Computational Methods for Oblivious Equilibrium*

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Abstract

Oblivious equilibrium is a new solution concept for approximating Markov perfect equilibrium in dynamic models of imperfect competition among heterogeneous firms. In this paper, we present algorithms for computing oblivious equilibrium and for bounding approximation error. We report results from computational case studies that serve to assess both efficiency of the algorithms and accuracy of oblivious equilibrium as an approximation to Markov perfect equilibrium. We also extend the definition of oblivious equilibrium, originally proposed for models with only firm-specific idiosyncratic random shocks, and our algorithms to accommodate models with industry-wide aggregate shocks. Our results suggest that, by using oblivious equilibrium to approximate Markov perfect equilibrium, it is possible to greatly increase the set of dynamic models of imperfect competition that can be analyzed computationally.

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1 Introduction

Recently, Ericson and Pakes (1995) (hereafter EP) introduced a framework for modeling a dynamic industry with heterogeneous firms. The stated goal of this work was to facilitate empirical research analyzing the effects of policy and environmental changes on things like market structure and consumer welfare in different markets. Due to the importance of dynamics in determining policy outcomes, and also because the EP model has proved to be quite adaptable and broadly applicable, the model has received much attention in the literature. Indeed, recent work has applied the framework to studying problems as diverse as advertising, auctions, collusion, consumer learning, environmental policy, firm mergers, industry dynamics, limit order markets, network externalities, and R&D investment.[1]

Despite this activity, there remain some substantial hurdles in the application of EP-type models. Because EP-type models are analytically intractable, their analysis relies on solving numerically for Markov perfect equilibrium (MPE) on a computer, and this computation suffers from the curse of dimensionality. In an EP-type model, at each time, each firm has a state variable that captures its competitive advantage. Though more general state spaces can be considered, we focus on the simple case where the firm state is an integer. The value of this integer can represent, for example, a measure of product quality, the firm's current productivity level, or its capacity. The *industry state* is a vector encoding the number of firms with each possible value of the firm state variable. At each time, each firm makes a decision based on its firm state and the industry state, and its subsequent firm state is determined by the current state, the current decision, and a random shock. Even if firms are restricted to symmetric strategies, the number of relevant industry states (and thus, the computer time and memory required for computing a MPE) becomes enormous very quickly as the numbers of firms and firm states grows. For a model with just 20 firms and 40 firm states, there are quadrillions of industry states. This renders commonly used dynamic programming algorithms infeasible in many problems of practical interest (see the related literature section for a discussion on some of these methods).

As a result, computational concerns have typically limited analysis to industries with just a few firms, much less than the real world industries the analysis is directed at. Such limitations have made it difficult to construct realistic empirical models, and application of the EP framework to empirical problems (the original intent) has been rare (see Gowrisankaran and Town (1997), Benkard (2004), Jenkins, Liu, Matzkin, and McFadden (2004), Ryan (2005), Collard-Wexler (2006)).

[1]See Berry and Pakes (1993), Gowrisankaran (1999), Fershtman and Pakes (2000), Judd, Schmedders, and Yeltekin (2002), Langohr (2003), Song (2003), Besanko and Doraszelski (2004), de Roos (2004), Besanko, Doraszelski, Kryukov, and Satterthwaite (2005), Fershtman and Pakes (2005), Goettler, Parlour, and Rajan (2005), Doraszelski and Markovich (2007), Markovich (2008), Noel (2008), and Schivardi and Schneider (2008), as well as Doraszelski and Pakes (2007) for an excellent survey.

In a recent paper (Weintraub, Benkard, and Van Roy (2008b)) we introduced a new notion of equilibrium for EP-type models called *oblivious equilibrium* (OE) that has the attractive feature that it is not subject to the curse of dimensionality. In an OE, each firm makes decisions based only on its own firm state and the long run average industry state that will prevail in equilibrium, while ignoring the current industry state. Because firms' decisions are not a function of the industry state, computing an OE only requires solving single dimensional dynamic programming problems. The main result of Weintraub, Benkard, and Van Roy (2008b) is an asymptotic theorem that establishes conditions under which OE well-approximates MPE asymptotically as the market size grows. Intuitively, in a large market the random evolution of individual firms will average out, such that variations in the normalized industry state are small in equilibrium. Given this, firms can make near-optimal decisions based on the average equilibrium industry state rather than the current industry state.

However, while the results of Weintraub, Benkard, and Van Roy (2008b) motivate consideration of OE as a solution concept, they are incomplete. First, to accommodate practice, we need methods for computing OE. In this paper we develop an algorithm for computing OE, and demonstrate its efficiency through computational experiments. A nice feature of the algorithm is that, unlike existing methods (see Section 2 for references), there is no need to place a-priori restrictions on the number of firms in the industry or the number of allowable states per firm. These are determined by the algorithm as part of the equilibrium solution. We find that the algorithm is typically able to compute OE in less than a minute even for industries with thousands of firms, a task that is far beyond what is computationally feasible for MPE computation with common dynamic programming algorithms.

Second, while the asymptotic theorem in our previous paper provides conditions for which OE approximates MPE well as the market size increases, it does not guarantee a good approximation in empirical applications whose market size is observed and fixed. Thus, again to accomodate practice, once an OE is obtained for a particular problem instance, we need methods to verify its accuracy as an approximation to MPE. To address this issue, in this paper we derive bounds on the approximation error that can be computed for each problem instance. Specifically, we measure error in terms of the expected incremental value that an individual firm in the industry can capture by unilaterally deviating from the OE strategy to an optimal Markov best response. The bounds can be computed using an efficient simulation-based algorithm that requires knowledge only of the OE strategies. Our algorithm for bounding approximation error is novel and represents a significant contribution. Not only is it important for the practical use of our model and solution concepts, but we believe the ideas should generalize to a broader class of games and large scale stochastic control problems.

The bounds on the approximation error give us the ability to evaluate whether OE provides close approximations to MPE in problem instances of practical interest. The third contribution of the paper is to show through a computational study that OE does indeed offer useful approximations for many relevant industries. Specifically, we find that the approximation is often good for industries involving hundreds of firms, and in some cases even tens of firms. These results support the conclusion that OE can be useful in empirical applications.

Finally, it is natural to think about extending the basic notion of OE in many directions, and we provide one important extension here. In the model of Weintraub, Benkard, and Van Roy (2008b), all the random shocks were assumed to be idiosyncratic across firms. However, in many problems of practical interest, it is important to incorporate shocks that are common to all firms in a market. These "aggregate" shocks are important, for example, when analyzing the dynamic effects of industry-wide business cycles (see Dunne, Roberts, and Samuelson (1988) and Davis, Haltiwanger, and Schuh (1998)). As a fourth contribution, we extend the model and also the computational algorithms and error bounds discussed earlier to accomodate aggregate shocks. We show through a computational study that OE can also offer useful approximations for many relevant models that incorporate aggregate shocks.

Our results suggest that OE opens the door to a much broader range of applications for EP-type models. Indeed, our algorithms have already been applied in an empirical study of R&D investment in the Korean electric motor industry (Xu 2006) and in a study of the impact of advertising regulation in the cigarette industry (Qi 2008). We have also done a computational study to determine conditions under which an industry becomes fragmented or remains concentrated as the market size grows (Weintraub, Benkard, and Van Roy 2005a). These studies would not have been possible using exact computation of MPE.

It is worth mentioning that, in practice, problem instances will arise in which the bounds suggest that OE may not offer an accurate approximation. Our hope is that in such cases it will be possible to extend the basic notion of OE further in order to improve the approximation at the expense of computational cost. It is encouraging that already OE has proven useful in many cases, and this motivates further work to design extensions that address additional cases. Some work along these lines is presented in Weintraub, Benkard, and Van Roy (2007) and further discussed in the conclusions.

Finally, we note that, while our emphasis is on the use of OE as an approximation to MPE, in many cases OE may also provide an appealing behavioral model on its own. If observing the industry state and designing strategies that keep track of it are costly and do not lead to significant increases in profit, firms may be better off using oblivious strategies.

The paper is organized as follows. In Section 2 we discuss related literature. In Section 3 we describe the dynamic industry model. In Section 4 we introduce the concept of oblivious strategies and oblivious equilibrium. Note that Sections 3 and 4 primarily restate assumptions and definitions from our previous work, and are included here for completeness. In Section 5 we provide a new method for computing OE. In Section 6 we derive several novel error bounds. In Section 7 we report the results of the computational experiments. In Section 8 we extend OE to treat a model with aggregate shocks. Section 9 presents conclusions and a discussion of future research directions.

2 Related Literature

We discuss the relation between our approach and some past work (see also Weintraub, Benkard, and Van Roy (2008b)).

In the past literature on EP models, MPE are usually computed using iterative dynamic programming algorithms (e.g., Pakes and McGuire (1994)). However, as discussed above, computational requirements grow with the number of industry states, making dynamic programming infeasible in many problems of practical interest. With this motivation, Judd (1998), Pakes and McGuire (2001), and Doraszelski and Judd (2006) have proposed methods that accelerate MPE computation, that are capable of addressing models with several additional firms. In this paper, we take a different tack and consider algorithms that can efficiently deal with any number of firms but aim to compute an approximation rather than an exact MPE and to bound its error. Our view is that for most models of practical interest, exact computation of MPE is unlikely to ever become feasible, and given that, approximations may offer the best available guidance for policy and strategy decisions.

OE-type approximations are based on state aggregation. Each firm predicts expected discounted profits based on partial information about the current state and uses a piecewise constant approximation to the value function. The broader approximate dynamic programming literature makes use of other families of functions such as linear combinations of arbitrary basis functions and nonlinearly parameterized approximators. An alternative approach to the one offered in this paper is to make use of such approximations and various approximate dynamic programming algorithms to address EP-type models (see Farias, Saure, and Weintraub (2008) and de Farias and Van Roy (2003)).

In a theory literature related to our work, several papers have studied dynamic rational expectations equilibrium models of competition, but without explicitly modeling agents' heterogeneity. For example, Deaton and Laroque (1996) and Routledge, Seppi, and Spatt (2000) use this type of model to study the impact of speculative storage in commodity price dynamics. Since heterogeneity among agents is not explicitly modeled, the rational expectations equilibrium is described by a low-dimensional dynamic programming problem, that does not suffer from the curse of dimensionality.

Most closely related to our work is Hopenhayn (1992) who develops a competitive equilibrium model of industry dynamics with heterogeneous firms. In every period firms receive productivity shocks, make entry and exit decisions, and compete in a perfectly competitive product market.[2] The industry holds a continuum of firms, each of which garners an infinitesimal fraction of the market, and productivity shocks generate heterogeneity among firms. The model is tractable because the industry state is constant over time, implicitly assuming a law of large numbers holds. This assumption is based on the same intuition that motivates our consideration of OE. However, there are some notable differences between our approach and Hopenhayn (1992). Our goal is to apply our model directly to data, matching such industry statistics as the number of firms, the market shares of leading firms, the level of markups, and the correlation between investment and firm size. Thus, we are forced to consider models that more closely reflect real world industries that have finite numbers of firms, with strictly positive market shares.

For asymptotically large markets we have shown that OE in EP-type models coincide with equilibria of Hopenhayn-type models (see Weintraub, Benkard, and Van Roy (2008a)), so in that sense the two concepts are similar. However, since OE is an equilibrium concept that is applied directly to a finite industry, while Hopenhayn-type models consider an infinite number of firms, the former offers superior approximations than the latter for finite markets. Our computational results show that OE can provide accurate approximations even in industries with tens of firms. In these cases, a model with a continuum of firms is likely to provide less accurate approximations. Also, note that our algorithms to compute OE, the error bounds, and the extension to aggregate shocks are all novel in the context of analyzing industry dynamics.

Finally, the notion that strategies can remain effective by considering the aggregate behavior of the competitors only, is related to the literature in aggregative games (see Dindos and Mezzetti (2006) and Novshek (1985)). These are static games for which equilibrium strategies can be represented as simple functions of the aggregate competitors' behavior, in particular, the sum of everybody else's actions. This differs from our context, where equilibrium strategies are in reality very complex, but can sometimes be well approximated using simple functions.

[2] Luttmer (2007) and Melitz (2003) extended Hopenhayn's model to a setting with monopolistic competition in the product market.

3 A Dynamic Model of Imperfect Competition

In this section we formulate a model of an industry in which firms compete in a single-good market. The model is identical to the one in Weintraub, Benkard, and Van Roy (2008b), which in turn, is close in spirit to Ericson and Pakes (1995). We restate it here for completeness. The model is general enough to encompass numerous applied problems in economics (see above for examples). The basic model includes only idiosyncratic shocks. In Section 8 we extend the model to incorporate aggregate shocks that are common to all firms.

3.1 Model and Notation

The industry evolves over discrete time periods and an infinite horizon. We index time periods with non-negative integers $t \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \geq 0\}$. We adopt a convention of indexing by t variables that are $\mathcal{F}_t$-measurable.

Each firm that enters the industry is assigned a unique positive integer-valued index. The set of indices of incumbent firms at time t is denoted by S_t. Firm heterogeneity is reflected through firm states. To fix an interpretation, we will refer to a firm's state as its quality level. However, firm states might more generally reflect productivity, capacity, the size of its consumer network, or any other aspect of the firm that affects its profits. At time t, the quality level of firm $i \in S_t$ is denoted by $x_{it} \in \mathbb{N}$.

We define the *industry state* s_t to be a vector over quality levels that specifies, for each quality level $x \in \mathbb{N}$, the number of incumbent firms at quality level x in period t. We define the state space $\overline{\mathcal{S}} = \left\{ s \in \mathbb{N}^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. Though in principle there are a countable number of industry states, we will also consider an extended state space $\mathcal{S} = \left\{ s \in \Re_+^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. For each $i \in S_t$, we define $s_{-i,t} \in \mathcal{S}$ to be the state of the *competitors* of firm i; that is, $s_{-i,t}(x) = s_t(x) - 1$ if $x_{it} = x$, and $s_{-i,t}(x) = s_t(x)$, otherwise.

In each period, each incumbent firm earns profits on a spot market. A firm's single period expected profit $\pi(x_{it}, s_{-i,t})$ depends on its quality level $x_{it} \in \mathbb{N}$ and its competitors' state $s_{-i,t} \in \mathcal{S}$.

The model also allows for entry and exit. In each period, each incumbent firm $i \in S_t$ observes a positive real-valued sell-off value ϕ_{it} that is private information to the firm. If the sell-off value exceeds the value of continuing in the industry then the firm may choose to exit, in which case it earns the sell-off value and then ceases operations permanently.

If the firm instead decides to remain in the industry, then it can invest to improve its quality level. If a firm invests $\iota_{it} \in \Re_+$, then the firm's state at time $t+1$ is given by,

$$x_{i,t+1} = \max\left(0, x_{it} + h(\iota_{it}, \zeta_{i,t+1})\right),$$

where the function h captures the impact of investment on quality and $\zeta_{i,t+1}$ reflects uncertainty in the outcome of investment. Uncertainty may arise, for example, due to the risk associated with a research and development endeavor or a marketing campaign. Note that this specification is very general as h may take on either positive or negative values (e.g., allowing for positive depreciation). We denote the unit cost of investment by d.

In each period new firms can enter the industry by paying a setup cost κ. Entrants do not earn profits in the period that they enter. They appear in the following period at state $x^e \in \mathbb{N}$ and can earn profits thereafter.[3]

Each firm aims to maximize expected net present value. The interest rate is assumed to be positive and constant over time, resulting in a constant discount factor of $\beta \in (0, 1)$ per time period.

In each period, events occur in the following order:

1. Each incumbent firms observes its sell-off value and then makes exit and investment decisions.
2. The number of entering firms is determined and each entrant pays an entry cost of κ.
3. Incumbent firms compete in the spot market and receive profits.
4. Exiting firms exit and receive their sell-off values.
5. Investment outcomes are determined, new entrants enter, and the industry takes on a new state s_{t+1}.

3.2 Model Primitives

Our model above allows for a wide variety of applied problems. To study any particular problem it is necessary to further specify the primitives of the model, including the profit function π, the distribution of the sell-off value ϕ_{it}, the investment impact function h, the distribution of the investment uncertainty ζ_{it}, the unit investment cost d, the entry cost κ, and the discount factor β.

Note that in most applications the profit function would not be specified directly, but would instead result from a deeper set of primitives that specify a demand function, a cost function, and a static equilibrium concept.

[3]Note that it would not change any of our results to assume that the entry state was a random variable.

3.3 Assumptions

We make several assumptions about the model primitives, beginning with the profit function.

Assumption 3.1. *For all $s \in \mathcal{S}$, $\pi(x, s)$ is increasing in x. Further,* $\sup_{x,s} \pi(x, s) < \infty$.

The assumption is natural. It ensures that increases in quality lead to increases in profit and that profits are bounded.

We also make assumptions about investment and the distributions of the private shocks:

Assumption 3.2.

1. *The random variables $\{\phi_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and have finite expectations and well-defined density functions with support $\Re_+$.*
2. *The random variables $\{\zeta_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and independent of $\{\phi_{it} | t \geq 0, i \geq 1\}$.*
3. *For all ζ, $h(\iota, \zeta)$ is nondecreasing in ι.*
4. *For all $\iota > 0$, $\mathcal{P}[h(\iota, \zeta_{i,t+1}) > 0] > 0$.*
5. *There exists a positive constant $\overline{h} \in \mathbb{N}$ such that $|h(\iota, \zeta)| \leq \overline{h}$, for all (ι, ζ). There exists a positive constant $\overline{\iota}$ such that $\iota_{it} \leq \overline{\iota}, \ \forall i, \forall t$.*
6. *For all $k \in \{-\overline{h}, \ldots, \overline{h}\}$, $\mathcal{P}[h(\iota, \zeta_{i,t+1}) = k]$ is continuous in ι.*
7. *The transitions generated by $h(\iota, \zeta)$ are unique investment choice admissible.*

Again the assumptions are natural and fairly weak. Assumptions 3.2.1 and 3.2.2 imply that investment and exit outcomes are idiosyncratic conditional on the state. Assumption 3.2.3 and 3.2.4 imply that investment is productive. Note that positive depreciation is neither required nor ruled out. Assumption 3.2.5 places a finite bound on how much progress can be made or lost in a single period through investment. Assumption 3.2.6 ensures that the impact of investment on transition probabilities is continuous. Assumption 3.2.7 is an assumption introduced by Doraszelski and Satterthwaite (2007) that ensures a unique solution to the firms' investment decision problem. In particular, it ensures the firms' investment decision problem is strictly concave or that the unique maximizer is a corner solution. The assumption is used to guarantee existence of an equilibrium in pure strategies, and is satisfied by many of the commonly used specifications in the literature.

We assume that there are an asymptotically large number of potential entrants who play a symmetric mixed entry strategy. This results in a Poisson-distributed number of entrants (see Weintraub, Benkard, and Van Roy (2008b) for a derivation of this result). Our associated modeling assumptions are as follows:

Assumption 3.3.

1. *The number of firms entering during period t is a Poisson random variable that is conditionally independent of $\{\phi_{it}, \zeta_{it} | t \geq 0, i \geq 1\}$, conditioned on s_t.*
2. *$\kappa > \beta \cdot \bar{\phi}$, where $\bar{\phi}$ is the expected net present value of entering the market, investing zero and earning zero profits each period, and then exiting at an optimal stopping time.*

We denote the expected number of firms entering at industry state s_t, by $\lambda(s_t)$. This state-dependent entry rate will be endogenously determined, and our solution concept will require that it satisfies a zero expected discounted profits condition. Modeling the number of entrants as a Poisson random variable has the advantage that it leads to simpler dynamics. However, our results can accommodate other entry processes as well. Assumption 3.3.2 ensures that the sell-off value by itself is not sufficient reason to enter the industry.

3.4 Equilibrium

As a model of industry behavior we focus on pure strategy Markov perfect equilibrium (MPE), in the sense of Maskin and Tirole (1988). We further assume that equilibrium is symmetric, such that all firms use a common stationary investment/exit strategy. In particular, there is a function ι such that at each time t, each incumbent firm $i \in S_t$ invests an amount $\iota_{it} = \iota(x_{it}, s_{-i,t})$. Similarly, each firm follows an exit strategy that takes the form of a cutoff rule: there is a real-valued function ρ such that an incumbent firm $i \in S_t$ exits at time t if and only if $\phi_{it} \geq \rho(x_{it}, s_{-i,t})$. In Weintraub, Benkard, and Van Roy (2008b) we show that there always exists an optimal exit strategy of this form even among very general classes of exit strategies. Let $\mathcal{M}$ denote the set of exit/investment strategies such that an element $\mu \in \mathcal{M}$ is a pair of functions $\mu = (\iota, \rho)$, where $\iota : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an investment strategy and $\rho : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an exit strategy. Similarly, we denote the set of entry rate functions by Λ, where an element of Λ is a function $\lambda : \mathcal{S} \rightarrow \Re_+$.

We define the value function $V(x, s|\mu', \mu, \lambda)$ to be the expected net present value for a firm at state x when its competitors' state is s, given that its competitors each follows a common strategy $\mu \in \mathcal{M}$, the entry rate function is $\lambda \in \Lambda$, and the firm itself follows strategy $\mu' \in \mathcal{M}$. In particular,

$$V(x, s|\mu', \mu, \lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x, s_{-i,t} = s\right],$$

where i is taken to be the index of a firm at quality level x at time t, τ_i is a random variable representing the time at which firm i exits the industry, and the subscripts of the expectation indicate the strategy followed by firm i, the strategy followed by its competitors, and the entry rate function. In an abuse of notation, we will use the shorthand, $V(x, s|\mu, \lambda) \equiv V(x, s|\mu, \mu, \lambda)$, to refer to the expected discounted value of profits when firm i follows the same strategy μ as its competitors.

An equilibrium to our model comprises of an investment/exit strategy $\mu = (\iota, \rho) \in \mathcal{M}$, and an entry rate function $\lambda \in \Lambda$ that satisfy the following conditions:

1. Incumbent firm strategies represent a MPE:

$$\sup_{\mu' \in \mathcal{M}} V(x, s|\mu', \mu, \lambda) = V(x, s|\mu, \lambda) \qquad \forall x \in \mathbb{N},\ \forall s \in \overline{\mathcal{S}}. \tag{3.1}$$

2. At each state, either entrants have zero expected discounted profits or the entry rate is zero (or both):

$$\begin{aligned} \textstyle\sum_{s \in \overline{\mathcal{S}}} \lambda(s) \left(\beta E_{\mu,\lambda} \left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s \right] - \kappa \right) = 0 & \\ \beta E_{\mu,\lambda} \left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s \right] - \kappa \leq 0 & \qquad \forall s \in \overline{\mathcal{S}} \\ \lambda(s) \geq 0 & \qquad \forall s \in \overline{\mathcal{S}}. \end{aligned}$$

In Weintraub, Benkard, and Van Roy (2008b), we show that the supremum in part 1 of the definition above can always be attained simultaneously for all x and s by a common strategy μ'.

Doraszelski and Satterthwaite (2007) establish existence of an equilibrium in pure strategies for a closely related model. We do not provide an existence proof here because it is long and cumbersome and would replicate this previous work. With respect to uniqueness, in general we presume that our model may have multiple equilibria.[4]

Dynamic programming algorithms can be used to optimize firm strategies and equilibria to our model can be computed via their iterative application. Stationary points of such iterations are MPE, but there is no guarantee of convergence. Nevertheless, such methods have proven to be effective in practical contexts and have consequently seen broad use. However, they require compute time and memory that grow proportionately with the number of relevant industry states, which is intractable in many applications. This difficulty motivates our alternative approach.

4 Oblivious Equilibrium

We now formally define the concept of oblivious equilibrium as in Weintraub, Benkard, and Van Roy (2008b), restated here for completeness. The motivation for considering OE is that, when there are a large number of firms, simultaneous changes in individual firm quality levels can average out such that the industry state remains roughly constant over time. In this setting, each firm can potentially make near-optimal

[4]Doraszelski and Satterthwaite (2007) also provide an example of multiple equilibria in their closely related model.

decisions based only on its own quality level and the long run average industry state. With this motivation, we consider restricting firm strategies so that each firm's decisions depend only on the firm's quality level. We call such restricted strategies *oblivious* since they involve decisions made without full knowledge of the circumstances — in particular, the state of the industry.

Let $\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\Lambda} \subset \Lambda$ denote the set of oblivious strategies and the set of oblivious entry rate functions. Since each strategy $\mu = (\iota, \rho) \in \tilde{\mathcal{M}}$ generates decisions $\iota(x,s)$ and $\rho(x,s)$ that do not depend on s, with some abuse of notation, we will often drop the second argument and write $\iota(x)$ and $\rho(x)$. Similarly, for an entry rate function $\lambda \in \tilde{\Lambda}$, we will denote by λ the real-valued entry rate that persists for all industry states.

Note that if all firms use a common strategy $\mu \in \tilde{\mathcal{M}}$, the quality level of each evolves as an independent transient Markov chain. Let the k-period transition sub-probabilities of this transient Markov chain be denoted by $P_\mu^k(x,y)$. Then, the expected time that a firm spends at a quality level x is given by $\sum_{k=0}^{\infty} P_\mu^k(x^e, x)$, and the expected lifespan of a firm is $\sum_{k=0}^{\infty} \sum_{x \in \mathbb{N}} P_\mu^k(x^e, x)$. Denote the expected number of firms at quality level x at time t by $\tilde{s}_t(x) = E[s_t(x)]$. The following result offers an expression for the long-run expected industry state when dynamics are governed by oblivious strategies and entry rate functions.

Lemma 4.1. *Let Assumption 3.2 hold. If firms make decisions according to an oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and enter according to an oblivious entry rate function $\lambda \in \tilde{\Lambda}$, and the expected time that a firm spends in the industry is finite, then*

$$\lim_{t \to \infty} \tilde{s}_t(x) = \lambda \sum_{k=0}^{\infty} P_\mu^k(x^e, x), \tag{4.1}$$

for all $x \in \mathbb{N}$.

We omit the proof, that is straightforward. To abbreviate notation, we let $\tilde{s}_{\mu,\lambda}(x) = \lim_{t\to\infty} \tilde{s}_t(x)$ for $\mu \in \tilde{\mathcal{M}}$, $\lambda \in \tilde{\Lambda}$, and $x \in \mathbb{N}$. For an oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and an oblivious entry rate function $\lambda \in \tilde{\Lambda}$ we define an *oblivious value function*

$$\tilde{V}(x|\mu', \mu, \lambda) = E_{\mu'} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \left(\pi(x_{ik}, \tilde{s}_{\mu,\lambda}) - d\iota_{ik} \right) + \beta^{\tau_i - t} \phi_{i,\tau_i} \middle| x_{it} = x \right].$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x and follows oblivious strategy μ', under the assumption that its competitors' state will be $\tilde{s}_{\mu,\lambda}$ for all time. Note that only the firm's own strategy μ' influences the firm's state trajectory because neither the profit function nor the strategy μ' depends on the industry state. Hence, the subscript in the expectation only reflects this dependence. Importantly, however, the oblivious value function remains a function of the competitors' strategy μ and the entry rate λ through the expected industry state $\tilde{s}_{\mu,\lambda}$. Again, we abuse notation by using $\tilde{V}(x|\mu,\lambda) \equiv \tilde{V}(x|\mu,\mu,\lambda)$ to refer to the oblivious value function when firm i follows the same strategy μ as its competitors.

We now define a new solution concept: an *oblivious equilibrium* consists of a strategy $\mu \in \tilde{\mathcal{M}}$ and an entry rate function $\lambda \in \tilde{\Lambda}$ that satisfy the following conditions:

1. Firm strategies optimize an oblivious value function:

$$\sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}(x|\mu',\mu,\lambda) = \tilde{V}(x|\mu,\lambda), \qquad \forall x \in \mathbb{N}. \tag{4.2}$$

2. Either the oblivious expected value of entry is zero or the entry rate is zero (or both):

$$\lambda\left(\beta\tilde{V}(x^e|\mu,\lambda) - \kappa\right) = 0$$
$$\beta\tilde{V}(x^e|\mu,\lambda) - \kappa \leq 0$$
$$\lambda \geq 0.$$

It is straightforward to show that OE exists under mild technical conditions. Furthermore, if the entry cost is not prohibitively high relative to single period profits, then an OE with a positive entry rate exists. We omit the proof of this for brevity. With respect to uniqueness, we have been unable to find multiple OE in any of the applied problems we have considered, but similarly with the case of MPE, we have no reason to believe that in general there is a unique OE.[5]

Finally, in Weintraub, Benkard, and Van Roy (2008b) we show that when strategies and entry rate functions are oblivious, the Markov process $\{s_t : t \geq 0\}$ admits a unique invariant distribution. Moreover, we show that, when firms play OE strategies, the invariant distribution of the industry state is such that the number of firms in each state $x \in \mathbb{N}$ is given by a Poisson random variable with mean $\tilde{s}_{\mu,\lambda}(x)$, independent across states $x \in \mathbb{N}$.

[5] However, since oblivious strategies rule out strategies that are dependent on competitors' states, there are likely to be fewer OE than there are MPE.

5 Algorithm

In this section we propose an algorithm to solve for OE.

Algorithm 1 (below) is designed to compute an OE with a positive entry rate. It starts with two extreme entry rates: $\underline{\lambda} = 0$ and $\overline{\lambda} = \frac{1}{\kappa}\left(\frac{\sup_{x,s}\pi(x,s)}{1-\beta} + \bar{\phi}\right)$. Under mild assumptions, any oblivious equilibrium entry rate must lie between these two extremes. The algorithm searches over entry rates between these two extremes for one that leads to an OE. For each candidate entry rate λ, an inner loop (steps 6-10) computes an OE firm strategy for that fixed entry rate. Strategies are updated "smoothly" (step 9).[6] If the termination conditions of both the inner and outer loops are satisfied with $\epsilon_1 = \epsilon_2 = 0$, we have an OE. Small values of ϵ_1 and ϵ_2 allow for small errors associated with limitations of numerical precision.

Algorithm 1 Oblivious Equilibrium Solver

1: $\underline{\lambda} := 0$; $\overline{\lambda} := \frac{1}{\kappa}\left(\frac{\sup_{x,s}\pi(x,s)}{1-\beta} + \bar{\phi}\right)$
2: $\mu(x) := 0$ for all x
3: $n := 0$
4: **repeat**
5:     $\lambda := (\underline{\lambda} + \overline{\lambda})/2$
6:     **repeat**
7:         Choose $\mu^* \in \tilde{\mathcal{M}}$ to maximize $\tilde{V}(x|\mu^*, \mu, \lambda)$ simultaneously for all $x \in \mathbb{N}$
8:         $\Delta := \|\mu^* - \mu\|_\infty$; $n := n+1$
9:         $\mu := \mu + (\mu^* - \mu)/(n^\gamma + N)$
10:     **until** $\Delta \leq \epsilon_1$
11:     **if** $\beta\tilde{V}(x^e|\mu, \lambda) - \kappa \geq 0$ **then**
12:         $\underline{\lambda} := \lambda$
13:     **else**
14:         $\overline{\lambda} := \lambda$
15:     **end if**
16: **until** $|\beta\tilde{V}(x^e|\mu, \lambda) - \kappa| \leq \epsilon_2$

The algorithm is easy to program and computationally efficient. In each iteration of the inner loop, the optimization problem to be solved is a one dimensional dynamic program. The state space in this dynamic program is the set of quality levels a firm can achieve. In principle, there could be an infinite number of them. However, beyond a certain quality level the optimal strategy for a firm is not to invest, so its quality cannot increase to beyond that level. In the numerical experiments we present in Section 7, the state space never had more than one hundred states per firm. The exact number of states is determined during execution of the algorithm.

While there are alternative algorithms to solve for OE, we chose the one previously described, motivated

[6] The parameters γ and N were set after some experimentation to speed up convergence to the values $2/3$ and zero, respectively.

by a few observations:

1. For $\lambda \in \tilde{\Lambda}$, let $\mu^* \in \tilde{\mathcal{M}}$ be the OE strategy associated with that entry rate. In our numerical experiments, we observed that $\tilde{V}(x^e|\mu^*, \lambda)$ is decreasing in λ. Therefore, the line search method proposed provides an efficient way to find an entry rate that satisfies the zero oblivious expected value condition and yields an OE.[7]

2. For given $\lambda \in \tilde{\Lambda}$ we use a myopic best response algorithm to find the OE strategy associated with that entry rate. While in many cases, including ours, the theoretical convergence properties of these algorithms are not well understood (Fudenberg and Levine 1998), positive practice experience supports their use. Moreover, our computational experiments showed that a smooth update of strategies (step 9) speeds up convergence.

Whether this algorithm is guaranteed to terminate in a finite number of iterations remains an open issue. However, in the numerical experiments we present in the next section, it always terminated in less than 30 seconds.[8]

6 Error Bounds

In this section we derive expressions that can be computed via simulation and that bound approximation error associated with a particular OE. While the asymptotic results in Weintraub, Benkard, and Van Roy (2008b) provide conditions under which the approximation will work well as the market size grows, the error bound can be used to evaluate the OE as an approximation of MPE for a particular set of model primitives.

To bound approximation error, we first need to define what is meant by *approximation error*. Consider an oblivious strategy and entry rate function $(\tilde{\mu}, \tilde{\lambda}) \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$. We assume that the initial industry state s_0 is sampled from the invariant distribution of $\{s_t : t \geq 0\}$. Hence, s_t is a stationary process; s_t is distributed according to its invariant distribution for all $t \geq 0$. We will quantify approximation error at each firm state $x \in \mathbb{N}$ by

$$E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t|\mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})\right].$$

[7] Note that there are potentially many alternative methods for searching over entry rates for an OE. For example, one alternative would be to start at an arbitrary entry rate and then implement small increments and decrements to the entry rate until an entry rate is found that leads to an OE.

[8] The algorithm was programmed in C++ and the experiments were executed on a UNIX shared machine with a CPU Intel 2.66GHz and 32 GB of RAM.

The expectation is over the invariant distribution of s_t. Hence, approximation error is the amount by which a firm at state $x \in \mathbb{N}$ can improve its expected net present value by unilaterally deviating from the OE strategy $\tilde{\mu}$, and instead following an optimal (non-oblivious) best response. Recall that a MPE requires that the expression in square brackets equals zero for all states (x, s). Approximation error instead considers the benefit of deviating to an optimal strategy starting from each firm state x, averaged over the invariant distribution of industry states. It would not be possible to obtain useful bounds point-wise. This is because in an OE firms may be making poor decisions in states that are far from the expected state. Offsetting this effect is the fact that these states have very low probability of occurrence, so they have a small impact on expected discounted profits. The idea is that when approximation error is small MPE strategies and entry rates at relevant states should be well approximated by oblivious ones. In Section 7 we present computational results that support this point.[9]

The next theorem provides two bounds on the approximation error. Recall that $\tilde{s}$ is the long run expected state in OE ($E[s_t]$). Let $a_x(y)$ be the expected discounted sum of an indicator of visits to state y for a firm starting at state x that uses strategy $\tilde{\mu}$. Let $[x]^+ = \max(x, 0)$ and $\underline{x}(k,t) = [x - (k-t)\overline{h}]^+$. Finally, let $|\Delta|(s) = \sup_{y \in \mathbb{N}} |\pi(y,s) - \pi(y,\tilde{s})|$ and $\Delta_A(s) = \sup_{y \in A} (\pi(y,s) - \pi(y,\tilde{s}))$.

Theorem 6.1. *Let Assumptions 3.1, 3.2, and 3.3 hold. Then, for any OE $(\tilde{\mu}, \tilde{\lambda})$ and firm state $x \in \mathbb{N}$,*

$$E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t|\mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})\right] \leq \frac{2}{1-\beta} E[|\Delta|(s_t)], \tag{6.1}$$

and

$$\begin{aligned} E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t|\mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})\right] &\leq \sum_{k=t}^{\infty} \beta^{k-t} E\left[\left[\Delta_{\{\underline{x}(k,t), \ldots, x+(k-t)\overline{h}\}}(s_k)\right]^+\right] \\ &+ \sum_{y \in \mathbb{N}} a_x(y) \left(\pi(y,\tilde{s}) - E\left[\pi(y,s_t)\right]\right) . \end{aligned} \tag{6.2}$$

The derivation of these bounds can be found in the Appendix. It is worth mentioning that the result can be generalized a great deal. In particular, many of the prior assumptions can be dropped; for instance, most alternative entry processes will not change the result.

The first bound is simpler so we will use it to provide an explanation of the main steps of the derivation here. First, we compare the value functions in the definition of approximation error through the OE value

[9] The AME property defined in Weintraub, Benkard, and Van Roy (2008b) for their asymptotic result required that the approximation error converged to zero as market size grows for all states $x \in \mathbb{N}$.

function. Formally,

$$
\begin{aligned}
E\left[V(x,s_t|\mu^*,\tilde{\mu},\tilde{\lambda}) - V(x,s_t|\tilde{\mu},\tilde{\lambda})\right] &= E\left[V(x,s_t|\mu^*,\tilde{\mu},\tilde{\lambda}) - \tilde{V}(x|\tilde{\mu},\tilde{\lambda})\right] \\
&+ E\left[\tilde{V}(x|\tilde{\mu},\tilde{\lambda}) - V(x,s_t|\tilde{\mu},\tilde{\lambda})\right] ,
\end{aligned} \tag{6.3}
$$

where $\mu^* \in \mathcal{M}$ is a Markovian (non-oblivious) best response to an OE $(\tilde{\mu},\tilde{\lambda})$ for a firm that is keeping track of the industry state. We now explain how we bound the first expectation in the right-hand side above. A similar argument can be used for the second one. First, we observe that because $\tilde{\mu}$ and $\tilde{\lambda}$ attain an OE, for all x,

$$
\tilde{V}(x|\tilde{\mu},\tilde{\lambda}) = \sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}(x|\mu',\tilde{\mu},\tilde{\lambda}) = \sup_{\mu' \in \mathcal{M}} \tilde{V}(x|\mu',\tilde{\mu},\tilde{\lambda}),
$$

where the last equation follows because there will always be an optimal oblivious strategy when optimizing an oblivious value function even if we consider more general strategies (a key feature of oblivious strategies). Hence,

$$
V(x,s|\mu^*,\tilde{\mu},\tilde{\lambda}) - \tilde{V}(x|\tilde{\mu},\tilde{\lambda}) \leq V(x,s|\mu^*,\tilde{\mu},\tilde{\lambda}) - \tilde{V}(x|\mu^*,\tilde{\mu},\tilde{\lambda}).
$$

Note that in the right-hand side of the above inequality both value functions are evaluated at the same set of strategies. This allows us to compare $V(x,s|\mu^*,\tilde{\mu},\tilde{\lambda})$ with $\tilde{V}(x|\tilde{\mu},\tilde{\lambda})$ by only taking into consideration the difference between single-period profits (actual versus oblivious). Because strategies are the same the terms associated to expected investments and sell-off value cancel out. Hence, using the previous inequality we obtain that the difference between value functions can be bounded by a discounted sum of expected differences between actual and oblivious single-period profits as required by the error bounds. To obtain a bound that does not depend on μ^*, we use the fact that under OE strategies firms' trajectories are independent.

By doing a more careful account on profits' differences and using the fact that a firm can change by at most $\overline{h}$ quality units per time period we obtain the second bound that is tighter. Note that the right-hand-side of the second bound depends on the initial firm state x, whereas the right-hand-side of the first bound does not.

Both bounds can be easily estimated via simulation algorithms. Computing the bounds involves computing expectations over the industry state s_t under its invariant distribution. Once the OE has been computed, the industry state has a known distribution, namely, the product form of Poisson random variables with mean $\tilde{s}$ (see Weintraub, Benkard, and Van Roy (2008b)). In particular, note that the bounds are not a function of the true MPE or even of the optimal non-oblivious best response strategy. Computing either of these strategies could require solving a high-dimensional dynamic program.

If the dynamics of the model include depreciation, that is, there is a positive probability the quality level of the firm goes down even if investment is arbitrarily large, tighter bounds can be derived. Let $\Delta(y,s) = \pi(y,s) - \pi(y,\tilde{s})$. Let $\hat{\mu}$ be a strategy such that the firm never exits the industry and invests an infinite amount at every state, and let $\{\hat{x}_t : t \geq 0\}$ be a process that describes the state evolution of a firm that uses strategy $\hat{\mu}$. We have the following result that we prove in the Appendix.

Theorem 6.2. *Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that, for all $s \in \overline{\mathcal{S}}$, the function $\Delta(y,s)^+$ is nondecreasing in y. Then, for all OE $(\tilde{\mu}, \tilde{\lambda})$, and firm state $x \in \mathbb{N}$,*

$$E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t|\mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} E_{\hat{\mu},\tilde{\mu},\tilde{\lambda}}\left[\Delta(\hat{x}_k, s_k)^+ | \hat{x}_t = x\right] + \sum_{y \in \mathbb{N}} a_x(y)\left(\pi(y,\tilde{s}) - E\left[\pi(y, s_t)\right]\right) . \tag{6.4}$$

As before, s_k is distributed according to the invariant distribution for all $k \geq 0$. The expectation over $\hat{x}_k$ can be written in closed form for the model in Section 7.1 facilitating its computation (see the Appendix for this result). If there is depreciation and $\Delta(y,s)^+$ is nondecreasing in y, as is the case in the model we introduce below, bound (6.4) is generally tighter than bound (6.2). The latter takes a maximum over achievable states in the first sum. The former takes an expectation with respect to $\hat{\mu}$ and because of depreciation, larger achievable states have smaller weights, reducing the magnitude of the bound.

Even tighter bounds can be derived for industries where there is no exit of incumbent firms and no entry of new firms. These bounds are described in the Appendix.

7 Computational Experiments

In this section we conduct some computational experiments to evaluate how OE performs in practice. We begin with the model to be analyzed. The model is similar to Pakes and McGuire (1994). However, it differs in the entry and exit processes, in the demand system, and in that we do not consider an aggregate shock.[10] The model satisfies Assumptions 3.1, 3.2, and 3.3.

7.1 The Computational Model

SINGLE-PERIOD PROFIT FUNCTION. We consider an industry with differentiated products, where each firm's state variable represents the quality of its product. There are m consumers in the market. In period t,

[10] In Section 8 we extend the model to incorporate aggregate shocks.

consumer j receives utility u_{ijt} from consuming the good produced by firm i given by:

$$u_{ijt} = \theta_1 \ln(\frac{x_{it}}{\psi} + 1) + \theta_2 \ln(Y - p_{it}) + \nu_{ijt} \ , \ i \in S_t, \ j = 1, \ldots, m,$$

where Y is the consumer's income, p_{it} is the price of the good produced by firm i, and ψ is a scaling factor. ν_{ijt} are i.i.d. random variables distributed Gumbel that represent unobserved characteristics for each consumer-good pair. There is also an outside good that provides consumers zero utility. We assume consumers buy at most one product each period and that they choose the product that maximizes utility. Under these assumptions our demand system is a classical logit model.

Let $N(x_{it}, p_{it}) = \exp(\theta_1 \ln(\frac{x_{it}}{\psi} + 1) + \theta_2 \ln(Y - p_{it}))$. Then, the expected market share of each firm is given by:

$$\sigma(x_{it}, s_{-i,t}, p_t) = \frac{N(x_{it}, p_{it})}{1 + \sum_{j \in S_t} N(x_{jt}, p_{jt})} \ , \ \forall i \in S_t \ .$$

We assume that firms set prices in the spot market. If there is a constant marginal cost c, the Nash equilibrium of the pricing game satisfies the first-order conditions,

$$Y - p_{it} + \theta_2(p_{it} - c)(\sigma(x_{it}, s_{-i,t}, p_t) - 1) = 0 \ , \ \forall i \in S_t \ . \tag{7.1}$$

There is a unique Nash equilibrium in pure strategies, denoted p_t^* (Caplin and Nalebuff (1991)). Expected profits are given by:

$$\pi_m(x_{it}, s_{-i,t}) = m\sigma(x_{it}, s_{-i,t}, p_t^*)(p_{it}^* - c) \ , \ \forall i \in S_t \ .$$

SELL-OFF PRICE. ϕ_{it} are i.i.d. exponential random variables with mean K.

TRANSITION DYNAMICS. A firm's investment is successful with probability $\frac{a\iota}{1+a\iota}$, in which case the quality of its product increases by one level. The firm's product depreciates one quality level with probability δ, independently each period. Note that our model differs from Pakes and McGuire (1994) here because the depreciation shocks in our model are idiosyncratic. Combining the investment and depreciation processes, it follows that the transition probabilities for a firm in state x that does not exit and invests ι are given by:

$$\mathcal{P}\left[x_{i,t+1} = y \middle| x_{it} = x, \iota\right] = \begin{cases} \frac{(1-\delta)a\iota}{1+a\iota} & \text{if } y = x + 1 \\ \frac{(1-\delta)+\delta a\iota}{1+a\iota} & \text{if } y = x \\ \frac{\delta}{1+a\iota} & \text{if } y = x - 1 \ . \end{cases}$$

7.2 Numerical Results: Behavior of the Bound

Our first set of results investigate the behavior of the approximation error bound under several different model specifications. A wide range of parameters for our model could reasonably represent different real world industries of interest. In practice the parameters would either be estimated using data from a particular industry or chosen to reflect an industry under study. We begin by investigating a particular set of representative parameter values. Following Pakes and McGuire (1994) we fix $a = 3$ and $\delta = 0.7$. Additionally, we fix marginal cost at $c = 0.5$, income at $Y = 1$, $\theta_2 = 0.5$, and $\psi = 1$. The discount factor is $\beta = 0.95$. The entry cost is $\kappa = 35$ and the entry state is $x^e = 10$. The average sell-off value is $K = 10$. In this case, $\beta \cdot \overline{\phi} < \kappa$, so the sell-off value by itself is not sufficient reason to enter the industry (Assumption 3.3.2). Additionally, both sell-off values and entry costs are substantially larger than marginal costs, consistent with empirical evidence.

In our computational experiments we found that the most important parameter affecting the approximation error bounds was θ_1, which determines the importance that consumers place on product quality. If θ_1 is small, the degree of vertical differentiation between products is small. This reduces the impact of changes in the industry state on profits, making the MPE strategies less sensitive to the industry state. Additionally, when θ_1 is small it turns out that the invariant distribution $\tilde{s}$ is very "light-tailed". Oblivious strategies work well in this case, and the approximation error bound is small. If θ_1 is large, we get the reverse implications and the approximation error bound is larger.

Based on these experiments, here we consider two different values of θ_1 and the investment cost d, (θ_1, d): $(0.1,\ 0.1)$ and $(0.5,\ 0.5)$. The former ("Low") is a situation where the level of vertical differentiation is low and it is inexpensive to invest to improve quality. The latter ("High") is the opposite. As a point of comparison, if a firm increases its state from $x = 10$ to $x = 20$, its single-period profits increase by 7% and 40% respectively in the two cases (holding competitors constant).

For each set of parameters, we use the approximation error bound in Theorem 6.2 to compute an upper bound on the percentage error in the value function, $\frac{E[\sup_{\mu' \in \mathcal{M}} V(x,s|\mu',\tilde{\mu},\tilde{\lambda}) - V(x,s|\tilde{\mu},\tilde{\lambda})]}{E\left[V(x,s|\tilde{\mu},\tilde{\lambda})\right]}$, where $(\tilde{\mu}, \tilde{\lambda})$ are the OE strategy and entry rate, respectively, and the expectations are taken with respect to s.[11] We estimate the expectations using simulation.[12] We compute the previously mentioned percentage approximation error

[11] While we are not able to show that $\Delta(y, s)^+$ is nondecreasing in y, we check it computationally for all sampled states s in the simulation.

[12] The expected value function is estimated with a relative precision of 1% and a confidence level of 98%. The bound is estimated with a relative precision of at most 10% and a confidence level of 98% (in cases where the bound is very small it is difficult to achieve better precision than this). Note that the percentage approximation error bound depends on the state x so for the purposes of this section we consider the percentage bound evaluated at the entry state. For the computations we took the maximum achievable state, $\tilde{x}_{max}$, to be a state such that the expected number of visits of a firm using $\tilde{\mu}$ was at most 10^{-5}. In computing the bounds, we assumed that the maximum achievable state under the best response (non-oblivious) strategy was also $\tilde{x}_{max}$.

bound for different market sizes. As the market size increases, the expected number of firms increases and the approximation error bound decreases.

In Figure 1 (see the Appendix for all tables and figures) we present the percentage approximation error bound as a function of the expected number of firms for the two levels of vertical differentiation (the two curves are obtained by varying the market size). For the low vertical differentiation case it takes around 150 firms to bring the bound down to 3%, and 250 firms to bring it to 2%. For the high case it takes around 200 firms to bring the bound to 4% and 700 firms to bring it to 2%.

When the level of vertical differentiation is high, the number of firms required to have a good approximation is large, requiring hundreds and even thousands of firms. The approximation would be better if the industry state s were always close to its mean, $\tilde{s}$. One aspect of the model that interferes with this is the Poisson entry process, that leads to a large amount of variability in the number of firms inside the industry. Recall that we chose to model the entry process this way because it simplified the dynamics. However, the expressions for the approximation error bounds remain correct for a wide range of entry models. To investigate this issue further, as an alternative, we tried using an entry process where the number of entrants each period is "almost deterministic", but still satisfies a zero profits condition.[13] This entry process implies a smaller variability in the number of firms.

Figure 2 presents the results with the new entry process. In the case of low vertical differentiation, the approximation error bound is around 3% with just 60 firms, around 2% with 125 firms, and around 1% with 500 firms. When the level of vertical differentiation is high the approximation error bound is around 3% when there are 125 firms and around 2% for 350 firms.

Going one step further in reducing the variability of the industry dynamics, we tried shutting down entry and exit altogether and considered an industry with a fixed number of firms. We used the error bound in Corollary B.2. See Figure 3 for the results.[14] For the low case the approximation error bound is less than 0.5% with just 5 firms, while for the high case it is 2% for 20 firms, and around 1% with 100 firms.

Most economic applications would involve from less than ten to several hundred firms. These results show that the approximation error bound may sometimes be small (<2%) in these cases, though this would depend on the model and parameter values for the industry under study.

[13] Note that the zero profits condition typically requires a fractional number of entrants to be satisfied exactly, so to accommodate this we instead randomized the number of entrants between the two neighboring integers. For example, if the equilibrium entry rate is 2.5, then the number of entrants is 2 or 3 with probability 0.5. Allowing for fractional numbers ensures existence of equilibrium. Note that with this entry process $\{s_t : t \geq 0\}$ also admits a unique invariant distribution.

[14] Since now there is no entry state, we report the percentage error bound evaluated at the most visited state.

7.3 Closeness to MPE Economic Indicators of Interest

Having gained some insight into what features of the model lead to low values of the approximation error bound, the question arises as to what value of the error bounds is required to obtain a good approximation of economic indicators like the ones researchers are usually interested on. To shed light on this issue we compare long-run statistics for the same industry primitives under OE and MPE strategies. A major constraint on this exercise is that it requires the ability to actually compute the MPE. With the current methods we are able to compute MPE for industries with a maximum of five to ten firms. Because we require the ability to compute equilibria for many different parameter values, to keep computation manageable we use four firms here. We therefore limit our analysis to the case of a fixed number of firms (no entry and exit), because only for that case were the approximation error bounds small under oblivious strategies (with only four firms). We use the same parameter specifications as in the previous subsection. Because of computational constraints in computing the MPE, we also impose a maximum state that a firm can reach of $x_{max} = 15$, at which point investment is assumed to have no further effect. The market size is fixed, $m = 30$.[15]

Recall that under OE strategies, the industry state is described by an ergodic Markov process (see Weintraub, Benkard, and Van Roy (2008b)). Under our assumptions, this is also true under MPE strategies (see also Ericson and Pakes (1995)). Therefore, both systems have a well defined invariant distribution that describe their long-run behavior. We compare the expected values of several economic statistics of interest with respect to the OE and the MPE invariant distributions. The quantities compared are: average investment, average producer surplus, average consumer surplus, average share of the largest firm (C1), and average share of the largest two firms (C2). Table 1 reports these statistics for a wide range of parameters. The table also reports the maximum value (across all states) and weighted average value (according to the invariant distribution) of the approximation percentage error bound, as well as the maximum and weighted average of the actual benefit from deviating and keeping track of the industry state (the actual difference $\frac{E[\sup_{\mu' \in \mathcal{M}} V(x,s|\mu',\tilde{\mu},\tilde{\lambda}) - V(x,s|\tilde{\mu},\tilde{\lambda})]}{E[V(x,s|\tilde{\mu},\tilde{\lambda})]}$). Note that the the latter quantity should always be smaller than the approximation error bound. In the results below we concentrate on the maximum value of the error bound across all states.

The table is separated into two groups. The first five rows correspond to industries with a relatively low cost of investment (low value of d relative to θ_1). In these industries the industry state tends to have a symmetric distribution (see Figure 4) reflecting a rich investment process. The last five rows of the table correspond to industries with a relatively high cost of investment. In these industries the industry state tends to be skewed (see Figure 5), reflecting low levels of investment.

[15]The code used to compute MPE was generously provided by Uli Doraszelski.

From the computational experiments we conclude the following:

1. When the bound is less than 1% the long-run quantities estimated under OE and MPE strategies are very close. The relation between long-run economic indicators and the error bound can be seen more clearly in Figure 6 where we plot the results for the industries with a relatively high cost of investment. We observe that if the error bound is small, the differences in long-run average producer and consumer surplus under OE and MPE strategies are small. As the level of vertical differentiation increases, the error bounds increase, and the differences increase as well. The results suggest that error bounds provide a useful indicator of whether economic quantities of interest are being well approximated.

2. Performance of the approximation depends on the richness of the equilibrium investment process. When the bound is between 1-20% and there is a rich investment process, the long-run quantities estimated under OE and MPE strategies are still quite close. When the bound is above 1% and there is little investment, the long-run quantities can be quite different on a percentage basis (5% to 20%), but still remain fairly close in absolute terms (see Table 2).

3. The performance bound is not tight. For a wide range of parameters the performance bound is as much as 10 to 20 times larger than the actual benefit from deviating.

The previous results suggest that economic indicators of interest under MPE strategies are well-approximated with OE strategies when the approximation error bound is small (less than 1-2% and in some cases even up to 20 %). These results, together with those from Subsection 7.2, demonstrate that the OE approximation significantly expands the range of applied problems that can be analyzed computationally.

8 Aggregate Shocks

Our base model does not allow for shocks to firm profitability that are common across firms. In some contexts, for example when studying how industry dynamics evolve over a business cycle, it is important to account for aggregate shocks. In this section, we extend OE and our computational methods to allow for aggregate shocks at the expense of greater modeling and computational complexity.

In Section 8.1 we introduce a dynamic industry model with aggregate shocks. In Section 8.2 we extend the notion of OE to accommodate aggregate shocks. Error bounds for this model are introduced in Section 8.3. We provide computational experiments in Section 8.4.

8.1 Model with Aggregate Profit Shocks

In this section we extend the model in Section 3 to incorporate a profit shock, z_t, that is common to all firms in the industry. z_t might represent a common demand shock, a common shock to input prices, or a common technology shock. These common shocks will serve to generate periods over which profits are high (or low) for all firms in the industry simultaneously.

The following assumption defines the aggregate shock process and replaces our earlier assumptions on the profit function.

Assumption 8.1. *Let $Z = \{z_t \in A : t \geq 0\}$ be a finite and ergodic Markov chain. Single-period profits for firm i at time t are given by $\pi(x_{it}, s_{-i,t}, z_t)$. For all z, $\pi(x, s, z)$ satisfies Assumption 3.1. Additionally,* $\sup_{x,s,z} \pi(x, s, z) < \infty$.

In this model, a strategy is a function $\mu(x, s, z)$ that depends on the firm's own state, the competitors' state and the level of the aggregate shock. An entry rate is a function, $\lambda(s, z)$, that depends on the industry state and the level of the aggregate shock. To formalize these notions, let $\mathcal{M}_z$ denote the set of exit/investment strategies such that an element $\mu \in \mathcal{M}_z$ is a pair of functions $\mu = (\iota, \rho)$, where $\iota : \mathbb{N} \times \mathcal{S} \times A \to \Re_+$ is an investment strategy and $\rho : \mathbb{N} \times \mathcal{S} \times A \to \Re_+$ is an exit strategy. We denote the set of entry rate functions by Λ_z, where an element of Λ_z is a function $\lambda : \mathcal{S} \times A \to \Re_+$.

Define the value function, $V(x, s, z|\mu', \mu, \lambda)$, to be the expected net present value for a firm at state x when its competitors' state is s, and the value of the aggregate shock is z, given that its competitors each follow a common strategy $\mu \in \mathcal{M}_z$, the entry rate function is $\lambda \in \Lambda_z$, and the firm itself follows strategy $\mu' \in \mathcal{M}_z$. Because this definition is analogous to the one in Section 3.4, we omit the details here for brevity. An equilibrium in this model comprises an investment/exit strategy $\mu = (\iota, \rho) \in \mathcal{M}_z$, and an entry rate function $\lambda \in \Lambda_z$ such that:

1. Incumbent firm strategies represent a MPE:

$$\sup_{\mu' \in \mathcal{M}_z} V(x, s, z|\mu', \mu, \lambda) = V(x, s, z|\mu, \lambda) \qquad \forall x \in \mathbb{N},\ \forall s \in \overline{\mathcal{S}},\ \forall z \in A.$$

2. At each state, either entrants have zero expected profits or the entry rate is zero (or both):

$$\begin{gathered} \textstyle\sum_{s \in \overline{\mathcal{S}},\ z \in A} \lambda(s, z) \left(\beta E_{\mu,\lambda} \left[V(x^e, s_{-i,t+1}, z_{t+1}|\mu, \lambda)|s_t = s,\ z_t = z\right] - \kappa\right) = 0 \\ \beta E_{\mu,\lambda} \left[V(x^e, s_{-i,t+1}, z_{t+1}|\mu, \lambda)|s_t = s,\ z_t = z\right] - \kappa \leq 0, \quad \forall s \in \overline{\mathcal{S}}, \forall z \in A \\ \lambda(s, z) \geq 0, \quad \forall s \in \overline{\mathcal{S}},\ \forall z \in A. \end{gathered}$$

8.2 Oblivious Equilibrium With Aggregate Shocks

In this section we extend the notion of OE to incorporate aggregate shocks. Recall that an OE was based on the idea that when there are a large number of firms (and no aggregate shocks), simultaneous changes in individual firm quality levels can average out such that in the long-run the industry state remains roughly constant over time. Because the aggregate shocks are likely to be of first order importance to strategies, in extending the notion of oblivious strategies to this model it makes sense to make strategies a function of the current value of the shock. In this case, even if there are a large number of firms, the industry state will not necessarily be close to a constant state; it will vary with the aggregate shock. However, we can still take advantage of averaging effects among many firms to restrict firm's strategies so that they do not depend on the industry state. Actually, if there are many firms and the time period is large, because of averaging effects, firms should be able to accurately predict the industry state based on the entire history of realizations of the aggregate shock. This is computationally impractical; instead, we will allow firms to predict the industry state based on a finite set of statistics that depend on the history of realizations of the shock.

Based on this motivation, we will restrict firm strategies so that each firm's decisions depend only on the firm's quality level, the current value of the aggregate shock, and a finite set of statistics that depend on the history of realizations of the aggregate shock. We call such restricted strategies *extended oblivious strategies*. To convey this dependence we introduce a $N-$dimensional Markov chain $\{w_t \in \mathcal{W} = \mathcal{W}_1 \times ... \times \mathcal{W}_N : t \geq 0\}$. We make the following assumption.

Assumption 8.2. *We assume that* $\{w_t : t \geq 0\}$ *is a finite Markov chain adapted to the filtration generated by* $\{z_t : t \geq 0\}$. $\{w_t : t \geq 0\}$ *has a single recurrent class* $\tilde{\mathcal{W}} \subseteq \mathcal{W}$ *and admits a unique invariant distribution. For all* $t \geq 0$, $w_t(1) = z_t$.

Note that Assumption 8.2 together with Assumptions 3.2, 3.3, and 8.1 imply that, when firms use Markov strategies, $\{(s_t, w_t) : t \geq 0\}$ is also a Markov chain that admits a unique invariant distribution.

Let $\tilde{\mathcal{M}}_z$ and $\tilde{\Lambda}_z$ denote the set of extended oblivious strategies and the set of extended oblivious entry rate functions. If firm i uses strategy $\mu \in \tilde{\mathcal{M}}_z$ then at time period t, firm i takes action $\mu(x_{it}, w_t)$, where x_{it} is the state of firm i at time t. Similarly, if $\lambda \in \tilde{\Lambda}_z$, then at time t, the entry rate is equal to $\lambda(w_t)$. Since by the second part of Assumption 8.2, $w_t(1) = z_t$, firms keep track of the current level of the aggregate shock when making decisions with extended oblivious strategies. The state variables $w_t(2), ..., w_t(N)$ allow firms to incorporate additional information about the history of realizations of the aggregate shock into the strategies. As we show in Section 8.4, accounting for past shocks will generally improve firms' decisions. As a consequence, strategies that depend on past shocks may provide a more appealing behavioral model.

It is worth mentioning here, though, that past aggregate shocks are not payoff-relevant (hence, they do not influence MPE strategies), so allowing extended oblivious strategies to depend on them may give rise to extended oblivious equilibria that are poor approximations to MPE.

Different extended oblivious strategies can be defined depending on the specification of w_t. We provide a few examples below. Both examples satisfy Assumption 8.2.

Example 8.1. *Suppose that for $j \in \{1, ..., N\}$, $w_t(j) = z_{t-j+1}$. Hence, $w_t = \{z_t, z_{t-1}, ..., z_{t-N+1}\}$; the aggregate shocks statistics correspond to the last realizations of the shock.*

One disadvantage of the previous scheme is that realizations of the shock that appear in a certain window of time influence the strategy, but if a realization occurs even slightly outside this window, it has no influence. With this motivation we introduce an alternative scheme based on exponentially weighted averages of past shocks.

Example 8.2. *Suppose that $w_t(1) = z_t$ and that for $j \in \{2, ..., N\}$, $w_{t+1}(j) = \alpha_j g_j(z_t) + (1 - \alpha_j) w_t(j)$ and $w_0(j) = g_j(z_0)$, where $\alpha_j \in [0, 1]$ and $g_j : A \to \Re$.*[16]

Suppose that all firms use a common strategy $\mu \in \tilde{\mathcal{M}}_z$ and that entry occurs according to the entry rate function $\lambda \in \tilde{\Lambda}_z$. We assume that (s_0, w_0) is distributed according to the invariant distribution of $\{(s_t, w_t) : t \geq 0\}$. Hence, (s_t, w_t) is a stationary process.

Firms predict the industry state based on the current realization of w_t. Accordingly, for all $w \in \tilde{\mathcal{W}}$, we define $\tilde{s}_{\mu,\lambda}(w) = E[s_t | w_t = w]$. In words, $\tilde{s}_{\mu,\lambda}(w)$ is the long-run expected industry state when dynamics are governed by extended oblivious strategy μ and extended oblivious entry rate function λ, conditional on the current realization of w_t being w.

With some abuse of notation we define an extended oblivious value function as,

$$(8.1) \quad \tilde{V}(x, w | \mu', \mu, \lambda) = E_{\mu'} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \left(\pi(x_{ik}, \tilde{s}_{\mu,\lambda}(w_k), z_k) - d\iota_{ik} \right) + \beta^{\tau_i - t} \phi_{i,\tau_i} \middle| x_{it} = x, w_t = w \right] .$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x, when the aggregate shocks statistics have value w, and the firm follows extended oblivious strategy μ'. The firm assumes that its competitors' state will be $\tilde{s}_{\mu,\lambda}(w_k)$ for all time periods k.

An *extended oblivious equilibrium* consists of a strategy $\mu \in \tilde{\mathcal{M}}_z$ and an entry rate function $\lambda \in \tilde{\Lambda}_z$ that satisfy the following conditions:

[16]In principle, $\mathcal{W}_j$ is an uncountable set that takes values between $\underline{a}_j = \min_{a \in A} g_j(a)$ and $\overline{a}_j = \max_{a \in A} g_j(a)$. However, for computational purposes we could assume that $\mathcal{W}_j$ is a finite grid contained in $[\underline{a}_j, \overline{a}_j]$ and we could approximate the values of $w_t(j)$ with its closest element in the grid.

1. Firm strategies optimize an extended oblivious value function:

$$\sup_{\mu' \in \tilde{\mathcal{M}}_z} \tilde{V}(x, w|\mu', \mu, \lambda) = \tilde{V}(x, w|\mu, \lambda), \qquad \forall x \in \mathbb{N},\ w \in \tilde{\mathcal{W}}.$$

2. Either the oblivious expected value of entry is zero or the entry rate is zero (or both):

$$\sum_{w \in \tilde{\mathcal{W}}} \lambda(w) \left(\beta E \left[\tilde{V}(x^e, w_{t+1}|\mu, \lambda) \middle| w_t = w \right] - \kappa \right) = 0,$$
$$\beta E \left[\tilde{V}(x^e, w_{t+1}|\mu, \lambda) \middle| w_t = w \right] - \kappa \leq 0, \quad \forall w \in \tilde{\mathcal{W}},$$
$$\lambda(w) \geq 0, \quad \forall w \in \tilde{\mathcal{W}}.$$

In the Appendix, we also provide an algorithm for computing an extended OE. Note that the state space of the firm's dynamic programming problem scales with the number of firm states and with the size of $\tilde{\mathcal{W}}$, the recurrent class of the aggregate shock statistics process. As the set $\tilde{\mathcal{W}}$ becomes richer, more computation time and memory is needed.

8.3 Error Bounds

We derive error bounds for this model. As before, approximation error is the amount by which a firm at state $x \in \mathbb{N}$ can improve its expected net present value by unilaterally deviating from the extended OE strategy, and instead following an optimal (non-oblivious) best response.

Because an optimal strategy for a firm that unilaterally deviates from an extended OE strategy depends on the aggregate shock statistics, (since its competitors are using extended OE strategies), we introduce extended Markov strategies. We define $\mathcal{M}_{ze}$ and Λ_{ze} as the set of extended Markov strategies and extended entry rate functions, respectively. An extended Markov strategy is a function of the firm's own state, the industry state, the aggregate shock, and the aggregate shock statistics. If firm i uses strategy $\mu \in \mathcal{M}_{ze}$ then at time period t, firm i takes action $\mu(x_{it}, s_{-i,t}, w_t)$. Similarly, if $\lambda \in \Lambda_{ze}$, then the entry rate at time t is $\lambda(s_t, w_t)$.[17]

For extended Markov strategy $\mu', \mu \in \mathcal{M}_{ze}$ and extended entry rate function $\lambda \in \Lambda_{ze}$, with some abuse

[17]Recall that $w_t(1) = z_t$, hence, strategies are a function of z_t.

of notation, we define the extended value function,

$$(8.2) \quad V(x,s,w|\mu',\mu,\lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}\left(\pi(x_{ik},s_{-i,k},z_k)-d\iota_{ik}\right)+\beta^{\tau_i-t}\phi_{i,\tau_i}\middle|x_{it}=x,s_{-i,t}=s,w_t=w\right],$$

where i is taken to be the index of a firm at quality level x at time t. The extended value function generalizes the value function defined in Section 3.4 allowing for dependence on extended strategies. We use this value function to evaluate the *actual* expected discounted profits garnered by a firm that uses an extended Markov strategy.

Consider an extended oblivious strategy and entry rate $(\tilde{\mu},\tilde{\lambda})\in\tilde{\mathcal{M}}_z\times\tilde{\Lambda}_z$. We assume the initial state (s_0,w_0) is sampled from the invariant distribution of $\{(s_t,w_t):t\geq 0\}$. Hence, (s_t,w_t) is a stationary process, it is distributed according to its invariant distribution for all $t\geq 0$. To abbreviate, let $\tilde{s}=\tilde{s}_{\tilde{\mu},\tilde{\lambda}}$. With some abuse of notation, let $\Delta_A(s,w)=\sup_{y\in A}\left(\pi(y,s,w(1))-\pi(y,\tilde{s}(w),w(1))\right)$ and let $\Delta(y,s,w)=\pi(y,s,w(1))-\pi(y,\tilde{s}(w),w(1))$. We have the following result that we prove in the Appendix.

Theorem 8.1. *Let Assumptions 8.1, 8.2, 3.2, and 3.3 hold. Then, for any extended OE $(\tilde{\mu},\tilde{\lambda})$, and firm state $x\in\mathbb{N}$,*

$$(8.3) \quad E\left[\sup_{\mu'\in\mathcal{M}_{ze}}V(x,s_t,w_t|\mu',\tilde{\mu},\tilde{\lambda})-V(x,s_t,w_t|\tilde{\mu},\tilde{\lambda})\right] \leq \sum_{k=t}^{\infty}\beta^{k-t}E\left[\left[\Delta_{\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}(s_k,w_k)\right]^+\right] + E\left[E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}\left(\pi(x_{ik},\tilde{s}(w_k),z_k)-\pi(x_{ik},s_{-i,k},z_k)\right)\middle|x_{it}=x,s_{-i,t}=s_t,w_t\right]\right].$$

Suppose that, for all $s\in\overline{\mathcal{S}}$ and $w\in\mathcal{W}$, the function $\Delta(y,s,w)^+$ is nondecreasing in y. Then, for any extended OE $(\tilde{\mu},\tilde{\lambda})$, and firm state $x\in\mathbb{N}$,

$$(8.4) \quad E\left[\sup_{\mu'\in\mathcal{M}_{ze}}V(x,s_t,w_t|\mu',\tilde{\mu},\tilde{\lambda})-V(x,s_t,w_t|\tilde{\mu},\tilde{\lambda})\right] \leq \sum_{k=t}^{\infty}\beta^{k-t}E_{\hat{\mu},\tilde{\mu},\tilde{\lambda}}\left[\Delta(\hat{x}_k,s_k,w_k)^+\middle|\hat{x}_t=x\right] + E\left[E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}\left(\pi(x_{ik},\tilde{s}(w_k),z_k)-\pi(x_{ik},s_{-i,k},z_k)\right)\middle|x_{it}=x,s_{-i,t}=s_t,w_t\right]\right].$$

Note that $\hat{x}_k$ in the second bound is controlled by strategy $\hat{\mu}$, that is, a strategy in which the firm never exits the industry and invests an infinite amount at every state. Recall that (s_k, w_k) is distributed according to the invariant distribution for all $k \geq 0$. The bound can be computed using simulation. As before, these bounds are quite general and do not rely on many of the detailed modeling assumptions. Finally, bounds that hold for fixed values of the aggregate shock (as opposed to averaging over them) can also be obtained.

8.4 Computational Experiments

In this section we present computational experiments for the model with aggregate shocks. First, we define the aggregate shocks model. Then, we characterize extended OE for different specifications of w_t. Finally, we study the behavior of the error bounds.

We consider the model presented in Section 7 with the parameters used for the case of high level of vertical differentiation and the "almost deterministic" entry process. We assume there is an aggregate shock to demand, z_t, that affects the market size in the following way:

$$\pi_m(x_{it}, s_{-i,t}, z_t) = z_t m\sigma(x_{it}, s_{-i,t}, p_t^*)(p_{it}^* - c) \ , \ \forall i \in S_t \ .$$

We assume that z_t takes three values: $L = 0.8$, $M = 1$, and $H = 1.2$. Hence, L is an unfavorable state of the economy for all firms in which the total market size is reduced by 20%. On the contrary, H is a favorable state of the economy for all firms in which the total market size is increased by 20%.

We consider two different transition matrices for the Markov process that describes the evolution of z_t:

$$\begin{bmatrix} 0.6 & 0.4 & 0.0 \\ 0.2 & 0.6 & 0.2 \\ 0.0 & 0.4 & 0.6 \end{bmatrix} \qquad \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.3 & 0.4 & 0.3 \\ 0.2 & 0.4 & 0.4 \end{bmatrix}$$

The first row and column of each matrix correspond to state L, the second to state M, and the third to state H. The matrix in the left represents a relatively more persistent aggregate shock in which the most likely value of the shock next period is its value in the current period. In contrast, the matrix in the right represents a relatively less persistent shock. We denote these cases as mp and lp, respectively.

We compute extended OE for the specification of w_t described in Example 8.1 for the cases $N = 1, 2, 3$. If $N = 1$, firms keep track of the current value of the shock only. If $N = 2$ and $N = 3$, firms keep track of the current and past realization of the shock, and of the current and two previous realizations, respectively. Similarly to OE, the algorithm presented in the Appendix to compute extended OE is not guaranteed to terminate. However, for the examples below it terminated in less than one minute for the case $N = 1$ and less than 30 minutes for the case $N = 3$.

In Figure 7 we present the results obtained for the case $N = 1$ for both transition matrices mp and lp. For all values of w we show: (i) $\tilde{n}(w) = \sum_x \tilde{s}_{\mu,\lambda}(x, w)$, the conditional expected total number of firms when dynamics are governed by extended OE strategies and the current value of the shock is w; and (ii) $\lambda(w)$, the extended OE entry rate. In both cases, the entry rate is large at state H and zero in the two other states. In addition, the average number of firms is larger at state H. Results are intuitive because H is the most profitable state for firms, and hence, attracts more of them. The extended OE outcome is consistent with the empirical fact that entry is pro-cyclical with the business cycle (e.g., see Campbell (1998)).

It is interesting to observe some properties of extended OE if $N > 1$. First, since z_t is a Markov process, it follows that under extended oblivious strategies, s_t is conditionally independent of z_t, conditioned on z_{t-1}. Therefore, $\tilde{s}_{\mu,\lambda}(w_t)$ does not vary with the values of z_t, it only varies with the values of $z_{t-1}, ..., z_{t-N+1}$. Second, again because z_t is Markov, $w_{t+1} = \{z_{t+1}, z_t, \ldots, z_{t-N+2}\}$ is conditionally independent of z_{t-N+1}, conditioned on $z_t, \ldots, z_{t-N+2}$. Likewise, $\tilde{s}_{\mu,\lambda}(w_{t+1})$ is conditionally independent of z_{t-N+1}, conditioned on $z_t, \ldots, z_{t-N+2}$. Recall that investment and entry rate decisions at time period t in OE are made in a way that optimizes discounted future profits based on the future trajectory $\{(x_\tau, w_\tau)|\tau > t\}$, assuming industry states $\tilde{s}_{\mu,\lambda}(w_\tau)$ at each time τ. As such, investment and entry decisions made at time t in an extended OE do not depend on z_{t-N+1}. For example, for $N = 2$ an extended OE strategy and entry rate are functions of z_t only.[18]

In Table 3 we compare the extended OE entry rate functions for the different values of N in the case mp. As expected, entry rates change when going from $N = 1$ to $N = 2$. In the latter, there is entry at state M, while in the former there is not. Also, note that when $N = 3$ there could be entry for all values of z_t. However, interestingly, the larger entry rates are obtained in states $\{z_t = H, z_{t-1} = M\}$ and $\{z_t = M, z_{t-1} = L\}$.[19] Again consistent with the empirical evidence mentioned above, these are situations in which the industry has evolved from a less favorable state to a more favorable one attracting more entrants. Even-though strategies change significantly as N increases, implied long-run averages change less so. The long-run average entry rate decrease from 4.3 in the case $N = 1$ to 4.0 in the case $N = 3$. The long-run average number of firms decreases from 113.5 in the case $N = 1$ to 112.7 in the case $N = 3$.

Having analyzed extended OE, we now study the behavior of the error bound in the two cases mp and lp and for the different values of N. We consider error bound (8.4) evaluated at the entry state and in

[18]Note, however, that extended OE with $N = 2$ are different to extended OE with $N = 1$.

[19]Note that state $\{z_t = 3, z_{t-1} = 1\}$ is not possible given the transition matrix mp.

percentage terms relative to the expected value function.[20] Results can be found in Figure 8. We observe that both the persistence of the shock and the number of realizations of the shock firms keep track of have a first order impact in the magnitude of the error bound. If the shock is less persistent, history becomes irrelevant more rapidly and firms can make relatively better decisions by just keeping track of one or few values of the shock. This is reflected in the results; error bounds for the case lp are smaller than for the case mp. Also, error bounds decrease significantly as N increases and firms keep track of more shock values; the additional information allows firms to predict better the conditional industry state and make better decisions.

Note that if $N = 3$ error bounds are close to 5%. In the example, the industry accommodates 113 firms on average. Therefore, the magnitude of extended OE error bounds are not too different from those from OE with the same number of firms (see Figure 2). We conclude that by keeping track of few shock values the accuracy of the approximation based on extended OE in an industry with aggregate shocks can be comparable to that of OE in an industry without aggregate shocks.

The results show that extended OE further extends the set of applied problems that can be analyzed computationally. First, extended OE provide a computationally feasible approach to approximating dynamic behavior in industries with aggregate shocks. Second, extended OE provide sensible results that qualitatively coincide with patterns observed in industry data.

9 Conclusions and Future Research

The goal of this paper has been to increase the set of applied economic problems that can be addressed using Ericson and Pakes (1995)-style dynamic models of imperfect competition. Due to the curse of dimensionality, existing dynamic programming methods have limited application of these models to industries with a small number of firms and a small number of states per firm. As an alternative, we proposed a method for approximating MPE behavior using an OE, where firms make decisions only based on their own state and the long run average industry state.

We introduced a simple algorithm to compute an OE. A nice feature of the method is that there is no need to place a' priori restrictions on the number of firms in the industry or the set of states that a firm can reach. As a result, computational considerations place very few constraints on model details. To facilitate using OE in practice, we derived approximation error bounds that indicate how good the approximation is in any particular problem instance. These approximation error bounds are quite general and thus can

[20] While we are not able to show that $\Delta(y, s, w)^+$ is nondecreasing in y, we check it computationally for all sampled states (s, w) in the simulation.

be used in a wide class of models. Through computational experiments, we showed that OE often yields a good approximation of MPE behavior for industries like those that empirical researchers would like to study. We also extended the notion of OE, our algorithms and error bounds for a model with an aggregate shock common to all firms. Through computational experiments, we showed that the extended notion of OE further expands the set of dynamic industries that can be analyzed. Note that even though the emphasis in this paper is on using OE to approximate industry dynamics, OE can also be used to approximate equilibria in more general stochastic games (see Weintraub, Benkard, and Van Roy (2005b) and Abhishek, Adlakha, Johari, and Weintraub (2007)).

Together, these methods provide a toolkit that facilitates the application of dynamic oligopoly models to a wide range of empirical problems. Computation of OE is light enough that it is even feasible to embed OE computation into an estimation algorithm (see Xu (2006) for an example). The estimation algorithm searches over model parameters and computes OE for many candidate sets of parameters. It stops when it finds the set of parameters that closest match a set of statistics generated by the OE of the model and those observed in the data. Examples of statistics that we might want to match in such an algorithm include the number of firms in the industry; sales, profits, prices, and/or market shares; entry and exit rates; rates of investment, the correlation between investment and market share, etc. The estimated model can then be used for many purposes including, for example, to evaluate the effects of a proposed policy change on industry structure, prices and welfare. For example, one might want to use such a technique to evaluate the effects of a carbon tax or cap-and-trade system on various polluting industries (Ryan 2005). Our hope is that through the use of OE, empirical researchers will be able to utilize richer industry models than was previously possible, and to tackle important problems that were previously intractable.

While we believe that the concept of OE will be useful in applications on its own, there are also some important extensions (see Weintraub, Benkard, and Van Roy (2007)). In order to capture short run transitional dynamics that may result, for example, from shocks or policy changes, we have developed a nonstationary notion of OE in which every firm knows the industry state in the initial period but does not update this knowledge after that point. Additionally, in ongoing research, we are working on an extended notion of OE that allows for there to be a set of “dominant firms”, whose firm states are always monitored by every other firm. This extension trades off increased computation time and memory for a better behavioral model and a better approximation to MPE behavior. Our hope is that the dominant firm OE will provide better approximations for more concentrated industries.

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A Proofs

Proof of Theorem 6.1. We derive bound (6.2). The derivation of bound (6.1) is similar. Let $\mu^* \in \mathcal{M}$ be a Markovian (non-oblivious) best response to an OE $(\tilde{\mu}, \tilde{\lambda})$ for a firm that is keeping track of the industry state. We have that:

$$\begin{aligned} E[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})] &= E[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda})] \\ &+ E[\tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})] \end{aligned} \tag{A.1}$$

In the following proposition we prove a bound for the first term above.

Proposition A.1. *For all* $x \in \mathbb{N}$,

$$E\left[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} E\left[\left[\Delta_{\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}(s_k)\right]^+\right].$$

Proof. Because $\tilde{\mu}$ and $\tilde{\lambda}$ attain an OE, for all x,

$$\tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}(x|\mu', \tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \mathcal{M}} \tilde{V}(x|\mu', \tilde{\mu}, \tilde{\lambda}),$$

where the last equation follows because there will always be an optimal oblivious strategy when optimizing an oblivious value function even if we consider more general strategies. It follows that,

$$V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \leq E_{\mu^*, \tilde{\mu}, \tilde{\lambda}}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}) - \pi(x_{ik}, \tilde{s})\right)\Big| x_{it} = x, s_{-i,t} = s\right].$$

The equation can be rewritten as:

$$V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \leq \sum_{k=t}^{\infty} \beta^{k-t} \sum_{\substack{y \in \mathbb{N} \\ s' \in \overline{\mathcal{S}}}} P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s' \mid x_{it} = x, s_{-i,t} = s]\left(\pi(y, s') - \pi(y, \tilde{s})\right), \tag{A.2}$$

where $P_{\mu^*, \tilde{\mu}, \tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s' \mid x_{it} = x, s_{-i,t} = s]$ is the probability firm i, currently in state x with competitors in state s, will be in state y and s', respectively, $k - t$ periods from now.

We can write:

$$P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s' \mid x_{it} = x, s_{-i,t} = s]$$
$$= P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s] P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid x_{it} = x, s_{-i,t} = s]$$
$$= P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s] P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s].$$

The last equation follows because rival firms use strategy $\tilde{\mu}$, that only depends on their own state, and the entry rate is $\tilde{\lambda}$ independent of the industry state. Substituting into equation (A.2), using Fubini's theorem, and the fact that the quality level of a firm can change by at most $\overline{h}$ units per time period:

(A.3)
$$V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \le \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s]$$
$$\times \sum_{y \in \{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}} P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s] \left(\pi(y, s') - \pi(y, \tilde{s})\right)$$
$$\le \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \left[\max_{y \in \{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}} \left(\pi(y, s') - \pi(y, \tilde{s})\right) \right]^+ .$$

Note that we need to take the positive part of $\max_y (\pi(y, s') - \pi(y, \tilde{s}))$ to get the last inequality, because this term can be negative and $\sum_y P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s]$ can be less than one. Let $q(s)$ be the invariant distribution of $\{s_t : t \ge 0\}$, where s_t is the industry state at time t when every firms uses strategy $\tilde{\mu}$ and the entry rate is $\tilde{\lambda}$. Therefore, for any $k \ge t$:

(A.4)
$$q(s') = \sum_{s \in \overline{\mathcal{S}}} q(s) P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s].$$

Multiplying equations (A.3) by $q(s)$, summing over all $s \in \overline{\mathcal{S}}$, and using Fubini's theorem we have:

$$\sum_{s \in \overline{\mathcal{S}}} q(s) \left(V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \right) \le$$
$$\sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} \sum_{s \in \overline{\mathcal{S}}} q(s) P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \left[\max_{y \in \{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}} \left(\pi(y, s') - \pi(y, \tilde{s})\right) \right]^+ .$$

Finally, by using equation (A.4), we obtain:

$$E\left[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \right] \le \sum_{k=t}^{\infty} \beta^{k-t} E\left[\left[\Delta_{\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}(s_k) \right]^+ \right] ,$$

where s_k is a random vector distributed according to q, for all $k \geq 0$. □

The first term in equation (A.1) is bounded by the previous proposition. Let us analyze the second term:

$$\begin{aligned}\tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) - V(x, s|\tilde{\mu}, \tilde{\lambda}) &= E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, \tilde{s}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{it} = x, s_{-i,t} = s\right] \\ &= \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \\ &\quad \times \sum_{y \in \mathbb{N}} P_{\tilde{\mu}}[x_{ik} = y \mid x_{it} = x]\left(\pi(y, \tilde{s}) - \pi(y, s')\right) .\end{aligned}$$

The last equation follows by using Fubini's theorem and because under oblivious strategies firms' trajectories are independent. Multiplying each term by $q(s)$, summing over all $s \in \overline{\mathcal{S}}$ and interchanging sums in the right hand side using Fubini we obtain:

$$E[\tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})] = \sum_{k=t}^{\infty} \beta^{k-t} \sum_{y \in \mathbb{N}} P_{\tilde{\mu}}[x_{ik} = y \mid x_{it} = x]\left(\pi(y, \tilde{s}) - E\left[\pi(y, s_t)\right]\right) ,$$

Finally, interchanging the sums:

$$\text{(A.5)} \qquad E[\tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})] = \sum_{y \in \mathbb{N}} a_x(y)\left(\pi(y, \tilde{s}) - E\left[\pi(y, s_t)\right]\right) .$$

Bound (6.2) follows by equations (A.1), (A.5), and the proposition. □

Proof of Theorem 6.2. By equation (A.3):

$$\begin{aligned}V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \leq &\sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \\ &\times \sum_{y \in \mathbb{N}} P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s]\left[\pi(y, s') - \pi(y, \tilde{s})\right]^{+} .\end{aligned}$$

It is simple to observe that, for all $k \geq t, x \in \mathbb{N}$, $s, s' \in \overline{\mathcal{S}}$, and $\mu \in \mathcal{M}$,

$$P_{\hat{\mu}}[x_{ik} = \cdot \mid x_{it} = x] \geq P_{\mu,\tilde{\mu},\tilde{\lambda}}[x_{ik} = \cdot \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s],$$

in the first order stochastic dominance sense. Therefore, because $[\pi(y, s') - \pi(y, \tilde{s})]^{+}$ is nondecreasing in

y,

$$\sum_{y \in \mathbb{N}} P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s] \left[\pi(y, s') - \pi(y, \tilde{s})\right]^+ \\ \leq \sum_{y \in \mathbb{N}} P_{\hat{\mu}}[x_{ik} = y \mid x_{it} = x] \left[\pi(y, s') - \pi(y, \tilde{s})\right]^+ .$$

Hence,

$$\text{(A.6)} \quad V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \;\leq\; \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \\ \times \sum_{y \in \mathbb{N}} P_{\hat{\mu}}[x_{ik} = y \mid x_{it} = x] \Delta(y, s')^+ .$$

Multiplying by $q(s)$, summing over all $s \in \overline{\mathcal{S}}$, and using Fubini's theorem we obtain:

$$\sum_{s \in \overline{\mathcal{S}}} q(s) \left(V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda})\right) \\ \leq \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} \sum_{s \in \overline{\mathcal{S}}} q(s) P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] E_{\hat{\mu}} \left[\Delta(\hat{x}_k, s')^+ | \hat{x}_t = x\right] .$$

Because $\hat{x}_k$ and s_k are independent processes, we get:

$$E\left[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} E_{\hat{\mu},\tilde{\mu},\tilde{\lambda}} \left[\Delta(\hat{x}_k, s_k)^+ | \hat{x}_t = x\right] .$$

The rest of the proof is analogous to Theorem 6.1. □

Proof of Theorem 8.1. We derive bound (8.3). A similar argument together with ideas from the proof of Theorem 6.2 can be used to derive bound (8.4). Let μ^* be an optimal extended (non-oblivious) best response to an extended OE $(\tilde{\mu}, \tilde{\lambda})$ for a firm that is keeping track of the industry state. Hence, $\mu^* \in \mathcal{M}_{ze}$ is such that $\sup_{\mu' \in \mathcal{M}_{ze}} V(x, s, w|\mu', \tilde{\mu}, \tilde{\lambda}) = V(x, s, w|\mu^*, \tilde{\mu}, \tilde{\lambda}),\ \forall x, s, w.$

We have that:

$$\text{(A.7)} \quad E[V(x, s_t, w_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - V(x, s_t, w_t|\tilde{\mu}, \tilde{\lambda})] = E[V(x, s_t, w_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x, w_t|\tilde{\mu}, \tilde{\lambda})] \\ + E[\tilde{V}(x, w_t|\tilde{\mu}, \tilde{\lambda}) - V(x, s_t, w_t|\tilde{\mu}, \tilde{\lambda})]$$

First, let us bound the first term in the right hand side of the previous equation.

Because $\tilde{\mu}$ and $\tilde{\lambda}$ attain an extended OE, for all x, w,

$$\tilde{V}(x, w|\tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \tilde{\mathcal{M}}_z} \tilde{V}(x, w|\mu', \tilde{\mu}, \tilde{\lambda}) = \sup_{\mu' \in \mathcal{M}_{ze}} \tilde{V}(x, w|\mu', \tilde{\mu}, \tilde{\lambda}),$$

where the last equation follows because there will always be an optimal extended oblivious strategy when optimizing an extended oblivious value function even if we consider extended Markovian strategies that keep track of the industry state. It follows that,

$$\begin{aligned}
&V(x, s, w|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x, w|\tilde{\mu}, \tilde{\lambda}) \leq \\
&\quad E_{\mu^*,\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}, z_k) - \pi(x_{ik}, \tilde{s}(w_k), z_k)\right)\Big| x_{it} = x, s_{-i,t} = s, w_t = w\right] \\
&\quad = \sum_{k=t}^{\infty} \beta^{k-t} \sum_{\substack{y, \in \mathbb{N} \\ s' \in \overline{\mathcal{S}}, w' \in \tilde{\mathcal{W}}}} P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s', w_k = w' \mid x_{it} = x, s_{-i,t} = s, w_t = w] \\
&\qquad\qquad \times \left(\pi(y, s', w'(1)) - \pi(y, \tilde{s}(w'), w'(1))\right),
\end{aligned}$$

where we abbreviate $\tilde{s} = \tilde{s}_{\mu,\lambda}$. We can write:

$$\begin{aligned}
&P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y, s_{-i,k} = s', w_k = w' \mid x_{it} = x, s_{-i,t} = s, w_t = w] \\
&\quad = P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', w_k = w', x_{it} = x, s_{-i,t} = s, w_t = w] \\
&\qquad \times P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s', w_k = w' \mid x_{it} = x, s_{-i,t} = s, w_t = w]
\end{aligned}$$

Additionally,

$$\begin{aligned}
&P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s', w_k = w' \mid x_{it} = x, s_{-i,t} = s, w_t = w] \\
&\quad = P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k} = s', w_k = w' \mid s_{-i,t} = s, w_t = w],
\end{aligned}$$

because under extended OE strategies, $(s_{-i,k}, w_k)$ is independent of x_{it}, conditional on $(s_{-i,t}, w_t)$. Replacing and using Fubini's theorem we obtain:

$$V(x,s,w|\mu^*,\tilde{\mu},\tilde{\lambda}) - \tilde{V}(x,w|\tilde{\mu},\tilde{\lambda}) \leq \sum_{k=t}^{\infty}\beta^{k-t}\sum_{\substack{s'\in\overline{\mathcal{S}}\\ w'\in\tilde{\mathcal{W}}}} P_{\tilde{\mu},\tilde{\lambda}}[s_{-i,k}=s', w_k=w' \mid s_{-i,t}=s, w_t=w] \times \left[\max_{y\in\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}\left(\pi(y,s',w'(1)) - \pi(y,\tilde{s}(w'),w'(1))\right)\right]^+ .$$

Finally, multiplying by $q(s,w)$, the invariant distribution of $\{(s_t,w_t) : t \geq 0\}$, summing over all (s,w), and using Fubini we get:

$$\text{(A.8)} \qquad E[V(x,s_t,w_t|\mu^*,\tilde{\mu},\tilde{\lambda}) - \tilde{V}(x,w_t|\tilde{\mu},\tilde{\lambda})] \leq \sum_{k=t}^{\infty}\beta^{k-t}E\left[\left[\Delta_{\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}(s_k,w_k)\right]^+\right] .$$

Now, let us bound the second term in equation (A.7). We have that,

$$\tilde{V}(x,w|\tilde{\mu},\tilde{\lambda}) - V(x,s,w|\tilde{\mu},\tilde{\lambda}) = E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}\left(\pi(x_{ik},\tilde{s}(w_k),z_k) - \pi(x_{ik},s_{-i,k},z_k)\right)\Big| x_{it}=x, s_{-i,t}=s, w_t=w\right]$$

Hence,

$$\text{(A.9)} \quad E[\tilde{V}(x,w_t|\tilde{\mu},\tilde{\lambda}) - V(x,s_t,w_t|\tilde{\mu},\tilde{\lambda})] = E\left[E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}\left(\pi(x_{ik},\tilde{s}(w_k),z_k) - \pi(x_{ik},s_{-i,k},z_k)\right)\Big| x_{it}=x, s_{-i,t}=s_t, w_t\right]\right] .$$

The result follows by equations (A.7), (A.8), and (A.9). □

B Additional Error Bounds

B.1 Error Bound for Model in Section 7.1

In this section we consider the industry model in Section 7.1 where firms can change their state by at most one quality level per time period, and the product depreciates one quality level with probability δ independently each period. We can derive the following error bound.

Corollary B.1. *Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that, for all $s \in \overline{\mathcal{S}}$, the function $\Delta(y,s)^+$ is nondecreasing in y. Then, for any OE $(\tilde{\mu}, \tilde{\lambda})$, firm state $x \in \mathbb{N}$,*

$$E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t|\mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t|\tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} \sum_{y \in \{x, \ldots, x+(k-t)\}} \binom{k-t}{y-x} (1-\delta)^{y-x} \delta^{(k-t)-(y-x)} E\left[\Delta(y, s_t)^+\right] + \sum_{y \in \mathbb{N}} a_x(y) \left(\pi(y, \tilde{s}) - E\left[\pi(y, s_t)\right]\right) \ .$$

Proof. Recall that $\hat{\mu}$ is a strategy such that the firm never exits the industry and invests an infinite amount at every state. Hence, under $\hat{\mu}$:

$$P_{\hat{\mu}}[x_{i,t+1} = y \big| x_{it} = x] = \begin{cases} 1-\delta & \text{if } y = x+1 \\ \delta & \text{if } y = x \ . \end{cases}$$

Moreover, under strategy $\hat{\mu}$, for $k \geq t$:

$$P_{\hat{\mu}}[x_{ik} = y \mid x_{it} = x] = \begin{cases} \binom{k-t}{y-x} (1-\delta)^{y-x} \delta^{(k-t)-(y-x)} & \text{if } y - x \leq k - t \text{ and } y \geq x \\ 0 & \text{otherwise} \ . \end{cases}$$

Replacing in equation (A.6):

$$V(x, s|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda}) \ \leq \ \sum_{k=t}^{\infty} \beta^{k-t} \sum_{s' \in \overline{\mathcal{S}}} P_{\tilde{\mu}, \tilde{\lambda}}[s_{-i,k} = s' \mid s_{-i,t} = s] \times \sum_{y \in \{x, \ldots, x+(k-t)\}} \binom{k-t}{y-x} (1-\delta)^{y-x} \delta^{(k-t)-(y-x)} \Delta(y, s')^+.$$

Multiplying by $q(s)$, summing over all $s \in \overline{\mathcal{S}}$, and using Fubini's theorem we obtain:

$$E\left[V(x, s_t|\mu^*, \tilde{\mu}, \tilde{\lambda}) - \tilde{V}(x|\tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} \sum_{y \in \{x, \ldots, x+(k-t)\}} \binom{k-t}{y-x} (1-\delta)^{y-x} \delta^{(k-t)-(y-x)} E\left[\Delta(y, s_t)^+\right] \ .$$

The rest of the proof is analogous to Theorem 6.1 □

B.2 Error Bounds for Industries with No Exit and No Entry

Consider an industry that at time $t = 0$ starts with a positive number of incumbent firms and where there is a constant sell-off value equal to zero and a very high entry cost. As a result, in this industry there will be no exit of incumbent firms and no entry of new firms; the number of firms in the industry will remain constant. Error bounds tighter than the ones in Theorems 6.1 and 6.2 can be derived in this case.

Corollary B.2. *Let Assumptions 3.1, 3.2, and 3.3 hold. Suppose that at time $t = 0$ there is a positive number of incumbent firms. Suppose that the sell-off value $\phi_{it} = 0$, $\forall i, t$, and that the entry cost $\kappa > \frac{\sup_{x,s} \pi(x,s)}{1-\beta} + \overline{\phi}$. Then, for any OE $(\tilde{\mu}, \tilde{\lambda})$ and firm state $x \in \mathbb{N}$,*

$$E\left[\sup_{\mu' \in \mathcal{M}} V(x, s_t | \mu', \tilde{\mu}, \tilde{\lambda}) - V(x, s_t | \tilde{\mu}, \tilde{\lambda})\right] \leq \sum_{k=t}^{\infty} \beta^{k-t} E\left[\Delta_{\{\underline{x}(k,t),\ldots,x+(k-t)\overline{h}\}}(s_t)\right] + \sum_{y \in \mathbb{N}} a_x(y) \left(\pi(y, \tilde{s}) - E\left[\pi(y, s_t)\right]\right) . \tag{B.1}$$

Proof. The proof is analogous to the proof of bound (6.2). The only difference is that in equation (A.3), $\sum_y P_{\mu^*,\tilde{\mu},\tilde{\lambda}}[x_{ik} = y \mid s_{-i,k} = s', x_{it} = x, s_{-i,t} = s] = 1$ (as opposed to ≤ 1), so there is no need to take the positive part of $\Delta_{\cdot}(s')$ to get the inequality. □

Note that compared to the bound (6.2) here we do not need to take the positive part of $\Delta_{\cdot}(s_k)$. A similar relaxation is valid for a bound like (6.4).

C Algorithm for Computing Extended OE

In this section we introduce an algorithm to compute extended OE. Throughout this section we only consider aggregate shock statistics w that are elements of the recurrent class $\tilde{\mathcal{W}}$.

We introduce the following algorithm to compute an extended OE. At each iteration of the algorithm, we (1) compute the expected industry state conditional on the shock statistics, $\tilde{s}_{\mu,\lambda}(w)$ (step 5); (2) we compute the strategy that maximizes the extended oblivious value function (step 6)[21]; and (3) we compute new entry rates depending on the extent of the violation of the zero-profit conditions (step 8). Strategies and entry rates are updated "smoothly" (steps 12 and 13). The parameters N_1, N_2, γ_1, and γ_2 are set after some experimentation to speed up convergence. If the termination condition of the outer loop is satisfied with $\epsilon_1 = \epsilon_2 = 0$, we have an extended OE. Small values of ϵ_1 and ϵ_2 allow for small errors associated with limitations of numerical precision.

[21] We implement this with a Gauss-Seidel version of value iteration.

For initialization, let $(\tilde{\mu}, \tilde{\lambda})$ be an OE where single-period profits are given by $E\left[\pi(x, s, z_t)\right]$, and the expectation is taken with respect to the invariant distribution of z_t. Let $\tilde{V}$ be the respective value function.

Algorithm 2 Extended Oblivious Equilibrium Solver

1: $\lambda(w) := \tilde{\lambda}$, for all w
2: $\mu(x, w) := \tilde{\mu}(x)$, for all x, w
3: $n := 0$
4: **repeat**
5:     Compute $\tilde{s}_{\mu,\lambda}(w)$, for all w
6:     Choose $\mu^* \in \tilde{\mathcal{M}}_z$ to maximize $\tilde{V}(x, w|\mu^*, \mu, \lambda)$ simultaneously for all x, w
7:     **for all** w **do**
8:         $\lambda^*(w) := \lambda(w)\left(\beta E\left[\tilde{V}(x^e, w_{t+1}|\mu^*, \mu, \lambda)\middle| w_t = w\right]/\kappa\right)$
9:     **end for**
10:     $\Delta_1 :=\| \mu - \mu^* \|_\infty$, $\Delta_2 := \|\lambda - \lambda^*\|_\infty$
11:     $n := n + 1$
12:     $\mu := \mu + (\mu^* - \mu)/(n^{\gamma_1} + N_1)$
13:     $\lambda := \lambda + (\lambda^* - \lambda)/(n^{\gamma_2} + N_2)$
14: **until** $\Delta_1 \leq \epsilon_1$ and $\Delta_2 \leq \epsilon_2$

We finish by suggesting a way of computing $\tilde{s}_{\mu,\lambda}(w)$ (step 5 in the algorithm). Let $p(x, w, y, w') = P_{\mu,\lambda}[x_{i,t+1} = y, w_{t+1} = w' \mid x_{it} = x, w_t = w]$. The probability that the firm exits from a state (x, w) is one minus the sum of transition probabilities from that state. Let $\tilde{s}(x, w)$ be the x component of $\tilde{s}_{\mu,\lambda}(w)$. Let $r(x, w)$ be the product of $\tilde{s}(x, w)$ and the steady state probability that the shock process is in state w, $q(w)$. Then, $r(x, w)$ satisfies the balance equations:

$$r(x, w) = \sum_{(y,w')} r(y, w')p(y, w', x, w) + \mathbf{1}(x = x^e) \sum_{w'} \lambda(w')q(w')p(w', w),$$

where $p(w', w) = P[w_{t+1} = w \mid w_t = w']$ and $\mathbf{1}$ is the indicator function. We can obtain $r(x, w)$ by solving this set of balance equations. We can also obtain steady state probabilities of the shock process by solving another set of balance equations. From these two objects, we obtain $\tilde{s}(x, w)$.

D Tables and Figures

Table 1: Comparison of MPE and OE strategies (4 firms, no entry and exit)

Parameters		Long Run Statistics (% Diff)					Perf Bound (% Diff)		Actual (% Diff)	
θ_1	d	Inv.	Prod Surp	Cons Surp	C1	C2	Max Diff	Weighted Avg	Max Diff	Weighted Avg
0.10	0.10	−0.26	−0.01	−0.02	0.03	0.03	0.14	0.13	0.08	0.07
0.30	0.30	−0.13	0.06	0.08	0.08	0.16	1.67	1.22	0.04	0.01
0.50	0.50	−0.11	0.20	0.28	0.18	0.50	6.64	3.61	0.21	0.06
0.70	0.70	−2.21	0.40	0.15	1.08	2.09	18.85	8.35	1.60	0.67
0.85	0.70	−2.19	0.23	−0.28	1.37	2.10	30.80	9.64	1.80	0.20
0.15	0.27	3.54	0.14	0.2	1.22	0.46	0.36	0.35	0.1	0.1
0.20	0.35	4.18	0.29	0.42	1.93	1.03	0.81	0.77	−0.09	−0.05
0.30	0.55	9.28	0.93	1.31	5.10	2.45	1.96	1.85	0.26	0.25
0.40	0.80	21.02	2.10	2.93	11.58	4.12	3.01	2.92	0.30	0.29
0.50	1.00	18.62	3.30	4.33	15.69	5.94	6.29	5.86	0.32	0.30

Long run statistics and value functions simulated with a relative precision of 1.0% and a confidence level of 99%. Error bound simulated with a relative precision of at most 10% and a confidence level of 99%.

Table 2: Comparison of MPE and OE Investment (4 firms, no entry and exit)

Parameters		Investment		
θ_1	d	MPE	OE	% Diff
0.10	0.10	0.752	0.754	−0.26
0.30	0.30	0.754	0.755	−0.13
0.50	0.50	0.741	0.742	−0.11
0.70	0.70	0.694	0.709	−2.21
0.85	0.70	0.748	0.765	−2.19
0.15	0.27	0.192	0.185	3.54
0.20	0.35	0.261	0.250	4.18
0.30	0.55	0.238	0.216	9.28
0.40	0.80	0.168	0.133	21.02
0.50	1.00	0.195	0.158	18.62
Investment simulated with a relative precision of 1.0% and a confidence level of 99%.				

Table 3: Extended OE entry rate functions for case mp and different values of N. For $N = 1, 2$, entry rate is a function of z_t, $\lambda(z_t)$. For $N = 3$, entry rate is a function of (z_t, z_{t-1}), $\lambda(z_t, z_{t-1})$. Only non-zero values are reported.

N	Entry rate function
1	$\lambda(H) = 17.3$
2	$\lambda(M) = 3.5,\ \lambda(H) = 9.9$
3	$\lambda(L, L) = 0.6,\ \lambda(M, M) = 2.8,\ \lambda(H, H) = 4.6,\ \lambda(M, L) = 11.7,\ \lambda(H, M) = 14.4$

Figure 1: Percentage approximation error bound for Poisson entry process for different market sizes.

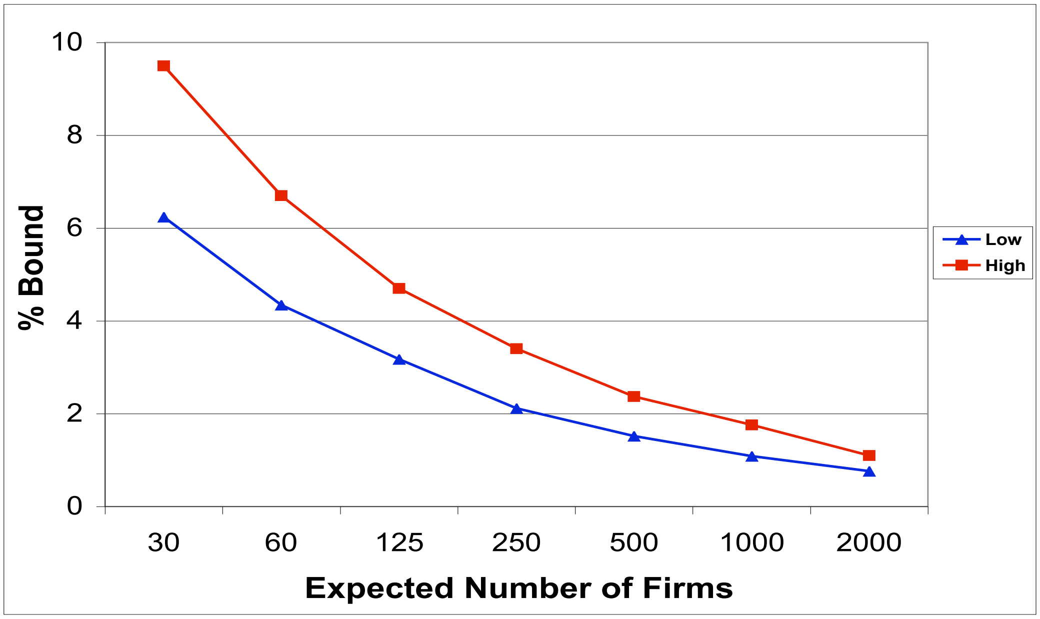

Figure 2: Percentage approximation error bound for deterministic entry process for different market sizes.

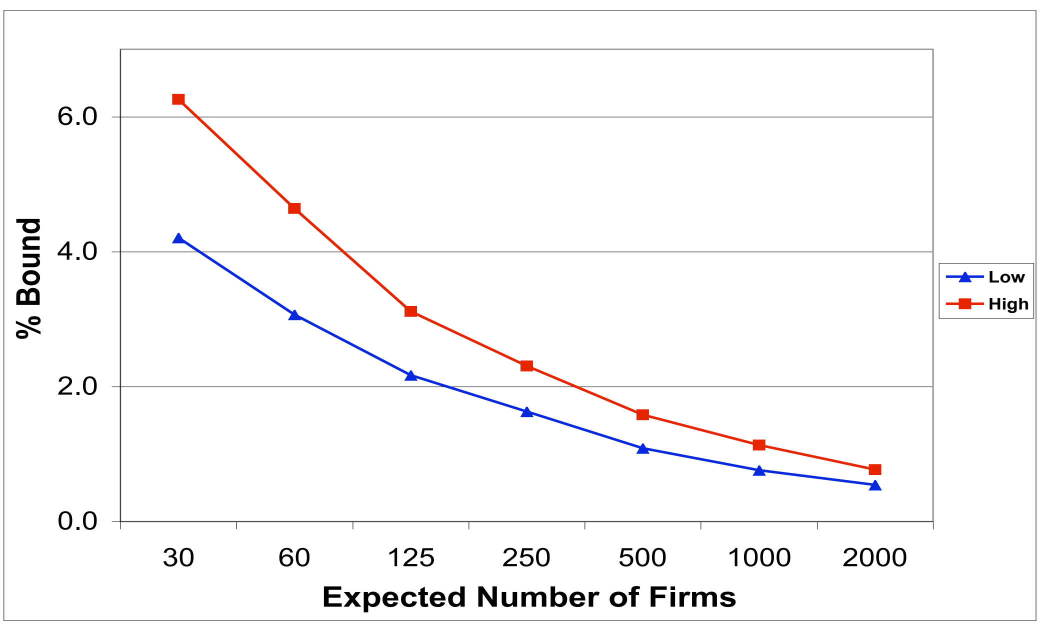

Figure 3: Percentage approximation error bound for fixed number of firms.

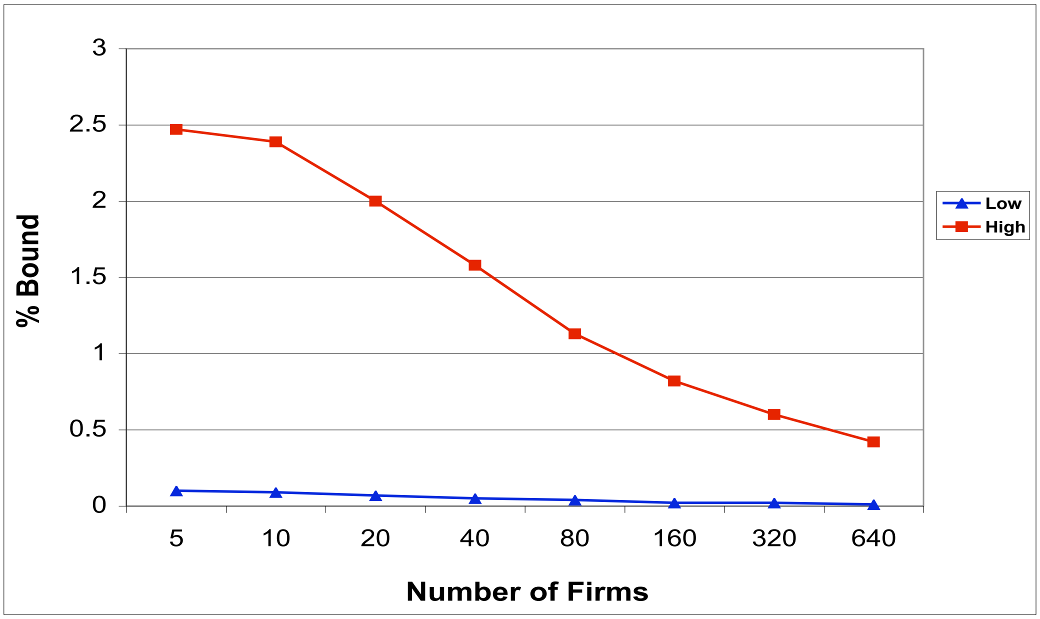

Figure 4: Average industry state for $\theta_1 = 0.5$ and $d = 0.5$.

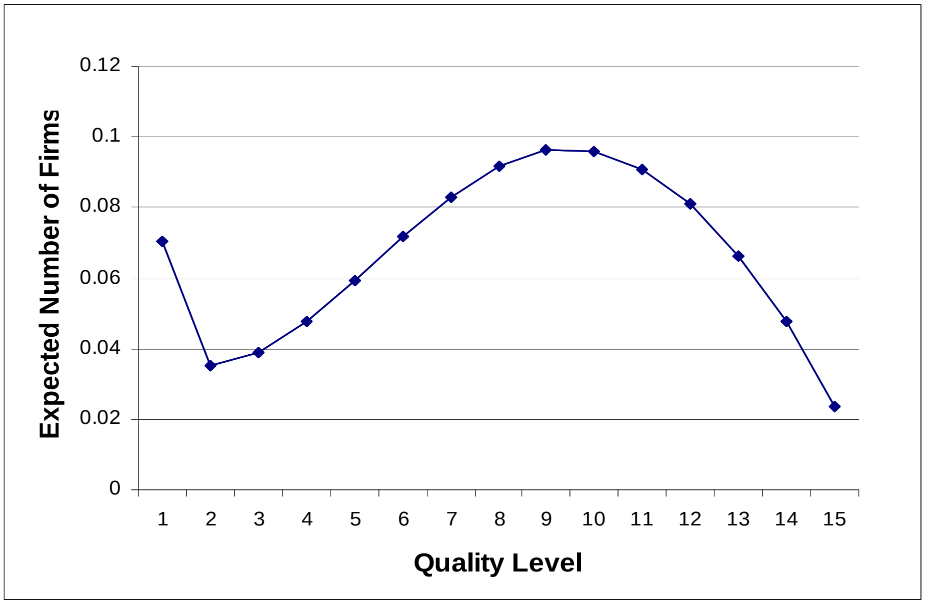

Figure 5: Average industry state for $\theta_1 = 0.4$ and $d = 0.8$.

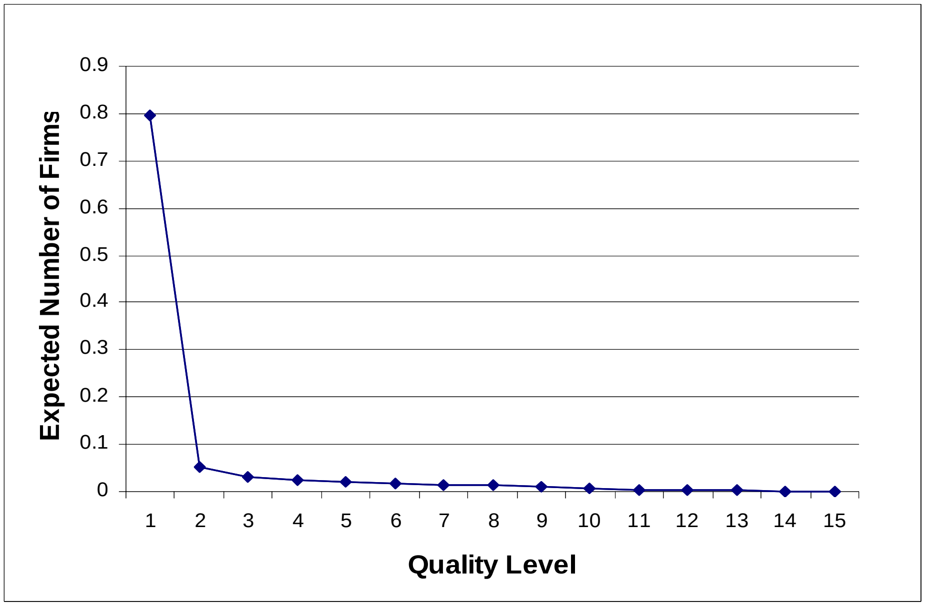

Figure 6: Percentage differences between OE and MPE long-run average consumer and producer surplus and percentage error bound. The four levels of vertical differentiation corresponds to the first four rows of the bottom part of Table 1, that is, to industries with a relatively high cost of investment. As the level of vertical differentiation increases from 1 to 4, θ_1 and d increase.

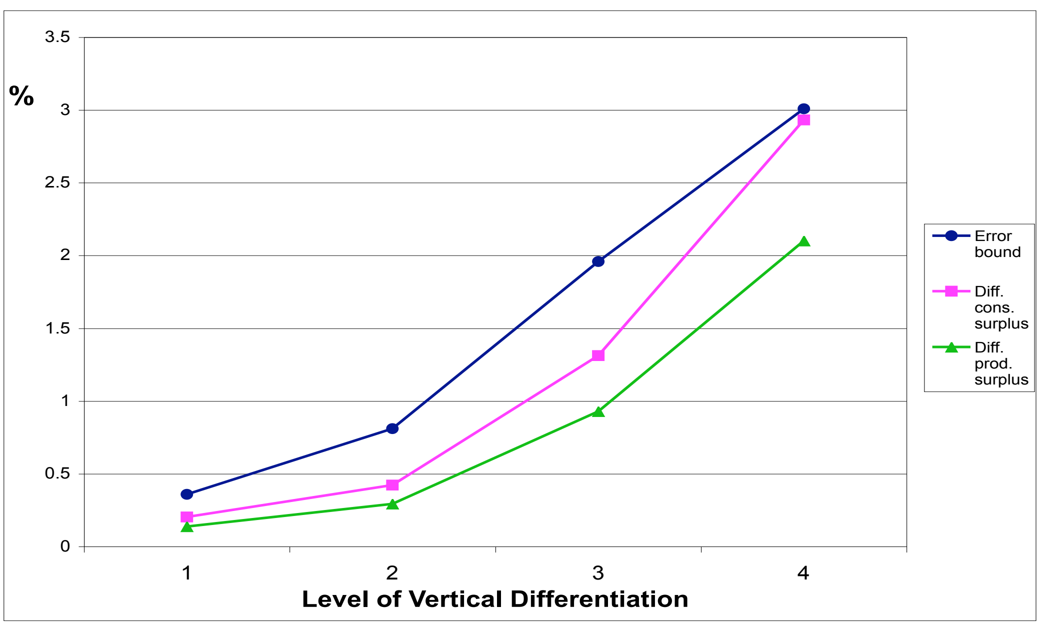

Figure 7: Expected number of firms and entry rate for extended OE.

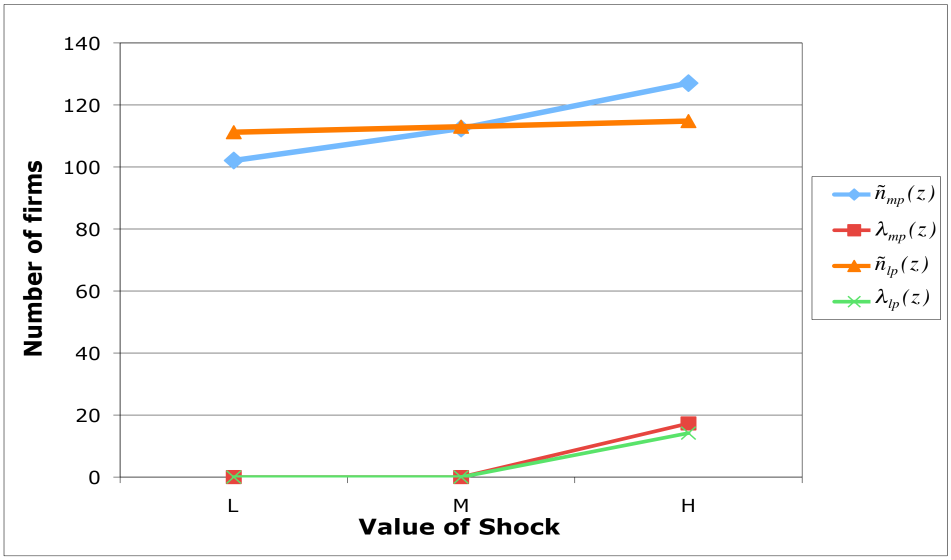

Figure 8: Percentage approximation error bound for extended OE.