Competitive Targeted Advertising over Networks

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Recent advances in information technology have allowed firms to gather vast amounts of data regarding consumers' preferences and the structure and intensity of their social interactions. This paper examines a game-theoretic model of competition between firms which can target their marketing budgets to individuals embedded in a social network. We provide a sharp characterization of the optimal targeted advertising strategies and highlight their dependence on the underlying social network structure. Furthermore, we provide conditions under which it is optimal for the firms to asymmetrically target a subset of the individuals and establish a lower bound on the ratio of their payoffs in these asymmetric equilibria. Finally, we find that at equilibrium firms invest inefficiently high in targeted advertising and the extent of the inefficiency is increasing in the centralities of the agents they target. Taken together, these findings shed light on the effect of the network structure on the outcome of marketing competition between the firms.

Key words: Social networks, competition, targeted advertising, targeting.

1. Introduction

It is widely accepted that word-of-mouth plays a central role in the propagation of brand or product information, and thus it is a first order consideration in the design and implementation of a marketing strategy. Moreover, marketers nowadays have access to and can take advantage of vast amounts of data on the pattern and intensity of social interactions between consumers. The advent of the Internet as a prominent communication and advertising platform has enabled firms to implement targeted advertising campaigns and direct their efforts to certain subsets of the population. The natural question that arises in this setting is how a firm can use the wealth of available information along with targeting technologies to increase the awareness about its products. The recent acquisitions of Buddy Media, Vitrue, and Wildfire Interactive by Salesforce, Oracle, and Google respectively point to the indisputable fact that social media marketing emerges as a viable alternative to traditional advertising and tech giants are striving to obtain a competitive advantage in the new landscape.

The focus in this paper is on prescribing the best way a firm can exploit word-of-mouth and its knowledge over the social network structure of consumers when devising a targeted advertising campaign. At the core of the model lies an *information externality* that arises endogenously due to communication: information obtained by an agent in the network can be passed along to her peer group and thus word of mouth communication may amplify the effect of a firm's marketing efforts. As a consequence, an optimal targeting strategy may involve allocating disproportionate fraction of the marketing budget to certain agents that play a central role in the word of mouth process anticipating that they will pass the relevant information to the rest. Indeed, strategies of similar flavor have been applied in practice, however in an ad hoc and heuristic way. For example, Klout provides social media analytics to measure a user's influence in her social network. The service scrapes social network data and assigns individuals a "Klout" score, which presumably reflects their influence. Then, it connects businesses with individuals of high score with the intention of influencing the latter to spread good publicity for the former in exchange for free merchandise and other perks. Our goal is to provide a systematic characterization of optimal targeting strategies and identify qualitative insights that lead to their success.

We develop a novel model of targeted advertising in the presence of competition and explore its impact on the evolution of *brand awareness* for the competing firms. The level of brand awareness for a firm, defined as the fraction of consumers that name the firm first when asked about a product category, evolves dynamically as agents communicate with their peers over time. Firms can influence the word-of-mouth communication process by targeting advertising funds to specific individuals. Our choice of brand awareness as the firm's metric of performance for a marketing campaign is motivated by the fact that growing brand awareness is the primary goal of devising a marketing campaign especially in a new market. Moreover, awareness is easier to measure and relate to marketing efforts and it is positively correlated with sales and profits.

The word-of-mouth process along with the marketing strategies of the firms define a dynamical system that tracks the evolution of the brand awareness levels for the competing firms. Our goal is to characterize the limiting behavior of this system as a function of the underlying network structure and the marketing strategies. First, we show that our objective is well defined since the agents' awareness levels converge to a fixed vector. Next, we provide a sharp characterization of the limit as a function of the underlying network structure and the firms' advertising efforts. It turns out that the average brand awareness converges (almost surely) to a weighted sum of the advertising funds that firms allocate to individuals, where the weights are given by a notion of *centrality* in the underlying network structure. Armed with a characterization of the limiting behavior of the system, we proceed to the problem of devising optimal marketing strategies for the firms over the social network structure. To this end, we show that the fraction of a firm's marketing budget that is targeted to an individual is increasing function in her centrality.

In the second part of the paper we model the competition between two firms as a game over the network of agents. First, firms simultaneously choose their marketing strategies, i.e., how to allocate their marketing budgets to the agents. Secondly, agents obtain information both from their peers and the firms over time until their awareness levels converge to a limiting vector. We are interested in characterizing the equilibrium strategies for the firms and derive qualitative insights about the relation of the network structure with the level of competition between them. We provide conditions that guarantee the existence of asymmetric equilibria in this ex-ante symmetric environment in which the two firms target different subsets of individuals.

Furthermore, we examine the extent to which equilibrium behavior leads to differences in the effectiveness of their marketing campaigns even when they have equal budgets. To this end, we provide a bound on the maximum value that the ratio of the firms' equilibrium payoffs can take over all networks. Finally, by endogenizing the firms' marketing budgets we study the joint optimization of marketing levels and their allocation to agents. We show that marketing levels at equilibrium are inefficiently high and the extent of inefficiency increases with the centrality of the agents that firms target.

A novel feature of our model is that we introduce competition between firms over a social network structure. To the best of our knowledge, there is very little work that attempts to capture the interaction between competition and network structure. In a recent contribution, Goyal and Kearns (Goyal et al. (2015)) examine a contagion model where two competing firms simultaneously choose their seed sets and then agents adopt one of two technologies according to a stochastic diffusion process related to the general threshold model. Their main goal is to study the level of inefficiency in the use of available resources, i.e., the number of seeds available, at equilibrium. They also introduce the notion of a *budget multiplier* that measures the extent to which ex-ante imbalances in players' budgets get amplified at equilibrium and identify a certain property on the adoption dynamics which guarantees that the budget multiplier remains bounded. In contrast to their paper, our focus is on explicitly characterizing the impact of the social network structure on the equilibrium outcomes of the competition between the firms and on the resulting dynamics of their levels of brand awareness as well as on prescribing optimal (network-dependent) targeted advertising strategies. Fazeli and Jadbabaie (2012) consider a similar setting as Goyal et al. (2015) with the difference that agents determine which technology to adopt based on the outcome of a local coordination game with their peers and they provide a lower and an upper bound on the proportion of product adoptions. Finally, Chasparis and Shamma (2014) explore a setting where agents' proclivities towards buying the products of firms evolve over time as a convex combination of the proclivities of their neighbors as well as an external influence that intends to capture the firms' advertising efforts. In their model, the optimal policy involves targeting a single individual due to the fact that the marginal returns to advertising are constant and agents update their proclivities linearly.

Closely related to our paper is the work by Carlos Lever (Lever (2010), Lever (2012)) that studies voting competitions where an agent's opinion may be influenced by the opinions of her peers. Lever extends DeGroot's consensus model (DeGroot (1974)) by considering two "persuaders" that may allocate their campaigning budgets to influence the voters' original opinions. He shows that the equilibrium of the game between the two "persuaders" is always symmetric and their campaigning efforts are allocated in proportion to the agents' consensus weights. In contrast, we model word-of-mouth as a process of exchanging messages and we assume that firms can influence this process by targeting their marketing budgets. Agents' original information is irrelevant in our context, agents do not reach consensus, i.e., even at the limit there is divergence in the agents' brand awareness levels, and we identify conditions under which firms allocate their budgets asymmetrically at equilibrium (and as a result they enjoy different returns in their marketing investments). We show that the equilibrium levels of marketing are too high and the extent of the inefficiency increases with the centrality of influential agents.

Our modeling framework is also related to the literature on Colonel Blotto games (e.g., Friedman (1958), Roberson (2011)). A typical Colonel Blotto game involves two budget constrained agents that strategically allocate resources across multiple contests. Importantly, Colonel Blotto games are zero-sum. A critical difference between our model and Colonel Blotto games (apart from the network structure) is the presence of a third, passive agent ("the status-quo option"). This difference implies that the underlying game is no longer zero-sum and has interesting and quite realistic implications such as the existence of asymmetric equilibria in which firms target different subsets of the population even when they are symmetric. To the best of our knowledge, the Colonel Blotto literature has mainly focused on zero-sum games between two agents (in which equilibria are symmetric when agents are ex-ante symmetric).

There is a sizable literature on awareness formation models that describe the growth and decay of a brand's awareness level over time as a function of its advertising efforts (see for example Dodson and Muller (1978) and Naik et al. (1998)). Change in the level of brand awareness is driven by the amount of advertising efforts directed to the unaware segment, word-of-mouth communication, and "forgetting" effects. Recently, Naik et al. (2008) extended this model by incorporating competition among brands. Unlike these models which assume a continuum of agents, we consider a society envisaged as a social network of n agents and study the effectiveness of *targeted* marketing campaigns in the presence of competition that explicitly takes into account the underlying network structure.

Our work also contributes to a recent stream of papers that examine the dynamics of opinion formation and information exchange among agents that are embedded in a social network. DeGroot (1974) introduced a tractable framework to study the interaction among agents in which their *beliefs* about an underlying state (e.g., the quality of a product) are modeled as continuous variables and communication with one's peers takes the form of a simple linear update. On the other extreme, Acemoglu et al. (2014) consider a dynamical model of information exchange among Bayesian agents. Their goal is to identify conditions under which dispersed information is aggregated through the communication process and agents "learn".

We build on the *binary voter model* (Clifford and Sudbury (1973)) in which the state (opinion) of an individual is a binary variable and each time communication takes place, the agent simply adopts the opinion of the communicating party. In recent work, Yildiz et al. (2014) consider an extension where a subset of agents are "stubborn", i.e., their opinion is fixed to one of the two values. In our model, the state of an individual is a continuous variable which is updated by exchanging messages in favor of the options available to the agent. The content of the message depends on the sender's state while the receiving agent updates her own state based on its content. Our awareness level update process is more general than both the rule used in DeGroot (1974) and the one suggested by the voter model and it can be thought of as bridging the gap between the "naive" learning rules and the often intractable Bayesian learning.

Our results illustrate a close connection between optimal marketing strategies in the presence of competition and the centrality of agents in the underlying network structure. In particular, we show that it is optimal for the firms to allocate their advertising funds to agents in proportion to a notion of network centrality that is very closely related to *Bonacich* centrality which is widely used as a sociological measure of influence (the two notions differ because of the presence of the competing firms in our setting). Ballester et al. (2006) and more recently Candogan et al. (2012) also note a link between centralities and economic outcomes. Unlike these models that feature exogenous local payoff complementarities and no competition, in our setting externalities among the agents arise *endogenously* due to word-of-mouth. Finally, there is some work that identifies the merits of targeted advertising (e.g., Iyer et al. (2005)) but does not account for word-of-mouth word-of-mouth over a network, which is the focus of our paper.

2. Model

The society $\mathcal{G}(\mathcal{N}, \mathcal{E})$ consists of a set $\mathcal{N} = \{1, \cdots, n\}$ of agents embedded in a social network represented by the adjacency matrix $\mathcal{E}$. The ij-th entry of $\mathcal{E}$, denoted by e_{ij}, represents the relative frequency of communication of agent i with agent j. We assume that $e_{ij} \in [0,1]$ for all i,j and we normalize $e_{ii} = 0$ for all i. We also assume that $\sum_i e_{ij} = 1$ for all j. Note that we do not impose any symmetry assumption on the e_{ij}'s (although our analysis remains valid in the case when $e_{ij} = e_{ji}$). We study the process of awareness formation in a market with two firms, A and B.[1]

The state of the marketing competition at time k can be summarized by the vector of awareness levels for brands A and B at k denoted by $\mathbf{x}^{\mathbf{A}}[k]$ and $\mathbf{x}^{\mathbf{B}}[k]$. In particular, $x_i^A[k] \in [0,1]$ represents the *top-of-mind* awareness of agent i for brand A, i.e., the frequency with which agent i ranks brand A as her first choice in its respective market. Finally, we do not impose that $x_i^A[k] + x_i^B[k] = 1$ to allow for the possibility that agent i is unaware of brands A, B (as they may be new in the market) or there is an incumbent brand, the *status quo* (short-handed by SQ), that the agent associates with the market. For this, we let $x_i^{SQ}[k] = 1 - x_i^A[k] - x_i^B[k]$.

Communication takes place at discrete time periods $k = \{1, 2, \ldots, \infty\}$. At time period k an agent receives information from her peer group with probability $\alpha \in (0,1)$ and directly from the competing firms with probability $1-\alpha$ (the analysis readily extends to the case when agents are heterogeneous with respect to parameter α, i.e., individual i receives information from her peer group with probability α_i). For simplifying the exposition and focusing on the effects we set out to study, we only discuss the case when $\alpha_i = \alpha_j = \alpha$, for all i,j. In other words, parameter α captures the relative importance of word-of-mouth communication in the process of building brand awareness. The peer group of a given agent i is defined by $\mathcal{N}_i = \{j \in \mathcal{N} | e_{ij} > 0\}$. Communication takes the form of a message that can take one of three values, A, B, or SQ. When agent i communicates with agent j (which occurs with probability αe_{ij} at time period k), then the message i receives is a function of agent j's state, i.e., the values of $x_j^A[k], x_j^B[k]$, and $x_j^{SQ}[k]$. A message generating function q maps an agent's state to a message. We assume that:

$$q(x_j^A[k], x_j^B[k], x_j^{SQ}[k]) = \begin{cases} A, & \text{with probability} \quad x_j^A[k], \\ B, & \text{with probability} \quad x_j^B[k], \\ SQ, & \text{with probability} \quad 1 - x_j^A[k] - x_j^B[k]. \end{cases}$$

In other words, $q(x_j^A[k], x_j^B[k], x_j^{SQ}[k])$ is the (random) message that agent j sends to agent i when they communicate and agent j's state is given by $x_j^A[k], x_j^B[k]$, and $x_j^{SQ}[k]$. Finally, if the agent obtains information directly from the firms (which occurs with probability $1-\alpha$), then she receives message A, B or SQ, with probability that is a function of the advertising funds that the firms allocate on that individual. Similar to above, q_{firms} maps the budgets firms allocate to agent i to a message that agent i receives with probability $(1-\alpha)$ at a given time period.

$$q_{firms}(b_i(A), b_i(B)) = \begin{cases} A, & \text{with probability} \quad h(b_i(A), b_i(B)), \\ B, & \text{with probability} \quad h(b_i(B), b_i(A)), \\ SQ, & \text{with probability} \quad 1 - h(b_i(A), b_i(B)) - h(b_i(B), b_i(A)), \end{cases}$$

where function h is such that $h:[0,C]\times[0,C]\to[0,1]$ and $h(x,y)+h(y,x)\le 1$, and C denotes the total marketing budget available to each of the two firms. Note that the analysis carries through when the two firms do not have the same marketing budgets. As our goal is to study the effect of the network structure on the marketing competition, we assume that the two firms are ex-ante symmetric. For the remainder of the paper we normalize C to 1 and focus on functions for which $h(0,x)=0$ for all x and the inequality is strict, i.e., $h(x,y)+h(y,x)<1$. As a concrete example, consider the following class of functions known as *contest success functions* (see Skaperdas (1996) and Blavatskyy (2010)):

$$h(x,y)=\frac{x^s}{x^s+y^s+\delta}\text{ with }0<\delta\le 1\text{ and }s>0.$$

At the beginning of the horizon, $k=0$, the state of the marketing competition is given by arbitrary vectors $\mathbf{x^A}[0]$ and $\mathbf{x^B}[0]$. Let $m_i[k]$ denote the content of the message agent i receives at time period $k>0$. Then, agent i's awareness of brand A is given by:

$$x_i^A[k]=\frac{\mathbf{1}_{m_i[k]=A}}{k}+\frac{k-1}{k}x_i^A[k-1],$$

where $\mathbf{1}_{\cdot}$ is the indicator function. In other words, $x_i^A[k]$ is equal to the fraction of messages in favor of A that agent i received by time k. We are imposing the following assumptions:

Assumption 1. *The communication network defined by adjacency matrix $\mathcal{E}$ is connected, i.e., for every pair of individuals i,j, there exists a communication path from i to j.*

Assumption 2. *The function $h(x,y)$ (recall that $h:[0,1]\times[0,1]\to[0,1]$) is twice continuously differentiable in x and y, and strictly increasing in x for all y.*

The two competing firms are interested in maximizing the average awareness of their brands in the population of agents. As we show in the next section, this is a well-defined optimization problem as the state of the marketing competition (i.e., vectors $\mathbf{x^A},\mathbf{x^B}$) converges (almost surely) in the limit for every allocation of marketing budgets from the firms.

2.1. Discussion of the model

As already mentioned in the introduction, the goal of this paper is to study how information about (new) brands propagates in a social network in the presence of competition. We focus on two channels through which agents obtain information: word of mouth, i.e., communicating with their peers, and direct advertising from the firms. There is plenty of empirical evidence that consumers learn about brands, new products or technologies by exchanging information with their peers as well as by the advertising efforts of firms (see for Godes and Mayzlin (2004)). The amount of advertising funds that a brand allocates to an individual determines the *intensity* of marketing efforts targeted towards the individual over time. Assumption 2 implies that the returns to advertising, i.e., the change in the level of brand awareness, is increasing with the amount of funds spent on an individual which is natural in our setting. Finally, as will become evident in what follows the two firms are not only competing against each other but they are also competing against the status quo option (SQ). The latter can be interpreted as an incumbent brand that does not engage in targeted advertising and passively allocates its marketing budget uniformly to all the agents. Alternatively, it can be seen as the decay in awareness due to forgetting over time (Mahajan et al. (1984)).

Moreover, we assume that word-of-mouth communication is achieved through the exchange of simple messages that can take one of three values in the benchmark model: A, B, or SQ. The message generating function q imposes the following intuitive assumption in the environment: the higher an agent's awareness level is for option i, the more likely she is to provide information about i. Finally, a consumer's response to the marketing efforts by firms A and B is governed by function h that takes as arguments the advertising funds that A and B allocate on the consumer.

The firms' objective is to maximize the average long-run awareness about their brands. Although we do not directly incorporate purchasing decisions from the agents into the model, we believe that this objective captures the essence of marketing competition between firms, as profits are typically positively correlated to brand awareness and it is often the case that firms invest in marketing not to promote a particular product but rather to build a customer pool that engages in repeat purchases with the firm. Furthermore, this objective is easier to measure and relate to the firm's marketing efforts than other metrics of performance, e.g., sales, that depend on several other factors such as retail availability or income shocks.

A consumer in our model behaves according to a given *rule of thumb*: first, she treats all information she receives in the same way as opposed to putting different weights depending on the identity of the sender. Moreover, when she engages in communication with her peers she passes a message regarding a brand with probability equal to her level of awareness. Although it would be interesting to extend the present model along these directions, we believe that our current formulation not only leads to a tractable analysis but it is also sufficient for the purposes of studying the effect of network structure on the outcomes of competition in targeted advertising.

3. Asymptotic behavior

We begin our analysis by characterizing the limiting behavior of the agents' brand awareness. First, we show that the vector of awareness levels converges almost surely and second we relate this limit to the properties of the underlying social network structure. Armed with this characterization, we study the problem that firms face: how to allocate their advertising funds to agents so as to maximize the average awareness towards their brands.

We define the vector $\mathbf{y^A}[k] \in [0,1]^{(n+3)\times 1}$ such that $y_i^A[k] = x_i^A[k]$ for $i \in \mathcal{N}$, $y_{n+1}^A[k] = 1$ and $y_{n+2}^A[k] = y_{n+3}^A[k] = 0$. The first n entries of the vector $y^A[k]$ correspond to the awareness levels of the corresponding agents about brand A. The $(n+1)$-th and $(n+2)$-th entries correspond to firms A and B respectively, and the $(n+3)$-th entry corresponds to the status quo option. Similarly, we define $\mathbf{y^B}[k]$, where $y_i^B[k] = x_i^B[k]$ for $i \in \mathcal{N}$, $y_{n+1}^B[k] = y_{n+3}^B[k] = 0$, and $y_{n+2}^B[k] = 1$. In other words, to simplify the exposition of results we construct vectors $\mathbf{y^A}[k]$ and $\mathbf{y^B}[k]$ by adding three "dummy" agents corresponding to A, B, and SQ that have fixed awareness levels (0 or 1).

Next we prove that both vectors of awareness levels converge almost surely to (deterministic) limits that can be characterized as a function of the network structure and the marketing allocations of the firms. Before stating the result, we define matrices W and V as:

$$[W]_{ij} = \begin{cases} -1 & i=j, i \in \mathcal{N} \\ \alpha e_{ij} & j \in \mathcal{N}_i, i \in \mathcal{N} \\ (1-\alpha)\ h_i^A & j = n+1, i \in \mathcal{N} \\ (1-\alpha)\ h_i^B & j = n+2, i \in \mathcal{N} \\ (1-\alpha)\ (1-h_i^A - h_i^B) & j = n+3, i \in \mathcal{N} \\ 0 & otherwise. \end{cases} \quad \text{and} \quad [V]_{ij} = \begin{cases} \dfrac{[W]_{ij}}{-[W]_i} & j \neq i,\ [W]_{ii} \neq 0 \\ 0 & j \neq i,\ [W]_{ii} = 0 \\ 0 & j = i,\ [W]_{ii} \neq 0 \\ 1 & j = i,\ [W]_{ii} = 0. \end{cases} \tag{1}$$

where $h_i^A = h(b_i(A), b_i(B)), h_i^B = h(b_i(B), b_i(A))$, and $[W]_i = -\sum_{j \in \mathcal{N}, j \neq i}[W]_{ij}$. Then, we obtain:

Proposition 1. *The awareness level vectors $y^A[k], y^B[k]$ converge almost surely to vectors y_{lim}^A, y_{lim}^B respectively. Moreover, the limiting vectors y_{lim}^A and y_{lim}^B are unique and satisfy:*

$$y_{lim}^A = \lim_{k\to\infty} V^k y^A[0], \quad y_{lim}^B = \lim_{k\to\infty} V^k y^B[0].$$

To gain intuition on Proposition 1, note that matrix V can be thought of as the transition matrix of a Markov chain with three absorbing states (corresponding to the two firms and the status quo, i.e., entries $n+1$, $n+2$, and $n+3$). Thus, matrix $\lim_{k\to\infty} V^k$ has non-zero elements only in columns $n+1$, $n+2$ and $n+3$. Moreover, it is straightforward to see that the i-th element of vector y_{lim}^A is equal to *the probability that the random walk defined by matrix V is absorbed by state $n+1$ (as opposed to being absorbed by states $n+2$ or $n+3$) when it is initiated at node i*. Similarly, the i-th element of vector y_{lim}^B is equal to the probability that the random walk defined by matrix V is absorbed by state $n+2$. Finally, since $\lim_{k\to\infty} V^k$ has non-zero elements only in the last three columns, the limiting awareness levels for the agents are independent of their values at time $k=0$, i.e., they do not depend on vectors $\mathbf{x^A}[0]$ and $\mathbf{x^B}[0]$.

In the rest of the section, we provide a characterization of the limiting value for the average awareness level for each of the two firms. In the remainder of the paper, we refer to the limiting average awareness level for firm i as its payoff. Let $m^A[k] = \frac{1}{n}\sum_{i=1}^n y_i^A[k]$, and $m^B[k] = \frac{1}{n}\sum_{i=1}^n y_i^B[k]$

be the average awareness levels at time period k. Then, it is straightforward to see that they converge almost surely to the unique values m^A_{lim}, m^B_{lim} defined as:

$$m^A_{lim} = \frac{1}{n}\sum_{i=1}^{n}[y^A_{lim}]_i, \quad m^B_{lim} = \frac{1}{n}\sum_{i=1}^{n}[y^B_{lim}]_i.$$

We can partition matrix V as

$$\begin{bmatrix} \alpha\mathcal{E} & E \\ 0 & I_3 \end{bmatrix},$$

where I_3 is the 3×3 identity matrix, $E \in \mathbb{R}^{n\times 3}$ is such that $[E]_{i1} = (1-\alpha)\ h(b_i(A), b_i(B))$ and $[E]_{i2} = (1-\alpha)\ h(b_i(B), b_i(A))$, for all $i \in \mathcal{N}$, and $\mathcal{E}$ is the adjacency matrix that captures the communication patterns among the agents. Given this partitioning, and denoting $[.]_i$ as the i-th column of its argument, we obtain:

Theorem 1. *The payoff for firm A converges almost surely to:*

$$m^A_{lim} = \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)[\mathbf{1}'(I-\alpha\mathcal{E})^{-1}]_i\ h(b_i(A), b_i(B)).$$

Theorem 1 provides a sharp characterization of the firms' payoffs as a function of the network structure captured by the *centrality* measure below and the allocations chosen by the two firms.

Definition 1. The *absorption* centrality c_i of agent i is defined as

$$c_i = [\mathbf{1}'(I-\alpha\mathcal{E})^{-1}]_i.$$

Agent i's absorption centrality c_i is equal to the expected number of visits to node i before absorption at either node $n+1$, $n+2$, or $n+3$ for a random walk with transition probability matrix V that started at a node other than i uniformly at random. Note that the c_i's are non-negative (strictly positive for a connected network) reflecting the fact that externalities in our modeling framework are positive. Furthermore, they sum up to $n/(1-\alpha)$ due to the way communication takes place in our model (recall that α denotes the probability that an agent receives information from her peer group as opposed to the competing firms at a given time period). Thus, Theorem 1 states that *the limiting average awareness levels are a weighted sum of the firms' marketing efforts where the weights are given by the absorption centralities of the agents.*

The notion of absorption centrality is such that connections to agents with high centrality contribute more to the centrality of an agent than connections to agents with low centrality. Therefore, calculating an agent's absorption centrality index takes the entire network structure into account, as opposed, for example, to *degree centrality* that is defined as the number of links incident to an agent and requires only local information. Moreover, Theorem 1 captures the fact that when firms devise their optimal marketing strategies they should account for *multiplier effects*: the return to their marketing strategies depends not only on the agents they directly target but also on how well connected the latter are with other central agents.

The model studied here is quite flexible and can be extended in multiple ways. As an example, we close the section by discussing a simple way to introduce heterogeneity in the agents' preferences. Assume that each agent i has a bias towards firm A or B given by parameters r_i^A and r_i^B respectively. Agent i updates her awareness level in favor of A with probability r_i^A upon receiving an "A" message (similarly for B) while with probability $1-r_i^A$ she ignores the message.

The following proposition that can be shown using similar arguments as in Theorem 1 describes how the firms' payoffs incorporate the agents' preferences captured by vectors $\{r_i^A\}_{i\in\mathcal{N}}$ and $\{r_i^B\}_{i\in\mathcal{N}}$. Let R denote the $n\times n$ diagonal matrix with $R_{ii}=r_i$. Then,

Proposition 2. *Firm A's payoff converges almost surely to:*

$$m_{lim}^A = \frac{1}{n}\sum_{i=1}^{n}(1-\alpha)r_i[\mathbf{1}'(I-\alpha R\mathcal{E})^{-1}]_i\ h(b_i(A),b_i(B)).$$

Note that the expressions for m_{lim}^A and m_{lim}^B are similar to those in Theorem 1 (the adjacency matrix $\mathcal{E}$ is now multiplied by diagonal matrix R).

4. The marketing competition game

Next, we study the equilibria of the marketing competition game between firms A and B. Before describing the game, we turn our attention to the optimization problem that a firm faces, i.e., how to optimally allocate its advertising funds to the agents given the allocation of its competitor. Then, we build on this and study the equilibria of the competition game between the two firms.

4.1. Optimal budget allocation

We characterize the best response of a firm to the budget allocation of its competitor, i.e., assuming that firm B has already chosen to allocate its advertising funds according to vector $\{b_i(B)\}_{i\in\mathcal{N}}$ we provide an expression for the optimal budget allocation for firm A.

Proposition 3. *Suppose that $h(x,y)$ is strictly concave in x for all $y\in[0,1]$. Then, firm A's optimization problem has a unique solution $\{b_i^\star(A)\}_{i\in\mathcal{N}}$ which satisfies:*

$$b_i^\star(A) = \begin{cases} 0 & \text{if } \gamma \geq \frac{1}{n}c_i(1-\alpha)\frac{\partial h(x,b_i(B))}{\partial x}\Big|_{x=0}, \\ b_i(A)\ \text{s.t.}\ \frac{1}{n}c_i(1-\alpha)\frac{\partial h(x,b_i(B))}{\partial x}\Big|_{b=b_i(A)} = \gamma & \text{if } \gamma < \frac{1}{n}c_i(1-\alpha)\frac{\partial h(x,b_i(B))}{\partial x}\Big|_{x=0}, \end{cases}$$

and $\sum_{i\in\mathcal{N}} b_i^\star(A) = 1$.

Proposition 3 essentially states that firm A allocates positive marketing budget on an individual i if she is *central* enough in the social network structure, as implied by her absorption centrality index c_i. This observation implies the following corollary, i.e., advertising funds targeted to an individual are increasing with the individual's centrality.

Corollary 1. *Suppose that agents $i,j \in \mathcal{N}$ are such that $b_i(B) = b_j(B)$ and $c_i \geq c_j$. Furthermore, suppose that h is strictly concave in x for all y. Then, $b_i^\star(A) \geq b_j^\star(A)$.*

4.2. Equilibria of the competition game

Firms A and B engage in the following two-stage game.

Allocating the advertising funds: Firms A and B choose how to allocate their marketing budgets that are both normalized to 1 so as to maximize their payoffs m^A_{lim}, m^B_{lim}.

Information exchange: Consumers obtain information from their peers and the competing firms according to the process outlined in previous sections.

Let $\mathcal{B}_A$ and $\mathcal{B}_B$ denote the set of pure strategies for firms A and B respectively, i.e.,

$$\mathcal{B}_A = \{b^A \in \mathbb{R}^n \big| b^A \geq 0, \sum_{i\in\mathcal{N}} b_i(A) \leq 1\}, \text{ and } \mathcal{B}_B = \{b^B \in \mathbb{R}^n \big| b^B \geq 0, \sum_{i\in\mathcal{N}} b_i(B) \leq 1\}.$$

Firms A and B may also use mixed strategies, i.e., Borel probability measures over sets $\mathcal{B}_A$ and $\mathcal{B}_B$ respectively. With a slight abuse of notation, we let $m^A_{lim}(\sigma^A,\sigma^B)$, $m^B_{lim}(\sigma^A,\sigma^B)$ denote the payoffs for firms A, B, when A, B employ (mixed) strategies σ^A, σ^B respectively. Then,

Definition 2. A pair of mixed strategies $\sigma = (\sigma^A, \sigma^B)$ is a Nash equilibrium for the marketing competition game defined above if and only if:

$$m^A_{lim}(\sigma^A,\sigma^B) \geq m^A_{lim}(\tilde{\sigma}^A,\sigma^B) \text{ for all } \tilde{\sigma}^A, \text{ and } m^B_{lim}(\sigma^A,\sigma^B) \geq m^B_{lim}(\sigma^A,\tilde{\sigma}^B) \text{ for all } \tilde{\sigma}^B.$$

The game defined above is guaranteed to have a mixed Nash equilibrium by Glicksberg (1952), since the firms' payoff functions are continuous in their strategies. Furthermore, note that by incorporating SQ into our modeling framework, which captures either an incumbent (passive) firm or the decay effects of marketing efforts, we have $h(x,y) + h(y,x) < 1$, which implies that the game described above is *not* zero-sum. This is the most interesting case since it implies that there may exist asymmetric pure-strategy equilibria as well. In the remainder of the section, we characterize the equilibria of the marketing competition game under different conditions on function h. We show that a symmetric equilibrium always exists and, more interestingly, we provide conditions on h and the network structure under which asymmetric equilibria also exist.

Case 1: Advertising efforts are strategic complements

First, we consider the environment defined by Assumption 3 below.

Assumption 3. *Function h is strictly concave in x and its mixed partial derivative is non-negative for all $x,y \in [0,1]$, i.e.,*

$$\frac{\partial^2 h(x,y)}{\partial x \partial y} \geq 0.$$

Note that a Nash equilibrium in *pure strategies* is guaranteed to exist under assumption 3 (Glicksberg (1952)). Furthermore, we show that all pure-strategy equilibria are *symmetric*, i.e., the marketing budget that firm A allocates to individual i is the same as the one that B allocates to i for all individuals.

PROPOSITION 4. *All pure-strategy Nash equilibria of the marketing competition game under Assumption 3 are symmetric, i.e., if (b^A, b^B) is an equilibrium, then $b_i(A) = b_i(B)$ for all $i \in \mathcal{N}$.*

Assumption 3 describes an environment where the firms' budget allocations on an individual are strategic complements. Increasing a firm's advertising effort exerts a positive externality on its competitor and thus no asymmetric allocations, i.e., allocations where there exists at least an agent i for which $b_i(A) \neq b_i(B)$, can be at equilibrium.

Case 2: Advertising efforts are strategic substitutes

The second case involves an environment where the dynamics of information exchange are governed by function h, such that advertising efforts are strategic substitutes, i.e., $\frac{\partial^2 h(x,y)}{\partial x \partial y} < 0$, and increasing a firm's budget allocation on an individual exerts a negative externality on its competitor. We let $\mathcal{A}_A^\sigma$ and $\mathcal{A}_B^\sigma$ denote the *activation sets* of firms A and B respectively for equilibrium σ, i.e., the sets of agents that receive a positive fraction of the marketing budget from the respective firm at σ. Then, we obtain

PROPOSITION 5. *Suppose that h is strictly concave in x and $\frac{\partial^2 h(x,y)}{\partial x \partial y} < 0$ for all $x, y \in [0,1]$. Furthermore, suppose that h satisfies the following:*

$$\frac{\partial^2 h(x+\alpha, x)}{\partial x^2} < \frac{\partial^2 h(x, x+\alpha)}{\partial x \partial y} < 0, \text{ for all } 0 \le x \le 1-\alpha \text{ and } \alpha > 0. \tag{2}$$

Then, the game has a unique and symmetric pure-strategy Nash equilibrium.

This result can be best understood in terms of the interplay between the "diminishing returns" to a firm's advertising efforts which are captured by $\frac{\partial^2 h}{\partial x^2}$ and the degree of strategic substitutability between the advertising efforts of the two firms which is captured by $\frac{\partial^2 h}{\partial x \partial y}$. When Equation (2) holds, firms find it optimal to spread their budgets across the agents since the diminishing returns effect dominates. Thus, the equilibrium ends up being unique and symmetric. Proposition 6 specializes the results of Propositions 4 and 5 to contest success functions.

PROPOSITION 6. *Suppose that h takes the form*

$$h(x,y) = \frac{f(x)}{f(x) + f(y) + \delta},$$

with $f(\cdot)$ concave, $f(0) = 0$, and $\delta > 0$. Then, the game has only symmetric equilibria.

Proposition 6 directly implies that when $h(x,y)=\frac{x^s}{x^s+y^s+\delta}$ with $s\le 1$ then the marketing competition game has only symmetric equilibria. Deviating from the conditions of Propositions 4 and 5 makes the existence of pure strategy asymmetric equilibria possible. The following proposition provides sufficient conditions on the network that guarantee the existence of asymmetric equilibria for functions

$$h(x,y)=\frac{x^s}{x^s+y^s+\delta},$$

with $s>1, 0<\delta\le 1$. Note that Proposition 6 implies that equilibria are symmetric for $s\le 1$. Without loss of generality we assume that $c_1\ge c_2\ge\cdots\ge c_n$.

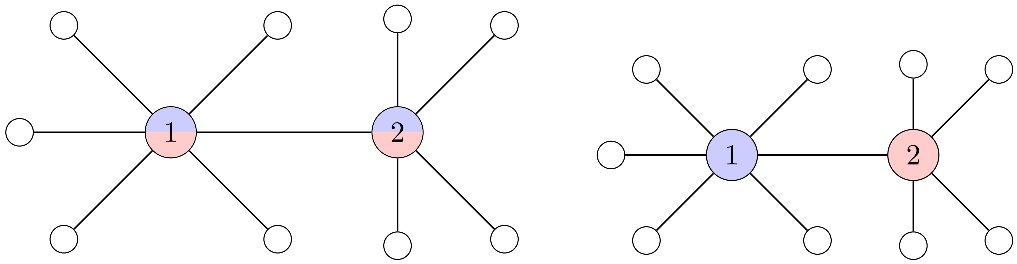
Proposition 7. *Suppose that*

$$h(x,y)=\frac{x^s}{x^s+y^s+\delta},$$

with $s>1$ and $0<\delta\le 1$ and the agents' absorption centralities are such that $\frac{c_1}{c_{\bar{k}}}\le\beta$ for $\bar{k}$ and β that depend on s and δ. Then, the game does not have symmetric equilibria in pure strategies, however it has an asymmetric equilibrium.

Proposition 7 states that for a large subclass of contest success functions and for networks that have a sufficiently large number of agents with high centrality, i.e., centrality close to c_1, the only equilibria in pure strategies are asymmetric and firms find it optimal to target different subsets of the population. There may also exist symmetric equilibria in mixed strategies as we show in the example below. As a side note, the proof of existence of asymmetric equilibria under the conditions of Proposition 7 is constructive. In particular, we show that the strategy profile that involves one of the firms optimizing its marketing budget allocation in the absence of competition and the competitor best responding to this allocation is an asymmetric equilibrium of the game. This strategy profile would also be an equilibrium of a Stackelberg version of the game in which one of the firms commits to a marketing strategy first and then the competitor follows (which is not the case for the symmetric equilibria in mixed strategies). We close the section by providing an example in which we explicitly construct symmetric and asymmetric equilibria.

Example: We consider the class of functions $h(x,y)=\dfrac{x^s}{x^s+y^s+1}$, for $s>0$. Proposition 6 implies that for $s\le 1$ all pure-strategy equilibria are symmetric whereas for $s>1$ asymmetric equilibria may also exist. We describe a symmetric and an asymmetric equilibrium when $s=2$ and a network with two hub nodes and 11 agents in total as depicted in Figure 1. For $\alpha=0.5$, the nodes centralities are given by: $c_1=4.96,\quad c_2=4.26,\quad c_3=c_4=c_5=c_6=c_7=1.41,\quad c_8=c_9=c_{10}=c_{11}=1.43$. The following are equilibria of the marketing competition game:



(a) Symmetric equilibrium. (b) Asymmetric equilibrium: The firms split the two hub subnetworks.

Figure 1 The symmetric and one asymmetric equilibrium for a network with two connected hubs.

(a) Symmetric equilibrium (in mixed strategies): Both firms allocate their entire budget to agent 1 with probability

$$p_1 = \frac{3c_1 - 2c_2}{c_1 + c_2} \approx 0.69,$$

and to agent 2 with probability $p_2 = 1 - p_1$.

(b) Asymmetric: Firms A and B allocate their budgets as follows:

$$b[A] = [1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^T \text{ and}$$
$$b[B] = [0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^T.$$

The asymmetric equilibrium described above recovers a realistic feature of marketing competition whereby firms offering substitutable products design their advertising campaigns to cater to different consumer segments thus softening the competition between them.

5. How does the network structure affect competition?

This section explores the question of how the network structure affects the outcomes of targeting under competition. Our goal is to provide intuition on what networks may lead to large differences in the marketing campaigns of the two firms. Specifically, we are interested in

$$\Pi = \max_{\mathcal{E},(\sigma^A,\sigma^B)} \frac{m^A_{lim}(\sigma^A,\sigma^B)}{m^B_{lim}(\sigma^A,\sigma^B)},$$

where $\sigma = (\sigma^A, \sigma^B)$ is an equilibrium of the marketing competition game when the adjacency matrix corresponding to the social network structure is given by $\mathcal{E}$. Let c_i denote the centrality of agent i under $\mathcal{E}$ (agents are ordered with respect to their centralities). We focus our attention to functions $h(.)$ for which asymmetric equilibria exist.

Proposition 8. *There exists $\bar{n}$ such that for $n \geq \bar{n}$ the ratio Π of payoffs is bounded as*

$$\Pi \geq \frac{c_1}{c_2},$$

when $c_1 \leq \min_{x\in(0,1]} \dfrac{h(1,0) - h(1-x,0)}{h(x,1)} c_2.$

The following corollary of Proposition 8 specializes the result for contest success functions.

COROLLARY 2. *Consider the class of functions* $h(x,y) = \dfrac{x^s}{x^s+y^s+1}$. *There exists* $\bar{n}(s)$ *such that*

$$\Pi = 1 \quad \textit{for } s \le 1 \quad \textit{and} \quad \Pi \ge \frac{3}{2} \quad \textit{for } s \ge \frac{3}{2}, \quad \textit{for } n \ge \bar{n}(s).$$

At this point, it would be instructive to describe the main idea behind the lower bound in Proposition 8 and Corollary 2. Consider a network with two connected hubs (see for example Figure 1). Then, if firm A allocates its entire budget to agent 1 and firm B to agent 2, ratio Π is equal to c_1/c_2. However, for this allocation to be an equilibrium, the ratio c_1/c_2 cannot be too large (as then firm B would find it profitable to deviate by allocating a fraction of its marketing budget to agent 1). The claim follows by maximizing c_1/c_2 subject to the constraint that firm B finds it optimal to allocate its entire budget to agent 2 when firm A places its budget to agent 1. Networks that feature agents with sufficiently high centralities may lead to equilibria where the returns on marketing efforts for two ex-ante symmetric firms are substantially different.

We conclude this section with a brief discussion on the stability properties of the equilibria we identified above. In particular, we are interested in providing some insight on which allocations are more likely to emerge as the outcome of the competition game in the context of the bound in Proposition 8. One can see that the equilibria described as part of Proposition 8 and Corollary 2 would also be equilibria of a Stackelberg version of the game in which one of the firms commits to a marketing strategy first (this would not be the case for the symmetric equilibria). Moreover, one can also see that the asymmetric equilibria that lead to large differences in the marketing efforts of the two firms are stable to small perturbations for the network structures of Proposition 8. In particular, an equilibrium is stable to small perturbations if, when firm i's strategy is perturbed by a sufficiently small ϵ, then (i) the best response for firm $j \neq i$ does not change; and (ii) firm i is worse off for the perturbed strategy profile. Consider a network with two hubs, agents 1 and 2, that have centrality c_1 and c_2 respectively (see the example in Figure 1). In this case, when c_1, c_2 are sufficiently larger than the centralities of the rest of the agents, firms A and B allocate their marketing budgets exclusively to agents 1 and 2. The symmetric equilibrium involves mixed strategies, i.e., firms allocate their marketing budget to 1 with probability p_1 and to 2 with probability $1-p_1$. It is straightforward then to see that this symmetric equilibrium is not stable to small perturbations. On the other hand, the asymmetric equilibrium we described above that is such that firm A allocates its budget to agent 1 whereas B to 2, satisfies both conditions for a perturbation with a sufficiently small ϵ when $c_1 > c_2$. Thus, for this class of networks that may lead to large values for Π, asymmetric equilibria are more likely to emerge as the competition outcome.

6. Endogenizing the marketing budgets

Our analysis so far has assumed that marketing budgets for both firms are fixed and given. This section relaxes this assumption and explores the implications of having firms optimize jointly over both the size of their marketing budgets as well as how to allocate them to the agents. In particular, we assume that the marginal cost of increasing a firm's marketing budget is constant and equal to ξ. To allow for tractable analysis and simplify the exposition, we focus on the case when

$$h(x,y) = \frac{x}{x+y+1}.$$

Then, firm A's payoff corresponding to budget allocation decisions $\{b_i(A)\}_{i\in\mathcal{N}}$ and $\{b_i(B)\}_{i\in\mathcal{N}}$ is given by

$$\frac{1}{n}(1-\alpha)\sum_{i=1}^{n} c_i \cdot \frac{b_i(A)}{b_i(A)+b_i(B)+1} - \xi\sum_{i=1}^{n} b_i(A).$$

This richer framework allows us to explore the value of targeting technologies compared to mass advertising strategies as well as clearly illustrate the impact of the network structure on a firm's payoff. We present our results regarding these questions in what follows. Our findings highlight that both the value of targeting as well as firms' payoffs depend critically on the structure of the agents' interactions.

The value of targeting: Implementing a targeted advertising strategy may involve costs above and beyond the actual marketing budget. Thus, it is worthwhile to compare the potential benefits of targeting with the payoff corresponding to engaging in a mass advertising strategy. In particular, we compare firms' payoffs at equilibrium in the following two settings:

- *Mass advertising:* Firms optimize over the size of their marketing budgets assuming that they allocate them uniformly over all agents. This setting represents an environment where firms do not use (potentially costly) targeting technologies and engage in mass advertising (which corresponds to allocating an equal fraction of their budgets to all agents). Specifically, firm A's optimization problem takes the following form (when firm B's total budget is equal to $\mathcal{B}_B$)

$$\max_{\mathcal{B}_A} \frac{1}{n}(1-\alpha)\frac{1/n\cdot\mathcal{B}_A}{1/n\cdot\mathcal{B}_A + 1/n\cdot\mathcal{B}_B + 1}\sum_{i=1}^{n} c_i - \xi\mathcal{B}_A. \qquad (3)$$

A game between the two competing firms is defined naturally as follows: each firm chooses its marketing budget in order to optimize its respective payoff given the action of its competitor and assuming that marketing budgets are allocated uniformly to the agents.

- *Targeted advertising:* Each firm optimizes over both the size of its marketing budget and how to allocate it to the set of agents taking into account the actions of its competitor.

Our first result states that targeting is always beneficial for the firms compared to mass advertising at equilibrium.

Proposition 9. *Equilibrium payoffs are at least as high when firms compete in targeted advertising as when they compete in mass advertising.*

Proposition 9 and Proposition 10 below imply that targeting technologies increase firms' payoffs even under competition: the positive effect of being able to target influential agents dominates the potentially negative effect of increasing the competition for such agents. Intuitively, mass advertising is inefficient when the network structure is heterogeneous in terms of the agents' centralities since a large fraction of the firms' marketing budgets is allocated to agents with relatively low centralities (note that when all agents have the same centrality, mass and targeted advertising lead to the same payoffs). The proposition below formalizes this intuition and, more broadly, establishes that the value of targeting increases in the presence of highly central agents. To simplify exposition, we let $\hat{c}_i = \frac{1}{n}(1-\alpha)c_i$.

Proposition 10. *Consider two networks $\mathcal{G}(\mathcal{N},\mathcal{E})$, $\mathcal{G}'(\mathcal{N},\mathcal{E}')$ for which the agents' centralities are given by sequences $\{c_i\}_{i=1}^n$ and $\{c_i'\}_{i=1}^n$ respectively. Assume that $\sum_{i=1}^n c_i = \sum_{i=1}^n c_i'$, i.e, the extent to which agents engage in word-of-mouth is fixed in both networks. Furthermore, assume that there exists $1 \le k < n$ such that*

$$c_i \ge c_i' \text{ for } 1 \le i \le k, \quad c_i \le c_i' \text{ for } i > k, \quad \text{ and } \sum_{i=1}^k c_i > \sum_{i=1}^k c_i',$$

and $c_n, c_n' \ge \xi$. Then, a firm's equilibrium payoff when firms compete in targeted advertising in network $\mathcal{G}$ is higher than in $\mathcal{G}'$.

Proposition 10 states that firms obtain higher payoffs at equilibrium for networks that feature relatively more central agents. Noting that the payoff for a firm when both competitors engage in mass advertising depends only on the sum of the agents' centralities, we conclude that the relative value of targeting compared to mass advertising is higher in $\mathcal{G}$ than in $\mathcal{G}'$.

Are marketing levels efficient? The next question that arises naturally in this setting is whether firms choose their marketing budgets efficiently. Proposition 10 implies that their aggregate payoff increases in the presence of highly central agents but it does not provide any intuition on how it compares with the payoff they could have obtained by jointly optimizing their marketing efforts, i.e., if they chose their actions (marketing budgets and their allocation to the agents) so as to maximize their aggregate payoff.

In what follows, we characterize the optimal marketing level corresponding to a single firm that does not face competition and compare it with the marketing levels corresponding to the equilibrium of the game between the two firms. We find that firms spend inefficiently high amounts on marketing as they do not internalize the negative externality they exert on each other. Furthermore and quite importantly, the extent of inefficiency increases with the heterogeneity in the network structure.

THEOREM 2. *The optimal level of marketing, i.e., the level of marketing that corresponds to the firms maximizing their aggregate payoff, is given by*

$$\mathcal{B}^* = \sum_{i=1}^{k} \sqrt{\frac{\hat{c}_i}{\xi}} - k,$$

where $k = \max\{j \in \mathcal{N} | \hat{c}_j > \xi\}$*. At the unique symmetric equilibrium when firms compete in targeting, the aggregate level of marketing, i.e., the sum of the firms' marketing budgets, is higher and it is given by*

$$2 \cdot \mathcal{B}^{eq} = \sum_{i=1}^{k} \sqrt{\frac{\hat{c}_i}{\xi}} \sqrt{b_i^{eq}(A) + 1} - k,$$

where $b_i^{eq}(A) = \frac{1}{8}\left[\left(\frac{\hat{c}_i}{\xi} - 4\right) + \sqrt{\left(\frac{\hat{c}_i}{\xi} - 4\right)^2 + 16\left(\frac{\hat{c}_i}{\xi} - 1\right)}\right]$*, and* $\mathcal{B}^{eq}$ *denotes the marketing budget of each firm at equilibrium. Furthermore, the firms' aggregate payoff at equilibrium is given by*

$$2 \cdot \Pi_{eq} = \sum_{i=1}^{k} \left(c_i + \xi - \sqrt{c_i \xi}\left(\frac{1}{\sqrt{x_i^{eq} + 1}} + \sqrt{x_i^{eq} + 1}\right)\right),$$

where Π_{eq} *denotes the equilibrium payoff of each firm while the payoff in a monopoly is given by*

$$\Pi_{monopoly} = \sum_{i=1}^{k} \left(c_i + \xi - 2\sqrt{c_i \xi}\right).$$

Theorem 2 clearly illustrates the effect of the underlying network structure on the marketing levels and firms' payoffs at equilibrium and how they compare with the levels in a monopoly. As the centrality of influential agents increases, both the efficient and equilibrium levels of marketing increase. Interestingly, the extent of inefficiency due to the firms competing also increases, i.e., the firms' payoffs in the presence of competition decrease relative to the monopoly levels as the underlying network structure has relatively more central agents. We make this formal in the following corollary of Theorem 2.

COROLLARY 3. *Consider two networks* $\mathcal{G}(\mathcal{N}, \mathcal{E})$, $\mathcal{G}'(\mathcal{N}, \mathcal{E}')$ *for which the agents' centralities are given by sequences* $\{c_i\}_{i=1}^n$ *and* $\{c'_i\}_{i=1}^n$ *respectively. Furthermore, assume that*

$$k = \max\{j \in \mathcal{N} | \hat{c}_j > \xi\} = \max\{j \in \mathcal{N} | \hat{c}_j{}' > \xi\},$$

and $\sum_{i=1}^k c_i = \sum_{i=1}^k c'_i$*, i.e, the sum of the centralities of the agents that firms find optimal to target is the same in both networks. Let*

$$c_i \geq c'_i \text{ for } 1 \leq i \leq \ell, \quad c_i \leq c'_i \text{ for } \ell < i \leq k, \quad \text{and } \sum_{i=1}^{\ell} c_i > \sum_{i=1}^{\ell} c'_i.$$

The difference between the monopoly payoff and the firms' aggregate payoff at equilibrium, i.e.,

$$\Pi_{monopoly} - 2 \cdot \Pi_{eq} = \sum_{i=1}^{k} \sqrt{c_i \xi} \left(\frac{1}{\sqrt{x_i^{eq}+1}} + \sqrt{x_i^{eq}+1} - 2 \right),$$

in network $\mathcal{G}$ is higher than in $\mathcal{G}'$. Similarly, the difference between the firms' aggregate marketing level at equilibrium and the optimal marketing level, i.e.,

$$2 \cdot \mathcal{B}^{eq} - \mathcal{B}^{*} = \sum_{i=1}^{k} \sqrt{\frac{c_i}{\xi}} \left(\sqrt{x_i^{eq}+1} - 1 \right),$$

is higher in $\mathcal{G}$ than in $\mathcal{G}'$.

7. Conclusions

The paper studies a model of marketing competition between two firms over social networks. We provide a characterization of the optimal marketing strategies for the two competitors and clearly illustrate that network centrality is an important metric of an agent's influence over her peers. Furthermore, we highlight the value of targeted advertising and how the latter depends on the structure of the underlying network. Finally, we show that equilibrium behavior over networks may lead to large differences in the payoffs for the two firms. Although agents in our model behave according to a simple rule-of-thumb (e.g., they put equal weight to all information they receive), we believe that our model captures many of the essential features of targeted advertising and brand awareness formation: advertising efforts have diminishing returns, a competitor's advertising may have both positive and negative effects on a firm's awareness level, and awareness may decay over time due to "forgetting".

Our main focus in this paper is to study the word-of-mouth process and how competing firms can influence it. Thus, we chose to use the limiting awareness level about its brand as the firm's objective. Extending our analysis to incorporate purchasing decisions from the agents and pricing decisions from the firms is an interesting direction for future research. In particular, consider a setting where consumers choose whether and when to purchase one of the competing products depending on their beliefs about their (originally unknown) quality and the respective prices. Furthermore, consumers can generate and pass information to their peers from their own experimentation with the product. Firms in this richer framework can control the rate of learning through both advertising and their pricing decisions and the objective would be to devise optimal combined pricing and marketing strategies for the two competitors (for related work, refer to Bergemann and Valimaki (2000), Acemoglu et al. (2014), and Ifrach et al. (2014).)

Appendix

Proof of Proposition 1

The proof of Proposition 1 is based on Lemmas 1 and 2. Lemma 1 characterizes the expectation of awareness levels for firms A, B at time k as a function of the history up to (and excluding) k.

LEMMA 1. *Given a society $\mathcal{G}(\mathcal{N},\mathcal{E})$, budget allocation decisions $\{b_i(A), b_i(B)\}_{i\in\mathcal{N}}$, and the history of information exchange up to time period k, $\{m[l]\}_{l<k}$, the expected value of the agents' awareness levels at time k is given by:*

$$\mathbb{E}[y^A[k]\big|\{m[l]\}_{l<k}] = (I + D[k-1]W)\, y^A[k-1], \text{ and } \mathbb{E}[y^B[k]\big|\{m[l]\}_{l<k}] = (I + D[k-1]W)\, y^B[k-1],$$

where I is the identity matrix, W is given in Equation (1), and $D[k-1]$ is a diagonal matrix with:

$$[D[k-1]]_{ii} = \begin{cases} 1/k & \text{if } i \in \mathcal{N}, \\ 1 & \text{if } i \in \{n+1, n+2, n+3\}. \end{cases}$$

Proof: First, we study the dynamics of an agent's awareness level about firm A. Note that given the history of message exchanges $\{m[l]\}_{l<k}$, the awareness level of agent i at iteration $k-1$ is deterministic and equal to $x_i^A[k-1]$. For a given agent $i \in \mathcal{N}$ and iteration k:

$$\mathbb{E}[x_i^A[k]|\{m[l]\}_{l<k}] = \alpha \mathbb{E}[x_i^A[k]|s_i[k] \in \mathcal{N}_i, \{m[l]\}_{l<k}] + (1-\alpha)\mathbb{E}[x_i^A[k]|s_i[k] \notin \mathcal{N}_i, \{m[l]\}_{l<k}], \tag{4}$$

where Equation (4) follows from the fact that the sender of the message to i at time k, which we denote by $s_i[k]$, belongs to her peer group with probability α and with probability $(1-\alpha)$ the message comes directly from A, B, or SQ. The first term of Equation (4) can be rewritten as:

$$\begin{aligned} \mathbb{E}[x_i^A[k]\big|s_i[k] \in \mathcal{N}_i, \{m[l]\}_{l<k}] &= \frac{k-1}{k} x_i^A[k-1] + \frac{1}{k} \sum_{j\in\mathcal{N}_i} (\mathbb{P}(s_i[k] = j)\; x_j^A[k-1] \\ &= \frac{k-1}{k} x_i^A[k-1] + \frac{1}{k} \sum_{j\in\mathcal{N}_i} e_{ij}\; x_j^A[k-1], \end{aligned}$$

whereas the second as $\mathbb{E}[x_i^A[k]\big|s_i[k] \notin \mathcal{N}_i, \{m[l]\}_{l<k}] = \frac{k-1}{k} x_i^A[k-1] + \frac{1}{k} h(b_i(A), b_i(B))$. Similarly, we obtain an expression for $\mathbb{E}[x_i^B[k]|\{m[l]\}_{l<k}]$. The lemma follows from simple algebra. ■

For the rest of the proof, we only study $y^A[k]$ (the analysis for $y^B[k]$ is identical). The next lemma describes the evolution of the awareness level vector from one time period to the next.

LEMMA 2. *The awareness level vector satisfies*

$$y^A[k] = y^A[k-1] + D[k-1]\left(W y^A[k-1] + n[k]\right). \tag{5}$$

Moreover, if $\{\mathcal{F}[l]\}_{l\geq 0}$ is the family of σ fields with $\mathcal{F}[l] = \sigma(\{m[l]\}_{l<k})$, then $\mathbb{E}[n[k]|\mathcal{F}[k-1]] = 0$ almost surely and $\mathbb{E}[||n[k]||^2|\mathcal{F}[k-1] \leq \sup_{l<k} K(1 + ||y[l]||^2)$ a.s., for all $k \geq 1$, and constant $K > 0$.

Proof: Expression (5) follows from Lemma 1 for some function $n[k]$. Next, note that $D[k-1]|\mathcal{F}[k-1]$ is invertible a.s. for all $k \geq 1$, since it is a diagonal matrix with non-zero entries a.s. Then, we can rewrite

Equation (5) as $n[k] = D^{-1}[k-1]\left(y^A[k] - (I + D[k-1]W)y^A[k-1]\right)$. By taking expectations on both sides conditional on the σ-field $\mathcal{F}[k-1]$ we have:

$$\begin{aligned}\mathbb{E}[n[k]|\mathcal{F}[k-1]] &= \mathbb{E}[D^{-1}[k-1]\left(y[k] - (I + D[k-1]W)y^A[k-1]\right)|\mathcal{F}[k-1]]\\ &= \mathbb{E}[D^{-1}[k-1]y^A[k]|\mathcal{F}[k-1]] - \mathbb{E}[D^{-1}[k-1]\ (I + D[k-1]W)y^A[k-1]|\mathcal{F}[k-1]]\\ &= \mathbb{E}[D^{-1}[k-1]|\mathcal{F}[k-1]]\mathbb{E}[y^A[k]|\mathcal{F}[k-1]] - \mathbb{E}[(D^{-1}[k-1] + W)y^A[k-1]|\mathcal{F}[k-1]] = 0,\end{aligned}$$

where the last equality follows from Lemma 1. For a given history of message exchanges $\{m[l]\}_{l<k}\}$, vector $y_i^A[k]$ has the following probability distribution:

$$y_i^A[k]|\{m[l]\}_{l<k} = \begin{cases} \frac{k-1}{k} y_i^A[k-1] + \frac{1}{k} & \text{w.p. } \frac{\alpha}{n}\sum_{j\in\mathcal{N}_i} e_{ij} y_j^A[k-1] + \frac{1-\alpha}{n} h(b_i(A), b_i(B)),\\ \frac{k-1}{k} y_i^A[k-1] & \text{otherwise.}\end{cases}$$

Using this we bound the expected value of $||n(k)||^2$ by:

$$\begin{aligned}\mathbb{E}[||n[k]||^2|\{m[l]\}_{l<k}] &\le ||Wy[k-1]||^2 + ||(I+W)y[k-1] + I||^2 + ||(I+W)y[k-1]||^2\\ &\le (3||W||^2 + 2||I||^2)||y[k-1]||^2 + ||I||^2.\end{aligned}$$

We obtain $\mathbb{E}[||n[k]||^2|\{m[l]\}_{l<k}] \le K(1 + y[k-1]) \le \sup_{l<k} K(1 + ||y[l]||^2)$, where $K = \max(3||W||^2 + 2||I||^2, ||I||^2)$. Finally, we can extend this for any sequence with non-zero measure, i.e., $\mathbb{E}[||n[k]||^2|\mathcal{F}[k-1] \le K(1 + y[k-1])$ *a.s.* ■

Lemma 2 and Borkar (2008, Ch. 7) imply that the awareness level vector $y^A[k]$ tracks a time-independent ordinary differential equation of the form:

$$\frac{\nabla z^A(t)}{dt} = Wz^A(t) \quad \forall t > 0, \tag{6}$$

with the initial condition $z^A(0) = y^A(0)$. The last three entries of vector $z^A(0)$ are 1, 0, and 0 respectively, and these elements are fixed for all $t > 0$, since the last three rows of W in Equation (1) are zero vectors. Since the communication network is connected for all agents, the dynamical system described by Equation (6) converges to a well defined limit, and has a unique solution which does not depend on the initial conditions. If we denote the limit of the differential equation in Equation (6) by z_{lim}^A, then according to Borkar (2008, Ch. 7) the agents' awareness levels converge almost surely to $\lim_{k\to\infty} y^A[k] = z_{lim}^A$. Moreover, due to the specific structure of the differential equation, the awareness level can be characterized by $z_{lim}^A = \lim_{t\to\infty} exp(W^t)z^A[0]$, where $exp(.)$ is the matrix exponential of its argument. While this equation uniquely characterizes the limit of agents' awareness levels, it is not particularly insightful. However, note that the dynamical system described by (6) defines a continuous time Markov Chain. We define the corresponding jump matrix V (Norris (1998, Ch. 2)) as in Expression (1) and use results from the Markov Chain literature (see Norris (1998, Ch. 3)) to show that $\lim_{t\to\infty} exp(W^t) = \lim_{k\to\infty} V^k$. Combining the above we conclude that $\lim_{k\to\infty} y^A[k] = \lim_{k\to\infty} V^k y^A[0]$.

Proof of Theorem 1

First, we study the average awareness level for firm A. Given that $V = \left[\begin{smallmatrix}\alpha\mathcal{E} & E\\ 0 & I_3\end{smallmatrix}\right]$, we obtain:

$$\lim_{k\to\infty}(V)^k = \lim_{k\to\infty}\begin{bmatrix}\alpha^k\mathcal{E}^k & \left(\sum_{l=0}^k \alpha^l\mathcal{E}^l\right)E\\ 0 & I_3\end{bmatrix} = \begin{bmatrix}0 & (I-\alpha\mathcal{E})^{-1}E\\ 0 & I_3\end{bmatrix}. \tag{7}$$

Moreover, using Proposition 1 and noting that $y_{n+1}^A[0] = 1, y_{n+2}^A[0] = 0, y_{n+3}^A[0] = 0$, we obtain $[y_{lim}^A]_{1:n} = (I-\alpha\mathcal{E})^{-1}[E]_1$, where $[E]_1$ is the first column of E. Finally, denoting by $\mathbf{1}$ the all ones vector, we get: $m_{lim}^A = \frac{1}{n}\mathbf{1}'(I-\alpha\mathcal{E})^{-1}[E]_1 = \frac{1}{n}\sum_{i\in\mathcal{N}}(1-\alpha)[\mathbf{1}'(I-\alpha\mathcal{E})^{-1}]_i\ h(b_i(A), b_i(B))$. The proof for the average awareness level about brand B follows from a similar argument.

Proof of Proposition 3

The proof builds on the KKT conditions for firm A's optimization problem. Specifically, if

$$\gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0},$$

for a given i, then $b_i^\star(A) > 0$. Our claim follows from the fact that h is assumed to be strictly concave in x and $\lambda_i \geq 0$. Moreover, if $b_i^\star(A) > 0$, then $\lambda_i = 0$ due to the complementary slackness condition. Therefore,

$$\gamma = \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)} \text{ for such } i.$$

On the other hand, if

$$\gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0},$$

then $b_i^\star(A) = 0$, since otherwise $\lambda_i > 0$, which would violate the complementary slackness condition. Therefore, we have:

$$b_i^\star(A) = \begin{cases} 0 & \text{if } \gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \\ b_i(A) \text{ s.t. } \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{b=b_i(B)} = \gamma & \text{if } \gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}, \end{cases}$$

Moreover, $\sum_{i\in\mathcal{N}} b_i^\star(A) = 1$ has to hold.

Proof of Corollary 1

Note that if $b_i^\star(A) = 0$, then $\gamma \geq \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(c,b_i(B))}{\partial x}\right|_{x=0}$ by Proposition 3. Since $c_i \geq c_j$, $b_i(B) = b_j(B)$, and $h(x,y)$ is increasing in x, then it must be that $\gamma \geq \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$. Therefore, $b_j^\star(A) = b_j^\star(A) = 0$. On the other hand, if $b_i^\star(A) > 0$, then $\gamma < \frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=0}$ by Proposition 3. For agent j, we either have $\gamma \geq \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$ or $\gamma < \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=0}$. If the former condition holds, then $b_j^\star(A) = 0$, i.e., $b_i^\star(A) > b_j^\star(A)$. If the latter condition holds, then:

$$\frac{1}{n}c_i(1-\alpha)\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)} = \gamma = \frac{1}{n}c_j(1-\alpha)\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=b_j^\star(A)},$$

by Proposition 3. The above equality implies that: $c_i\left.\frac{\partial h(x,b_i(B))}{\partial x}\right|_{x=b_i^\star(A)} = c_j\left.\frac{\partial h(x,b_j(B))}{\partial x}\right|_{x=b_j^\star(A)}$. The claim follows since $h(x,y)$ is increasing and strictly concave in x for all y, and thus $b_i^\star(A) \geq b_j^\star(B)$.

Proof of Proposition 4

The game has a symmetric equilibrium in pure strategies since it a symmetric game with compact, convex strategy spaces and continuous, quasiconcave utility functions (see Cheng et al. (2004)). In what follows, we show that the game under the assumptions in the proposition does not have asymmetric equilibria. Assume for the sake of contradiction that in a given equilibrium σ, the activation set of firm A is a strict subset of the activation set of firm B, i.e., $\mathcal{A}_A^\sigma \subset \mathcal{A}_B^\sigma$. Then, there exist i and j such that $i \notin \mathcal{A}_A^\sigma$ and $i \in \mathcal{A}_B^\sigma$, and $j \in \mathcal{A}_A^\sigma, \mathcal{A}_B^\sigma$. Furthermore, firm A allocates strictly larger budget to agent j than firm B (since both firms have equal budgets). If we denote by $b(A), b(B)$ the budget allocation vectors of firms A and B at this equilibrium, then $b_i(A) = 0$, $b_i(B), b_j(B), b_j(A) > 0$, and $b_j(A) > b_j(B)$. From the KKT conditions, the following must hold:

$$c_i \frac{\partial h(0, b_i(B))}{\partial x} \leq c_j \frac{\partial h(b_j(A), b_j(B))}{\partial x}, \tag{8}$$

$$c_i \frac{\partial h(b_i(B), 0)}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{9}$$

Using the fact that $\frac{\partial h(x,y)}{\partial x} > 0$, $c_i > 0 \ \forall\, i \in \mathcal{N}$, and dividing both sides of Equation (8) by the corresponding sides of Equation (9) we obtain

$$\frac{\partial h(0, b_i(B))}{\partial x} \frac{\partial h(b_j(B), b_j(A))}{\partial x} \leq \frac{\partial h(b_j(A), b_j(B))}{\partial x} \frac{\partial h(b_i(B), 0)}{\partial x}. \tag{10}$$

We first note that:

$$\frac{\partial h(b_i(B), 0)}{\partial x} < \frac{\partial h(0,0)}{\partial x} \leq \frac{\partial h(0, b_i(B))}{\partial x}, \tag{11}$$

where the first inequality follows from the fact that $h(x,y)$ is strictly concave in x, and the second inequality follows from the fact that the mixed partial derivative is non-negative. Since $b_j(A) > b_j(B)$ and following the same argument as above, we show that:

$$\frac{\partial h(b_j(A), b_j(B))}{\partial x} < \frac{\partial h(b_j(B), b_j(B))}{\partial x} \leq \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{12}$$

From Eqs. (11) and (12) we obtain

$$\frac{\partial h(b_i(B), 0)}{\partial x} \frac{\partial h(b_j(A), b_j(B))}{\partial x} < \frac{\partial h(0, b_i(B))}{\partial x} \frac{\partial h(b_j(B), b_j(A))}{\partial x}.$$

However, this contradicts with Equation (10), and $b(A)$ and $b(B)$ cannot satisfy the KKT conditions. Therefore, $b(A), b(B)$ cannot be equilibrium budget allocations, and $\mathcal{A}_A \not\subset \mathcal{A}_B$.

Next to reach as contradiction, we assume that in an equilibrium, there exists $i \in \mathcal{N}$ such that $i \notin \mathcal{A}_A$ and $i \in \mathcal{A}_B$, and also $j \in \mathcal{N}$ such that $j \in \mathcal{A}_A$ and $j \notin \mathcal{A}_B$. If we denote $b(A), b(B)$ as the budget allocation vectors of firms A, B, then $b_i(A) = b_j(B) = 0$, and $b_j(A), b_i(B) > 0$.

Due to the KKT conditions, the following holds for i and j:

$$c_i \frac{\partial h(0, b_i(B))}{\partial x} \leq c_j \frac{\partial h(b_j(A), 0)}{\partial x}, c_j \frac{\partial h(0, b_j(A))}{\partial x} \leq c_i \frac{\partial h(b_i(B), 0)}{\partial x}. \tag{13}$$

Once again, note that:

$$\frac{\partial h(b_j(A), 0)}{\partial x} < \frac{\partial h(0,0)}{\partial x} \leq \frac{\partial h(0, b_j(A))}{\partial x}, \tag{14}$$

where the first inequality follows from the fact that $h(x,y)$ is strictly concave in x, and the second from the fact that the mixed derivative is non negative. Similarly, we can also show that:

$$\frac{\partial h(b_i(B), 0)}{\partial x} < \frac{\partial h(0,0)}{\partial x} \leq \frac{\partial h(0, b_i(B))}{\partial x}. \tag{15}$$

However, Eqs. (13), (14), (15) lead to a contradiction. Therefore, at any equilibrium $\mathcal{A}_A = \mathcal{A}_B$, i.e., the activation sets of A and B are equal. Let's assume that given $\mathcal{A}_A = \mathcal{A}_B$, the budget allocations are asymmetric, i.e., there exists an $i \in \mathcal{N}$ for which $b_i(A) > b_i(B)$. Since the budgets for A and B are equal, there exists $j \in \mathcal{N}$ such that $b_j(B) > b_j(A)$. From the KKT conditions, the following must hold for i and j:

$$c_i \frac{\partial h(b_i(A), b_i(B))}{\partial x} = c_j \frac{\partial h(b_j(A), b_j(B))}{\partial x}, c_i \frac{\partial h(b_i(B), b_i(A))}{\partial x} = c_j \frac{\partial h(b_j(B), b_j(A))}{\partial x}. \tag{16}$$

However, this leads to the following

$$\frac{\partial h(b_i(A), b_i(B))}{\partial x} < \frac{\partial h(b_i(B), b_i(B))}{\partial x} \leq \frac{\partial h(b_i(B), b_i(A))}{\partial x},$$

where the first inequality follows from the fact that $h(x, y)$ is strictly concave in x, and the second inequality follows from the fact that the mixed derivative is non-negative. By noting the fact that $b_j(B) > b_j(A)$, following the same argument as above, and some algebra, we reach another contradiction. Thus, $b(A) = b(B)$, i.e., equilibria are symmetric under the assumptions of the proposition.

Proof of Proposition 5

As before, the game has a symmetric equilibrium in pure strategies since it a symmetric game with compact, convex strategy spaces and continuous, quasiconcave utility functions. It is straightforward to show that the symmetric equilibrium is unique. Finally, note that Equation (2) in the proposition implies that

$$\frac{\partial h(t_1, t_2)}{\partial x} < \frac{\partial h(t_2, t_1)}{\partial x}, \text{ for } 0 \leq t_2 < t_1 \leq 1.$$

Using this and similar arguments as in Proposition 4 we show that there exist no asymmetric equilibria.

Proof of Proposition 6

For $h(x, y) = \frac{f(x)}{f(x)+f(y)+\delta}$ with $f(\cdot)$ concave we have

$$\frac{\partial h(t_1, t_2)}{\partial x} < \frac{\partial h(t_2, t_1)}{\partial x}, \text{ for } 0 \leq t_2 < t_1 \leq 1.$$

Given this the proof of the proposition follows from the proofs of Propositions 4 and 5.

Proof of Proposition 7

Assume that $\bar{k}$ satisfies the following inequality

$$\bar{k} \geq 2 \left(\frac{\delta(s-1)}{s+1} \right)^{-1/s}.$$

Assume for the sake of contradiction that there exists a symmetric equilibrium. First, we consider the case when in this equilibrium agents 1 to $\bar{k}$ receive a positive fraction of the firms' budgets. Then, there will be at least two agents i and j with $c_i \geq c_j$ for which $b_i(A) < \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$ and $b_j(A) < \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$ since the total budget for a firm is normalized to one. However, note that moving ϵ fraction of A's budget from j to i increases her payoff which implies that the allocation cannot be at equilibrium. Similarly, if there is at least one agent out of the first $\bar{k}$ that does not receive any budget from the firms then there is a profitable deviation that involves one of the firms moving some budget to that agent.

We claim that there exists an asymmetric equilibrium in which firm A allocates its budget to agents 1 to ℓ with $\ell \leq \left(\frac{\delta(s-1)}{s+1}\right)^{-1/s} \leq \bar{k}/2$ and firm B to agents $\ell + 1$ to $\bar{k}$. The candidate equilibrium is such that firm A's allocation is the solution to the following optimization problem

$$\max_{\{b_i(A)\}_{1 \leq i \leq \ell}} \sum_{i=1}^{\ell} c_i \frac{b_i(A)^s}{b_i(A)^s + \delta}, \text{ s.t. } \sum_{1 \leq i \leq \ell} b_i(A) = 1,$$

i.e., firm A allocates its budget as if there was no competition. In turn, firm B best responds to A's allocation. To complete the proof we have to show that firm B has no profitable deviation. Note that the solution to the optimization problem above is such that $b_i(A) \geq \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$ for $i \leq \ell - 1$ since function $\frac{x^s}{x^s+\delta}$ is convex for $x < \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$. Moreover, since firm A is allocating some budget to agent ℓ it has to be the case that

$$c_1 \frac{b_1(A)^{s-1}}{(\delta + b_1(A)^s)^2} = c_\ell \frac{b_\ell(A)^{s-1}}{(\delta + b_\ell(A)^s)^2}.$$

Further note that function $\frac{x^{s-1}}{(\delta+x^s)^2}$ is increasing for $x < \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$ and decreasing for $x > \left(\frac{\delta(s-1)}{s+1}\right)^{1/s}$.This implies that

$$\frac{b_\ell(A)^{s-1}}{(\delta + b_\ell(A)^s)^2} = \frac{c_1}{c_\ell} \frac{b_1(A)^{s-1}}{(\delta + b_1(A)^s)^2} \geq \frac{b_1(A)^{s-1}}{(\delta + b_1(A)^s)^2} \geq \frac{1}{(1+\delta)^2},$$

which consequently implies a lower bound $\bar{b}$ for $b_\ell(A)$. Note that if $\beta \leq 1 + \frac{\bar{b}^s}{1+\delta}$ firm B has no profitable deviation and the allocation described above is an (asymmetric) equilibrium of the game.

Proof of Proposition 8

To show a lower bound for Π, we provide an adjacency matrix and a corresponding equilibrium that yields a ratio of payoffs equal to c_1/c_2. Consider adjacency matrix $\mathcal{E}$:

$$\mathcal{E}_{11} = \mathcal{E}_{12} = 0, \mathcal{E}_{13} = \cdots = \mathcal{E}_{1n} = \frac{1}{n-2}, \quad \mathcal{E}_{21} = \mathcal{E}_{22} = 0, \mathcal{E}_{23} = \cdots = \mathcal{E}_{2n} = \frac{1}{n-2},$$

$$\mathcal{E}_{31} = \cdots = \mathcal{E}_{n1} = \frac{\lambda}{1+\lambda}, \mathcal{E}_{32} = \cdots = \mathcal{E}_{n2} = \frac{1}{1+\lambda} \quad \mathcal{E}_{ij} = 0, \text{ for } i, j > 2,$$

where $\lambda > 1$ is a constant. In particular, agents 1 and 2 are weighted disproportionately more compared to agents $3, \cdots, n$ in $\mathcal{E}$ and agent 1 is weighted λ times as much as agent 2. From the definition of the centrality vector ($\mathbf{c} = [\mathbf{1}'(I - \alpha\mathcal{E})^{-1}]$) we obtain that:

$$\frac{c_1}{c_2} = \frac{(n-2)\alpha \frac{\lambda}{1+\lambda} + 2\alpha^2 \frac{\lambda}{1+\lambda} + 1 - \alpha^2}{(n-2)\alpha \frac{1}{1+\lambda} - 2\alpha^2 \frac{\lambda}{1+\lambda} + 1 + \alpha^2} \text{ and } \frac{c_3}{c_2} = \frac{1 + \alpha \frac{2}{n-2}}{(n-2)\alpha \frac{1}{1+\lambda} - 2\alpha^2 \frac{\lambda}{1+\lambda} + 1 + \alpha^2}.$$

We choose λ so that $\frac{c_1}{c_2} = \min_{x \in (0,1]} \frac{h(1,0) - h(1-x,0)}{h(x,1)}$ (this is always possible) and we consider the following allocation: $b_1(A) = 1 = b_2(B)$ and $b_2(A) = \cdots = b_n(A) = b_1(B) = b_3(B) = \cdots = b_n(B) = 0$. Then, it is straightforward to see that neither of the two firms has any profitable deviation that involves moving advertising funds between agents 1 and 2. Thus, for the allocation to be an equilibrium we need to make sure that firm B (and as a consequence firm A) has no incentive to deviate by moving advertising funds to agents $3, \cdots, n$. This holds for a sufficiently large population of agents, since as we increase the number of agents n, the ratio c_3/c_2 decreases. Thus, the proposed allocation is an equilibrium and

$$\Pi \geq \frac{c_1 h(1,0)}{c_2 h(1,0)} = c_1/c_2.$$

Proof of Corollary 2

First, note that $\min_{x \in (0,1]} \frac{h(1,0) - h(1-x,0)}{h(x,1)} = \frac{3}{2}$, for $h(x,y) = \dfrac{x^s}{x^s + y^s + 1}$ and $s \geq 3/2$. Furthermore, note that from Proposition 5 the game has only symmetric equilibria when $s \leq 1$.

Proof of Proposition 9

The KKT conditions for the firms' optimization problems imply the following: when firms employ targeting technologies they only allocate advertising funds to agents such that $\hat{c}_i > \xi$, where $\hat{c}_i = \frac{1}{n}(1-\alpha)c_i$. Furthermore, we obtain that at the unique symmetric equilibrium firms allocate to agent i with $\hat{c}_i > \xi$ budget x_i that is the unique positive solution to the equation below:

$$\frac{x_i+1}{(2x_i+1)^2} = \frac{\xi}{\hat{c}_i}. \tag{17}$$

On the other hand, when firms engage in mass advertising, their marketing budget is positive if and only if $\sum_{i=1}^n \hat{c}_i > n\xi$. If this is the case, firms allocate budget x to each of the agents, where x is the positive solution of the following equation:

$$\frac{x+1}{(2x+1)^2} = n\frac{\xi}{\sum_{i=1}^n \hat{c}_i}. \tag{18}$$

Expanding (17) and (18), we conclude that firms are always better off when they employ targeting technologies. In particular, (17) implies that the payoff of a firm under targeting is given by

$$\Pi_{targeting} = \sum_{i=1}^n \left[\hat{c}_i \frac{x_i}{2x_i+1} - \xi x_i\right] = \sum_{i=1}^n \xi \frac{x_i^2}{x_i+1},$$

where x_i is the solution of equation (17). Similarly, when firms engage in mass advertising (no targeting) we have, $\Pi_{mass} = n\xi x^2/(x+1)$. Note that from (17) and (18) we obtain that:

$$\xi\frac{x_i^2}{x_i+1} = \frac{\hat{c}_i}{4} - \xi\frac{4x_i+1}{4x_i+4} \quad \text{and} \quad n\xi\frac{x^2}{x+1} = \sum_{i=1}^n \frac{\hat{c}_i}{4} - n\xi\frac{4x+1}{4x+4}.$$

We have two cases to consider. First, suppose that $nx < \sum_{i=1}^n x_i$, i.e., a firm's marketing budget is higher when it employs a targeting technology. Then, the claim follows by noting that function $x^2/(x+1)$ is convex and increasing. On the other hand, if $nx \geq \sum_{i=1}^n x_i$ the claim follows from the concavity of function $(4x+1)/(x+1)$.

Proof of Proposition 10

Consider the agents' centralities corresponding to networks $\mathcal{G}$ and $\mathcal{G}'$ and let Π, Π' denote the corresponding equilibrium payoffs for a firm. Then, according to the proof of Proposition 9 we have:

$$\Pi = \sum_{i=1}^n \xi\frac{x_i^2}{x_i+1} \quad \text{and} \quad \Pi' = \sum_{i=1}^n \xi\frac{x_i'^2}{x_i'+1}, \tag{19}$$

where

$$x_i = \frac{1}{8}\left[\left(\frac{\hat{c}_i}{\xi}-4\right) + \sqrt{\left(\frac{\hat{c}_i}{\xi}-4\right)^2 + 16\left(\frac{\hat{c}_i}{\xi}-1\right)}\right] \text{ and } x_i' = \frac{1}{8}\left[\left(\frac{\hat{c}_i'}{\xi}-4\right) + \sqrt{\left(\frac{\hat{c}_i'}{\xi}-4\right)^2 + 16\left(\frac{\hat{c}_i'}{\xi}-1\right)}\right]. \tag{20}$$

From (20), we obtain that $\xi x_i^2/(x_i+1)$ is convex in $\hat{c}_i$ for all i. This in combination with the assumptions of the Proposition imply that $\Pi \geq \Pi'$.

Proof of Theorem 2

A firm that does not face any competition allocates budget x_i^* to agent i with $\hat{c}_i > \xi$ such that

$$\hat{c}_i \frac{1}{(x_i^*+1)^2} = \xi \quad \text{or} \quad x_i^* = \sqrt{\frac{\hat{c}_i}{\xi}} - 1,$$

and therefore $\mathcal{B}^* = \sum_{i=1}^k \sqrt{\frac{\hat{c}_i}{\xi}} - k$. Similarly, we obtain that in the presence of competition each firm spends B^{eq} at equilibrium such that:

$$2 \cdot \mathcal{B}^{eq} = \sum_{i=1}^k \sqrt{\frac{\hat{c}_i}{\xi}} \sqrt{b_i^{eq}(A)+1} - k,$$

where x_i^{eq} is the positive solution of equation $(2x_i^{eq}+1)^2 = \frac{\hat{c}_i}{\xi}(x_i^{eq}+1)$. Firms spend more on aggregate when they compete, i.e.,

$$2 \cdot \mathcal{B}^{eq} - B^* = \sum_{i=1}^k \sqrt{\frac{\hat{c}_i}{\xi}} \left(\sqrt{x_i^{eq}+1} - 1 \right).$$

Furthermore, we obtain that

$$\Pi_{monopoly} = \sum_{i=1}^k \hat{c}_i \frac{x_i^*}{x_i^*+1} - \xi x_i^* = \sum_{i=1}^k \left(\hat{c}_i + \xi - 2\sqrt{\hat{c}_i \xi} \right),$$

and that the aggregate payoff of firms at equilibrium is given by

$$2 \cdot \Pi_{eq} = \sum_{i=1}^k \hat{c}_i \frac{2x_i^{eq}}{2x_i^{eq}+1} - 2\xi x_i^{eq} = \sum_{i=1}^k \left(\hat{c}_i + \xi - \sqrt{\hat{c}_i \xi} \left(\frac{1}{\sqrt{x_i^{eq}+1}} + \sqrt{x_i^{eq}+1} \right) \right).$$

Finally,

$$\Pi_{monopoloy} - 2 \cdot \Pi_{eq} = \sum_{i=1}^k \sqrt{\hat{c}_i \xi} \left(\left(\frac{1}{\sqrt{x_i^{eq}+1}} + \sqrt{x_i^{eq}+1} \right) - 2 \right),$$

which is always non-negative for $x_i^{eq} \geq 0$.

Proof of Corollary 3

The proof of the corollary follows directly from the fact that functions

$$f(\hat{c}_i) = \sqrt{\hat{c}_i \xi} \left(\left(\frac{1}{\sqrt{x_i^{eq}+1}} + \sqrt{x_i^{eq}+1} \right) - 2 \right),$$

and

$$f(\hat{c}_i) = \sqrt{\frac{\hat{c}_i}{\xi}} \left(\sqrt{x_i^{eq}+1}) - 1 \right),$$

are convex in $\hat{c}_i$ when x_i^{eq} is given by expression (20).

Endnotes

1. Although many of our results extend in a straightforward way to any finite number of competitors, i.e., Proposition 1 and Theorem 1, we chose to focus on the case of two competitors.

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