Extensions to Oblivious Equilibrium

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1 Introduction

In a recent paper, Weintraub, Benkard, and Van Roy (2007b) propose an approximation method for analyzing Ericson and Pakes (1995)-style dynamic models of imperfect competition. In that paper, we defined a new notion of equilibrium, *oblivious equilibrium* (henceforth, OE), in which each firm is assumed to make decisions based only on its own state and knowledge of the long run average industry state, but where firms ignore current information about competitors' states. The great advantage of OE is that they are much easier to compute than are Markov perfect equilibria (henceforth, MPE). Moreover, we showed that an OE provides meaningful approximations of *long-run* Markov perfect dynamics of an industry with many firms if, alongside some technical requirements, the equilibrium distribution of firm states obeys a *light-tail condition*.

To facilitate using OE in practice, in Weintraub, Benkard, and Van Roy (2007a) we provide a computational algorithm for solving for OE, and approximation bounds that can be computed to provide researchers with a numerical measure of how close OE is to MPE in their particular application. We also provided computational evidence supporting the conclusion that OE often yields good approximations of MPE behavior for industries like those that empirical researchers would like to study.

While our computational results suggest that OE will be useful in many applications on its own, we believe that a major contribution of OE will be as a starting point with which to build even better approximations. As a matter of fact, in Weintraub, Benkard, and Van Roy (2007a) we extend our base model as well as algorithms for computing OE and error bounds to incorporate aggregate shocks common to all firms. Such an extension is important, for example, when analyzing the dynamic effects of industry-wide business cycles. In this note we introduce other important extensions to OE.

First, in order to capture short run transitional dynamics that may result, for example, from shocks or policy changes, we develop a nonstationary notion of OE in which every firm knows the industry state in the initial period but does not update this knowledge after that point. OE offers a way to approximate *long-run* Markov perfect industry dynamics with many firms, and it could be used by analysts interested in long-run economic indicators, such as long-run average investment. These quantities are independent of the initial state of the industry. In other cases, analysts may be interested in the short-run dynamic behavior of an industry starting from a given initial condition of interest; for example, they may want to asses how an industry would evolve over a few years after a policy or environmental change, such as a merger. With this motivation, in Section 3 we introduce a new equilibrium concept, *nonstationary oblivious equilibrium*, that is based on the same idea as oblivious equilibrium but it offers a way to approximate *short-run* Markov perfect industry dynamics with many firms.

Second, we develop an extended notion of nonstationary oblivious equilibrium that allows for there to be a set of "dominant firms", whose firm states are always monitored by every other firm. Our hope is that the dominant firm nonstationary OE will provide better approximations for more concentrated industries. A light-tail condition that precludes highly concentrated industries is needed for approximations based on oblivious strategies to be accurate (see Weintraub, Benkard, and Van Roy (2007b)). However, in many industries there are few dominant firms that have a significant market share. With this motivation, in Section 4 we extend the notion of nonstationary oblivious equilibrium and introduce a new equilibrium concept in which each firm keeps track of the state of few dominant firms.

These extensions together with the extended version of OE that incorporates aggregate shocks (see Weintraub, Benkard, and Van Roy (2007a)) trade off increased computation time for a better behavioral model and a better approximation to MPE behavior. In each case, we (i) define a new equilibrium concept that serves as an approximation of MPE and dramatically reduces the computational complexity; and (ii) we derive error bounds that can be used to indicate how good the approximation is in any particular problem under study. In future research we will explore through computational experiments how these approximation methods work in practice.

The rest of the note is organized as follows. First, in Section 2 we introduce our dynamic industry model. In Section 3 we introduce nonstationary oblivious equilibrium as a way to approximate short-run industry dynamics. Section 4 introduces a new equilibrium concept in which firms keep track of few dominant firms. We conclude in Section 5. All proofs and mathematical arguments can be found in Section 6.

2 A Dynamic Model of Imperfect Competition

In this section we formulate a model of an industry in which firms compete in a single-good market. Our model is based on Weintraub, Benkard, and Van Roy (2007b) and our base model includes only idiosyncratic shocks.

2.1 Model and Notation

The industry evolves over discrete time periods and an infinite horizon. We index time periods with non-negative integers $t \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \geq 0\}$. We adopt a convention of indexing by t variables that are $\mathcal{F}_t$-measurable.

Each firm that enters the industry is assigned a unique positive integer-valued index. The set of indices of incumbent firms at time t is denoted by S_t. At each time $t \in \mathbb{N}$, we denote the number of incumbent firms as n_t.

Firm heterogeneity is reflected through firm states. To fix an interpretation, we will refer to a firm's state as its quality level. However, firm states might more generally reflect productivity, capacity, the size of its consumer network, or any other aspect of the firm that affects its profits. At time t, the quality level of firm $i \in S_t$ is denoted by $x_{it} \in \mathbb{N}$.

We define the *industry state* s_t to be a vector over quality levels that specifies, for each quality level $x \in \mathbb{N}$, the number of incumbent firms at quality level x in period t. We define the state space $\overline{\mathcal{S}} = \left\{ s \in \mathbb{N}^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. Though in principle there are a countable number of industry states, we will also consider an extended state space $\mathcal{S} = \left\{ s \in \Re_+^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. This will allow us, for example, to consider derivatives of functions with respect to the industry state. For each $i \in S_t$, we define $s_{-i,t} \in \mathcal{S}$ to be the state of the *competitors* of firm i; that is, $s_{-i,t}(x) = s_t(x) - 1$ if $x_{it} = x$, and $s_{-i,t}(x) = s_t(x)$, otherwise. Similarly, $n_{-i,t}$ denotes to the number of competitors of firm i.

In each period, each incumbent firm earns profits on a spot market. A firm's single period expected profit $\pi(x_{it}, s_{-i,t})$ depends on its quality level x_{it} and its competitors' state $s_{-i,t}$.

The model also allows for entry and exit. In each period, each incumbent firm $i \in S_t$ observes a positive real-valued sell-off value ϕ_{it} that is private information to the firm. If the sell-off value exceeds the value of continuing in the industry then the firm may choose to exit, in which case it earns the sell-off value and then ceases operations permanently.

If the firm instead decides to remain in the industry, then it can invest to improve its quality level. If a firm invests $\iota_{it} \in \Re_+$, then the firm's state at time $t+1$ is given by,

$$x_{i,t+1} = x_{it} + w(\iota_{it}, \zeta_{i,t+1}),$$

where the function w captures the impact of investment on quality and $\zeta_{i,t+1}$ reflects uncertainty in the outcome of investment. Uncertainty may arise, for example, due to the risk associated with a research and development endeavor or a marketing campaign. Note that this specification is very general as w may take on either positive or negative values (e.g., allowing for positive depreciation). We denote the unit cost of investment by d.

In each period new firms can enter the industry by paying a setup cost κ. Entrants do not earn profits in the period that they enter. They appear in the following period at state $x^e \in \mathbb{N}$ and can earn profits thereafter.[1]

Each firm aims to maximize expected net present value. The interest rate is assumed to be positive and constant over time, resulting in a constant discount factor of $\beta \in (0, 1)$ per time period.

In each period, events occur in the following order:

1. Each incumbent firms observes its sell-off value and then makes exit and investment decisions.
2. The number of entering firms is determined and each entrant pays an entry cost of κ.
3. Incumbent firms compete in the spot market and receive profits.
4. Exiting firms exit and receive their sell-off values.
5. Investment outcomes are determined, new entrants enter, and the industry takes on a new state s_{t+1}.

2.2 Model Primitives

The model as specified is general enough to encompass numerous applied problems in economics. Indeed, a blossoming recent literature on EP-type models has applied similar models to advertising, auctions, collusion, consumer learning, environmental policy, international trade policy, learning-by-doing, limit order markets, mergers, network externalities, and other applied problems. To study any particular applied problem it is necessary to further specify the primitives of the model, including:

[1] Note that it would not change any of our results to assume that the entry state was a random variable.

profit function	π
sell-off value distribution	$\sim \phi_{it}$
investment impact function	w
investment uncertainty distribution	$\sim \zeta_{it}$
unit investment cost	d
entry cost	κ
discount factor	β

Note that in most applications the profit function would not be specified directly, but would instead result from a deeper set of primitives that specify a demand function, a cost function, and a static equilibrium concept.

2.3 Assumptions

We make several assumptions about the model primitives, beginning with the profit function.

Assumption 2.1.

1. *For all $s \in \mathcal{S}$, $\pi(x, s)$ is increasing in x.*
2. *For all $x \in \mathbb{N}$ and $s \in \mathcal{S}$, $\pi(x, s) > 0$, and $\sup_{x,s} \pi(x, s) < \infty$.*

The assumptions are natural. Assumption 2.1.1 ensures that increases in quality lead to increases in profit. Assumption 2.1.2 ensures that profits are positive and bounded. We also make assumptions about investment and the distributions of the private shocks:

Assumption 2.2.

1. *The variables $\{\phi_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and have finite expectations and well-defined density functions with support $\Re_+$.*
2. *The random variables $\{\zeta_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and independent of $\{\phi_{it} | t \geq 0, i \geq 1\}$.*
3. *For all ζ, $w(\iota, \zeta)$ is nondecreasing in ι.*
4. *For all $\iota > 0$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) > 0] > 0$.*
5. *There exists a positive constant $\overline{w} \in \mathbb{N}$ such that $|w(\iota, \zeta)| \leq \overline{w}$, for all (ι, ζ). There exists a positive constant $\bar{\iota}$ such that $\iota_{it} < \bar{\iota}$, $\forall i, \forall t$.*
6. *For all $k \in \{-\overline{w}, \dots, \overline{w}\}$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) = k]$ is continuous in ι.*
7. *The transitions generated by $w(\iota, \zeta)$ are unique investment choice admissible .*

Again the assumptions are natural and fairly weak. Assumptions 2.2.1 and 2.2.2 imply that investment and exit outcomes are idiosyncratic conditional on the state. Assumption 2.2.3 and 2.2.4 imply that investment is productive. Note that positive depreciation is neither required nor ruled out. Assumption 2.2.5 places a finite bound on how much progress can be made or lost in a single period through investment. Assumption 2.2.6 ensures that the impact of investment on transition probabilities is continuous. Assumption 2.2.7 is an assumption introduced by Doraszelski and Satterthwaite (2003) that ensures a unique solution to the firms' investment decision problem. It is used to guarantee existence of an equilibrium in pure strategies, and is satisfied by many of the commonly used specifications in the literature.

We assume that there are a large number of potential entrants who play a symmetric mixed entry strategy. In that case the number of actual entrants is well approximated by the Poisson distribution (see Weintraub, Benkard, and Van Roy (2007b) for a derivation of this result). This leads to the following assumptions:

Assumption 2.3.

1. *The number of firms entering during period t is a Poisson random variable that is conditionally independent of $\{\phi_{it}, \zeta_{it} | t \geq 0, i \geq 1\}$, conditioned on s_t.*
2. *$\kappa > \beta \cdot \bar{\phi}$, where $\bar{\phi}$ is the expected net present value of entering the market, investing zero and earning zero profits each period, and then exiting at an optimal stopping time.*

We denote the expected number of firms entering at industry state s_t, by $\lambda(s_t)$. This state-dependent entry rate will be endogenously determined, and our solution concept will require that it satisfies a zero expected profit condition. Modeling the number of entrants as a Poisson random variable has the advantage that it leads to simpler dynamics. However, our results can accommodate other entry processes as well. Assumption 2.3.2 ensures that the sell-off value by itself is not sufficient reason to enter the industry.

2.4 Equilibrium

As a model of industry behavior we focus on pure strategy Markov perfect equilibrium (MPE), in the sense of Maskin and Tirole (1988). We further assume that equilibrium is symmetric, such that all firms use a common stationary investment/exit strategy. In particular, there is a function ι such that at each time t, each incumbent firm $i \in S_t$ invests an amount $\iota_{it} = \iota(x_{it}, s_{-i,t})$. Similarly, each firm follows an exit strategy that takes the form of a cutoff rule: there is a real-valued function ρ such that an incumbent firm $i \in S_t$ exits at time t if and only if $\phi_{it} \geq \rho(x_{it}, s_{-i,t})$. In Weintraub, Benkard, and Van Roy (2007b) we show that there always exists an optimal exit strategy of this form even among very general classes of exit strategies. Let $\mathcal{M}$ denote the set of exit/investment strategies such that an element $\mu \in \mathcal{M}$ is a pair of functions $\mu = (\iota, \rho)$, where $\iota : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an investment strategy and $\rho : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an exit strategy. Similarly, we denote the set of entry rate functions by Λ, where an element of Λ is a function $\lambda : \mathcal{S} \rightarrow \Re_+$.

We define the value function $V(x, s|\mu', \mu, \lambda)$ to be the expected net present value for a firm at state x when its competitors' state is s, given that its competitors each follows a common strategy $\mu \in \mathcal{M}$, the entry rate function is $\lambda \in \Lambda$, and the firm itself follows strategy $\mu' \in \mathcal{M}$. In particular,

$$V(x, s|\mu', \mu, \lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x, s_{-i,t} = s\right],$$

where i is taken to be the index of a firm at quality level x at time t, τ_i is a random variable representing the time at which firm i exits the industry, and the subscripts of the expectation indicate the strategy followed by firm i, the strategy followed by its competitors, and the entry rate function. In an abuse of notation, we will use the shorthand, $V(x, s|\mu, \lambda) \equiv V(x, s|\mu, \mu, \lambda)$, to refer to the expected discounted value of profits when firm i follows the same strategy μ as its competitors.

An equilibrium to our model comprises an investment/exit strategy $\mu = (\iota, \rho) \in \mathcal{M}$, and an entry rate function $\lambda \in \Lambda$ that satisfy the following conditions:

1. Incumbent firm strategies represent a MPE:

$$\sup_{\mu' \in \mathcal{M}} V(x, s|\mu', \mu, \lambda) = V(x, s|\mu, \lambda) \qquad \forall x \in \mathbb{N},\ \forall s \in \overline{\mathcal{S}}. \tag{2.1}$$

2. At each state, either entrants have zero expected profits or the entry rate is zero (or both):

$$\begin{aligned} \textstyle\sum_{s\in\overline{\mathcal{S}}} \lambda(s)\left(\beta E_{\mu,\lambda}\left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s\right] - \kappa\right) &= 0 & \\ \beta E_{\mu,\lambda}\left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s\right] - \kappa &\leq 0 & \forall s \in \overline{\mathcal{S}} \\ \lambda(s) &\geq 0 & \forall s \in \overline{\mathcal{S}}. \end{aligned}$$

In Weintraub, Benkard, and Van Roy (2007b), we show that the supremum in part 1 of the definition above can always be attained simultaneously for all x and s by a common strategy μ'.

Doraszelski and Satterthwaite (2003) establish existence of an equilibrium in pure strategies for a closely related model. We do not provide an existence proof here because it is long and cumbersome and would replicate this previous work. With respect to uniqueness, in general we presume that our model may have multiple equilibria.[2]

Dynamic programming algorithms can be used to optimize firm strategies, and equilibria to our model can be computed via their iterative application. However, these algorithms require compute time and memory that grow proportionately with the number of relevant industry states, which is often intractable in contexts of practical interest. This difficulty motivates our alternative approach.

[2] Doraszelski and Satterthwaite (2003) also provide an example of multiple equilibria in their closely related model.

3 Nonstationary Oblivious Equilibrium

Oblivious equilibrium (OE), offers a way to approximate *long-run* Markov perfect industry dynamics with many firms. In this section we introduce a new equilibrium concept, nonstationary oblivious equilibrium (NOE), that can be used to approximate the *short-run* dynamic behavior of an industry with many firms starting from an initial state of interest. In Section 3.1 we define NOE, in Section 3.2 we derive bounds to asses approximation error, and in Section 3.3 we introduce an algorithm to compute NOE.

3.1 Nonstationary Oblivious Equilibrium

In this section we define a nonstationary oblivious equilibrium. Recall that an oblivious equilibrium was based on the idea that when there are a large number of firms (and no aggregate shocks), simultaneous changes in individual firm quality levels can average out such that in the long-run the industry state remains roughly constant over time. Using a similar idea we introduce a method to approximate short-run Markov perfect equilibrium behavior of an industry that starts from a given state of interest. If there are a large number of firms (and no aggregate shocks), the industry state starting from a given initial state roughly follows a deterministic trajectory. In this setting, each firm can potentially make near-optimal decisions based only on its own quality level and by knowing the deterministic trajectory followed by the industry state. With this motivation, we consider restricting firm strategies so that each firm's decisions depend only on the firm's quality level and the time period. We call such restricted strategies *nonstationary oblivious* since they involve decisions made without full knowledge of the circumstances — in particular, the state of the industry. Note that nonstationary oblivious strategies differ from oblivious strategies because they depend on the time period. To simplify notation we assume that the industry is at the initial state of interest at time period $t = 0$.

3.1.1 Nonstationary Oblivious Strategies and Entry Rate Functions

Let $\tilde{\mathcal{M}}_{ns} = \tilde{\mathcal{M}}^{\infty} \subset \mathcal{M}^{\infty}$ and $\tilde{\Lambda}_{ns} = \tilde{\Lambda}^{\infty} \subset \Lambda^{\infty}$ denote the set of nonstationary oblivious strategies and the set of nonstationary oblivious entry rate functions.[3] A nonstationary oblivious strategy is a sequence of oblivious strategies. Hence, if $\mu \in \tilde{\mathcal{M}}_{ns}$ is a nonstationary oblivious strategy, then $\mu = \{\mu_0, \mu_1, ...\}$, where for each time period $t \geq 0$, $\mu_t \in \tilde{\mathcal{M}}$ is an oblivious strategy. For example, if firm i uses strategy $\mu \in \tilde{\mathcal{M}}_{ns}$ then at time period t, firm i takes action $\mu_t(x_{it})$, where x_{it} is the state of firm i at time t (so

[3] Recall that $\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\Lambda} \subset \Lambda$ denote the set of oblivious strategies and the set of oblivious entry rate functions; $\mathcal{M}$ and Λ are the set of Markov strategies and the set of entry rate functions (see Weintraub, Benkard, and Van Roy (2007b)). The set X^{∞} is the infinite cross product of X, i.e. $X^{\infty} = X \times X \times X....$

the action depends both on the time period and the state). In a nonstationary oblivious equilibrium firms will make decisions assuming that the industry state evolves deterministically. Moreover, firms will assume the industry state at time period t is the *expected* industry state after t time periods of evolution given the competitors' strategy and starting from the industry state of interest. The (only) purpose of the dependence of nonstationary oblivious equilibrium strategies on the time period is to make strategies a function of the expected industry state.

A nonstationary oblivious entry rate function is a sequence of oblivious entry rate functions. Hence, if $\lambda \in \tilde{\Lambda}_{ns}$ is a nonstationary oblivious entry rate function, then $\lambda = \{\lambda_0, \lambda_1, ...\}$ where for every period $t \geq 0$, λ_t is real-valued.

3.1.2 Nonstationary Oblivious Equilibrium

Note that if all firms use a common strategy $\mu \in \tilde{\mathcal{M}}_{ns}$, the quality level of each evolves as an independent transient non-homogenous Markov chain. Let the transition sub-probabilities of this transient Markov chain for period t be denoted by $P_{\mu_t}(x, y)$. If there were an infinite number of firms, though each evolves stochastically, the percentage of firms that transition from any given quality level to another would be deterministic. Similarly, the percentage of firms that exit would be deterministic. Motivated by this fact, for $\mu \in \tilde{\mathcal{M}}_{ns}$, $\lambda \in \tilde{\Lambda}_{ns}$, and $s \in \overline{\mathcal{S}}$ we define the following sequence of industry states:

(3.1)
$$\tilde{s}_{t+1}(x) = \begin{cases} \sum_{y \in \mathbb{N}} P_{\mu_t}(y, x)\tilde{s}_t(y) + \lambda_t & \text{if } x = x^e \\ \sum_{y \in \mathbb{N}} P_{\mu_t}(y, x)\tilde{s}_t(y) & \text{otherwise,} \end{cases}$$

where $\tilde{s}_0 = s \in \overline{\mathcal{S}}$. Note that $\tilde{s}_t$ is the expected industry state at time t given strategy μ. It can be computed through matrix multiplication. For all $x \in \mathbb{N}$, we let $\tilde{s}_{(\mu,\lambda,s),t}(x) = \tilde{s}_t(x)$, where for all $t \geq 0$, $\tilde{s}_t(x)$ is given by equation (3.1).

For nonstationary oblivious strategies $\mu', \mu \in \tilde{\mathcal{M}}_{ns}$, a nonstationary oblivious entry rate function $\lambda \in \tilde{\Lambda}_{ns}$, and an initial industry state s, we define a *nonstationary oblivious value function* for period t

(3.2)
$$\tilde{V}_t(x|\mu', \mu, \lambda, s) = E_{\mu'}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, \tilde{s}_{(\mu,\lambda,s),k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x\right].$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x at time t and follows nonstationary oblivious strategy μ', under the assumption that, for all $t \geq 0$, its competitors' state will be given by $\tilde{s}_{(\mu,\lambda,s),t}$ at time t. Note that even though the firm's state trajectory only depends on the firm's own strategy μ', the nonstationary oblivious value function remains a function of the competitors' strategy μ and the entry rate λ through the expected industry state trajectory $\tilde{s}_{(\mu,\lambda,s),\cdot}$. We abuse notation by using $\tilde{V}_t(x|\mu,\lambda,s) \equiv \tilde{V}_t(x|\mu,\mu,\lambda,s)$ to refer to the nonstationary oblivious value function when firm i follows the same strategy μ as its competitors.

We now define a new solution concept: an $s-$*nonstationary oblivious equilibrium* consists of a strategy $\mu \in \tilde{\mathcal{M}}_{ns}$ and an entry rate function $\lambda \in \tilde{\Lambda}_{ns}$ that satisfy the following conditions:

1. Firm strategies optimize a nonstationary oblivious value function:

$$\sup_{\mu' \in \tilde{\mathcal{M}}_{ns}} \tilde{V}_0(x|\mu',\mu,\lambda,s) = \tilde{V}_0(x|\mu,\lambda,s), \qquad \forall x \in \mathbb{N}. \tag{3.3}$$

2. At every period of time, either the nonstationary oblivious expected value of entry is zero or the entry rate is zero (or both) . For all $t \geq 0$,

$$\lambda_t \left(\beta \tilde{V}_{t+1}(x^e|\mu,\lambda,s) - \kappa\right) = 0$$
$$\beta \tilde{V}_{t+1}(x^e|\mu,\lambda,s) - \kappa \leq 0$$
$$\lambda_t \geq 0.$$

Note that the optimization of $\tilde{V}_0$ implies, by dynamic programming principles, that firms optimize $\tilde{V}_t$ for all $t \geq 0$. However, for computational purposes we will consider a finite time horizon.

Suppose we are mostly interested in the behavior of the industry in the interval between time periods $t = 0$ and $t = \underline{T}$. We choose a large enough time period $\overline{T}$, for which $\overline{T} := \min\{t|\beta^{t-\underline{T}}\tilde{V}(x^e + t\bar{w}) < \epsilon_0\} \approx 0$, where $\tilde{V}$ is the (stationary) OE value function. The idea is that we estimate expected discounted profits from a distant time period onwards with the OE value function.We make all computations assuming there is a finite time horizon of length $\overline{T}$; we assume that after $\overline{T}$ periods the industry dies off. This simplification should not have a significant impact on the behavior of the industry for the time periods of interest between $t = 0$ and $t = \underline{T}$. After this reduction computing a NOE is simple; it requires solving finite-horizon one-dimensional dynamic programming problems.

We use an $s-$nonstationary oblivious equilibrium to approximate MPE short-run behavior of an industry that starts from a given initial state s. In the next section, we provide error bounds that are useful to asses the accuracy of the approximation for any given applied problem.

3.2 Error Bounds

We derive error bounds. To bound approximation error, we first define what is meant by approximation error in this context. Because an optimal strategy for a firm that unilaterally deviates from a NOE strategy depends on the time period (since its competitors are using nonstationary strategies), we introduce nonstationary Markov strategies. We define $\mathcal{M}_{ns} = \mathcal{M}^\infty$ and $\Lambda_{ns} = \Lambda^\infty$ as the set of nonstationary Markov strategies and entry rate functions, respectively. A nonstationary Markov strategy is a sequence of Markov strategies. Hence, if $\mu \in \mathcal{M}_{ns}$ is a nonstationary Markov strategy, then $\mu = \{\mu_0, \mu_1, ...\}$, where for all $t \geq 0$, $\mu_t \in \mathcal{M}$ is a Markov strategy. Similarly, a nonstationary entry rate function is a sequence of entry rate functions. Hence, if $\lambda \in \Lambda_{ns}$ is a nonstationary entry rate function, then $\lambda = \{\lambda_0, \lambda_1, ...\}$ where for all $t \geq 0$, $\lambda_t \in \Lambda$ is an entry rate function. For nonstationary Markov strategies $\mu', \mu \in \mathcal{M}_{ns}$ and nonstationary entry rate function $\lambda \in \Lambda_{ns}$, we define the nonstationary value function,

$$V_t(x, s|\mu', \mu, \lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, s_{-i,k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x, s_{-i,t} = s\right],$$

where i is taken to be the index of a firm at quality level x at time t. In an abuse of notation, we will use the shorthand, $V_t(x, s|\mu, \lambda) \equiv V_t(x, s|\mu, \mu, \lambda)$. The nonstationary value function generalizes the value function defined in Section 2 allowing for dependence on nonstationary Markov strategies. We use this value function to evaluate the *actual* expected discounted profits garner by a firm that uses a nonstationary Markov strategy. Suppose we are interested in the short-run dynamic behavior of an industry that starts at state $s \in \overline{\mathcal{S}}$. Let $(\tilde{\mu}, \tilde{\lambda})$ be a NOE. We quantify approximation error at each state $x \in \mathbb{N}$ by

$$\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right).$$

Hence, approximation error is the amount by which a firm in state x with competitors in state s can improve its expected discounted profits by unilaterally deviating from the nonstationary oblivious strategy $\tilde{\mu}$ to an optimal nonstationary (non-oblivious) Markov strategy. We introduce our error bound. We denote $[x]^+ = \max(x, 0)$.

Theorem 3.1. *Let Assumptions 2.1, 2.2, and 2.3 hold. Let* $\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}$ *and* $\tilde{\lambda} \in \tilde{\Lambda}_{ns}$ *be an*

$s-$nonstationary oblivious equilibrium. Then, for all $x \in \mathbb{N}$,

$$\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) \leq E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\infty} \beta^k \left[\max_{x' \in \{\underline{x}(k),\ldots,x+k\overline{w}\}} \left(\pi(x', s_{-i,k}) - \pi(x', \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right)\right]^+ \Big| s_{-i,0} = s\right] + E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right], \tag{3.4}$$

where $\underline{x}(k) = \max(x - k\overline{w}, 0)$.

The proof can be found in Section 6. Given a nonstationary oblivious equilibrium, the error bound can be computed using simulation. It requires simulating the industry evolution under nonstationary oblivious equilibrium strategies starting from the initial industry state s. It is worth mentioning that the result can be generalized a great deal. In particular, many of the prior assumptions can be dropped; for instance, most alternative entry processes will not change the result.

3.3 Algorithms and Computations

In this section we introduce algorithms to compute nonstationary oblivious equilibria and error bounds.

3.3.1 Algorithm for Computing Nonstationary OE

We introduce an algorithm for computing an $s-$nonstationary oblivious equilibrium. Suppose we are mostly interested in the behavior of the industry in the interval between time periods $t = 0$ and $t = \underline{T}$. Let $\tilde{V}, \tilde{\mu}, \tilde{\lambda}$ be the (stationary) OE value function, strategy and entry rate, respectively. Let $\overline{T} := \min\{t|\beta^{t-\underline{T}}\tilde{V}(x^e + t\bar{w}) < \delta\}$, where δ is a predetermined precision. The idea is that we estimate expected discounted profits from a distant time period onwards with the OE value function. We assume there is a finite time horizon of length $\overline{T} + 1$ and that at $\overline{T} + 1$ firms garner profits according to the OE value function (step 6). Additionally, we define x_{max} as the largest quality level a firm can ever reach. We let $x_{max} := x^e + \overline{T}\overline{w}$.

At each iteration of the algorithm, we (1) compute the strategies that maximize the nonstationary oblivious value functions (step 10) and (2) we compute new entry rates depending on the extent of the violation of the zero-profit conditions (step 16). Strategies and entry rates are updated "smoothly" (steps 20 and 21). The parameters N_1, N_2, γ_1, and γ_2 are set after some experimentation to speed up convergence.[4]

[4]For computing OE we use $\gamma_1 = 2/3$ and $N_1 = N_2 = 0$. Some numerical experiments suggest that choosing $\gamma_2 = 0$ speeds up convergence.

If $\epsilon_0 = 0$ and the termination condition of the outer loop is satisfied with $\epsilon_1 = \epsilon_2 = 0$, we have an $s-$nonstationary oblivious equilibrium. Small values of ϵ_0, ϵ_1, and ϵ_2 allow for small errors associated with limitations of numerical precision.

Algorithm 1 $s-$Nonstationary Oblivious Equilibrium Solver

$\lambda_t := \tilde{\lambda}$, for all t.
$\mu_t := \tilde{\mu}$, for all t.
$n := 1$.
repeat
  Compute $\tilde{s}_{(\mu,\lambda,s),t}$ for $t \in \{0, ..., \overline{T}+1\}$.
  Define $\tilde{V}_{\overline{T}+1}(x|\mu^*, \mu, \lambda, s) := \tilde{V}(x)$, for all x and $\mu^* \in \tilde{\mathcal{M}}_{ns}$.
  $\Delta_0 := 0$; $\Delta_1 := 0$.
  $t := \overline{T}$.
  repeat
    Choose $\mu_t^* \in \tilde{\mathcal{M}}$ to maximize $\tilde{V}_t(x|\mu^*, \mu, \lambda, s)$ simultaneously for all x.
    $\psi_t = \beta \tilde{V}_{t+1}(x^e|\mu^*, \mu, \lambda, s)) - \kappa$
    $\Delta_0 = \max(\Delta_0, \psi_t)$.
    if $\lambda_t > \epsilon_0$ **then**
      $\Delta_1 = \max(\Delta_1, -\psi_t)$.
    end if
    $\lambda_t^* := \lambda_t(\beta \tilde{V}_{t+1}(x^e|\mu^*, \mu, \lambda, s))/\kappa$.
    Let $t := t - 1$.
  until $t = 0$.
  $\Delta_2 :=\| \mu - \mu^* \|_\infty$.
  $\mu := \mu + (\mu^* - \mu)/(n^{\gamma_1} + N_1)$.
  $\lambda := \lambda + (\lambda^* - \lambda)/(n^{\gamma_2} + N_2)$.
  $n := n + 1$.
until $\Delta_0 \leq \epsilon_1$ and $\Delta_1 \leq \epsilon_1$ and $\Delta_2 \leq \epsilon_2$.

3.3.2 Algorithm for Computing Error Bounds

In this section, we introduce an algorithm to compute error bounds. Given an $s-$NOE $(\tilde{\mu}, \tilde{\lambda})$ and a respective trajectory $s_{(\tilde{\mu},\tilde{\lambda},s),\cdot}$, the bound is computed by doing the following:

1. Starting from state s, simulate trajectories of the firm and industry state under strategies $(\tilde{\mu}, \tilde{\lambda})$.

2. Using the results from the simulations, estimate both expected sums in the error bound with a predetermined precision.

3. The infinite sum should be truncated at $\overline{T}$.

TO BE COMPLETED.

4 Dominant Firms

In Weintraub, Benkard, and Van Roy (2007b) we argued that approximations based on oblivious strategies do not offer accurate representations of MPE behavior if there are few dominant firms in the industry, even if the number of other firms is large. A strategy that does not keep track of the dominant firms will not perform well. To overcome this limitation, in this section we extend the notion of NOE and introduce an approximation method where each firm keeps track of its own state and the state of few dominant firms. In Section 4.1 we define the new equilibrium concept; and we derive bounds to asses approximation error in Section 4.2.

4.1 Nonstationary Partially Oblivious Equilibrium

In this section we extend the concept of nonstationary oblivious equilibrium and let firms keep track of the state of few firms (which we call *dominant firms*) and make an estimate based on averages for the state of all other firms (which we call *fringe firms*). We call this new concept *nonstationary partially oblivious equilibrium*, hereafter NPOE.

Let $\overline{S}_0 = \{i_1, i_2, ..., i_n\} \subseteq S_0$ be the set of indices associated to the dominant firms. A natural way of selecting $\overline{S}_0$ may be to choose the n largest firms from the initial state at $t = 0$. In this approximation, the identity of the n dominant firms does not change over time; firms always keep track of the same set of firms indexed by $\overline{S}_0$ and all new entrants become part of the fringe. This is consistent with the fact that in reality the identity of the dominant firms in a specific industry is unlikely to change in the short-term.

We incorporate this information by extending the firm's state and include a binary variable to indicate whether a firm is dominant or not. Hence, the state of firm i at time t is $\overline{x}_{it} = (x_{it}, 1)$ if $i \in \overline{S}_0$, and $\overline{x}_{it} = (x_{it}, 0)$ if $i \notin \overline{S}_0$.

4.1.1 Nonstationary Partially Oblivious Strategies

Similarly to a nonstationary oblivious strategy, a nonstationary partially oblivious strategy is a function of the firm's state (which in this case also indicates whether the firm is dominant or not) and the time period. However, firms also keep track of the state of firms $i \in \overline{S}_0$. Let y_t be a vector that represents the state of the dominant firms at time t. Formally, $y_t = (x_{i_1 t}, x_{i_2 t}, ..., x_{i_n t})$. If during the period of analysis, firm $i \in \overline{S}_0$ exits the industry, then $x_{it} = -1$ from there on.

While there may be few dominant firms on a given industry, so it is computationally feasible to optimize over strategies that are a function of the dominant firms' state, it is likely that the number of other firms (fringe firms) may be large. Hence, strategies cannot be a function of the fringe firms' state; instead, firms will make an estimate based on averages. Actually, if there are many firms, because of averaging effects, firms should be able to accurately predict the fringe firms' state for a given time period based on the entire history of dominant firms' state.[5] This is computationally impractical; instead, we will allow firms to predict the fringe firms' state based on a finite set of statistics that depend on the entire evolution of the dominant firms' state.[6]

Based on this motivation, we will restrict firm strategies so that each firm's decisions depend only on the firm's state, the time period, the current state of the dominant firms, and a finite set of statistics that depend on the history of realizations of the dominant firms' state. We call such restricted strategies *nonstationary partially oblivious strategies*. To convey this dependence, we define the sequence $\{w_t \in \mathcal{W} = \mathcal{W}_1 \times ... \times \mathcal{W}_N : t \geq 0\}$ where $w_t(1) = y_t$, for all $t \geq 0$, and $\mathcal{W}_j$ are countable sets.

To formally define a nonstationary partially oblivious strategy and a nonstationary partially oblivious entry rate, we first define $\tilde{\mathcal{M}}_p$ and $\tilde{\Lambda}_p$ as the set of partially oblivious strategies and the set of partially oblivious entry rate functions, respectively. If firm i uses strategy $\mu_t \in \tilde{\mathcal{M}}_p$ at time period t, then firm i takes action $\mu_t(\overline{x}_{it}, w_t)$, where $\overline{x}_{it}$ is the state of firm i at time t (which indicates the firm's quality level and whether it is dominant or not). Similarly, if for time period t, the entry rate function is $\lambda_t \in \tilde{\Lambda}_p$, the entry rate is equal to $\lambda_t(w_t)$. Since $w_t(1) = y_t$, firms keep track of the current dominant firms' state when making decisions with partially oblivious strategies. The state variables $w_t(2), ..., w_t(N)$ allow firms to incorporate additional information about the history of realizations of the dominant firms' state into the strategies. This information could be useful to better predict the average fringe firm state conditional on observing w_t. We provide some examples below. In general, accounting for past information will generally improve a firms decisions. It is worth mentioning here, though, that past information is not payoff-relevant (hence, it does not influence MPE strategies), so allowing partially oblivious strategies to depend on that may give rise to NPOE that are poor approximations to MPE.

Now, let $\tilde{\mathcal{M}}_{pns} = \tilde{\mathcal{M}}_p^\infty$ and $\tilde{\Lambda}_{pns} = \tilde{\Lambda}_p^\infty$ denote the set of nonstationary partially oblivious strategies and the set of nonstationary partially oblivious entry rate functions. A nonstationary partially oblivious strategy is a sequence of partially oblivious strategies. Hence, if $\mu \in \tilde{\mathcal{M}}_{pns}$ is a nonstationary partially oblivious strategy, then $\mu = \{\mu_0, \mu_1, ...\}$, where for each time period $t \geq 0$, $\mu_t \in \tilde{\mathcal{M}}_p$ is a partially oblivious strategy. For example, if firm i uses strategy $\mu \in \tilde{\mathcal{M}}_{pns}$ then at time period t, firm i takes action $\mu_t(\overline{x}_{it}, w_t)$. A

[5] Because firms keep track of the state of the dominant firms, even if there are a large number of firms, the fringe firms' state will not necessarily follow a deterministic trajectory like in the case of NOE; the trajectory will depend on the dominant firms' evolution.

[6] This idea is similar to OE with aggregate shocks; see Weintraub, Benkard, and Van Roy (2007a).

nonstationary partially oblivious entry rate function is a sequence of partially oblivious entry rate functions. Hence, if $\lambda \in \tilde{\Lambda}_{pns}$ is a nonstationary oblivious entry rate function, then $\lambda = \{\lambda_0, \lambda_1, ...\}$ where for every period $t \geq 0$, $\lambda_t \in \tilde{\Lambda}_p$, and the entry rate is $\lambda_t(w_t)$. Note that because the firm's state indicates whether the firm is dominant or not, this specification allows for different strategies for dominant and fringe firms, which is likely to be the case in equilibrium.

We make the following assumption over the state statistics w_t.

Assumption 4.1. *Suppose that firms use strategy $\mu \in \tilde{\mathcal{M}}_{pns}$ and enter according to $\lambda \in \tilde{\Lambda}_{nps}$. Then, $\{w_t : t \geq 0\}$ is a non-homogeneous Markov process adapted to the filtration generated by $\{y_t : t \geq 0\}$.*

Different partially oblivious strategies can be defined depending on the specification of w_t. We provide a few examples below. Both examples satisfy Assumption 4.1.

Example 4.1. *Suppose that for $j \in \{1, ..., K\}$, $w_t(j) = y_{t-j+1}$. Hence, $w_t = \{y_t, y_{t-1}, ..., y_{t-K+1}\}$; the statistics correspond to the last realizations of the dominant firms' state. In this case $\mathcal{W}$ is the set of feasible K-tuples of consecutive dominant firms' state.*

One disadvantage of the previous scheme is that realizations of the dominant firms' state that appear in a certain window of time influence the strategy, but if a realization occurs even slightly outside this window, it has no influence. With this motivation we introduce an alternative scheme based on exponentially weighted averages of past states.

Example 4.2. *Suppose that $w_t(1) = y_t$ and that for $j \in \{2, ..., K\}$, $w_{t+1}(j) = \alpha_j g_j(y_t) + (1 - \alpha_j)w_t(j)$ and $w_0(j) = 0$, where $\alpha_j \in [0, 1]$ and $g_j : \mathbb{N}^n \to \Re^n$.*[7]

4.1.2 Fringe Firms' Expected State

In a NPOE firms will make decisions assuming that the fringe firms' state at time period t is the *expected* fringe firms' state after t time periods of evolution given the competitors' strategy, starting from the industry state of interest, and conditional on the current value of the state statistics w_t. Formally, suppose that firms use strategy $\mu \in \tilde{\mathcal{M}}_{pns}$ and enter according to $\lambda \in \tilde{\Lambda}_{nps}$. Let z_t be a vector over quality levels that specifies, for each quality level $x \in \mathbb{N}$, the *number of fringe firms* that are at quality level x in period t. Suppose that the initial state of the industry at $t = 0$ is $s = (y_0, z_0)$. Note that under our assumptions, $\{(w_t, z_t) : t \geq 0\}$ is a non-homogeneous Markov process. We define $\tilde{z}_{(\mu,\lambda,s),t}(w) = E[z_t | w_t = w]$. Now, suppose that dominant

[7] In principle, $\mathcal{W}_j$ is an uncountable set that takes values between $\underline{a}_j = \min_{a \in A} g_j(a)$ and $\overline{a}_j = \max_{a \in A} g_j(a)$. However, for computational purposes we could assume that $\mathcal{W}_j$ is a finite grid contained in $[\underline{a}_j, \overline{a}_j]$ and we could approximate the values of $w_t(j)$ with its closest element in the grid.

firm i deviates from common strategy μ and uses strategy $\mu' \in \tilde{\mathcal{M}}_{pns}$ instead. Because fringe firms keep track of the state of dominant firms, this deviaiton will affect the evolution of z_t. Hence, for this case, we define $\tilde{z}_{(i,\mu',\mu,\lambda,s),t}(w) = E[z_t | w_t = w]$. The vectors $\tilde{z}_{(\mu,\lambda,s),t}(w)$ and $\tilde{z}_{(i,\mu',\mu,\lambda,s),t}(w)$ can be obtained by solving a system of balance equations. [COMPLETE]

4.1.3 Nonstationary Partially Oblivious Value Function

With some abuse of notation, let $\pi(\overline{x}_{it}, y_t, z_t)$ be the single-period profits for a firm in state $\overline{x}_{it}$, if the dominant firms' state is y_t, and the fringe firms' state is z_t.[8] Suppose that firm i uses nonstationary partially oblivious strategies $\mu' \in \tilde{\mathcal{M}}_{pns}$ and its competitors use strategy $\mu \in \tilde{\mathcal{M}}_{pns}$. Entry occurs according to nonstationary partially oblivious entry rate function $\lambda \in \tilde{\Lambda}_{pns}$, and the initial industry state is s. We define a *nonstationary partially oblivious value function* for dominant firm $i \in \overline{S}_0$ and period t

(4.1)

$$\tilde{V}_{it}(\overline{x}, w|\mu', \mu, \lambda, s) \quad = \quad E_{\mu',\mu}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(\overline{x}_{ik}, y_k, \tilde{z}_{(i,\mu',\mu,\lambda,s),k}(w_k)) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| \overline{x}_{it} = \overline{x}, w_t = w\right],$$

This value function should be interpreted as the expected net present value of firm i at state $\overline{x}$ at time t, when the dominant firms' statistics have value w, and firm i follows nonstationary partially oblivious strategy μ'. Competitors use strategy μ and enter according to λ. The firm assumes that the fringe firms' state will be $\tilde{z}_{(i,\mu',\mu,\lambda,s),k}(w_k)$ in time period k, for all $k \geq 0$. Note that because the deviant firm i is dominant, it needs to take this into account when computing $\tilde{z}$. Similarly, we define a *nonstationary partially oblivious value function* for fringe firm $i \notin \overline{S}_0$ and period t

(4.2)

$$\tilde{V}_{it}(\overline{x}, w|\mu', \mu, \lambda, s) \quad = \quad E_{\mu',\mu}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(\overline{x}_{ik}, y_k, \tilde{z}_{(\mu,\lambda,s),k}(w_k)) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| \overline{x}_{it} = \overline{x}, w_t = w\right].$$

This value function is similar to the previous one, but the deviant firm i is not dominant, hence, its evolution does not affect the evolution of other fringe firms (μ' does not affect $\tilde{z}$). In an abuse of notation, we define $\tilde{V}_{it}(\overline{x}, w|\mu, \lambda, s) = \tilde{V}_{it}(\overline{x}, w|\mu, \mu, \lambda, s)$.

[8]Note that if firm i is dominant, it needs to subtract itself from y_t.

4.1.4 Nonstationary Partially Oblivious Equilibrium

We now define a new solution concept: an $s-$ *nonstationary partially oblivious equilibrium* consists of a strategy $\mu \in \tilde{\mathcal{M}}_{pns}$ and an entry rate function $\lambda \in \tilde{\Lambda}_{pns}$ that satisfy the following conditions:

1. Firm strategies optimize a nonstationary partially oblivious value function. For all $i \in \overline{S}_0$,

   (4.3)
   $$\sup_{\mu' \in \tilde{\mathcal{M}}_{pns}} \tilde{V}_{i0}((x,1), w|\mu', \mu, \lambda, s) = \tilde{V}_{i0}((x,1), w|\mu, \lambda, s), \qquad \forall x \in \{y : y = x_{i0}, i \in \overline{S}_0\}, w = w_0,$$

   and, for all $i \notin \overline{S}_0$,

   $$\sup_{\mu' \in \tilde{\mathcal{M}}_{pns}} \tilde{V}_{i0}((x,0), w|\mu', \mu, \lambda, s) = \tilde{V}_{i0}((x,0), w|\mu, \lambda, s), \qquad \forall x \in \mathbb{N}, w = w_0. \tag{4.4}$$

2. At every period of time, either the nonstationary partially oblivious expected value of entry is zero or the entry rate is zero (or both) . For all $t \geq 0$ and $i \notin \overline{S}_0$,

   $$\begin{gathered} \sum_{w \in \mathcal{W}} \lambda_t(w) \left(\beta E\left[\tilde{V}_{i,t+1}((x^e, 0), w_{t+1}|\mu, \lambda, s) \Big| w_t = w \right] - \kappa \right) = 0, \\ \beta E\left[\tilde{V}_{i,t+1}((x^e, 0), w_{t+1}|\mu, \lambda, s) \Big| w_t = w \right] - \kappa \leq 0, \quad \forall w \in \mathcal{W}, \\ \lambda_t(w) \geq 0, \quad \forall w \in \mathcal{W}. \end{gathered}$$

Note that the optimization of $\tilde{V}_{i0}$ implies, by dynamic programming principles, that firms optimize $\tilde{V}_{it}$ for all $t \geq 0$. However, like in NOE, for computational purposes we consider a finite time horizon. Note that to derive a NPOE it is enough to consider one dominant firm (starting from all dominant firms' initial states) and one fringe firm. New entrants become part of the fringe. Finally, if $n = 0$, a NPOE is a NOE.

4.2 Error Bounds

TO BE COMPLETED.

4.3 Algorithms and Computations

We introduce an algorithm for computing an $s-$nonstationary partially oblivious equilibrium. Suppose we are mostly interested in the behavior of the industry in the interval between time periods $t = 0$ and $t = \underline{T}$. Let $\tilde{V}, \tilde{\mu}, \tilde{\lambda}$ be the (stationary) OE value function, strategy and entry rate, respectively. Let $\overline{T} := \min\{t | \beta^{t-\underline{T}} \tilde{V}(x^e + t\bar{w}) < \delta\}$, where δ is a predetermined precision. The idea is that we estimate expected discounted profits from a distant time period onwards with the OE value function. We assume there is a finite time horizon of length $\overline{T}+1$ and that at $\overline{T}+1$ firms garner profits according to the OE value function (step 6). Additionally, we define x_{max} as the largest quality level a firm can ever reach. We let $x_{max} := x^e + \overline{T}\overline{w}$.

At each iteration of the algorithm, we (1) compute the strategies that maximize the nonstationary partially oblivious value functions (step 10) and (2) we compute new entry rates depending on the extent of the violation of the zero-profit conditions (step 16). Strategies and entry rates are updated "smoothly" (steps 20 and 21). The parameters N_1, N_2, γ_1, and γ_2 are set after some experimentation to speed up convergence.[9]

If $\epsilon_0 = 0$ and the termination condition of the outer loop is satisfied with $\epsilon_1 = \epsilon_2 = 0$, we have an $s-$nonstationary partially oblivious equilibrium. Small values of ϵ_0, ϵ_1, and ϵ_2 allow for small errors associated with limitations of numerical precision.

Before introducing the algorithm, we define for $i \in \overline{S}_0$,

$$\tilde{V}_{it}(\overline{x}, w|\mu'', \mu', \mu, \lambda, s) \;=\; E_{\mu'',\mu}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(\overline{x}_{ik}, y_k, \tilde{z}_{(i,\mu',\mu,\lambda,s),k}(w_k)) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle|\overline{x}_{it} = \overline{x}, w_t = w\right],$$

This value function should be interpreted as the expected net present value of firm i at state $\overline{x}$ at time t, when the dominant firms' statistics have value w, and firm i follows nonstationary partially oblivious strategy μ''. Competitors use strategy μ and enter according to λ. The firm assumes that the fringe firms' state will be $\tilde{z}_{(i,\mu',\mu,\lambda,s),k}(w_k)$ in time period k, for all $k \geq 0$. Note that firm i may be using a different strategy (μ'') than the one used to compute $\tilde{z}$ (μ'). While in equilibrium this will never be the case, this value function will be used in the algorithm.

5 Closing Remarks

In this note we studied several important extensions of oblivious equilibrium, including an approximation method to analyze the short-run dynamic behavior of an industry and more sophisticated approximation methods where each firm keeps track of the state of few dominant firms.

Several theoretical and computational issues will be studied in future research:

1. We will explore bounds to asses approximation error for the dominant firms approximation.

2. Additionally, we plan to combine the ideas of Weintraub, Benkard, and Van Roy (2007a) and Section

[9] For computing OE we use $\gamma_1 = 2/3$ and $N_1 = N_2 = 0$. Some numerical experiments suggest that choosing $\gamma_2 = 0$ speeds up convergence.

Algorithm 2 $s-$Nonstationary Partially Oblivious Equilibrium Solver

$\lambda_t(w) := \tilde{\lambda}$, for all t, w.
$\mu_t(\overline{x}, w) := \tilde{\mu}(x)$, for all $t, \overline{x}, x, w$.
Define $\tilde{V}_{i,\overline{T}+1}(\overline{x}, w|\mu^*, \mu, \lambda, s) := \tilde{V}(x)$, for all $i, \overline{x}, x, w, \mu, \mu^* \in \tilde{\mathcal{M}}_{pns}$, and $\tilde{\Lambda}_{pns}$.
$n := 1$.
repeat
  Compute $\tilde{z}_{(\mu,\lambda,s),t}(w)$ for $t \in \{0, ..., \overline{T}\}$ and for all w.
  $\Delta_0 := 0$; $\Delta_1 := 0$.
  $t := \overline{T}$.
  repeat
    Take $i \notin \overline{S}_0$. Choose $\mu_t^* \in \tilde{\mathcal{M}}_p$ to maximize $\tilde{V}_{it}((x, 0), w|\mu^*, \mu, \lambda, s)$ simultaneously for all x, w.
    for all w **do**
      $\psi_t(w) = \beta\tilde{V}_{i,t+1}((x^e, 0), w|\mu^*, \mu, \lambda, s)) - \kappa$.
      $\Delta_0 = \max(\Delta_0, \psi_t(w))$.
      if $\lambda_t(w) > \epsilon_0$ **then**
        $\Delta_1 = \max(\Delta_1, -\psi_t(w))$.
      end if
      $\lambda_t^*(w) := \lambda_t(w)(\beta\tilde{V}_{i,t+1}((x^e, 0), w|\mu^*, \mu, \lambda, s))/\kappa$.
    end for
    Let $t := t - 1$.
  until $t = 0$.
  //For $i \in \overline{S}_0$, choose $\mu_t^* \in \tilde{\mathcal{M}}_p$ to maximize $\tilde{V}_{it}((x, 1), w|\mu^*, \mu, \lambda, s)$ simultaneously for all x, w.
  Take $i \in \overline{S}_0$. $\mu' := \mu$.
  repeat
    $t := \overline{T}$.
    repeat
      Choose $\mu_t^* \in \tilde{\mathcal{M}}_p$ to maximize $\tilde{V}_{it}((x, 1), w|\mu^*, \mu', \mu, \lambda, s)$ simultaneously for all x, w.
      Let $t := t - 1$.
    until $t = 0$.
    $\mu' := \mu^*$.
    Compute $\tilde{z}_{(i,\mu',\mu,\lambda,s),t}(w)$ for $t \in \{0, ..., \overline{T}\}$ and for all w.
  until $\parallel \mu^* - \mu' \parallel_\infty < \epsilon_2$
  $\Delta_2 := \parallel \mu - \mu^* \parallel_\infty$.
  $\mu := \mu + (\mu^* - \mu)/(n^{\gamma_1} + N_1)$.
  $\lambda := \lambda + (\lambda^* - \lambda)/(n^{\gamma_2} + N_2)$.
  $n := n + 1$.
until $\Delta_0 \le \epsilon_1$ and $\Delta_1 \le \epsilon_1$ and $\Delta_2 \le \epsilon_2$.

4 to incorporate an aggregate shock to an approximation method where each firm keeps track of the state of few dominant firms.

3. In a partially oblivious equilibrium, firms always keep track of the same (dominant) firms. In the future we plan to explore approximation methods where each firm keeps track of the current dominant firms in the industry (whose identity might change over time). Additionally, we will study how to build more sophisticated approximations where firms keep track of additional industry statistics, such as the total number of firms in the industry. One complication of these approximations, though, is that the processes that describe the evolution of many of these industry statistics are not Markov.

4. We will do a large set of computational experiments, considering industries with different levels of market concentration, to study how our approximation methods perform in practice.

6 Proofs and Mathematical Arguments

6.1 Proofs and Mathematical Arguments for Section 3

Proof of Theorem 3.1. *Let Assumptions 2.1, 2.2, and 2.3 hold. Let $\tilde{\mu} \in \tilde{\mathcal{M}}_{ns}$ and $\tilde{\lambda} \in \tilde{\Lambda}_{ns}$ be an $s-$nonstationary oblivious equilibrium. Then, for all $x \in \mathbb{N}$,*

$$\begin{aligned}
\sup_{\mu' \in \mathcal{M}_{ns}} V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) \leq \\
E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\infty} \beta^k \left[\max_{x' \in \{\underline{x}(k),\ldots,x+k\overline{w}\}} \left(\pi(x', s_{-i,k}) - \pi(x', \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right)\right]^+ \Big| s_{-i,0} = s\right] \\
+ \; E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i} \beta^k \left(\pi(x_{ik}, \tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k}) - \pi(x_{ik}, s_{-i,k})\right) \Big| x_{i0} = x, s_{-i,0} = s\right],
\end{aligned}$$

where $\underline{x}(k) = \max(x - k\overline{w}, 0)$.

Proof. First, let us write,

$$\begin{aligned}
V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right) &= V_0\left(x, s|\mu', \tilde{\mu}, \tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) \\
&+ \tilde{V}_0\left(x|\tilde{\mu}, \tilde{\lambda}, s\right) - V_0\left(x, s|\tilde{\mu}, \tilde{\lambda}\right). \qquad (6.1)
\end{aligned}$$

Because $\tilde{\mu}$ and $\tilde{\lambda}$ attain an $s-$nonstationary oblivious equilibrium, for all x,

$$\tilde{V}_0\left(x|\tilde{\mu},\tilde{\lambda},s\right) = \sup_{\mu'\in\tilde{\mathcal{M}}_{ns}} \tilde{V}_0\left(x|\mu',\tilde{\mu},\tilde{\lambda},s\right) = \sup_{\mu'\in\mathcal{M}_{ns}} \tilde{V}_0\left(x|\mu',\tilde{\mu},\tilde{\lambda},s\right) \ ,$$

where the last equation follows because there will always be an optimal nonstationary oblivious strategy when optimizing a nonstationary oblivious value function even if we consider more general strategies. Let $\mu^*\in\mathcal{M}_{ns}$ be such that $\sup_{\mu'\in\mathcal{M}_{ns}} V_0\left(x,s|\mu',\tilde{\mu},\tilde{\lambda}\right) = V_0\left(x,s|\mu^*,\tilde{\mu},\tilde{\lambda}\right)$, for all $x\in\mathbb{N}$. We have,

$$\text{(6.2)}\quad V_0\left(x,s|\mu^*,\tilde{\mu},\tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu},\tilde{\lambda},s\right) \leq E_{\mu^*,\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i}\beta^k\left(\pi(x_{ik},s_{-i,k}) - \pi(x_{ik},\tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right)\Big| x_{i0}=x, s_{-i,0}=s\right] \ .$$

Competitors of firm i are using nonstationary oblivious strategies, therefore, their evolution is not affected by firm i's evolution. Hence,

$$\text{(6.3)}\quad V_0\left(x,s|\mu^*,\tilde{\mu},\tilde{\lambda}\right) - \tilde{V}_0\left(x|\tilde{\mu},\tilde{\lambda},s\right) \leq E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\infty}\beta^k\left[\max_{x'\in\{\underline{x}(k),\ldots,x+k\overline{w}\}}\left(\pi(x',s_{-i,k}) - \pi(x',\tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k})\right)\right]^+\Big| s_{-i,0}=s\right] \ ,$$

where we used Assumption 2.2.6 to restrict the possible quality levels a firm can reach at every time period.

On the other hand,

$$\text{(6.4)}\quad \tilde{V}_0\left(x|\tilde{\mu},\tilde{\lambda},s\right) - V_0\left(x,s|\tilde{\mu},\tilde{\lambda}\right) = E_{\tilde{\mu},\tilde{\lambda}}\left[\sum_{k=0}^{\tau_i}\beta^k\left(\pi(x_{ik},\tilde{s}_{(\tilde{\mu},\tilde{\lambda},s),k}) - \pi(x_{ik},s_{-i,k})\right)\Big| x_{i0}=x, s_{-i,0}=s\right] \ .$$

The result follows by expressions (6.1), (6.3), and (6.4). □

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