Spatial Pricing in Ride-Sharing Networks

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Abstract

We explore spatial price discrimination in the context of a ride-sharing platform that serves a network of locations. Riders are heterogeneous in terms of their destination preferences and their willingness to pay for receiving service. Drivers decide whether and where to provide service so as to maximize their expected earnings, given the platform's pricing and compensation policy. Our findings highlight the impact of the demand pattern on the platform's prices, profits, and the induced consumer surplus. In particular, we establish that profits and consumer surplus at the equilibrium corresponding to the platform's optimal pricing and compensation policy are maximized when the demand pattern is "balanced" across the network's locations. In addition, we show that they both increase monotonically with the balancedness of the demand pattern (as formalized by its structural properties). Furthermore, if the demand pattern is not balanced, the platform can benefit substantially from pricing rides differently depending on the location they originate from. Finally, we consider a number of alternative pricing and compensation schemes that are commonly used in practice and explore their performance for the platform.

Keywords: Ride-sharing, Revenue management, Network flows, Spatial price discrimination.

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§We are grateful to Yiling You for excellent research assistance. We thank the Area editor and Associate editor, and two anonymous referees for their valuable comments. In addition, we thank Dan Adelman, Nick Arnosti, Baris Ata, John Birge, Rene Caldentey, Kimon Drakopoulos, Austan Goolsbee, Yash Kanoria, Azarakhsh Malekian, Garrett van Ryzin, Peng Shi, Jim Smith, Gabriel Weintraub, and seminar participants at INFORMS, Lyft Data Science, Uber MODS, the Denver Research Meeting on Networks, Matching, and Platforms, Imperial College London, University of Paris 1, UCSD, Kellogg, Duke, Wharton, INSEAD, CMU, MSR New England, UCLA, UIC, USC, UPF, Michigan Ross, MIT, and the University of Toronto for very helpful suggestions.

1 Introduction

Ride-sharing platforms such as Lyft and Uber are in the process of disrupting the transportation industry by better matching the supply of drivers with the demand for rides. Interestingly, they do not employ any drivers but rather they operate as two-sided markets between riders and independent contractors that provide service as drivers. One of the main tools that such platforms have at their disposal to facilitate the matching between drivers and riders is their pricing and compensation policy. In fact, Uber has received a lot of praise but also criticism on how their pricing works.[1]

The design of a pricing policy for a platform may be challenging as prices need to serve a dual role: match supply and demand in *time* and also in *space*. Most of the attention so far has largely been focused on the first role, i.e., how to employ surge/dynamic pricing techniques to mitigate the impact of temporal demand fluctuations on service and, consequently, on profits at a given location. Equally important, however, is the second role, i.e., the fact that the platform serves demand across a *network of interconnected locations*. In particular, the price set by the platform at a location not only determines the level of service at that location but also affects the supply of drivers at all other locations. Thus, even in the absence of any temporal demand fluctuations, the platform can benefit substantially from setting different prices across the network.

Our goal in this paper is to complement the existing literature, which has mainly addressed the problem of dealing with temporal demand fluctuations at a given location by focusing squarely on the demand pattern for rides across a network's locations and its impact on the platform's prices, profits, and consumer surplus. To this end, we consider a time-invariant environment that ensures that our analytical findings isolate the impact of the demand pattern's spatial structure and study how the platform should price rides depending on where they originate from.

We expect that in practice ride-sharing platforms will benefit from using a combination of spatial price discrimination to account for long-term predictable demand patterns across a network of locations and surge-pricing techniques to smooth out short-term demand fluctuations at a given location. For instance, if during the morning rush hour riders consistently demand rides from suburban to downtown locations, setting different prices for rides originating from the suburbs versus those from downtown may lead to gains for both the platform and the riders. If, in addition, there are short-term fluctuations in supply and demand, e.g., due to the weather conditions, temporary surge pricing may help to further mitigate the mismatch between the supply of drivers and the demand for rides.

This paper is among the very first to explicitly account for the spatial dimension of a ride-sharing platform's pricing problem. Our contributions can be briefly summarized as follows. First, we develop a tractable model to study a platform operating on a network of locations that may differ

[1] The merits and the potential shortcomings of Uber's pricing algorithm have been extensively covered in the press. For example, see: `https://www.technologyreview.com/s/529961/in-praise-of-efficient-price-gouging/`, and `http://fortune.com/2015/01/26/uber-caps-surge-pricing-during-blizzard-but-people-still-complain/`.

in both the size of their potential demand and the destination preferences of riders (we refer to them jointly as the network's *demand pattern*). Importantly, the model features drivers who endogenously determine not only whether and where to join the platform and start providing service but also where to relocate themselves when they are idle. Second, we provide a characterization of how the demand pattern affects the platform's prices, profits, and induced consumer surplus. Our findings illustrate that the demand pattern's "balancedness," which captures the extent to which the mass of riders who demand a ride leaving a location differs from the mass of those requesting a ride going to that same location, succinctly summarizes the profit potential of a given network of locations for the platform. Finally, we explore the benefits and limitations of a number of pricing and compensation schemes through a combination of analytical results and simulations on real-world networks. We describe our contributions in detail below.

Main Contributions. We introduce a tractable model that is tailored to exploring the interplay between the demand pattern across the locations of a ride-sharing network and the platform's prices and profits (as well as the resulting consumer surplus). Our model economy is comprised of a set of interconnected locations and populations of riders that seek transportation from one such location to another. Both the distribution of the riders' willingness to pay for a ride as well as their aggregate origin-destination preferences are assumed to be known. The objective of the platform is to maximize its aggregate profits. To this end, it sets the price that a rider has to pay and the compensation given out to the corresponding driver for each ride that it facilitates. These prices and compensations may differ depending on where the ride originates from. Drivers decide whether and where to join the platform and where to relocate themselves when they are idle, with the goal of maximizing their expected earnings over the time they provide service, by taking into account the prices and compensations set by the platform. Thus, while determining its pricing and compensation policy, the platform needs to carefully consider its impact on drivers' entry and relocation decisions. To the best of our knowledge, ours is among the very first models in the literature to explicitly focus on the fact that the platform sets prices to serve demand across a set of interconnected locations while the supply of drivers behaves strategically.

Our first main contribution is to identify a property of demand patterns, which we call balancedness, that, to some extent, is the right property to consider when evaluating the profit potential of a given network for the platform. Informally, a demand pattern is *balanced* if the potential demand for rides at each location is roughly the same as the potential demand for rides with this location as their destination. We establish that the profits the platform can generate are higher the more balanced the underlying demand pattern is and, consequently, the profit potential of a network is highest when the demand pattern is balanced.[2] In addition, similarly to profits, we show that con-

[2]To be more specific, for a given demand pattern and any *strongly* balanced demand pattern (Section 3 provides a formal definition), we consider the set of demand patterns that can be written as a convex combination of the two. Then, in Theorem 1, we establish that the closer the demand pattern is to the strongly balanced one, the higher are the profits the platform can generate. We say that demand pattern A is *more balanced* than demand pattern B if A lies in the convex

sumer surplus in the induced equilibrium is also monotonic with respect to the balancedness of the underlying demand pattern.

Furthermore, the extent to which the underlying demand pattern is balanced has implications on the potential benefits of employing spatial price discrimination. In particular, when the demand pattern is balanced, there is no need for the platform to price discriminate: it is optimal to set prices for rides to be equal across the network's locations. By contrast, in the presence of demand imbalances, it is beneficial to set different prices for rides depending on the location they originate from. Informally, given that riders are price sensitive, the platform leverages spatial price discrimination to ensure that the pattern of *the demand that gets served* becomes more balanced.

In particular, we focus on settings where the number of riders at each location is set to be the same (and normalized to one) and establish that locations that are relatively more attractive as destinations are likely to receive a large number of incoming rides, and, consequently, they end up with an excess supply of drivers. The platform "subsidizes" rides originating from such locations in order to induce more demand. This, in turn, allows for a better utilization of drivers who find themselves in such locations after completing a ride. On the other hand, the platform finds it optimal to set higher prices for rides originating from locations that are relatively less popular as destinations (and, consequently, end up with a shortage of supply). These locations may also feature higher compensation levels for drivers. The combination of higher prices (which reduce demand) and higher compensation levels (which increase the incentives of drivers to provide service at these locations) leads to a better matching of supply with demand at these locations. It is worthwhile to note that, even though supply and demand are determined endogenously in the context of our model, we are able to identify locations that end up with an excess or shortage of supply under the platform's optimal prices and compensations, directly from the primitives of the economy (specifically, the demand pattern).

Finally, we consider alternative pricing and compensation schemes that are commonly used in practice, and explore their implications on platform's profits. First, we study a compensation scheme that features fixed commission rates paid to drivers. We identify classes of demand patterns for which employing such a scheme is without loss of optimality, i.e., it can generate the same profits for the platform as a scheme that features different compensations depending on a ride's origin. However, we construct a simple example that shows that, in general, using a fixed commission rate leads to significant profit losses for the platform. Beyond the fact that solving for the optimal fixed commission rate may be computationally challenging, our result points to the potential shortcomings of this scheme.

Second, we provide convex programming formulations to compute the platform's optimal pricing policy when it is restricted to setting the same price at all of the network's locations and when it can set different prices depending on both a ride's origin and its destination. We compare these pricing schemes with origin based price discrimination, and illustrate their differences using data

combination of B and a strongly balanced demand pattern.

that represents demand imbalances in real-world networks. We find that pricing rides differently depending on where they originate from may lead to significantly higher profits relative to using the same price across the network, especially when the demand pattern is highly unbalanced. On the other hand, the additional gain from setting prices that depend on both the origin and the destination of a ride seems to be less significant.

Implications and Economic Insights. In summary, our modeling framework and analysis yield a number of novel insights into the operations of ride-sharing platforms. On the descriptive side, we show that the balancedness of a demand pattern can serve as a measure of the profit potential of a given geographical region for the platform. On the prescriptive side, we show that a platform should consider using spatial price discrimination especially when the underlying demand pattern is highly unbalanced, as in such settings spatial pricing yields significant profit gains. In addition, we establish that a platform should offer relatively low/high prices respectively for rides originating from attractive/unattractive destinations for riders as determined by their aggregate origin-destination preferences. Doing so allows for better matching supply with demand and, consequently, maximizing profits.

Another set of prescriptive insights relate to whether and when different pricing and compensation schemes should be employed in practice. In particular, we show that explicitly compensating drivers with a fixed ratio of the revenues they generate, generally leads to a significant loss for the platform. This finding may help explain recent efforts at both Uber and Lyft to implement a richer compensation scheme by offering compensation "boosts" to drivers depending on the locations they provide service at.[3] Finally, we illustrate that pricing rides based on where they originate from can lead to a significant improvement in a platform's profits. On the other hand, pricing rides depending on both their origin and destination seems to lead to additional gains that, although more modest, may still be significant in practice given the volume that such platforms operate.

Related Literature. Our work is related to the burgeoning literature that explores the design and operations of online marketplaces. Allon et al. (2012) study the role of a platform in improving the operational efficiency of large-scale service marketplaces. More recent work has provided insights into how product-sharing platforms may affect an individual's decision to own (Benjaafar et al. (2018)), how a host's experience may explain her earnings at AirBnb (Li et al. (2017)), how reducing search costs may lead to inefficiencies in online matching markets (Horton (2018), Arnosti et al. (2016), Kanoria and Saban (2017)), and how information may be disclosed so as to induce experimentation (Papanastasiou et al. (2016)). The overview article by Azevedo and Weyl (2016) highlights the research opportunities provided by the increasing popularity of digital markets.

In the context of ride-sharing platforms, Banerjee et al. (2015), Cachon et al. (2017), and Castillo

[3] For more details, see https://www.uber.com/drive/atlanta/resources/driver-partner-earnings-boost/ and http://therideshareguy.com/how-does-uber-earnings-boost-work/.

et al. (2018) explore the potential benefits of state-contingent pricing when demand for rides is stochastic. Relatedly, Gurvich et al. (2018), Taylor (2017), and Tang et al. (2017) also explore how stochasticity in market conditions may affect the platform's pricing and compensation decisions. By contrast, we focus squarely on the spatial structure of the underlying demand and endogenous supply. In particular, we abstract away from short-term supply/demand fluctuations and isolate the impact of the network structure on the platform's decision making and the drivers' equilibrium behavior. Thus, we complement the existing work that mainly focuses on the temporal aspect of such an environment, by exploring its spatial dimension.[4]

Also related is a set of papers that explore the use of design levers other than pricing to match supply with demand. Ozkan and Ward (2017) propose a linear programming based approach to determine the matching between drivers and potential riders and establish its asymptotic optimality. Afeche et al. (2018) explore how platforms can optimally accept ride requests and reposition drivers within a two-location network. Hu and Zhou (2016) provide conditions for optimal matching to follow a priority hierarchy. In a complementary direction, Yang et al. (2016) consider a model motivated in part by ride-sharing services in which agents compete for time-varying location-specific resources. In addition and parallel to the literature on ride-sharing platforms, a recent series of papers considers the operations of large-scale bike-share systems. Kabra et al. (2018) build a structural demand model for the Vélib' bike-share system in Paris. Henderson et al. (2016) study the allocation of bikes and docks across a city in the context of an ongoing collaboration with the NYC Bike Share.

Finally, our work shares some modeling features with Lagos (2000) who studies a time-invariant model of the taxi industry.[5] Prices and the aggregate supply of taxis are fixed and exogenously given and the paper's main objective is to illustrate that the drivers' behavior may result in search frictions. By contrast, we take the perspective of a ride-sharing platform and explore how it can optimize its profits by appropriately pricing demand and compensating drivers at different locations. Apart from the fact that prices are determined by the platform, in our work the supply of drivers is also endogenous.

2 Model

We consider an infinite horizon discrete time model of a ride-sharing network with n locations equidistant from one another. Getting from a location to any other location takes one period.[6] Demand for rides is time-invariant. In particular, every period a continuum of potential riders of mass θ_i seek rides originating from location i. The fraction of riders at location i who wish to go to loca-

[4] Banerjee et al. (2017) also recognize the importance of taking the network structure into account when devising pricing policies for shared vehicle systems. However, their approach is mostly algorithmic whereas our study provides analytical insights on the interplay between the platform's pricing, the demand pattern, and the drivers' incentives.

[5] Lagos (2003) and Buchholz (2017) build on Lagos (2000) and study empirically the effect of taxi regulations on the overall efficiency of the market using data from New York City.

[6] Subsection 5.1 discusses how our findings extend when distances between different locations may not be the same. In addition, for simplicity, throughout the paper we ignore the cost (of fuel) for traveling between locations.

tion j is given by the ij-th entry of matrix $\mathbf{A}$, denoted by α_{ij}, where $\sum_j \alpha_{ij} = 1$ for all i (thus, the total mass of riders who would like a ride from i to j in each time period is $\theta_i \alpha_{ij}$). Note that $\mathbf{A}$ can also be viewed as a weighted adjacency matrix associated with the ride-sharing network, where there is a directed edge from location i to j when $\alpha_{ij} > 0$. We call $(\mathbf{A}, \boldsymbol{\theta})$ the network's *demand pattern* and we make the following assumption throughout the paper:[7]

Assumption 1. The network's demand pattern $(\mathbf{A}, \boldsymbol{\theta})$ is such that:

(i) For every location i, the mass of riders who wish to take a ride originating from i is strictly positive, i.e., $\theta_i > 0$ for all i.

(ii) Each component of the directed graph defined by adjacency matrix $\mathbf{A}$ is strongly connected.

Riders are heterogeneous in terms of their willingness to pay for a ride. Specifically, if the price for receiving service is set to p, the induced demand for rides between locations i and j (at a given time period) is given by $\theta_i \alpha_{ij}(1 - F(p))$. Here, $F(\cdot)$, can be viewed as the (empirical) cumulative distribution of the riders' willingness to pay, which we assume to be the same for all origin-destination pairs, and has support $[0, \bar{z}]$ (where with some abuse of notation $\bar{z} = \infty$ allows for settings with unbounded support).[8] Finally, we assume that riders who do not get assigned to a driver in the period they seek service from the platform, e.g., because of excess demand for rides at their locations, use other means of transportation and leave the platform.

Drivers participating in the platform can provide rides originating from location i at a given time period only if they are located at i at that period. We assume that the platform can assign a ride originating from location i to any driver present at this location, and drivers cannot reject the rides they are assigned to.[9] In particular, if the supply of drivers at a location is lower than the demand at that location, then all drivers are assigned to rides. Otherwise, we assume that each driver at the aforementioned location has equal probability, given by the ratio of demand to supply, of getting matched to a rider. Drivers who do not get assigned to a ride decide where to continue providing service, i.e., stay at the location where they are currently at or relocate to a location of their choice. In both cases, we assume that drivers are available again for service at the beginning of the *next* time

[7] Assumption 1(ii) implies that a driver can potentially start from any location i and reach any other location j within the same component of the network defined by $\mathbf{A}$, after serving a sequence of rides.

[8] The assumption that the riders' willingness to pay follows the same distribution irrespective of where they are located or they intend to go allows us to isolate the impact of the riders' origin-destination preferences, i.e., matrix $\mathbf{A}$, on the platform's profits. That said, our approach and formulation readily extend to settings where there is heterogeneity along this dimension as well.

[9] We could have also allowed for the following modeling feature: drivers decide whether to make themselves available at the location they end up upon completing a ride or remain unavailable and relocate to another more attractive (in terms of expected earnings) location. Although such a feature could seemingly affect the equilibrium outcome, it turns out that at the optimal prices and compensations drivers always find it optimal to be available for new rides at all locations. This can be shown by noting that if no service is being provided at a location, then the platform can improve profits by offering a high price/compensation that induces some service at this location. On the other hand, if some drivers find it in their best interest to offer service at this location, others also have a (weak) incentive to wait, and relocate only in case of not being matched to a rider. To simplify the exposition and reduce the notational burden, we omit the aforementioned feature from the model, but we emphasize that doing so is without loss of optimality.

period (unless they exit the platform altogether as we explain shortly).[10] A driver who is assigned to a ride at location i has probability α_{ij} of serving a ride with location j as its destination (recall that α_{ij} fraction of the induced demand at i wishes to go to location j).

Upon completing a ride or relocating to a location of her choice (both of which, as we mentioned above, take one period), each driver on the platform exits with probability $(1-\beta)$, where $\beta \in (0,1)$ is meant to capture the fact that a driver provides service for a limited amount of time in expectation (e.g., a shift that lasts eight hours in expectation),[11] which we refer to as her *lifetime*.[12] We assume that there is an infinite supply of potential drivers who may enter the platform and start providing service if their participation constraint is met. In particular, each driver has an outside option that amounts to lifetime earnings equal to a positive scalar w (for example, w can be thought as the average wage for low-skilled labor). A driver enters the platform if her expected lifetime earnings from the platform is at least w.[13] Note that drivers' lifetime earnings depend on their entry location, and drivers enter only at locations that yield the highest expected lifetime earnings for them (as we detail in the next subsection). We emphasize that a key feature of the model is that drivers endogenously determine both whether and where to enter the platform and also where to relocate themselves when they are not assigned to a rider, with the objective of maximizing their earnings.

The platform facilitates the matching of riders to drivers. Its objective is to maximize the flow rate of profits by choosing prices and compensations $\{p_i, c_i\}_{i=1}^n$, where p_i denotes the price that a rider has to pay for a ride that originates from location i, and c_i is the corresponding compensation for the driver. Note that given that the demand for rides is time-invariant, we restrict attention to pricing and compensation policies that are also time-invariant.

At this point, it is worthwhile to briefly summarize the key features of our modeling framework:

(1) Drivers decide whether and where to provide service so as to maximize their expected lifetime earnings. Given that they have an outside option that amounts to lifetime earnings of w, they only participate in the platform if compensations are such that in expectation their lifetime earnings are at least w.

[10]As a convention, when a driver who did not get assigned to a ride at location i decides to stay at i, we say that the driver relocates to i. Thus, any driver who is at i in the beginning of a period either provides service or relocates.

[11]We assume that β is exogenous and it is same for all drivers, implying that their shifts last the same in expectation. A very interesting direction for future work would be to assume that drivers react to the state of the system when deciding whether to exit, i.e., parameter β is a function of the drivers' expectations about their future earnings. Such a modeling extension would likely necessitate a considerably different analysis and perhaps would be more meaningful in a non-stationary model (unlike ours).

[12]To obtain our results, we make the assumption that each driver arriving at a location (after completing a ride or relocating) has the same probability, equal to $(1-\beta)$, of exiting the platform, but our results do not depend on the exact process by which these exits happen. One way to formalize the process of drivers exiting the platform is to view them as points on a circle with a circumference equal to their mass. Then, the drivers who exit the platform are those who lie in an interval of length $(1-\beta)$ times the mass starting at a point drawn uniformly at random from the circle. We are also agnostic to the specifics of the matching mechanism between riders and drivers. Following a similar idea, i.e., representing the drivers as points on a circle and constructing intervals starting at randomly drawn points on the circle, it is possible to ensure that each driver at a location has equal probability of being assigned to a rider going to a given destination, as well as (when the supply of drivers is higher than the induced demand) remaining unmatched.

[13]Throughout the main body of the paper, we assume that drivers have the same outside option (reservation wage). Appendix C.4 discusses how we could incorporate heterogeneity in the drivers' reservation wages.

(2) Given that we are interested in studying the potential benefits of spatial price discrimination, we consider a platform that sets prices/compensations $\{p_i, c_i\}_{i=1}^n$ differently for rides that originate from different locations. Importantly, in our baseline model, we do not consider pricing policies that depend on the destination of a ride (in addition to its origin). Arguably, pricing policies that depend on the location a ride originates from appear to be common in practice.[14] For completeness, Appendix C.3 discusses how our results are affected when the platform may set the price/compensation for a ride as a function of both the ride's origin and its destination. In addition, in Subsection 4.2, we report computational results that compare the profits generated by the platform when it sets prices that depend on a ride's origin versus both its origin and destination, using data representative of demand imbalances in real-world networks.

(3) As we are interested in illustrating how a platform may maximize its profits over a network of locations (thus, we place emphasis on the spatial dimension of the platform's pricing problem), we assume that the demand pattern is time-invariant.

2.1 Equilibrium

This subsection considers the equilibrium outcome induced by the pricing and compensation policy $\{p_i, c_i\}_{i=1}^n$ set by the platform. Since the demand for rides and prices/compensations are time-invariant, throughout the paper we focus on a time-invariant equilibrium outcome, where at a given location, the same mass of drivers enter, provide service, and relocate at every time period. Before we provide a formal definition of the equilibrium concept, we describe how prices determine the flow of riders and drivers across locations.

Riders request a ride at location i if their willingness to pay is at least as high as the price p_i set by the platform. Thus, the induced demand for rides at i is given by $\theta_i(1 - F(p_i))$. We let δ_i denote the mass of (new) drivers who choose to *enter* the platform at a given period and begin providing service at location i. We denote by y_{ij} the mass of drivers at location i who decide to relocate to location j upon not getting assigned to a rider at i (and y_{ii} stands for the mass of drivers who decide to stay at location i). Finally, we let x_i denote the mass of drivers at location i at the beginning of a period, which is given by the following expression

$$x_i = \beta \left[\sum_j \alpha_{ji} \min\{x_j, \theta_j(1 - F(p_j))\} + \sum_j y_{ji} \right] + \delta_i. \tag{1}$$

[14] For example, Uber's surge pricing used to be based solely on the rider's pickup location (see https://help.uber.com/h/e9375d5e-917b-4bc5-8142-23b89a440eec). In November 2016, Uber's rider app started requiring riders to provide their destination, and the display of origin-specific surge multipliers was replaced with fare estimates (see http://time.com/4554138/uber-app-update-2016/). Even when the platform provides riders with such upfront fare estimates, these still appear to depend only on the supply-demand imbalances at the origin and not at the destination (see https://help.uber.com/h/4efa31c0-1123-48a7-b9b1-6e968a62fd6e).

The first summation in Equation (1) is equal to the mass of drivers who find themselves located at i after completing a ride that started in the previous time period, given that the term $\min\{x_j, \theta_j(1 - F(p_j))\}$ is equal to the total demand the platform serves at location j (α_{ji} fraction of which has location i as its destination). The second summation is equal to mass of the drivers who did not get a ride at j in the previous time period (e.g., due to an excess supply of drivers at j) and chose to relocate to i. Since $1-\beta$ fraction of drivers exit after completing a ride or relocating to a location, we scale both summations by β. The total mass of drivers relocating from j matches the excess supply at j, i.e., $\{y_{jk}\}_{k=1}^{n}$ are such that:

$$\sum_k y_{jk} = \max\{x_j - \theta_j(1 - F(p_j)), 0\}.$$

Motivated by this observation, we say that j is a *location with excess supply* if $\sum_k y_{jk} > 0$.

It is convenient to associate each location i with the expected future earnings for a driver located at i at the beginning of a period. In particular, if we let V_i denote the expected future earnings for a driver at location i, we have

$$V_i = \min\left\{\frac{\theta_i(1 - F(p_i))}{x_i}, 1\right\}\left(c_i + \sum_k \alpha_{ik}\beta V_k\right) + \left(1 - \min\left\{\frac{\theta_i(1 - F(p_i))}{x_i}, 1\right\}\right)\beta \overline{V}, \tag{2}$$

where $\overline{V} = \max_j V_j$. The first term in (2) corresponds to the case where the driver is assigned to a ride at i, which happens with probability $\min\left\{\frac{\theta_i(1-F(p_i))}{x_i}, 1\right\}$. In this case, the driver earns c_i for providing service and takes a rider to her destination, which is location k with probability α_{ik}. After dropping off the rider at k, the driver continues providing service with probability β and has expected future earnings of V_k. The second term corresponds to the case where the driver does not get assigned to a ride and chooses to relocate to one of the locations that maximize her expected future earnings. Note that if a driver enters the platform at location i, V_i also captures her expected lifetime earnings from the platform. Motivated by this observation, we alternatively refer to V_i as the lifetime earnings for drivers who enter at location i.

Using the notation above, we formally introduce our equilibrium concept.

Equilibrium. An equilibrium under the vector of prices and compensations $\{p_i, c_i\}_{i=1}^{n}$ is a tuple $\{\delta_i, x_i, y_{ij}\}_{i,j=1}^{n}$ with $\delta_i, x_i, y_{ij} \geq 0$ for all $i, j \in \{1, \ldots, n\}$ such that:

(i) The expected lifetime earnings for a driver who enters the platform at i is given by (2). Given that there exists an infinite supply of potential drivers who can enter the platform and provide service, there cannot exist a location for which $V_i > w$ at equilibrium (since, otherwise, additional drivers would find it optimal to enter, thus leading to a decrease in V_i). In other words, $V_i \leq w$ for all i at equilibrium. On the other hand, provided that there is entry, we should have $V_i = w$ for at least one location i; since if $V_i < w$ for all i, then no driver would find it optimal to enter. Thus, we have $\overline{V} = \max_i V_i = w$ at equilibrium. Moreover, given that drivers choose where to enter at/relocate to so as to maximize their earnings, there should only be entry at/relocation to locations with the highest expected lifetime earnings. In other words,

$$V_i = \overline{V} = w \text{ for all } i \text{ such that } \delta_i + \sum_j y_{ji} > 0. \tag{3}$$

Equation (3) captures the drivers' incentive-compatibility constraints. We refer to locations with $\delta_i + \sum_j y_{ji} > 0$ as *entry points*.

(ii) The mass of drivers at location i at the beginning of a time period is given by (1), i.e.,

$$x_i = \beta \left[\sum_j \alpha_{ji} \min\{x_j, \theta_j(1 - F(p_j))\} + \sum_j y_{ji} \right] + \delta_i,$$

with $\sum_k y_{jk} = \max\{x_j - \theta_j(1 - F(p_j)), 0\}$, for every location j. We refer to this set of constraints as the *equilibrium flow constraints*.

Proposition 1 states that an equilibrium as defined above exists for any given vector of prices and compensations $\{p_i, c_i\}_{i=1}^n$ set by the platform.

Proposition 1. *An equilibrium tuple $\{\delta_i, x_i, y_{ij}\}_{i,j=1}^n$ exists under any given vector of prices and compensations $\{p_i, c_i\}_{i=1}^n$.*

It is worthwhile to note that there may exist multiple equilbria corresponding to the same vector of prices and compensations $\{p_i, c_i\}_{i=1}^n$, which generate different profits for the platform (we provide an example in Appendix C.1). That said, as we establish in what follows, all equilibria corresponding to the *optimal* vector of prices and compensations generate the same profits, prices, and aggregate entry of drivers.

2.2 The Platform's Optimization Problem

We conclude this section by stating the platform's optimization problem. The platform determines the tuple $\{p_i, c_i\}_{i=1}^n$, i.e., prices for riders and compensations for drivers for rides that originate from each of the n locations. Its objective is to maximize the aggregate flow rate of profits across the n locations subject to the drivers' equilibrium constraints. Specifically, the optimization problem takes the following form:

$$\max_{\{p_i, c_i, \delta_i, x_i, y_{ij}\}_{i,j=1}^n} \quad \sum_{i=1}^n \min\{x_i, \theta_i(1 - F(p_i))\} \cdot (p_i - c_i)$$

$$s.t. \;\; \{\delta_i, x_i, y_{ij}\}_{i,j=1}^n \text{ is an equilibrium under } \{p_i, c_i\}_{i=1}^n, \tag{4}$$

where in the objective $(p_i - c_i)$ is the platform's profit margin for a ride originating from location i and $\min\{x_i, \theta_i(1 - F(p_i))\}$ is equal to the total demand of riders that the platform serves at i.

As a first step towards a tractable analysis, we provide an alternative optimization formulation in which we relax the drivers' incentive-compatibility constraints. Subsequently, in Lemma 1, we establish that this is without loss of optimality for any absolutely continuous and strictly increasing distribution $F(\cdot)$ for the riders' willingness to pay. In particular, we focus on the following optimization formulation:

$$\begin{aligned} \max_{\{p_i,\delta_i,x_i,y_{ij},d_i\}_{i,j=1}^n} \quad & \sum_i p_i d_i - w \sum_i \delta_i \\ s.t.\ \ & d_i = \Big(1 - F(p_i)\Big)\theta_i, \text{ for all } i \\ & x_i = \beta \left[\sum_j \alpha_{ji} d_j + \sum_j y_{ji} \right] + \delta_i, \text{ for all } i \\ & \sum_j y_{ij} = x_i - d_i, \text{ for all } i \\ & p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j. \end{aligned} \tag{5}$$

The objective function in (5) consists of two terms: the first is equal to the platform's aggregate revenue flow rate $\sum_i p_i d_i$, where d_i denotes the demand that the platform serves at location i. It can be readily seen that $d_i = \min\{x_i, \theta_i(1 - F(p_i))\}$.[15] Thus, this term coincides with the corresponding revenue term in the objective function of (4).

On the other hand, the second term $w \sum_i \delta_i$ captures the platform's cost rate. Note that the two formulations use different ways to express the cost for serving the platform's induced demand. Specifically, in (4) a cost (compensation) c_i is assigned to each ride whereas in the context of (5) the platform incurs cost w (equal to the outside option) for every driver entering the platform to provide service. In light of the drivers' incentive-compatibility constraints, the latter is a lower bound on the platform's cost at equilibrium, which in turn implies that assuming that the induced demand is the same under both formulations, the objective value of (5) is an upper bound for that of (4). In addition, the constraints in (5) correspond to the equilibrium flow constraints as stated in Subsection 2.1, whereas the drivers' incentive-compatibility constraints as expressed in (3) are relaxed in (5).

The preceding discussion implies that the optimal value for Problem (5) is an upper bound on

[15]The first and third equality constraints, together with the non-negativity constraints in (5), imply that $d_i = \theta_i(1 - F(p_i)) \leq x_i$, i.e., the supply of drivers at each location is sufficient to satisfy the induced demand. This observation implies that $d_i = \min\{x_i, \theta_i(1 - F(p_i))\}$.

It is worthwhile to point out that the condition $\theta_i(1-F(p_i)) \leq x_i$ does not necessarily hold at all feasible solutions of (4). Thus, (5) implicitly imposes an additional constraint on the prices that the platform can choose. However, it is straigthforward to see that this constraint can also be imposed in (4) without loss of optimality, since if this constraint is violated at a feasible solution of (4), another feasible solution with a higher objective value can be obtained by increasing p_i.

the profits the platform can generate using prices that depend on a ride's origin. Lemma 1 below states that there exist compensations $\{c_i\}_{i=1}^n$ that support the optimal solution to optimization problem (5) as an equilibrium, thus solving (5) yields the optimal prices for the platform.

Lemma 1. *Consider optimization problem (5). Suppose that Assumption 1 holds and $F(\cdot)$ is an absolutely continuous distribution that is strictly increasing over its domain. We have:*

(i) If $\{p_i, \delta_i, x_i, y_{ij}, d_i\}_{i,j=1}^n$ is a feasible solution for (5), such that $d_i > 0$ for every i, then there exist compensations $\{c_i\}_{i=1}^n$ such that the tuple $\{\delta_i, x_i, y_{ij}\}_{i,j=1}^n$ constitutes an equilibrium under $\{p_i, c_i\}_{i=1}^n$. Furthermore, the (per-period) cost incurred by the platform under these compensations is equal to $w \sum_i \delta_i$.

(ii) If, in addition, $(1-\beta)w < \bar{z}$, any optimal solution $\{p_i^\star, \delta_i^\star, x_i^\star, y_{ij}^\star, d_i^\star\}_{i,j=1}^n$ for (5) is such that $d_i^\star > 0$ for all i, where recall that $\bar{z}$ denotes the upper bound on the riders' willingness to pay. Conversely, if $(1-\beta)w \geq \bar{z}$, any optimal solution for (5) is such that $\delta_i^\star = d_i^\star = 0$ for all i.

Lemma 1 establishes that when the drivers' outside option is not too high, it is optimal for the platform to set prices and compensations such that it serves some demand at all locations. In addition, given any optimal solution for (5), there exists a set of compensations such that the same solution is optimal for (4) as well.[16] On the other hand, when $(1-\beta)w \geq \bar{z}$, it is always optimal to set $\delta_i^\star = d_i^\star = 0$ for any solution for (5), which, in turn, implies that there is no demand served in any optimal solution for (4). In that sense, it is sufficient to work directly with optimization problem (5) as we can always construct an optimal solution for (4) from an optimal solution for (5).

Although optimization problem (5) is non-convex for a general cumulative distribution function $F(\cdot)$, it can be rewritten as a convex optimization problem for distributions for which the platform's profits are concave in the induced demand $\{d_i\}_{i=1}^n$ (this holds for a number of commonly used distributions, e.g., uniform, exponential, and Pareto). To simplify exposition, in the remainder of the paper (with the exception of Subsection 5.2), we restrict attention to the case where the riders' willingness to pay is uniformly distributed in $[0, 1]$.[17] For this case, we can rewrite the platform's problem as:

$$\begin{aligned} \max_{\{p_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - w\sum_i \delta_i \\ \text{s.t.} \quad & \sum_j y_{ij} = \beta\left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\right] + \delta_i - (1-p_i)\theta_i, \text{ for all } i, \\ & p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \\ & p_i \leq 1, \text{ for all } i, \end{aligned} \tag{6}$$

[16] It is worthwhile to note that there may be several vectors of compensations for the same prices that constitute an equilibrium and lead to the same profits for the platform. For example, we construct such compensations in the proofs of Lemma 1 and in Proposition 3.

[17] Subsection 5.2 establishes the robustness of our main findings to this assumption by reporting computational results on the case when the riders' willingness to pay follows distributions other than uniform.

where we substitute $F(p_i) = p_i$, $d_i = (1-p_i)\theta_i$, and $x_i = \beta\Big[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\Big] + \delta_i$ (and impose $p_i \leq 1$) to obtain a cleaner formulation. It is straightforward to see that the resulting formulation, i.e., optimization problem (6), is convex (in particular, it is a quadratic problem with linear constraints) and, consequently, it can be solved in a computationally efficient way.

3 Spatial Pricing and the Platform's Profits

Our main goal in Section 3 is to explore how the demand pattern $(\mathbf{A}, \boldsymbol{\theta})$ affects the prices set by the platform and, consequently, its profits and consumer surplus.[18] To this end, in Subsection 3.1 we introduce a notion that captures how "balanced" the demand pattern is across the network's locations and establish that demand balancedness is closely related to the profit potential of a network. In particular, we show that the closer a demand pattern is to being balanced (in a way that we formalize in terms of the pattern's structural properties), the higher are the platform's profits. Furthermore, a similar insight holds for consumer surplus (when the platform uses profit maximizing prices and compensations). This set of results clearly showcases how imbalances in the demand for rides across a network may affect a platform's operations and profits. Furthermore, they illustrate how spatial price discrimination may be helpful (at least partially) in dealing with those imbalances.

Subsection 3.2 illustrates the results in Subsection 3.1 in the context of a class of networks that range from the *star* (where a central location is disproportionately more likely to be the destination of any given ride) to the *complete* network (where demand is balanced across locations). For this class of networks, we show that profits and consumer surplus increase as the underlying network gets closer to the complete network, i.e., as the associated demand pattern becomes more balanced.

3.1 Profits and Consumer Surplus

The goal of this subsection is to characterize how the underlying demand pattern shapes the platform's optimal pricing policy and profits. Our first step toward this goal is to formalize the notion of a "balanced" demand pattern.

Definition 1 (Balanced Demand Pattern)**.** Demand pattern $(\mathbf{A}, \boldsymbol{\theta})$ is balanced for a given β if

$$(\beta \mathbf{A}^T - \mathbf{I})\boldsymbol{\theta} \leq 0. \tag{7}$$

Furthermore, if inequality (7) holds for every $\beta \in (0,1)$, we say that the demand pattern is *strongly* balanced.

[18] Throughout the section we restrict attention to prices and compensations that depend on the origin of a ride, i.e., we assume that the platform optimizes over $\{p_i, c_i\}_{i=1}^n$. Thus, when we say *optimal prices* and *compensations* we refer to the optimal tuple $\{p_i, c_i\}_{i=1}^n$ derived by solving (4). Section 4 considers a larger class of pricing/compensation schemes.

The notion of "balancedness" introduced in Definition 1 is closely related to a set of flow balance constraints *when the entire potential demand is served.*[19] In words, a demand pattern is said to be balanced if, at each location i, the potential demand for rides θ_i weakly exceeds $\beta[\mathbf{A}^T\boldsymbol{\theta}]_i = \beta\sum_j \theta_j\alpha_{ji}$, i.e., the supply of drivers that would be available at i after completing rides when all potential demand is served at every location. As an illustrative example, note that when $\boldsymbol{\theta} = \mathbf{1}$, the *complete* network in which all destinations are equally likely for a ride originating from any location in the network, i.e., $\alpha_{ij} = 1/(n-1)$ for all i, j with $i \neq j$, is balanced—in fact, strongly balanced. On the other hand, the *star* network, where rides originating from any location $i \neq 1$ have location 1 as their destination, and rides originating from location 1 are equally likely to have any other location $i \neq 1$ as their destination, is an example of an unbalanced demand pattern (for sufficiently high β).

Proposition 2 establishes that Definition 1 succinctly characterizes the set of demand patterns for which the platform can achieve its maximal profits out of all demand patterns with the same population of riders at each location. Recall that throughout our analysis in Section 3 we assume that the riders' willingness to pay is uniformly distributed in $[0, 1]$.

Proposition 2. *The platform's optimal prices and compensations satisfy the following properties:*

(a) If $(1-\beta)w \geq 1$, it is optimal for the platform not to serve any demand at any of the network's locations.

(b) If $(1-\beta)w < 1$, we obtain the following for the platform's optimal prices and compensations

(i) The profits corresponding to a balanced demand pattern are the highest among those achieved by any demand pattern with the same vector of potential riders $\boldsymbol{\theta}$.

(ii) Under a balanced demand pattern, the platform maximizes its profits by setting the same price at all locations. The optimal price is given by

$$p_i^\star = \frac{1}{2} + \frac{(1-\beta)w}{2}, \text{ for all } i.$$

In addition, it is optimal to offer the same compensation for drivers at all locations, i.e.,

$$c_i^\star = (1-\beta)w, \text{ for all } i.$$

[19]The relationship between "balancedness" and flow balance can be best illustrated in the case of strongly balanced demand patterns. Note that, since $\mathbf{A}$ and $\boldsymbol{\theta}$ have nonnegative entries, inequality (7) holds for any $\beta \in (0,1)$ if and only if it holds for $\beta = 1$. Therefore, to characterize strong balancedness, it suffices to consider (7) for $\beta = 1$. Given that matrix $\mathbf{A}$ is row stochastic, if (7) holds for $\beta = 1$, then it must hold with equality, i.e., $\theta_i = [\mathbf{A}^T\boldsymbol{\theta}]_i$ for all i. If the entire potential demand for rides, i.e., $\boldsymbol{\theta}$, is served, each location i has θ_i units of supply leaving to serve rides (outflow), and $[\mathbf{A}^T\boldsymbol{\theta}]_i$ units of supply arriving after completing a ride (inflow). The discussion above implies that these quantities must be equal for (7) to hold (when $\beta = 1$). In other words, a network is strongly balanced if and only if the flow balance conditions hold at each location when the entire potential demand for rides, i.e., $\boldsymbol{\theta}$, is served. When $\beta < 1$, given that a mass of new drivers enters the platform and replaces those that exit, the results that follow rely on a weaker notion of balanced demand patterns (where inequality (7) can be strict).

The corresponding equilibrium outcome $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ *is such that:*

$$x_i^\star = \theta_i(1-p_i^\star) = \theta_i\left(\frac{1}{2} - \frac{(1-\beta)w}{2}\right) \text{ and } \delta_i^\star = x_i^\star - \beta\sum_j \alpha_{ji}x_j^\star, \text{ for all } i,$$

whereas $y_{ij}^\star = 0$ *for all* i, j.

(iii) Under a balanced demand pattern drivers are never idle, i.e., for the length of time they provide service on the platform they always get assigned to a ride.

The characterization in Proposition 2 is a function of the drivers' outside option w, i.e., the cost of the platform's labor supply. As expected, the platform finds it optimal not to provide any service when w takes large values. In contrast, when w is sufficiently small, the platform's optimal prices and compensations induce an equilibrium outcome in which drivers are always busy.

As a side remark, note that the optimal price $p_i^\star$ as prescribed in Proposition 2 is equal to the price that maximizes the profits the platform generates from location i, assuming that the cost of a ride for the platform is equal to $(1-\beta)w$. In turn, the latter quantity is the per-period compensation rate that guarantees that a driver makes w during her lifetime (in expectation) given that she is always busy in the induced equilibrium, i.e., she is assigned to a rider at every time period.

Unbalanced Demand Patterns. Although setting the same price at all locations turns out to maximize profits when the demand pattern is balanced, this is not necessarily the case for demand patterns for which inequality (7) does not hold. As we illustrate in the discussion that follows, the platform finds it optimal to set prices differently depending on a location's relative likelihood of being the destination for a requested ride. For the remainder of the section, we make the following assumption.

Assumption 2. All locations have the same mass of potential riders, which we normalize to one, i.e., $\boldsymbol{\theta} = \mathbf{1}$. Furthermore, the drivers' outside option w is equal to one.

Assuming that all locations have an equal mass of potential riders allows us to squarely focus on supply and demand imbalances that arise from riders' destination preferences and not from ex-ante differences in the mass of potential riders at the network's locations. We start by providing a characterization of the optimal prices and compensations for the platform.

Proposition 3. *Suppose that Assumption 2 holds and consider an optimal solution* $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ *to optimization problem* (6)*. Let* $\{\lambda_i^\star\}_{i=1}^n$ *denote a set of optimal dual variables corresponding to the equality constraints in* (6)*. Then, the optimal prices take the following form:*

$$p_i^\star = \frac{1 + \lambda_i^\star - \beta\sum_j \alpha_{ij}\lambda_j^\star}{2}. \tag{8}$$

Also, let k be an entry point and ℓ be a location with excess supply for this optimal solution. Then,

$$1 - \frac{\beta}{2} \leq p_k^\star \leq 1 - \frac{\beta^2}{2} \qquad \textit{and} \qquad \frac{1}{2} \leq p_\ell^\star \leq \frac{1+\beta}{2} - \frac{\beta^2}{2}. \tag{9}$$

Finally, the tuple $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ with $x_i^\star = \sum_j \beta[\alpha_{ji}(1-p_j^\star) + y_{ji}^\star] + \delta_i^\star$ constitutes an equilibrium under $\{p_i^\star, c_i^\star\}_{i=1}^n$ where

$$c_i^\star = \lambda_i^\star - \beta \sum_j \alpha_{ij} \lambda_j^\star. \tag{10}$$

As seen from (8) and (10), prices and compensations clearly reflect the marginal value that the platform assigns to an additional driver at each location of the network. Specifically, the dual variable λ_i corresponding to the equilibrium flow constraint for drivers at location i (i.e., the equality constraint in (6)) can be thought of as the marginal value of supply at i for the platform. The price for a ride leaving i turns out to be an affine function of these dual variables (as captured by the term $\lambda_i^\star - \beta \sum_j \alpha_{ij} \lambda_j^\star$ in (8)), i.e., the difference between the value of supply at i and the average value of supply at the destination reached after completing a ride that originated from i (scaled by β). Naturally, these dual variables take smaller values at locations with excess supply (as each additional driver is less valuable at such locations), and larger values at entry points. Consequently, the platform finds it optimal to offer lower prices to riders at locations with excess supply (as this is where additional supply has less value). This can also be seen from (9) by noting that $1 - \beta/2 > (1+\beta)/2 - \beta^2/2$, for $\beta \in (0,1)$. Offering low prices at the aforementioned locations, in turn, increases the demand that the platform serves at these locations, thereby allowing for a higher utilization of drivers who find themselves there after completing a ride.

On the other hand, the platform finds it optimal to set higher prices for rides originating from locations for which the corresponding dual variables have higher values, i.e., locations where the value of supply is higher. In addition, the platform also offers higher compensations to drivers who get assigned to a ride originating from these locations, thus increasing their incentives to provide service there. As a side remark, note that under the combination of prices and compensations provided in Proposition 3, the platform has a positive profit margin (bounded below by $\beta^2/2$ as can be seen from Lemma A.1 in Appendix A) for every ride that it facilitates. Moreover, as is clear from Expression (9) the set of entry points does not overlap with the set of locations with excess supply under the optimal prices and compensations.

Complementary to Proposition 3, Corollary 1 provides a (partial) characterization of entry points and locations with excess supply in terms of the primitives of the demand pattern, i.e., matrix $\mathbf{A}$. The corollary further suggests that imbalances in the riders' destination preferences drive the platform's optimal pricing decisions. To state the corollary, we let $\kappa_i(\mathbf{A}) = \sum_j \alpha_{ji}$, i.e., $\kappa_i(\mathbf{A})$ is equal to matrix $\mathbf{A}$'s i-th column sum (note that since $\boldsymbol{\theta}$ is normalized to $\mathbf{1}$, κ_i is also equal to the total mass of riders who wish to reach destination i).

Corollary 1. *Suppose that Assumption 2 holds and consider the demand pattern $(\mathbf{A}, \mathbf{1})$. Then, in the equilibrium induced by the profit maximizing prices and compensations:*

(i) If $\kappa_i(\mathbf{A}) > 1/\beta^3$, location i has excess supply.

(ii) If $\kappa_i(\mathbf{A}) < \beta$, location i is an entry point.

This result suggests that when location i is a relatively popular destination for rides (i.e., when κ_i is large), the incoming supply of drivers exceeds the demand induced by the price set at i under the optimal pricing/compensation policy. Consequently, such a location ends up with an excess supply of drivers, and, as we show in Proposition 3, the platform finds it optimal to offer lower prices for rides originating from there. On the other hand, when κ_i is relatively small, the converse holds; thus, serving demand at i necessitates additional entry of drivers and prices are set to higher levels.

Platform's Profits. Given that balanced demand patterns lead to the highest profits for the platform (as shown in Proposition 2), intuitively it can be expected that the more balanced a demand pattern is, the higher profits the platform can generate. Theorem 1 below formalizes this intuition.[20] We state our result, by using the shorthand notation $\Pi(\mathbf{A}, \mathbf{1})$ to denote the profits corresponding to the platform's optimal pricing/compensation policy for demand pattern $(\mathbf{A}, \mathbf{1})$.

Theorem 1. *Suppose that Assumption 2 holds and consider a strongly balanced demand pattern $(\mathbf{D}, \mathbf{1})$ and any other demand pattern $(\mathbf{F}, \mathbf{1})$ (not necessarily balanced). Define the family of demand patterns parameterized by ξ such that:*

$$\mathbf{F}^{\xi} = \xi \mathbf{D} + (1 - \xi)\mathbf{F}. \tag{11}$$

Then, $\Pi(\mathbf{F}^{\xi}, \mathbf{1})$ is (weakly) increasing in ξ.

Theorem 1 allows for comparing unbalanced demand patterns in terms of their profit potential for the platform as it establishes that the more unbalanced a demand pattern is, the lower the corresponding profits for the platform are (for demand patterns that belong to the convex combination of a strongly balanced demand pattern and another arbitrary one). Thus, Theorem 1 and Proposition 2 jointly illustrate the impact of the underlying demand pattern on profits.

We complement Theorem 1 by providing an additional result that allows for comparisons between demand patterns in terms of their profit potential. To this end, we define sets $S_1(\mathbf{A}) = \left\{ i \middle| \sum_j \alpha_{ji} > 1/\beta^3 \right\}$ and $S_2(\mathbf{A}) = \left\{ i \middle| \beta \leq \sum_j \alpha_{ji} \leq 1/\beta^3 \right\}$ corresponding to demand pattern $(\mathbf{A}, \mathbf{1})$. Note that according to Corollary 1 the former corresponds to a set of locations with excess supply, whereas locations that do not belong to either of the two sets are entry points.

[20] Although we state Theorem 1 for $\boldsymbol{\theta} = \mathbf{1}$, we confirmed computationally that the theorem holds for a general $\boldsymbol{\theta}$ vector.

Proposition 4. *Suppose that Assumption 2 holds and consider demand patterns* $(\mathbf{A}, \mathbf{1})$ *and* $(\mathbf{A}', \mathbf{1})$ *for which the following hold:*

(i) $S_1(\mathbf{A}) = S_1(\mathbf{A}')$ *and* $S_2(\mathbf{A}) = S_2(\mathbf{A}')$.

(ii) $\sum_{i \in S_1(\mathbf{A})}(\beta^2 \kappa_i(\mathbf{A}') - \kappa_i(\mathbf{A})) \geq \sum_{i \in S_2(\mathbf{A})}(\kappa_i(\mathbf{A}) - \beta^2 \kappa_i(\mathbf{A}'))^+$.

Then, the optimal profits corresponding to demand pattern $(\mathbf{A}, \mathbf{1})$ *are higher than those corresponding to* $(\mathbf{A}', \mathbf{1})$.

Proposition 4 considers demand patterns $(\mathbf{A}, \mathbf{1})$ and $(\mathbf{A}', \mathbf{1})$ that have the same sets of entry points and locations with excess supply as identified by Corollary 1; i.e., $\kappa_i(\mathbf{A}') > 1/\beta^3$ if and only if $\kappa_i(\mathbf{A}) > 1/\beta^3$ and, similarly, $\kappa_i(\mathbf{A}') < \beta$ if and only if $\kappa_i(\mathbf{A}) < \beta$. For such demand patterns, the inequality in (ii) implies that $\sum_{i \in S_1} \kappa_i(\mathbf{A}') \geq \sum_{i \in S_1} \kappa_i(\mathbf{A})$ with $S_1 = S_1(\mathbf{A}) = S_1(\mathbf{A}')$, i.e., the aggregate of the column sums for locations in S_1 are larger for $\mathbf{A}'$ than for $\mathbf{A}$. Given that the potential demand for rides leaving any location is the same across the network under Assumption 2 (i.e, when $\boldsymbol{\theta} = \mathbf{1}$), the column sum corresponding to a location captures the potential demand in the network for rides with this location as their destination. Thus, a higher aggregate for the column sums corresponding to locations in S_1 implies higher imbalance in these locations between demand for rides with origin versus destination in S_1. In this sense, inequality (ii) intuitively implies that demand pattern $(\mathbf{A}', \mathbf{1})$ is less balanced than $(\mathbf{A}, \mathbf{1})$. Thus, similarly to Theorem 1, Proposition 4 suggests that more balanced demand patterns lead to higher profits for the platform.[21]

We conclude our discussion on profits by providing two bounds on the difference between the profits corresponding to a general demand pattern and a balanced one.

Proposition 5. *Suppose that Assumption 2 holds and consider any strongly balanced demand pattern* $(\mathbf{D}, \mathbf{1})$ *and any other demand pattern* $(\mathbf{F}, \mathbf{1})$ *(not necessarily balanced). Then,*

$$\Pi(\mathbf{D}, \mathbf{1}) - \Pi(\mathbf{F}, \mathbf{1}) \leq \frac{\beta^2}{2}(\boldsymbol{\lambda}^\star)^T \Big(\mathbf{1} - \mathbf{F}^T \mathbf{1}\Big), \tag{12}$$

where $\{\lambda_i^\star\}_{i=1}^n$ *denote a set of optimal dual variables corresponding to the equality constraints in (6) when the demand pattern is* $(\mathbf{F}, \mathbf{1})$.

Finally, restricting attention to unbalanced demand patterns for which each location satisfies one of the two conditions of Corollary 1 yields the following corollary, which provides a bound that depends only on the modeling primitives and, thus, may be easier to interpret.

[21] Consider demand patterns $(\mathbf{A}, \mathbf{1})$ and $(\mathbf{C}, \mathbf{1})$ whose optimal profits cannot be directly compared using Theorem 1 and Proposition 4 and assume that there exists $(\mathbf{B}, \mathbf{1})$ such that $(\mathbf{A}, \mathbf{1})$ can be written as a convex combination of $(\mathbf{B}, \mathbf{1})$ and a strongly balanced demand pattern and, in addition, $(\mathbf{C}, \mathbf{1})$ and $(\mathbf{B}, \mathbf{1})$ can be compared using Proposition 4, i.e., $S_1(\mathbf{C}) = S_1(\mathbf{B})$ and $S_2(\mathbf{C}) = S_2(\mathbf{B})$ and $\sum_{i \in S_1(\mathbf{B})}(\beta^2 \kappa_i(\mathbf{C}) - \kappa_i(\mathbf{B})) \geq \sum_{i \in S_2(\mathbf{B})}(\kappa_i(\mathbf{B}) - \beta^2 \kappa_i(\mathbf{C}))^+$. Then, the profits corresponding to $(\mathbf{A}, \mathbf{1})$ are higher than those corresponding to $(\mathbf{C}, \mathbf{1})$. Thus, we can leverage Theorem 1 and Proposition 4 to compare the profit potential of a rich set of demand patterns.

Corollary 2. *Suppose that Assumption 2 holds and consider any strongly balanced demand pattern* $(\mathbf{D}, \mathbf{1})$ *and any other demand pattern* $(\mathbf{F}, \mathbf{1})$ *(not necessarily balanced). Further, assume that* $(\mathbf{F}, \mathbf{1})$ *is such that the column sum corresponding to a location* i *satisfies either* $\kappa_i(\mathbf{F}) < \beta$ *(i.e.,* i *is an entry point) or* $\kappa_i(\mathbf{F}) > 1/\beta^3$ *(i.e.,* i *has excess supply). Then, we have*

$$\Pi(\mathbf{D}, \mathbf{1}) - \Pi(\mathbf{F}, \mathbf{1}) \leq \frac{\beta^2}{2}(1-\beta) \sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (\kappa_i(\mathbf{F}) - 1). \tag{13}$$

To gain some intuition on the bound on the profit difference between balanced and unbalanced demand patterns given in Expression (13), note that when $\boldsymbol{\theta} = \mathbf{1}$ the column sum κ_i associated with location i measures the extent to which the location is unbalanced; i.e., it measures how large the demand for rides with this location as destination (captured by the column sum κ_i corresponding to the location) is relative to the demand for rides for which it is an origin (which is equal to one since $\theta_i = 1$). Expression (13) implies that the profit difference can be upper bounded in terms of $\sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (\kappa_i(\mathbf{F}) - 1)$, which can be viewed as an aggregate measure of how unbalanced the demand pattern $(\mathbf{F}, \mathbf{1})$ is. Intuitively, Corollary 2 suggests that the profits for the platform under the optimal pricing and compensation policy can be substantially lower than those under a balanced demand pattern only when demand pattern $(\mathbf{F}, \mathbf{1})$ is highly unbalanced.

Consumer Surplus. In the final part of the subsection, we leverage the dual formulation of the platform's profit maximization problem (6) and establish a result analogous to Theorem 1 for aggregate consumer surplus. First, we provide the definition of aggregate consumer surplus for a given vector of prices $\mathbf{p}$, assuming that there is sufficient supply to meet the induced demand, i.e., the mass of riders who get assigned to drivers at location i is equal to $(1 - p_i)$.

Definition 2 (Consumer Surplus)**.** Consider the vector of prices $\mathbf{p}$ set by the platform and assume that the induced demand is satisfied. Then, when the riders' willingness to pay for a ride follows the uniform distribution, aggregate consumer surplus, denoted by CS, is given by:[22]

$$CS = 1/2(\mathbf{1} - \mathbf{p})^T(\mathbf{1} - \mathbf{p}).$$

We emphasize that consumer surplus at the profit maximizing prices takes a particularly simple form when the riders' willingness to pay is uniformly distributed; i.e., it is equal to half the platform's aggregate profits. As a direct consequence of that, Corollary 3 states that aggregate consumer

[22]For a general differentiable cumulative value distribution $F(\cdot)$ and mass of potential riders θ_i, the consumer surplus at location i under price p_i can be expressed as follows: $CS_i = \theta_i \int_{p_i}^{\infty} (v - p_i) f(v) dv$, where $f(z) = \frac{dF(z)}{dz}$. In turn, the consumer surplus for the entire network is given by $\sum_i CS_i$. When $\theta_i = 1$ for all i and the value distribution (distribution of the willingness to pay) is uniform, we obtain the expression in Definition 2.

surplus at the profit maximizing prices decreases as the demand pattern across the n locations becomes less balanced. It is worthwhile to note that the losses in consumer surplus are not uniformly distributed in the network as, in general, the platform sets different prices for different locations (this is illustrated in Figure 2).

Corollary 3. *Suppose that Assumption 2 holds. Consider a strongly balanced demand pattern* $(\mathbf{D}, \mathbf{1})$ *and any other demand pattern* $(\mathbf{F}, \mathbf{1})$ *(not necessarily balanced). Then, the aggregate consumer surplus under the platform's optimal prices for demand pattern* $(\mathbf{F}^{\xi}, \mathbf{1})$ *is (weakly) increasing in* ξ.

3.2 Star-to-Complete Networks

The present subsection focuses on the family of *star-to-complete* networks, which provides a simplified setting to illustrate the results obtained above on the way optimal profits, consumer surplus, and prices vary as a function of the demand pattern. In particular, our goal is to study how these quantities change as the underlying network shifts from being a star to being a complete network.

Formally, we consider a class of demand patterns $(\mathbf{A}^{\xi}, \mathbf{1})$ on $n \geq 3$ locations parameterized by scalar $\xi \in [0, 1]$. The relative frequencies of rides originating from location i and ending at j are succinctly summarized by matrix $\mathbf{A}^{\xi}$ such that

$$\mathbf{A}^{\xi} = \xi \mathbf{A}^{C} + (1 - \xi)\mathbf{A}^{S},$$

where

$$\mathbf{A}^{C} = \begin{pmatrix} 0 & 1/(n-1) & \cdots & 1/(n-1) \\ 1/(n-1) & 0 & \cdots & 1/(n-1) \\ \vdots & \vdots & \ddots & \vdots \\ 1/(n-1) & 1/(n-1) & \cdots & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}^{S} = \begin{pmatrix} 0 & 1/(n-1) & \cdots & 1/(n-1) \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}. \tag{14}$$

In other words, $\mathbf{A}^{\xi}$ is a convex combination of $\mathbf{A}^{C}$ (the complete network) in which all destinations are equally likely for a ride originating from any location in the network, i.e., $\alpha_{ij} = 1/(n-1)$ for all i, j with $i \neq j$, and $\mathbf{A}^{S}$ (the star network) where rides originating from any location $i \neq 1$ have location 1 as their destination, and rides originating from location 1 are equally likely to have any other location $j \neq 1$ as their destination. For that reason, with some abuse of terminology, we refer to location 1 as the center and to the rest of the locations as the leaves.

Intuitively, parameter ξ captures how "balanced" the demand pattern described by (14) is. Note that the sum $\sum_j \mathbf{A}^{\xi}_{j\ell}\theta_j = \sum_j \mathbf{A}^{\xi}_{j\ell}$ is equal to the aggregate mass of potential riders with location ℓ as their destination (recall that we assume $\boldsymbol{\theta} = \mathbf{1}$). For $\xi < 1$, location 1 is a relatively more attractive destination than the rest (in the sense that it is the destination for the largest mass of potential riders, i.e., $\sum_j \mathbf{A}^{\xi}_{j1} > \sum_j \mathbf{A}^{\xi}_{j\ell}$ for all $\ell \neq 1$). Furthermore, as ξ decreases, the mass of potential riders going to location 1 increases, and, consequently, the difference between the attractiveness of the center as a destination and the rest of the locations also increases. Conversely, when $\xi = 1$, the demand pattern is strongly balanced across the network's locations since every location is a destination for an equal mass of potential riders, i.e., for any location ℓ, we have $\sum_j \mathbf{A}^{\xi}_{j1} = \sum_j \mathbf{A}^{\xi}_{j\ell}$.

Proposition 6 below provides a characterization of the platform's optimal prices as a function of scalar ξ.

Proposition 6. *Suppose that Assumption 2 holds. In addition, assume that the demand pattern across the n locations is given by $(\mathbf{A}^{\xi}, \mathbf{1})$. Then, under the optimal prices:*

(i) The platform's profits and consumer surplus are increasing in ξ.

(ii) Prices at the leaves are identical to each other and are higher than the price at the center. Furthermore, the price at the center is increasing whereas prices at the leaves are decreasing in ξ.

(iii) The demand served at the center (at the leaves) is decreasing (increasing) in ξ.

The first part of Proposition 6 follows directly from Theorem 1 and Corollary 3. In addition, in Appendix B we provide a closed-form characterization of the optimal prices, profits, and consumer surplus corresponding to demand pattern $(\mathbf{A}^{\xi}, \mathbf{1})$, which, in turn, establishes the remaining parts of the proposition. Figure 1 illustrates the platform's optimal profits, consumer surplus, and prices as a function of ξ. As a final remark, note that although, in general, there may exist multiple compensation vectors that support the same prices and profits for the platform at equilibrium, expressions (8) and (10) imply that for the compensation policy provided in Proposition 3, we have $p_i^\star = 1/2 + c_i^\star/2$; i.e., prices and compensations follow the same trend as ξ increases.

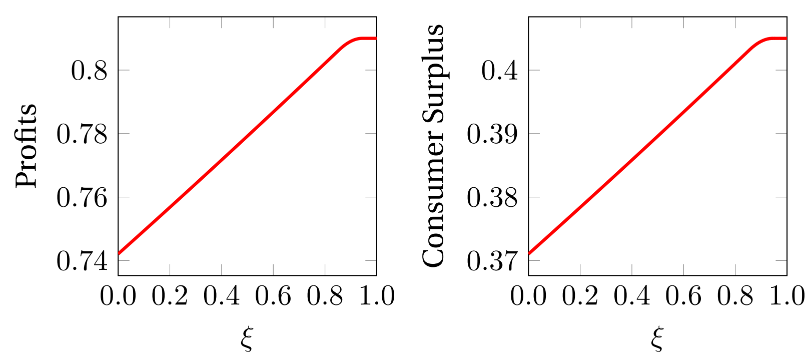


Figure 1: Profits and induced consumer surplus corresponding to the platform's optimal prices/compensations $\{p_i^\star, c_i^\star\}$ for the class of star-to-complete networks with $n = 4$ locations, $w = 1$, and $\beta = 0.9$.

As implied by the proof of the proposition and illustrated in Figure 1, there are three regimes for the platform's optimal prices as a function of ξ. In the first, starting with $\xi = 0$, the demand pattern takes the form of a star-like structure with most riders requesting a ride to location 1. In this

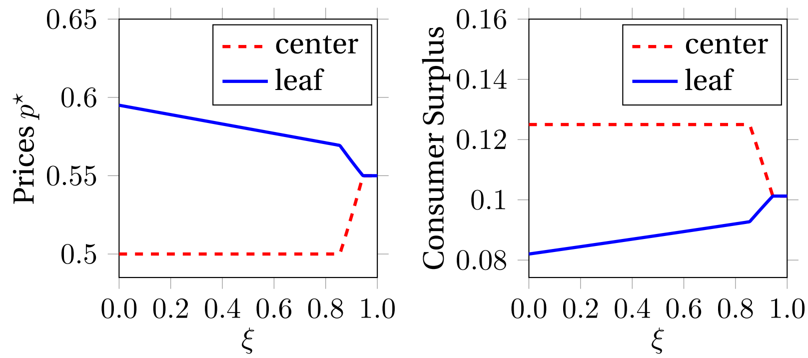


Figure 2: Prices and induced consumer surplus corresponding to the platform's optimal prices/compensations $\{p_i^\star, c_i^\star\}$ at the center and a leaf location for the class of star-to-complete networks with $n = 4$ locations, $w = 1$, and $\beta = 0.9$.

regime, the price at the center (location 1) is equal to $1/2$, which is the price that would maximize the platform's profits at that location, assuming that serving a rider was costless for the platform. On the other hand, the price at the leaves is higher. Intuitively, when ξ is small, location 1 is a considerably more attractive destination to riders than the rest. Then, the platform finds it optimal to set a relatively high price at the leaves and, consequently, limit the incoming supply of drivers to the center. Despite this, at equilibrium, there is an excess supply of drivers at the center and as a result riders leaving that location get a "subsidy." That is, the price at the center is lower than the rest of the network, and thus a higher fraction of location 1's overall demand is served. Finally, there is positive probability that a driver does not get assigned to a ride at the center, i.e., the equilibrium induced by the optimal prices is such that the supply of drivers at the center exceeds the mass of riders who are willing to pay for a ride.

At the other extreme, i.e., when ξ takes large values, the demand pattern is balanced across the network's locations. Therefore, there is no need to use prices as an instrument to deal with supply/demand imbalances. As a result, the optimal price induces an equilibrium in which drivers are never idle and prices across the network are equal. This regime is the one in which the platform maximizes its profits. It is also the regime in which aggregate consumer surplus is maximized (as we establish in Proposition 2 and Corollary 3 respectively).

Finally, in the third regime, which corresponds to intermediate values of ξ, the platform still limits the number of rides to the center by setting a higher price at the leaves. In addition, riders at the center are again favored by a relatively lower price. However, unlike in the first regime, here the equilibrium induced by the platform's optimal prices is such that no driver is ever idle at any of the network's locations.

In sum, this class of demand patterns illustrates that when a subset of locations are relatively more attractive as destinations than others, the platform sets out to balance the (endogenous) supply of available drivers with rider demand by setting lower prices and serving more demand at these "popular" destinations and, conversely, setting relatively higher prices in the rest of the network.

4 Alternative Pricing/Compensation Schemes

So far, we have considered a pricing scheme that features different prices and compensations depending on the location a ride originates from. The present section studies alternative pricing and compensation schemes, variants of which are used in practice. In particular, we first explore the widely adopted scheme that compensates drivers with a fixed ratio of the revenues they generate for the platform. Then, we formulate the platform's pricing problem and simulate its profits for demand patterns that capture imbalances in real-world networks under three pricing schemes: having the same price per ride across the network (much like in the taxi industry), price discriminating riders based on the origin of their ride, and, finally, discriminating them based on both their origin and desired destination. The main goal of this section is to shed light on the benefits of spatial price discrimination in practice, i.e., setting the price/compensation for a ride as an explicit function of its origin and/or destination. Throughout this section we again impose the assumption that the riders' willingness to pay is uniformly distributed.

4.1 Fixed Commission Rates

The ride-sharing industry has predominantly adopted fixed commission rates, i.e., the compensation for a driver is a fixed ratio of the total fare (typically in the order of 75%–80%) paid by the rider.[23] Here, we discuss whether and when such a scheme (which may be simpler to communicate to drivers) performs well relative to the compensation scheme we studied in Section 3, which is such that a compensation for a ride depends on its origin. In particular, in this section we restrict attention to a setting where the compensation c_i for a driver who gets assigned to a ride at location i is given by γp_i, where p_i is the price for the ride and $\gamma \in [0, 1]$ is the fraction of the fare that is given out to the driver. We study the platform's profits when it optimally determines the vector of prices $\{p_i\}_{i=1}^n$ and the fraction γ of the fare that is given to the driver as compensation.

Solving for the optimal prices and commission rate γ analytically is, in general, a challenging task. This is mainly due to the fact that the set of vectors that satisfy the equilibrium conditions is non-convex.[24] Despite the fact that the platform's optimization problem under the assumption that the commission rate is the same across locations is non-convex, there are instances where it is tractable. For instance, in a setting with a single location, one can implement the optimal solution $(p_1^\star, c_1^\star)$ described in Section 3 by setting $\gamma = c_1^\star / p_1^\star$. In other words, in a single location, the drivers' incentive-compatibility constraints can be relaxed and the platform's optimization problem simplifies to that studied in Section 3. More generally, we identify two classes of networks for which

[23]The following webpage explains Lyft's commission structure: https://help.lyft.com/hc/en-us/articles/213815618.

[24]This can be readily illustrated in a setting with a single location, location 1, with $\theta_1 = 1$. Note that if there is any demand served at location 1 the following condition has to be satisfied for drivers to enter and provide service: $\gamma p_1(1 - p_1)/x_1 + \beta V_1 = w = V_1$ or, equivalently, $\gamma p_1(1 - p_1)/x_1 = w(1 - \beta)$. Thus, the (p_1, x_1, γ) tuples that satisfy this constraint can be written as solutions to a nonlinear equation and, consequently, their set is non-convex.

the equilibrium described in Section 3 can be implemented with a fixed commission rate leading to the same profits and demand served at each location. In particular, we first state that there is no loss by imposing a fixed commission rate when the underlying demand pattern is balanced (see Definition 1).

Corollary 4. *Consider a balanced demand pattern . Then, using a fixed commission rate is without loss of optimality for the platform.*

This corollary follows directly from the optimal solution to optimization problem (6) as given in Proposition 2. In this solution, all locations feature the same price and compensation, i.e., $p_i = p_j$ and $c_i = c_j$. Thus, the equilibrium outcome can be implemented using a fixed commission rate $\gamma = c_i/p_i$.

In addition, Proposition 7 identifies another class of demand patterns, which we call *two-type* demand patterns, for which there is also no profit loss when using a fixed commission rate.

Definition 3. We say that $(\mathbf{A}, \boldsymbol{\theta})$ belongs to the class of *two-type* demand patterns if the network's locations can be partitioned into two subsets $\mathcal{N}_1, \mathcal{N}_2$ such that:

(i) $\theta_i = \theta_j$ for every $i, j \in \mathcal{N}_1$ or $i, j \in \mathcal{N}_2$,

(ii) $\sum_{k\in\mathcal{N}_1} \alpha_{ik} = \sum_{k\in\mathcal{N}_1} \alpha_{jk}$ and $\sum_{k'\in\mathcal{N}_2} \alpha_{ik'} = \sum_{k'\in\mathcal{N}_2} \alpha_{jk'}$, for all $i, j \in \mathcal{N}_\ell$ and $\ell \in \{1, 2\}$,

(iii) $\sum_{k\in\mathcal{N}_1} \alpha_{ki} = \sum_{k\in\mathcal{N}_1} \alpha_{kj}$ and $\sum_{k'\in\mathcal{N}_2} \alpha_{k'i} = \sum_{k'\in\mathcal{N}_2} \alpha_{k'j}$, for all $i, j \in \mathcal{N}_\ell$ and $\ell \in \{1, 2\}$.

Essentially, networks that belong to the class of two-type demand patterns are such that their locations can be partitioned into two subsets $\mathcal{N}_1, \mathcal{N}_2$, so that any two locations in the same subset look exactly the same in terms of their population of riders (item (i)), the demand for rides leaving the locations towards destinations in $\mathcal{N}_1$ and $\mathcal{N}_2$ (item (ii)), and the incoming demand from origins in $\mathcal{N}_1$ and $\mathcal{N}_2$ (item (iii)). Given the definition above, we establish the following proposition.

Proposition 7. *Consider a two-type demand pattern and assume that $w = 1$. Then, using a fixed commission rate is without loss of optimality for the platform.*

The following corollary follows directly from the definition of two-type demand patterns and Proposition 7 (for the sake of brevity, we omit the proof).

Corollary 5. *Consider the class of star-to-complete networks. Then, using a fixed commission rate is without loss of optimality for the platform.*

Although Corollary 4, Proposition 7, and Corollary 5 provide some justification for the widespread use of fixed commission rates, their optimality, as we argue next, is not guaranteed for richer network structures. In particular, in Figure 3 we provide a simple example that illustrates the potential drawbacks of such a compensation scheme. We show that although the network consists of only

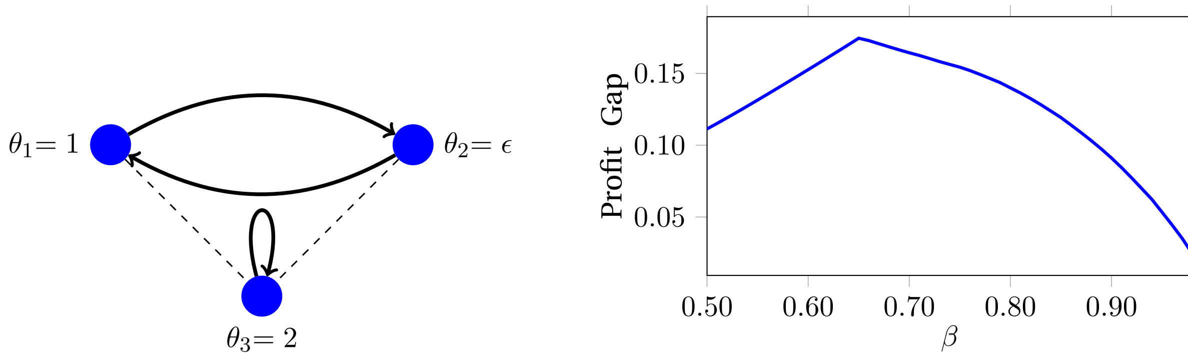


Figure 3: A simple network with three locations. The total demand at each location is equal to $\theta_1 = 1, \theta_2 = \epsilon \ll 1$, and $\theta_3 = 2$ respectively. Finally, $\alpha_{12} = \alpha_{21} = \alpha_{33} = 1$, i.e., riders at location 1 want to go to location 2 and vice versa, whereas rides that originate from location 3 end up at the same location. The plot on the right illustrates the profit gap as a function of parameter β when drivers' compensation per ride is a fixed fraction of the fare. For example, the gap in profits is 16.5$\%$ for $\beta = 0.7$, 14$\%$ for $\beta = 0.8$, and 9$\%$ for $\beta = 0.9$.

three locations, the gap in the profits corresponding to fixed commission rates compared to those generated with the compensation scheme of Section 3 can reach 10–15$\%$ under reasonable modeling parameters.[25] To some extent, this is the simplest example that violates the conditions of Corollary 4 and Proposition 7, i.e., it does not belong to the class of two-type demand patterns and it is not balanced. This further underscores the need to carefully account for the network structure imposed by the demand pattern when evaluating the trade-offs associated with different pricing/compensation schemes.

4.2 Comparing Different Pricing Schemes

Having established the shortcomings of fixed commission rates, in the present subsection we compare the profits generated by the platform under the following three pricing schemes:

(i) **Single price.** The platform sets the same price p for all rides irrespective of their origin or destination (the platform may use different compensations for drivers depending on a ride's origin). The optimal price p for a ride and the vector of compensations $\{c_i\}_{i=1}^n$ can be obtained by solving optimization problem (6) with the additional constraint that $p_i = p$ for all i.[26]

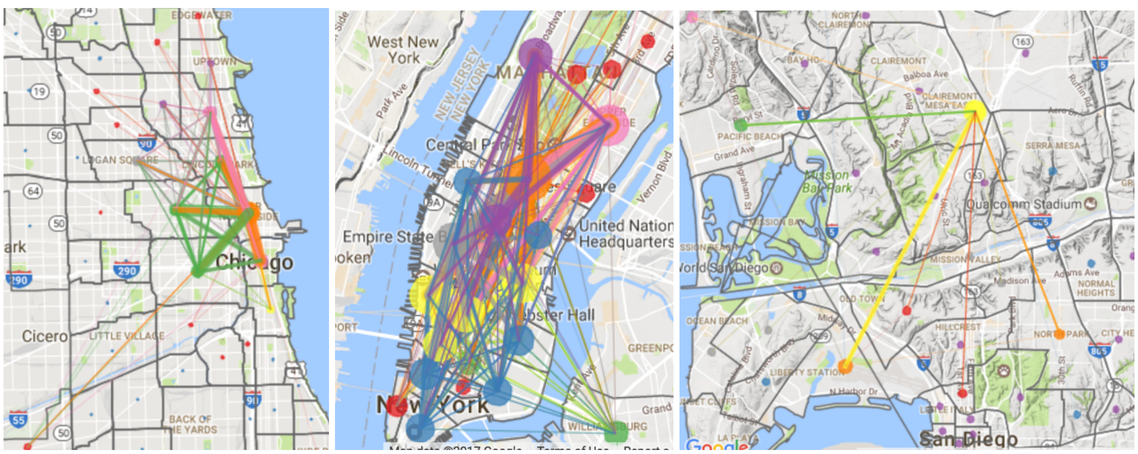
(ii) **Origin pricing.** The platform optimizes over $\{p_i, c_i\}_{i=1}^n$ by solving (6), where p_i and c_i denote the price and compensation for a ride that originates from location i regardless of its destination.

[25] We provide additional details on how we compute the platform's optimal profits for the network depicted in Figure 3 in Appendix C.2.

[26] Note that a quick inspection of the proof of Lemma 1 directly implies that as long as the platform can choose potentially different compensations for each of the locations, solving optimization problem (6) with the additional constraint that $p_i = p$ for all i generates the optimal solution for the platform, since the optimal single price can be supported by appropriately chosen compensations.

(iii) **Origin-destination pricing.** The platform optimizes over $\{p_{ij}, c_{ij}\}_{i,j=1}^{n}$, i.e., the price and compensation for a ride may be a function of both its origin and its destination. It turns out that the platform's decision problem can be formulated as a convex program similarly to (6). We provide the details and an extensive discussion in Appendix C.3.

The single price scheme is a special case of origin pricing, which itself is a special case of origin-destination pricing. Thus, profits for the platform are highest for origin-destination pricing and lowest when the platform uses the same price across the network. Next, we quantify the benefits of pricing rides differently depending on where they originate from (origin pricing) relative to using the same price for the entire network (single price). Furthermore, we explore whether setting prices as a function of both the origin and the destination of a ride generates any additional value for the platform relative to origin pricing.[27]



(a) Chicago (b) New York City (c) San Diego

Figure 4: Flows of realized rides by Uber for the cities of Chicago, New York, and San Diego (Source: Uber Newsroom).

The discussion that follows is based on real-world networks/demand patterns, which we obtained from Uber Newsroom.[28] The data include information about the flow of riders in different neighborhoods in a number of major US cities. In particular, for each of Chicago, New York City, and San Diego the data available on the Newsroom include:

- A partition of the city into a number of different locations/neighborhoods. Each such location corresponds to a node in Figure 4. For our simulations, we assume that these different locations are equidistant.

[27]We allow for general w and $\boldsymbol{\theta}$, i.e., we do not restrict attention to $w = 1$ and $\boldsymbol{\theta} = 1$. The optimization formulations corresponding to the pricing schemes we discuss here are all valid for general w and $\boldsymbol{\theta}$.

[28]The data on Chicago, New York City, and San Diego, was obtained from the following URLs: https://newsroom.uber.com/wp-content/uploads/2014/07/uber_chi_connectome_.html, https://newsroom.uber.com/wp-content/uploads/2014/07/uber_nyc_connectome_.html, and https://newsroom.uber.com/wp-content/uploads/2014/07/uber_sd_connectome_.html.

- For each location/node, we also obtain information on the total number of rides that had this location as their origin (represented in the figure by the size of the corresponding node). We use this information as our vector $\boldsymbol{\theta}$. An important caveat here is that we do not have information on uncensored demand but only on realized rides. That said, the realized rides leaving a location can be seen as a reasonable proxy for the potential demand at that location (at least in the context of our numerical study given that our results remain the same when we scale the vector of $\boldsymbol{\theta}$'s by the same factor).
- Finally, we also obtain information on the "weight" of the edges connecting pairs of locations. These weights (which are proportional to the thickness of the edges in Figure 4) give us our second primitive, the matrix $\mathbf{A}$ of origin-destination preferences.

Thus, we can map this information to the primitives of our model, i.e., vector $\boldsymbol{\theta}$ and matrix $\mathbf{A}$ respectively, as we describe above, and obtain the platform's profits under the single price, origin pricing, and origin-destination pricing schemes by solving the corresponding optimization problems.[29] We emphasize that this exercise is only meant to illustrate our findings on data resembling the demand imbalances in real-world networks. To the extent that imbalances in the number of realized rides leaving different locations is a good proxy for the corresponding imbalances in the potential demand for rides at these locations, our numerical study captures the differences in profits for the platform associated with different pricing schemes reasonably well. A comprehensive empirical study of a ride-sharing platform and its interaction with drivers and riders is outside the scope of the present paper (in part due to the fact that the data we have at our disposal is not sufficient for this purpose).[30]

Figure 5 illustrates how the profits corresponding to the three pricing schemes compare to one another as a function of w, the drivers' outside option, for the cities of Chicago, New York, and San Diego. Note that for small values of w, there is little difference in the performance of the three schemes. This is a natural consequence of the fact that when labor is inexpensive (w takes small values) compensating drivers does not have a considerable impact on the platform's profits (note that when $w = 0$ it is optimal to set the same price at each location, i.e., the price that maximizes the platform's profits at a location when serving demand is costless). On the other hand, for large values of w, i.e., $w \geq 1/(1-\beta)$, it is not profitable for the platform to serve any demand (labor is so expensive that, even if a driver remained busy throughout her time on the platform, the revenues she would generate would be lower than her outside option). Thus, in Figure 5 (as well as Figure 6) we report our findings restricting attention to the regime where w is not too large.

[29] To further test the robustness of our analytical findings, we computed numerically the profits corresponding to the three pricing schemes in over 500 instances, where both the vector $\boldsymbol{\theta}$ and matrix $\mathbf{A}$ were generated at random. The simulation results were qualitatively the same as those obtained based on the data from Uber Newsroom; hence, they are omitted from the paper.

[30] For one, the data we obtained from Uber Newsroom includes only aggregate information about the realized rides. We do not have any information about the uncensored demand at each location, the prices set by the platform, the average time that drivers spent providing service, etc.

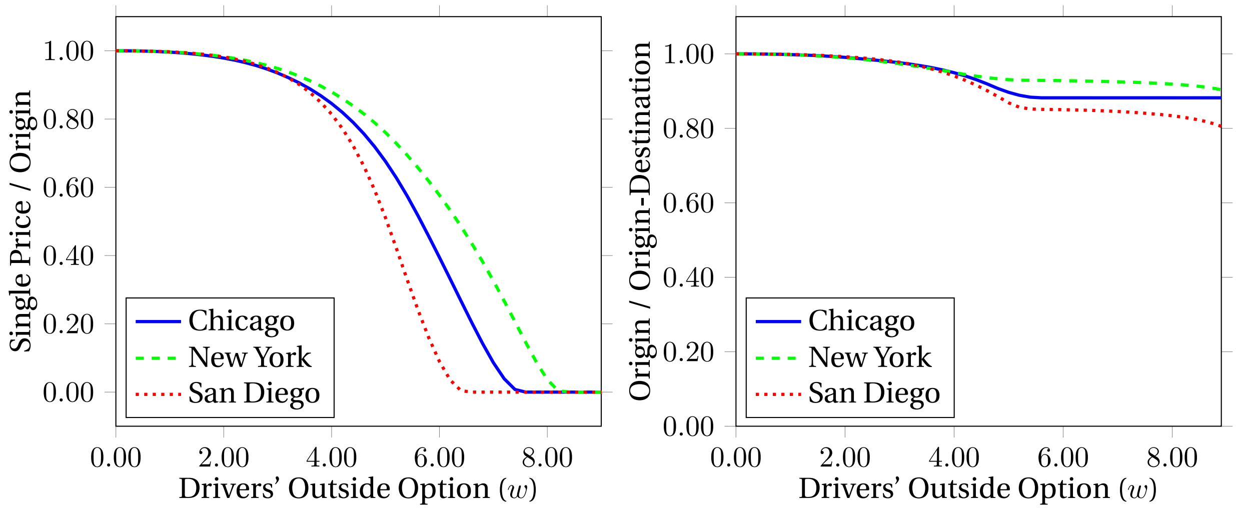


Figure 5: Ratio of profits corresponding to the optimal single price over those corresponding to the origin pricing scheme (*Left*) and ratio of profits corresponding to the origin pricing scheme over those corresponding to the origin-destination pricing scheme (*Right*) for the cities of Chicago, New York, and San Diego as a function of the drivers' outside option w. In both cases, we set $\beta = 0.9$.

As can be seen in Figure 5, as w increases, so does the benefit of using origin pricing relative to a single price, and of using origin-destination pricing relative to origin pricing. For example, for San Diego, the ratio of the profits the platform can generate by using a single price to those it can generate by using the optimal origin pricing scheme is equal to 0.93 for $w = 3$ and is only equal to 0.51 for $w = 5$. It is worthwhile to note that the benefit from using prices that depend on the ride's origin relative to using the same price at all locations is the highest in San Diego (interestingly, as Figure 4 suggests, the demand pattern in San Diego has a star-like structure, i.e., it is unbalanced). In addition, New York City has a more balanced structure and it is associated with the smallest benefit from origin pricing among the three cities we consider. This is in line with our theoretical results that illustrate that spatial price discrimination can serve as a valuable tool for the platform to mitigate supply/demand imbalances across the network's locations.

On the other hand, the additional gain from setting prices that depend on both the origin and the destination of a ride appears to be more modest. One way to explain this is by resorting to Proposition C.1 in Appendix C.3: even though the platform can set n^2 prices (one for each origin-destination pair), the vector of optimal origin-destination prices can be expressed compactly in terms of n dual variables associated with the constraints in the optimization problem for finding the optimal origin-destination prices.[31] In summary, the numerical results depicted in Figure 5 provide

[31] Importantly, when the riders' willingness to pay is different for different origin-destination pairs, origin-destination pricing may yield more substantial benefits to the platform. This can be seen, for instance, by considering a star network with three locations and assuming that at the leaves the mass of potential riders is close to zero (therefore, most of the profits are generated by the center of the star). If the riders at the center who want to go to different leaf locations have the same willingness to pay, it is optimal for the platform to set the same price for both destinations. On the other hand, if they have very different willingness to pay distributions, the pricing problem at the center effectively decouples for riders with different destinations and the platform can improve its profits by offering different prices. We note that the

evidence in favor of using spatial price discrimination and suggest that the benefits are higher when the underlying demand pattern is unbalanced. Furthermore, they also illustrate that using different prices for rides originating from different locations (origin pricing) yields significantly higher profits for the platform relative to using the same price at all locations. On the other hand, the gain from using origin-destination pricing relative to origin pricing is more modest (though still non-negligible and potentially quite relevant in practice, at least for high values for the outside option w).

We conclude the subsection by discussing another natural pricing scheme, which sets prices such that the supply of drivers at each location of the network exactly matches the induced demand, i.e., the market for rides clears at each location (note that matching the supply of drivers with rider demand is always optimal in a single isolated location). In particular, we explore the performance of the following pricing scheme:

(iv) **Local market clearing pricing.** Similar to the origin pricing scheme, the platform optimizes over $\{p_i, c_i\}_{i=1}^n$, where p_i and c_i denote the price and compensation for a ride that originates from location i with the additional constraint that the supply of drivers is equal to the induced demand at each location, i.e., $x_i = 1 - p_i$ is added as a constraint to (6).

Imposing the additional constraint that the induced demand is always equal to the available supply of drivers at each location (and thus, effectively, drivers are always busy) clearly leads to weakly lower profits for the platform than the optimal origin pricing scheme. As can be seen in Figure 6, the difference in profits is more pronounced when the drivers' outside option w takes small values (i.e., labor is inexpensive) as, then, it may be beneficial for the platform to induce an excess supply of drivers at a subset of locations. On the other hand, when labor is relatively expensive, having an excess supply of drivers at any of the network's locations is costly for the platform. Thus, the market clearing pricing scheme that guarantees that drivers remain busy throughout the time they provide service performs reasonably well. Finally, note that setting the same price at all of the network's locations performs well precisely when the market clearing pricing rule does not (when w takes small values).

5 Extensions

This section discusses the natural extension of our benchmark model to the case of networks in which the distances between different pairs of locations may not be equal. In addition, we provide simulation results that illustrate the robustness of our findings to the assumption that the riders' willingness to pay follows the uniform distribution.

convex optimization problem in Appendix C.3 readily extends to this setting after expressing the demand for each origin-destination pair appropriately.

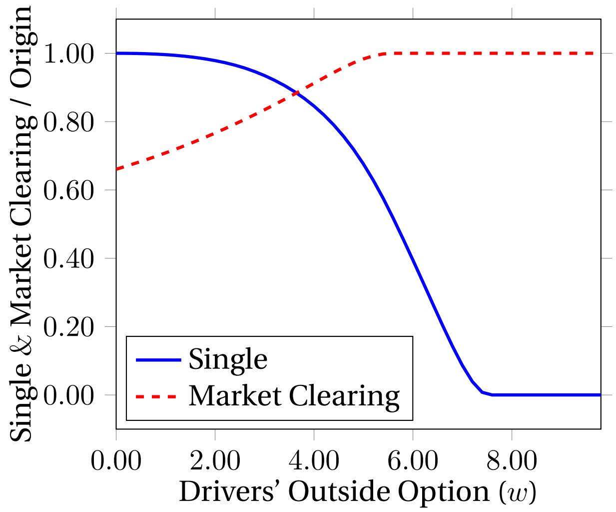


Figure 6: Ratio of profits corresponding to the optimal single price and local market clearing over those corresponding to the origin pricing scheme respectively for the city of Chicago as a function of the drivers' outside option w (here, $\beta = 0.9$).

5.1 Unequal Distances

We consider a network in which the distance between locations $i \neq j$ is given by a positive integer ζ_{ij} with $\zeta_{ij} = \zeta_{ji}$. We also follow the convention that $\zeta_{ii} = 1$ for all i. For simplicity, we assume that the time it takes to complete a ride from i to j is equal to the distance between the two locations, i.e., ζ_{ij}. In addition, we interpret the willingness to pay, price p_i, and compensation c_i for a ride originating from location i in a per unit of time (distance) basis. In other words, a driver who gets assigned to a ride from location i to location j earns $\zeta_{ij} \cdot c_i$ whereas the rider pays to the platform a fare equal to $\zeta_{ij} \cdot p_i$. Finally, a driver exits the platform upon completing a ride from i to j (or relocating from i to j) with probability $(1 - \beta^{\zeta_{ij}})$. This is a natural generalization of our benchmark formulation to the case where locations may not be equidistant, given that $(1 - \beta^{\zeta_{ij}})$ is equal to the probability that a driver would have exited the platform after ζ_{ij} time periods in the model we study in Section 3. In this case, the resulting optimization problem for the platform can be written in a similar way to (6):

$$
\begin{aligned}
\max_{\{p_i, \delta_i, y_{ij}\}_{i,j=1}^n} \quad & \sum_i \sum_j \alpha_{ij} \zeta_{ij} p_i (1 - p_i) \theta_i - w \sum_i \delta_i \\
s.t. \quad & (1 - p_i)\theta_i + \sum_j y_{ij} - \sum_j \beta^{\zeta_{ij}} \left[\alpha_{ji} (1 - p_j) \theta_j + y_{ji} \right] - \delta_i = 0, \text{ for all } i \\
& p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j \\
& p_i \leq 1, \text{ for all } i,
\end{aligned} \tag{15}
$$

where we relax the drivers' incentive-compatibility constraints and we only require that the platform incurs cost equal to w for each driver upon her entry. Optimization problem (15) is convex and, consequently, can be solved efficiently to derive the optimal prices, assuming that the platform has explicit control over the flow of drivers. In general, this is not possible since we assume that drivers choose whether and where to provide service to maximize their expected lifetime earnings. Proposition 3 establishes that when the network's locations are equidistant from one another there exist compensations such that the optimal solution to optimization problem (15) can be implemented as an equilibrium leading to the same profits for the platform, i.e., the relaxation of the drivers' incentive-compatibility constraints is without loss of optimality. However, it is not clear whether this holds for the case where distances between different pairs of locations may not be the same (the proof of Proposition 3 relies on the fact that locations are equidistant from one another). As Proposition 8 establishes, the same holds under Assumption 2 for the case of unequal distances.[32]

Proposition 8. *Suppose that Assumption 2 holds and consider the optimal solution* $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ *for optimization problem (15). Also, let* $\{\lambda_i^\star\}_{i=1}^n$ *denote the optimal dual variables corresponding to the equality constraints in (15). Then, the tuple* $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ *with* $x_i^\star = \sum_j \beta^{\zeta_{ij}}[\alpha_{ji}(1-p_j^\star)\theta_j + y_{ji}^\star] + \delta_i^\star$ *constitutes an equilibrium under (per unit of distance) prices and compensations* $\{p_i^\star, c_i^\star\}_{i=1}^n$*, where*

$$c_i^\star = \frac{1}{\sum_j \alpha_{ij}\zeta_{ij}} \cdot \left(\lambda_i^\star - \sum_j \alpha_{ij}\beta^{\zeta_{ij}}\lambda_j^\star\right). \tag{16}$$

In addition, the expected future earnings for a driver at location i *are equal to the corresponding dual variable, i.e.,* $V_i = \lambda_i^\star$*.*

Implementing the optimal prices and compensations $\{p_i^\star, c_i^\star\}_{i=1}^n$ involves solving convex optimization problem (15) and its dual to obtain the prices and compensations as given by Proposition 8. The per unit of time/distance compensation $c_i^\star$ at the optimal solution for a driver who accepts a ride at location i can be perhaps best understood when we rewrite (16) as

$$c_i^\star \sum_j \alpha_{ij}\zeta_{ij} = \left(V_i - \sum_j \alpha_{ij}\beta^{\zeta_{ij}} V_j\right).$$

Note that the left-hand side of the expression above is equal to the expected compensation associated with a ride leaving location i, since a driver earns $c_i^\star\zeta_{ij}$ for completing a ride from i to j and

[32] In the case of unequal distances, the expressions involved in the equilibrium definition are scaled appropriately according to the ζ_{ij}'s. In particular, Expression (2) turns into:

$$V_i = \min\left\{\frac{(1-p_i)}{x_i}, 1\right\} \sum_j \alpha_{ij}\left(c_i\zeta_{ij} + \beta^{\zeta_{ij}} V_j\right) + \left(1 - \min\left\{\frac{(1-p_i)}{x_i}, 1\right\}\right)\overline{V}_i,$$

where $\overline{V}_i = \max_k \beta^{\zeta_{ik}} V_k$. In addition, Expression (1) becomes: $x_i = \sum_j \beta^{\zeta_{ij}}\left[\alpha_{ji}\min\{x_j, \theta_j(1-p_j)\} + y_{ji}\right] + \delta_i$.

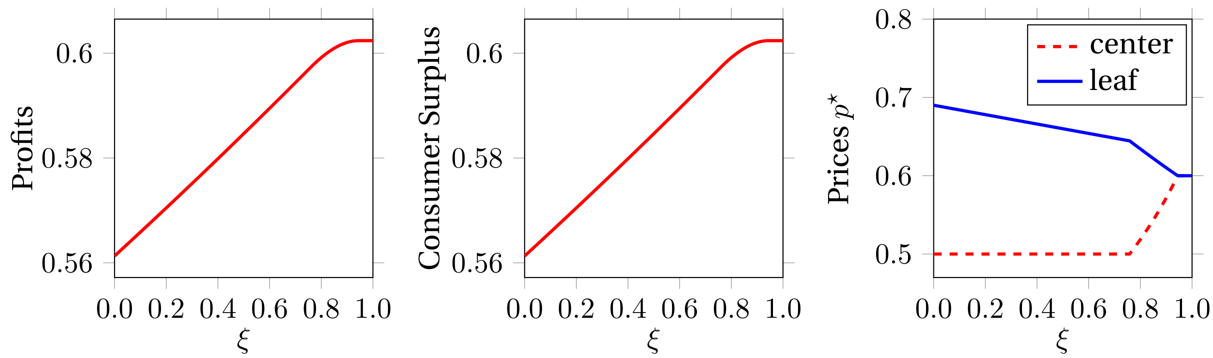


Figure 7: Profits, consumer surplus, and prices corresponding to the platform's optimal origin pricing policy for the class of star-to-complete networks with $n = 4$ locations, $w = 1$, and $\beta = 0.9$. Here, the riders' willingness to pay is distributed according to the exponential distribution with parameter $\lambda = 2$. Note that, in this case, the mean is equal to that of a uniform distribution in $[0, 1]$.

the fraction of rides leaving i with j as their destination is equal to α_{ij}. On the other hand, the right-hand side of the expression captures the difference between the expected future earnings in the origin (i.e., V_i) and the average of the expected future earnings at the destinations for such rides discounted by the time to reach the destination (i.e., $\sum_j \alpha_{ij} \beta^{\zeta_{ij}} V_j$).

5.2 Robustness

We close this section by arguing that the qualitative nature of our results is robust to the assumption that the riders' willingness to pay is uniformly distributed (which was made to simplify the analysis and exposition). Figures 7 and 8 illustrate simulation results for the profits, induced consumer surplus, and prices at different locations corresponding to the platform's optimal origin pricing policy. The underlying networks belong to the class of star-to-complete networks and the riders' willingness to pay follows the exponential and Pareto distributions respectively with the same mean as in the uniform distribution. The figures clearly showcase that our insights regarding profits and induced consumer surplus are robust to the assumptions on the distributions of the riders' willingness to pay and they are consistent with the findings in Subsection 3.2. In addition, we computationally tested and verified the robustness of our theoretical findings with respect to other modeling primitives and, in particular, the drivers' outside option w and the vector of potential riders θ (for the sake of brevity we omit these computational results from the paper).

6 Concluding Remarks

This paper explores the benefits of spatial price discrimination for a ride-sharing platform that serves a network of locations. Potential riders at different locations have possibly different destination preferences, which induce a demand pattern across the network's locations. Given the prices

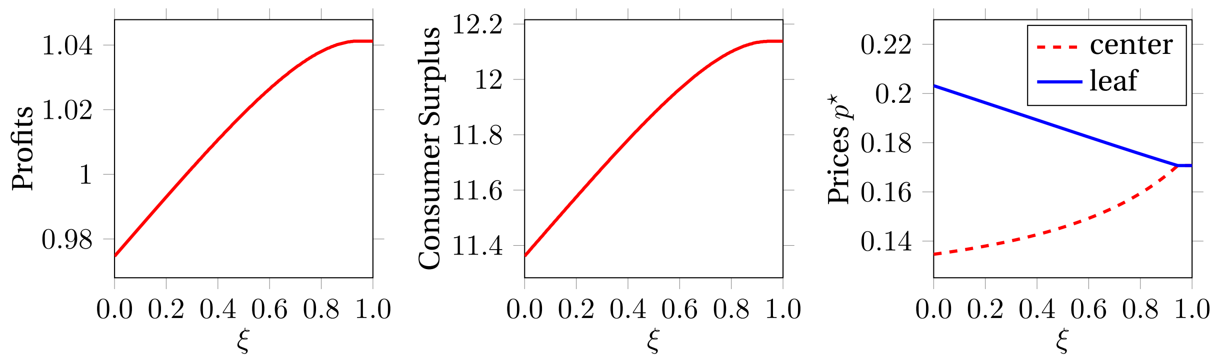


Figure 8: Profits, consumer surplus, and prices corresponding to the platform's optimal origin pricing policy for the class of star-to-complete networks with $n = 4$ locations, $w = 1$, and $\beta = 0.9$. Here, the riders' willingness to pay is distributed according to the Pareto distribution with parameters $\alpha = 1 + \sqrt{2}$ and $x_m = \frac{\sqrt{2}}{2(1+\sqrt{2})}$. Note that, in this case, the mean is equal to that of a uniform distribution in $[0, 1]$.

and compensations set by the platform, drivers decide whether to join the platform and, if so, where to locate themselves so as to maximize their expected lifetime earnings. Therefore, when setting its prices for riders and compensations for drivers, the platform must take into account the drivers' endogenous decision making at the induced equilibrium.

We establish that both profits for the platform and aggregate consumer surplus are maximized when the demand pattern is balanced across the network's locations. Moreover, in this case, prices for riders and compensations for drivers are the same irrespective of where the ride originates from. On the other hand, when a subset of locations are relatively more popular as destinations than others (after setting the potential demand at every location to be the same), i.e., when the demand across the network is unbalanced, the platform finds it optimal to set the price for a ride differently depending on where it originates from, as a way to better balance the demand for rides with the (endogenous) supply of drivers across the network. In particular, prices are lower for rides leaving popular destinations as these are typically the locations that feature an excess supply of drivers at the induced equilibrium. Finally, both the profits the platform can generate by optimizing its prices and compensations and the induced consumer surplus increase with the balancedness of the underlying demand pattern.

Our findings complement the recent focus on exploring the use of surge pricing as a way to address short-term demand fluctuations over time and they highlight that spatial pricing, i.e., setting the price of a ride as a function of where it originates from, may be an effective tool to match the demand across the ride-sharing network's locations with the supply of drivers. As a way to best isolate the impact of the network structure on equilibrium outcomes, we make a number of assumptions, most notably that the demand is time-invariant. In practice, we expect that ride-sharing platforms would use a combination of spatial pricing to account for long-term predictable demand patterns, e.g., weekday commuting, and surge-pricing techniques to address short-term demand fluctuations that may be more challenging to forecast. Therefore, an interesting avenue for future research would be to empirically estimate the benefits of such tools using real-world datasets and to relate their relative performance to the geography of the corresponding regions. More broadly, as urban centers become denser we believe that it may be worth exploring how such spatial pricing techniques may be employed to combine ride-sharing with a city's transportation infrastructure as a way to alleviate congestion and ensure a more efficient utilization of resources.

Appendix A: Dual Formulation and Auxiliary Results

Recall from Section 2 that, when the riders' willingness to pay is uniformly distributed in $[0, 1]$, the platform's profit maximization problem can be written as:

$$\begin{aligned} \max_{\{p_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - w\sum_i \delta_i \\ s.t. \quad & \sum_j y_{ij} = \beta\left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\right] + \delta_i - (1-p_i)\theta_i \\ & p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \end{aligned} \tag{17}$$

where we relaxed the constraint that $p_i \leq 1$ for all i (we subsequently establish that this is without loss of optimality for the results in Appendix A—see Remark 1). Next, we state Proposition A.1 that provides a characterization of the dual of optimization problem (17). In what follows, $\boldsymbol{\Theta}$ denotes the $n \times n$ diagonal matrix whose i-th diagonal element is equal to θ_i.

Proposition A.1. *Suppose that the drivers' outside option w is equal to one. The dual of Problem* (17) *is given as follows:*

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & \frac{1}{4}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big)^T \boldsymbol{\Theta}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big) \\ s.t. \quad & \lambda_i \geq \beta\lambda_j, \text{ for all } i, j, \\ & \lambda_i \leq 1, \text{ for all } i. \end{aligned} \tag{18}$$

The primal and dual optimization problems, i.e., Problems (17) *and* (18)*, satisfy strong duality. Finally, the vector of optimal prices* $\mathbf{p}^\star$ *in* (17) *and the vector of optimal dual variables* $\boldsymbol{\lambda}^\star$ *satisfy:*

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta\mathbf{A}\boldsymbol{\lambda}^\star}{2}, \tag{19}$$

and the platform's optimal profits are equal to $(\mathbf{1} - \mathbf{p}^\star)^T\boldsymbol{\Theta}(\mathbf{1} - \mathbf{p}^\star)$.

Proof. Optimization problem (17) is a quadratic maximization problem with a concave objective function and affine constraints. Thus, Slater's condition is satisfied for (17) and, consequently, strong duality also holds. Substituting $w = 1$, we obtain that the Lagrangian of Problem (17) is given by:

$$L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}) = \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \mathbf{1}^T\boldsymbol{\delta} + \boldsymbol{\lambda}^T(\boldsymbol{\delta} + \beta\mathbf{Y}^T\mathbf{1} + \beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p})) - \boldsymbol{\lambda}^T(\mathbf{Y}\mathbf{1} + \boldsymbol{\Theta}(1-\mathbf{p})). \tag{20}$$

By strong duality we obtain

$$\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} \min_{\boldsymbol{\lambda}} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda}} \max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda}).$$

Let $g(\boldsymbol{\lambda}) := \max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$. The dual problem, which has the same optimal objective value as the primal one, is given by $\min_{\boldsymbol{\lambda}} g(\boldsymbol{\lambda})$. Furthermore, the feasibility of the primal problem implies that both the primal and the dual optimal objectives are bounded and the corresponding optimal solutions exist.

Next, we consider expression $\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$ for some fixed $\boldsymbol{\lambda}$. First, observe that

$$\frac{\partial L}{\partial y_{ij}} = \beta\lambda_j - \lambda_i.$$

Given that the Lagrangian is linear in $\mathbf{Y}$, it follows that $g(\boldsymbol{\lambda}) = \infty$, if $\lambda_i < \beta\lambda_j$. Moreover, if in the optimal solution $y_{ij} > 0$, then $\lambda_i = \beta\lambda_j$. Similarly,

$$\frac{\partial L}{\partial \delta_i} = -1 + \lambda_i,$$

and hence $g(\boldsymbol{\lambda}) = \infty$, if $\lambda_i > 1$ as the Lagrangian is linear in δ_i. Moreover, if in the optimal solution $\delta_i > 0$, then $\lambda_i = 1$. These observations imply that $g(\boldsymbol{\lambda}) < \infty$ only when $\lambda_i \leq 1$ for all i and $\lambda_i \geq \beta\lambda_j$ for all i, j. Thus, we can rewrite the dual problem as follows:

$$\begin{aligned} \min_{\boldsymbol{\lambda}} g(\boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda}} \quad & \max_{\mathbf{p}\geq 0} \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) + \boldsymbol{\lambda}^T\Big[\beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \boldsymbol{\Theta}(1-\mathbf{p})\Big] \\ s.t. \quad & \lambda_i \geq \beta\lambda_j, \text{ for all } i, j, \\ & \lambda_i \leq 1, \text{ for all } i, \end{aligned} \tag{21}$$

where in the objective function we replace $\max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}\geq 0} L(\mathbf{p},\mathbf{Y},\boldsymbol{\delta},\boldsymbol{\lambda})$ with:

$$\max_{\mathbf{p}\geq 0} \mathbf{p}^T\boldsymbol{\Theta}(1-\mathbf{p}) + \boldsymbol{\lambda}^T\Big[\beta\mathbf{A}^T\boldsymbol{\Theta}(1-\mathbf{p}) - \boldsymbol{\Theta}(1-\mathbf{p})\Big], \tag{22}$$

since, as mentioned above, in the optimal solution, $\delta_i > 0$ implies $\lambda_i = 1$ and $y_{ij} > 0$ implies $\lambda_i = \beta\lambda_j$; thus, we can remove the terms that involve $\boldsymbol{\delta}$ and $\mathbf{Y}$.

Ignoring the non-negativity constraint on the vector of prices for a moment, the first order optimality conditions of the optimization problem in the right hand side of (21) suggest that:

$$2\mathbf{p} - \mathbf{1} + \beta\mathbf{A}\boldsymbol{\lambda} - \boldsymbol{\lambda} = \mathbf{0},$$

or equivalently $\mathbf{p} = \frac{\mathbf{1}+\boldsymbol{\lambda}-\beta\mathbf{A}\boldsymbol{\lambda}}{2}$. Using the fact that matrix $\mathbf{A}$ is row-stochastic and $\lambda_i \geq \beta\lambda_j$ for all i, j yields:

$$\boldsymbol{\lambda} - \beta\mathbf{A}\boldsymbol{\lambda} \geq \boldsymbol{\lambda} - \left(\beta \max_k \lambda_k\right)\mathbf{A}\mathbf{1} \geq \boldsymbol{\lambda} - \left(\beta \max_k \lambda_k\right)\mathbf{1} \geq \mathbf{0}.$$

Thus, it follows that $\mathbf{0} \leq \frac{\mathbf{1}+\boldsymbol{\lambda}-\beta\mathbf{A}\boldsymbol{\lambda}}{2}$ and the non-negativity constraint in the right hand side of (21) can be relaxed without affecting the optimal solution, i.e., the optimal solution is interior. By strong duality, it follows that the primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ satisfies

$$(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star) \in \arg\max_{(\mathbf{p},\mathbf{Y},\boldsymbol{\delta})} L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}^\star),$$

for the optimal dual solution $\boldsymbol{\lambda}^\star$. Thus, the vector $\mathbf{p}^\star$ that solves (22) for $\boldsymbol{\lambda} = \boldsymbol{\lambda}^\star$ is also equal to the vector of optimal prices in (17). That is,

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta\mathbf{A}\boldsymbol{\lambda}^\star}{2},$$

as stated in the proposition. Using the characterization for the vector of optimal prices $\mathbf{p} = (\mathbf{1} + \boldsymbol{\lambda} - \beta\mathbf{A}\boldsymbol{\lambda})/2$ derived above (for $\boldsymbol{\lambda}$ such that $\lambda_i \geq \beta\lambda_j$ for all i, j), we then conclude that the dual problem can be rewritten as

$$\begin{aligned}
\min_{\boldsymbol{\lambda}\in\mathbb{R}^n} \quad & \frac{1}{4}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big)^T \boldsymbol{\Theta}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}\Big) \\
s.t. \quad & \lambda_i \geq \beta\lambda_j, \text{ for all } i, j, \\
& \lambda_i \leq 1, \text{ for all } i.
\end{aligned}$$

Finally, strong duality and Expression (19) directly imply that the platform's optimal profits, i.e., the value of the objective function of optimization problem (17) at the optimal vector of prices $\mathbf{p}^\star$, are given as

$$\frac{1}{4}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}^\star\Big)^T \boldsymbol{\Theta}\Big(\mathbf{1} - (\mathbf{I} - \beta\mathbf{A})\boldsymbol{\lambda}^\star\Big) = (\mathbf{1} - \mathbf{p}^\star)^T \boldsymbol{\Theta}(\mathbf{1} - \mathbf{p}^\star).$$

□

In the remainder of Appendix A we state and prove two lemmas that establish a number of properties for the optimal dual vector $\boldsymbol{\lambda}^\star$ and the optimal vector of prices $\mathbf{p}^\star$. The lemmas are used in the analysis that follows in Appendix B.

Lemma A.1. *Suppose that the drivers' outside option w is equal to one. Then,*

(a) In the optimal solution to optimization problem (17) *we have* $\sum_i \delta_i^\star > 0$.

(b) The following hold for the optimal dual vector $\boldsymbol{\lambda}^\star$ *of Problem* (17) *:*

(i) $\lambda_i^\star \in [\beta, 1]$ *for all* i.

(ii) $\lambda_i^\star = 1$ *if* $\delta_i^\star > 0$.

(iii) $\lambda_i^\star = \beta$ *and* $\lambda_j^\star = 1$, *if* $y_{ij}^\star > 0$.

Proof. Given that $w = 1$, in any optimal solution for (17), there exists location k such that $\delta_k^\star > 0$, since otherwise serving some demand at k would lead to a solution with a higher value for the objective function. To see this, note that setting $p_k = 1 - \epsilon$, $\delta_k = \epsilon(1-\beta^2)$, and $y_{ik} = \beta\alpha_{ki}(1-p_k)\theta_k$ for all i and for some $\epsilon \ll 1$ is a feasible solution for (17) and generates positive profits for the platform (by contrast, setting $\delta_i = 0$ for all i generates zero profits).

For part (b), recall from the proof of Proposition A.1 that the primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ satisfies

$$(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star) \in \arg\max_{(\mathbf{p},\mathbf{Y},\boldsymbol{\delta})} L(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta}, \boldsymbol{\lambda}^\star),$$

for the optimal dual solution $\boldsymbol{\lambda}^\star$. In addition, again from the proof of Proposition A.1, we have $\lambda_i^\star = 1$ when $\delta_i^\star > 0$, which establishes part (b)(ii) of the lemma. Also, by part (a) we have that there exists location i such that $\delta_i^\star > 0$, which together with $\lambda_i^\star \geq \beta\lambda_j^\star$ from the feasibility constraints of the dual, establishes part (b)(i) of the lemma. Finally, noting that $\lambda_i^\star = \beta\lambda_j^\star$ when $y_{ij}^\star > 0$ in combination with part (b)(i), establishes part (b)(iii) of the lemma. □

Lemma A.2. *Suppose that the drivers' outside option w is equal to one. The following set of inequalities hold for all i*

$$\frac{1+\lambda_i^\star - \beta}{2} \leq p_i^\star \leq \frac{1+\lambda_i^\star - \beta^2}{2}.$$

Proof. From Equation (19) the optimal vector of prices $\mathbf{p}^\star$ and the corresponding optimal dual vector $\boldsymbol{\lambda}^\star$ satisfy $(I - \beta\mathbf{A})\boldsymbol{\lambda}^\star = (2\mathbf{p}^\star - 1)$. Restricting attention to the i-th row of the vectors in this equation, we obtain

$$\lambda_i^\star - \beta\sum_j \alpha_{ij}\lambda_j^\star = 2p_i^\star - 1. \tag{23}$$

Note that since $\lambda_j^\star \in [\beta, 1]$ (Lemma A.1) and $\mathbf{A}$ is a row stochastic matrix, we get

$$\beta^2 = \beta\sum_j(\alpha_{ij}\beta) \leq \beta\sum_j \alpha_{ij}\lambda_j^\star \leq \beta\sum_j \alpha_{ij} = \beta. \tag{24}$$

Using (23) and the inequalities in (24), we obtain

$$\lambda_i^\star - \beta \leq 2p_i^\star - 1 \leq \lambda_i^\star - \beta^2, \tag{25}$$

which, by rearranging terms, concludes the proof of the lemma. □

Finally, we conclude Appendix A with the following remark:

Remark 1. Lemmas A.1 and A.2 imply that $p_i^\star < 1$ for all i, when w is equal to one; thus, Problem (6) is equivalent to Problem (17), i.e., relaxing the constraint $\mathbf{p} \leq \mathbf{1}$ for Problem (17) is without loss of optimality (assuming that $w = 1$). Moreover, $\boldsymbol{\lambda}^\star$ characterized in Appendix A is a vector of optimal dual multipliers for Problem (6) (together with the multipliers for the constraint $\mathbf{p} \leq \mathbf{1}$ which are equal to zero, since the inequality is strict in the optimal solution).

Appendix B: Proofs

Proof of Proposition 1

To establish the existence of an equilibrium, we construct an auxiliary normal form game with finitely many players, and convex and compact strategy spaces. We start by introducing some notation. Let M, L denote large constants such that $M \gg \max\{w, \sum_i \theta_i\}$ and $L \gg M$. Note that if an equilibrium exists, then the following must hold for all i

$$V_i = \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\left(c_i + \beta\sum_j \alpha_{ij}V_j\right) + \left(1 - \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\right)\beta w.$$

Collecting the terms involving V_i, this is equivalent to

$$V_i = \frac{\min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\left(c_i + \beta\sum_{j\neq i}\alpha_{ij}V_j\right) + \left(1 - \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\right)\beta w}{\left(1 - \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\beta\alpha_{ii}\right)}. \tag{26}$$

We let function $R_i(x_i, V_{-i})$ denote the right hand side of Expression (26), i.e.,

$$R_i(x_i, V_{-i}) \equiv \frac{\min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\left(c_i + \beta\sum_{j\neq i}\alpha_{ij}V_j\right) + \left(1 - \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\right)\beta w}{\left(1 - \min\left\{\frac{(1-F(p_i))\theta_i}{x_i}, 1\right\}\beta\alpha_{ii}\right)}.$$

Next, we construct the auxiliary game. In this (auxiliary) normal form game, we assume that there exist 5 types of agents:

- Type 1: For each $i \in \{1, \cdots, n\}$ we have a type 1 agent, who chooses her strategy δ_i from the strategy space $S_i^1 = [0, M]$. The payoff function for the type 1 agent i is given by

$$u_i^1 = -\delta_i|V_i - w|.$$

- Type 2: For each pair $(i, j) \in \{1, \cdots, n\} \times \{1, \cdots, n\}$ we have a type 2 agent, who chooses her strategy y_{ij} from the strategy space $S_{ij}^2 = [0, M]$. The payoff function for the type 2 agent associated with the pair of locations (i,j) is given by

$$u_{ij}^2 = -y_{ij}|V_j - w|.$$

- Type 3: For each $i \in \{1, \cdots, n\}$ we have a type 3 agent, who chooses her strategy V_i from the strategy space $S_i^3 = [0, M]$. The payoff function for the type 3 agent i is given by

$$u_i^3 = -|V_i - R(x_i + e_i, V_{-i})|,$$

  where, e_i denotes the action of the type 5 agent associated with location i, which is defined below.

- Type 4: For each $i \in \{1, \cdots, n\}$ we have a type 4 agent, who chooses her strategy x_i from the strategy space $S_i^4 = [0, L]$. The payoff function for the type 4 agent i is given by

$$u_i^4 = -\left| x_i - \delta_i - \beta \sum_j \alpha_{ji} \min\{x_j, \theta_j(1 - F(p_j))\} - \beta \sum_j y_{ji} \right|.$$

- Type 5: For each $i \in \{1, \cdots, n\}$ we have a type 5 agent, who chooses her strategy e_i from the strategy space $S_i^5 = [0, L]$. The payoff function for the type 5 agent i is given by

$$u_i^5 = -e_i(1 - \mathbf{1}_{V_i \geq w}) + \mathbf{1}_{V_i \geq w}(V_i - w)e_i.$$

Note that in this construction the strategy spaces of all agents are convex and compact. The payoffs of agents of types 1,2, 3, and 4 are continuous, whereas the payoffs of agents of type 5 are upper semi-continuous. Moreover, for each agent other than agents of type 4, the payoff function is concave in her strategy. For a type 4 agent, it is straightforward to see that the payoffs are quasi concave in the agent's strategy.

In addition, let $\bar{u}_i^5 = \max_{e_i} u_i^5 = \max_{e_i} -e_i(1 - \mathbf{1}_{V_i \geq w}) + \mathbf{1}_{V_i \geq w}(V_i - w)e_i$. Note that

$$\bar{u}_i^5 = \begin{cases} 0 & \text{if } V_i < w \\ 0 & \text{if } V_i = w \\ L(V_i - w) & \text{if } V_i > w. \end{cases} \tag{27}$$

Thus, it follows that $\bar{u}_i^5$ is a lower semi-continuous function of strategies of agents other than the type 5 agent i; i.e., the agent who chose action e_i. Similarly, for type 1,2,3, and 4 agents, maximizing each agent's payoff over her own action, yields a continuous function, since their payoff functions are continuous. Together with the upper semi-continuity of payoffs, convex and compact strategy spaces, and (quasi)concavity of each agent's payoff in her own actions, this implies that the game as constructed above has a pure strategy Nash equilibrium (this is a direct consequence of Dasgupta and Maskin (1986)—refer to Theorem 2 and the corresponding corollary).

Next, we argue that the Nash equilibrium of this auxiliary game corresponds to an equilibrium of the original game. First, note that the payoffs of type 1 and 2 agents imply that at a Nash equilibrium we have

$$\delta_i = 0, y_{ji} = 0, \qquad \text{when } V_i \neq w. \tag{28}$$

Similarly, note that the payoff function for a type 4 agent implies that

$$x_i = \delta_i + \beta \sum_j \alpha_{ji} \min\{x_j, \theta_j(1 - F(p_j))\} + \beta \sum_j y_{ji}, \tag{29}$$

which is always feasible since $x_i \in [0, L]$, $\delta_i, y_{ji} \in [0, M]$, and $L \gg M$.

Finally, consider agent i of type 5. For the sake of contradiction, let $V_i > w$, which, consequently, implies that $e_i = L$. Also, note that as $x_i \to \infty$, we have $R_i(x_i, V_{-i}) \to \beta w$. Thus, for sufficiently large L, it follows that $R_i(x_i+e_i, V_{-i}) \leq w$ for any V_{-i}, since $L \gg M$. On the other hand, the payoff function of type 3 agents implies that at such an equilibrium, we also have $V_i = R_i(x_i + e_i, V_{-i}) \leq w$. Hence, we obtain a contradiction to $V_i > w$, and conclude that at the Nash equilibrium of the auxiliary game, we always have $V_i \leq w$.

Finally observe that given that $V_i \leq w$, the equilibrium action of type 3 agent i is given by

$$V_i = R_i(x_i + e_i, V_{-i}) \leq w.$$

Suppose that $e_i > 0$ at the equilibrium. Note that in this case $V_i = w$ (as for $V_i < w$ the corresponding type 5 agent finds it optimal to set $e_i = 0$). Using the fact that $V_i = w$, it can be readily checked that another equilibrium can be constructed by setting $\delta_i' = \delta_i + e_i$, $e_i' = 0$, and $x_i' = x_i + e_i$ while keeping all other actions the same. Thus, without loss of generality, we can restrict attention to equilibria where $e_i = 0$ for all i. In this case, the condition $V_i = R_i(x_i + e_i, V_{-i}) \leq w$ reduces to

$$V_i = R_i(x_i, V_{-i}) \leq w. \tag{30}$$

Expressions (28), (29), and (30) coincide with the equilibrium conditions for the original game studied in the paper. Hence, we conclude that an equilibrium always exists; thus, the claim follows. □

Proof of Lemma 1

Consider a feasible solution$\{p_i, \delta_i, x_i, y_{ij}, d_i\}_{i,j=1}^n$ for Problem (5), such that $d_i = \theta_i(1 - F(p_i)) > 0$ for every i. Note that if the compensation structure, the supply of drivers, and the demand are such that $V_i = w$ for all locations i, then it directly follows that the corresponding tuple $\{\delta_i, x_i, y_{ij}\}_{i,j=1}^n$ can be supported at equilibrium. Thus, part (i) of the lemma follows from constructing a compensation structure $\{c_i\}_{i=1}^n$ that guarantees that $V_i = w$ for all i. In particular, let

$$c_i = \frac{\beta\Big[\sum_j \alpha_{ji}(1-F(p_j))\theta_j + \sum_j y_{ji}\Big] + \delta_i}{(1-F(p_i))\theta_i} \cdot w(1-\beta), \text{ for all } i. \tag{31}$$

First, note that the compensation structure is well-defined, i.e., $c_i < \infty$, since, by assumption, $d_i = (1 - F(p_i))\theta_i > 0$ for all i. In addition, note that the probability that a given driver is assigned to a ride at location i is equal to $\theta_i(1-F(p_i))/x_i$ with $x_i = \beta\Big[\sum_j \alpha_{ji}(1-F(p_j))\theta_j + \sum_j y_{ji}\Big] + \delta_i$. Thus, the expected earnings for a single time period for a driver located at i are equal to $w(1 - \beta)$. Since, this holds for all locations i, it follows that the expected lifetime earnings V_i corresponding to any location i are given by $\sum_j \beta^j(1 - \beta)w = w$, which establishes that feasible solution $\{p_i, \delta_i, x_i, y_{ij}, d_i\}_{i,j=1}^n$ can be supported as an equilibrium when compensations are given by (31).

In addition, the cost incurred by the platform under these compensations per period is equal to:

$$\sum_i c_i \cdot (1-F(p_i))\theta_i = \sum_i \Big[\beta\Big(\sum_j \alpha_{ji}(1-F(p_j))\theta_j + \sum_j y_{ji}\Big) + \delta_i\Big] \cdot w(1-\beta) = \sum_i x_i(1-\beta)w = \sum_i \delta_i w,$$

where the last equality follows from the fact that $\sum_i \delta_i = \sum_i x_i(1 - \beta)$, i.e., the mass of drivers entering the platform to provide service at every time period is equal to the mass of drivers that are leaving.

For part (ii), first note that the expected revenues that a driver can generate for the platform during the time she provides service are upper bounded by $\bar{z}/(1-\beta)$, which corresponds to the case when she is never idle and rides are priced at the maximum willingness to pay. Thus, when $w \geq \bar{z}/(1 - \beta)$, the cost incurred per driver is higher than the revenue she generates and, consequently, it is optimal for the platform not to serve any demand and set $\delta_i^\star = d_i^\star = 0$.

For the remainder of the proof, we assume that $w < \bar{z}/(1-\beta)$. Then, any optimal solution for (5) would be such that $\sum_i d_i^\star > 0$, since otherwise serving some demand at a location i, e.g., by setting $\delta_i^\star = \epsilon(1 - \beta^2) \ll 1$, $p_i^\star = F^{-1}(\bar{z} - \epsilon/\theta_i)$, and $y_{ji}^\star = \beta\alpha_{ij}\theta_i(1 - F(p_i^\star))$, would lead to a solution with a higher value for the objective function. To complete the proof of part (ii), it remains to show that if it is optimal to have $\sum_i d_i^\star > 0$, then there exists an optimal solution in which it is optimal to have $d_i^\star > 0$ for all i. Assume, by way of contradiction, that this is not the case. Then, consider an optimal solution $\{p_i^\star, \delta_i^\star, x_i^\star, y_{ij}^\star, d_i^\star\}_{i,j=1}^n$ for (5) and partition the network into those locations for which $d_i^\star > 0$ and those for which $d_i^\star = 0$. By Assumption 1, the network is strongly connected; thus, there must exist i, j such that $d_i^\star > 0, d_j^\star = 0$ and $\alpha_{ij} > 0$. Thus, we have $x_j^\star > 0$ and $\sum_k y_{jk}^\star > 0$.

Next, we show that serving some demand at j (weakly) improves the platform's profits. In particular, consider a new feasible solution in which price p'_j is such that $\theta_j(1 - F(p'_j)) = \epsilon_j$ for some $0 < \epsilon_j < x_j^\star$. Also, the remaining excess supply at location j is routed as in the original solution, i.e., $y'_{jk} = y_{jk}^\star(x_j^\star - \epsilon_j)/x_j^\star$ for all k. On the other hand, mass ϵ_j of drivers is assigned to riders at location j and relocate according to the riders' destination preferences, e.g., $\alpha_{j\ell} \cdot \epsilon_j$ of them end up at location ℓ. In the next time period, after $(1 - \beta)$ fraction of them leaves the platform, the remaining drivers are routed in the new feasible solution such that mass $\beta\epsilon_j y_{jk}^\star/x_j^\star$ is sent to each location k. Thus, the total mass of drivers who relocate from j to k (directly or after being assigned to a rider, moving to some location ℓ, and then routed to k) is given by

$$y_{jk}^\star(x_j^\star - \epsilon_j)/x_j^\star + \beta\epsilon_j y_{jk}^\star/x_j^\star.$$

To ensure that $x'_k = x_k^\star$, the mass of drivers that enter at location k and start providing service should increase by:

$$\beta y_{jk}^\star - \beta\Big(y_{jk}^\star(x_j^\star - \epsilon_j)/x_j^\star + \beta\epsilon_j y_{jk}^\star/x_j^\star\Big) = \beta\epsilon_j y_{jk}^\star/x_j^\star - \beta^2\epsilon_j y_{jk}^\star/x_j^\star,$$

since only β fraction of the drivers who relocate to k provide service in the subsequent time period. Thus, the total increase in the mass of drivers that enter the platform at every time period is given by:

$$\sum_k \delta'_k - \sum_k \delta_k^\star = (\beta - \beta^2)\epsilon_j.$$

For the new solution to generate (weakly) higher profits than the original one it should be the case that the increase in revenues is higher than the higher cost associated with entry, i.e.,

$$\epsilon_j p'_j \geq \epsilon_j \beta(1 - \beta)w.$$

Note that setting such a price p'_j is feasible as the riders' willingness to pay is distributed according to an atomless $F(\cdot)$ with support in $[0, \bar{z}]$ where $\bar{z} > w(1 - \beta) > w\beta(1 - \beta)$. □

Proof of Proposition 2

Part (a) of the proposition follows directly from observing that when the riders' willingness to pay is upper bounded by 1, then the expected revenues that a driver can generate for the platform during the time she provides service are upper bounded by $1/(1 - \beta)$. Thus, it is optimal for the platform not to serve any demand when $1/(1 - \beta) \leq w$.

For the remainder of the proof, we assume that $1/(1 - \beta) > w$ and focus on establishing part (b) of the proposition. First, note that the solution to

$$\max_{\mathbf{p}\geq 0} \sum_i (1-p_i)\theta_i p_i - (1-\beta)\sum_i (1-p_i)\theta_i w, \tag{32}$$

is an upper bound to the maximum profit in any given network. To see this, note that the problem above corresponds to one in which the incentive-compatibility constraints for the drivers and the flow constraints at each location are relaxed. In addition, the second summation is a lower bound on the total cost that the platform would have to incur in order to satisfy demand $\theta_i(1-p_i)$ at each location i since at least $\sum_i \theta_i(1-p_i)$ drivers need to be providing service at the platform at any given time period and

$$(1-\beta)\sum_i (1-p_i)\theta_i,$$

drivers leave in each time period.

Next, note that the solution to the optimization problem above is given by

$$p_i^\star = \frac{1}{2} + \frac{(1-\beta)w}{2}, \text{ for all } i.$$

Let $y_{ij}^\star = 0$ for all i, j and vector $\boldsymbol{\delta}^\star$ that takes the following form:

$$\delta_i^\star = (1-p_i^\star)\theta_i - \beta\sum_j \alpha_{ji}(1-p_j^\star)\theta_j = (1-p_i^\star)\Big(\theta_i - \beta\sum_j \alpha_{ji}\theta_j\Big),$$

where the second equality is a consequence of $p_i^\star = p_j^\star$ for all i, j.

The fact that demand pattern $(\mathbf{A}, \boldsymbol{\theta})$ is balanced implies that $\boldsymbol{\delta}^\star \geq 0$. In addition, by construction $(\mathbf{p}^\star, \boldsymbol{\delta}^\star, \mathbf{Y}^\star)$ satisfies the equality constraints and the rest of the non-negativity constraints of Problem (6), thus is is feasible.

Next, we argue that $(\mathbf{p}^\star, \boldsymbol{\delta}^\star, \mathbf{Y}^\star)$ as constructed above is in fact optimal for optimization problem (6). Noting that $\delta_i^\star = (1-p_i^\star)\left(\theta_i - \beta\sum_j \alpha_{ji}\theta_j\right)$, we obtain that:

$$w\sum_i \delta_i^\star = w\sum_i (1-p_i^\star)\left(\theta_i - \beta\sum_j \alpha_{ji}\theta_j\right) = w\sum_i (1-p_i^\star)(1-\beta)\theta_i, \tag{33}$$

where the second equality follows from the fact that $p_i^\star = p_j^\star$ for all i, j which, in turn, implies that

$$\beta\sum_i (1-p_i^\star)\sum_j \alpha_{ji}\theta_j = \beta(1-p_i^\star)\sum_i\sum_j \alpha_{ji}\theta_j = \beta(1-p_i^\star)\sum_j \theta_j.$$

Note that this implies that the value of the objective function of (6) for $(\mathbf{p}^\star, \boldsymbol{\delta}^\star, \mathbf{Y}^\star)$ is equal to the optimal value of the objective function of (32). Given that the optimal value of (32) is an upper bound for the optimal value of (6), we conclude that $(\mathbf{p}^\star, \boldsymbol{\delta}^\star, \mathbf{Y}^\star)$ is indeed optimal for (6).

To conclude the proof, first note that $y_{ij}^\star = 0$ for all i, j, which implies that the supply of drivers and demand for rides at location i are equal for all i, i.e.,

$$x_i^\star = \beta \sum_j \alpha_{ji}\theta_j(1 - p_j^\star) + \delta_i^\star = \theta_i(1 - p_i^\star),$$

and, as a consequence, drivers always get assigned to rides. In turn, this implies that we can write

$$\delta_i^\star = x_i^\star - \beta \sum_j \alpha_{ji} x_j^\star.$$

Furthermore, setting compensations to be such that $c_i^\star = w(1-\beta)$ for all i guarantees that:

$$V_i = w, \text{ for all } i,$$

since drivers always get assigned to rides, i.e., they are never idle. Thus, $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an equilibrium under $\{p_i^\star, c_i^\star\}_{i=1}^n$.

Finally, the optimal value of Problem (32) is an upper bound on the profits the platform can generate for any demand pattern. We established above that when the demand pattern is balanced, using prices and compensations $\{p_i^\star, c_i^\star\}_{i=1}^n$ generates profits equal to the optimal value of Problem (32), which implies part (b)(i) of the proposition.

This completes the proof of the claim. In particular, we established that when the demand pattern is balanced there exists a profit maximizing solution for the platform in which the optimal price and compensation at location i are given by $p_i^\star = 1/2 + (1-\beta)w/2$ and $c_i^\star = (1-\beta)w$ respectively, and $\beta \mathbf{A}^T d^\star \leq d^\star$, i.e., drivers always get assigned to rides. □

Proof of Proposition 3

The first part of the proposition, i.e., the part that relates to the prices set by the platform, follows directly from Proposition A.1, which establishes the characterization of the price vector $\{p_i^\star\}_{i=1}^n$ as a function of the dual variables $\{\lambda_i\}_{i=1}^n$, and Lemmas A.1 and A.2. In particular, note that the dual variables corresponding to an entry point k and a location with excess supply ℓ are equal to $\lambda_k = 1$ and $\lambda_\ell = \beta$. Substituting these values for the prices as given in Lemma A.2 yields the desired result.

It remains to show that the compensations defined by Equation (10) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$, i.e., the optimal solution to Problem (6), as an equilibrium under price vector $\{p_i^\star\}_{i=1}$. First, note that the compensations defined in (10) are the solution to Equation (34) below, i.e., the equation that describes the drivers' expected earnings, when we set $V_i = \lambda_i^\star$ for all i:

$$V_i = \frac{1-p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \left(c_i^\star + \beta V_j \right) + \left(1 - \frac{1-p_i^\star}{x_i^\star} \right) \max_k \beta V_k. \tag{34}$$

To see this, first consider the case where $x_i^\star > (1-p_i^\star)$, which, in turn, implies that $y_{ij}^\star > 0$ for some j. Recall from Lemma A.1 that $\lambda_i^\star = \beta$ when $y_{ij}^\star > 0$ for some j and $\max_k \lambda_k^\star = 1$. Thus, we can rewrite (34) as:

$$\begin{aligned} V_i &= \frac{1-p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \left(c_i^\star + \beta \lambda_j^\star \right) + \left(1 - \frac{1-p_i^\star}{x_i^\star} \right) \max_k \beta \lambda_k^\star \\ &= \frac{1-p_i^\star}{x_i^\star} c_i^\star + \frac{1-p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \beta \lambda_j^\star + \left(1 - \frac{1-p_i^\star}{x_i^\star} \right) \max_k \beta \lambda_k^\star \\ &= \frac{1-p_i^\star}{x_i^\star} \lambda_i^\star + \left(1 - \frac{1-p_i^\star}{x_i^\star} \right) \lambda_i^\star = \lambda_i^\star, \end{aligned}$$

where the equality in the last line follows directly from the definition of compensation $c_i^\star$ (Equation (10)) and the fact that $\lambda_i^\star = \max_k \beta \lambda_k^\star = \beta$. The claim for the case where $x_i^\star = (1-p_i^\star)$ follows immediately from (34) and the definition of $c_i^\star$.

Therefore, V_i's as defined here satisfy Equation (2) (as it is equivalent to (34) when $x_i^\star \geq (1-p_i^\star)$). In addition, they satisfy the drivers' incentive-compatibility constraints, i.e., Equation (3), since $\lambda^\star = V_i = 1 = w$, when $\delta_i^\star > 0$ and $\lambda_i^\star \leq 1$ for all i. Finally, condition (ii) in the equilibrium definition, i.e., Equation (1), is satisfied trivially as $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is feasible for Problem (6) and $x_i^\star \geq (1-p_i^\star)$. Thus, we conclude that the compensations defined by Equation (10) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ as an equilibrium under price vector $\{p_i^\star\}_{i=1}$ and expected future earnings for a driver at location i given by $V_i = \lambda_i^\star$. □

Proof of Corollary 1

The corollary follows directly from Lemmas A.1 and A.2. In particular, note that the supply of drivers at any location i is lower bounded by:

$$x_i \geq \beta \sum_j \alpha_{ji} (1 - p_j^\star) \geq \beta \sum_j \alpha_{ji} \Big(1 - \frac{1}{2} - \frac{1-\beta^2}{2} \Big) \geq \kappa_i \frac{\beta^3}{2},$$

where the second inequality follows by using $p_j^\star \leq \frac{1+\lambda_j^\star - \beta^2}{2}$ from Lemma A.2 and $\lambda_j^\star \leq 1$ for all j (from Lemma A.1). Thus, the assumption that $\kappa_i > 1/\beta^3$ for the first part of the corollary implies that the supply of drivers at location i is greater than 1/2 and, consequently, location i has excess supply (note again by Lemmas A.1 and A.2, it holds that $p_i^\star \geq 1/2$, thus the maximum mass of riders that the platform finds optimal to serve at location i is 1/2).

For the second part of the corollary, assume by way of contradiction that there exists location i such that $\kappa_i(\mathbf{A}) < \beta$, but location i is not an entry point, i.e., $\delta_i + \sum_j y_{ji} = 0$. Then, the supply of drivers at location i is upper bounded by:

$$x_i \leq \beta \sum_j \alpha_{ji}(1 - p_j^\star) \leq \beta \sum_j \alpha_{ji}\Big(1 - \frac{1}{2}\Big) \leq \kappa_i \frac{\beta}{2}.$$

Thus, since $\kappa_i(\mathbf{A}) < \beta$ we obtain that the supply of drivers at i is strictly less than $\beta^2/2$. Given that $p_i^\star \leq 1 - \beta^2/2$ (from Proposition 3), this leads to a contradiction, since there has to be additional entry at location i to satisfy the demand induced by setting $p_i^\star \leq 1 - \beta^2/2$ (otherwise, the platform can generate strictly higher profits by increasing $p_i^\star$ as doing so does not violate the feasibility constraints). Thus, we conclude that if $\kappa_i < \beta$, location i has to be an entry point. $\square$

Proof of Theorem 1

Consider a strongly balanced demand pattern $(\mathbf{D}, \mathbf{1})$. In addition, let $\Pi(\mathbf{F}^\xi, \mathbf{1})$ denote the optimal profits for the platform when the underlying demand pattern is given by $(\xi\mathbf{D} + (1-\xi)\mathbf{F}, \mathbf{1})$. Assumption 2 and Lemma 1 imply that the profits can be obtained as follows:

$$\begin{aligned} \Pi(\mathbf{F}^\xi, \mathbf{1}) = \max_{\mathbf{p},\mathbf{Y},\boldsymbol{\delta}} \quad & \mathbf{p}^T(1 - \mathbf{p}) - \mathbf{1}^T\boldsymbol{\delta} \\ s.t. \quad & \mathbf{Y}\mathbf{1} - \boldsymbol{\delta} + (\mathbf{1} - \mathbf{p}) - \beta\mathbf{Y}^T\mathbf{1} - \beta(\xi\mathbf{D} + (1-\xi)\mathbf{F})^T(\mathbf{1} - \mathbf{p}) = 0 \\ & \mathbf{p}, \boldsymbol{\delta}, \mathbf{Y} \geq 0, \end{aligned} \tag{35}$$

where we omit the constraint $\mathbf{p} \leq \mathbf{1}$ which is without loss of optimality as established in Appendix A (Remark 1).

To establish the theorem, it suffices to show that $\Pi(\mathbf{F}^\xi, \mathbf{1})$ is (weakly) increasing for $\xi \in [0, 1]$. Since $\mathbf{F}$ is an arbitrary matrix, it suffices to prove the claim for $\xi = 0$, i.e., show that at $\xi = 0$, $\Pi(\mathbf{F}^\xi, \mathbf{1})$ is (weakly) increasing in ξ. This is due to the fact that $\Pi(\mathbf{F}^\xi, \mathbf{1})$ is (weakly) increasing in ξ for some $\xi \in [0, 1]$ if and only if $\Pi(\hat{\mathbf{F}}^{\xi'}, \mathbf{1})$ is increasing in ξ' at $\xi' = 0$, where $\hat{\mathbf{F}} = \xi\mathbf{D} + (1-\xi)\mathbf{F}$.

Note that, since the constraints are affine, Slater's condition and, consequently, strong duality hold. Moreover, the objective and constraints are continuous in ($\mathbf{p}$, $\mathbf{Y}$,$\boldsymbol{\delta}$, ξ) and convex in ($\mathbf{p}$, $\mathbf{Y}$,$\boldsymbol{\delta}$). In addition, the upper contour sets at the objective are compact.[33] Thus, it follows that $\Pi(\mathbf{F}^\xi, \mathbf{1})$ is (Hadamard) differentiable in ξ (for more details see Bonnans and Shapiro (2013)).

[33] Compactness in $\mathbf{p}, \boldsymbol{\delta}$ readily follows from the quadratic payoff structure and $\boldsymbol{\delta} \geq \mathbf{0}$.To see compactness in the space of $(\mathbf{p}, \mathbf{Y}, \boldsymbol{\delta})$, note that for arbitrary large $\mathbf{Y}$ feasibility implies also large $\mathbf{p}$ and $\boldsymbol{\delta}$ and, consequently, arbitrarily small value for the objective function.

Thus, it follows from Bonnans and Shapiro (2013) and Milgrom and Segal (2002) (Corollary 5) that for some primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$, and dual optimal solution $\boldsymbol{\lambda}^\star$ of (35) at $\xi = 0$, we have

$$\left.\frac{\partial_+ \Pi(\mathbf{F}^\xi, \mathbf{1})}{\partial \xi}\right|_{\xi=0} = \beta \boldsymbol{\lambda}^\star (\mathbf{D} - \mathbf{F})^T (1 - \mathbf{p}^\star). \tag{36}$$

The proof follows directly from the following technical lemma (which we prove below).

Lemma B.1. *For any permutation matrix* $\mathbf{H}$*, and any primal-dual pair of optimal solutions* $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ *and* $\boldsymbol{\lambda}^\star$ *associated with* $\xi = 0$ *in* (35) *the following holds:*

$$\boldsymbol{\lambda}^\star (\mathbf{H} - \mathbf{F})^T (1 - \mathbf{p}^\star) \geq 0.$$

To complete the proof of Theorem 1, note that by the Birkhoff–von Neumann theorem, we obtain that any doubly-stochastic matrix belongs to the convex hull of permutation matrices. Thus, Lemma B.1 above implies that $\boldsymbol{\lambda}^\star (\mathbf{D} - \mathbf{F})^T (1 - \mathbf{p}^\star) \geq 0$ for any strongly balanced demand pattern $(\mathbf{D}, \mathbf{1})$ (note that $(\mathbf{D}, \mathbf{1})$ being strongly balanced implies that $\mathbf{D}$ is a doubly-stochastic matrix). Using this observation together with (36), we conclude that $\left.\frac{\partial_+ \Pi(\mathbf{F}^\xi, \mathbf{1})}{\partial \xi}\right|_{\xi=0} \geq 0$, which in turns implies that the theorem holds. □

Proof of Lemma B.1

Lemma B.1 follows from a series of claims that we state and prove below.

Lemma B.2. *Let* i, j *be such that* $\lambda_j^\star \leq \lambda_i^\star$*. Then, we have* $(1 - p_i^\star)/(1 - p_j^\star) \leq 1/\beta$*.*

Proof. The following inequality follows directly from the lower and upper bounds on $p_i^\star$ and $p_j^\star$ respectively obtained by Lemma A.2:

$$\frac{1 - p_i^\star}{1 - p_j^\star} \leq \frac{1 + \beta - \lambda_i^\star}{1 - \lambda_j^\star + \beta^2}. \tag{37}$$

Furthermore, the assumption that $\lambda_j^\star \leq \lambda_i^\star$ implies

$$\frac{1 - p_i^\star}{1 - p_j^\star} \leq \frac{1 + \beta - \lambda_j^\star}{1 - \lambda_j^\star + \beta^2}.$$

Recall that by Lemma A.1, we have $\lambda_i^\star \in [\beta, 1]$ for all i. Let $h(x) = \frac{1+\beta-x}{1-x+\beta^2}$ and note that $h(x)$ is increasing for $x \in [\beta, 1]$. Therefore,

$$\frac{1 + \beta - \lambda_j^\star}{1 - \lambda_j^\star + \beta^2} \leq \frac{1}{\beta},$$

as can be seen by setting $\lambda_j^\star$ to one, which completes the proof of the lemma. □

Lemma B.3. *Consider renaming the locations such that* $\lambda_1^\star \le \lambda_2^\star \le \ldots \le \lambda_n^\star$*, which is without loss of generality. Let* σ *be a permutation of the locations such that* $1 - p^\star_{\sigma(1)} \ge 1 - p^\star_{\sigma(2)} \ge \ldots \ge 1 - p^\star_{\sigma(n)}$*. Then, for all* i *we have*

$$\frac{1 - p_i^\star}{\beta} \ge 1 - p^\star_{\sigma(i)}.$$

Proof. We prove the claim by contradiction. Let i be the smallest index such that $(1-p_i^\star)/\beta < 1-p^\star_{\sigma(i)}$. By the definition of the permutation σ, for every $j < i$, we have:

$$1 - p^\star_{\sigma(j)} \ge 1 - p^\star_{\sigma(i)} > \frac{1 - p_i^\star}{\beta}. \tag{38}$$

However, Lemma B.2 implies that we have $(1 - p_k^\star)/(1 - p_i^\star) \le 1/\beta$ when $\lambda_k^\star \ge \lambda_i^\star$. Since, by Expression (38) we have $\frac{1-p^\star_{\sigma(j)}}{1-p_i^\star} > \frac{1}{\beta}$, this implies that $\sigma(j) \le i - 1$ for all $j < i$. Similarly, Expression (38) also suggests that $\frac{1-p^\star_{\sigma(i)}}{1-p_i^\star} > \frac{1}{\beta}$, and hence $\sigma(i) \le i - 1$. Therefore, we must have $\sigma(j) \le i - 1$ for all $j \le i$, which leads to a contradiction as $\sigma(\cdot)$ defines a permutation (and hence a bijection). □

Finally, using Lemmas A.1 and B.3 we complete the proof of Lemma B.1, i.e., we show that for a permutation matrix $\mathbf{H}$ and a primal-dual pair of optimal solutions $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ and $\boldsymbol{\lambda}^\star$ the following holds:

$$\boldsymbol{\lambda}^\star(\mathbf{H} - \mathbf{F})^T(1 - \mathbf{p}^\star) \ge 0.$$

In particular, consider a permutation matrix $\mathbf{H}$, and define a function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f(\boldsymbol{\lambda}) = \boldsymbol{\lambda}^T(\mathbf{H}^T - \mathbf{F}^T)(1 - \mathbf{p}^\star)$. We prove the claim by showing that $f(\boldsymbol{\lambda}^\star) \ge 0$. First, let $\mathbf{1}$ denote the vector of ones and note that $f(\mathbf{1}) = 0$, since both $\mathbf{F}$ and $\mathbf{H}$ are row-stochastic matrices (hence, $\mathbf{1}^T\mathbf{H}^T = \mathbf{1}^T\mathbf{F}^T = \mathbf{1}^T$). Therefore, to prove the claim it suffices to show that $f(\boldsymbol{\lambda}^\star) \ge f(\mathbf{1}) = 0$. We show this by constructing a sequence of vectors $\{\boldsymbol{\lambda}^\star = \boldsymbol{\lambda}^1, \boldsymbol{\lambda}^2, \ldots, \boldsymbol{\lambda}^n = \mathbf{1}\}$ such that $f(\boldsymbol{\lambda}^\star) = f(\boldsymbol{\lambda}^1) \ge f(\boldsymbol{\lambda}^2) \ge \ldots \ge f(\mathbf{1}) = 0$, which readily implies the claim.

To construct such a sequence, assume without loss of generality that $\lambda_1^\star \le \lambda_2^\star \le \ldots \le \lambda_n^\star$, and define set S_i as $S_i = \{j : j \le i\}$. In addition, for each $2 \le k \le n$, define η_k as $\eta_k = \lambda_k^\star - \lambda_{k-1}^\star$. In words, η_k is the difference between the k-th and $(k-1)$-th smallest entry of $\boldsymbol{\lambda}^\star$, and thus $\eta_k \ge 0$. Given a set of indices S, we use $\mathbf{e}_S$ to denote the indicator vector of S, i.e., $(\mathbf{e}_S)_i = 1$ if $i \in S$, and zero otherwise. We define the following sequence of vectors:

$$\begin{aligned}
\boldsymbol{\lambda}^1 &= \boldsymbol{\lambda}^\star \\
\boldsymbol{\lambda}^k &= \boldsymbol{\lambda}^{k-1} + \eta_k \mathbf{e}_{S_{k-1}} \qquad \text{for all } k \text{ such that } 2 \le k \le n.
\end{aligned}$$

Note that by construction, for each $\boldsymbol{\lambda}^k$ we have that $\lambda_i^k = \lambda_k^\star$ if $i \le k$, and $\lambda_i^k = \lambda_i^\star \ge \lambda_k^\star$ if $i > k$.

Next, we show that for the sequence of vectors defined above, we have

$$f(\boldsymbol{\lambda}^\star) = f(\boldsymbol{\lambda}^1) \ge f(\boldsymbol{\lambda}^2) \ge \ldots \ge f(\mathbf{1}).$$

Since $f(\cdot)$ is a linear function of $\boldsymbol{\lambda}$, it follows that $f(\boldsymbol{\lambda}^k) = f(\boldsymbol{\lambda}^{k-1}) + f(\eta_k \mathbf{e}_{S_{k-1}})$. Thus, to prove the claim it suffices to show that $f(\eta_k \mathbf{e}_{S_{k-1}}) \leq 0$ for all $2 \leq k \leq n$. To this end, using the definition of $f(\cdot)$, we can express $f(\eta_k \mathbf{e}_{S_{k-1}})$ as follows:

$$
\begin{aligned}
f(\eta_k \mathbf{e}_{S_{k-1}}) &= \eta_k \left(\mathbf{e}_{S_{k-1}}^T \mathbf{H}^T (1 - \mathbf{p}^\star) - \sum_{j \in S_{k-1}} \left[\mathbf{F}^T (1 - \mathbf{p}^\star) \right]_j \right) \\
&\leq \eta_k \left(\sum_{1 \leq j \leq k-1} (1 - p^\star_{\sigma(j)}) - \sum_{1 \leq j \leq k-1} \left[\mathbf{F}^T (1 - \mathbf{p}^\star) \right]_j \right)
\end{aligned}
$$

where σ is a permutation of vertices such that $1 - p^\star_{\sigma(1)} \geq 1 - p^\star_{\sigma(2)} \geq \ldots \geq 1 - p^\star_{\sigma(n)}$.

Recall that, by Lemma A.1, we have that $\lambda^\star_n = 1$. Note that, if $\lambda^\star_k = 1$, then $\eta_j = 0$ for all $j \geq k+1$ and the inequality $f(\eta_j \mathbf{e}_{S_{j-1}}) \leq 0$ trivially holds. Therefore, it suffices to show that the above inequality also holds for every k such that $\lambda^\star_{k-1} < 1$. However, if $\lambda^\star_j < 1$ for a some location j, then it follows that $\delta^\star_j = y^\star_{ij} = 0$ for all i (as shown in Lemma A.1). On the other hand, feasibility implies that the total supply of available drivers at location j should be greater than the demand that the platform serves at j, i.e., $\beta[\mathbf{A}^T (1 - \mathbf{p}^\star)]_j \geq 1 - p^\star_j$. Hence, for every k such that $\lambda^\star_{k-1} < 1$ we have:

$$
\sum_{1 \leq j \leq k-1} (1 - p^\star_{\sigma(j)}) - \sum_{1 \leq j \leq k-1} \left[\mathbf{F}^T (1 - \mathbf{p}^\star) \right]_j \leq \sum_{1 \leq j \leq k-1} (1 - p^\star_{\sigma(j)}) - \frac{1}{\beta} \sum_{1 \leq j \leq k-1} 1 - p^\star_j.
$$

By Lemma B.3, we have $1 - p^\star_{\sigma(j)} \leq \frac{1 - p^\star_j}{\beta}$ for all j. Therefore,

$$
\sum_{1 \leq j \leq k-1} (1 - p^\star_{\sigma(j)}) - \frac{1}{\beta} \sum_{1 \leq j \leq k-1} 1 - p^\star_j \leq 0.
$$

The bound above together with the fact that $\eta_k \geq 0$ implies that $f(\eta_k \mathbf{e}_{S_{k-1}}) \leq 0$ for all $2 \leq k \leq n$, and thus $f(\boldsymbol{\lambda}^\star) \geq f(\boldsymbol{\lambda}^2) \geq \ldots \geq f(\mathbf{1}) = 0$. Therefore, we have established that, for every permutation matrix $\mathbf{H}$, we have $\boldsymbol{\lambda}^\star (\mathbf{H} - \mathbf{F})^T (1 - \mathbf{p}^\star) \geq 0$, which completes the proof of the lemma. □

Proof of Proposition 4

Consider the optimal prices $\{p'_i\}_{i=1}^n$ corresponding to demand pattern $(\mathbf{A}', \mathbf{1})$. To establish that the platform can generate higher profits under demand pattern $(\mathbf{A}, \mathbf{1})$ than under $(\mathbf{A}', \mathbf{1})$ we consider the profits that correspond to prices $\{p'_i\}_{i=1}^n$ in the two demand patterns, assuming that the platform serves the induced demand in both cases. Given the latter, it's sufficient to compare the costs associated with serving the demand or, more specifically, compare the mass of drivers who do not get assigned to a ride (since the mass of drivers who get assigned to rides is the same in both cases).

First, note that the bounds on the optimal prices provided in Lemma A.2 apply to the locations in S_1 and S_2 regardless of whether the demand pattern is $(\mathbf{A}, \mathbf{1})$ or $(\mathbf{A}', \mathbf{1})$. This implies that the demand served at location i, which recall that we denote by d_i, satisfies:

$$\beta^2/2 \leq d_i = 1 - p_i \leq 1/2, \tag{39}$$

where we use Lemmas A.1 and A.2.

For a location i in set $S_1(\mathbf{A})$ and demand pattern $(\mathbf{A}, \mathbf{1})$, we have that the excess supply of drivers at i is equal to:

$$\beta \sum_j \alpha_{ji} d_j - d_i \geq \beta^3/2 \sum_j \alpha_{ji} - 1/2 = \beta^3/2\kappa_i(\mathbf{A}) - 1/2,$$

where the first inequality follows from (39). Similarly, for $(\mathbf{A}', \mathbf{1})$ we have $\beta \sum_j \alpha'_{ji} d_j - d_i \geq \beta^3/2\kappa_i(\mathbf{A}') - 1/2$. Sets $S_1(\mathbf{A}) = S_1(\mathbf{A}')$ correspond to locations with excess supply under the two demand patterns. The difference in the excess supply of drivers at a location $i \in S_1(\mathbf{A})$ corresponding to the two demand patterns (under prices $\{p'_i\}_{i=1}^n$) can be bounded below from:

$$\left(\beta \sum_j \alpha'_{ji} d_j - d_i\right) - \left(\beta \sum_j \alpha_{ji} d_j - d_i\right) \geq \sum_j \alpha'_{ji} \frac{\beta^3}{2} - \sum_j \alpha_{ji} \frac{\beta}{2} = \frac{\beta}{2}\Big(\beta^2 \kappa_i(\mathbf{A}') - \kappa_i(\mathbf{A})\Big), \tag{40}$$

where again we use (39). Similarly, for the set of locations in $S_2(\mathbf{A}) = S_2(\mathbf{A}')$ we have that the difference of excess supply under the two demand patterns can be bounded as follows:

$$\left(\beta \sum_j \alpha_{ji} d_j - d_i\right)^+ - \left(\beta \sum_j \alpha'_{ji} d_j - d_i\right)^+ \leq \left(\sum_j \alpha_{ji} \frac{\beta}{2} - \sum_j \alpha'_{ji} \frac{\beta^3}{2}\right) = \frac{\beta}{2}\Big(\kappa_i(\mathbf{A}) - \beta^2 \kappa_i(\mathbf{A}')\Big)^+, \tag{41}$$

where the inequality follows from (39) and the fact that for any two real numbers a, b it holds that $a^+ - b^+ \leq (a-b)^+$. Finally, for any remaining location, i.e., $i \notin S_1(\mathbf{A}) \cup S_2(\mathbf{A})$, we have $\kappa_i(\mathbf{A}) < \beta$ and $\kappa_i(\mathbf{A}') < \beta$. By Corollary 1 it follows that i is an entry point under both demand patterns and, consequently, does not feature any excess supply. Expressions (40) and (41) along with the assumptions of the proposition establish the claim since the difference in the cost the platform has to incur in order to serve the demand induced by prices $\{p'_i\}_{i=1}^n$ under the two demand patterns is equal to:

$$\sum_{i \in S_1(\mathbf{A})} \left(\left(\sum_j \alpha'_{ji} d_j - d_i\right) - \left(\sum_j \alpha_{ji} d_j - d_i\right)\right) + \sum_{i \in S_2(\mathbf{A})} \left(\left(\sum_j \alpha'_{ji} d_j - d_i\right)^+ - \left(\sum_j \alpha_{ji} d_j - d_i\right)^+\right),$$

which, from the discussion above, is greater than zero. Thus, we conclude that the platform can generate (weakly) higher profits under $\mathbf{A}$ than under $\mathbf{A}'$, as claimed. $\square$

Proof of Proposition 5

Let $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ and $\boldsymbol{\lambda}^\star$ be a primal-dual pair of optimal solutions for demand pattern $(\mathbf{F}, \mathbf{1})$. Similarly, let $(\hat{\mathbf{p}}, \hat{\mathbf{Y}}, \hat{\boldsymbol{\delta}})$ and $\hat{\boldsymbol{\lambda}}$ be a primal-dual pair of optimal solutions for demand pattern $(\mathbf{D}, \mathbf{1})$. Then, strong duality implies the following

$$\begin{aligned}
\Pi(\mathbf{F},\mathbf{1}) &= L(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star, \boldsymbol{\lambda}^\star) \\
&\geq L(\hat{\mathbf{p}}, \hat{\mathbf{Y}}, \hat{\boldsymbol{\delta}}, \boldsymbol{\lambda}^\star) \\
&= \hat{\mathbf{p}}^T(\mathbf{1}-\hat{\mathbf{p}}) - \mathbf{1}^T\hat{\boldsymbol{\delta}} + (\boldsymbol{\lambda}^\star)^T(\hat{\boldsymbol{\delta}} + \beta\hat{\mathbf{Y}}^T\mathbf{1} + \beta\mathbf{F}^T(\mathbf{1}-\hat{\mathbf{p}})) - (\boldsymbol{\lambda}^\star)^T(\hat{\mathbf{Y}}\mathbf{1} + (\mathbf{1}-\hat{\mathbf{p}})) \\
&= \Pi(\mathbf{D},\mathbf{1}) + (\boldsymbol{\lambda}^\star)^T(\hat{\boldsymbol{\delta}} + \beta\hat{\mathbf{Y}}^T\mathbf{1} + \beta\mathbf{F}^T(\mathbf{1}-\hat{\mathbf{p}}) - \hat{\mathbf{Y}}\mathbf{1} - (\mathbf{1}-\hat{\mathbf{p}})) \\
&= \Pi(\mathbf{D},\mathbf{1}) + (\boldsymbol{\lambda}^\star)^T(\beta\mathbf{F}^T(\mathbf{1}-\hat{\mathbf{p}}) - \beta\mathbf{D}^T(\mathbf{1}-\hat{\mathbf{p}})) \\
&= \Pi(\mathbf{D},\mathbf{1}) + \beta(\boldsymbol{\lambda}^\star)^T(\mathbf{F}-\mathbf{D})^T(\mathbf{1}-\hat{\mathbf{p}}),
\end{aligned} \tag{42}$$

where the first line follows from the saddle point characterization of the optimal solution, and the second one follows since the primal optimal solution $(\mathbf{p}^\star, \mathbf{Y}^\star, \boldsymbol{\delta}^\star)$ maximizes the Lagrangian for the optimal vector of dual multipliers $\boldsymbol{\lambda}^\star$. In the third line, we provide the expression for the Lagrangian explicitly, whereas the fourth one uses the fact that

$$\Pi(\mathbf{D},\mathbf{1}) = \hat{\mathbf{p}}^T(\mathbf{1}-\hat{\mathbf{p}}) - \mathbf{1}^T\hat{\boldsymbol{\delta}}.$$

The feasibility of vector $(\hat{\mathbf{p}}, \hat{\mathbf{Y}}, \hat{\boldsymbol{\delta}})$ for the platform's optimization problem corresponding to $(\mathbf{D}, \mathbf{1})$ requires

$$\hat{\boldsymbol{\delta}} + \beta\hat{\mathbf{Y}}^T\mathbf{1} + \beta\mathbf{D}^T(\mathbf{1}-\hat{\mathbf{p}}) - \hat{\mathbf{Y}}\mathbf{1} - (\mathbf{1}-\hat{\mathbf{p}}) = 0.$$

Combining this observation with the fourth line, we obtain the fifth. Finally, rearranging terms yields the last line. As a side remark, note the bound we obtain in (42) holds for any pair of demand patterns (since, so far, we have not used the fact that $(\mathbf{D}, \mathbf{1})$ is balanced).

Next, we specialize (42) for a strongly balanced demand pattern $(\mathbf{D}, \mathbf{1})$. Note that Proposition 2 implies that $\hat{\mathbf{p}} = \left(1 - \frac{\beta}{2}\right)\mathbf{1}$. Thus, the bound we provide in (42) can be rewritten as follows

$$\Pi(\mathbf{D},\mathbf{1}) - \Pi(\mathbf{F},\mathbf{1}) \leq \frac{\beta^2}{2}(\boldsymbol{\lambda}^\star)^T(\mathbf{D}-\mathbf{F})^T\mathbf{1}.$$

Finally, using the fact that $\mathbf{D}$ is doubly stochastic, since the corresponding demand pattern is strongly balanced, we obtain

$$\Pi(\mathbf{D},\mathbf{1}) - \Pi(\mathbf{F},\mathbf{1}) \leq \frac{\beta^2}{2}(\boldsymbol{\lambda}^\star)^T(\mathbf{1}-\mathbf{F}^T\mathbf{1}), \tag{43}$$

which concludes the proof of the proposition.

Proof of Corollary 2

We consider demand patterns $(\mathbf{F}, \mathbf{1})$ for which $\kappa_i(\mathbf{F}) > 1/\beta^3$ or $\kappa_i(\mathbf{F}) < \beta$. Note that Lemma A.1 and Corollary 1 imply that $\lambda_i^\star = \beta$ for i such that $\kappa_i(\mathbf{F}) > 1/\beta^3$ and $\lambda_i^\star = 1$ for i such that $\kappa_i(\mathbf{F}) < \beta$. Thus, the bound provided in (43) can be rewritten as

$$\begin{aligned}
\Pi(\mathbf{D}, \mathbf{1}) - \Pi(\mathbf{F}, \mathbf{1}) &\leq \frac{\beta^2}{2} (\boldsymbol{\lambda}^\star)^T (\mathbf{1} - \mathbf{F}^T \mathbf{1}) \\
&= \frac{\beta^2}{2} \left(\sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} \lambda_i^\star (1 - \kappa_i(\mathbf{F})) + \sum_{i \left| \kappa_i(\mathbf{F}) < \beta \right.} \lambda_i^\star (1 - \kappa_i(\mathbf{F})) \right) \\
&= \frac{\beta^2}{2} \left(\beta \sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (1 - \kappa_i(\mathbf{F})) + \sum_{i \left| \kappa_i(\mathbf{F}) < \beta \right.} (1 - \kappa_i(\mathbf{F})) \right) \\
&= \frac{\beta^2}{2} \left(\sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (1 - \kappa_i(\mathbf{F}))(\beta - 1) \right) \\
&= \frac{\beta^2}{2} (1 - \beta) \sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (\kappa_i(\mathbf{F}) - 1),
\end{aligned}$$

where the equality in the fourth line uses the fact that $\sum_i (1 - \kappa_i(\mathbf{F})) = 0$ or equivalently

$$\sum_{i \left| \kappa_i(\mathbf{F}) < \beta \right.} (1 - \kappa_i(\mathbf{F})) = - \sum_{i \left| \kappa_i(\mathbf{F}) > 1/\beta^3 \right.} (1 - \kappa_i(\mathbf{F})).$$

This concludes the proof of the corollary.

□

Proof of Corollary 3

The corollary is a consequence of the fact that the expression for the profits corresponding to the platform's optimal prices, i.e., the value of the objective function in (17), is equal to the expression for the consumer surplus (up to a constant factor) induced under the same prices (this can be seen from Proposition A.1 and Definition 2). Then, invoking Theorem 1 directly yields the result. □

Proof of Proposition 6

Assume that the demand pattern across the n locations is given by $(\mathbf{A}^\xi, \mathbf{1})$. We provide a closed form characterization of the platform's optimal prices and profits as a function of ξ. Then, the proof of the proposition follows directly from this characterization. In particular, we show the following:

(a) If $\xi \in \left[0, \max\left(\frac{(n-1)}{2(1-\beta)\beta(n-2)}\left(\beta(1-2\beta)+\sqrt{\frac{\beta^2(n-1)+4\beta-4}{n-1}}\right), 0\right)\right]$, then optimal prices are given as:

$$p_1 = \frac{1}{2} \qquad \text{and} \qquad p_2 = \cdots = p_n = \frac{1}{2} + \frac{1-\beta^2(1-\xi+\xi/(n-1))+\xi\beta(n-2)/(n-1)}{2}w,$$

where location 1 is the center of the star and locations $2, \cdots, n$ are the leaves. In addition, the platform's profits as a function of ξ for this range are equal to

$$\Pi(\mathbf{A}^{\xi}, \mathbf{1}) = \frac{n}{4} - (n-1)\left(\frac{1-\beta^2(1-\xi+\xi/(n-1))+\xi\beta(n-2)/(n-1)}{2}\right)^2$$
$$- w\left(\frac{1}{2} - \frac{1-\beta^2(1-\xi+\frac{\xi}{n-1})+\xi\beta\frac{n-2}{n-1}}{2}\right)\left((1-\beta^2)(n-1)-\beta^2 n\xi-(n-2)\xi\beta\right).$$

(b) If $\xi \in \left[\max\left(\frac{(n-1)}{2(1-\beta)\beta(n-2)}\left(\beta(1-2\beta)+\sqrt{\frac{\beta^2(n-1)+4\beta-4}{n-1}}\right), 0\right), \frac{\beta(n-1)-1}{\beta(n-2)}\right]$, then optimal prices are given as:

$$p_2 = \cdots = p_n = \frac{1}{2} + \frac{\beta Z(1+\beta Z+\beta w)+w(n-1)-w\beta\xi(n-2)}{2(n-1)+2\beta^2 Z^2},$$

where $Z = (\xi(n-2)-(n-1))$ and

$$p_1 = 1 - \beta((1-\xi)(n-1)+\xi)(1-p_2).$$

In addition, the platform's profits for this range are equal to

$$p_1(1-p_1) + (n-1)p_2(1-p_2) - w(1-\beta)\left((n-1)(1-p_2)+(1-p_1)\right).$$

(c) Finally, if $\xi \in \left[\frac{\beta(n-1)-1}{\beta(n-2)}, 1\right]$, then optimal prices are all equal, i.e.,

$$p_1 = \cdots = p_n = \frac{1}{2} + \frac{(1-\beta)w}{2}.$$

The platform's profits are equal to

$$n\left(\frac{1}{2} - \frac{(1-\beta)w}{2}\right)^2.$$

Proof. First, recall that in any optimal solution we must have $d_i = (1-p_i)$. Thus, we can rewrite problem (6) as follows

$$\begin{aligned} \max_{\{d_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i (1-d_i)\, d_i - w\sum_i \delta_i \\ s.t. \quad & \sum_j y_{ij} = \beta\Big[\sum_j \alpha_{ji} d_j + \sum_j y_{ji}\Big] + \delta_i - d_i \text{ for all } i \\ & \delta_i, y_{ij} \geq 0, \text{ for all } i, j. \end{aligned} \tag{44}$$

Note that in Problem (44) we relax the constraints $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p} \leq \mathbf{1}$ (equivalently, $\mathbf{d} \leq \mathbf{1}$ and $\mathbf{d} \geq \mathbf{0}$), which is without loss of optimality, since the resulting optimal prices do not violate the constraints, as we argue subsequently. This is a convex optimization problem with affine constraints, so the Karush–Kuhn–Tucker conditions are both necessary and sufficient for optimality. In particular, let $\gamma_i (i = 1, \cdots, n)$ and $\omega_{ij} (i, j = 1, \cdots, n)$, denote the dual variables corresponding to the inequality constraints $-\delta_i \leq 0$ and $-y_{ij} \leq 0$ respectively, and $\lambda_i (i = 1, \cdots, n)$ denote those corresponding to the equality constraints in optimization problem (44). Then, the corresponding KKT conditions can be written as:

(i) $1 - 2d_i + \beta \sum_j \lambda_j \alpha_{ij} - \lambda_i = 0$ for all i,

(ii) $-w + \lambda_i + \gamma_i = 0$ for all i,

(iii) $-\lambda_i + \beta\lambda_j + \omega_{ij} = 0$ for all i, j,

(iv) $\lambda_i \left(\beta\Big[\sum_j \alpha_j d_j + \sum_j y_{ji}\Big] + \delta_i - d_i - \sum_j y_{ij}\right) = 0$,

(v) $\gamma_i \delta_i = 0 = \omega_{ij} y_{ij}$, for all i, j,

along with primal feasibility and the non-negativity of γ_i, ω_{ij}. Using these conditions, we establish the optimality of Cases (a) and (b) by constructing a pair of primal-dual solutions. The optimality of Case (c) follows directly from Proposition 2.

Case (a): First, we provide values for the primal variables. Using the expressions for p_1 and $p_2, \cdots, p_n$ as stated in Case (a) above, we have $d_1 = \frac{1}{2}$ and $d_i = 1 - p_i$ for $i \geq 2$. In addition, we let $\delta_1 = 0$ and

$$\delta_i = d_i - \beta\frac{x_1}{n-1} - \beta d_i \xi \frac{n-2}{n-1} = \left(1 - \beta\xi\frac{n-2}{n-1} - \beta^2\left((1-\xi) + \frac{\xi}{n-1}\right)\right) d_i.$$

In addition, we let $y_{i1} = 0$, $y_{ij} = 0$ and $y_{1i} = \beta((1-\xi) + \xi/(n-1))d_i - \frac{1}{2(n-1)}$ for $2 \leq i, j \leq n$. Note that, under the conditions of Case (a), this is a feasible solution. In particular, note that the supply of drivers reaching the center of the star is at least $1/2$. Then, it is straightforward to see that if we let $\lambda_i = w$ for $i \geq 2$, $\lambda_1 = \beta w$, $\gamma_1 = (1-\beta)w$, $\gamma_i = 0$ for $i \geq 2$, and $\omega_{ij} = \lambda_i - \beta\lambda_j$, for i, j, the KKT conditions are satisfied.

Case (b): Similarly, we provide values for the primal variables. Using the expressions for p_1 and $p_2, \cdots, p_n$ as stated in Case (b) above as well as the expression for Z, we have that if $d_1 = 1 - p_1$ and $d_i = 1 - p_i$ for $i \geq 2$, then $d_1 = -\beta Z d_2$. Noting that the entry is only at the leaves (and thus $\delta_2, \cdots, \delta_n > 0$), we obtain $\gamma_2 = \cdots = \gamma_n = 0$ and $\lambda_2 = \cdots = \lambda_n = w$. Using the definition of λ_i for $i \geq 2$, we note that λ_1 must be equal to $1 - 2d_1 + \beta(n-1)\lambda_i\alpha_{ij} = 1 + 2\beta Z d_i + \beta w$ for $i \geq 2$. To show optimality, we simply need to check that the equality $2d_i = 1 + \beta(n-2)\lambda_i\alpha_{ij} + \beta\lambda_1\alpha_{i1} - \lambda_i$ is satisfied for all $i, j \neq 1$ (all other conditions are satisfied). Note that $\alpha_{i1} = \xi/(n-1) + (1-\xi) = -Z/(n-1)$ for $i \neq 1$; thus, we can rewrite the right hand side as:

$$1 + \beta(n-2)\lambda_i\alpha_{ij} + \beta\lambda_1\alpha_{i1} - \lambda_i = 1 + \beta(n-2)w\frac{\xi}{n-1} - \beta\frac{Z}{n-1}(1 + 2\beta Z d_i + \beta w) - w,$$

for $i, j \neq 1$. Multiplying by $(n-1)$ and rearranging terms yields

$$\left(2(n-1) + 2\beta^2 Z^2\right) d_i = (n-1) + \left(\xi\beta(n-2) - \beta^2 Z - (n-1)\right) w - \beta Z.$$

By adding and subtracting $\beta^2 Z^2$ from the left-hand side, we obtain the desired expression. Therefore, we have shown that the solution induced by these prices is optimal. □

Proof of Corollary 4

The claim follows directly from Proposition 2, which establishes that when the underlying demand pattern is balanced, the platform maximizes its profits by setting the same price at all locations, i.e., $p_i^\star = p^\star = 1/2 + (1-\beta)w/2$ for all i, and, in addition, the optimal solution can be supported by the same compensation for drivers at all locations, i.e., $c_i^\star = c^\star = (1-\beta)w$, for all i. Thus, the platform maximizes its profits by setting $p_i^\star = p^\star = 1/2 + (1-\beta)w/2$ for all i and using fixed commission rate $\gamma^\star = c^\star/p^\star$, which implies that for every ride a driver completes, she earns $\gamma^\star p^\star = c^\star$. □

Proof of Proposition 7

Before establishing the proposition, we state and prove two lemmas.

Lemma B.4. *Consider a demand pattern $(\mathbf{A}, \boldsymbol{\theta})$, and assume that $w = 1$. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be an optimal solution to Problem (6). Then, if $\delta_i^\star + \sum_j y_{ji}^\star > 0$ for all i, the solution to Problem (6) can be implemented using a commission rate that is fixed across the network's locations with the same prices $\{p_i^\star\}_{i=1}^n$ and $\gamma^\star = \frac{2(1-\beta)}{2-\beta}$.*

Proof. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be an optimal solution to Problem (6). Recall that for any arbitrary $\boldsymbol{\theta}$ the vector of optimal prices $\{p_i^\star\}_{i=1}^n$ and the vector of optimal dual variables $\boldsymbol{\lambda}^\star$ must satisfy:

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta \mathbf{A} \boldsymbol{\lambda}^\star}{2},$$

as given in Equation (19). Furthermore, as all locations are entry points, we must have $\lambda_i^\star = 1$ for all i (see Lemma A.1). Given that $\mathbf{A}$ is row-stochastic, we have $p_i^\star = 1 - \beta/2$ for all i. In addition, since all locations are entry points, we obtain that the supply in the optimal solution must satisfy $x_i^\star = (1 - p_i^\star)\theta_i$ at all locations.

To complete the proof, we show that the optimal solution to Problem (6) is in fact an equilibrium under the same vector of prices $\{p_i^\star\}_{i=1}^n$ and fixed commission rate $\gamma^\star$ as defined in the statement of the lemma. To that end, consider the recursion given by

$$V_i = \frac{(1-p_i^\star)\theta_i}{x_i^\star}\gamma p_i^\star + \frac{(1-p_i^\star)\theta_i}{x_i^\star}\beta \sum_j \alpha_{ij} V_j + \beta \left(1 - \frac{(1-p_i^\star)\theta_i}{x_i^\star}\right) \overline{V} \text{ for all } i. \tag{45}$$

Using the fact that $p_i^\star = 1 - \beta/2$ and $x_i^\star = (1 - p_i^\star)\theta_i$ at all locations, we can rewrite the expression above as $V = \gamma \left(1 - \frac{\beta}{2}\right) \mathbf{1} + \beta \mathbf{A} V$. Using $\gamma = \gamma^\star$, it is straightforward to see that $V_i = 1$ for all i is in fact a solution to the above system. Therefore, the fixed commission rate $\gamma^\star$ and the vector of prices $\mathbf{p}^\star$ given above constitute an equilibrium as claimed. □

Lemma B.5. *Consider a network with two locations, demand pattern $(\mathbf{A}, \boldsymbol{\theta})$, and $w = 1$. Then, the optimal solution to Problem (6) can be implemented using a fixed commission rate.*

Proof. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^2$ be an optimal solution to Problem (6). We will show that there exists a $\gamma^\star$ such that the profit of the platform using a fixed commission rate with parameters $\{p_i^\star\}_{i=1}^n$ and $\gamma = \gamma^\star$ is equal to the optimal profit corresponding to the solution to Problem (6). Note that if $\delta_i^\star = 0$ for all i, then the claim follows trivially since the platform provides no service. Thus, for the remainder of the proof we assume that there exists location i with $\delta_i^\star > 0$. Since the network has only two locations, we know that it must be the case that either one or both are entry points in the optimal solution to Problem (6). If both locations are entry points, the result follows by Lemma B.4. Therefore, it suffices to show the result for the case in which only one location is an entry point.

To that end, assume without loss of generality that $\delta_1 > 0$, i.e., location 1 is an entry point. First, we show that there exists a γ such that the expected earnings of the drivers when prices, entry, and relocation are given by $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^2$ satisfy $V_1 = 1$ and $V_2 \leq 1$, that is, they satisfy the equilibrium conditions under a fixed commission rate. Second, we establish that this implies that the profits under a fixed commission rate are equal to those of the optimal solution to Problem (6).

Let $q_i^\star = \frac{\theta_i(1-p_i^\star)}{x_i^\star}$ denote the probability of accepting a ride at location i, where $x_i^\star$ is as defined by Equation (1). Noting that $q_1^\star = 1$ (since Lemma A.1 implies that when $\delta_i^\star > 0$ for some location i, then $y^\star_{ij} = 0$ for all j and $q^\star_i = 1$), we obtain the following for a fixed $\gamma \in (0,1)$:

$$V_1 = \gamma p^\star_1 + \beta\alpha_{11}V_1 + \beta\alpha_{12}V_2 \tag{46}$$
$$V_2 = \gamma q^\star_2 p^\star_2 + \beta(1 - q^\star_2\alpha_{22})V_1 + \beta q^\star_2\alpha_{22}V_2. \tag{47}$$

Next, we show that $p^\star_1 \geq p^\star_2$ and, as a consequence, $0 \leq (p^\star_1 - p^\star_2) \leq p^\star_1 - q^\star_2 p^\star_2$. Assume by way of contradiction that $p^\star_2 > p^\star_1$. Recall that the vector of optimal prices $\mathbf{p}^\star$ and the vector of optimal dual variables $\boldsymbol{\lambda}^\star$ must satisfy

$$\mathbf{p}^\star = \frac{\mathbf{1} + \boldsymbol{\lambda}^\star - \beta\mathbf{A}\boldsymbol{\lambda}^\star}{2},$$

from Equation (19). By subtracting the equation for $p^\star_1$ from that of $p^\star_2$, we must have:

$$\begin{aligned} 0 \; < \; 2(p^\star_2 - p^\star_1) \quad &= \quad \lambda^\star_2 - \lambda^\star_1 + \beta\lambda^\star_2\left(\alpha_{12} - \alpha_{22}\right) + \beta\lambda^\star_1\left(\alpha_{11} - \alpha_{21}\right) \\ &= \quad \left(\lambda^\star_2 - \lambda^\star_1\right) + \beta\left(\lambda^\star_2 - \lambda^\star_1\right)\left(\alpha_{12} - \alpha_{22}\right) \\ &= \quad \left(\lambda^\star_2 - \lambda^\star_1\right)\left(1 + \beta\left(\alpha_{12} - \alpha_{22}\right)\right), \end{aligned}$$

where the second equality follows from the fact that A is row-stochastic and thus $\alpha_{11} = 1 - \alpha_{12}$ and $\alpha_{21} = 1 - \alpha_{22}$. Note that $\left(1 + \beta\left(\alpha_{12} - \alpha_{22}\right)\right) > 0$ for $\beta < 1$. In addition, recall that at any optimal solution we must have $\beta \leq \lambda^\star_i \leq 1$ for all i, and $\lambda^\star_i = 1$ if i is an entry entry point (see Lemma A.1). Therefore, we have that $\left(\lambda^\star_2 - \lambda^\star_1\right) \leq 0$ and thus $\left(\lambda^\star_2 - \lambda^\star_1\right)\left(1 + \beta\left(\alpha_{12} - \alpha_{22}\right)\right) \leq 0$, which is a contradiction implying that $p^\star_1 \geq p^\star_2$.

Subtracting the Expression (47) from (46) yields

$$\begin{aligned} 0 \leq \gamma\left(p^\star_1 - q^\star_2 p^\star_2\right) \quad &= \quad V_1 - \beta\alpha_{11}V_1 - \beta\alpha_{12}V_2 - V_2 + \beta(1 - q^\star_2\alpha_{22})V_1 + \beta q^\star_2\alpha_{22}V_2 \\ &= \quad V_1\left(1 - \beta\alpha_{11} + \beta(1 - q^\star_2\alpha_{22})\right) - V_2\left(1 + \beta\alpha_{12} - \beta q^\star_2\alpha_{22}\right) \\ &= \quad \left(V_1 - V_2\right)\left(1 + \beta - \beta\alpha_{11} - \beta q^\star_2\alpha_{22}\right), \end{aligned}$$

where in the last equality we used the fact that $\mathbf{A}$ is row-stochastic and thus $\alpha_{12} = 1 - \alpha_{11}$. Note that this implies that $V_1 \geq V_2$ for *any* fixed γ, provided that the rest of $\{p^\star_i, \delta^\star_i, y^\star_{ij}\}^n_{i,j=1}$ remain fixed. We conclude the first step by noting that for $\{x^\star_i, \delta^\star_i, y^\star_{ij}\}^n_{i,j=1}$ to be an equilibrium under a fixed commission rate with $\{p^\star_i\}^n_{i=1}$and $\gamma = \gamma^\star$ it suffices to set $\gamma^\star$ so that $V_1 = 1$.

The second step involves establishing that the profits corresponding to the two solutions, i.e., the optimal solution to Problem (6) and its implementation using a fixed commission rate, are equal. This follows from noting that Expressions (46) and (47) when $V_1 = 1$ imply that

$$\gamma p^\star_1(1 - p^\star_1)\theta_1 = (1 - p^\star_1)\theta_1 - \beta\alpha_{11}(1 - p^\star_1)\theta_1 - \beta\alpha_{12}V_2(1 - p^\star_1)\theta_1$$
$$\gamma p^\star_2(1 - p^\star_2)\theta_2 = (V_2 - \beta)x^\star_2 + \beta\alpha_{22}(1 - p^\star_2)\theta_2 - \beta\alpha_{22}(1 - p^\star_2)\theta_2 V_2,$$

which, in turn, using the fact that $x_2^\star = \beta(\alpha_{12}(1-p_1^\star)\theta_1 + \alpha_{22}(1-p_2^\star)\theta_2)$ yields:

$$\gamma(p_1^\star(1-p_1^\star)\theta_1 + p_2^\star(1-p_2^\star)\theta_2) = (1-p_1^\star)\theta_1 - \beta\alpha_{11}(1-p_1^\star)\theta_1 - \beta x_2^\star + \beta\alpha_{22}(1-p_2^\star)\theta_2 = \delta_1^\star,$$

where the last equality following from the first constraint in Problem 6. Finally, given that the profits corresponding to the two solutions can be written as $\sum_i p_i^\star(1-p_i^\star)\theta_i - \delta_1^\star$ and $\sum_i p_i^\star(1-p_i^\star)\theta_i - \gamma^\star\left(\sum_i p_i^\star(1-p_i^\star)\theta_i\right)$ respectively, we conclude that the two solutions lead to equal profits. □

Proof of Proposition 7: We reduce the platform's pricing problem with a two-type demand pattern to an equivalent pricing problem in a network with only two locations such that each location aggregates all locations belonging to the same type. We show that the optimal solution in this two-location network can be constructed using the optimal solution to the original problem. We then exploit the fact that there exists a $\gamma^\star$ such that the optimal solution for the two-location network can be implemented using a fixed commission rate (Lemma B.5) to finally argue that $\{x_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an equilibrium under a fixed commission rate with $\{p_i^\star\}_{i=1}^n$and $\gamma = \gamma^\star$ that achieves the optimal profit.

In particular, we define a network with two locations for which $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{A}}$ are defined as follows:

- $\hat{\theta}_1 = \sum_{i\in\mathcal{N}_1}\theta_i$ and $\hat{\theta}_2 = \sum_{j\in\mathcal{N}_2}\theta_j$.
- $\hat{\alpha}_{11} = \frac{\sum_{i\in\mathcal{N}_1}\sum_{i'\in\mathcal{N}_1}\alpha_{ii'}}{|\mathcal{N}_1|}$, $\hat{\alpha}_{12} = \frac{\sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2}\alpha_{ij}}{|\mathcal{N}_1|}$, $\hat{\alpha}_{21} = \frac{\sum_{j\in\mathcal{N}_2}\sum_{i\in\mathcal{N}_1}\alpha_{ji}}{|\mathcal{N}_2|}$, and $\hat{\alpha}_{22} = \frac{\sum_{j\in\mathcal{N}_2}\sum_{j'\in\mathcal{N}_2}\alpha_{jj'}}{|\mathcal{N}_2|}$.

That is, location 1 aggregates the locations in $\mathcal{N}_1$ and location 2 aggregates those in $\mathcal{N}_2$. The demand at each location corresponds to the total demand of the locations in each of the two sets and the transition probabilities represent the average probability that a ride originating from one of the sets has as its destination a location in the same/different set.

We can now relate the optimal solution to Problem (6) in the original network with that in the two-location network as follows. Let $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ be the optimal solution to Problem (6) in the original network, and let $x^\star$ denote the associated vector of supply. Since the objective function in Problem (6) is concave, we can establish that there is a symmetric solution in the original network, i.e., a solution that features the same price for all locations belonging to the same subset, i.e., $p_i^\star = p_{i'}^\star$ for all $i, i' \in \mathcal{N}_1$ and $p_j^\star = p_{j'}^\star$ for all $j, j' \in \mathcal{N}_2$. To see why, assume by way of contradiction that this is not the case, i.e., there does not exist an optimal solution that is symmetric. Then, if $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an optimal solution to (6), consider tuple $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ such that:

- $p_i' = \frac{1}{|\mathcal{N}_1|}\sum_{k\in\mathcal{N}_1} p_k^\star$ for all $i \in \mathcal{N}_1$ and $p_j' = \frac{1}{|\mathcal{N}_2|}\sum_{k\in\mathcal{N}_2} p_k^\star$ for all $j \in \mathcal{N}_2$,
- $\delta_i' = \frac{1}{|\mathcal{N}_1|}\sum_{k\in\mathcal{N}_1}\delta_k^\star$ and $\delta_j' = \frac{1}{|\mathcal{N}_2|}\sum_{k\in\mathcal{N}_2}\delta_k^\star$, and
- $y_{ij}' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\sum_{k\in\mathcal{N}_1}\sum_{\ell\in\mathcal{N}_2} y_{k\ell}^\star$ and $y_{ji}' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\sum_{\ell\in\mathcal{N}_2}\sum_{k\in\mathcal{N}_1} y_{\ell k}^\star$ for all $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$. Also, $y_{ii'}' = \frac{1}{|\mathcal{N}_1|^2}\sum_{k\in\mathcal{N}_1}\sum_{k'\in\mathcal{N}_1} y_{kk'}^\star$ and $y_{jj'}' = \frac{1}{|\mathcal{N}_2|^2}\sum_{k\in\mathcal{N}_2}\sum_{k'\in\mathcal{N}_2} y_{kk'}^\star$ for all $i, i' \in \mathcal{N}_1$ and $j, j' \in \mathcal{N}_2$.

Note that the concavity of the objective function implies that the profits corresponding to tuple $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ are at least as high as those corresponding to $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ assuming that there is enough supply to satisfy the entire induced demand $\sum_{i=1}^n (1-p_i')\theta_i$. To see that the latter is true, consider locations that belong to $\mathcal{N}_1$ (a similar argument holds for locations in $\mathcal{N}_2$). Note that by construction the supply of drivers and the induced demand under $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$ is the same at each of the locations in $\mathcal{N}_1$. Thus, it suffices to establish that $\sum_{i\in\mathcal{N}_1} x_i' \geq \sum_{i\in\mathcal{N}_1}(1-p_i')\theta_i$. To this end, we have

$$\begin{aligned}\sum_{i\in\mathcal{N}_1} x_i' &= \beta \sum_{i\in\mathcal{N}_1} \Big[\sum_{k\in\mathcal{N}_1} (\alpha_{ki}(1-p_k')\theta_k + y_{ki}) + \sum_{k\in\mathcal{N}_2} (\alpha_{ki}(1-p_k')\theta_k + y_{ki}) \Big] + \sum_{i\in\mathcal{N}_1} \delta_i \\ &= \sum_{i\in\mathcal{N}_1} x_i^\star \geq \sum_{i\in\mathcal{N}_1} (1-p_i^\star)\theta_i = \sum_{i\in\mathcal{N}_1} (1-p_i')\theta_i,\end{aligned}$$

where the equalities follow from the definition of two-type demand patterns and the construction of tuple $\{p_i', \delta_i', y_{ij}'\}_{i,j=1}^n$. The inequality $\sum_{i\in\mathcal{N}_1} x_i' \geq \sum_{i\in\mathcal{N}_1}(1-p_i')\theta_i$ follows directly from the fact that in an optimal solution the available supply of drivers has to be greater than the induced demand. Thus, it follows that there exists a symmetric optimal solution, i.e., we can assume that $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is symmetric.

Next, we define a solution $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ for the two-location network that generates the same profits as $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ in the original network as follows:

- $\hat{p}_1 = p_i^\star$ for $i \in \mathcal{N}_1$ and $\hat{p}_2 = p_j^\star$ for $j \in \mathcal{N}_2$.
- $\hat{\delta}_1 = \sum_{k\in\mathcal{N}_1} \delta_k^\star = |\mathcal{N}_1|\delta_i^\star$ for $i \in \mathcal{N}_1$, and $\hat{\delta}_2 = \sum_{k\in\mathcal{N}_2} \delta_k^\star = |\mathcal{N}_2|\delta_j^\star$ for $j \in \mathcal{N}_2$.
- $\hat{y}_{12} = \sum_{i\in\mathcal{N}_1}\sum_{j\in\mathcal{N}_2} y_{ij}^\star$ and $\hat{y}_{21} = \sum_{j\in\mathcal{N}_2}\sum_{i\in\mathcal{N}_1} y_{ji}^\star$.

It is straightforward that $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ is in fact an optimal solution to Problem (6) in the two-location case. By way of contradiction, suppose that there exists another solution $\{\tilde{\mathbf{p}}, \tilde{\boldsymbol{\delta}}, \tilde{\mathbf{Y}}\}$ in the two-location network that generates higher profits for the platform. Then, the following is a solution to Problem (6) that generates higher profits for the platform than $\{p_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ in the original network, which leads to a contradiction:

- $p_i'' = \tilde{p}_1$ for all $i \in \mathcal{N}_1$ and $p_j'' = \tilde{p}_2$ for all $i \in \mathcal{N}_2$,
- $\delta_i'' = \frac{1}{|\mathcal{N}_1|}\tilde{\delta}_1$ for all $i \in \mathcal{N}_1$ and $\delta_j'' = \frac{1}{|\mathcal{N}_2|}\tilde{\delta}_2$ for all $i \in \mathcal{N}_2$, and
- $y_{ij}'' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\tilde{y}_{12}$ and $y_{ji}'' = \frac{1}{|\mathcal{N}_1||\mathcal{N}_2|}\tilde{y}_{21}$ for all $i \in \mathcal{N}_1$ and $j \in \mathcal{N}_2$. Also, $y_{ii'}'' = 0$ and $y_{jj'}'' = 0$ for all $i, i' \in \mathcal{N}_1$ and $j, j' \in \mathcal{N}_2$.

The optimality of $\{\hat{\mathbf{p}}, \hat{\boldsymbol{\delta}}, \hat{\mathbf{Y}}\}$ in the two-location network and Lemma B.5 imply that there exists a $\gamma^\star$ such that $\{\hat{x}_i, \hat{\delta}_i, \hat{y}_{ij}\}_{i=1}^n$ is an equilibrium under a fixed commission rate $\gamma^\star$ and prices $\{\hat{p}_i\}_{i=1}^n$ in the two-location network. To complete the proof, it suffices to argue that $\{x_i^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is an equilibrium under a fixed commission rate with $\{p_i^\star\}_{i=1}^n$ and $\gamma = \gamma^\star$ that achieves the optimal profit. However, this follows from the mapping between the solutions in the original and two-location networks—a driver entering the platform to provide service makes exactly the same profit in both cases. □

Proof of Proposition 8

Let $\psi_i(i = 1, \cdots, n)$, $\phi_i(i = 1, \cdots, n)$, $\omega_{ij}(i, j = 1, \cdots, n)$, $\nu_i(i = 1, \cdots, n)$ denote the dual variables corresponding to the inequality constraints $-p_i \leq 0, -\delta_i \leq 0, -y_{ij} \leq 0, p_i - 1 \leq 0$, respectively, and $\lambda_i(i = 1, \cdots, n)$ denote those corresponding to the equality constraints in optimization problem (15). Then, the corresponding KKT conditions imply the following for the optimal solution to (15) under Assumption 2 (note that the platform's optimization problem is a convex program with affine constraints and, therefore, Slater's condition holds):

(1) Taking the derivative of the objective function and constraints with respect to δ_i yields $-1 = -\lambda_i^\star - \phi_i^\star$. Given that $\phi_i^\star \geq 0$ we have that $\lambda_i^\star \leq 1$. In addition, from complementary slackness we obtain that $\lambda_i^\star = 1$ when $\delta_i^\star > 0$.

(2) Furthermore, the derivative with respect to y_{ij} yields $0 = \lambda_i^\star - \beta^{\zeta_{ij}}\lambda_j^\star - \omega_{ij}^\star$. Note that since $\omega_{ij}^\star \geq 0$ we have $\lambda_i^\star \geq \beta^{\zeta_{ij}}\lambda_j^\star$ and from complementary slackness we have that if $y_{ij}^\star > 0$, i.e., if it is optimal to relocate excess supply from location i to j, then $\lambda_i^\star = \beta^{\zeta_{ij}}\lambda_j^\star$.

Next, we establish that the compensations defined by Equation (16) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$, i.e., the optimal solution to Problem (15), as an equilibrium under price vector $\{p_i^\star\}_{i=1}$. First, note that the compensations defined in (16) are the solution to Equation (48) below, i.e., the equation that describes the drivers' expected earnings, when we set $V_i = \lambda_i^\star$ for all i:

$$V_i = \frac{1 - p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \left(c_i^\star \zeta_{ij} + \beta^{\zeta_{ij}} V_j \right) + \left(1 - \frac{1 - p_i^\star}{x_i^\star} \right) \max_k \beta^{\zeta_{ik}} V_k. \tag{48}$$

To see this, first consider the case where $x_i^\star > (1 - p_i^\star)$, which, in turn, implies that $y_{ij}^\star > 0$ for some j. Then, we can rewrite (48) by setting $V_j = \lambda_j^\star$ in the right hand side as follows:

$$\begin{aligned}
V_i &= \frac{1 - p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \left(c_i^\star \zeta_{ij} + \beta^{\zeta_{ij}} \lambda_j^\star \right) + \left(1 - \frac{1 - p_i^\star}{x_i^\star} \right) \max_k \beta^{\zeta_{ik}} \lambda_k^\star \\
&= \frac{1 - p_i^\star}{x_i^\star} c_i^\star \sum_j \alpha_{ij} \zeta_{ij} + \frac{1 - p_i^\star}{x_i^\star} \sum_j \alpha_{ij} \beta^{\zeta_{ij}} \lambda_j^\star + \left(1 - \frac{1 - p_i^\star}{x_i^\star} \right) \max_k \beta^{\zeta_{ik}} \lambda_k^\star \\
&= \frac{1 - p_i^\star}{x_i^\star} \lambda_i^\star + \left(1 - \frac{1 - p_i^\star}{x_i^\star} \right) \lambda_i^\star = \lambda_i^\star,
\end{aligned}$$

where the equality in the last line follows directly from the definition of compensation $c_i^\star$ (Equation (16)) and the fact that $\lambda_i^\star = \max_k \beta^{\zeta_{ik}} \lambda_k^\star$ from item (2) above. The claim for the case where $x_i^\star = (1 - p_i^\star)$ follows immediately from (48) and the definition of $c_i^\star$.

Therefore, the V_i's as defined here satisfy Equation (48) (which is equivalent to Equation (2) when the compensations and the terms involving β are appropriately scaled with the ζ_{ij}'s). In addition, the drivers' incentive-compatibility constraints, i.e., Equation (3), are satisfied as well, since $\lambda_i^\star = V_i = 1 = w$, when $\delta_i^\star > 0$ and $\lambda_i^\star \leq 1$ for all i. Finally, condition (ii) in the equilibrium definition, i.e., Equation (1) (appropriately scaled to incorporate the ζ_{ij}'s), is satisfied trivially as $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ is feasible for Problem (16) and $x_i^\star \geq (1 - p_i^\star)$.

Thus, we conclude that the compensations defined by Equation (16) can support $\{\delta_i^\star, x_i^\star, y_{ij}^\star\}_{i,j=1}^n$ as an equilibrium under price vector $\{p_i^\star\}_{i=1}$ and expected future earnings for a driver at location i given by $V_i = \lambda_i^\star$.[34] □

Appendix C: Supporting Material

C.1 Nonuniqueness of Equilibria

In this subsection, we illustrate that the equilibrium need not be unique. As a simple example, consider a network consisting of a single location with $\alpha_{11} = 1$, and let $c_1 = (1 - \beta)w$ and $p_1 = 1/2$. Suppose that the riders' willingness to pay is uniformly distributed in $[0, 1]$. Then, we can construct a continuum of equilibria. In particular, any tuple $\{\delta_1, x_1, y_{11}\}$ with

$$\delta_1 \leq \theta_1(1 - F(1/2))(1 - \beta),$$

and $x_1 = \delta_1/(1 - \beta)$ and $y_{11} = 0$, constitutes an equilibrium under $(p_1, c_1) = (1/2, (1 - \beta)w)$. This is straightforward to see, since for any such δ_1 drivers are indifferent between entering or not and upon entering they always get assigned to a ride. Moreover, the profits for the platform corresponding to these equilibria are not the same. In particular, the flow rate of profits for the platform is given by

$$p_1 x_1 - w\delta_1 = \left(\frac{1}{2(1 - \beta)} - w \right) \delta_1.$$

□

C.2 Derivations for the Example in Figure 3

Example. Consider the network depicted in Figure 3 and assume that $w = 1$. Then, restrict attention to location 3 and let γ take some fixed value in $[0, 1]$. For a driver to find it optimal to enter and

[34] Recall that in the case of unequal distances, the expressions involved in the equilibrium definition are scaled appropriately according to the ζ_{ij}'s as we discuss in Subsection 5.1 (footnote 32).

provide service at location 3, i.e., for any demand to be served at location 3, it has to be the case that the price p_3 set by the platform at location 3 satisfies:

$$w = V_3 = \min\{1, (1-p_3)/x_3\} \cdot \gamma p_3 + \beta V_3 \leq \frac{\gamma p_3}{1-\beta}, \tag{49}$$

where x_3 denotes the supply of drivers who provide service at location 3. The right hand side of the inequality is equal to the expected lifetime earnings of a driver when the probability of getting assigned to a rider at location 3 is equal to one (and, therefore, it is an upper bound to the earnings that a driver can make by providing service at 3 when the compensation per ride is equal to γp_3). Expression (49) further implies that, when w is normalized to one, we must have $p_3 \geq (1-\beta)/\gamma$ for any demand to be served at location 3. Thus, in such solutions, the platform's optimal profits at location 3 are equal to the solution to the following problem:

$$\max_{p_3 \in [0,1]} (1-\gamma)\theta_3 p_3 (1-p_3), \text{ subject to } p_3 \geq \frac{1-\beta}{\gamma},$$

which implies that for fixed γ the optimal price $p_3^\star(\gamma)$ is equal to $\max\{1/2, (1-\beta)/\gamma\}$ whereas the platform's optimal profits from location 3 as a function of γ, which we denote by $\Pi_3^\star(\gamma)$, take the following form:

$$\Pi_3^\star(\gamma) = \begin{cases} 0 & \text{if } \gamma \leq 1-\beta, \\ (1-\gamma)\theta_3 \frac{1-\beta}{\gamma}\left(1 - \frac{1-\beta}{\gamma}\right) & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/4(1-\gamma)\theta_3 & \text{otherwise.} \end{cases}$$

To complete the description of the equilibrium outcome for location 3 for a fixed γ, we have

$$p_3^\star(\gamma) = \begin{cases} 1 & \text{if } \gamma \leq 1-\beta, \\ (1-\beta)/\gamma & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/2 & \text{otherwise} \end{cases}$$

where with some abuse of notation $p_3^\star(\gamma)$ denotes the optimal price for the platform at location 3 as a function of γ. Similarly,

$$\delta_3^\star(\gamma) = \begin{cases} 0 & \text{if } \gamma \leq 1-\beta, \\ (1-\beta)(1-p_3^\star(\gamma))\theta_3 & \text{if } \gamma \in (1-\beta, \min\{2(1-\beta), 1\}), \\ 1/4 \cdot \gamma\theta_3 & \text{otherwise} \end{cases}$$

and $x_3^\star(\gamma) = \delta_3^\star(\gamma)/(1-\beta)$, whereas $y_{33}^\star(\gamma) = x_3^\star(\gamma) - (1-p_3^\star(\gamma))\theta_3$ and $y_{i3}^\star(\gamma) = y_{3i}^\star(\gamma) = 0$ for $i \neq 3$.

Furthermore, for the subgraph consisting of locations 1 and 2 a similar analysis when $\theta_2 = \epsilon \to 0$ yields the following for the optimal profits in the subgraph as a function of γ (which we denote by $\Pi^\star_{1,2}(\gamma)$):

$$\Pi^\star_{1,2}(\gamma) = \begin{cases} 0 & \text{if } \gamma \le 1-\beta^2, \\ (1-\gamma)\theta_1 \frac{1-\beta^2}{\gamma}\left(1 - \frac{1-\beta^2}{\gamma}\right) & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/4(1-\gamma)\theta_1 & \text{otherwise.} \end{cases}$$

To complete the description of the equilibrium outcome for the subgraph for a fixed γ, we have

$$p^\star_1(\gamma) = \begin{cases} 1 & \text{if } \gamma \le 1-\beta^2, \\ (1-\beta^2)/\gamma & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/2 & \text{otherwise} \end{cases}$$

and $p^\star_2(\gamma) = 0$. Also,

$$\delta^\star_1(\gamma) = \begin{cases} 0 & \text{if } \gamma \le 1-\beta^2, \\ (1-\beta^2)(1-p^\star_1(\gamma))\theta_1 & \text{if } \gamma \in (1-\beta^2, \min\{2(1-\beta^2), 1\}), \\ 1/4 \cdot \gamma\theta_1 & \text{otherwise} \end{cases}$$

and $\delta^\star_2(\gamma) = 0$. Finally, $x^\star_1(\gamma) = \delta^\star_1(\gamma)/(1-\beta^2), x^\star_2(\gamma) = \beta(1-p^\star_1(\gamma))$, $y^\star_{21}(\gamma) = \beta(1-p^\star_1(\gamma))$, and $y^\star_{11}(\gamma) = x^\star_1(\gamma) - (1-p^\star_1(\gamma))\theta_1$, which completes the description of the equilibrium outcome for locations 1,2 for a fixed γ.

To see this, note that if any demand is served at the subgraph consisting of locations 1 and 2, it has to be the case that $\delta_1 > 0 = \delta_2$, i.e., drivers enter at location 1. Moreover, a driver who gets assigned to a ride at 1, completes it at location 2, and then returns to 1 without earning additional compensation. Thus, the supply of new drivers who enter the platform at location 1 in each time period is equal to $(1-\beta^2)x^\star_1(\gamma)$ given that β^2-fraction of the drivers who provide service at 1 return back to this location and continue providing service. Thus, the platform's problem at the subgraph consisting of locations 1 and 2 is essentially equivalent to the problem at location 3 when the fraction of drivers continuing to provide service changes from β to β^2. Finally, the platform's optimal choice of γ is the value that maximizes $\Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma)$. Let Π_{opt} denote the corresponding optimal objective of (6) for the same network. In Figure 3, we illustrate the profit gap between Π_{opt} and $\max_\gamma \Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma)$, i.e., $1 - (\max_\gamma(\Pi^\star_{1,2}(\gamma) + \Pi^\star_3(\gamma))/\Pi_{opt}$ for different values of β.

C.3 Origin-Destination Pricing

Our analysis so far has restricted attention to pricing/compensation policies that depend only on the origin of a requested ride, i.e., the platform optimizes over the tuple $\{p_i, c_i\}_{i=1}^n$ where p_i, c_i are the price and compensation respectively corresponding to a ride that originates from location i (irrespective of its destination). Here, we consider the extension of origin-destination pricing policies, i.e., we allow the platform to optimize over $\{p_{ij}, c_{ij}\}_{i,j=1}^n$ where p_{ij} (c_{ij}) denotes the price (compensation) for a ride from i to j. Finally, similarly to the rest of our analysis, we assume that the distribution of the riders' willingness to pay is the same for all origin-destination pairs.

Similar to optimization problem (6) we obtain

$$\begin{aligned} \max_{\{p_{ij},\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i \sum_j \alpha_{ij} p_{ij}(1-p_{ij})\theta_i - w\sum_i \delta_i \\ s.t. \quad & \sum_j \alpha_{ij}(1-p_{ij})\theta_i + \sum_j y_{ij} - \beta \sum_j \left[\alpha_{ji}(1-p_{ji})\theta_j + y_{ji}\right] - \delta_i = 0, \text{ for all } i \\ & p_{ij}, \delta_i, y_{ij} \geq 0, \text{ for all } i, j \\ & p_{ij} \leq 1, \text{ for all } i, j, \end{aligned} \tag{50}$$

for determining the optimal origin-destination pricing policy when the drivers' incentive-compatibility constraints are relaxed and we only require that each incoming driver is given a one-time payment of w. Next, note that a result analogous to Lemma 1 holds in the case of origin-destination pricing as well, thus enabling us to work directly with optimization problem (50) (the proof is essentially the same as that of Lemma 1 after restricting attention to the case where the riders' willingness to pay is uniformly distributed in $[0,1]$ and it is, therefore, omitted).

Optimization problem (50) is clearly convex, therefore, deriving the optimal origin-destination pricing policy is computationally efficient. In addition, the Karush–Kuhn–Tucker conditions imply the following proposition that provides a characterization of the optimal prices as a function of the set of dual variables corresponding to the equality constraints in (50).

Proposition C.1. *Suppose that Assumption 2 holds and consider the optimal solution* $\{p_{ij}^\star, \delta_i^\star, y_{ij}^\star\}_{i,j=1}^n$ *to optimization problem* (50) *and let* $\{\lambda_i^\star\}_{i=1}^n$ *denote the set of optimal dual variables corresponding to the equality constraints in Problem* (50). *Then,*

$$p_{ij}^\star = \frac{1 + \lambda_i^\star - \beta\lambda_j^\star}{2}. \tag{51}$$

Furthermore, $\lambda_i^\star \in [\beta, 1]$ *for all* i *with* $\lambda_i^\star = \beta$ *when* $y_{ij}^\star > 0$*, i.e., location* i *has an excess supply of drivers, and* $\lambda_i^\star = 1$ *when* $\delta_i^\star + \sum_j y_{ji}^\star > 0$*, i.e., location* i *is an entry point.*

Proof. First, we relax the constraints $p_{ij} \geq 0$ and $p_{ij} \leq 1$ in optimization problem (50) (which as we establish subsequently is without loss of optimality). Then, we let $\phi_i (i = 1, \cdots, n), \omega_{ij}(i, j = 1, \cdots, n)$, denote the dual variables corresponding to the inequality constraints $-\delta_i \leq 0, -y_{ij} \leq 0$, respectively, and $\lambda_i (i = 1, \cdots, n)$ denote those that correspond to the equality constraints for optimization problem (50). The Karush–Kuhn–Tucker conditions associated with Problem (50) yield the following conditions for optimality (note that the platform's optimization problem is a convex program with affine constraints and, therefore, Slater's condition holds):

(1) Taking the derivative with respect to p_{ij} we obtain

$$\alpha_{ij} - 2\alpha_{ij}p^\star_{ij} = -\alpha_{ij}\lambda^\star_i + \lambda^\star_j\beta\alpha_{ij}.$$

(2) Similarly, the derivative with respect to δ_i yields $-1 = -\lambda^\star_i - \phi^\star_i$. Given that $\phi^\star_i \geq 0$ we have that $\lambda^\star_i \leq 1$ and, by complementary slackness, $\lambda^\star_i = 1$ when $\delta^\star_i > 0$.

(3) Finally, the derivative with respect to y_{ij} yields $0 = \lambda^\star_i - \beta\lambda^\star_j - \omega^\star_{ij}$. Note that since $\omega^\star_{ij} \geq 0$ we have $\lambda^\star_i \geq \beta\lambda^\star_j$ and by complementary slackness we have that when $y^\star_{ij} > 0$, i.e., if it is optimal to relocate excess supply from location i to j, then $\lambda^\star_i = \beta\lambda^\star_j$.

Since there is at least one location, say i, with $\delta^\star_i > 0$ (the equilibrium features some entry of new drivers under Assumption 2 as we have established in Lemma A.1) we have that $\lambda^\star_i = 1$ and, consequently, from the KKT conditions involving y_{ij} we obtain that $\lambda^\star_j \geq \beta\lambda^\star_i = \beta$ for all j. Also, the KKT conditions involving δ_i imply that $\lambda^\star_j \leq 1$ for all j, therefore, overall we have $\lambda^\star_j \in [\beta, 1]$ for all j. Moreover, note that the KKT conditions involving y_{ij} imply that when $y^\star_{ij} > 0$ then $\lambda^\star_i = \beta\lambda^\star_j$, which in light of the fact that $\lambda^\star_i \in [\beta, 1]$ implies that $\lambda^\star_i = \beta$ and $\lambda^\star_j = 1$. Finally, from item (1) above we obtain

$$p^\star_{ij} = \frac{1 + \lambda^\star_i - \beta\lambda^\star_j}{2}.$$

In light of the fact that $\lambda^\star_i \in [\beta, 1]$ for all i, prices $\{p^\star_{ij}\}^n_{i,j=1} \in (0, 1)$ and, therefore, relaxing constraints $p_{ij} \geq 0$ and $p_{ij} \leq 1$ is without loss of optimality, which completes the proof of the claim. □

The characterization we provide in Proposition C.1, which is a direct generalization of (8), yields the following interesting observation: although, in principle, the platform has n^2 decision variables when determining the optimal origin-destination pricing policy, it turns out that the optimal prices can be written as a function of only the n dual variables. In addition, $\lambda^\star_i$ can be interpreted as the marginal benefit that the platform derives from having an additional unit of supply at location i. Therefore, the price for a ride from location i to j reflects the relative "values" for the supply of drivers in the two locations, i and j. Intuitively, the value of supply is higher in entry points than it is in locations with excess supply. This is clearly illustrated in the characterization we provide in Proposition C.1: rides from entry points to locations with excess supply are more expensive than those going the opposite direction. Similar to our baseline model, the platform leverages its pricing to deal with imbalances in the demand pattern by incentivizing drivers (through accepting rides) to relocate from locations with excess supply to entry points.

C.4 Heterogeneity in the Drivers' Outside Option

Throughout the paper, we assume that the drivers have the same outside option, which amounts to lifetime earnings equal to w. This allows us to focus on the questions we are mostly interested in; i.e., how imbalances in the demand and its destination preferences across a network of locations may affect a platform's pricing policy and profits.

That said, we describe below how we could accommodate heterogeneity in the outside option among the platform's potential drivers. Suppose that the mass of potential drivers that could join and start providing service for the platform at each time period is Δ and their outside option (reservation wage) is distributed with CDF $G(\cdot)$; i.e., $\Delta \cdot G(w)$ is the total mass of drivers that would be willing to join the platform when their expected lifetime earnings by participating are equal to w. In the remainder of this appendix, we consider optimizing over prices and compensations $\{p_i, c_i\}_{i=1}^n$, where p_i denotes the price that a rider has to pay for a ride that originates from location i, and c_i is the corresponding compensation for the driver (as in the main body of the paper). Then, the platform's optimization problem would be written as (this is an extension of optimization problem (6) incorporating heterogeneity in the reservation wages—note that the relaxation implied by Lemma 1 applies here as well):

$$
\begin{aligned}
\max_{w,\{p_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - w\cdot\Delta\cdot G(w) \\
s.t. \quad & \sum_j y_{ij} = \beta\left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\right] + \delta_i - (1-p_i)\theta_i, \text{ for all } i, \\
& p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \\
& p_i \leq 1, \text{ for all } i, \\
& \sum_i \delta_i \leq \Delta \cdot G(w),
\end{aligned}
\tag{52}
$$

where in the objective function we substitute $w\sum_i \delta_i = w\cdot\Delta\cdot G(w)$, which is a consequence of the fact that the platform will set w so that the total mass of drivers $\sum_i \delta_i$ that would enter each time period is precisely equal to the drivers the platform needs.

Note that although in general optimization problem (52) is non-convex for a general $G(\cdot)$, it is convex for a number of distributions (similar to our discussion in the paper for the riders' willingness to pay distribution $F(\cdot)$). For example, if the distribution of drivers' reservation wages is uniform with support in $[0, \bar{w}]$, we could rewrite (52) as:

$$\begin{aligned}
\max_{w,\{p_i,\delta_i,y_{ij}\}_{i,j=1}^n} \quad & \sum_i p_i(1-p_i)\theta_i - \Delta/\bar{w}\cdot w^2 \\
s.t. \quad & \sum_j y_{ij} = \beta\left[\sum_j \alpha_{ji}(1-p_j)\theta_j + \sum_j y_{ji}\right] + \delta_i - (1-p_i)\theta_i, \text{ for all } i, \\
& p_i, \delta_i, y_{ij} \geq 0, \text{ for all } i, j, \\
& p_i \leq 1, \text{ for all } i, \\
& \sum_i \delta_i \leq \Delta/\bar{w}\cdot w, \\
& w \leq \bar{w}.
\end{aligned} \tag{53}$$

Note that imposing $w \leq \bar{w}$ above (which is necessary given our assumption on the support of the reservation wages) is without any loss of optimality as the platform would never find it optimal to set the wage w higher than $\bar{w}$ (even in the absence of the $w \leq \bar{w}$ constraint). The resulting optimization problem (53) is convex and upon solving it we obtain the equilibrium wage w the platform would find optimal to induce as a function of its demand pattern.

Moreover, note that in the context of the model of the main body, where w is fixed and there is free-entry, drivers' surplus is equal to zero (as their expected lifetime earnings at equilibrium are equal to their outside option). On the other hand, if drivers are heterogeneous in their outside option, i.e., their outside options are distributed according to $G(w)$, we obtain the following expression for their surplus when expected earnings in the platform at equilibrium are equal to w:

$$\Delta \int_0^w (w-x)g(x)dx,$$

where recall that Δ denotes the total mass of drivers who are willing to provide service at every time period. When outside options are distributed uniformly in $[0, \bar{w}]$, we can rewrite the expression for the drivers' surplus as follows

$$\Delta/\bar{w} \int_0^w (w-x)dx = \frac{\Delta}{2\bar{w}}\cdot w^2,$$

i.e., drivers' surplus is increasing with the prevailing equilibrium wage in the platform.

Although drivers' surplus is increasing with the equilibrium wage induced by the platform's pricing policy, this wage and, consequently, drivers' surplus do not satisfy a monotonicity property as a function of the demand pattern's balancedness. In particular, as the following figure illustrates, equilibrium wage/drivers' surplus may increase or decrease as the network becomes more balanced (unlike platforms' profits and consumer surplus that always increase with balancedness even under heterogeneity in the drivers' reservation wages as also illustrated in the figure).

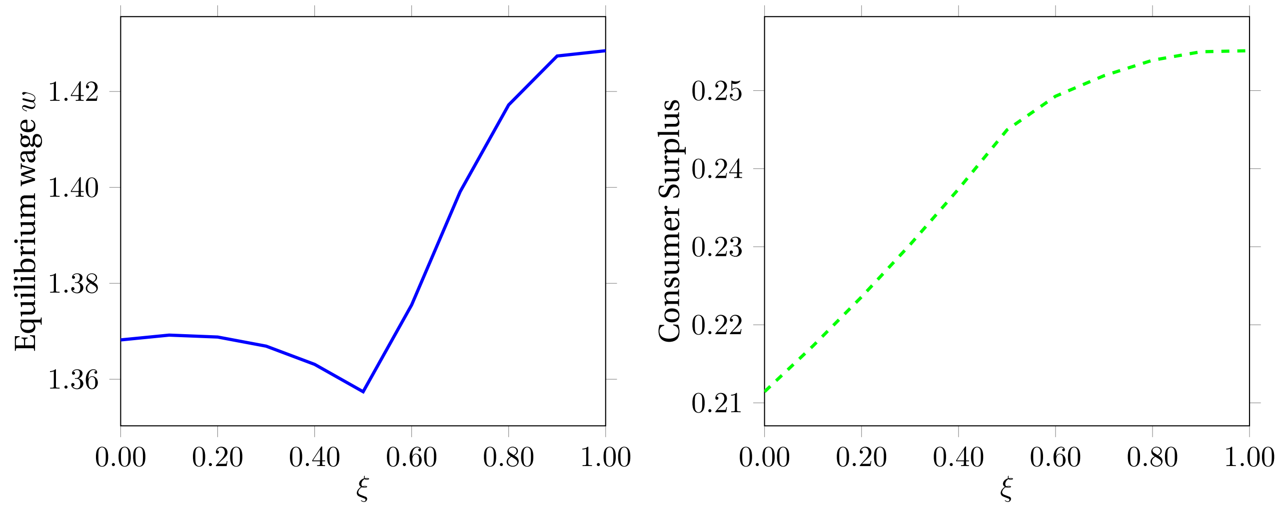


Figure 9: Induced equilibrium wage w and consumer surplus corresponding to the platform's optimal origin pricing policy for the class of star-to-complete networks with $n = 4$ locations, $\boldsymbol{\theta} = \mathbf{1}$, and $\beta = 0.9$. Here, the riders' willingness to pay is uniformly distributed in $[0, 1]$ and the drivers' reservation wage is uniformly distributed in $[0, 5]$. Finally, $\Delta = 0.5$.

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