Online Appendix to "A Theory of Endogenous Commitment," by Guillermo Caruana and Liran Einav: Algorithm Description

Here we describe the algorithm, which is essential for the proof of Theorem 1 in the paper. In the proof we also refer to the limiting version of the algorithm, that is, as the fineness of the grid $\varphi(g)$ goes to zero. Since the switching cost technology is continuous, the limiting version is identical to the finite version of the algorithm, with the only changes affecting parts 2 and 4, in which $\text{next}_i(t)$ and $\text{prev}_i(t)$ are replaced by t. A Matlab code for the limiting version of the algorithm is available at *http://www.stanford.edu/~leinav*.

In the end of this appendix we prove that the algorithm terminates in a small and finite number of steps, for any grid. Finally, in what follows, if p is one player we use $\sim p$ to denote the other player. Given a particular game (Π, C, g) the algorithm steps are described below.

Initialization: Set $m = 0$ (stage counter, starting from the end); $t_0^* = T$ (the last critical time encountered); $V_0(a,p) = \Pi$ (continuation value of player p at profile a just after t_m^*); $AM_0(a,p) = 0$ (an indicator function; it equals one iff there is an active switch at time t_m^* by player p from profile a); and $IM = \{(a,p)|a \in A,\, p = 1,2\}$ (the set of inactive moves).

Update (m, V_m, AM_m)**:**

1. $m = m + 1$
2. Find the next critical time t_m^*, and the action a^* and player p^* associated with it. This is done by comparing the potential benefits and costs for each move. We use some auxiliary definitions:
   (a) We use some auxiliary definitions:
      i. Let $q(a,p)$ be the first player who switches out of a if player $\sim p$ is the first who moves. More precisely, let
      $$q(a,p) = \begin{cases} \sim p & \text{if } AM_{m-1}(a, \sim p) \neq \varnothing \\ p & \text{if } AM_{m-1}(a, \sim p) = \varnothing \text{ and } AM_{m-1}(a,p) \neq \varnothing \\ \varnothing & \text{otherwise} \end{cases}$$
      ii. Let $SM_{m-1}(a,p)$ be the longest ordered set of action profiles $(a^0, a^1, ..., a^{k-1}, a^k)$ such that $a^0 = a$ and, for $i > 0$
      $$a^i = \begin{cases} (a^{i-1}_{\sim q(a,q)}, AM_{m-1}(a^{i-1}, q(a,q))) & \text{if } i \text{ is odd and } AM_{m-1}(a^{i-1}, q(a,q))) \neq \varnothing \\ (a^{i-1}_{q(a,q)}, AM_{m-1}(a^{i-1}, \sim q(a,q))) & \text{if } i \text{ is even and } AM_{m-1}(a^{i-1}, q(a,q)) \neq \varnothing \end{cases}$$
      This defines the sequence of consecutive switches within stage $m-1$ that start at a and ends at a profile from which there is no active move. We denote this final node by $\overline{SM}_{m-1}(a,p)$. The sequence is finite, contains up to three switches, and is solely a function of AM_{m-1}.
      iii. Given $SM_{m-1}(a,p) = (a^0, ..., a^k)$, define $FS_{m-1}(a,p) = \sum_{i=1}^{k} I(a_p^{i-1} \neq a_p^i)$ where $I(\cdot)$ is the indicator function (FS_{m-1} computes the number of switches by player p in the $SM_{m-1}(a,p)$ sequence).
      iv. Let $\Delta V_{m-1}(a,p) \equiv V_{m-1}(\overline{SM}_{m-1}((a'_p, a_{\sim p}), p)) - V_{m-1}(\overline{SM}_{m-1}(a,p))$. This difference in values stands for the potential benefits of each move at profile a by player p.
   (b) Now, compute the critical time associated with each move. This involves four different cases, as shown below. The first is when the move gives negative value. The second is a case in which if player p does not move, he will be moving at his next turn (because the other player will move to a profile in which player p prefers to move). This means that player p prefers to move right away, rather than delaying his move, so the critical time kicks in immediately before the next

critical time. The third case is the "standard" case, in which the critical time is the last time at which the cost of switching is less than its benefit. The last case is similar, but takes into account that the move involves an extra immediate switch at the next period.

$$t_m(a,p)^1 = \begin{cases} 0 \\ \quad \text{if } \Delta V_{m-1}(a,p) < 0 \\ \text{prev}_p(t^*_{m-1}) \\ \quad \text{if } \Delta V_{m-1}(a,p) \geq 0 \text{ and } FS_{m-1}(a,p) > 0 \\ \max\left\{t \in g_p, t < t^*_{m-1} | C_p(a_p \to a'_p, t) \leq \Delta V_{m-1}(a,p)\right\} \\ \quad \text{if } \Delta V_{m-1}(a,p) \geq 0 \text{ and } FS_{m-1}(a,p) = 0 \text{ and } FS_{m-1}((a'_p, a_{\sim p}), p) = 0 \\ \max\left\{t \in g_p, t < t^*_{m-1} | C_p(a_p \to a'_p, t) + C_p(a'_p \to a_p, \text{next}_p(t)) \leq \Delta V_{m-1}(a,p)\right\} \\ \quad \text{if } \Delta V_{m-1}(a,p) \geq 0 \text{ and } FS_{m-1}(a,p) = 0 \text{ and } FS_{m-1}((a'_p, a_{\sim p}), p) > 0 \end{cases}$$

The next critical time is the one associated with the move that maximizes the above, out of the moves that are not active yet.

$$(a^*, p^*) = \underset{(a,p) \in IM}{\arg\max} \{t_m(a,p)\}$$

(c) Given (a^*, p^*):
Abort if $|p^*| > 1$.[2] Equal critical times for different players (*the solution is not grid invariant*).
If not, set $t^*_m = t_m(a^*, p^*)$
Abort if $t^*_m = 0$ (*a player is indifferent between two actions at* $t = 0$)
If not, set $p^*_m = p^*$

3. Update the set of active moves. First, activate the move associated with the new critical time. Second, deactivate moves by the other player that originate from the same action profile, but only if $m = 2$ or if we are in the early part of the game. The third case involves a move whose destination is the origin of the new active move. Such a move is deleted and reevaluated in the next iteration. Finally, the rest of the moves remain as they were before.

$$AM_m(a,p) = \begin{cases} 1 & \text{if } (a,p) \in (a^*, p^*) \\ 0 & \text{if } (a,p) \in (a^*, \sim p^*) \text{ and } (m = 2 \text{ or } AM_{m-1}(a', p) = 1) \\ 0 & \text{if } (a,p) \in ((a^*_{p^*}, \sim a^*_{\sim p^*}), \sim p^*) \\ AM_{m-1}(a,p) & \text{otherwise} \end{cases}$$

4. Compute the continuation values of the players just after t^*_m. This is done by using the value at the terminal node of an active sequence of consecutive moves (as defined in part 2), and subtracting the switching costs incurred by the player along this sequence. These switching costs are incurred just after t^*_m. First, define the following mapping

$$V^{new}(V^{old}, AM, t, \overline{p})(a,p) = V^{new}(V^{old}, SM(AM), t, \overline{p})(a,p) = V^{old}(\overline{SM}(a, \overline{p})) - CC(SM(a, \overline{p}), t, a, p)$$

where CC is recursively defined as follows:

$$CC(SM(a,\overline{p}), t, a, p) = \begin{cases} 0 & \text{if } \overline{SM}(a, \overline{p}) = (a) \\ CC(SM(a^1, \sim p), \text{next}_{\sim p}(t), a^1, p) & \text{if } a_p = a^1_p \text{ where } (a^0, ..., a^k) = SM(a, \overline{p}) \\ CC(SM(a^1, p), \text{next}_p(t), a^1, p) + \\ \quad + C_p(a_p \to a^1_p, \text{next}_p(t)) & \text{if } a_p \neq a^1_p \text{ where } (a^0, ..., a^k) = SM(a, \overline{p}) \end{cases}$$

Now, compute the continuation values by $V_m = V^{new}(V_{m-1}, AM_{m-1}, t^*_m, p^*_m)$.

[1] Note that by having weak inequalities within the max operator we implicitly assume that a player switches whenever he is indifferent between switching or not.

[2] arg max is a correspondence. This is why we use '∈' rather than equalities in part 3 of the algorithm. Given the way we construct $t_m(a,p)$, the multiple solutions must be associated with a unique p^* for any finite grid. In the limiting case, this is the only generic case. This is why the algorithm may abort in non-generic cases.

5. Let $IM = \{(a,p)|AM_m(a,p) = 0 \text{ and } AM_m((a'_p, a_{\sim p}), p) = 0\}$.

6. **Terminate** if $\#IM = 0$ (all moves are active), and let $\overline{m} = m$, $t^*_{\overline{m}+1} = 0$. Otherwise, go to part 1.

Output: The essential information of the algorithm consists of the number of stages of the game, $\overline{m}$, the critical points that define the end of each stage, $(t^*_m)_{m=0}^{\overline{m}}$, and the strategies at every stage

$$S_g(p,a,m) = \begin{cases} a_p & \text{if} \quad AM_m(a,p) = 0 \\ a'_p & \text{if} \quad AM_m(a,p) = 1 \end{cases}$$

Nevertheless, for practical reasons we define the output of the algorithm to be

$$(t^*_m, S_g(p,a,m), V_m, AM_m)_{m=0}^{\overline{m}}$$

In the limiting case, we use the notation $S(p,a,m)$ instead of $S_g(p,a,m)$.

Lemma 1 *For any* (Π, C, g)*, the algorithm ends in a finite number of stages, and in particular* $\overline{m} \leq 8$.

Proof. The algorithm finishes when $\#IM = 0$. Observe that:

1. If $AM_m(a,p) = 1$ then $AM_m((a'_p, a_{\sim p}), p) = 0$ and vice versa, thus $\#IM = 0$ implies that $\#AM = 4$.

2. Whenever $\exists p, m$ s.t.$\sum_a AM_m(a,p) = 2$ we get into a "termination phase" (which corresponds to Lemma 2 in the paper) and the algorithm is guaranteed to terminate within at most two more stages. It can be verified that $\sum_a AM_{m+1}(a, \sim p) = 2$ and that both active moves by player $\sim p$ are in the same direction. Therefore, player p's two moves immediately become active at stage $m+2$, without any deletion of an active move by player $\sim p$, terminating the algorithm.

3. $\#AM$ is non-decreasing in m: each iteration adds an active move $(AM(a^*, p^*))$ and may potentially remove at most one active move.[3]

4. For $m > 2$, and before reaching the "termination phase," an active move (a,p) is deleted only when $(a,p) \in ((a^*_{p^*}, a'^*_{\sim p^*}), \sim p^*)$. In particular, at stage m, a deleted move must belong to player $\sim p^*_m$.

5. Observations 2 and 4 imply that once $\#AM = 2$ the algorithm terminates within at most 3 stages. If the two active moves are by the same player then we can use observation 2. If they are by different players, observation 4 guarantees that in the next stage one player will have 2 active moves.

Using all the above, all we need to show is that it is not possible to have an infinite sequence of stages with only one active move in each of them. That is, such that any move that becomes active at stage m, becomes inactive at stage $m+1$. Suppose, toward contradiction, that such an infinite sequence exists. Without loss of generality, consider $m = 2$, in which $AM_2(a,p) = 1$ for some (a,p), and $AM_2(\widetilde{a}, p') = 0$ for any $(\widetilde{a}, p') \neq (a,p)$. If (a,p) is deleted at $m = 3$, it must be that the new active move is such that $AM_3((\widetilde{a}_p, a_{\sim p}), \sim p) = 1$. Similarly, we obtain that $AM_4(a', p) = 1$ and that $AM_5((a_p, a'_{\sim p}), \sim p) = 1$. This gives the following contradiction. By $AM_2(a,p) = 1$ we know that $V_3(a, \sim p) = V_3((a'_p, a_{\sim p}), \sim p)$. By $AM_3((a'_p, a_{\sim p}), \sim p) = 1$ we know that $V_3((a'_p, a_{\sim p}), \sim p) < V_3(a', \sim p) - C_{\sim p}(a_{\sim p} \to a'_{\sim p}, t)$ for any $t < t^*_3$. It is easy to see that $t^*_4 < t^*_3$, so the above implies that $V_5(a, \sim p) = V_3((a'_p, a_{\sim p}), \sim p) < V_3(a', \sim p) - C_{\sim p}(a_{\sim p} \to a'_{\sim p}, t^*_4) = V_5((a'_p, a_{\sim p}), \sim p)$, while by $AM_4(a', p) = 1$ we also know that $V_5(a', \sim p) = V_5((a_p, a'_{\sim p}), \sim p)$. The two last equations imply that $\Delta V_5(a', \sim p) > \Delta V_5((a_p, a'_{\sim p}), \sim p)$, which is a contradiction to the fact that $(a^*, p^*) = ((a_p, a'_{\sim p}), \sim p)$ at $m = 5$. This, together with observation 5 above, also shows that $\overline{m} \leq 8$. ■

Remark 1 *In fact, it can be shown that* $\overline{m} \leq 7$ *because a deletion at* $m = 2$ *according to* $(a,p) \in (a^*, \sim p^*)$ *and* $m = 2$ *implies that there can be only one (rather than two) additional deletions later on.*

[3] Whenever the arg max is not a singleton, then it is easy to see that we add two active moves by the same player, thus we are done by observation 2 above.