ON LOW-DEPTH QUANTUM ALGORITHMS FOR ROBUST MULTIPLE-PHASE ESTIMATION

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Abstract. This paper is an algorithmic study of quantum phase estimation with multiple eigenvalues. We present robust multiple-phase estimation (RMPE) algorithms with Heisenberg-limited scaling that are particularly suitable for early fault-tolerant quantum computers in the following senses: (1) a minimal number of ancilla qubits are used, (2) an imperfect initial state with a significant residual is allowed, (3) the prefactor in the maximum runtime can be arbitrarily small given that the residual is sufficiently small and a gap among the dominant eigenvalues is known in advance. Even if the eigenvalue gap does not exist, the proposed RMPE algorithms are able to achieve the Heisenberg limit while maintaining the aforementioned benefits (1) and (2). In addition, our method handles both the integer-power model, where the unitary $U$ is given as a black box with only integer powers accessible, and the real-power model, where the unitary $U$ is defined through a Hamiltonian $H$ with $U = \exp(-2\pi i H)$.

1. Introduction

This paper concerns the quantum phase estimation (QPE) problem with multiple eigenvalues. For a unitary matrix $U$, let $\{(e^{-2\pi i \lambda_s}, |\psi_s\rangle)\}$ be the eigenpairs of $U$ with $\lambda_s \in [0, 1]$. Suppose that $|\psi\rangle$ is an initial quantum state of the form

$$|\psi\rangle = \sum_{s=1}^{S} c_s |\psi_s\rangle + c_{\text{res}} |\psi_{\text{res}}\rangle,$$

where $S$ is the number of dominant eigenvalues and $c_{\text{res}} |\psi_{\text{res}}\rangle$ is the residual. Here by dominant, we mean that the overlaps between these eigenstates $|\psi_s\rangle$ and $|\psi\rangle$ are bounded from below by a constant $\beta$, i.e., $\min_{1 \leq s \leq S} |c_s|^2 \geq \beta > 0$. On the other hand, the energy of the residual $|c_{\text{res}}|^2$ is bounded from above by a constant $\omega$ less than $\beta$, i.e., $|c_{\text{res}}|^2 \leq \omega < \beta$. The goal is to estimate the set $\Lambda \equiv \{\lambda_s\}_{s=1}^{S}$ of dominant eigenvalues up to a prescribed accuracy $\epsilon$.

The difficulty level of multiple eigenvalue estimation depends on the gap between the dominant eigenvalues. As the goal is to recover the eigenvalues up to accuracy $\epsilon$, one can assume without loss of generality that the gap between any two dominant eigenvalues is at least $O(\epsilon)$ because these eigenvalues can be merged otherwise. When the gap is bounded from below by a constant independent of $\epsilon$, we refer to it as the gapped case. When the gap can be as small as $O(\epsilon)$, we refer to it as the gapless case. The problem becomes more challenging when the gap gets smaller.

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There are several key complexity metrics for evaluating the performance of multiple-phase algorithms.

1. The number of ancilla qubits required. The smaller, the better.
2. The amount of residual $\omega$ allowed in the initial state $|\psi\rangle$, i.e., $\omega$ is the maximum $|c_{\text{res}}|^2$ allowed.
3. The maximum runtime $T_{\text{max}}$. It is defined as the maximum depth of the quantum circuits used by the algorithm.
4. The total runtime $T_{\text{total}}$, i.e., the sum of the circuit depths over all executions. It has been shown in [11,37,38], for example, that $T_{\text{total}}$ has a lower bound named the Heisenberg limit $T_{\text{total}} = \Omega(\epsilon^{-1})$.
5. Finally, the minimum gap between the eigenvalues allowed.

Among these metrics, a small $T_{\text{max}}$ is particularly important for early fault-tolerant quantum devices since these devices typically have a relatively short coherence time.

Based on these metrics, an ideal phase estimation algorithm should meet the following requirements:

1. Using a small number of (even a single) ancilla qubits.
2. Allowing the initial state $|\psi\rangle$ to be inexact and ideally $\omega$ to be proportional or even close to $\beta$.
3. Satisfying $T_{\text{max}} = O(\epsilon^{-1})$ with the prefactor ideally proportional to $\omega/\beta$.
4. Achieving the Heisenberg-limited scaling $T_{\text{total}} = \tilde{O}(\epsilon^{-1})$, where $\tilde{O}$ means omitting the poly-logarithmic term.
5. Allowing the minimum gap to be on the same order of $\epsilon$.

The design of multiple eigenvalue estimation algorithms depends closely on the representation of $U$. In the most general model, $U$ is represented by a quantum circuit or even a black box model, such as a quantum approximate optimization algorithm (QAOA) or variational quantum eigensolver (VQE). This model only allows for access to integer powers $U^j$ of $U$ for $j \in \mathbb{Z}$, $j \geq 0$, and we refer to it as the integer-power model. A different model is $U = e^{-2\pi i H}$ where $H$ is a quantum Hamiltonian with eigenvalues in $[0,1]$. Here $U$ is often implemented with Trotterization or the splitting method, where one of the main applications is to find the energy of the ground state and a few low-lying excited states of $H$. This model allows for access to $U^t := e^{-2\pi i H t}$ for any $t \in \mathbb{R}_+$, and we refer to it as the real-power model.

1.1. Related work. As a key primitive of quantum algorithms, quantum phase estimation has attracted a lot of research activities in the past few decades.

1.1.1. Single eigenvalue. Several existing methods can be applied when there is only one dominant eigenvalue, i.e., $S = 1$. The early algorithms require a perfect eigenstate, i.e., $c_{\text{res}} = 0$. One of the most fundamental algorithms is the Hadamard test, as shown in Figure 1(a), which utilizes the $I$ and $S$ gates after the controlled-$U$ gate. The Hadamard test provides estimations of $\text{Re} \langle \psi | U | \psi \rangle$ and $\text{Im} \langle \psi | U | \psi \rangle$, respectively, from the probability of getting $|0\rangle$ while measuring the ancilla qubit. For the Hadamard test, one needs $O(\epsilon^{-2})$ repetitions to reach precision $\epsilon$, leading to a total gate complexity $T_{\text{total}} = O(\epsilon^{-2})$.

A quadratic improvement proposed by Kitaev [15,16] uses measurements of $\langle \psi | U^j | \psi \rangle$ for $j = 0, 1, 2, \ldots, J$ with $J = O(\log(\epsilon^{-1}))$, as illustrated by the circuit in Figure 1(b). The total runtime $T_{\text{total}} = O(\epsilon^{-1})$ of Kitaev’s method achieves the Heisenberg limit ([11,37,38]).
However, the original version of Kitaev’s method only applies to perfect eigenstates, i.e., $c_{\text{res}} = 0$.

Another example that reaches the Heisenberg limit is the QPE algorithm with quantum Fourier transform (QFT) \cite{3}, which only involves a single execution but needs more ancilla qubits and a deeper circuit. Many alternatives have also been proposed in the recent literature \cite{6,13,18,21,22,28,30}. For a more comprehensive overview about the QPE algorithms for a single eigenvalue, we refer to the detailed discussions in \cite{6,22,27}.

(a) (b)

Figure 1. (a) The circuit for the Hadamard test. H is the Hadamard gate. Concerning the $I/S^+$ gate, we use $I$ (the identity) for the real part of $\langle \psi | U | \psi \rangle$, and $S^+$ (the Hermitian conjugate of the phase gate $S$) for the imaginary part. (b) The circuit used in the Kitaev algorithm estimates the real and imaginary parts of $\langle \psi | U^{2^j} | \psi \rangle$ for multiple integer values of $j$.

As we mentioned earlier, for early fault-tolerant quantum devices, besides using a small number of (or even only a single) qubits and reaching the Heisenberg limit $T_{\text{total}} = O \left( \epsilon^{-1} \right)$ for the case $S = 1$ and $|c_{\text{res}}|^2 > 0$, it is also desired to allow the prefactor in $T_{\text{max}} = O \left( \epsilon^{-1} \right)$ to depend on the magnitude of $c_{\text{res}}$. The authors of \cite{6} proposed QCELS, an optimization subroutine that works with $|c_{\text{res}}|^2 \leq 0.29$ and allows the prefactor of $T_{\text{max}} = O \left( \epsilon^{-1} \right)$ to scale as $\Theta(|c_{\text{res}}|)$. In a recent work \cite{26}, we showed that a modified version of the robust phase estimation (RPE) algorithm \cite{1,14,34} can work with $|c_{\text{res}}|^2 \leq 2\sqrt{3} - 3 \approx 0.464$ and it gives the near-optimal prefactor $\Theta(|c_{\text{res}}|)$.

1.1.2. Multiple eigenvalues. The work \cite{28} considers the problem of estimating multiple eigenvalues with a signal processing subroutine and adopts the matrix pencil method. The method can be sensitive to noise, and the Heisenberg-limited scaling is not achieved. The algorithm proposed in \cite{36} also estimates multiple eigenvalues based on time series analysis. The total cost is $O \left( \epsilon^{-6} \right)$, which is quite far from the Heisenberg limit. A more recent work \cite{9} extends the idea of the RPE algorithm to multiple eigenvalues and achieves the Heisenberg limit. However, the residual term $|\psi_{\text{res}}\rangle$ is not allowed in \cite{9}, and the prefactor power of $S$ is quite large.

The work \cite{7} extended the QCELS algorithm to the multiple eigenvalue setting. This extension achieves the Heisenberg limit with a single ancilla qubit and allows for a residual. The maximum circuit depth can also be reduced when the residual amount $\omega$ is small compared to $\beta$. However, the performance of this optimization technique relies on the existence of a spectral gap $\Delta$ between the multiple dominant eigenvalues, where the minimal runtime of the circuits is $\Omega(\Delta^{-1})$. Moreover, a grid search is generally needed to solve the optimization problem due to the highly complicated landscape, which leads to a classical cost exponential in $S$. 
Quantum subspace diagonalization methods have also been used for multiple eigenvalue estimation [4,17,23,25,29,35], where the eigenvalues are obtained by solving certain projected eigenvalue problems or singular value problems. Compared to the numerical performance demonstrated, the theoretical analysis of the subspace diagonalization methods is still rather preliminary and pessimistic [10].

1.2. Contributions. This paper conducts a comprehensive study of the multiple eigenvalue quantum phase estimation. Our main contribution is a family of robust multiple-phase estimation (RMPE) algorithms for both the gapless and the gapped cases and for both the real-power and integer-power models. These RMPE algorithms build on the overall structure of the method proposed in [9].

For the gapless case, the proposed RMPE algorithm estimates the dominant $S \geq 1$ eigenvalues for any residual overlap $|c_{\text{res}}|^2 < \beta$. It utilizes the measurement results from Hadamard tests to the unitary operators $U^{M_\ell}$ for an exponentially-growing sequence $\{M_\ell\}$, one for each step $\ell$. The estimated dominant eigenvalues of $U^{M_\ell}$ allow us to narrow down the intervals that contain the dominant eigenvalues of $U$, and the desired precision will be obtained after $O(\log(\epsilon^{-1}))$ steps. One key component is a simple but new signal processing routine that estimates eigenvalue locations from quantum measurements without any gap assumption. Another key component of the algorithm is the careful choice of $\{M_\ell\}$ to avoid potential collision of the intervals. To achieve this for the real-power model, we utilize non-integer $\{M_\ell\}$ that grows each time by a factor between 2 and 4. For the integer-power model, we leverage results from prime number theory for the choice of $\{M_\ell\}$ to avoid collisions. Both the real-power and integer-power versions can tolerate a residual $\omega$ up to $\beta$ and achieve the Heisenberg limit. In summary, this algorithm meets all five requirements, except that the prefactor of $T_{\text{max}}$ does not scale like $\omega/\beta$.

For the gapless case, we improve the algorithms with two key modifications. First, we adopt the ESPRIT algorithm [33], a different signal processing routine that allows for more accurate eigenvalue estimation when a gap is available. This allows us to build fairly accurate approximations to $\Lambda$ even at the initial step. Second, with this better initial estimate, the number of iterations of the algorithm can be significantly reduced, resulting in $T_{\text{max}}$ to scale like $\omega/\beta$. Both the real-power and integer-power versions can tolerate residual $\omega = O(\beta)$ and reach the Heisenberg limit. In summary, this algorithm meets the first four requirements mentioned above.

1.2.1. Comparisons. Compared with [9], our RMPE algorithms have the following advantages. First, the proposed algorithms allow an imperfect initial state with a residual term. Second, by using a simpler signal processing routine in both the gapped and gapless cases, we obtain a smaller prefactor in terms of the sparsity level $S$: the prefactor of $T_{\text{total}}$ is only quadratic in terms of the sparsity level $S$. In contrast, the dependency on $S$ obtained in [9] is $O(S^{12})$. Third, the proposed RMPE algorithms work in the real-power and integer-power models, while [9] only works with the real-power model. Furthermore, when the dominant eigenvalues present a gap, $T_{\text{max}}$ of our gapped algorithm has a prefactor of order $\omega/\beta$, which is a property not shared by the algorithm of [9].

Compared with [7], the proposed RMPE algorithms have two advantages. First, we address the gapless case that is not visited in [7]. Second, the classical computational complexity is much lower: our RMPE algorithms scale polynomially in $S$ while the optimization step of the method in [7] generally scales exponentially in $S$. 
1.3. Contents. The rest of the paper is organized as follows. Section 2 introduces the main structure of the proposed algorithms. Section 3 discusses the gapless case in detail. In Section 4, we present the easier gapped case while focusing on its difference from the gapless case. Both sections address both the real-power and integer-power models.

A few comments about the notations are in order here. For a measurable set \( T \) and helps improve the current estimation \( E \) of \( \Lambda \). Applying the Hadamard test to \( U \) and \( T, \eta \) for some integer \( K \) with input \( |\psi\rangle \) provides a noisy measurement \( \{y(k)\} \) of

\[
\langle \psi|U^k|\psi\rangle = \sum_{s=1}^{S} |c_s|^2 e^{-2\pi i \lambda_s k} + |c_{\text{res}}|^2 \langle \psi_{\text{res}}|U^k|\psi_{\text{res}}\rangle, \quad k = -K, \ldots, K.
\]

This is the Fourier transform of a probability measure \( f(x) \) defined as

\[
f(x) \equiv \sum_{s=1}^{S} |c_s|^2 \delta_{\lambda_s}(x) + \text{Residual},
\]

where Residual is a positive measure with mass \( |c_{\text{res}}|^2 \). \( \Lambda = \{\lambda_s\}_{s=1}^{S} \) can then be viewed as the dominant support of the measure \( f(x) \).

At the first step, from the noisy approximation \( y(k) \) to the Fourier coefficient \( f(k) \), one can first extract a rough estimation \( \hat{E}_0 \) of \( \Lambda \) using an appropriate signal processing routine (to be detailed in the following sections).

The algorithm then chooses a sequence of amplifying factors \( (m_1, m_2, \ldots, m_\ell) \). Define \( M_0 = 1 \) and \( M_\ell = m_1 m_2 \cdots m_\ell \). At the \( \ell \)-th step, we start with \( E_{\ell-1} \), the current approximation of \( \Lambda \). Applying the Hadamard test to \( U^{M_\ell}, \ldots, U^{M_\ell K} \) with input \( |\psi\rangle \) gives rise to noisy measurements \( \{y_\ell(k)\} \) of

\[
\langle \psi|U^{M_\ell k}|\psi\rangle = \sum_{s=1}^{S} |c_s|^2 e^{-2\pi i M_\ell \lambda_s k} + |c_{\text{res}}|^2 \langle \psi_{\text{res}}|U^{M_\ell k}|\psi_{\text{res}}\rangle,
\]

This can be viewed as the Fourier transform of a probability measure \( f_\ell(x) \)

\[
f_\ell(x) \equiv \sum_{s=1}^{S} |c_s|^2 \delta_{\lambda_s M_\ell}(x) + \text{Residual},
\]

where Residual is again a positive measure with mass \( |c_{\text{res}}|^2 \) and \( [x] \equiv (x \mod 1) \). The data \( y_\ell(k) \) is a noisy approximation to the Fourier coefficient \( f_\ell(k) \). Again with a signal processing subroutine, one can obtain an estimation \( Y_\ell \) of the dominant support of \( f_\ell(x) \), which is denoted by \( \Lambda_\ell = \{[\lambda_s M_\ell]\}_{s=1}^{S} \). This estimation \( Y_\ell \) provides information with an increased resolution and helps improve the current estimation \( E_{\ell-1} \) to an update \( E_\ell \).
More specifically, with the spectrum estimation $Y_\ell$, one divides it by $M_\ell$ to obtain the set $\frac{1}{M_\ell} Y_\ell$. From the property of the signal processing methods used, $\frac{1}{M_\ell} Y_\ell$ will be a union of at most $S$ disjoint intervals $\frac{1}{M_\ell} Y_\ell = \bigcup_{i=1}^{S_\ell} I_{\ell,i}'$ with $\Lambda \subset \bigcup_{i=1}^{S_\ell} (I_{\ell,i}' + \frac{q_{\ell,i}}{M_\ell})$ for properly chosen integers $q_{\ell,i}$. Here $S_\ell \leq S$ is the number of intervals in $\frac{1}{M_\ell} Y_\ell$, and $I_{\ell,i}'$ is the $i$-th disjoint interval in $\frac{1}{M_\ell} Y_\ell$. By carefully determining the amplifying factors $(m_1, m_2, \ldots, m_\ell)$, we show that at most one integer $q_{\ell,i}$ satisfies $(I_{\ell,i}' + \frac{q_{\ell,i}}{M_\ell}) \cap E_{\ell-1} \neq \emptyset$. Then we define $(I_{\ell,i}' + \frac{q_{\ell,i}}{M_\ell})$ as $I_{\ell,i}$ and form the updated estimation $E_\ell = \bigcup_{i=1}^{S_\ell} I_{\ell,i}$. The determination of $m_\ell$ will be discussed in detail in the following sections. Given that $m_\ell$ is chosen properly, the algorithm can be summarized as in Algorithm 1.

**Algorithm 1** Structure of robust multiple-phase estimation (RMPE) algorithm

**Input:** $\epsilon$: desired precision, $\beta$: the lower bound of dominant eigenvalues, $S$: the number of dominant spikes, $\omega$: upper bound for the residual in the initial state $|\psi\rangle$.

Set the initial estimation $E_{-1}$.

Set $M_0 = 1$, $\ell = 0$.

Calculate $\eta$, $K$, $N_{HR}$ and $\alpha$ according to $\epsilon$, $\beta$, $\omega$ and $S$.

**while** $\eta/M_\ell > \epsilon$ **do**

Choose an $m_\ell \geq 2$ according to Lemma 3.4 or Lemma 3.8 and set $M_\ell = M_{\ell-1} m_\ell$ if $\ell > 0$.

Run the circuit in Figure 1(a) with $U$ replaced by $U^{M_\ell k}$ for $N_{HR}$ times each for the real and imaginary parts and $0 \leq k \leq K$ and obtain the signal $y_\ell$.

Obtain $Y_\ell$ from a signal processing routine, and then several intervals $I_{\ell,1}', \ldots, I_{\ell,S_\ell}'$ from $\frac{1}{M_\ell} Y_\ell$. The choice of $m_\ell$ ensures that for each $I_{\ell,i}'$, only one integer $q_{\ell,i}$ gives $(I_{\ell,i}' + \frac{q_{\ell,i}}{M_\ell}) \cap E_{\ell-1} \neq \emptyset$.

Set $I_{\ell,i} = I_{\ell,i}' + \frac{q_{\ell,i}}{M_\ell}$ and $E_\ell = \bigcup_{i=1}^{S_\ell} I_{\ell,i}$

$\ell \leftarrow \ell + 1$.

**end while**

**Output:** The final estimation $E_\ell$ as an approximated support of $\Lambda$.

We will show in Section 3.2 and Section 3.3 that the determination of $K$, $N_{HR}$, $m_\ell$ and $\eta$ ensures that $E_\ell = \bigcup_{i=1}^{S_\ell} I_{\ell,i}$ enjoys the following properties:

1. $\{I_{\ell,i}\}$ are disjoint, and $\sum_{i=1}^{S_\ell} |I_{\ell,i}| \leq S \eta/M_\ell$.
2. For each $I_{\ell,i}$, the intersection $I_{\ell,i} \cap \Lambda \neq \emptyset$.
3. $\Lambda \subset E_\ell \subset B(\Lambda, \frac{\eta}{M_\ell})$ (real-power model) or $\Lambda \subset E_\ell \subset B_T(\Lambda, \frac{\eta}{M_\ell})$ (integer-power model).

Since the chosen $m_\ell$ satisfies $m_\ell \geq 2$, it can be deduced from the last property above that the while loop in Algorithm 1 ends in $O(\log(\frac{\omega}{\epsilon}))$ iterations with high probability. In Section 3.2, we show that for the real-power model, one can choose a proper $m_\ell \in [2,4]$ according to the previous estimations. On the other hand, for the integer-power model, one can choose an appropriate $m_\ell$ with the help of prime numbers. Detailed explanations and analyses are provided in Section 3.3. Comparing with the original version of Kitaev’s method, where $m_\ell = 2$ for all $\ell$, the adaptive calculation of these factors enables the proposed algorithm to address QPE problems with multiple dominant eigenvalues and a non-zero residual.

Here we discuss some specific aspects of the signal processing routine used to extract $Y_\ell$. At the $\ell$-th iteration, we implement $N_{HR}$ measurements of $f_\ell(k)$ for each $k \in \{0,1,\ldots,K\}$.
such that the averaged measurement result $y_\ell(k)$ satisfies

$$|y_\ell(k) - \hat{f}_\ell(k)| \leq \alpha$$

for all $k$ with high probability. The determination of parameters $N_{HR}$, $K$, and $\alpha$ will be elaborated in the following sections and is omitted for now. The problem of recovering $\Lambda_\ell = \{[\lambda_s M_\ell]\}_{s=1}^S$ from $\{y_\ell(k)\}_{k=0}^K$ has been extensively studied under the name line spectrum estimation, and plenty of established results for line spectrum estimations can be used. As explained earlier, Algorithm 1 requires that $Y_\ell$ satisfies the following three requirements.

1. $Y_\ell$ is a union of at most $S$ disjoint intervals such that each interval contains at least one dominant eigenvalue.
2. $\Lambda_\ell$ is a subset of $Y_\ell$.
3. $|Y_\ell| \leq S\eta$ for a parameter $\eta$ to be determined by the algorithm.

We provide a detailed description in Corollary 3.2 and show that these requirements can indeed be satisfied for both the gapless case (Section 3.1) and the gapped case (Section 4.1).

3. THE GAPLESS CASE

This section specializes Algorithm 1 to the gapless case, i.e., no gap is assumed among the dominant eigenvalues. The signal processing routine proposed in [19] satisfies the requirements 1–3 listed above for $Y_\ell$ in Section 2. After summarizing its main results in Section 3.1 for completeness, we discuss the real-power model in Section 3.2 and the integer-power model in Section 3.3, respectively. Figure 2 gives a graphical illustration of the gapless case algorithm.

Figure 2. Illustration of Algorithm 1 for the gapless case with $S = 3$. At each step, we maintain a union of at most $S$ intervals as an estimation of $\Lambda$, shown by the horizontal blue sticks in the diagram. The yellow intervals represent $M_\ell^{-1}Y_\ell$ and its translates by integer multiples of $M_\ell^{-1}$. The choice of $M_\ell$ ensures that only one integer $q_{i,s}$ satisfies $I_{e,i}^s + \frac{q_{i,s}M_\ell}{M_\ell} \cap E_{e,i-1} \neq \emptyset$. The union of these $I_{e,i}^s + \frac{q_{i,s}M_\ell}{M_\ell}$ then becomes the new estimation of $\Lambda$ and is used for the next update. Note that the number of intervals in $E_\ell$ may increase. Due to the low resolution, $\lambda_1$ and $\lambda_2$ are covered by the same interval in $E_0$. As the resolution increases, they belong to two different intervals in $E_1$ and $E_2$. 
3.1. Signal processing routine. Recall that $y_{\ell}$ are the averaged measurements result such that $|y_{\ell}(k) - \hat{f}_{\ell}(k)| \leq \alpha$, where $\hat{f}_{\ell}$ is defined in (3). Here $y_{\ell}(-k)$ is defined as the complex conjugate of $y_{\ell}(k)$ for $0 \leq k \leq K$, since $\hat{f}_{\ell}(-k)$ is the conjugate of $\hat{f}_{\ell}(k)$. Following [19], the spikes $\Lambda_{\ell}$ can be estimated by the following set:

$$(4) \quad X_{\ell} = \left\{ x : \left| \sum_{|k| \leq K} y_{\ell}(k) \hat{\phi}_{p}(k) e^{2\pi i k x} \right| > \frac{6\beta + 5\omega}{11} \phi_{s} \right\},$$

where $\sigma = \sqrt{\frac{1}{\pi} \log \frac{12}{\beta - \omega}}$, $\hat{\phi}_{p}(k) = \exp(-\pi(k\sigma/K)^2)$, $k \in \mathbb{Z}$ and $\phi_{s} = \sum_{k \in \mathbb{Z}} \hat{\phi}_{p}(k)$. The following theorem holds for this choice of $X_{\ell}$.

**Theorem 3.1.** Suppose $\alpha < \frac{\beta - \omega}{3}$ and $K \geq 3\tau$, then $\Lambda_{\ell} \subset X_{\ell}$ and $\max_{x \in X_{\ell}} \text{dist}(x, \Lambda_{\ell}) \leq \tau/K$, where $\tau = \frac{1}{\pi} \log \frac{12}{\beta - \omega}$.

Note that it is possible that $X_{\ell}$ is a disjoint union of more than $S$ intervals, and some of them may not contain a true spike. In the following corollary, we form a set $Y_{\ell}$ with the help of the estimations $X_{\ell}$ obtained above to meet the requirements listed in Section 2.

**Corollary 3.2.** Using the set $X_{\ell}$ we obtained from the above signal processing routine, we can construct a set $Y_{\ell}$ that satisfies the following properties when $\eta > 3\tau/K$:

1. $Y_{\ell} = \bigcup_{i=1}^{S} Y_{\ell,i}$ is the disjoint union of intervals, and $|Y_{\ell}| \leq S\eta$.
2. For each interval $Y_{\ell,i}$, the intersection $I_{\ell,i} \cap \Lambda_{\ell} \neq \emptyset$.
3. $\Lambda_{\ell} \subset Y_{\ell} \subset B_{T}(\Lambda_{\ell}, \eta)$.

**Proof.** All the sets in the proof are in a modulo-1 sense because $X_{\ell}$ can be viewed as a subset of $\mathbb{R}/\mathbb{Z}$. Since $X_{\ell}$ is the level set of a continuous function, it can be written as the disjoint union of finitely many intervals

$$X_{\ell} = \bigcup_{i=1}^{q} (a_{i}, b_{i}),$$

where $0 \leq a_{1} \leq b_{1} \leq a_{2} \leq \cdots \leq a_{q} \leq b_{q} \leq 1$. Let $J = \{ 1 \leq j \leq q : a_{j} - b_{j-1} < \tau/K \}$, where $a_{1} - b_{0} < \tau/K$ means $a_{1} + 1 - b_{q} < \tau/K$ since the intervals are in modulo-1 sense. Then we may define

$$Y_{\ell} = \left( \bigcup_{i=1}^{q} [a_{i}, b_{i}] \right) \cup \left( \bigcup_{j \in J} [b_{j-1}, a_{j}] \right).$$

In this way, we have $\Lambda_{\ell} \subset X_{\ell} \subset Y_{\ell}$, and $Y_{\ell} \subset B_{T}(X_{\ell}, \frac{\tau}{2T}) \subset B_{T}(\Lambda_{\ell}, \frac{3\tau}{2T}) \subset B_{T}(\Lambda_{\ell}, \frac{\tau}{2})$, which means $|Y_{\ell}| \leq 3S\tau/K \leq S\eta$.

By definition, $Y_{\ell}$ is the disjoint union of some intervals of the form $[a_{i_{n}}, b_{j_{n}}]$, where $a_{i_{n}} - b_{j_{n-1}} \geq a_{i_{n}} - b_{i_{n-1}} \geq \tau/K$ and similarly $a_{i_{n+1}} - b_{j_{n}} \geq \tau/K$. If it does not contain any spike, then $\frac{1}{2}(a_{i_{n}} + b_{i_{n}})$ will be at least $\tau/K$ away from any spike. However, we know $\frac{1}{2}(a_{i_{n}} + b_{i_{n}}) \in X_{\ell}$, which contradicts with Theorem 3.1.

We would like to emphasize that $K$ dictates the estimation accuracy obtained. Even if $\omega$ is zero, the estimation error is proportional to $1/K$. Without making $K$ larger, it is not possible to make the approximation more accurate in this signal-processing routine.
3.2. The real-power model. In this subsection, we aim to recover the eigenvalues of some Hamiltonian \( H \) assuming we have access to \( e^{-2\pi i H t} \ket{\psi} \) for \( t \in \mathbb{R}_+ \), and \( U^t \) is an abbreviation of \( e^{-2\pi i H t} \).

Without loss of generality, we can assume \( \Lambda \subset [0,0.9] \). Otherwise, one can prescale \( H \) appropriately. Hence the initialization \( E_{-1} \) is defined as \([0,0.9]\). In this way, the property \( q \) can be guaranteed at step \( \ell = 0 \). Otherwise, if the signal processing subroutine gives an interval \( I_{0,i} \) that is \([\gamma_1, \gamma_2] \mod 1 \) with \( \gamma_1 < 0 \) and \( \gamma_2 > 0 \), one cannot tell whether there is a true spike in \([0, \gamma_2] \) or \([1+\gamma_1, 1] \). However, this can be avoided under this assumption because the eigenvalues are estimated to error level \( \eta < 0.1 \) at step \( \ell = 0 \), and the spikes near 0.9 and 0 will not interfere with each other.

As previously mentioned, a vital step in Algorithm 1 is the determination of \( m_\ell \) such that for each \( I'_{\ell,i} \), only one integer \( q_{\ell,i} \) gives \( I'_{\ell,i} + \frac{q_{\ell,i}}{M_{\ell-1} m_\ell} \cap E_{\ell-1} \neq \emptyset \). Once such \( m_\ell \) is chosen, then the properties \( \[1 \[] \) and \( \[2 \] \) are satisfied due to Corollary 3.2.

Now assume that the properties \( \[1 \[] \) and \( \[2 \] \) are already satisfied at step \( \ell - 1 \). For any \( \lambda_s \in \Lambda \), since \( \lambda_s \in E_{\ell-1} \), there must be some \( \lambda_s \in I'_{\ell,i} \mod \frac{1}{M_{\ell-1} m_\ell} \). Thus there exists at least one \( q_\ell,i \) such that \( I'_{\ell,i} + \frac{q_\ell,i}{M_{\ell-1} m_\ell} \cap E_{\ell-1} \neq \emptyset \). Choose an arbitrary such \( q_\ell,i \) and let \( I_{\ell,i} = I'_{\ell,i} + \frac{q_\ell,i}{M_{\ell-1} m_\ell} \), then \( I_{\ell,i} \subset B(\Lambda, \frac{\eta}{2M_{\ell-1}}) \) as long as \( m_\ell \geq 2 \). We also have \( \Lambda \subset E_{\ell-1} \), which implies \( I_{\ell,i} \subset B(\Lambda, \frac{\eta}{2M_{\ell-1}}) \subset B(E_{\ell-1}, \frac{\eta}{2M_{\ell-1}}) \). Therefore, if the choice of \( m_\ell \) satisfies

\[
(5) \quad \left( B\left(E_{\ell-1}, \frac{\eta}{2M_{\ell-1}}\right) + \frac{q}{M_{\ell-1} m_\ell}\right) \cap E_{\ell-1} = \emptyset, \quad \forall q \in \mathbb{Z} \setminus \{0\},
\]

then we can deduce that \( \left( I_{\ell,i} + \frac{q}{M_{\ell-1}}\right) \cap E_{\ell-1} = \emptyset \) for any \( q \in \mathbb{Z} \setminus \{0\} \) and thus the choice of \( q_\ell,i \) is unique. Moreover, we have \( E_\ell = \bigcup_{i=1}^{S} I_{\ell,i} \subset B(\Lambda, \frac{\eta}{2M_{\ell-1}}) \subset B(E_{\ell-1}, \frac{\eta}{2M_{\ell-1}}) \). Summarizing the discussion above, we obtain the following lemma:

**Lemma 3.3.** Given the previous amplifying factor \( M_{\ell-1} \) and estimation \( E_{\ell-1} \) in Algorithm 1 if the choice of \( m_\ell \) satisfies (5), then the choice of \( q_\ell,i \) in Algorithm 1 is unique, and the corresponding \( E_\ell \) satisfies

\[
(6) \quad E_\ell \subset B\left(\Lambda, \frac{\eta}{2M_{\ell-1}}\right) \subset B\left(E_{\ell-1}, \frac{\eta}{2M_{\ell-1}}\right).
\]

Before establishing (5), we first prove the following lemma.

**Lemma 3.4.** Suppose \( \{[\theta_i - \frac{\zeta_i}{M}, \theta_i + \frac{\zeta_i}{M}]\}_{i=1}^{t} \) are \( t \) disjoint intervals with \( t \leq S \) and \( \sum_{i=1}^{t} \zeta_i \leq S\zeta \). If \( \zeta \leq \frac{1}{8S(2S-1)} \), then there must be some \( m \in \{2, 4\} \) such that

\[
(7) \quad \left[ \theta_i - \frac{\zeta_i}{M} - \frac{q}{Mm}, \theta_i + \frac{\zeta_i}{M} - \frac{q}{Mm} \right] \cap \left[ \theta_j - \frac{\zeta_j}{M}, \theta_j + \frac{\zeta_j}{M} \right] = \emptyset
\]

holds for all \( 1 \leq i, j \leq t \), and \( q \in \mathbb{Z} \setminus \{0\} \).

**Proof.** For fixed \( i \) and \( j \), we define the following set:

\[
(8) \quad R_{ij} := [2, 4] \cap \left\{ m : \exists q \in \mathbb{Z} \setminus \{0\} \text{ such that } \left| \theta_i - \theta_j - \frac{q}{Mm} \right| \leq \frac{1}{M}(\zeta_i + \zeta_j) \right\},
\]

which is the set of all \( m \)’s that violate (7).
First, consider the case $i = j$. From the bound of $\zeta$, we can deduce $\frac{1}{M_M} \geq \frac{1}{M_M} \geq \frac{2S^C_M^C}{M_M} \geq \frac{1}{M}(2\zeta_i)$, and thus $R_{ii} = \emptyset$ for every $i$.

For the case $i \neq j$, we can assume $\theta_i > \theta_j$ without loss of generality because clearly $R_{ij} = R_{ji}$. Notice that the existence of such $i$ and $j$ implies $t \geq 2$. In this case, the bound of $\zeta$ again gives $\theta_i - \theta_j + \frac{1}{M_M} \geq \frac{1}{M_M} \geq \frac{2S^C_M^C}{M_M} \geq \frac{1}{M}(\zeta_i + \zeta_j)$, which means that only positive $q$’s can violate (7). Therefore, we can rewrite (8) as

$$R_{ij} = [2, 4] \cap \left( \bigcup_{q=1}^{\infty} \left[ \frac{q}{M(\theta_i - \theta_j) + (\zeta_i + \zeta_j)} - \frac{q}{M(\theta_i - \theta_j) - (\zeta_i + \zeta_j)} \right] \right).$$

We consider two cases to estimate $|R_{ij}|$. In the case that $M(\theta_i - \theta_j) \leq \frac{1}{4} - (\zeta_i + \zeta_j)$, we have $\frac{q}{M(\theta_i - \theta_j) + (\zeta_i + \zeta_j)} \geq 4$ for every $q$, so $|R_{ij}| = 0$. Another case is that $M(\theta_i - \theta_j) > \frac{1}{4} - (\zeta_i + \zeta_j)$, then the maximal $q$ that $\frac{q}{M(\theta_i - \theta_j) + (\zeta_i + \zeta_j)} \leq 4$ is $q_{max} \leq 4(M(\theta_i - \theta_j) + (\zeta_i + \zeta_j))$. Therefore

$$|R_{ij}| \leq \sum_{q=1}^{q_{max}} \left| \frac{q}{M(\theta_i - \theta_j) - (\zeta_i + \zeta_j)} - \frac{q}{M(\theta_i - \theta_j) + (\zeta_i + \zeta_j)} \right|$$

$$= \frac{2(\zeta_i + \zeta_j)}{(M(\theta_i - \theta_j) - (\zeta_i + \zeta_j))(M(\theta_i - \theta_j) + (\zeta_i + \zeta_j))} \sum_{q=1}^{q_{max}} q$$

$$\leq \frac{2(\zeta_i + \zeta_j)}{(M(\theta_i - \theta_j) - (\zeta_i + \zeta_j))(M(\theta_i - \theta_j) + (\zeta_i + \zeta_j))} (4(M(\theta_i - \theta_j) + (\zeta_i + \zeta_j))^2$$

$$= 32(\zeta_i + \zeta_j) \left( 1 + \frac{2(\zeta_i + \zeta_j)}{M(\theta_i - \theta_j) - (\zeta_i + \zeta_j)} \right)$$

$$< 32(\zeta_i + \zeta_j) \left( 1 + \frac{2(\zeta_i + \zeta_j)}{\frac{1}{4} - 2(\zeta_i + \zeta_j)} \right)$$

$$= \frac{32(\zeta_i + \zeta_j)}{(1 - 8(\zeta_i + \zeta_j))}.$$

Taking all pairs $(i, j)$ into account, we have

$$\left| \bigcup_{1 \leq i, j \leq t} R_{ij} \right| = \left| \bigcup_{1 \leq i < j \leq t} R_{ij} \right| \leq \sum_{1 \leq i < j \leq t} \frac{32(\zeta_i + \zeta_j)}{(1 - 8(\zeta_i + \zeta_j))}$$

$$< \sum_{1 \leq i < j \leq t} \frac{32(\zeta_i + \zeta_j)}{(1 - 8S\zeta)} = \frac{32(t - 1)}{1 - 8S\zeta} \sum_{i=1}^{t} \zeta_i \leq \frac{32(S - 1)S\zeta}{1 - 8S\zeta} \leq 2 = 4 - 2,$$

which means that

$$\left[ 2, 4 \right] \setminus \left( \bigcup_{1 \leq i, j \leq t} R_{ij} \right) \neq \emptyset,$$

and any element $m$ of this set satisfies (7). \qed

Since $E_{t-1} = \bigcup_{i=1}^{S_{t-1}} I_{t-1, i}$ is the union of at most $S$ disjoint intervals and $|E_{t-1}| \leq \frac{S^q}{M_{t-1}}$, its neighborhood $B \left( E_{t-1}, \frac{q}{2M_{t-1}} \right)$ must be the union of at most $S$ disjoint intervals with
\[ B \left( E_{\ell-1}, \frac{\eta}{2M_{\ell-1}} \right) \leq \frac{2^S \eta}{M_{\ell-1}}. \] This is exactly the case in Lemma 3.4 when taking \( M = M_{\ell-1} \) and \( \zeta = \eta \). Therefore, the conclusion of Lemma 3.4 guarantees that we can constructively find an \( m_\ell \) such that
\[
\left( B \left( E_{\ell-1}, \frac{\eta}{2M_{\ell-1}} \right) + \frac{q}{M_{\ell-1} m_\ell} \right) \cap B \left( E_{\ell-1}, \frac{\eta}{2M_{\ell-1}} \right) = \emptyset, \quad \forall q \in \mathbb{Z} \setminus \{0\},
\]
which is stronger than (5). Thus the requirement on \( m_\ell \) in Algorithm 1 is satisfied, i.e., for each \( I_{\ell,i} \), only one integer \( \eta \) gives \((I_{\ell,i} + \frac{q}{M_{\ell-1}}) \cap E_{\ell-1} \neq \emptyset \) due to the choice of \( m_\ell \). According to Lemma 3.4, one can choose any \( \eta \leq \frac{1}{8S(25-1)} \). We thus obtain the following theorem.

**Theorem 3.5.** Define \( \tau = \frac{1}{\pi} \log \frac{12}{\beta - \omega} \) and let \( \alpha \) be a constant that satisfies \( \alpha < \frac{\beta - \omega}{3} \). For any \( \eta \leq \frac{1}{8S(25-1)} \), if \( K \geq 3\eta^{-1} \tau \) and
\[
N_{HR} = 2 \left[ \frac{4}{\alpha^2} \log \frac{4}{\rho} + \log \left( \lceil \log_2 \frac{\eta}{\epsilon} \rceil + 1 \right) + \log(K + 1) \right],
\]
and the signal processing algorithm in Section 3.1 is used for spectrum estimation in Algorithm 1, then with probability at least \( 1 - \rho \), the output \( E_L \) satisfies
\[
\Lambda \subset E_L \subset B(\Lambda, \epsilon).
\]

The maximum runtime and total runtime are
\[
T_{\text{max}} = O \left( \eta K \epsilon^{-1} \right), \quad T_{\text{total}} = O \left( \eta K^2 N_{HR} \epsilon^{-1} \right).
\]

**Proof.** For the \( N_{HR} \) defined above, one knows from Hoeffding’s inequality that for any \( \ell \) and \( k \),
\[
\mathbb{P} \left( \left| y_{\ell}(k) - \hat{f}_\ell(k) \right| < \alpha \right) > 1 - \frac{\rho}{\lceil \log_2 \frac{\eta}{\epsilon} \rceil + 1}(K + 1).
\]
Thus \( |y_{\ell}(k) - \hat{f}_\ell(k)| < \alpha \) is true for all \( \ell \) and \( k \) with probability at least \( 1 - \rho \) by the union bound, and the rest follows from Lemma 3.3 and Lemma 3.4. \( \square \)

**Corollary 3.6.** By choosing \( \eta = \frac{1}{8S(25-1)} \) and \( K = 3\eta^{-1} \tau \), the maximum runtime is \( T_{\text{max}} = O \left( \log \frac{1}{\beta - \omega} \epsilon^{-1} \right) \) and the total complexity is \( T_{\text{total}} = \tilde{O} \left( (\beta - \omega)^{-2} S^2 \epsilon^{-1} \right) \), which achieves the Heisenberg limit.

3.3. **The integer-power model.** This subsection shows that if \( U \) is given as a black box and only (positive) integer powers of \( U \) can be accessed, Algorithm 1 can be applied with the help of prime numbers. As for the initialization, we take \( E_{-1} = [0, 1] \). Similar with Section 3.3 we can verify the uniqueness of \( q_{\ell,i} \) in Algorithm 1 with help of the following lemma:

**Lemma 3.7.** Given integer \( M_{\ell-1} \) and the set \( E_{\ell-1} \), if the choice of integer \( m_\ell \) satisfies
\[
\left( B_\tau \left( E_{\ell-1}, \frac{\eta}{2M_{\ell-1}} \right) + \frac{q}{M_{\ell-1} m_\ell} \right) \cap E_{\ell-1} = \emptyset \quad \text{mod 1 for all } q \in \mathbb{Z} \text{ that } (m_\ell M_{\ell-1}) \mid q,
\]
then the choice of \( q_{\ell,i} \) in Algorithm 1 is unique, and the corresponding \( E_\ell \) satisfies
\[
E_\ell \subset B_\tau \left( \Lambda, \frac{\eta}{2M_{\ell-1}} \right) \subset B_\tau \left( E_{\ell-1}, \frac{\eta}{2M_{\ell-1}} \right).
\]
This lemma is the modulo-1 version of Lemma 3.3 which can be directly obtained from Lemma 3.3 since \( \{m_{\ell}\} \) are all integers and all the sets we consider here are in \( \mathbb{R}/\mathbb{Z} \). In what follows, we denote the \( i \)-th prime number by \( p_i \). Here we define \( p_0 = 1 \) to unify the notations. The factor \( m_{\ell} \) in the algorithm above can be chosen with the help of the following lemma.

**Lemma 3.8.** For any \( (\theta_1, \ldots, \theta_t) \in \mathbb{R}^t \) and \( \frac{t(t-1)}{2} + 1 \) prime numbers \( p_1 < p_2 < \cdots p_{\frac{t(t-1)}{2}+1} \), there is at least one \( p_i \in \{p_1, p_2, \ldots, p_{\frac{t(t-1)}{2}+1}\} \) such that

\[
\min_{1 \leq i < j \leq t} \left| \frac{\theta_i - \theta_j}{p_i} \right| \geq \frac{1}{2p_{\frac{t(t-1)}{2}+1}}.
\]

**Proof.** When \( i = j \), it is straightforward to see \( \left| \frac{\theta_i - \theta_j}{p_i} \right| < \frac{1}{2p_{\frac{t(t-1)}{2}+1}} \) when \( p_i \nmid k \), and the case \( t = 1 \) is thus verified. In the following, we assume \( t \geq 2 \) and prove

\[
\min_{1 \leq i < j \leq t} \left| \frac{\theta_i - \theta_j}{p_i} \right| \geq \frac{1}{2p_{\frac{t(t-1)}{2}+1}}
\]

by contradiction. Suppose there is some \( (\theta_1, \ldots, \theta_t) \in \mathbb{R}^t \) such that (19) does not hold. Then for any \( p_i \in \{p_1, p_2, \ldots, p_{\frac{t(t-1)}{2}+1}\} \), there are some \((i_i, j_i)\) such that

\[
\left| \frac{\theta_i - \theta_j}{p_i} \right| < \frac{1}{2p_{\frac{t(t-1)}{2}+1}}.
\]

Since there are at most \( \frac{t(t-1)}{2} \) different values that \( |\theta_i - \theta_j| \) can take, but there are \( \frac{t(t-1)}{2} + 1 \) different prime numbers \( p_i \), there must be some \( l \) and \( l' \) with \( l \neq l' \) such that \( |k_t/p_l - k_{t'/p_{l'}}| < \frac{1}{2p_{\frac{t(t-1)}{2}+1}} \).

Hence, \( k_t/p_l = k_{t'/p_{l'}} \), otherwise \( |k_t/p_l - k_{t'/p_{l'}}| \geq 1/p_lp_{t'} \geq \frac{1}{2p_{\frac{t(t-1)}{2}+1}} \), which contradicts with \( |k_t/p_l - k_{t'/p_{l'}}| < \frac{1}{2p_{\frac{t(t-1)}{2}+1}} \). But \( p_l \) and \( p_{l'} \) are different prime numbers, so \( k_t/p_l = k_{t'/p_{l'}} \) if and only if \( k_t = mp_l \) and \( k_{t'} = mp_{l'} \) for some integer \( m \), which contradicts \( p_l \nmid k_t \) and \( p_{l'} \nmid k_{t'} \). The contradiction indicates that (19) must hold.

**Remark 3.9.** The result also applies to mutually prime integers that are not necessarily prime themselves. Moreover, this procedure only involves classical computing with polynomial complexity in terms of \( t \), so it can be implemented efficiently.

Based on Lemma 3.8, one can show that there is at least one \( m_{\ell} \in \{p_1, p_2, \ldots, p_{\frac{t(t-1)}{2}+1}\} \) that satisfies the requirements for Algorithm 1 if \( \eta < \left(3S\frac{p_{\frac{t(t-1)}{2}+1}}{\frac{t(t-1)}{2}+1}\right)^{-1} \). Since \( p_n = O(n \log n) \) (12), the requirement for \( \eta \) is \( \eta = O\left(S^{-5} \log^{-2}(S)\right) \). More precisely, in 32 it was proved that \( p_n < n(\log n + 2 \log \log n) \) for \( n \geq 4 \), thus by direct calculation one knows \( p_n < 2n(\log n + 1) \) for any \( n \geq 1 \). Hence, it suffices to have \( \eta \leq \min\{\frac{1}{2}, 3S^{-5}(0.31 + 2 \log S)^2\} \).

To establish (18), we also need the following lemma:

**Lemma 3.10.** Suppose \( \eta < \left(3S\frac{p_{\frac{t(t-1)}{2}+1}}{\frac{t(t-1)}{2}+1}\right)^{-1} \), then for step \( \ell \) (\( \ell \geq 1 \)) in Algorithm 1 one can choose a prime number \( m_{\ell} \) such that

\[
\left( B_T(E_{\ell-1}, \frac{\eta}{M_{\ell-1}}) + \frac{q}{m_{\ell}M_{\ell-1}} \right) \cap B_T(E_{\ell-1}, \frac{\eta}{M_{\ell-1}}) = \varnothing \mod 1, \quad \text{if} \quad (m_{\ell}M_{\ell-1}) \nmid q,
\]

which guarantees the construction of \( E_{\ell} \) in the algorithm.

**Proof.** Since \( |E_{\ell-1}|_T \leq \frac{S\eta}{M_{\ell-1}} \), one has \( |M_{\ell-1}E_{\ell-1}|_T \leq S\eta \) and \( \left|M_{\ell-1}B_T(E_{\ell-1}, \frac{\eta}{M_{\ell-1}})\right| = |B_T(M_{\ell-1}E_{\ell-1}, \eta)|_T \leq 3S\eta \). Let \( B_T(M_{\ell-1}E_{\ell-1}, \eta) = \bigcup_{1 \leq i \leq t} \{\theta_i - \eta_i, \theta_i + \eta_i\} \), then \( \sum_{i=1}^t \eta_i \leq \frac{3}{2}S\eta \).
According to Lemma 3.8, one can choose \( m_\ell \in \{ p_1, p_2, \ldots, p_{s_\ell-1} + 1 \} \) such that
\[
\left| \theta_i - \theta_j - \frac{q}{m_\ell} \right| \geq \frac{1}{2p_{s_\ell-1}p_{s_\ell-1} + 1} > \frac{3S\eta}{2} \geq \eta_i + \eta_j, \quad \text{if } m_\ell \nmid q.
\]
This implies that
\[
\left( B_T(M_{\ell-1}E_{\ell-1}, \eta) + \frac{q}{m_\ell} \right) \cap B_T(M_{\ell-1}E_{\ell-1}, \eta) = \emptyset \mod 1, \quad \text{if } m_\ell \nmid q,
\]
which is equivalent to
\[
(21) \quad \left( B_T(E_{\ell-1}, \eta_{M_{\ell-1}}) + \frac{q}{m_\ell M_{\ell-1}} \right) \cap B_T(E_{\ell-1}, \eta_{M_{\ell-1}}) = \emptyset \mod 1, \quad \text{if } m_\ell \nmid q.
\]
Now we prove (20) inductively. For \( \ell = 1 \), it is implied by (21) since \( M_0 = 1 \). Suppose (20) already holds for \( \ell - 1 \), then from Lemma 3.7 we know \( E_{\ell-1} \subset B_T(E_{\ell-2}, \eta_{M_{\ell-2}}) \), and thus
\[
B_T(E_{\ell-1}, \eta_{M_{\ell-1}}) \subset B_T(E_{\ell-2}, \eta_{M_{\ell-1}}) \subset B_T(E_{\ell-2}, \eta_{M_{\ell-2}}).
\]
Therefore, from the induction hypothesis, one can deduce that
\[
(22) \quad \left( B_T(E_{\ell-1}, \eta_{M_{\ell-1}}) + \frac{q'}{M_{\ell-1}} \right) \cap B_T(E_{\ell-1}, \eta_{M_{\ell-1}}) = \emptyset \mod 1, \quad \text{if } M_{\ell-1} \nmid q',
\]
where we used \( m_{\ell-1}M_{\ell-2} = M_{\ell-1} \). If we let \( q = m_\ell q' \) in (22), then \( M_{\ell-1} \nmid q' \) is equivalent to \( m_\ell \nmid q, \quad (m_\ell M_{\ell-1}) \nmid q \), so (20) is proved combining (21) and (22).

Similar to Theorem 3.5, one directly obtains the following theorem by applying Hoeffding’s inequality and the union bound to Lemma 3.7 and Lemma 3.10.

**Theorem 3.11.** Define \( \tau = \frac{1}{\pi} \log \frac{12}{\beta - \omega} \) and let \( \alpha \) be a constant that satisfies \( \alpha < \frac{\beta - \omega}{3} \). For any \( \eta \leq \min \left\{ \frac{1}{6}, \frac{1}{3S^3(0.31 + 2\log S)} \right\} \), if \( K \geq 3\eta^{-1} \tau \) and
\[
N_{HR} = 2 \left\lceil \frac{4}{\alpha^2} \left( \log \frac{4}{\rho} + \log \left( \left\lceil \log_2 \frac{\eta}{\epsilon} \right\rceil + 1 \right) + \log(K + 1) \right) \right\rceil,
\]
and the signal processing algorithm in Section 3.1 is used for spectrum estimation in Algorithm 1, then with probability at least \( 1 - \rho \), the output \( E_L \) satisfies
\[
(24) \quad \Lambda \subset E_L \subset B_T(\Lambda, \epsilon).
\]
The maximum runtime and total runtime are
\[
(25) \quad T_{\text{max}} = O(\eta K \epsilon^{-1}), \quad T_{\text{total}} = O(\eta K^2 N_{HR} \epsilon^{-1}).
\]

**Corollary 3.12.** By choosing \( \eta = \min \left\{ \frac{1}{6}, \frac{1}{3S^3(0.31 + 2\log S)} \right\} \) and \( K = 3\eta^{-1} \tau \), the maximum runtime is \( T_{\text{max}} = \tilde{O} \left( \log \left( \frac{1}{\beta - \omega} \right) \epsilon^{-1} \right) \) and the total complexity is \( T_{\text{total}} = \tilde{O} \left( (\beta - \omega)^{-2} S^5 \epsilon^{-1} \right) \), which achieves the Heisenberg limit.
4. THE GAPPED CASE

This section specializes Algorithm 1 to the simpler gapped case, i.e., there is a minimum separation among the dominant eigenvalues defined as:

\[
\Delta = \min_{1 \leq s \leq s', \lambda_s - \lambda_{s'} - n} |.|
\]

We first review the famous signal processing routine ESPRIT in Section 4.1. In particular, the number of frequencies \( K \) needed only depends on \( S \) and \( \Delta \). With the help of this specific version of ESPRIT, we show that Algorithm 1 meets the requirements 1, 2, 3 and 4. In particular, the maximum runtime \( T_{\text{max}} \) can scale like \( \omega/\beta \). The real-power and integer-power models are investigated in Section 4.2 and Section 4.3, respectively. A graphical illustration of the algorithm for the gapless case is given in Figure 3.

**Figure 3.** Illustration of Algorithm 1 in the gapped case with \( S = 2 \). Based on prior knowledge of the separation between \( \lambda_1 \) and \( \lambda_2 \), one can obtain much tighter initial-stage estimations than the gapless case (see Figure 2) if the residual \( \omega \) is small. As a result, much fewer steps are needed to reach the desired precision \( \epsilon \), and the maximum runtime is reduced.

### 4.1. SIGNAL PROCESSING ROUTINE

Since a finite-size gap between the dominant eigenvalues is available, multiple signal processing algorithms can be used in this setting, such as the ones in [5, 20, 24]. Here we adopt a particular version of the ESPRIT algorithm discussed in [20]. Suppose that \( K \geq 4S \) is even. Recall that at the \( \ell \)-th step, the data \( y_\ell(k) \) collected from the Hadamard test results satisfy \( |y_\ell(k) - \hat{f}_\ell(k)| \leq \alpha \), where \( \hat{f}_\ell \) is defined in [3]. The first step in ESPRIT is to construct the following Hankel matrix.

\[
H_\ell = \begin{bmatrix}
y_\ell(0) & y_\ell(1) & \ldots & y_\ell(\frac{K}{2}) \\
y_\ell(1) & y_\ell(2) & \ldots & y_\ell(\frac{K}{2} + 1) \\
\vdots & \vdots & \ddots & \vdots \\
y_\ell(\frac{K}{2}) & y_\ell(\frac{K}{2} + 1) & \ldots & y_\ell(K)
\end{bmatrix}
\]

Then one applies the singular value decomposition (SVD) to \( H_\ell \) and obtains

\[
H_\ell = [U_\ell, U_\ell^\perp] \Sigma [V_\ell, V_\ell^\perp]^*.
\]
where \( U_\ell \) has \( S \) columns. Let the first \( K/2 \) rows of \( U_\ell \) be \( U_\ell^{(0)} \) and let its last \( K/2 \) rows be \( U_\ell^{(1)} \). The last step is to compute the eigenvalues \( \{\mu_1, \ldots, \mu_S\} \) of the matrix \((U_\ell^{(0)})^\dagger U_\ell^{(1)}\), where \((U_\ell^{(0)})^\dagger\) is the Moore-Penrose pseudo-inverse of \( U_\ell^{(0)} \). The output is the set
\[
\tilde{\Lambda}_\ell := \left\{-\frac{1}{2\pi} \arg(\mu_1), \ldots, -\frac{1}{2\pi} \arg(\mu_S)\right\}.
\]

The following theorem is proved in [20] for this particular version of ESPRIT:

**Theorem 4.1.** Suppose \( K \geq 4S \) is even and \( K \geq \frac{4}{\Delta} \). For any constant \( C \in (2, \frac{K\Delta}{2}) \), if
\[
\omega + \alpha \leq \frac{C\beta}{8(C-1)\sqrt{2S}} \sqrt{1 - \frac{2(C-1)S}{CK}},
\]
then
\[
\text{md}(\tilde{\Lambda}_\ell, \Lambda_\ell) \leq \frac{40S^2}{\beta} \sqrt{\frac{C^3(2 + K)}{(C-1)^3K}} \left(1 - \frac{2CS}{(C-1)K}\right)^{-1} (\omega + \alpha),
\]
where \( \text{md}(\cdot, \cdot) \) denotes the matching distance between two finite sets with the same cardinality.

We emphasize that the approximation error is controlled by the noise in the gapped case. The number of frequency \( K \) only needs to be proportional to \( S \), which is considered a fixed number. This means that the maximum depth of the circuit can be kept fixed. Only more repetitions are needed to bring down the noise level.

When \( \omega \) is sufficiently small compared with \( \beta \), [27] can always be satisfied by properly choosing \( \alpha \). The set \( Y_\ell \) needed in the main algorithm can then be defined as
\[
Y_\ell = B_T(\tilde{\Lambda}, \frac{\eta}{2}),
\]
with \( \eta \) satisfying
\[
\frac{80S^2}{\beta} \sqrt{\frac{C^3(2 + K)}{(C-1)^3K}} \left(1 - \frac{2CS}{(C-1)K}\right)^{-1} (\omega + \alpha) < \eta.
\]
This \( Y_\ell \) is guaranteed to satisfy the three properties stated in Corollary 3.2.

In the following sections, we provide modified versions of the algorithms described in Section 3 and show that when \( \omega \) is close to 0, one can further reduce the maximal runtime by a factor \( \Omega(\omega/\beta) \).

### 4.2. The real-power model.

The following theorem states the \( T_{\text{max}} \) and \( T_{\text{total}} \) bounds for our RMPE algorithm for the real-power model under the gapped case.

**Theorem 4.2.** Let \( \hat{\Delta} < \min\{\frac{1}{8S(2S-1)}, \Delta\} \), \( K > 4/\hat{\Delta} \) and \( \eta \) be any number such that
\[
\frac{80S^2}{\beta} \sqrt{\frac{C^3(2 + K)}{(C-1)^3K}} \left(1 - \frac{2CS}{(C-1)K}\right)^{-1} \omega < \eta < \frac{1}{8S(2S-1)},
\]
where \( C \) is a constant such that \( C \in (2, \frac{K\hat{\Delta}}{2}) \). Suppose that \( \alpha \) satisfy
\[
\alpha < \min\left\{\frac{C\beta}{8(C-1)\sqrt{2S}} \sqrt{1 - \frac{2(C-1)S}{CL}}, \frac{\beta}{80S^2} \sqrt{\frac{(C-1)^3K}{C^3(2 + K)}} \left(1 - \frac{2CS}{(C-1)K}\right) \eta\right\} - \omega,
\]
and $N_{HR}$ satisfy
\begin{equation}
N_{HR} = 2 \left[ \frac{4}{\alpha^2} \left( \log \frac{4}{\rho} + \log \left( \left\lceil \log_2 \frac{\eta}{\epsilon} \right\rceil + 1 \right) + \log(K + 1) \right) \right],
\end{equation}
and the ESPRIT routine described in Section 4.1 is used for spectrum estimation in Algorithm 1, then with probability at least $1 - \rho$, the output $E_L$ satisfies
\begin{equation}
\Lambda \subset E_L \subset B(\Lambda, \epsilon).
\end{equation}
The maximum runtime and total runtime are
\begin{equation}
T_{\text{max}} = O \left( \eta K \epsilon^{-1} \right), \quad T_{\text{total}} = O \left( \eta K^2 N_{HR} \epsilon^{-1} \right).
\end{equation}

\textit{Proof.} We will prove that Algorithm 1 will work at each step $\ell$. The procedure is almost identical to Theorem 3.5. We only need to check that after step $\ell - 1$ we can choose an $m_\ell \in [2, 4]$ that satisfies both (5) and the gap $\Delta_\ell := \min_{1 \leq i < j \leq S} |M_\ell \lambda_i - M_\ell \lambda_j|$ is greater than $\tilde{\Delta}$, so that ESPRIT can work within the error bound given in Theorem 4.1. In step $\ell - 1$ of Algorithm 1, we can already bound $\lambda_i$ in a interval with length less than $\eta / M_\ell$, which can be denoted by $\lambda_i \in \left[ \theta_i - \frac{\eta}{2M_\ell}, \theta_i + \frac{\eta}{2M_\ell} \right]$. Therefore, $\Delta_\ell \geq \tilde{\Delta}$ is equivalent to make the $S$ intervals
\begin{equation}
\left[ \theta_i - \frac{\eta}{2M_\ell}, \theta_i + \frac{\eta}{2M_\ell} \right] \quad (1 \leq i \leq S)
\end{equation}
disjoint in modulo-$\frac{1}{M_\ell}$ sense. Since $M_\ell \geq 2$, we can relax the term $\frac{\tilde{\Delta}}{m_\ell M_\ell}$ as $\frac{\tilde{\Delta}}{2M_\ell}$. Note that (5) is equivalent to say that the $S$ intervals
\begin{equation}
\left[ \theta_i - \frac{3\eta}{4M_\ell}, \theta_i + \frac{3\eta}{4M_\ell} \right] \quad (1 \leq i \leq S)
\end{equation}
are disjoint in modulo-$\frac{1}{M_\ell m_\ell}$ sense. Therefore, we conclude that we can find a proper $m_\ell$ by letting $M = M_\ell$ and $\zeta_i = \max\left\{ \eta + \frac{\tilde{\Delta}}{2}, \frac{3\eta}{4} \right\}$ in Lemma 3.4 since $\sum_{i=1}^{S} \zeta_i = S \max\left\{ \eta + \frac{\tilde{\Delta}}{2}, \frac{3\eta}{4} \right\} \leq \frac{1}{8S(2S - 1)}$.

After proving that the ESPRIT algorithm described in Section 4.1 can be used in Algorithm 1 given the existence of the spectral gap, the rest of the proof of the complexity bounds are the same as Theorem 3.5. \hfill \square

\textbf{Corollary 4.3.} By setting $\eta = \Theta(\frac{\omega}{\beta})$, the complexity bounds given in the theorem are
\begin{equation}
T_{\text{max}} = O \left( (\Delta^{-1} + S^2) S^2 \omega \beta^{-1} \epsilon^{-1} \right), \quad T_{\text{total}} = \tilde{O} \left( (\Delta^{-1} + S^2) S^2 \omega^{-1} \beta^{-1} \epsilon^{-1} \right).
\end{equation}
The prefactor of $T_{\text{max}}$ is proportional to $\omega \beta^{-1}$, the ratio between the energy of the residual in $|\psi\rangle$ and the energy lower bound of the dominant eigenmodes.

Therefore, the maximum circuit depth prefactor can be made small when this ratio is sufficiently small. However, $T_{\text{total}}$ will increase for small $\omega$, so there is a trade-off between maximal runtime and total runtime, and one should choose $\eta$ appropriately in practice.
4.3. The integer-power model. Similar to Section 3.3, one can implement the algorithm with only integer powers of $U$. The following theorem states the $T_{\text{max}}$ and $T_{\text{total}}$ bounds for our RMPE algorithm for the integer-power model under the gapped case.

**Theorem 4.4.** Let $\tilde{\Delta} < \min\{\frac{1}{2p_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}}, \Delta\}$, $K > 4/\tilde{\Delta}$ and $\eta$ be any number such that

$$80S^2 \frac{\beta}{\beta} \sqrt{C^3(2 + K)} \left(1 - \frac{2CS}{(C - 1)K}\right)^{-1} - \omega < \frac{1}{4Sp_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}},$$

where $C$ is a constant such that $C \in (2, \frac{K\Delta}{2})$. Let $\alpha$ satisfy

$$\alpha < \min\left\{\frac{C\beta}{8(S - 1)\sqrt{S}}, \frac{\beta}{80S^2} \sqrt{(C - 1)^3K} \left(1 - \frac{2CS}{(C - 1)K}\right)\right\} - \omega,$$

and

$$N_{HR} = 2 \left\lfloor \frac{4}{\alpha^2} \left(\log \frac{4}{\rho} + \log \left(\left\lceil \log_2 \frac{\eta}{\epsilon} \right\rceil + 1\right) + \log(K + 1)\right) \right\rfloor.$$  

Suppose ESPRIT is used for spectrum estimation in Algorithm 1 and $m_\ell$ is chosen according to Lemma 3.10, then with probability at least $1 - \rho$, the output $E_L$ satisfies

$$\Lambda \subset E_L \subset B(\Lambda, \epsilon).$$

The maximum runtime and total runtime are

$$T_{\text{max}} = O(\eta K \epsilon^{-1}), \quad T_{\text{total}} = O(\eta K^2 N_{HR} \epsilon^{-1}).$$

**Proof.** From ESPRIT’s assumptions and properties, $E_1$ already has $S$ disjoint intervals, each containing an actual spike. All steps in the proof of Lemma 3.10 go through except that one needs to check the assumptions for Theorem 4.4 for each iteration in Algorithm 1. In other words, one must check that the minimum separation among $\Lambda_\ell$ is bounded from below by $\tilde{\Delta}$. We prove this by enhancing the induction hypothesis in the proof of Lemma 3.10 with an additional condition:

$$\min_{1 \leq s, s' \leq S, n \in \mathbb{Z}} |M_\ell(\lambda_s - \lambda_{s'}) - n| \geq \tilde{\Delta}.$$  

When $\ell = 0$, this follows from the definition of $\tilde{\Delta}$. Now assume that this holds for $\ell - 1$. By the choice of $m_\ell$ in the proof of Lemma 3.10, one has

$$\min_{1 \leq s, s' \leq S, q \in \mathbb{Z}, m_\ell | q} \left|\theta_s - \theta_{s'} - \frac{q}{m_\ell}\right| \geq \frac{1}{2p_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}}.$$  

where $\theta_s$ is the center of the $s$-th interval in $E_{\ell-1}$, which is also the $s$-th element in $\Lambda_{\ell-1}$ defined in Theorem 4.4. By the property of $Y_\ell$, one has

$$\min_{1 \leq s, s' \leq S, q \in \mathbb{Z}, m_\ell | q} \left|M_{\ell-1}(\lambda_s - \lambda_{s'}) - \frac{q}{m_\ell}\right| \geq \min_{1 \leq s, s' \leq S, q \in \mathbb{Z}, m_\ell | q} \left|\theta_s - \theta_{s'} - \frac{q}{m_\ell}\right| - S\eta \geq \frac{1}{4p_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}}.$$  

Thus

$$\min_{1 \leq s, s' \leq S, q \in \mathbb{Z}, m_\ell | q} \left|M_{\ell}(\lambda_s - \lambda_{s'}) - q\right| \geq \frac{m_\ell}{4p_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}} \geq \frac{1}{2p_{\frac{1}{2}S(S-1)p_{\frac{1}{2}S(S-1)+1}}} \geq \tilde{\Delta}.$$  

By the induction hypothesis, one also has

\[ \min_{1 \leq s \leq s' \leq S, n \in \mathbb{Z}} |M_{\ell-1}(\lambda_s - \lambda_{s'}) - n| \geq \tilde{\Delta}, \]

which means

\[ \min_{1 \leq s \leq s' \leq S, q' \in \mathbb{Z}} |M_{\ell}(\lambda_s - \lambda_{s'}) - m\ell q'| \geq m\ell \tilde{\Delta} > \tilde{\Delta}, \]

and combining with (40) one obtains

\[ \min_{1 \leq s \leq s' \leq S, n \in \mathbb{Z}} |M_{\ell}(\lambda_s - \lambda_{s'}) - n| \geq \tilde{\Delta}. \]

The other steps in the proof of Lemma 3.10 can be used directly to show that the algorithm works, and the arguments for the complexity bounds are the same as the ones in Theorem 3.11. □

**Corollary 4.5.** When \( \omega \) is sufficient close to zero, one can take \( \eta = \Theta(\frac{\omega}{\beta}) \), then since \( K \) is a constant (recall that \( K \geq \frac{1}{\tilde{A}} \)) we have

\[ T_{\text{max}} = \tilde{O} \left( (\Delta^{-1} + S^4)S^2\omega\beta^{-1}e^{-1} \right), \quad T_{\text{total}} = \tilde{O} \left( (\Delta^{-1} + S^4)S^2\omega^{-1}\beta^{-1}e^{-1} \right), \]

Here the maximum circuit depth prefactor again scales like \( O(\omega\beta^{-1}) \).

5. Discussions

This paper proposed robust multiple-phase estimation (RMPE) algorithms for estimating multiple eigenvalues. The proposed algorithms address both the gapless case in Section 3 and the gapped case in Section 4 and for both the integer-power and real-power models.

When the problem is gapless, according to Section 3, the number of frequencies \( K \) needs to be increased if one needs an estimation with a smaller error level \( \eta \) from the signal processing routine, which prevents us from reducing the maximum runtime even when the magnitude of the residual \( \omega \) is close to zero. When a finite-size spectral gap is assumed, the number of frequencies \( K \) is a constant independent of \( \eta \). An immediate direction is to explore signal processing algorithms that can improve \( K \)’s dependence on \( \eta \) and the spectral gap, enabling us to reduce the prefactor in the maximum runtime when \( \omega \) is small.

Another relevant problem comes from the bound in Lemma 3.8.

\[ \max_{1 \leq t \leq \frac{(t-1)}{2} + 1, 1 \leq i \leq j \leq t} \min_{p_i|k} \left| \theta_i - \theta_j - \frac{k}{p_i} \right| \geq \frac{1}{2p_{t(t-1)}p_{t(t-1)+1}}, \]

for any \( \{\omega\}_{i=1}^{t} \). However, one can also show that

\[ \inf_{(\theta_i)_{i=1}^{t}} \max_{1 \leq t' \leq t} \min_{p_i|k} \left| \theta_i - \theta_j - \frac{k}{p_i} \right| = \gamma(t') > 0, \]

for any \( t' > t \). The conclusion of Lemma 3.8 is thus \( \gamma(t') \) is at least \( \frac{1}{2p_{t(t-1)}p_{t(t-1)+1}} \). It would be interesting to find a sharper estimation for \( \gamma(t') \) and, thereby, the optimal choice of \( t' \) in terms of the overall complexity.
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