one region of the parameter space, firm B's equilibrium profits may be higher in the restricted setting than in the setting with data tracking and forward-looking consumers. That is, Proposition 5 does not always hold in this setting. We

[78] Specifically, we numerically approximate the equilibrium (when it exists) for all combinations of $m \in \{0, 0.05, \ldots, 10\}$ and $r \in \{0.01, \ldots, 0.99\}$.

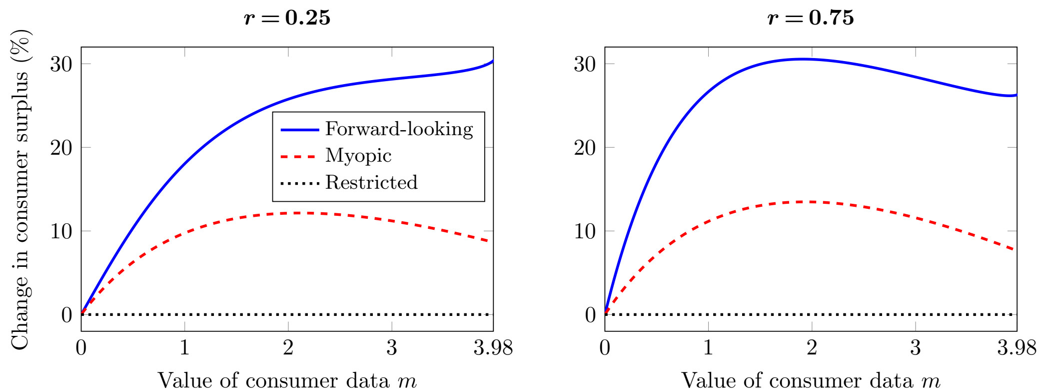

Figure 13 Percentage change in the equilibrium aggregate consumer surplus relative to the restricted setting for $r = 0.25$ (left) and $r = 0.75$ (right), as a function of the value of consumer data m, for $\bar{u} = 1$. We restrict the range of m to $[0, 3.98]$, for which we find an interior equilibrium in the forward-looking setting

illustrate this comparison in Figure 14, where we split the parameter space into two subsets, defined by whether firm B's equilibrium profits are lower in the restricted than in the forward-looking setting. Our computations show that firm B may be better off without data tracking when the value of consumer data (m) is low and the correlation between s and θ is relatively large (induced by high values of r). The intuition for this observation is the following: for high values of r, the distribution of consumer types is skewed toward types with high product 2 valuations, which correspond to consumers with stronger incentives to refrain from transacting with firm B in the first period to avoid revealing their type (i.e., the consumers that reduce demand to buy product 1 from firm B due to "privacy" concerns). Our computations show that when m is low, the benefits of data collection for firm B might not outweigh the demand reduction effect if the distribution of consumer types is considerably skewed toward types with stronger incentives to avoid transacting with firm B, i.e., if (m, r) belongs to the region shaded in grey in Figure 14. However, we still observe that if consumers are myopic, data tracking always results in higher profits for firm B.

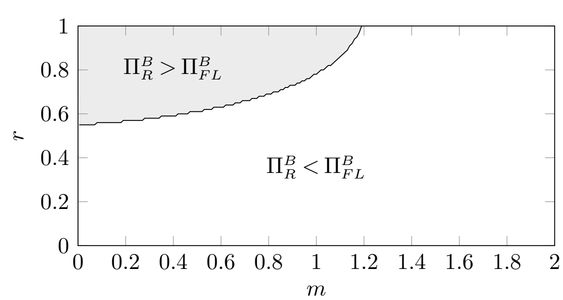

Figure 14 Numerical computations show that firm B's equilibrium profit is higher in a restricted economy than in the setting with data tracking and forward-looking consumers only when (m, r) belongs to the region shaded in grey. We restrict the chart to values of m lower than 2, since we observe that firm B makes a higher profit with data tracking regardless of the value of r for all $m > 1.2$

Appendix K: Arbitrary Distributions for θ

One of the key results of our analysis of Section 4 is that under competition, data tracking can increase consumer surplus even if consumers act myopically. As our baseline analysis assumes that the consumer's type follows a uniform distribution, we now investigate whether this conclusion holds more generally. To this end, we extend our model to consider a wider family of distributions for the consumer's type. Concretely, we relax the assumption that the consumer's type is uniformly distributed, by allowing an arbitrary distribution for the consumer's valuation for product 2; that is, we assume that θ follows a general distribution F that takes values in $[0,\infty)$. The rest of the model remains as given in Section 2. In particular, we maintain the assumption that the consumer's location in the Hotelling line for product 1, s, is uniformly distributed in $[0,1]$, and that s and θ are independent.

In Appendix K.1, we characterize the equilibrium outcomes for the restricted and myopic settings under the assumption that θ follows a general distribution F, which may belong to a wide class of distributions. Then, in Appendix K.2, we compare the aggregate consumer surplus for these two settings, and find that its relationship with data tracking is similar structure to the corresponding one in the baseline model. Concretely, we establish that for a large family of distributions, when the consumer acts myopically, consumer surplus is higher in the presence of data tracking as long as the value of consumer data (as measured by m) is lower than a threshold that depends on F (Proposition 20), which is in line with the conclusions of Theorem 2 and Proposition 7 of the baseline model. Furthermore, we characterize the class of distributions F for which this conclusion applies and illustrate that the aforementioned relationship holds quite broadly (Lemma 6).

Finally, we compare the individual consumer's utility in the myopic and restricted settings as a function of her type and find that this comparison is similar to the one in the baseline model as well: consumers with low values of θ tend to benefit the most from data tracking, while the opposite holds for high values of θ (Proposition 22).

K.1. Equilibrium Outcomes in the Restricted and Myopic Settings

In what follows, we characterize the equilibrium outcomes of the restricted and myopic settings as a function of the distribution F, and show that these outcomes display a similar structure as in Propositions 1 and 2. For the rest of this appendix, we assume that the distribution F satisfies the following mild conditions.

Assumption 2. *F has finite expectation and is supported on a subset of $[0,\infty)$. In addition, F admits a unique solution to the profit maximization problem for a monopolist that faces consumers with valuations distributed according to F. That is, there exists a unique solution p_F^* to*

$$\pi_F^* := \max_{p\geq 0} p\bar{F}(p), \tag{K.1}$$

where $\bar{F}(p)=\int_p^\infty dF$.

Assumption 2 is common in the literature and holds for a wide range of distributions; e.g., continuous distributions with increasing hazard rate, and distributions over finitely many types for which problem (K.1) has a single solution.

Before characterizing the equilibrium outcomes, we define the following notation for a distribution F that satisfies Assumption 2. We denote by π_F^* the value of problem (K.1), which corresponds to the maximum expected profit for a monopolist that sells a product to a consumer with valuation drawn from F. Moreover, we let S_F be the expected consumer surplus for such a consumer when offered the price p_F^*, i.e.,

$$S_F := \mathbb{E}\left[(\theta - p_F^*)^+\right],$$

where the expectation is taken with respect to F. Finally, let Δ_F denote the gap between the expected value of θ and the monopoly profit π_F^*, i.e.,

$$\Delta_F := \mathbb{E}[\theta] - \pi_F^*,$$

where the expectation is taken with respect to F. Note that Δ_F represents the gap between the expected profit of a monopolist that observes the consumer's valuation θ (who sets a price of θ) and its profit in a setting without this information (who sets price p_F^*). Indeed, in our model we can then interpret the quantity $m\Delta_F$ as a metric for the expected benefit of data tracking for firm B.

We now characterize the equilibrium of the restricted setting. Similar to Proposition 1, in equilibrium both firms set a price of $1/2$ for product 1, and the consumer buys from firm A if $s < 1/2$ (and from firm B otherwise). Since no information flows from the first to the second period, firm B's belief on θ when choosing the price for product 2 is equal to the prior distribution F, which leads firm B to set the price of product 2 as $p_2^{B,R} = mp_F^*$. Proposition 18 states this formally; we omit the proof of Proposition 18 as it follows the exact same argument of the proof of Proposition 1 (see Appendix E.1).

Proposition 18. *Suppose that $\theta \sim F$ where F satisfies Assumption 2. Then, for any $m > 0$, there exists an equilibrium in the restricted setting. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) Both firms set a price of $1/2$ for product 1.

(ii) The consumer buys product 1 from firm A if her type satisfies $s < 1/2$, and she buys from firm B if $s > 1/2$.

(iii) Firm B sets a price of $p_2^{B,R} = mp_F^*$ for product 2, which the consumer buys if $\theta > p_F^*$.

We now consider the setting where firm B has access to data tracking and the consumer acts myopically. In line with Proposition 2, data tracking drives both firms' product 1 prices to be lower than $1/2$. Moreover, incentivized to observe the consumer's type, firm B offers a discount relative to firm A when selling product 1. As in Proposition 2, we observe that, depending on the value of m, we may have an interior or a corner equlibrium. Proposition 19 formally describes the equilibrium outcomes in the myopic setting.

Proposition 19. *Suppose that $\theta \sim F$ where F satisfies Assumption 2. Then, for any $m > 0$, there exists an equilibrium in the myopic setting. In addition, the equilibrium path is essentially unique and takes the following form:*

(i) The firms' prices for product 1 are, respectively,

$$p_1^{A,M}(m) = \max\left\{\frac{1}{2} - \frac{m\Delta_F}{3}, 0\right\}, \quad p_1^{B,M}(m) = \max\left\{\frac{1}{2} - \frac{2m\Delta_F}{3}, -\frac{1}{2}\right\}. \tag{K.2}$$

(ii) The consumer buys product 1 from firm B if her type (s,θ) *satisfies* $s > p_1^{B,M} - p_1^{A,M} + 1/2$*. In that case, firm B perfectly observes the consumer's type and, in the second period, sets a price equal to* $m\theta$ *for product 2, which the consumer also buys.*

(iii) The consumer buys product 1 from firm A if her type (s,θ) *satisfies* $s < p_1^{B,M} - p_1^{A,M} + 1/2$*. In that case, firm B sets a price of* $p_2^{B,M} = mp_F^*$ *for product 2, which the consumer buys if* $\theta > p_F^*$*.*

(iv) Firm B faces a higher expected demand for product 1 than firm A. In particular, firm A's expected demand for product 1 is equal to $p_1^{A,M} < 1/2$*.*

The proof of Proposition 19 follows a backwards induction argument analogous to the one of Proposition 2 (see Appendix E.2).

K.2. Implications of Data Tracking on Consumer Surplus

We now focus on comparing the equilibrium outcomes described in Propositions 18 and 19 in terms of consumer surplus. In the baseline model with uniformly distributed types, we have shown that even if the consumer behaves myopically, consumer surplus can be higher in the presence of data tracking. Indeed, Theorem 2 and Proposition 7 (see Appendix D) imply that consumer surplus is higher with data tracking as long as $m \leq 7$. In what follows, we establish that a similar result holds for a wide range of distributions F. Concretely, Proposition 20 below establishes that for any distribution F that satisfies mild conditions (which we describe in Assumption 3), there is a threshold M_F such that if $m < M_F$ and the consumer acts myopically, then consumer surplus is higher in the prescence of data tracking. Before making this consumer surplus comparison, we introduce the aforementioned assumption about the distribution F.

Assumption 3. *F satisfies* $0 < S_F < \Delta_F$*.*

The conditions in Assumption 3 can be better explained in terms of the market for product 2 in the restricted setting, for which firm B acts as a monopolist that sets a unique price mp_F^* for all consumers, given the prior belief that $\theta \sim F$:

1. The condition $S_F < \Delta_F$ can be equivalently stated as $\mathbb{E}[\theta] > \pi_F^* + S_F$, where the expectation is taken with respect to F. In other words, the fact that firm B is a monopolist for product 2 must be socially inefficient; i.e., the sum of consumer surplus and firm B's profit associated with product 2 is less than the average valuation for that product.

2. The condition that $S_F > 0$ simply requires the expected consumer surplus given the option to buy product 2 from firm B to be positive.

We can now establish that, under Assumptions 2 and 3, we obtain a comparison between consumer surplus in the myopic and restricted settings that is structurally similar to the corresponding one in the baseline analysis. That is, consumer surplus is higher in the myopic than in the restricted setting as long as m is below a threshold, and the opposite is true if m exceeds the threshold.

Proposition 20. *Assume that* $\theta \sim F$ *where F satisfies Assumptions 2 and 3. Then, there exists* $M = M_F > 0$ *such that consumer surplus is higher in the myopic than in the restricted setting if and only if* $m < M_F$*. Moreover, this threshold is given by*

$$M_F = \begin{cases} \frac{9(\Delta_F - S_F)}{\Delta_F(6S_F - \Delta_F)}, & \text{if } 7\Delta_F \leq 12 S_F, \\ \frac{7}{8S_F}, & \text{otherwise.} \end{cases} \tag{K.3}$$

Proof of Proposition 20. Following a similar computation as in the proof of Claim 20 with the equilibrium prices described in Propositions 18 and 19, we have that the equilibrium consumer surplus in the restricted and the myopic settings is, respectively,

$$CS_F^R(m) = u - \frac{5}{8} + mS_F \tag{K.4}$$

and

$$CS_F^M(m) = \begin{cases} u - \frac{5}{8} + \frac{m}{2}\Delta_F + \frac{m^2}{18}\Delta_F^2 + \left(\frac{1}{2} - \frac{m\Delta_F}{3}\right) mS_F, & \text{if } m\Delta_F \le 3/2, \\ u + \frac{1}{4}, & \text{otherwise.} \end{cases} \tag{K.5}$$

Fix a distribution F that satisfies Assumptions 2 and 3 and, to ease notation, denote $S = S_F$ and $\Delta = \Delta_F$. Let $h(m) = CS_F^N(m) - CS_F^R(m)$; then, by equations (K.4) and (K.5) we have:

$$h(m) = \begin{cases} m\Delta\left(\frac{1}{2} + \frac{1}{18}m\Delta\right) - \left(\frac{1}{2} + \frac{1}{3}m\Delta\right) mS, & \text{if } m\Delta \le 3/2, \\ \frac{7}{8} - mS, & \text{otherwise.} \end{cases} \tag{K.6}$$

It remains to show that there exists a threshold $M > 0$ such that $h(m) > 0$ iff $m < M$. To do so, we have two cases to cover:

Case 1. Suppose that $h(m) > 0$ for all $m \in (0, \frac{3}{2\Delta}]$. Then, we define $M = \frac{7}{8S}$ ($M < \infty$ by Assumption 3) and it follows that $h(m) > 0$ if and only if $m < M$, as desired.

Case 2. Suppose instead that $h(m) \le 0$ for some $m \in (0, \frac{3}{2\Delta}]$. In this region, we can write $h(m) = m\tilde{h}(m)$, where

$$\tilde{h}(m) = \frac{\Delta - S}{2} + \frac{\Delta}{3}\left[\frac{\Delta}{6} - S\right] m.$$

Thus, $h(m)$ has the same sign as $\tilde{h}(m)$ if $m \in (0, \frac{3}{2\Delta}]$. Moreover, $\tilde{h}(0) = (\Delta - S)/2$ is positive by Assumption 3, and so $h(m) > 0$ for all m small enough. Since $\tilde{h}(m)$ is linear in m, it has only one root in $(0, \frac{3}{2\Delta}]$, which is

$$M = \frac{9(\Delta - S)}{\Delta(6S - \Delta)}.$$

It follows that $h(m) > 0$ for $m < M$, and vice versa for $M < m \le \frac{3}{2\Delta}$. Since $h(m)$ is everywhere continuous and decreasing for $m > \frac{3}{2\Delta}$, we have that $h(m) < 0$ in this region as well. Thus, h changes sign only at $m = M$. To conclude, the expression for the threshold M given in (K.3) follows by observing that, by the previous argument, case 1 applies if and only if $h(3/2\Delta) > 0$. Algebra shows that this is equivalent to $7\Delta > 12S$. □

K.2.1. A Characterization of the Distributions that Satisfy Assumption 3 Proposition 20 establishes necessary and sufficient conditions for data tracking to benefit myopic consumers, provided that F satisfies Assumptions 2 and 3. While Assumption 2 is common, Assumption 3 can in principle seem more restrictive. To better understand the extent to which Proposition 20 applies, we now characterize the family of distributions that satisfy Assumption 3.

Concretely, the following lemma shows that the distributions that satisfy Assumption 3 are those that assign positive probability to events where θ is either larger or smaller than the monopoly price p_F^* and, in addition, $p_F^* > 0$. Another way to state these conditions is as follows.

LEMMA 6. *Let F be a CDF that satisfies Assumption 2, and let $A \subseteq [0, \infty)$ be the support of F. Denote $A^+ = A \cap (0, \infty)$. Then F satisfies Assumption 3 if and only if* $\inf A^+ < p_F^* < \sup A^+$.

Proof of Lemma 6. First, notice that

$$\Delta_F - S_F = \mathbb{E}[\theta] - p_F^* \mathbb{P}[\theta \geq p_F^*] - \mathbb{E}\left[(\theta - p_F^*)^+\right] = \mathbb{E}[\theta I(\theta < p_F^*)].$$

Then, if $\inf A^+ < p_F^*$, $\mathbb{P}[\theta \in (0, p_F^*)] > 0$, and therefore $\Delta_F - S_F = \mathbb{E}[\theta I(\theta < p_F^*)] > 0$. Similarly, if $p_F^* < \sup A^+$, we have that $S_F = \mathbb{E}[(\theta - p_F^*)^+] > 0$.

The converse follows similarly. If $\Delta_F - S_F > 0$, then the event $\{\theta \in (0, p_F^*)\}$ occurs with positive probability, and so $\inf A^+ < p_F^*$. If $S_F > 0$ then $\{\theta > p_F^*\}$ occurs with positive probability, and thus $p_F^* < \sup A^+$. □

The conditions of Lemma 6 are satisfied by a wide range of distributions. In particular, Lemma 6 implies that Assumption 3 holds when θ is a continuous random variable that satisfies Assumption 2 and is supported in an interval of the form $[0, b]$ or $[0, \infty)$. We now illustrate the previous results with some examples.

Example 1. Uniform distribution. To confirm the result of the baseline model, assume that F is the uniform distribution in $[0, 1]$. In this case, we have that $\Delta_F = 1/4$ and $S_F = 1/8$. Thus, from (K.3) we have that $M_F = \frac{7}{8S_F} = 7$. That is, consumer surplus is higher in the myopic than in the restricted setting if and only if $m < 7$. Indeed, this is the same threshold as the one obtained in Proposition 7 (see Appendix D).

Example 2. Exponential distribution. We now assume that θ follows an exponential distribution with mean 1, and denote its CDF by F. We have that $p\bar{F}(p) = pe^{-p}$ for $p \geq 0$, and so $p_F^* = 1$. Since $p_F^* \in (0, \infty)$ and the exponential distribution is supported in $[0, \infty)$, it follows from Lemma 6 that Assumption 3 is satisfied. Indeed, this can also be verified by noting that $\pi_F^* = e^{-1}$ and therefore $\Delta_F = 1 - e^{-1}$. In addition, $S_F = \mathbb{E}[(\theta - 1)^+] = e^{-1}$. Thus, $\Delta_F > S_F > 0$.

Proposition 20 therefore applies (as Assumption 2 is clearly satisfied). One can then verify that $7\Delta_F > 12S_F$ and by (K.3) we have that $M_F = \frac{7}{8S_F} = \frac{7e}{8}$. Thus, we conclude that consumer surplus is higher in the myopic than in the restricted setting if and only if $m < \frac{7e}{8}$.

Example 3. Two-type distribution. Now we consider a discrete distribution that takes only two possible values, and show that the conclusions of Proposition 20 do not hold in this case, as Assumption 3 is not satisfied. Assume then that θ takes the values H and L with probabilities q and $1 - q$, respectively, where $H > L > 0$. The price p_F^* that solves (K.1) depends on the value of q and we have that $p_F^* = H$ if $q > L/H$ and $p_F^* = L$ if $q < L/H$. Assumption 3 fails in both cases since p_F^* is either the supremum or infimum of $\{L, H\}$. We now analyze these two cases separately and show that they result in opposite conclusions.[79]

Case 1. $q > L/H$. We have that $p_F^* = H$, $\pi_F^* = qH$, $\Delta_F = (1 - q)L$, and $S_F = 0$. Since $S_F = 0$, it follows from (K.4) and (K.5) that consumer surplus is higher under data tracking for all $m > 0$.

Case 2. $q < L/H$. In this case, $p_F^* = L$, $\pi_F^* = L$, $\Delta_F = q(H - L)$, and $S_F = q(H - L)$. Since $\Delta_F = S_F$, Assumption 3 is not satisfied. Moreover, defining $h(m)$ as in (K.6) in the proof of Proposition 20, we have that $h(m) = -\frac{5(m\Delta_F)^2}{18} < 0$ for all $m \in [0, \frac{3}{2\Delta_F}]$. For $m > \frac{3}{2\Delta_F}$ we have that $h(m) = \frac{7}{8} - m\Delta_F < 0$. Therefore, consumer surplus is higher in the restricted setting than in the myopic one for all $m > 0$. Another way to verify this is to note that, if we were to apply equation (K.3) to derive a threshold, we would obtain $M_F = 0$.

The last example illustrates that when Assumption 3 is not satisfied, consumer surplus is either always higher in the myopic setting (i.e, for all $m > 0$) or always higher in the restricted setting. We now prove this by formalizing the previous argument.

[79] Assumption 2 fails when $q = L/H$.

Proposition 21. *Assume that $\theta \sim F$ where F satisfies Assumption 2, but not Assumption 3. Then,*

1. *If $p_F^* = \inf A^+$, consumer surplus is lower with data tracking than in the restricted setting for all $m > 0$.*
2. *The opposite is true for all $m > 0$ if $p_F^* = \sup A^+$.*

Proof of Proposition 21. We use the same notation as in the proof of Proposition 20. First, if $p_F^* = \inf A^+$, then $S_F = \Delta_F$ by the proof of Lemma 6. Therefore, defining $h(m) = CS_F^N(m) - CS_F^R(m)$ as in the proof of Proposition 20, we have that

$$h(m) = \begin{cases} -\frac{5(m\Delta_F)^2}{18}, & \text{if } m\Delta_F \le 3/2, \\ \frac{7}{8} - m\Delta_F, & \text{otherwise.} \end{cases}$$

Note that $h(0) \le 0$ and h is continuous and strictly decreasing in m, and so $h(m) < 0$ for all $m > 0$. Thus, $CS_F^N(m) < CS_F^R(m)$ for all $m > 0$.

Now assume that $p_F^* = \sup A^+$. Then $S_F = 0$ by the proof of Lemma 6, and we have that for all $m > 0$,

$$h(m) = \begin{cases} m\Delta_F\left(\frac{1}{2} + \frac{1}{18} m\Delta_F\right), & \text{if } m\Delta_F \le 3/2. \\ \frac{7}{8}, & \text{otherwise.} \end{cases}$$

Observe that $h(m) > 0$, which implies that $CS_F^N(m) > CS_F^R(m)$ for all $m > 0$. □

K.2.2. Consumer Surplus Comparisons at the Individual Level In the previous section, we compared the equilibrium aggregate consumer surplus in the restricted setting and the setting with data tracking and myopic consumers. We now study this comparison at the individual level; that is, we compare the utility of a consumer as a function of her type in the restricted and the myopic settings.

In line with the discussion of Section 4, we establish that consumers with relatively low valuations for product 2 (i.e., low values of θ) are the ones that benefit from the presence of data tracking, while consumers with high values of θ tend to capture higher utility when data tracking is restricted. The inuition for this observation is the same as in Section 4: lower product 1 prices benefit all consumers, while only those with high values of θ bear the surplus loss associated with personalized product 2 prices when data tracking is available. The following proposition formalizes this comparison, which we also illustrate in Figure 15.

Proposition 22. *Fix $m > 0$ and a distribution F that satisfies Assumption 2 and let A denote the support of F. Suppose that the consumer's type is (s, θ) and let $U^R(s,\theta)$ and $U^M(s,\theta)$ denote the total utility of the consumer in the equilibrium of the restricted and myopic settings, respectively. Then, $U^M(s,\theta) > U^R(s,\theta)$ if and only if $(s,\theta) \in \mathcal{M}_1 \cup \mathcal{M}_2$, where these sets are defined by*

$$\mathcal{M}_1 = \left\{(s,\theta) \in [0,1] \times A : s < p_1^{A,M}(m) \text{ or } \theta < p_F^*\right\},$$
$$\mathcal{M}_2 = \left\{(s,\theta) \in [0,1] \times A : s > p_1^{A,M}(m), \theta < p_F^* + \min\left\{\frac{2}{3}\Delta_F, \frac{1}{m}\right\} - \frac{1}{m}\left(\frac{1}{2} - s\right)^+\right\},$$

with $p_1^{A,M}(m)$ as defined in (K.2).

Proof of Proposition 22. By Proposition 18, we can write the equilibrium utility in the restricted setting when the consumer's type is (s,θ) as

$$U^R(s,\theta) = u - \frac{1}{2} - \frac{1}{2}\min\{s, 1-s\} + m\left(\theta - p_F^*\right)^+. \tag{K.7}$$

On the other hand, by Propostion 19, the corresponding utility in the myopic setting is

$$U^M(s,\theta) = \begin{cases} u - \frac{1}{2}s - p_1^{A,M}(m) + m(\theta - p_F^*)^+, & \text{if } s < p_1^{A,M}(m), \\ u - \frac{1}{2}(1-s) - p_1^{B,M}(m), & \text{otherwise,} \end{cases} \tag{K.8}$$

where $p_1^{A,M}(m)$, $p_1^{B,M}(m)$ are the equilibrium prices for product 1 in the myopic setting as defined in (K.2). A straightforward comparison of expressions (K.7) and (K.8) yields the result. □

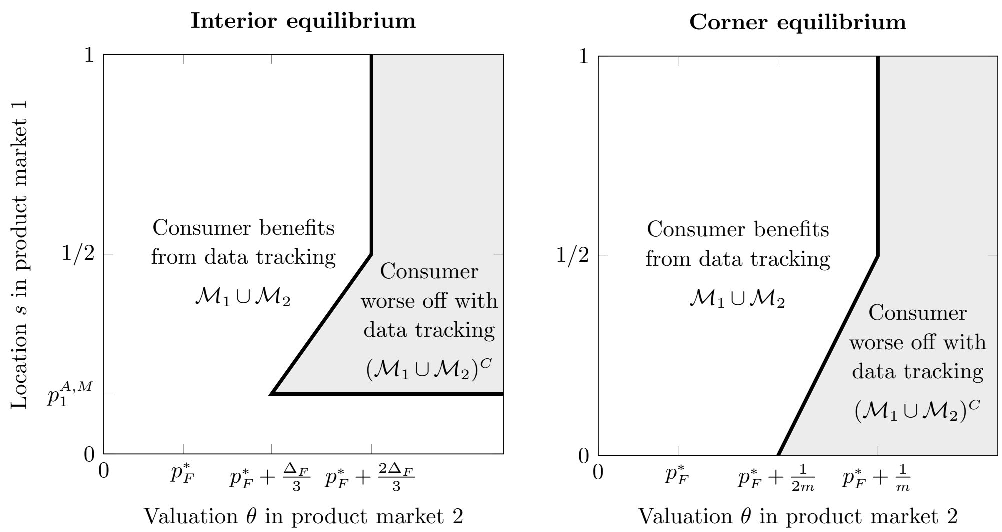

Figure 15 Comparison of the consumer's utility as a function of her type (s,θ) in the myopic and restricted settings for interior (left) and corner (right) equilibria, as given in Proposition 19. Consumers with high values of θ, i.e., whose types fall into the shaded region, are better off in the absence of data tracking

K.3. Implications of Data Tracking on Firms' Profits

For completeness, we also compare the equilibrium profits in the myopic and restricted settings for each firm, given the equilibrium outcomes described in Propositions 18 and 19. This comparison extends the one from the baseline model with uniform distributions (i.e., Propositions 5 and 6). As one would expect, in the presence of myopic consumers data tracking benefits firm B while it decreases firm A's profit.

PROPOSITION 23. *Assume that F satisfies Assumption 2 and that $\Delta_F > 0$. Then, for all $m > 0$, firm B benefits from data tracking when the consumer is myopic, and conversely for firm A.*

Proof of Proposition 23. We want to show that firm B's equilibrium profit is higher in the myopic than in the restricted setting and vice versa for firm A. By Proposition 18, it is straightforward to verify that the firms' equilibrium profits in the restricted setting are, respectively,

$$\Pi_R^A = \frac{1}{4}, \quad \Pi_R^B = \frac{1}{4} + m\pi_F^*.$$

Similarly, given the equilibrium outcomes described in Proposition 19, the firms' respective equilibrium profits in the myopic setting are

$$\Pi_M^A = \left(\max\left\{\frac{1}{2} - \frac{m}{3}\Delta_F, 0\right\}\right)^2, \quad \Pi_M^B = \begin{cases} m\pi_F^* + \left(\frac{1}{2} + \frac{m}{3}\Delta_F\right)^2, & \text{if } m\Delta_F \le 3/2, \\ m\pi_F^* + m\Delta_F - \frac{1}{2}, & \text{otherwise.} \end{cases}$$

A simple comparison of the respective profits of both firms yields the result. To see this, first consider the case where $m\Delta_F \leq 3/2$. Then, $\Pi_R^A = 1/4 > \left(\frac{1}{2} - \frac{1}{3}\Delta_F\right)^2 = \Pi_M^A$. For firm B, we have that

$$\Pi_M^B - \Pi_R^B = \left(\frac{1}{2} + \frac{1}{3}m\Delta_F\right)^2 - \frac{1}{4} > 0.$$

On the other hand, if $m\Delta_F > 3/2$, firm A's profit is zero in the myopic setting, but 1/4 in the restricted one. For firm B, we have that $\Pi_M^B - \Pi_R^B = m\Delta_F - 3/4 > 3/4$. □