

Markov Perfect Industry Dynamics with Many Firms - Technical Appendix

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A Proofs and Mathematical Arguments for Section 4: Long-Run Behavior and the Invariant Industry Distribution

Proof of Lemma A.3. *Let Assumptions 3.2 and 3.3 hold. Assume that firms follow a common oblivious strategy $\mu \in \tilde{\mathcal{M}}$, the expected entry rate is $\lambda \in \tilde{\Lambda}$, and the expected time that each firm spends in the industry is finite. Let $\{Z_x : x \in \mathbb{N}\}$ be a sequence of independent Poisson random variables with means $\{\tilde{s}_{\mu,\lambda}(x) : x \in \mathbb{N}\}$, and let Z be a Poisson random variable with mean $\sum_{x \in \mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. Then,*

- (a) $\{s_t : t \geq 0\}$ is an irreducible, aperiodic, and positive recurrent Markov process;
- (b) the invariant distribution of s_t is a product form of Poisson random variables;
- (c) for all x , $s_t(x) \Rightarrow Z_x$;
- (d) $n_t \Rightarrow Z$.

Proof. If every firm uses a strategy $\mu \in \tilde{\mathcal{M}}$ and entry is according to an entry rate function $\lambda \in \tilde{\Lambda}$, then $A = \{s_t : t \geq 0\}$ is clearly an irreducible Markov process. All states reach the state $\emptyset = \{0, 0, \dots\}$ with positive probability and all states can be reached from \emptyset as well. Moreover, state \emptyset is aperiodic, hence, A is aperiodic. Finally, A is positive recurrent because the expected time to come back from state \emptyset to itself is finite (Kleinrock 1975).

Now, let us write:

$$(A.1) \quad s_t(x) = \sum_{\tau=0}^t \sum_{i=1}^{W_\tau} \mathbf{1}_{\{X_{i,t-\tau}=x\}},$$

where W_τ are i.i.d. Poisson random variables with mean λ , the first sum is taken over all periods previous to (and including) t , the second sum is taken over the firms that entered the industry in each period, and

for each τ , $X_{i,t-\tau}$ are random variables that represent the state of firm i after $t - \tau$ periods inside the industry when using strategy μ . Since firms use oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and shocks are idiosyncratic their state evolutions are independent, so $\mathbf{1}_{\{X_{i,t-\tau}=x\}}$ are i.i.d. across i . It follows that $\sum_{i=1}^{W_\tau} \mathbf{1}_{\{X_{i,t-\tau}=x\}}$ is a filtered Poisson random variable, so it is a Poisson random variable. Thus $s_t(x)$, as a sum of independent Poisson random variables, is also Poisson. Given that the expected time a firm spends inside the industry is finite, using characteristic functions it is straightforward to show that $s_t(x) \Rightarrow Z_x, \forall x \in \mathbb{N}$. To show that $\{Z_x : x \in \mathbb{N}\}$ is a sequence of independent random variables note that by the filtering property of Poisson random variables, for all t , $\{s_t(x) : x \in \mathbb{N}\}$ is a sequence of independent random variables (Durrett 1996). By summing over $x \in \mathbb{N}$, we can show that $n_t \Rightarrow Z$. \square

Proof of Lemma A.4. *Let Assumptions 3.2 and 3.3 hold. Assume that firms follow a common oblivious strategy $\mu \in \tilde{\mathcal{M}}$, the expected entry rate is $\lambda \in \tilde{\Lambda}$, and the expected time that each firm spends in the industry is finite. Let $\{Y_n : n \in \mathbb{N}\}$ be a sequence of integer-valued i.i.d. random variables, each distributed according to $\tilde{s}_{\mu,\lambda}(\cdot) / \sum_{x \in \mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. Then, for all $n \in \mathbb{N}$,*

$$\left(x_{(1)t}, \dots, x_{(n)t} \mid n_t = n \right) \Rightarrow (Y_1, \dots, Y_n).$$

Proof. The proof relies on a well known result for Poisson processes; conditional on n arrivals on an interval $[0, T]$, the unordered arrival times have the same distribution as n i.i.d. uniform random variables in $[0, T]$.

Let us condition on $n_t = n$. $\{x_{(j)t} : j = 1, \dots, n\}$ are the random variables that represent the state of each of the n firms in the industry when they are sampled randomly. The expected time a firm spends inside the industry is finite, so the time a firm spends inside the industry is finite with probability one. A firm can increase its quality level by at most \bar{w} states each period. Therefore, for all $\epsilon > 0$, there exists a state z , such that, for all $j \in \{1, \dots, n\}$ and for all t , $\mathcal{P}[x_{(j)t} > z] < \frac{\epsilon}{n}$. Hence, $\mathcal{P}\left[\bigcup_{j=1}^n \{x_{(j)t} > z\} \mid n_t = n\right] < \epsilon$, for all t , so the sequence of random vectors $\{(x_{(1)t}, \dots, x_{(n)t} \mid n_t = n) : t \geq 0\}$ is tight. By Theorem 9.1 in Durrett (1996) and tightness, to prove the lemma it is enough to show that for all n , for all (z_1, \dots, z_n) :

$$\lim_{t \rightarrow \infty} \mathcal{P}\left[x_{(j)t} = z_j, j = 1, \dots, n \mid n_t = n\right] = \prod_{j=1}^n p(z_j),$$

where $p(\cdot)$ is the pmf $\tilde{s}_{\mu,\lambda}(\cdot) / \sum_{x \in \mathbb{N}} \tilde{s}_{\mu,\lambda}(x)$. Let \tilde{T}_j be the entry time period for firm (j) , and $T_j = t - \tilde{T}_j$

be its age. Then we can write:

$$\begin{aligned}
\text{(A.2)} \quad \mathcal{P} \left[x_{(j)t} = z_j, j = 1, \dots, n \mid n_t = n \right] &= \\
&\sum_{\substack{0 \leq t_1 < \infty, \dots, \\ 0 \leq t_n < \infty}} \mathcal{P}[x_{(j)t} = z_j, j = 1, \dots, n \mid T_1 = t_1, \dots, T_n = t_n, n_t = n] \\
&\quad \times \mathcal{P}[T_1 = t_1, \dots, T_n = t_n \mid n_t = n] \\
&= \sum_{\substack{0 \leq t_1 < \infty, \dots, \\ 0 \leq t_n < \infty}} \prod_{j=1}^n \mathcal{P}[x_{(j)t} = z_j \mid T_j = t_j] \cdot \mathcal{P}[T_1 = t_1, \dots, T_n = t_n \mid n_t = n] .
\end{aligned}$$

The last equation follows because the evolution of firms is independent across firms. Note that if any t_j has a value greater than t , then $\mathcal{P}[T_1 = t_1, \dots, T_n = t_n \mid n_t = n] = 0$. We can write

$$\begin{aligned}
\mathcal{P} \left[x_{(j)t} = z_j \mid T_j = t_j \right] &= \frac{\mathcal{P}[x_{(j)t} = z_j, T_j = t_j]}{\mathcal{P}[T_j = t_j]} \\
&= \frac{\mathcal{P}[T_j = t_j, X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \\
&= \frac{\mathcal{P}[T_j = t_j] \mathcal{P}[X_{j,t_j} = z_j]}{\mathcal{P}[T_j = t_j]} \\
\text{(A.3)} \quad &= \mathcal{P}[X_{j,t_j} = z_j] ,
\end{aligned}$$

where X_{j,t_j} is a random variable that represents a firm's state after t_j periods, conditional on having stayed in the industry. Note that for all k , $\{X_{j,k} : j \geq 1\}$ are i.i.d. The second to last equation follows because the evolution of a firm is independent of its entry time.

Now we show that

$$\lim_{t \rightarrow \infty} \mathcal{P}[T_1 = t_1, \dots, T_n = t_n \mid n_t = n] = \prod_{j=1}^n u[t_j] ,$$

for some pmf u . We derive this equation by invoking the relationship between n_t and a Poisson process.

Similarly to equation (A.1), we can write:

$$n_t = \sum_{\tau=0}^t \sum_{i=1}^{W_\tau} A_{i,t-\tau} ,$$

where $A_{i,t-\tau}$ are i.i.d. Bernoulli random variables that equal one if the firm is still in the industry after $t - \tau$ periods when using strategy μ and zero otherwise. Since $A_{i,t-\tau}$ are i.i.d., $n_{t,\tau} = \sum_{i=1}^{W_\tau} A_{i,t-\tau}$ is a filtered

Poisson random variable, and is therefore Poisson. Let us denote its mean by $\alpha_{t,\tau}$. It follows that n_t is a sum of independent Poisson random variables, so it is Poisson with mean $\sum_{\tau=0}^t \alpha_{t,\tau}$.

Consider $\{N(t) : t \geq 0\}$, an homogeneous Poisson process on the real line with rate 1. Note that $N(t)$ and n_t are equivalent in the sense that we can construct n_t using the process $\{N(s) : 0 \leq s \leq \sum_{\tau=0}^t \alpha_{t,\tau}\}$. For each $0 \leq \tau \leq t$, with some abuse of notation, let $N(\alpha_{t,\tau-1}, \alpha_{t,\tau-1} + \alpha_{t,\tau})$ be the total number of events of the Poisson process in the interval $[\alpha_{t,\tau-1}, \alpha_{t,\tau-1} + \alpha_{t,\tau}]$, where $\alpha_{t,-1} = 0$. Then, we can construct $n_t = \sum_{\tau=0}^t n_{t,\tau}$, by defining $n_{t,\tau} = N(\alpha_{t,\tau-1}, \alpha_{t,\tau-1} + \alpha_{t,\tau})$, for all τ .

Now, conditional on the event $N(\sum_{\tau=0}^t \alpha_{t,\tau}) = n$, the unordered arrival times of $N(t)$ have the same distribution as n i.i.d. uniform random variables in $[0, \sum_{\tau=0}^t \alpha_{t,\tau}]$ (Durrett (1996)). By the equivalence argument described above, conditional on $n_t = n$, the unordered arrival times of the n firms are i.i.d. discrete random variables with pmf:

$$v_t(\tau) = \frac{\alpha_{t,\tau}}{\sum_{j=0}^t \alpha_{t,j}}, 0 \leq \tau \leq t .$$

Recall that $\alpha_{t,\tau}$ is the expected number of firms that entered at time τ and are still inside the industry at time t . Since the entry rate is oblivious, all firms use the same oblivious strategy, and shocks are idiosyncratic, $\alpha_{t,\tau} = \tilde{\alpha}_{t-\tau}$, where $\tilde{\alpha}_{t-\tau}$ is the expected number of firms that entered the industry at time s , for any s , and are still inside the industry at time $s + t - \tau$. This suggests making a change of variable and defining:

$$u_t(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^t \tilde{\alpha}_j}, 0 \leq k \leq t .$$

$u_t(k)$ is the probability a random sampled firm from the industry at time t entered k periods ago, conditional on $n_t = n$. Taking the limit as t tends to infinity, we get that:

$$\lim_{t \rightarrow \infty} u_t(k) = u(k) = \frac{\tilde{\alpha}_k}{\sum_{j=0}^{\infty} \tilde{\alpha}_j}, 0 \leq k < \infty ,$$

provided that $\lim_{t \rightarrow \infty} E[n_t] = \sum_{j=0}^{\infty} \tilde{\alpha}_j < \infty$, which is true because the expected time that each firm spends in the industry is finite. $u(k)$ is the probability a random sampled firm, while the industry state is distributed according to its invariant distribution, entered k periods before the sampling period. Therefore,

$$\lim_{t \rightarrow \infty} \mathcal{P}[T_1 = t_1, \dots, T_n = t_n | n_t = n] = \prod_{j=1}^n u[t_j] .$$

Replacing the previous equation together with equation (A.3) in equation (A.2) we obtain:

$$\lim_{t \rightarrow \infty} \mathcal{P} [x_{(j)t} = z_j, j = 1, \dots, n \mid n_t = n] = \prod_{j=1}^n \sum_{0 \leq t < \infty} \mathcal{P}[X_{j,t} = z_j] u(t),$$

where the interchange between the infinite sum and the limit follows by the dominated convergence theorem. The sum yields the pmf $p(\cdot)$. The previous equation proves that, for all $n \in \mathbb{N}$, $(x_{(1)t}, \dots, x_{(n_t)t} \mid n_t = n) \Rightarrow (Y_1, \dots, Y_n)$, where Y_1, \dots, Y_n are i.i.d. random variables with pmf $p(\cdot)$ which does not depend on n .

To finish, consider a very large time period. Formally, suppose that s_0 is sampled from the invariant distribution of $\{s_t : t \geq 0\}$ (which is well defined by Lemma A.3). In this case, s_t is a stationary process; s_t is distributed according to the invariant distribution for all $t \geq 0$.

$$\tilde{s}_{\mu, \lambda}(x) = E[s_t(x)] = E \left[\sum_{j=1}^{n_t} \mathbf{1}_{\{x_{(j)t} = x\}} \right].$$

Conditioning on n_t , and considering that we already proved that $\{x_{(j)t} : j = 1, \dots, n\}$ are i.i.d. with pmf $p(\cdot)$, we conclude that $p(\cdot) = \tilde{s}_{\mu, \lambda}(\cdot) / \sum_{x \in \mathbb{N}} \tilde{s}_{\mu, \lambda}(x)$. □

References

- Durrett, R. (1996). *Probability: Theory and Examples* (Second ed.). Duxbury Press.
- Kleinrock, L. (1975). *Queueing Systems, Volume 1: Theory* (First ed.). Wiley-Interscience.