Industry Dynamics: Foundations For Models with an Infinite Number of Firms*

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February, 2009

Abstract

This paper explores the connection between two important threads of economic research offering different approaches to studying the dynamics of an industry with heterogeneous firms. *Elemental models* of the form pioneered by Ericson and Pakes (1995) capture the dynamics of a finite number of heterogeneous firms as they compete in an industry. Available algorithms can determine Markov perfect equilibrium behavior, though computational requirements become onerous when there are more than a few incumbent firms. *Aggregative models* of the form pioneered by Hopenhayn (1992), on the other hand, assume that firms are infinitesimal and infinite in number. The industry state in stationary equilibrium is constant due to averaging effects among firms, and this dramatically simplifies analysis and computation of equilibrium behavior. The main result of this paper provides conditions under which stationary equilibria of aggregative models approximate Markov perfect equilibria of elemental models arbitrarily well in asymptotically large markets. Our conditions require that the distribution of firm states in stationary equilibrium obeys a certain "light tail" condition. In addition, we explore the connection between stationary equilibria in aggregative models and oblivious equilibria in elemental models, as introduced in Weintraub, Benkard, and Van Roy (2008).

JEL classification numbers: E22, L11, L13

*We have had very helpful conversations with Daron Acemoglu, José Blanchet, Eduardo Engel, Juan Escobar, Hugo Hopenhayn, Chad Jones, Boyan Jovanovic, Andy Skrzypacz, and Assaf Zeevi. Correspondence: gweintraub@columbia.edu; lanierb@stanford.edu; bvr@stanford.edu.

1 Introduction

In this paper we explore the connection between two threads of economic research, each of which offers an approach to studying dynamics of an industry with heterogeneous firms. One is based on the model of Ericson and Pakes (1995), which captures interactions among individual firms as they enter, compete in, and eventually exit an industry. The state of each firm captures its competitive advantage and evolves over time, driven by investments and random shocks. The *industry state* is a histogram which provides the number of firms in each firm state. We refer to such models as *elemental* as they explicitly account for the evolution of individual firms, which can be viewed as basic elements of the industry. While this is an attractive feature, the analysis of elemental models typically involves computation of Markov perfect equilibria (MPE) using dynamic programming algorithms (see Doraszelski and Pakes (2007) for an excellent survey). Such analyses typically restrict the number of firms to a small number because the computational requirements become unmanageable when there are more than few of them.

The second approach, pioneered by Hopenhayn (1992), assumes that there are an infinite number of firms, each of which garners an infinitesimal market share. This leads to models with stationary equilibria (SE) in which the industry state is constant over time because of averaging effects among the infinite number of firms. We refer to such models as *aggregative* models as they average out idiosyncratic random shocks experienced by individual firms. Though the number of firms is infinite, SE in aggregative models can often be analyzed and computed efficiently. The simplification relative to MPE arises from the fact that individual firms need not keep track of the industry state since it is constant.

Because of their usefulness, an important literature on aggregative models has been developed in macroeconomics, international trade, and industrial organization, studying diverse dynamic phenomena such as the size distribution of firms (Luttmer 2007), the intra-industry effects of international trade (Melitz 2003), R&D investments (Klette and Kortum 2004), firms' technological learning (Mitchell 2000), and job turnover (Hopenhayn and Rogerson 1993) to name a few. While aggregative models have become increasingly popular in the recent literature, they are an idealization of real-world industries because they assume an infinite number of firms. Despite their importance, there is no rigorous justification in the literature for the use of aggregative models. This paper provides foundations for aggregative models, and justification for their use.

Specifically, we derive conditions under which aggregative models provide useful approximations of dynamic behavior in large but finite industries. In Section 4 we present our main result. We provide conditions under which aggregative models SE approximate elemental model MPE for large market sizes.[1] Our

[1]We consider elemental models with idiosyncratic shocks only.

conditions require that (1) the number of firms grows with the market size; and (2) that the distribution of firms across firm states exhibits a “light-tail” condition that we define precisely. Under these conditions one can justify use of an aggregative model as a tractable proxy of an elemental model. An aggregative model SE can be a poor approximation of an elemental model MPE if either of these conditions is violated.

Our result points out that an elemental model MPE with many firms is not necessarily well-approximated by an aggregative model SE. Problems arise when markets are highly concentrated, that is, when the market is dominated by a small fraction of relatively large firms. In this case it is inappropriate to treat the industry state as constant since dominant firms can exert significant influence on industry dynamics and therefore all firms should factor dominant firm states into their decisions. Our light-tail condition rules out such situations by ensuring that asymptotically large markets do not exhibit large concentration.

The light-tail condition is satisfied by many commonly used economic models. For example, it is satisfied by a model with a logit demand system and firms competing Nash in prices in the spot market if the SE average firm size is finite. Moreover, for many aggregative models (e.g., those that assume monopolistic competition (Dixit and Stiglitz 1977)), the mere existence of a SE with positive entry rates immediately implies the light-tail condition. In these cases, the use of an aggregative model as a tractable proxy of an elemental model is well justified.

An implication of our results is that, for asymptotically large markets, a simple strategy that ignores current market information can be close to optimal. In this sense, our results contribute to the vast and classic literature on the convergence to competitive equilibria (Roberts and Postlewaite (1976), Novshek and Sonnenschein (1978), Mas-Colell (1982), Mas-Colell (1983), Novshek and Sonnenschein (1983), Novshek (1985), Allen and Hellwig (1986a), Allen and Hellwig (1986b), and Jones (1987)). Roughly speaking, these papers establish conditions in different static models under which the set of oligopolistic Nash equilibria approaches, in some sense, the set of (Walrasian) competitive equilibria as the size of individual firms (or agents) becomes small relative to the size of the market. There are some notable differences with our work, though. Our interests lie in approximating dynamic firm behavior in large markets, not in showing that the product market is perfectly competitive in the limit. In particular, while the above papers study *static* models, in which the main strategic decisions are usually prices or quantities, we study *dynamic* models, in which the main decisions are, for example, investment, entry, and exit. Thus, while we show that firm investment, entry, and exit strategies become simple in markets with many firms, we do not rule out that a small fraction or even all firms may still have some degree of market power in the product market even in the limit. Indeed, in the examples provided in the paper the limit product market is given by monopolistic competition. Additionally, differently from the papers above, in our model the size of firms is endogenously determined in equilibrium through investment. For that reason, we impose a light-tail condition that controls for the appearance of dominant firms to obtain our asymptotic result.

Our result implies that if the light-tail condition is not satisfied, a model that averages out idiosyncratic shocks can provide a poor approximation to industry dynamics. In this sense, our work is related to Jovanovic (1987) and Gabaix (2008) that provide conditions for which idiosyncratic fluctuations can generate aggregate shocks even in an economy with a large number of firms. In particular, Gabaix (2008) argues that if the distribution of firm sizes is heavy-tailed, idiosyncratic shocks to large firms can lead to non-trivial aggregate shocks. He empirically shows that the movements of the 100 largest firms in the US appear to explain one third of variations in output.

There is a close connection between oblivious equilibria (OE) of elemental models (Weintraub, Benkard, and Van Roy (2008)) and SE of aggregative models. Similarly to SE of aggregative models, OE strategies have the advantage that they are a function only of the firm state. The main difference is that OE is a solution concept defined for elemental models with a finite number of firms. As such it can relate more directly to industry data such as the number of firms and the market share of leading firms. The light-tail condition and convergence result in this paper is similar to the main result in Weintraub, Benkard, and Van Roy (2008) where we established that, if a light-tail condition is satisfied along with some technical requirements, then OE provide close approximations of MPE when markets are asymptotically large.

In Section 5 we also prove that, under the appropriate light-tail condition, the set of elemental model OE approaches the set of aggregative model SE as the market size grows in the following sense: (1) for large markets, every OE is close to a SE; and (2) all sequences of strategies that approach a SE satisfy the OE conditions asymptotically. Point (1) corresponds to the upper-hemicontinuity and point (2) is related to the lower-hemicontinuity of the OE correspondence, respectively, at the point where the number of firms and market size become infinite, and when the the light-tail condition is satisfied.

2 An Elemental Model

In this section we formulate an elemental model in which firms compete in a single-good market. Our model is based on Weintraub, Benkard, and Van Roy (2008) which in turn is close in spirit to Ericson and Pakes (1995). We note that an important difference with Ericson and Pakes (1995) is that our model includes only idiosyncratic shocks. This simplification is important to relate our elemental model to an aggregative model.

2.1 Model and Notation

The industry evolves over discrete time periods and an infinite horizon. We index time periods with non-negative integers $t \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots\}$). All random variables are defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ equipped with a filtration $\{\mathcal{F}_t : t \geq 0\}$. We adopt a convention of indexing by t variables that are $\mathcal{F}_t$-measurable.

Each firm that enters the industry is assigned a unique positive integer-valued index. The set of indices of incumbent firms at time t is denoted by S_t. At each time $t \in \mathbb{N}$, we denote the number of incumbent firms as n_t.

Firm heterogeneity is reflected through firm states. To fix an interpretation, we will refer to a firm's state as its quality level. However, firm states might more generally reflect productivity, capacity, the size of its consumer network, or any other aspect of the firm that affects its profits. At time t, the quality level of firm $i \in S_t$ is denoted by $x_{it} \in \mathbb{N}$.

We define the *industry state* s_t to be a vector over quality levels that specifies, for each quality level $x \in \mathbb{N}$, the number of incumbent firms at quality level x in period t. We define the state space $\overline{\mathcal{S}} = \left\{ s \in \mathbb{N}^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. Though in principle there are a countable number of industry states, we will also consider an extended state space $\mathcal{S} = \left\{ s \in \Re_+^\infty \middle| \sum_{x=0}^{\infty} s(x) < \infty \right\}$. This will be useful to define the aggregative model and will allow us, for example, to consider derivatives of functions with respect to the industry state. For each $i \in S_t$, we define $s_{-i,t} \in \mathcal{S}$ to be the state of the *competitors* of firm i; that is, $s_{-i,t}(x) = s_t(x) - 1$ if $x_{it} = x$, and $s_{-i,t}(x) = s_t(x)$, otherwise. Similarly, $n_{-i,t}$ denotes the number of competitors of firm i.

In each period, each incumbent firm earns profits on a spot market. A firm's single period expected profit $\pi(x_{it}, s_{-i,t})$ depends on its quality level x_{it} and its competitors' state $s_{-i,t}$. Note that in most applied problems the profit function would not be specified directly, but would instead result from a deeper set of primitives that specify a demand function, a cost function, and a static equilibrium concept. An important parameter of the demand function, that we will focus on below, is the size of the relevant market, which we will denote as m.

The model also allows for entry and exit. In each period, each incumbent firm $i \in S_t$ observes a positive real-valued sell-off value ϕ_{it} that is private information to the firm. If the sell-off value exceeds the value of continuing in the industry then the firm may choose to exit, in which case it earns the sell-off value and then ceases operations permanently.

If the firm instead decides to remain in the industry, then it can invest to improve its quality level. If a firm invests $\iota_{it} \in \Re_+$, then the firm's state at time $t+1$ is given by,

$$x_{i,t+1} = \max\left(0, x_{it} + w(\iota_{it}, \zeta_{i,t+1})\right),$$

where the function w captures the impact of investment on quality and $\zeta_{i,t+1}$ reflects uncertainty in the outcome of investment. Uncertainty may arise, for example, due to the risk associated with a research and development endeavor or a marketing campaign. Note that this specification is very general as w may take on either positive or negative values (e.g., allowing for positive depreciation). We denote the unit cost of investment by d.

In each period new firms can enter the industry by paying a setup cost κ. Entrants do not earn profits in the period that they enter. They appear in the following period at state $x^e \in \mathbb{N}$ and can earn profits thereafter.

Each firm aims to maximize expected net present value. The interest rate is assumed to be positive and constant over time, resulting in a constant discount factor of $\beta \in (0,1)$ per time period.

In each period, events occur in the following order:

1. Each incumbent firms observes its sell-off value and then makes exit and investment decisions.
2. The number of entering firms is determined and each entrant pays an entry cost of κ.
3. Incumbent firms compete in the spot market and receive profits.
4. Exiting firms exit and receive their sell-off values.
5. Investment outcomes are determined, new entrants enter, and the industry takes on a new state s_{t+1}.

We make the same assumptions made in Weintraub, Benkard, and Van Roy (2008) about the model primitives. For completeness we state the assumptions in the Appendix. *Assumptions A.1, A.2, A.3 are kept throughout the paper unless otherwise explicitly noted.*

2.2 Equilibrium

We consider a notion of symmetric pure strategy Markov perfect equilibrium (MPE), which we now define. There is a function ι such that at each time t, each incumbent firm $i \in S_t$ invests an amount $\iota_{it} = \iota(x_{it}, s_{-i,t})$. Similarly, each firm follows an exit strategy that takes the form of a cutoff rule: there is a real-valued function ρ such that an incumbent firm $i \in S_t$ exits at time t if and only if $\phi_{it} \geq \rho(x_{it}, s_{-i,t})$.[2] Let $\mathcal{M}$ denote the set of exit/investment strategies such that an element $\mu \in \mathcal{M}$ is a pair of functions $\mu = (\iota, \rho)$, where $\iota : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an investment strategy and $\rho : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ is an exit strategy. Similarly, we denote the

[2] Weintraub, Benkard, and Van Roy (2008) show that there always exists an optimal exit strategy of this form even among very general classes of exit strategies.

set of entry rate functions by Λ, where an element of Λ is a function $\lambda : \mathcal{S} \to \Re_+$. We assume the number of entering firms each period t is a Poisson random variable with mean $\lambda(s_t)$ (see Assumption A.3 for a detailed description of the entry process).

We define the value function $V(x, s|\mu', \mu, \lambda)$ to be the expected net present value for a firm at state x when its competitors' state is s, given that its competitors each follows a common strategy $\mu \in \mathcal{M}$, the entry rate function is $\lambda \in \Lambda$, and the firm itself follows strategy $\mu' \in \mathcal{M}$. In particular,

$$V(x, s|\mu', \mu, \lambda) = E_{\mu',\mu,\lambda}\left[\sum_{k=t}^{\tau_i} \beta^{k-t} \left(\pi(x_{ik}, s_{-i,k}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i} \middle| x_{it} = x, s_{-i,t} = s\right],$$

where i is taken to be the index of a firm at quality level x at time t, τ_i is a random variable representing the time at which firm i exits the industry, and the subscripts of the expectation indicate the strategy followed by firm i, the strategy followed by its competitors, and the entry rate function. In an abuse of notation, we will use the shorthand, $V(x, s|\mu, \lambda) \equiv V(x, s|\mu, \mu, \lambda)$, to refer to the expected discounted value of profits when firm i follows the same strategy μ as its competitors.

An equilibrium to our model comprises an investment/exit strategy $\mu = (\iota, \rho) \in \mathcal{M}$, and an entry rate function $\lambda \in \Lambda$ that satisfy the following conditions:

1. Incumbent firm strategies represent a MPE:

$$\sup_{\mu' \in \mathcal{M}} V(x, s|\mu', \mu, \lambda) = V(x, s|\mu, \lambda) \qquad \forall x \in \mathbb{N},\ \forall s \in \overline{\mathcal{S}}. \tag{2.1}$$

2. At each state, either entrants have zero expected profits or the entry rate is zero (or both):

$$\begin{aligned} \textstyle\sum_{s\in\overline{\mathcal{S}}} \lambda(s)\left(\beta E_{\mu,\lambda}\left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s\right] - \kappa\right) &= 0 \\ \beta E_{\mu,\lambda}\left[V(x^e, s_{-i,t+1}|\mu, \lambda)|s_t = s\right] - \kappa &\leq 0 \qquad \forall s \in \overline{\mathcal{S}} \\ \lambda(s) &\geq 0 \qquad \forall s \in \overline{\mathcal{S}}. \end{aligned}$$

Weintraub, Benkard, and Van Roy (2008) show that the supremum in part 1 of the definition above can always be attained simultaneously for all x and s by a common strategy μ'. They also discuss that existence of MPE can be establish using similar arguments to previous work (like Doraszelski and Satterthwaite (2007)).

Dynamic programming algorithms can be used to optimize firm strategies, and equilibria to our model can be computed via their iterative application. However, these algorithms require compute time and memory that grow proportionately with the number of relevant industry states, which is often intractable in contexts of practical interest. Aggregative models alleviate this difficulty.

3 An Aggregative Model

In this section we formulate an aggregative model. The model we present is motivated by and very close in spirit to that proposed by Hopenhayn (1992). Both models involve an infinite number of firms, each of which garners an infinitesimal fraction of the market. However, there are some notable differences. We mention the most significant ones. First, unlike Hopenhayn's model, in which quality is real-valued, the quality level for each firm in our model is an integer. We note, however, that the key characteristic of an continuum of firms, that there an infinite number of infinitesimal firms, is present in our model. Second, Hopenhayn restricts a-priori the set of feasible quality levels to lie on a compact set (the interval $[0, 1]$), whereas we do not. Finally, in our model, firm investment decisions are endogenously determined, whereas Hopenhayn's model focusses on entry and exit and assumes firms' trajectories follow exogenous Markov processes.

Because of averaging effects among the infinite number of firms, in an aggregative model the industry state evolves deterministically. Further, following Hopenhayn (1992), we propose an equilibrium concept in which the state of the industry is constant over time, and each firm therefore makes decisions based only on its own quality level. This corresponds to the steady-state behavior of the industry. Because each firm's strategy in equilibrium depends only on its own quality level, the present value maximization problem amounts to a single-state-variable dynamic program, alleviating the curse of dimensionality. We now define the aggregative model and its associated equilibrium concept.

3.1 Single-Period Profit Function

The aggregative model represents an asymptotic regime within the elemental model above where the number of firms and the market size become infinite. In order to consider this regime, we need to make the dependence of firm profits on the market size explicit. Market size is assumed to enter the profit function via an underlying demand system; in particular, profit would typically increase with market size for a firm at a given state (x, s). To convey the dependence of profits on market size, we denote profit functions by π_m. We consider a sequence of markets indexed by market sizes $m \in \mathbb{N}$. All other model primitives except the market size are assumed to remain constant within this sequence.

We start by making the following assumption which applies to sequences of profit functions, in addition to Assumption A.1, which applies to individual profit functions. Let $\mathcal{S}_1 = \{f \in \mathcal{S} | \sum_{x \in \mathbb{N}} f(x) = 1\}$. With some abuse of notation, we define $\pi_m(x_{it}, f_{-i,t}, n_{-i,t}) \equiv \pi_m(x_{it}, n_{-i,t} \cdot f_{-i,t})$.

Assumption 3.1.

1. $\sup_{x \in \mathbb{N}, s \in \mathcal{S}} \pi_m(x, s) = O(m)$.[3]
2. *There exists a real-valued function $\overline{\pi}$ that satisfies Assumption A.1, such that, for all $x \in \mathbb{N}$, $f \in \mathcal{S}_1$, $c > 0$, and all sequences $\{n(m) | m \in \mathbb{N}\}$,*

$$\lim_{m \to \infty} \pi_m(x, f, n(m)) = \begin{cases} \overline{\pi}(x, cf) > 0 & \textit{if } \lim_m n(m)/m = c \in (0, \infty), \\ \infty & \textit{if } \lim_m n(m)/m = 0, \\ 0 & \textit{if } \lim_m n(m)/m = \infty. \end{cases}$$

3. *For all $f \in \mathcal{S}_1$ and $c > 0$, there exists $d, e > 0$, such that, $\pi_m(x, f, cm) \leq d\overline{\pi}(x, cf) + e$, for all $x \in \mathbb{N}$ and $m \in \mathbb{N}$.*

   Further,

$$\sup_{\mu \in \mathcal{M}} E_\mu \left[\sum_{k=t}^{\infty} \beta^{k-t} \sup_{f \in \mathcal{S}_1} \overline{\pi}(x_{ik}, cf) \right] < \infty \ .$$

4. $$\sup_{m \in \mathbb{N}, x \in \mathbb{N}, f \in \mathcal{S}_1, n > 0} \left| \frac{d \ln \pi_m(x, f, n)}{d \ln n} \right| < \infty.$$

Assumption 3.1.1, which states that profits increase at most linearly with market size, should hold for virtually all relevant classes of profit functions. It is satisfied, for example, if the total disposable income of the consumer population grows linearly in market size. Assumption 3.1.2 states that, if for a given normalized industry state, the number of firms grows proportionally with the market size, then profits converge to a positive number as the market size grows to infinity. If the number of firms increases slower than the market size, profits grow to infinity; if the number of firms increases faster than the market size, profits converge to zero. The second part of Assumption 3.1.3 guarantees expected discounted profits are finite in the limit model. The first part of Assumption 3.1.3 is required for technical reasons; together with the second part, it implies that expected discounted profits remain uniformly bounded over all market sizes when the number of firms and the market size grow to infinity at the same rate. Assumption 3.1.4 requires that profits are "smooth" with respect to the number of firms and, in particular, states that the relative rate of change of profit with respect to relative changes in the number of firms is uniformly bounded. Assumptions A.1 and 3.1 are satisfied, for example, if the single-period profit function is derived from a demand system given by a logit model and where the spot market equilibrium is Nash in prices. In this case, the limit profit function

[3] In this notation, $n(m) = O(h(m))$ denotes $\limsup_m \frac{n(m)}{h(m)} < \infty$.

$\overline{\pi}$ corresponds to a logit model of monopolistic competition (Besanko, Perry, and Spady 1990).[4] In Section 4.4 we discuss this model in detail.

3.2 Dynamic Model

The state space for our aggregative model is similar to that of the elemental model. Firm quality levels take on values in $\mathbb{N}$, and the state $\overline{s}_t$ of the industry at each time t is an element of $\mathcal{S}$. Note that this industry state can take on real values and is interpreted differently from that of the elemental model. In particular, if $\overline{s}_t(x)$ is nonzero for some x, there are an infinite number of firms at quality level x. The value of $\overline{s}_t(x)$ is interpreted as the ratio of the number of firms at quality level x to the market size m. Both of these quantities are infinite, but the interpretation as a ratio is useful because it suggests that for a large market size m, the product $m\overline{s}_t(x)$ is an approximation to the number of firms that would be at quality level x.

Unlike the elemental model, in an aggregative model, there is no need for a distinction between the industry state and the state of the competitors of a specific firm. This is because there are an infinite number of firms and the industry state $\overline{s}_t$ is normalized, so if one firm is removed from the industry, the change is not noticeable.

Profits in the aggregative model are defined by the single-period profit function $\overline{\pi} : \mathbb{N} \times \mathcal{S} \rightarrow \Re_+$ (see Assumption 3.1). The value $\overline{\pi}(x, \overline{s})$ is interpreted as the profit received by a firm at quality level x if the industry is in state $\overline{s}$. As previously discussed, relative to the profit function π_m of the elemental model, the aggregative model's profit function should be viewed as a limit: $\overline{\pi}(x, \overline{s}) = \lim_{m\rightarrow\infty} \pi_m(x, m\overline{s})$. Equivalently, for $c > 0$ and $f \in \mathcal{S}_1$, we have $\overline{\pi}(x, cf) = \lim_m \pi_m(x, f, cm)$.

Similarly with the elemental model, investment and exit decisions are generated by strategies ι and ρ. The exit and investment processes are the same as in the elemental model.

Entry is controlled by an entry rate function $\overline{\lambda} \in \Lambda$. In our aggregative model, an entry rate $\overline{\lambda}(\overline{s}_t)$ represents the ratio between the number of firms entering and the market size. Clearly, this is not a formal statement, since both the number of firms and the market size are infinite. The idea, however, is that for a large market size m, the product $m\overline{\lambda}(\overline{s}_t)$ is an approximation to the number of firms entering during the tth time period.

Because there are an infinite number of firms, though each evolves stochastically, the percentage of firms that transition from any given quality level to another is deterministic. Similarly, the percentage of firms that exit is deterministic. For a strategy $\mu \in \mathcal{M}$, let $\mathcal{P}_\mu[x_{i,t+1} = x | x_{i,t} = y, \overline{s}_t]$ be the probability that a firm

[4] More precisely, this single-period profit function satisfies the assumptions in the subset of industry states where it is strictly positive.

in state y with competitors in state $\overline{s}_t$ evolve to state x next period when using strategy μ. For a strategy $\mu \in \mathcal{M}$ and entry rate function $\overline{\lambda} \in \Lambda$, let $\overline{s}_{\mu,\overline{\lambda},t}$ be the industry state at time t when all firms use strategy μ and firms enter the industry according to $\overline{\lambda}$. For a strategy $\mu \in \mathcal{M}$ and entry rate function $\overline{\lambda} \in \Lambda$, the industry state evolves deterministically according to a difference equation:

$$\overline{s}_{\mu,\overline{\lambda},t+1}(x) = \begin{cases} \sum_{y \in \mathbb{N}} \mathcal{P}_\mu[x_{i,t+1} = x | x_{i,t} = y, \overline{s}_{\mu,\overline{\lambda},t}] \overline{s}_{\mu,\overline{\lambda},t}(y) + \overline{\lambda}(\overline{s}_{\mu,\overline{\lambda},t}) & \text{if } x = x^e \\ \sum_{y \in \mathbb{N}} \mathcal{P}_\mu[x_{i,t+1} = x | x_{i,t} = y, \overline{s}_{\mu,\overline{\lambda},t}] \overline{s}_{\mu,\overline{\lambda},t}(y) & \text{otherwise.} \end{cases} \tag{3.1}$$

Completely analogous with the context of the elemental model, we define value functions associated with our aggregative model: for all $x \in \mathbb{N}$ and $\overline{s} \in \mathcal{S}$,

$$\overline{V}(x, \overline{s} | \mu', \mu, \overline{\lambda}) = E_{\mu'} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \left(\overline{\pi}(x_{ik}, \overline{s}_{\mu,\overline{\lambda},k}) - d\iota_{ik} \right) + \beta^{\tau_i - t} \phi_{i,\tau_i} \middle| x_{it} = x, \overline{s}_{\mu,\overline{\lambda},t} = \overline{s} \right].$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x and follows strategy μ', under the assumption that the industry state evolve deterministically according to equation (3.1) starting from state $\overline{s}$. Again in an abuse of notation, we will use the shorthand, $\overline{V}(x, s|\mu, \overline{\lambda}) \equiv \overline{V}(x, s|\mu, \mu, \overline{\lambda})$, to refer to the expected discounted value of profits when firm i follows the same strategy μ as its competitors.

3.3 Equilibrium

As in Hopenhayn (1992), the trajectory of industry states is deterministic and we assume it reaches a steady state. Hence, the equilibrium concept need only ensure best response strategies and a zero-profit entry condition for a single industry state. We also require that the expected time each firm spends inside the industry is finite. More precisely, a *stationary equilibrium (SE)* of our aggregative model comprises an investment/exit strategy $\mu = (\iota, \rho) \in \mathcal{M}$ and an entry rate function $\overline{\lambda} \in \Lambda$ that satisfy the following conditions:

1. The industry state is constant and the expected time each firms spends inside the industry is finite. For all $t \geq 0$, $\overline{s}_{\mu,\overline{\lambda},t} = \overline{s}_{\mu,\overline{\lambda}}$, where[5]

$$\overline{s}_{\mu,\overline{\lambda}}(x) = \begin{cases} \sum_{y \in \mathbb{N}} \mathcal{P}_\mu[x_{i,t+1} = x | x_{i,t} = y, \overline{s}_{\mu,\overline{\lambda}}] \overline{s}_{\mu,\overline{\lambda}}(y) + \overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}}) & \text{if } x = x^e \\ \sum_{y \in \mathbb{N}} \mathcal{P}_\mu[x_{i,t+1} = x | x_{i,t} = y, \overline{s}_{\mu,\overline{\lambda}}] \overline{s}_{\mu,\overline{\lambda}}(y) & \text{otherwise,} \end{cases}$$

[5] Because there is a countable number of states, the system of equations below may not have a unique solution. If that is the case, we choose the minimum non-negative solution (Kemeny, Snell, and Knapp 1976).

$$\sum_{x \in \mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x) < \infty.$$

2. Firm strategies comprise a best response:

$$\sup_{\mu' \in \mathcal{M}} \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}} | \mu', \mu, \overline{\lambda}) = \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}} | \mu, \overline{\lambda}), \qquad \forall x \in \mathbb{N}.$$

3. The value of entry is zero or the entry rate is zero (or both):

$$\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}}) \left(\beta \overline{V}(x^e, \overline{s}_{\mu,\overline{\lambda}} | \mu, \overline{\lambda}) - \kappa \right) = 0$$
$$\beta \overline{V}(x^e, \overline{s}_{\mu,\overline{\lambda}} | \mu, \overline{\lambda}) - \kappa \leq 0$$
$$\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}}) \geq 0.$$

Since the industry state is constant, the only relevant elements of an equilibrium strategy are $\mu(x, \overline{s}_{\mu,\overline{\lambda}})$, $x \in \mathbb{N}$, and $\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})$. Using the fact that expected discounted profits are bounded (Assumption 3.1.3) together with Assumption A.2, it is possible to show that the supremum in part 2 of the definition can always be attained simultaneously for all x by a common strategy $\mu'(x, \overline{s}_{\mu,\overline{\lambda}})$ (see Bhattacharya and Majumdar (1989)).

In some models there will not exist any SE with positive entry, that is, such that $\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}}) > 0$. For example, consider an industry where single-period profits exhibit increasing returns to investment, so that every incumbent firm has incentives to grow arbitrarily large (for a fixed industry state). In this case, new entrants might not be able to recover the entry cost (even if it is arbitrarily small), because they will need to invest an arbitrarily large amount of resources to catch up with incumbents. Because we are interested in situations where competition between incumbent firms is actually observed, we assume throughout the paper that there always exists a SE with positive entry and focus on this case from here on.

4 Stationary Equilibrium Approximates MPE Asymptotically

Aggregative industry models are intended to approximate behavior of elemental models with large numbers of firms. In this section we formalize this notion. In particular, we establish that if a light-tail condition is satisfied, then aggregative model SE offer close approximations to elemental model MPE when markets are large.[6]

[6]Because numbers of firms are endogenously determined in equilibrium we instead do asymptotic analysis as the market size becomes large.

First, we introduce notation that will be useful for our asymptotic analysis. Then, we define the "asymptotic Markov equilibrium property" to formalize the sense in which the approximation becomes exact. After, we introduce the light-tail condition, and we prove the main result of the paper: if the light-tail condition is satisfied, a SE of the aggregative industry model approximates MPE of the elemental model asymptotically as the market size grows. Finally, we provide a concrete example to illustrate some of the main ideas in the analysis.

4.1 Notation

We index functions and random variables associated with market size m with a superscript (m). Let $V^{(m)}$ represent the value function when the market size is m. The random variable $s_t^{(m)}$ denotes the industry state at time t when every firm uses strategy $\mu^{(m)}$ and the entry rate is $\lambda^{(m)}$. We assume that the process $\{s_t^{(m)} : t \geq 0\}$ admits an invariant distribution that we denote by $q^{(m)}$. In order to simplify our analysis, we assume that the initial industry state $s_0^{(m)}$ is sampled from $q^{(m)}$. Hence, $s_t^{(m)}$ is a stationary process; $s_t^{(m)}$ is distributed according to $q^{(m)}$ for all $t \geq 0$. Note that this assumption does not affect long-run asymptotic results since for any initial condition the process approaches stationarity as time progresses.

4.2 Asymptotic Markov Equilibrium Property

Similarly to Weintraub, Benkard, and Van Roy (2008), we define the following concept to formalize the sense in which the approximation becomes exact.

Definition 4.1. *A sequence $\{(\mu^{(m)}, \lambda^{(m)})|m \in \mathbb{N}\} \in \mathcal{M} \times \Lambda$ possesses the asymptotic Markov equilibrium (AME) property if for all $x \in \mathbb{N}$,*

$$\lim_{m\to\infty} E_{\mu^{(m)},\lambda^{(m)}} \left[\sup_{\mu' \in \mathcal{M}} V^{(m)}(x, s_t^{(m)}|\mu', \mu^{(m)}, \lambda^{(m)}) - V^{(m)}(x, s_t^{(m)}|\mu^{(m)}, \lambda^{(m)}) \right] = 0 \, .$$

Recall that the process s_t is taken to be stationary, and therefore, this expectation does not depend on t. The definition of AME assesses approximation error at each firm state x in terms of the amount by which a firm at state x can increase its actual expected net present value by deviating from the strategy $\mu^{(m)}$, and instead following a best response strategy. Recall that a MPE requires that the expression in square brackets equals zero for all states (x, s). The AME property instead considers the benefit of deviating to an optimal strategy starting from each firm state x, averaged over the invariant distribution of industry states. If a sequence possesses the AME property, then, asymptotically firms make near-optimal decisions in states that have significant probability of occurrence. Hence, MPE strategies should be well-approximated in the set of relevant states.

4.3 Light-Tail Condition and Main Result

Let $\mu \in \mathcal{M}$ and $\overline{\lambda} \in \Lambda$ be a SE. In this case, recall that firms make decisions according to $\mu(x, \overline{s}_{\mu,\overline{\lambda}})$ in state x and the entry rate equals $\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})$ at every period. We want to show that the SE approximates MPE as the market size grows. For this purposes, we define the following sequence of strategies and entry rate functions: $\mu^{(m)}(x, s) = \mu(x, \overline{s}_{\mu,\overline{\lambda}})$, for all $x \in \mathbb{N}$, $s \in \overline{\mathcal{S}}$, and $m \in \mathbb{N}$, and $\lambda^{(m)}(s) = m\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})$, for all $s \in \overline{\mathcal{S}}$ and $m \in \mathbb{N}$. The strategies $\mu^{(m)}$ are the same for all m and correspond to the SE strategy evaluated at the constant state. The entry rate $\lambda^{(m)}$ takes into account the interpretation of the entry rate in an aggregative industry model as the ratio of entering firms to the market size. It is simple to show that with these strategies and entry rates the processes $\{s_t^{(m)} : t \geq 0\}$ admit an invariant distribution for all $m \in \mathbb{N}$.

Consider a sequence of industries indexed by the market size m; in industry m firms use strategy $\mu^{(m)}$ and enter according to $\lambda^{(m)}$. In this case, the expected number of firms grows proportionally with the market size, because the entry rate does. However, even in this case with a large number of firms, if the market tends to be concentrated – for example, if the market is usually dominated by a small fraction of relatively large firms — the AME property is unlikely to hold. To ensure the AME property, we need to impose a "light-tail" condition that rules out this kind of market concentration.

Note that $\frac{d \ln \pi_m(y,f,n)}{df(x)}$ is the semi-elasticity of one period profits with respect to the fraction of firms in state x. We define the *maximal absolute semi-elasticity function*:

$$g(x) = \sup_{m \in \mathbb{N}, y \in \mathbb{N}, f \in \mathcal{S}_1, n > 0} \left| \frac{d \ln \pi_m(y, f, n)}{df(x)} \right| .$$

For each x, $g(x)$ is the maximum rate of relative change of any firm's single-period profit that could result from a small change in the fraction of firms at quality level x. Since larger competitors tend to have greater influence on firm profits, $g(x)$ typically increases with x, and can be unbounded.

We introduce the light-tail condition. Let $\overline{f} = \overline{s}_{\mu,\overline{\lambda}} / \sum_{x \in \mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x)$. Hence, $\overline{f}$ is the normalized industry state; it is the vector of expected *fraction* of firms in each state. Let $\overline{x}$ be a random variable with probability mass function $\overline{f}$. The random variable $\overline{x}$ can be interpreted as the quality level of a firm that is randomly sampled from among all incumbents while the industry state $s_t^{(m)}$ is distributed according to its invariant distribution. We introduce the light-tail condition for this model.

Assumption 4.1. *For all quality levels x, $0 < g(x) < \infty$, and $\liminf_{x \to \infty} g(x) > 0$. Further, $E\left[g(\overline{x})\right] <$*

∞.

The first part of the assumption imposes some regularity conditions over the function g. The second part controls for the appearance of dominant firms. Put simply, it requires that states where a small change in the fraction of firms has a large impact on the profits of other firms, must have a small probability under the invariant distribution, so that the expected impact of a randomly sampled incumbent is finite. In practice this typically means that very large firms (and hence high concentration) rarely occur under the invariant distribution. We provide an example in Section 4.4.[7]

We now provide the main result of the paper.

Theorem 4.1. *Under Assumptions 3.1 and 4.1, the sequence* $\{(\mu^{(m)}, \lambda^{(m)})|m \in \mathbb{N}\}$ *possesses the AME property.*

Proof. Let $\mu^{*(m)}$ be an optimal (non-oblivious) best response to $(\mu^{(m)}, \lambda^{(m)})$ in industry m; in particular,

$$V^{(m)}(x, s|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) = \sup_{\mu \in \mathcal{M}} V^{(m)}(x, s|\mu, \mu^{(m)}, \lambda^{(m)}).$$

Let

$$\hat{V}^{(m)}(x, s) = V^{(m)}(x, s|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - V^{(m)}(x, s|\mu^{(m)}, \lambda^{(m)}) \geq 0.$$

The AME property, which we set out to establish, asserts that for all $x \in \mathbb{N}$, $\lim_{m\to\infty} E_{\mu^{(m)},\lambda^{(m)}}[\hat{V}^{(m)}(x, s_t^{(m)})] = 0$.

Let us write

$$\begin{aligned} \hat{V}^{(m)}(x, s) &= \Big(V^{(m)}(x, s|\mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}}|\mu, \overline{\lambda})\Big) \\ &\quad + \Big(\overline{V}(x, \overline{s}_{\mu,\overline{\lambda}}|\mu, \overline{\lambda}) - V^{(m)}(x, s|\mu^{(m)}, \lambda^{(m)})\Big) \\ &\equiv A^{(m)}(x, s) + B^{(m)}(x, s). \end{aligned}$$

To complete the proof, we will establish that $E_{\mu^{(m)},\lambda^{(m)}}[A^{(m)}(x, s_t^{(m)})]$ converges to zero. An analogous argument that we omit for brevity establishes that $E_{\mu^{(m)},\lambda^{(m)}}[B^{(m)}(x, s_t^{(m)})]$ converges to zero as well.

[7]We note that even if the expected time a firm spends inside the industry is finite and firms can only move a finite number of quality levels per time period, firms could reach arbitrarily large states with positive probability. For this reason, the light-tail condition is not directly satisfied. Also note that $\sum_{x\in\mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x) < \infty$ does not imply the light-tail condition. For example, suppose $\overline{s}_{\mu,\overline{\lambda}}(x) \propto 1/x^2$ and $g(x) = x$. Then, $\sum_{x\in\mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x) < \infty$, but $E[g(\overline{x})] = \infty$.

Recall that in a SE, $\overline{s}_{\mu,\overline{\lambda},t} = \overline{s}_{\mu,\overline{\lambda}}$, $\forall t$. Because μ and $\overline{\lambda}$ attain a SE, we have

$$\sup_{\mu' \in \mathcal{M}} \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}} | \mu', \mu, \overline{\lambda}) = \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}} | \mu, \overline{\lambda}), \qquad \forall x \in \mathbb{N}.$$

Therefore,

$$\begin{aligned} \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}} | \mu, \overline{\lambda}) &\geq E_{\mu^{*(m)},\mu^{(m)},\lambda^{(m)}} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \left(\overline{\pi}(x_{ik}, \overline{s}_{\mu,\overline{\lambda}}) - d\iota_{ik} \right) + \beta^{\tau_i - t} \phi_{i,\tau_i} \middle| x_{it} = x, s_{-i,t} = s \right] \\ &\equiv \overline{V}^{*(m)}(x, s | \mu, \overline{\lambda}) \,. \end{aligned} \tag{4.1}$$

Hence,

$$A^{(m)}(x, s) \leq V^{(m)}(x, s | \mu^{*(m)}, \mu^{(m)}, \lambda^{(m)}) - \overline{V}^{*(m)}(x, s | \mu, \overline{\lambda}) \,.$$

Let τ_i be the time at which firm i exits, and let $\Delta_{it}^{(m)} = |\pi_m(x_{it}, s_{-i,t}^{(m)}) - \overline{\pi}(x_{ik}, \overline{s}_{\mu,\overline{\lambda}})|$. It follows that

$$A^{(m)}(x, s) \quad \leq \quad E_{\mu^{*(m)},\mu^{(m)},\lambda^{(m)}} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \Delta_{ik}^{(m)} \;\middle|\; x_{it} = x, s_{-i,t}^{(m)} = s \right]$$

Letting $q^{(m)}$ be the invariant distribution of $s_t^{(m)}$ with the strategy $\mu^{(m)}$ and the entry rate $\lambda^{(m)}$,

$$E_{\mu^{(m)},\lambda^{(m)}} \left[A^{(m)}(x, s_t^{(m)}) \right] \quad \leq \quad E_{\mu^{*(m)},\mu^{(m)},\lambda^{(m)}} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} \Delta_{ik}^{(m)} \;\middle|\; x_{it} = x, s_{-i,t}^{(m)} \sim q^{(m)} \right]$$

By the triangle inequality,

$$\Delta_{ik}^{(m)} \leq |\pi_m(x_{ik}, s_{-i,k}^{(m)}) - \pi_m(x_{ik}, \tilde{s}^{(m)})| + |\pi_m(x_{ik}, \tilde{s}^{(m)}) - \overline{\pi}(x_{ik}, \overline{s}_{\mu,\overline{\lambda}})|,$$

where $\tilde{s}^{(m)} = E[s_t^{(m)}]$.

Using Assumption 4.1 and following a similar argument to the result in Weintraub, Benkard, and Van Roy (2008) that establishes that, under a light-tail condition, a sequence of oblivious equilibria posseses the AME property, one can show that

$$E_{\mu^{*(m)},\mu^{(m)},\lambda^{(m)}} \left[\sum_{k=t}^{\tau_i} \beta^{k-t} |\pi_m(x_{ik}, s_{-i,k}^{(m)}) - \pi_m(x_{ik}, \tilde{s}^{(m)})| \middle| \, x_{it} = x, s_{-i,t}^{(m)} \sim q^{(m)} \right] \to 0.$$

Note that $\tilde{s}^{(m)} = m\overline{s}_{\mu,\overline{\lambda}}$. Hence, by Assumptions 3.1.2, for all $x \in \mathbb{N}$, $|\pi_m(x, \tilde{s}^{(m)}) - \overline{\pi}(x, \overline{s}_{\mu,\overline{\lambda}})| \to 0$.

Assumption 3.1.3 together with the dominated convergence theorem imply that,

$$E_{\mu^{*(m)},\mu^{(m)},\lambda^{(m)}}\left[\sum_{k=t}^{\tau_i}\beta^{k-t}|\pi_m(x_{ik},\tilde{s}^{(m)})-\overline{\pi}(x_{ik},\overline{s}_{\mu,\overline{\lambda}})|\,\middle|\, x_{it}=x, s_{-i,t}^{(m)}\sim q^{(m)}\right]\to 0.$$

The result follows. □

Theorem 4.1 states that under the light-tail condition, a SE of the aggregative industry model approximates MPE of the elemental model asymptotically as the market size grows.[8] In the aggregative industry model it is assumed that the industry state is constant because there are an infinite number of firms. However, in an industry with a large but finite expected number of firms, the industry state will slightly move around its average. The light-tail condition guarantees that in large markets, these movements have a small impact on expected discounted profits.

4.4 Example: Logit Demand System with Price Competition

In this section we illustrate our result through an example. We consider an industry with differentiated products, where each firm's state variable represents the quality of its product. There are m consumers in the market. In period t, consumer j receives utility u_{ijt} from consuming the good produced by firm i given by:

$$u_{ijt}=\theta_1\ln(x_{it}+1)+\theta_2\ln(Y-p_{it})+\nu_{ijt}\ ,\ i\in S_t,\ j=1,\dots,m,$$

where Y is the consumer's income and p_{it} is the price of the good produced by firm i. ν_{ijt} are i.i.d. random variables distributed Gumbel that represent unobserved characteristics for each consumer-good pair. There is also an outside good that provides consumers zero utility. We assume consumers buy at most one product each period and that they choose the product that maximizes utility. Under these assumptions our demand system is a classical logit model.

We assume that firms set prices in the spot market. If there is a constant marginal cost b, there is a unique Nash equilibrium in pure strategies, denoted p_t^* (Caplin and Nalebuff 1991). Expected profits are given by:

$$\pi_m(x_{it},s_{-i,t})=m\sigma(x_{it},s_{-i,t},p_t^*)(p_{it}^*-b)\ \forall i\in S_t\ ,$$

where σ represents the market share function from the logit model.

[8] One can show that if the sequence $(\mu^{(m)},\lambda^{(m)})$ possesses the AME property, then it is also true that $\lim_{m\to\infty}E_{\mu^{(m)},\lambda^{(m)}}\left[\beta V^{(m)}(x^e,s_t^{(m)}|\mu^{(m)},\lambda^{(m)})\right]=\kappa$. Hence, entry rates at relevant states should also by well approximated.

One can show that, if $\lim_{m\to\infty} n(m)/m = \tilde{c}$, then, for all x,

$$\lim_{m\to\infty} \pi_m(x, f, n(m)) = \overline{\pi}(x, \tilde{c}f) = \frac{N(x,\tilde{p})}{\tilde{c}\sum_{y\in\mathbb{N}} f(y)N(y,\tilde{p})}(\tilde{p}-b),$$

where $\tilde{p} = (Y + b\theta_2)/(1+\theta_2)$ and $N(y,p) = (y+1)^{\theta_1}(Y-p)^{\theta_2}$. The limit profit function $\overline{\pi}$ corresponds to a logit model of monopolistic competition (Besanko, Perry, and Spady 1990).

Weintraub, Benkard, and Van Roy (2008) show that, in this model, the function $g(x)$ takes a very simple form, $g(x) \propto x^{\theta_1}$. Therefore, the light tail condition amounts to a simple condition on the equilibrium distribution of firm states. Under our assumptions, such a condition is equivalent to a condition on the equilibrium size distribution of firms. If $\theta_1 \leq 1$ then the light-tail condition is satisfied if $E\left[\overline{x}\right] < \infty$, i.e., if the average firm quality level is finite. This condition allows for relatively "fat-tailed" distributions. For example, if $\overline{x}$ is a log-normal distribution, then the condition is satisfied. On the other hand, if $\overline{x}$ is a Pareto distribution with parameter one (which does not have a finite first moment), then the condition would not be satisfied.

It is interesting to note that in this example, for a SE with positive entry rate, $(\mu, \overline{\lambda})$, it must be that $\overline{\pi}(x, \overline{s}_{\mu,\overline{\lambda}}) > 0, \forall x \in \mathbb{N}$. If not, firms would not be able to recover the entry cost. Hence,

$$\sum_{y\in\mathbb{N}} \overline{f}(y)N(y,\tilde{p}) = (Y-\tilde{p})^{\theta_2}\sum_{y\in\mathbb{N}} \overline{f}(y)(y+1)^{\theta_1} < \infty\ . \tag{4.2}$$

For this model, the light-tail assumption is satisfied if $E\left[\overline{x}^{\theta_1}\right] = \sum_{y\in\mathbb{N}} \overline{f}(y)y^{\theta_1} < \infty$, which is implied by expression (4.2). Hence, for this model, the light-tail condition is immediatly satisfied for SE with positive entry rates. The same observation is also obtained in others models of monopolistic competition à la Dixit and Stiglitz (1977). We have not been able to prove a more general result.

5 Relationship to Oblivious Equilibrium

There is a close connection between oblivious equilibria (OE) of elemental models, as introduced in Weintraub, Benkard, and Van Roy (2008), and SE of aggregative models. In elemental models, when there are a large number of firms, simultaneous changes in individual firm quality levels average out such that the normalized industry state remains roughly constant over time. In this setting, each firm can potentially make near-optimal decisions based only on its own quality level and the long run average industry state. With this motivation, Weintraub, Benkard, and Van Roy (2008) restrict firm strategies in an elemental model so that each firm's decisions depend only on the firm's quality level. Such restricted strategies are called *oblivious* strategies since they involve decisions made without full knowledge of the circumstances — in particular, the state of the industry. In an OE firms use oblivious strategies. In this section we show that under the appropriate light tail conditions the set of OE of an elemental model approaches the set of SE of the aggregative model as the market size grows.

5.1 Definition of Oblivious Equilibrium

Let $\tilde{\mathcal{M}} \subset \mathcal{M}$ and $\tilde{\Lambda} \subset \Lambda$ denote the set of oblivious strategies and the set of oblivious entry rate functions. Since each strategy $\mu = (\iota, \rho) \in \tilde{\mathcal{M}}$ generates decisions $\iota(x, s)$ and $\rho(x, s)$ that do not depend on s, with some abuse of notation, we will often drop the second argument and write $\iota(x)$ and $\rho(x)$. Similarly, for an entry rate function $\lambda \in \tilde{\Lambda}$, we will denote by λ the real-valued entry rate that persists for all industry states.

Suppose firms make decisions according to an oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and enter according to an oblivious entry rate function $\lambda \in \tilde{\Lambda}$. Denote the expected number of firms at quality level x at time t by $\tilde{s}_t(x) = E[s_t(x)]$.[9] To abbreviate notation, we let $\tilde{s}_{\mu,\lambda}(x) = \lim_{t\to\infty} \tilde{s}_t(x)$ for $\mu \in \tilde{\mathcal{M}}$, $\lambda \in \tilde{\Lambda}$, and $x \in \mathbb{N}$. The vector $\tilde{s}_{\mu,\lambda}$ is the long-run expected number of firms at each state when firms use an oblivious strategy μ and enter according to an oblivious entry rate function λ. For an oblivious strategy $\mu \in \tilde{\mathcal{M}}$ and an oblivious entry rate function $\lambda \in \tilde{\Lambda}$ we define an *oblivious value function*

$$\tilde{V}(x|\mu', \mu, \lambda) = E_{\mu'}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\pi(x_{ik}, \tilde{s}_{\mu,\lambda}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x\right].$$

This value function should be interpreted as the expected net present value of a firm that is at quality level x and follows oblivious strategy μ', under the assumption that its competitors' state will be $\tilde{s}_{\mu,\lambda}$ for all time. Again, we abuse notation by using $\tilde{V}(x|\mu,\lambda) \equiv \tilde{V}(x|\mu,\mu,\lambda)$ to refer to the oblivious value function when firm i follows the same strategy μ as its competitors.

Following Weintraub, Benkard, and Van Roy (2008) we define an *oblivious equilibrium (OE)* as a strategy $\mu \in \tilde{\mathcal{M}}$ and an entry rate function $\lambda \in \tilde{\Lambda}$ that satisfy the following conditions:

1. Firm strategies optimize an oblivious value function:

$$\sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}(x|\mu', \mu, \lambda) = \tilde{V}(x|\mu, \lambda), \qquad \forall x \in \mathbb{N}. \tag{5.1}$$

[9] Under our assumptions, it is simple to observe that for all oblivious strategies $\mu \in \tilde{\mathcal{M}}$ the expected time a firm spends inside the industry is finite. In turn, this implies that for all $\mu \in \tilde{\mathcal{M}}$ and $\lambda \in \tilde{\Lambda}$, $\sum_{x\in\mathbb{N}} \tilde{s}_t(x) < \infty$.

2. Either the oblivious expected value of entry is zero or the entry rate is zero (or both):

$$\lambda\left(\beta\tilde{V}(x^e|\mu,\lambda)-\kappa\right)=0$$
$$\beta\tilde{V}(x^e|\mu,\lambda)-\kappa\leq 0$$
$$\lambda\geq 0.$$

Note that an oblivious equilibrium of an elemental model is similar to an aggregative model SE. The long-run expected industry state in an OE plays a similar role as the constant state $\overline{s}_{\mu,\overline{\lambda}}$ in a SE. Recall that given the constant state $\overline{s}_{\mu,\overline{\lambda}}$, the only relevant elements of a SE strategy and entry rate function are $\mu(x,\overline{s}_{\mu,\overline{\lambda}}),\ x\in\mathbb{N}$, and $\overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})$, respectively. Hence, the relevant elements of a SE strategy and entry rate function constitute an oblivious strategy and oblivious entry rate function, respectively. An aggregative model SE can be understood as the OE of that model. The main difference between OE and aggregative model SE is that OE is defined for an elemental model with a finite number of firms and a finite market size.

Similarly to the case of SE, one can show that the supremum in part 1 of the definition above can always be attained simultaneously for all x by a common strategy μ'. As Weintraub, Benkard, and Van Roy (2008) argue, it is possible to show that an OE with positive entry rate always exists under mild technical conditions and provided that the entry cost is not prohibitively high.

5.2 Asymptotic Results for Oblivious Equilibria

Consider a sequence of elemental models indexed by the market size $m\in\mathbb{N}$. An elemental model with market size m has an associated profit function π_m. All other model primitives remain the same along the sequence. We let $(\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)})$ denote an oblivious equilibrium for market size m. To further abbreviate notation we denote the expected industry state associated with $(\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)})$ by $\tilde{s}^{(m)}\equiv\tilde{s}_{\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)}}$. The random variable $s_t^{(m)}$ denotes the industry state at time t when every firm uses strategy $\tilde{\mu}^{(m)}$ and the entry rate is $\tilde{\lambda}^{(m)}$. Like before, we assume $s_t^{(m)}$ is distributed according to its invariant distribution for all $t\geq 0$.[10]

It will be helpful to decompose $s_t^{(m)}$ according to $s_t^{(m)}=f_t^{(m)}n_t^{(m)}$, where $f_t^{(m)}$ is the random vector that represents the fraction of firms in each state and $n_t^{(m)}$ is the total number of firms, respectively. Similarly, let $\tilde{f}^{(m)}\equiv E[f_t^{(m)}]$ denote the expected fraction of firms in each state and $\tilde{n}^{(m)}\equiv E[n_t^{(m)}]=\sum_{x\in\mathbb{N}}\tilde{s}^{(m)}(x)$ denote the expected number of firms. It is easy to check that $\tilde{f}^{(m)}=\frac{\tilde{s}^{(m)}}{\tilde{n}^{(m)}}$. We also define $\tilde{T}^{(m)}\equiv\tilde{n}^{(m)}/\tilde{\lambda}^{(m)}$, which represents the expected time a firm spends inside the industry when using strategy $\tilde{\mu}^{(m)}$.

[10]Weintraub, Benkard, and Van Roy (2008) show that $\{s_t^{(m)}:t\geq 0\}$ admits an invariant distribution.

Weintraub, Benkard, and Van Roy (2008) show that if a light-tail condition is satisfied, then oblivious equilibria well-approximate MPE as the market size grows. For each m, let $\tilde{x}^{(m)} \sim \tilde{f}^{(m)}$, that is, $\tilde{x}^{(m)}$ is a random variable with probability mass function $\tilde{f}^{(m)}$. The random variable $\tilde{x}^{(m)}$ can be interpreted as the quality level of a firm that is randomly sampled from among all incumbents while the industry state is distributed according to its invariant distribution. They make the following assumption.

Assumption 5.1. *For all quality levels x, $0 < g(x) < \infty$, and $\liminf_{x\to\infty} g(x) > 0$. For all $\epsilon > 0$, there exists a quality level z such that*

$$E\left[g(\tilde{x}^{(m)})\mathbf{1}_{\{\tilde{x}^{(m)}>z\}}\right] \leq \epsilon,$$

for all market sizes m.

Assumption 5.1 is similar to Assumption 4.1. If $g(x)$ is increasing and unbounded, then if there exists $\gamma > 0$, such that, $\sup_m E[g\left(\tilde{x}^{(m)}\right)^{1+\gamma}] < \infty$, then the second part of Assumption 5.1 is satisfied.[11] The condition is slightly stronger than requiring uniformly bounded first moments of $g(\tilde{x}^{(m)})$. In the special case where there exists a random variable $\tilde{x}$, such that, $\tilde{x}^{(m)} = \tilde{x}$, for all m, then the second part of Assumption 5.1 is equivalent to $E\left[g(\tilde{x})\right] < \infty$. Weintraub, Benkard, and Van Roy (2008) show the following result.

Theorem 5.1. *Under Assumptions 3.1, and 5.1, the sequence $\{(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})|m \in \mathbb{N}\}$ of oblivious equilibria possesses the AME property.*

The result is the analog to Theorem 4.1 for a sequence of OE in elemental models.

Additionally, we prove the following result that we will use below. First, we define $\|f\|_{1,g} = \sum_x |f(x)|g(x)$. Note that if $f \in \mathcal{S}_1$ and X is a random variable with distribution f, then $E[g(X)] = \|f\|_{1,g}$. Let $A = \{f|f \in \mathcal{S}_1, \|f\|_{1,g} < \infty\}$ be a normed space endowed with the norm $\|\cdot\|_{1,g}$. If $f \in A$, we say f is light-tailed. Let $B = \{\tilde{f}^{(m)}|m \in \mathbb{N}\}$.

Theorem 5.2. *Suppose Assumptions 3.1, and 5.1 hold. Then,*

1. *The closure of $B \subseteq A$ is compact. Hence, the sequence of expected normalized industry states, $\{\tilde{f}^{(m)}|m \in \mathbb{N}\}$, has a convergent subsequence to an element in A, that is, to a light-tailed distribution.*
2. *Asymptotically, the expected number of firms $\tilde{n}^{(m)}$ and the OE entry rate $\tilde{\lambda}^{(m)}$ grow proportionally with the market size m.*
3. *The expected time inside the industry $\tilde{T}^{(m)}$ remains uniformly bounded over all market sizes.*

[11] In this case, the second part of Assumption 5.1 is equivalent to the uniform integrability of the sequence of random variables $\{g(\tilde{x}^{(m)}) : m \in \mathbb{N}\}$.

Theorem 5.2.1 establishes that in a light-tailed sequence of OE, $\{\tilde{f}^{(m)}|m \in \mathbb{N}\}$ has a subsequence that converges to a light-tailed distribution. If the sequence of expected normalized industry states $\{\tilde{f}^{(m)}|m \in \mathbb{N}\}$ has a unique accumulation point, then it converges. Additionally, by the second part of the theorem, the expected number of firms grows proportionally to the market size asymptotically. In the limit the resulting market structure shares an important characteristic with the market structure assumed in an aggregative industry model; in each quality level there will be an infinite number of firms. These observations underscore the close connection between elemental model OE and aggregative model SE. The proof of this result can be found in the Appendix.

5.3 Upper-Hemicontinuity

To further formalize the relation between elemental model OE and aggregative model SE, we show that under the appropriate light-tail conditions the set of OE of an elemental model approaches the set of SE of the aggregative industry model as the market size grows. More precisely, we first show that if a sequence of OE satisfies the light-tail condition, then it converges to a light-tailed aggregative model SE. Recall that aggregative model SE can be understood as the OE of the model. Therefore, the previous statement corresponds to the upper-hemicontinuity of the OE correspondence at the point where number of firms and market size become infinite, when the sequence of OE satisfies the light-tail condition.

Recall that Theorem 5.2 states that if the light-tail condition is satisfied, then the sequence $\tilde{f}^{(m)}$ is contained in a compact set and that $\tilde{n}^{(m)}$ grows proportionally to the market size. To simplify our analysis, we will further assume that $\tilde{f}^{(m)}$ and $\tilde{n}^{(m)}/m$ are "well-behaved" sequences in the sense that they have one accumulation point each.

Assumption 5.2. *The sequences $\{\tilde{f}^{(m)}|m \in \mathbb{N}\}$ and $\{\tilde{n}^{(m)}/m|m \in \mathbb{N}\}$ have one accumulation point each.*

Assumptions 3.1, 5.1, and 5.2 together with Theorem 5.2 imply that both sequences converge. Let $\tilde{f} \equiv \lim_m \tilde{f}^{(m)}$ (with convergence in the $\|\cdot\|_{1,g}$ norm), and $\tilde{c} \equiv \lim_m \tilde{n}^{(m)}/m$. Note that by Theorem 5.2, $\|\tilde{f}\|_{1,g} < \infty$.

To state the next result we need few more assumptions. Let $T^{(m)}$ be the random variable that represents the time a firm spends inside industry (m) when using strategy $\tilde{\mu}^{(m)}$. By definition, $\tilde{T}^{(m)} = E[T^{(m)}]$. In Theorem 5.2 we established that, under the light-tail condition, the sequence of random variables $T^{(m)}$ has uniformly bounded first moments, that is, $\sup_m E[T^{(m)}] < \infty$. For technical reasons, we introduce a slightly stronger assumption.

Assumption 5.3. *The sequence of random variables $\{T^{(m)}|m \in \mathbb{N}\}$ is uniformly integrable.*

Note that if there exists $\gamma > 0$, such that, $\sup_m E\left[\left(T^{(m)}\right)^{1+\gamma}\right] < \infty$, then Assumption 5.3 holds. The condition is slightly stronger than requiring uniformly bounded first moments of $\tilde{T}^{(m)}$.

Finally, we also need to strengthen Assumption 3.1.

Assumption 5.4. *Assumption 3.1 holds. Moreover, suppose that for all $c > 0$ and for all sequences $s^{(m)} = f^{(m)}n^{(m)}$ with $\lim_{m\to\infty} n^{(m)}/m = c$, there exists $d, e > 0$ and $k > 0$, such that, $\pi_m(x, s^{(m)}) \leq dx^k + e$, for all $x \in \mathbb{N}$ and $m \in \mathbb{N}$.*

The assumption imposes that the profit function does not grow faster than a polynomial as the quality level grows, when number of firms grows at the same rate as the market size. The assumption is used to show that the Bellman equation on a countable infinite space associated with the limit profit function $\overline{\pi}$ has a unique solution. Note that Assumption 5.4 does not imply Assumption 5.1. It is simple to construct examples that satisfy Assumption 5.4 that do not generate a light-tailed sequence of OE (e.g., cases where $\overline{\pi}(x, s)$ grows superlinearly in x). We have the following result.

Theorem 5.3. *Suppose Assumptions 5.1, 5.2, 5.3, and 5.4 hold. Then, the sequence of OE $\{(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})|m \in \mathbb{N}\}$ converges to a light-tailed aggregative model SE. Formally, there exists an aggregative model SE $(\mu, \overline{\lambda})$, such that, for all $x \in \mathbb{N}$, $\lim_m \tilde{\mu}^{(m)}(x) = \mu(x, \overline{s}_{\mu,\overline{\lambda}})$ and $\lim_m \tilde{\lambda}^{(m)}/m = \overline{\lambda}(s_{\mu,\overline{\lambda}})$. Morever, $\overline{s}_{\mu,\overline{\lambda}} = \tilde{c}\tilde{f}$.*

Proof. We start by stating preliminary lemmas that we prove in the Appendix.

Lemma 5.1. *Suppose Assumptions 5.1, 5.2, and 5.4 hold. Then, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \pi_m(x, \tilde{s}^{(m)}) = \overline{\pi}(x, \tilde{c}\tilde{f})$. Moreover, there exists $d, e > 0$ and $k > 0$, such that, $\overline{\pi}(x, \tilde{c}\tilde{f}) \leq dx^k + e$, for all $x \in \mathbb{N}$.*

Now, we state that the oblivious equilibrium value function $\tilde{V}^{(m)}$ and the oblivious equilibrium strategy $\tilde{\mu}^{(m)}$ converge to the optimal value function and optimal strategy, respectively, of a firm's dynamic programming problem with profits given by $\overline{\pi}(x, \tilde{c}\tilde{f})$. To abbreviate, with some abuse of notation, we let $\tilde{V}^{(m)}(x) \equiv \tilde{V}^{(m)}(x|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})$. For all $x \in \mathbb{N}$, let

$$\tilde{V}^{(\infty)}(x) = \sup_{\mu' \in \tilde{\mathcal{M}}} E_{\mu'}\left[\sum_{k=t}^{\tau_i} \beta^{k-t}\left(\overline{\pi}(x_{ik}, \tilde{c}\tilde{f}) - d\iota_{ik}\right) + \beta^{\tau_i - t}\phi_{i,\tau_i}\middle| x_{it} = x\right],$$

and let $\tilde{\mu}^{(\infty)} \in \tilde{\mathcal{M}}$ be the strategy that achieves the maximum (which are well defined by Assumption A.2 and 3.1.3, and the results in Bhattacharya and Majumdar (1989)). Hence, the value function $\tilde{V}^{(\infty)}$ and the strategy $\tilde{\mu}^{(\infty)}$ are the optimal value function and optimal strategy, respectively, of a firm's dynamic programming problem with profits given by $\overline{\pi}(x, \tilde{c}\tilde{f})$.

Lemma 5.2. *Suppose Assumptions 5.1, 5.2, and 5.4 hold. Then, for all* $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{V}^{(m)}(x) = \tilde{V}^{(\infty)}(x)$, *and* $\lim_{m\to\infty} \tilde{\mu}^{(m)}(x) = \tilde{\mu}^{(\infty)}(x)$.

We prove one final lemma, about the expected time firms spend inside the industry. Like in the proof of Theorem 5.2, we define, for all $x \in \mathbb{N}$, $\tilde{T}_x^{(m)}$ as the expected number of visits a firm makes to state x when using strategy $\tilde{\mu}^{(m)}$. Note that $\tilde{T}^{(m)} = \sum_{x\in\mathbb{N}} \tilde{T}_x^{(m)}$. Similarly, we define $\tilde{T}_x^{(\infty)}$ and $\tilde{T}^{(\infty)}$ as the expected number of visits a firm makes to state x and the expected time the firm spends inside the industry, respectively, when using strategy $\tilde{\mu}^{(\infty)}$. In the next lemma, we show that the expected number of visits to a state under oblivious equilibrium strategies $\tilde{\mu}^{(m)}$ converges to the expected number of visits under strategy $\tilde{\mu}^{(\infty)}$.

Lemma 5.3. *Suppose Assumptions 5.1, 5.2, 5.3, and 5.4 hold. Then, for all* $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{T}_x^{(m)} = \tilde{T}_x^{(\infty)}$. *Moreover,* $\lim_{m\to\infty} \tilde{T}^{(m)} = \tilde{T}^{(\infty)} < \infty$.

We use the previous lemmas to prove the theorem.

Proof of Theorem 5.3.

Lemmas 5.1 and 5.2 establish that, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{\mu}^{(m)}(x) = \tilde{\mu}^{(\infty)}(x)$ and $\lim_{m\to\infty} \tilde{V}^{(m)}(x) = \tilde{V}^{(\infty)}(x)$. It is simple to observe that $\tilde{n}^{(m)} = \tilde{\lambda}^{(m)}\tilde{T}^{(m)}$. By Assumption 5.2 and Lemma 5.3, $\lim_m \tilde{\lambda}^{(m)}/m = \tilde{c}/\tilde{T}^{(\infty)} \equiv \tilde{\lambda}^{(\infty)}$. We prove that if $\mu(x, \overline{s}_{\mu,\overline{\lambda}}) \equiv \tilde{\mu}^{(\infty)}(x)$, for all $x \in \mathbb{N}$, and $\overline{\lambda}(s_{\mu,\overline{\lambda}}) \equiv \tilde{\lambda}^{(\infty)}$, then $(\mu, \overline{\lambda})$ constitute an aggregative model SE.

First, note that because $\lim_m \tilde{V}^{(m)}(x^e) = \tilde{V}^{(\infty)}(x^e)$ and $\tilde{V}^{(m)} = \kappa/\beta$, for all m, it must be that $\tilde{V}^{(\infty)}(x^e) = \kappa/\beta$. Second, recall that the value function $\tilde{V}^{(\infty)}$ and the strategy $\tilde{\mu}^{(\infty)}$ are the optimal value function and optimal strategy, respectively, of a firm's dynamic programming problem with profits given by $\overline{\pi}(x, \tilde{c}\tilde{f})$.

Therefore, to establish that $(\mu, \overline{\lambda})$ defined above constitute an aggregative model SE it is enough to show that $\overline{s}_{\mu,\overline{\lambda}} = \tilde{c}\tilde{f}$. It is simple to observe that the vector $\overline{s}_{\mu,\overline{\lambda}} = \overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})(\tilde{T}_x^{(\infty)})_{x\in\mathbb{N}} = \tilde{c}/\tilde{T}^{(\infty)} \cdot (\tilde{T}_x^{(\infty)})_{x\in\mathbb{N}}$. By Lemma 5.3, $\tilde{T}_x^{(\infty)} = \lim_m \tilde{T}_x^{(m)}$ and $\tilde{T}^{(\infty)} = \lim_m \tilde{T}^{(m)}$. Because $\tilde{f}^{(m)} = (\tilde{T}_x^{(m)})_{x\in\mathbb{N}}/\tilde{T}^{(m)}$, we have that $\tilde{f} = (\tilde{T}_x^{(\infty)})_{x\in\mathbb{N}}/\tilde{T}^{(\infty)}$. Hence, $\overline{s}_{\mu,\overline{\lambda}} = \tilde{c}\tilde{f}$ as needed. □

5.4 Lower-Hemicontinuity

In this section we show that all sequences of oblivious strategies that approach a light-tailed SE satisfy the OE conditions asymptotically. This result is related to the lower-hemicontinuity of the OE correspondence at the point where number of firms and market size become infinite, when OE in the aggregative model satisfies the light-tail condition. We begin with some definitions.

Definition 5.1. *A sequence* $\{(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})|m \in \mathbb{N}\} \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$ *possesses the asymptotic oblivious equilibrium (AOE) property, if for all* $x \in \mathbb{N}$,

$$\lim_{m\to\infty} \sup_{\mu' \in \tilde{\mathcal{M}}} \tilde{V}^{(m)}(x|\mu', \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) - \tilde{V}^{(m)}(x|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = 0 \quad \textit{and}$$

$$\lim_{m\to\infty} \beta \tilde{V}^{(m)}(x^e|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = \kappa \, .$$

The AOE property requires that the OE conditions are satisfied asymptotically.

Definition 5.2. *We say that the sequence* $\{(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})|m \in \mathbb{N}\} \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$ *of oblivious strategies and entry rate functions converges to a SE* $(\mu, \overline{\lambda})$ *if for all* $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{\mu}^{(m)}(x) = \mu(x, \overline{s}_{\mu,\overline{\lambda}})$, $\lim_{m\to\infty} \tilde{\lambda}^{(m)}/m = \overline{\lambda}(\overline{s}_{\mu,\overline{\lambda}})$, $\lim_{m\to\infty} \| \tilde{s}_{\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)}} / \sum_{x\in\mathbb{N}} \tilde{s}_{\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)}}(x) - \overline{s}_{\mu,\overline{\lambda}} / \sum_{x\in\mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x) \|_{1,g} = 0$, *and*
$\lim_{m\to\infty} \sum_{x\in\mathbb{N}} \tilde{s}_{\tilde{\mu}^{(m)},\tilde{\lambda}^{(m)}}(x)/m = \sum_{x\in\mathbb{N}} \overline{s}_{\mu,\overline{\lambda}}(x)$.

The definition establishes a norm under which a sequence of oblivious strategies and entry rate functions converges: strategies, entry rates, associated vector of expected fraction of firms, and expected number of firms should converge in an appropriate sense. We have the following result.

Theorem 5.4. *Suppose Assumption 5.4 holds. Suppose* $(\mu, \overline{\lambda})$ *is an aggregative model SE that satisfies Assumption 4.1. Let* $\{(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})|m \in \mathbb{N}\} \in \tilde{\mathcal{M}} \times \tilde{\Lambda}$ *be a sequence of oblivious strategies and entry rate functions that converges to* $(\mu, \overline{\lambda})$. *Then,* $(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})$ *possesses the AOE property.*

Proof. The argument in Lemma 5.1 establishes that $\lim_m \pi_m(x, \tilde{s}_{\tilde{\mu}(m),\tilde{\lambda}^{(m)}}) = \overline{\pi}(x, \overline{s}_{\mu,\overline{\lambda}})$. Using a similar argument to Lemma 5.2 one can show that, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \sup_{\mu'\in\tilde{\mathcal{M}}} \tilde{V}^{(m)}(x|\mu', \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}}|\mu, \overline{\lambda})$. Similarly, since, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{\mu}^{(m)}(x) = \mu(x, \overline{s}_{\mu,\overline{\lambda}})$, it follows that $\lim_{m\to\infty} \tilde{V}^{(m)}(x|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = \overline{V}(x, \overline{s}_{\mu,\overline{\lambda}}|\mu, \overline{\lambda})$. Hence, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \sup_{\mu'\in\tilde{\mathcal{M}}} \tilde{V}^{(m)}(x|\mu', \tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) - \tilde{V}^{(m)}(x|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = 0$ and $\lim_{m\to\infty} \beta \tilde{V}^{(m)}(x^e|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) = \kappa$. Therefore, $(\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)})$ possesses the AOE property. □

The result establishes a weaker property than lower-hemicontinuity of the OE correspondence at the point where number of firms and market size become infinite, because we only require that the sequences of strategies possess the AOE property (hence, that the OE conditions are satisfied asymptotically); we do not require that they are sequences of OE. On the other hand, what is shown is stronger than lower-hemicontuinity which would only require that there *exists* a sequence of OE that converges to the SE.

6 Conclusions

In this paper we provide foundations for a class of aggregative models that have become popular in the recent literature in the fields of macroeconomics, international trade, and industrial organization. Our approach serves to unify two separate threads of economic research, each of which offers an alternative approach to analyzing dynamics of an industry with heterogeneous firms. In our main result, we provided conditions under which aggregative model SE provide meaningful approximations of elemental model MPE as the market size grows. Our conditions require that the stationary equilibrium distribution of firm states exhibits a light tail condition that we define precisely. Similar conditions guarantee that the set of OE of elemental models approach the set of SE fo the aggregative model as the market size grows.

As the light tail condition is a condition on equilibrium outcomes, an interesting open question for future research is to find conditions over the model primitives that guarantee all stationary equilibria are light-tailed. In Section 4.4 we show that for a class of monopolistic competition models, the light-tail condition is satisfied for SE with positive entry rates. In these cases, SE provide meaningful approximations of MPE without further qualifications. A result along these lines may be true for a more general class of models. However, we leave the study of this conjecture for future work.

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A Appendix: Assumptions

We make several assumptions about the model primitives, beginning with the profit function. An industry state $s \in \mathcal{S}$ is said to *dominate* $s' \in \mathcal{S}$ if for all $x \in \mathbb{N}$, $\sum_{z \geq x} s(z) \geq \sum_{z \geq x} s'(z)$. We will denote this relation by $s \succeq s'$. Intuitively, competition associated with s is no weaker than competition associated with s'.

Assumption A.1.

1. *For all* $s \in \mathcal{S}$*,* $\pi(x, s)$ *is increasing in* x*.*
2. *For all* $x \in \mathbb{N}$ *and* $s, s' \in \mathcal{S}$*, if* $s \succeq s'$ *then* $\pi(x, s) \leq \pi(x, s')$*.*
3. *For all* $x \in \mathbb{N}$ *and* $s \in \mathcal{S}$*,* $\pi(x, s) > 0$*, and* $\sup_{x,s} \pi(x, s) < \infty$*.*

4. *For all $x \in \mathbb{N}$, $y \in \mathbb{N}$, and $s \in \mathcal{S}$, $\pi(x, s)$ is differentiable with respect to $s(y)$. Further, for any $x \in \mathbb{N}$, $y \in \mathbb{N}$, $s \in \mathcal{S}$, and $h \in \mathcal{S}$ such that $s + \gamma h \in \mathcal{S}$ for $\gamma > 0$ sufficiently small, if*

$$\sum_{y \in \mathbb{N}} h(y) \left| \frac{\partial \ln \pi(x, s)}{\partial s(y)} \right| < \infty,$$

then

$$\frac{d \ln \pi(x, s + \gamma h)}{d\gamma}\Big|_{\gamma=0} = \sum_{y \in \mathbb{N}} h(y) \frac{\partial \ln \pi(x, s)}{\partial s(y)}.$$

The assumptions are natural. Assumption A.1.1 ensures that increases in quality lead to increases in profit. Assumption A.1.2 states that strengthened competition cannot result in increased profit. Assumption A.1.3 ensures that profits are positive and bounded. The first part of Assumption A.1.4 requires partial differentiability of the profit function with respect to each $s(y)$. Profit functions that are "smooth", such as ones arising from random utility demand models like the logit model, will satisfy this assumption. The second part of Assumption A.1.4 is technical and essentially requires that the profit function is Fréchet differentiable.

We also make assumptions about investment and the distributions of the private shocks:

Assumption A.2.

1. *The variables $\{\phi_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and have finite expectations and well-defined density functions with support $\Re_+$.*
2. *The random variables $\{\zeta_{it} | t \geq 0, i \geq 1\}$ are i.i.d. and independent of $\{\phi_{it} | t \geq 0, i \geq 1\}$.*
3. *For all ζ, $w(\iota, \zeta)$ is nondecreasing in ι.*
4. *For all $\iota > 0$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) > 0] > 0$.*
5. *There exists a positive constant $\overline{w} \in \mathbb{N}$ such that $|w(\iota, \zeta)| \leq \overline{w}$, for all (ι, ζ). There exists a positive constant $\bar{\iota}$ such that $\iota_{it} < \bar{\iota}$, $\forall i, \forall t$.*
6. *For all $k \in \{-\overline{w}, \ldots, \overline{w}\}$, $\mathcal{P}[w(\iota, \zeta_{i,t+1}) = k]$ is continuous in ι.*
7. *The transitions generated by $w(\iota, \zeta)$ are unique investment choice admissible .*

Again the assumptions are natural and fairly weak. Assumptions A.2.1 and A.2.2 imply that investment and exit outcomes are idiosyncratic conditional on the state. Assumption A.2.3 and A.2.4 imply that investment is productive. Note that positive depreciation is neither required nor ruled out. Assumption A.2.5 places a finite bound on how much progress can be made or lost in a single period through investment. Assumption A.2.6 ensures that the impact of investment on transition probabilities is continuous. Assumption A.2.7 is an assumption introduced by Doraszelski and Satterthwaite (2007) that ensures a unique solution to the firms' investment decision problem. It is used to guarantee existence of an equilibrium in pure strategies, and is satisfied by many of the commonly used specifications in the literature.

We assume that there are a large number of potential entrants who play a symmetric mixed entry strategy. In that case one can show that the number of actual entrants is well approximated by the Poisson distribution (see Weintraub, Benkard, and Van Roy (2008) for a derivation of this result). This leads to the following assumptions:

Assumption A.3.

1. *The number of firms entering during period t is a Poisson random variable that is conditionally independent of $\{\phi_{it}, \zeta_{it} | t \geq 0, i \geq 1\}$, conditioned on s_t.*
2. *$\kappa > \beta \cdot \bar{\phi}$, where $\bar{\phi}$ is the expected net present value of entering the market, investing zero each period, and then exiting at an optimal stopping time.*

We denote the expected number of firms entering at industry state s_t, by $\lambda(s_t)$. This state-dependent entry rate will be endogenously determined, and our solution concept will require that it satisfies a zero expected profit condition. Modeling the number of entrants as a Poisson random variable has the advantage that it leads to more elegant asymptotic results. Assumption A.3.2 ensures that the sell-off value by itself is not sufficient reason to enter the industry.

B Appendix: Proofs

Proof of Theorem 5.2. *Suppose Assumptions 3.1, and 5.1 hold. Then,*

1. *The closure of $B \subseteq A$ is compact. Hence, the sequence of expected normalized industry states, $\{\tilde{f}^{(m)} | m \in \mathbb{N}\}$, has a convergent subsequence to an element in A, that is, to a light-tailed distribution.*
2. *Asymptotically, the expected number of firms $\tilde{n}^{(m)}$ and the OE entry rate $\tilde{\lambda}^{(m)}$ grow proportionally with the market size m.*
3. *The expected time inside the industry $\tilde{T}^{(m)}$ remains uniformly bounded over all market sizes.*

Proof. Part (1). It is simple to show that the space A is complete. Using Assumption 5.1, it is straightforward to prove that B is a totally bounded subset of A. Therefore, the closure of B is compact (Marsden and Hoffman 1993). The sequence $\{\tilde{f}^{(m)} | m \in \mathbb{N}\}$ has a convergent subsequence to an element in A by the Bolzano-Weierstrass Theorem.

Part (2). That the expected number of firms $\tilde{n}^{(m)}$ grows proportionally with the market size m follows by Assumptions 3.1.2 and 5.1, and the OE-zero profit condition at the entry state. If the number of firms were to grow slower than the market size, profits would blow up and the zero profit condition at the entry state would not be met. On the other hand, if the number of firms were to grow faster than the market size, profits would be driven down to zero and firms would not recover the entry cost. Additionally, we know that $\tilde{n}^{(m)} = \tilde{\lambda}^{(m)}\tilde{T}^{(m)}$. Hence, by part (3) of the theorem it must be that the OE entry rate also grows proportionally with the market size m.

Part (3). Suppose, for the sake of a contradiction, that the expected time inside the industry $\tilde{T}^{(m)}$ does not remain uniformly bounded over all market sizes. Because under our assumptions, for all m, $\tilde{T}^{(m)} < \infty$, this implies that $\limsup_{m\to\infty} \tilde{T}^{(m)} = \infty$. We will prove that in this case, for all $z \in \mathbb{N}$, $\limsup_{m\to\infty} \sum_{x\geq z} \tilde{f}^{(m)}(x) = 1$. Since by the first part of Assumption 5.1, $\liminf_{x\to\infty} g(x) > 0$, this contradicts the second part of Assumption 5.1 (light-tail assumption).

We define, for all $x \in \mathbb{N}$, $\tilde{T}_x^{(m)}$ as the expected number of visits a firm makes to state x when using strategy $\tilde{\mu}^{(m)}$. Note that $\tilde{T}^{(m)} = \sum_{x\in\mathbb{N}} \tilde{T}_x^{(m)}$.

Weintraub, Benkard, and Van Roy (2008) show that for all $x \in \mathbb{N}$, $\sup_m \tilde{V}^{(m)}(x|\tilde{\mu}^{(m)}, \tilde{\lambda}^{(m)}) < \infty$. Recall that by Assumption A.2.1, the sell-off value has support in $\Re^+$. Hence, for all $x \in \mathbb{N}$, each time a firm visits state x, there is a probability uniformly bounded away from zero over all market sizes, that the firm will exit the industry. The exit process from state x can be represented as a geometric random variable. It follows that, for all $x \in \mathbb{N}$, $\sup_m \tilde{T}_x^{(m)} < \infty$.

We can write $\tilde{f}^{(m)}(x) = \tilde{T}_x^{(m)}/\tilde{T}^{(m)}$. Therefore, for all $z \in \mathbb{N}$,

$$\sum_{x\geq z} \tilde{f}^{(m)}(x) = \frac{\sum_{x\geq z} \tilde{T}_x^{(m)}}{\sum_{x\in\mathbb{N}} \tilde{T}_x^{(m)}} .$$

Because for all $x \in \mathbb{N}$, $\sup_m \tilde{T}_x^{(m)} < \infty$ and, by assumption, $\limsup_{m\to\infty} \tilde{T}^{(m)} = \infty$, it must be that, for all $z \in \mathbb{N}$, $\limsup_m \sum_{x\geq z} \tilde{T}^{(m)}(x) = \infty$. Hence, for all $z \in \mathbb{N}$, $\limsup_m \sum_{x\geq z} \tilde{f}^{(m)}(x) = 1$, and the second part of Assumption 5.1 (light-tail assumption) is violated. Therefore, it must be that the expected time inside the industry $\tilde{T}^{(m)}$ remains uniformly bounded over all market sizes. □

Proof of Lemma 5.1. *Suppose Assumptions 5.1, 5.2, and 5.4 hold. Then, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \pi_m(x, \tilde{s}^{(m)}) = \overline{\pi}(x, \tilde{c}\tilde{f})$. Moreover, there exists $d, e > 0$ and $k > 0$, such that, $\overline{\pi}(x, \tilde{c}\tilde{f}) \leq dx^k + e$, for all $x \in \mathbb{N}$.*

Proof.

$$\begin{aligned} \pi_m(x, \tilde{s}^{(m)}) - \overline{\pi}(x, \tilde{c}\tilde{f}) &= \Big(\pi_m(x, \tilde{f}^{(m)}, \tilde{n}^{(m)}) - \pi_m(x, \tilde{f}, \tilde{n}^{(m)})\Big) \\ &+ \Big(\pi_m(x, \tilde{f}, \tilde{n}^{(m)}) - \overline{\pi}(x, \tilde{c}\tilde{f})\Big) \end{aligned} \tag{B.1}$$

The second term trivially converges to zero by Assumptions 3.1.2 and 5.2, and Theorem 5.2. For the first term, consider that by Assumption 5.2 and Theorem 5.2, $\lim_{m\to\infty} \parallel \tilde{f}^{(m)} - \tilde{f} \parallel_{1,g} = 0$. Then, by Assumptions A.1.3 and A.1.4, and Lemma A.10 in Weintraub, Benkard, and Van Roy (2008), it follows that $\lim_{m\to\infty} \pi_m(x, \tilde{f}^{(m)}, \tilde{n}^{(m)})/\pi_m(x, \tilde{f}, \tilde{n}^{(m)}) = 1$. By Assumption 3.1.2, $\sup_m \pi_m(x, \tilde{f}, \tilde{n}^{(m)}) < \infty$. Therefore, $\lim_{m\to\infty} \pi_m(x, \tilde{f}^{(m)}, \tilde{n}^{(m)}) - \pi_m(x, \tilde{f}, \tilde{n}^{(m)}) = 0$. That there exists $d, e > 0$ and $k > 0$, such that, $\overline{\pi}(x, \tilde{c}\tilde{f}) \leq dx^k + e$, for all $x \in \mathbb{N}$, follows directly from Assumption 5.4. $\square$

Proof of Lemma 5.2. *Suppose Assumptions 5.1, 5.2, and 5.4 hold. Then, for all* $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{V}^{(m)}(x) = \tilde{V}^{(\infty)}(x)$, *and* $\lim_{m\to\infty} \tilde{\mu}^{(m)}(x) = \tilde{\mu}^{(\infty)}(x)$.

Proof. We prove convergence of the value functions. The proof of convergence of the strategy functions is analogous. The proof follows two main steps. First, we show that $\tilde{V}^{(m)}$ lies on a compact set. Then, we prove that the limit of any convergent subsequence of $\tilde{V}^{(m)}$ must be $\tilde{V}^{(\infty)}$. This implies that, for all $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{V}^{(m)}(x) = \tilde{V}^{(\infty)}(x)$.

Weintraub, Benkard, and Van Roy (2008) show that $\overline{v}(x) \equiv \sup_m \tilde{V}^{(m)}(x) < \infty$, for all x. Therefore, $V^{(m)}(x) \in [0, \overline{v}(x)]$, for all m, x. By Tychonoff Theorem (Royden 1988), the product set $\times_{x\in\mathbb{N}}[0, \overline{v}(x)]$ is compact in the topology of pointwise convergence (or the product topology).

Suppose $\tilde{V}^{(m_n)}$ is a converging subsequence of $\tilde{V}^{(m)}$. Let, for all $x \in \mathbb{N}$, $\tilde{v}^{(\infty)}(x)$ be the (pointwise) limit of $\tilde{V}^{(m_n)}(x)$. We prove that, for all $x \in \mathbb{N}$, $\tilde{v}^{(\infty)}(x) = \tilde{V}^{(\infty)}(x)$. That is, the limit of any convergent subsequence of $\tilde{V}^{(m)}$ is $\tilde{V}^{(\infty)}$. Let us define the following sequence of dynamic programming operators for value functions $V \leq \overline{v}$:

$$
\begin{aligned}
F^{(m)}V(x) &= \pi_m(x, \tilde{s}^{(m)}) + E\left[\max\left\{\phi_{it}, \sup_{\iota\geq 0}\left(-d\iota + \beta E_{\mu,\lambda}\left[V(x_{i,t+1})|x_{it} = x, \iota_{it} = \iota\right]\right)\right\}\right] \\
&\equiv \pi_m(x, \tilde{s}^{(m)}) + QV(x) \ , \qquad \text{(B.2)}
\end{aligned}
$$

for all $x \in \mathbb{N}$. The operator $F^{(\infty)}$ is defined as above, but with the profit function $\overline{\pi}(x, \tilde{c}\tilde{f})$.

Using Assumption A.2, the operator Q can be written as:

$$
\begin{aligned}
QV(x) &= \sup_{\substack{\iota\in[0,\overline{\iota}] \\ \rho\in[0,\overline{v}(x)]}} \left(-d\iota + \beta \sum_{y\in[[x-\overline{w}]^+, x+\overline{w}]} \mathcal{P}\left[x_{i,t+1} = y \middle| x_{i,t} = x, \iota_{it} = \iota\right] V(y)\right) \\
&\times \ \mathcal{P}[\phi_{it} < \rho] \ + \ E[\phi_{it} \middle| \phi_{it} \geq \rho]\mathcal{P}[\phi_{it} \geq \rho] \\
&\equiv \sup_{\substack{\iota\in[0,\overline{\iota}] \\ \rho\in[0,\overline{v}(x)]}} f_x(\iota, \rho, V) \ .
\end{aligned}
$$

It is simple to check that, by Assumption A.2, the operator f_x is continuous in the topology of pointwise convergence. Hence, by Berge's maximum theorem, the operator $QV(x)$ is continuous in the topology of pointwise convergence. Additionally, by Lemma 5.1, for all $x \in \mathbb{N}$, $\lim_m \pi_m(x, \tilde{s}^{(m)}) = \overline{\pi}(x, \tilde{c}\tilde{f})$. Therefore, for all $x \in \mathbb{N}$, $\lim_{n\to\infty} F^{(m_n)}\tilde{V}^{(m_n)}(x) = F^{(\infty)}\tilde{v}^{(\infty)}(x)$. Additionally, $\tilde{V}^{(m_n)}$ is the oblivious equilibrium value function, therefore, for all $n \in \mathbb{N}$, and for all $x \in \mathbb{N}$, it must solve Bellman's equation: $F^{(m_n)}\tilde{V}^{(m_n)}(x) = \tilde{V}^{(m_n)}(x)$ (because of Assumptions A.1.3 and A.2, and the results in Bertsekas (2001)). We conclude that, for all $x \in \mathbb{N}$, $F^{(\infty)}\tilde{v}^{(\infty)}(x) = \tilde{v}^{(\infty)}(x)$. Moreover, using the fact that $\overline{\pi}(x, \tilde{c}\tilde{f})$ does not grow faster than a polynomial as x grows (by Lemma 5.1), Theorem 6.10.4 in Puterman (1994) implies that $\tilde{V}^{(\infty)}$ is the unique solution of $F^{(\infty)}v(x) = v(x)$, for all $x \in \mathbb{N}$. Therefore, for all $x \in \mathbb{N}$, $\tilde{v}^{(\infty)}(x) = \tilde{V}^{(\infty)}(x)$. That is, the limit of any convergent subsequence of $\tilde{V}^{(m)}$ must be $\tilde{V}^{(\infty)}$. The result follows. □

Proof of Lemma 5.3. *Suppose Assumptions 5.1, 5.2, 5.3, and 5.4 hold. Then, for all* $x \in \mathbb{N}$, $\lim_{m\to\infty} \tilde{T}_x^{(m)} = \tilde{T}_x^{(\infty)}$. *Moreover,* $\lim_{m\to\infty} \tilde{T}^{(m)} = \tilde{T}^{(\infty)} < \infty$.

Proof. First we prove that $\lim_{m\to\infty} \tilde{T}_x^{(m)} = \tilde{T}_x^{(\infty)}$. Let $P_\mu^t(x, y)$ be the probability that a firm in state x will be in state y, t time periods from now when using strategy μ. The expected number of visits to state x can be written as $\tilde{T}_x^{(m)} = \sum_{t=0}^{\infty} P_{\tilde{\mu}^{(m)}}^t(x^e, x) = \sum_{t=0}^{T} P_{\tilde{\mu}^{(m)}}^t(x^e, x) + \sum_{t>T} P_{\tilde{\mu}^{(m)}}^t(x^e, x)$. Using Assumption A.2 and Lemma 5.2, it is simple to show that, for all $t,\ x \in \mathbb{N}$, $\lim_{m\to\infty} P_{\tilde{\mu}^{(m)}}^t(x^e, x) = P_{\tilde{\mu}^{(\infty)}}^t(x^e, x)$. Clearly, for all $x, t, m \in \mathbb{N}$, $P_{\tilde{\mu}^{(m)}}^t(x^e, x) \le \mathcal{P}[T^{(m)} \ge t]$. If a firm is in state x after t time periods, it must be inside the industry after t time periods. It is simple to show that Assumption 5.3 implies that $\lim_{T\to\infty} \sup_m \sum_{t>T} \mathcal{P}[T^{(m)} \ge t] = 0$. Therefore, $\lim_{T\to\infty} \sup_m \sum_{t>T} P_{\tilde{\mu}^{(m)}}^t(x^e, x) = 0$. It follows that $\lim_{m\to\infty} \tilde{T}_x^{(m)} = \sum_{t=0}^{\infty} P_{\tilde{\mu}^{(\infty)}}^t(x^e, x) = \tilde{T}_x^{(\infty)}$.

Now we prove that $\lim_{m\to\infty} \tilde{T}^{(m)} = \tilde{T}^{(\infty)} < \infty$. Provided the limit exists, $\tilde{T}^{(\infty)}$ is finite, because $\tilde{T}^{(m)}$ remains uniformly bounded over all market sizes. Note that $\tilde{T}^{(m)} = \sum_{x\in\mathbb{N}} \tilde{T}_x^{(m)}$ and $\tilde{f}^{(m)}(x) = \tilde{T}^{(m)}(x)/\tilde{T}^{(m)}$. Assumption 5.1 implies $\lim_{z\to\infty} \sup_m \sum_{x>z} \tilde{f}^{(m)}(x) = 0$. This together with $\sup_m \tilde{T}^{(m)} < \infty$ implies $\lim_{z\to\infty} \sup_m \sum_{x>z} \tilde{T}_x^{(m)} = 0$. Therefore, $\lim_{m\to\infty} \tilde{T}^{(m)} = \sum_{x\in\mathbb{N}} \tilde{T}_x^{(\infty)} = \tilde{T}^{(\infty)} < \infty$. □