18.112: Functions of a Complex Variable

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Introduction

The course syllabus can be found on Stellar, and we were also given a paper copy in class.

This class will assign nine problem sets, roughly due every week except for the three weeks with exams. The two midterms will be quiz-style – they're only going to be 80 minutes long, since that's how much time we have for class, and they'll be on October 3rd and November 5th. (The questions will be pretty basic for the quizzes.) The final exam will take place during finals week, date and location to be determined. Grading is 30% homework (one problem set dropped), 30% final, and 20% for each midterm.

Collaboration is allowed, but we're encouraged to solve the problems ourselves first. If we do talk with others, we should indicate all collaborators on the homework itself. As per policy, late homework won't be accepted, but certain situations can be accommodated e.g. from S^3.

1 September 5, 2019

Let's start with some motivation for why we care about functions of a complex variable.

Fact 1 (Notation)

$\mathbb{R}$ will denote the reals, $\mathbb{Z}$ will denote the integers, and $\mathbb{N}$ will denote the natural (positive) integers throughout this class.

Here, $\mathbb{R} \supset \mathbb{Z} \supset \mathbb{N}$, and we'll be dealing with a superset of the real numbers: $\mathbb{C}$, the complex numbers. We can identify elements of $\mathbb{C}$ naturally as ordered pairs of real numbers: in other words, $\mathbb{C}$ can be thought of as a 2-dimensional vector space over $\mathbb{R}$. To review, any complex number z can be written as $x + yi$ for real numbers x and y, where $i = \sqrt{-1}$.

Remark 2. *Mathematics defies English here: we say "I is" instead of "I am."*

But why do we even care about $\mathbb{C}$ in the first place – are there any applications for studying this?

Definition 3

Prime numbers are the set of natural numbers whose only divisors are 1 and themselves. The **prime counting function**

$$\pi(x) = \text{number of primes } p : p < x$$

counts the number of primes less than a real number $x \in \mathbb{R}_{>0}$.

One important result in number theory is the following result:

Theorem 4 (Prime number theorem)
Asymptotically,

$$\pi(x) \sim \frac{x}{\log x}.$$

That is, the limit

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

Obviously, this theorem statement has nothing to do with complex numbers (technically, integers are all we really need), but it turns out that the right way to think about this theorem is to use the **Riemann zeta function**

$$\zeta(s) = \sum_{n\geq 1}^{\infty} \frac{1}{n^s}.$$

(For example, we know that $\zeta(2) = \frac{\pi^2}{6}$.) The important change of perspective here is that we can replace s with an arbitrary complex number, and there are a lot of interesting things that come out of this! Obviously, $\zeta(s)$ isn't convergent everywhere, but it will turn out to be convergent for $\mathrm{Re}(s) > 1$, and we'll study this a bit more later: it turns out we can prove the prime number theorem using the complex Riemann zeta function.

One good way to start studying functions of a complex variable is to restrict our attention to some "nice" functions. When we have an arbitrary continuous function $f : \mathbb{R} \to \mathbb{R}$, we generally like it a lot when we can consider the derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

If this quantity f' exists, then we call our function **differentiable**. Let's see if we can copy this over to complex numbers:

Definition 5
Fix a point $z \in \mathbb{C}$. Given a function $f : \mathbb{C} \to \mathbb{C}$, define the **derivative**

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h},$$

if it exists.

We should be a bit more specific here with what we want: since h is not just a real number anymore, what does it mean for h to go to 0? To answer this, we need to think about the idea of **convergence**.

Definition 6
A sequence of complex numbers $\{z_n\}$ **converges** to z if

$$\lim_{n\to\infty} |z_n - z| = 0.$$

(Note that this last statement can be interpreted as an expression in the real numbers by looking at the real and imaginary parts at the same time.) Let's try to introduce a few more real analysis concepts into our language here, as this will make later study easier:

Definition 7

$\{z_n\}$ is a **Cauchy sequence** if

$$|z_n - z_m| \to 0 \text{ as } n, m \to \infty.$$

Definition 8

A set Ω is **open** if every point has an open neighborhood contained in Ω. A set is closed if its complement is open.

Since $\mathbb{C}$ can be treated similarly to $\mathbb{R}^2$, we can also use the following definition:

Definition 9

A set Ω is **compact** if it is closed and bounded.

Example 10

$\overline{D}(z_0, r)$, the closed disk centered at z_0 with radius r, is compact. However, $D(z_0, r)$, the open disk, is not compact, though it is bounded.

The reason we care about compact sets is that **every sequence whose elements are in a compact set has a convergent subsequence.**

Definition 11

Ω is **connected** if we cannot write it as a disjoint union $\Omega_1 \sqcup \Omega_2$ of open sets. Ω is **path-connected** if any two points x, y in Ω can be connected by a path that lies inside Ω. (We'll call open connected sets **regions**.)

In $\mathbb{R}^n$, being open and connected is the same as being path-connected, but this isn't true in a general topological space.

Definition 12

Let Ω be an open subset of $\mathbb{C}$ (which can be $\mathbb{C}$ itself). A function $f : \Omega \to \mathbb{C}$ is **holomorphic at z** if the derivative exists at z, and f is **holomorphic on Ω** if f is holomorphic at every $z \in \Omega$.

Demanding that f be holomorphic is actually pretty strong: h can approach 0 from basically any angle in the complex plane, and we need to arrive at the same answer in all directions.

Example 13

The function $f(z) = z^n$ is holomorphic for all n: expanding out $(z + h)^n$ with the binomial theorem is still valid, so the power rule still holds and the derivative exists: $f'(z) = nz^{n-1}$. However, $g(z) = \overline{z}$ is not holomorphic:

$$\frac{g(z+h) - g(z)}{h} = \frac{\overline{h}}{h},$$

and notice that h and $\overline{h}$ will have the same norm as h approaches zero, but the arguments will be different (depends on the angle of approach). Thus, $g'(z)$ does not exist.

This means that we somehow want functions that involve only z, not $\overline{z}$: how do we quantify this? Notice that when we have a function $f : \mathbb{C} \to \mathbb{C}$, we can think of this as taking ordered pairs $(x, y) \in \mathbb{R}^2$ to other ordered pairs $(u, v) \in \mathbb{R}^2$. So let's try to study f in this new context!

Theorem 14 (Cauchy-Riemann equations)
Let Ω be an open set, and let $f : \Omega \to \mathbb{C}$ be a function where $z = x + iy$ and $f(z) = u + iv$. Then f is holomorphic on Ω if and only if u and v are differentiable, satisfying

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Notice that u and v are both functions of x and y, so we have four partial derivatives, and all four of them are being used here.

Proof. We can get expressions in these partial derivatives by looking at specific limits: for example,

$$\lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(x + iy + h) - f(x + iy)}{h} = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Similarly, if we go along the imaginary axis,

$$\lim_{\substack{h \in \mathbb{R} \\ h \to 0}} \frac{f(x + i(y + h)) - f(x + iy)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y} = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

But if the limit exists, these two are both equal to $f'(x + iy)$, so we can equate real and imaginary parts:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{1}{i}\frac{\partial u}{\partial y} = i\frac{\partial v}{\partial x}.$$

□

We can state this equivalently in the following way:

Proposition 15
A function f is holomorphic if and only if

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}.$$

Proof. If f is holomorphic, we know that

$$f(z + h) = f(z) + f'(z)h + h\phi(h),$$

where $\phi(h) \to 0$ as $h \to 0$. (This can also be written as $df = f'(z)dz$.) We want to (intuitively) write down something that looks like

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \overline{z}}d\overline{z}$$

with the multivariable chain rule. To do this, note that

$$df = d(u + iv) = du + idv = u_x dx + u_y dy + iv_x dx + iv_y dy,$$

but we can also relate x and y to z and $\overline{z}$:

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i},$$

which is equivalently (in differential form)

$$dx = \frac{1}{2}dz + \frac{1}{2}d\overline{z}, \quad dy = \frac{1}{2i}dz - \frac{1}{2i}d\overline{z}.$$

So we can actually use the multivariable chain rule to say that

$$\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z},$$

and substituting in the values of $\frac{\partial x}{\partial z}$ and $\frac{\partial y}{\partial z}$ from above, we find that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

Similarly, we find that

$$\frac{\partial f}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\frac{\partial f}{\partial y}\right).$$

The Cauchy-Riemann equations tell us that this last quantity should be 0 (by directly writing out $f = u + iv$), so $\frac{\partial f}{\partial \overline{z}} = 0$ if and only if f is holomorphic, which yields the result we want. □

This hopefully gives us a better understanding of why we "don't want $\overline{z}$ in our holomorphic functions."

2 September 10, 2019

Last time, we introduced a class of functions of the form $f : \Omega \to \mathbb{C}$ (where Ω is an open subset) which we call **holomorphic**. (Recall that we're implicitly associating $\mathbb{C}$ with $\mathbb{R}^2$ in a lot of the analysis definitions.) Holomorphic functions are those that are differentiable and satisfy the Cauchy-Riemann equations; these equations can be written as $\frac{\partial f}{\partial \overline{z}} = 0$, meaning that our function should depend essentially on z but not $\overline{z}$. As an example, recall that polynomials in z are holomorphic, but polynomials in $\overline{z}$ are not.

The reason we care about holomorphic functions is that we can define a derivative in the same way as we do for real-valued functions: for instance,

$$p(z) = \sum_{i=0}^{n} a_i z^i \implies p'(z) = \sum_{i=0}^{n} i a_i z^{i-1}.$$

Definition 16

The exponential function is defined for complex numbers in a similar way as it is for real numbers:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Before we can use this, we need to be careful to ensure that the series exists:

Proposition 17

Let $\{a_n\}$ be a sequence of complex numbers. Then $\sum_{n=0}^{\infty} a_n$ is convergent if $\sum_{n=0}^{\infty} |a_n|$ is convergent.

(This is because the real and imaginary parts must converge independently.) So now note that $\sum_{n=0}^{\infty} \frac{|z|^n}{n!}$ converges (by the ratio test for $|z| \in \mathbb{R}$), so e^z is indeed defined for all $z \in \mathbb{C}$.

We claim that e^z is indeed holomorphic, but we postpone this until later!

Example 18

Consider the function

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, \quad |z| < 1.$$

Notice that the area of convergence is now an open disk (in $\mathbb{C}$) rather than just an interval (in $\mathbb{R}$), which gives us more freedom. Notably, if we want to show that this function is holomorphic, we need to make the open disk our region Ω rather than all of $\mathbb{C}$. Ultimately, the idea is that the inverse of any polynomial will stay holomorphic, as long as we avoid the regions where the denominator is 0.

Definition 19

A complex **power series** takes the form

$$\sum_{n=0}^{\infty} a_n z^n,$$

where all coefficients $a_n \in \mathbb{C}$.

Definition 20

Given a sequence of real numbers $\{a_i\}$, define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \left(\sup_{m\ge n} \{a_m\} \right).$$

$\liminf$ is defined similarly, but with $\inf$ instead of $\sup$.

The inner term here is the smallest upper bound after dropping the first n terms, which gives us a decreasing sequence. The limit of a decreasing sequence must exist! So the $\limsup$ of a sequence a_n will always exist (though it can be $\pm\infty$). Notably, if $\lim_{n\to\infty} a_n$ exists, it will take on the same value as $\limsup_{n\to\infty} a_n$.

But it's possible that a function doesn't have a limit but does have a helpful $\limsup$: for example, if we have two sequences $\{a_i\}$ and $\{b_i\}$ with different limits A and B, we can alternate them. Then the sequence $(a_1, b_1, a_2, b_2, \cdots)$ does not have a limit, but the $\limsup$ will be $\max(A, B)$.

Theorem 21 (Hadamard formula)

Define R such that

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n} \in [0, \infty].$$

Then $\sum_{n\ge 0} a_n x^n$ is convergent whenever $|z| < R$ and divergent whenever $|z| > R$.

Convergence where $|z| = R$ is a bit more subtle: it's possible that the series is convergent at some points and divergent at others. Also, note that the ratio test gives the same answer as the Hadamard formula if the limit of ratios exists!

Example 22
For our exponential function e^z,

$$\frac{1}{R} = \limsup_{n\to\infty} \left(\frac{1}{n!}\right)^{1/n} = 0,$$

so $R = \infty$. Similarly, for the function $\frac{1}{1-z}$,

$$\frac{1}{R} = \lim_{n\to\infty} 1 = 1,$$

so $R = 1$.

One other way to show convergence in this last case: the partial sums for the geometric series $\sum_{n=0}^{N} z^n = \frac{1-z^{N+1}}{1-z}$ approach $\frac{1}{1-z}$ if $|z| < 1$, because $z^{N+1} \to 0$.

So now let's return to the question from earlier in the class: do we have a nice notion of differentiability here?

Theorem 23
Define $f(z) = \sum_{n\geq 0} a_n z^n$ for $z \in D_0(R)$ (the open disk of convergence). Then f is holomorphic, and

$$f'(z) = \sum_{n\geq 0} n a_n z^{n-1}.$$

In addition, the radius of convergence for $f'(z)$ is the same as the radius of convergence for $f(z)$.

To show this last fact, notice that Hadamard's formula tells us that the radius for f' satisfies

$$\frac{1}{R} = \limsup_{n\to\infty} |(n+1)a_{n+1}|^{1/n} = \limsup_{n\to\infty} |na_n|^{1/n}$$

(this last equality holds because the ratio of the exponents goes to 1, and both limits exist). Then $\lim_{n\to\infty} n^{1/n} = 1$ (we can see this via $n = e^{\log n} \implies n^{1/n} = e^{\log n/n}$), so

$$\limsup_{n\to\infty} |na_n|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n},$$

which is the radius of convergence for f.

Proof sketch that f' is holomorphic. The whole point is that if we just have a polynomial, we can differentiate term by term. So the work we need to do is to show that adding in a limit doesn't change too much: indeed, because the higher-order terms go to 0 for all z in the open disk, everything is okay. $\square$

We can then compute derivatives as usual: e^z is its own derivative, as we expect. Notice, though, that if a power series has a radius of convergence R, and its derivative also has radius of convergence R, we can repeat this arbitrarily!

Corollary 24
Power series of the form in Theorem 23 are infinitely differentiable:

$$f^{(k)}(z) = (f^{(k-1)})' = \sum_{n\geq 0}^{\infty} n(n-1)\cdots(n-k+1)a_n z^{n-k}.$$

We can also generalize our power series to be centered around any $z_0 \in \mathbb{C}$: this just replaces z with $(z - z_0)$ everywhere. But still, the open set where we're defining our function is still an open disk $D_{z_0}(R)$: is there a way to generalize this?

Definition 25

A function on an open set $\Omega \subseteq \mathbb{C}$ is called **analytic** if for every $z_0 \in \Omega$, there exists a power series $g(z) = \sum_{n \geq 0} a_n (z - z_0)^n$ with radius of convergence $R > 0$, such that $f(z) = g(z)$ for all $z \in D_{z_0}(R)$.

Intuitively, this means that at every point, f is equal to its power series, though the radius of convergence is allowed to vary. This is great, because Ω can now be an open set that isn't just shaped like a disk!

Corollary 26

By Theorem 23, any analytic function is holomorphic, since we just check convergence at every point.

Example 27

The function $f(z) = \frac{1}{1 - z}$ is holomorphic at all $z \neq 1$, so f is analytic on the domain $\mathbb{C} - \{1\}$.

Remember that the definition of holomorphic is pretty simple: it's differentiable, and $\frac{\partial}{\partial \overline{z}} f = 0$. When we work with real-valued functions, this isn't that powerful: if $f'(x)$ exists, there's no real reason that $f''(x)$ should exist. What's striking is that **holomorphic functions are always analytic!** This also tells us that if $f'(z)$ exists for a complex-valued function, then $f''(z)$ must automatically exist!

To prepare for that proof, let's try to discuss a notion of integration. Given a function f', does there exist an antiderivative?

Definition 28

Fix a function $f : \Omega \to \mathbb{C}$ (not necessarily holomorphic). A function $F : \Omega \to \mathbb{C}$ is called a **primitive** of f if it is holomorphic and $F' = f$.

This is equivalent to the indefinite integral for real-valued functions! Remember that the indefinite integral is related to the definite integral by the Fundamental Theorem of Calculus in the real-valued case: here's a notion of definite integration as well.

Definition 29

A **parametrized smooth curve** γ on Ω is defined by a function $z : [a, b] \to \Omega \subseteq \mathbb{C}$, where $a, b \in \mathbb{R}$ (we have a finite interval), such that z is **smooth**: both the real and imaginary parts are infinitely differentiable functions, and $z'(t) \neq 0$ for all $t \in [a, b]$. A curve is **closed** if $z(a) = z(b)$.

Basically, we trace out a path in Ω, and the derivative $z'(t)$ is the tangent vector. This can't be zero, or our path will stop moving! We can make a slight generalization as well:

Definition 30

A **piecewise smooth curve** is a curve where we can divide $[a, b]$ into a finite number of pieces, so that z is smooth on each piece.

For example, we can draw our curve so that there are cusps or turns. Two curves that are the same "shape" but differently parametrized should be treated equivalently, though:

Definition 31

Given two functions $z : [a, b] \to \Omega$ and $z' : [c, d] \to \Omega$, define an equivalence relation $\sim$ such that $z \sim z'$ if we can make an (invertible) change of variables $s : [a, b] \to [c, d]$ with $z = z' \circ s$ and $z' = z \circ s^{-1}$.

For example, this allows us to change $[a, b]$ to $[2a, 2b]$, and it allows us to reverse the orientation of a curve. We can denote the actual curve via γ, even if there are many different ways of representing it.

Now we're ready to discuss the definite integral:

Definition 32

The integral along a curve is defined as

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt,$$

where γ is being represented by the function $z : [a, b] \to \Omega$.

Intuitively, here $dz = z'(t)dt$, just like the substitution in calculus! Notably, the chain rule means that **the value of this integral depends only on the equivalence class of z.**

Example 33

Consider the closed curve γ corresponding to the unit circle: we can parametrize via $z(t) = e^{it}$ on the interval $[0, 2\pi]$. (Notice that this means that our curves have an orientation: here we go around counterclockwise.) Then

$$\int_\gamma f(z)dz = \int_\gamma \frac{1}{z}dz = \int_0^{2\pi} \frac{1}{e^{it}} \cdot ie^{it} = 2\pi i.$$

Theorem 34

If a function f has a primitive F, then

$$\int_\gamma f(z)dz = F(z(b)) - F(z(a)).$$

This follows from the definition and the fundamental theorem of calculus.

Corollary 35

If a primitive F exists for a function f,

$$\int_\gamma f = 0$$

for any closed curve γ.

So $f(z) = \frac{1}{z}$ doesn't actually have an antiderivative! In particular, $\log z$ isn't so easy to define in the complex plane anymore.

Corollary 36

Let Ω be connected. If $f' = 0$, then f is constant.

Proof. By definition, f is a primitive of f', so $\int_\gamma f'(z)dz = f(B) - f(A) = 0$ by Theorem 34 for any A, B. So $f(B) = f(A)$ as long as A and B are connected. □

This turns out to be useful, and it'll also be helpful for some homework problems!

3 September 12, 2019

We've been considering some questions that are analogous to those in one-variable calculus for a function $f : \Omega \to \mathbb{C}$. At first, we tried to think about a derivative, and that led us to the definition of a function being **holomorphic**. Recently, we started thinking about the analog of an integral: we started by defining a **primitive** F satisfying $F'(z) = f$, corresponding to the indefinite integral. Definite integrals are a bit less clear: if we have something like $\int_a^b f(x)dx$ (for a real-valued function), there's only one way to go from a to b. But there are many paths from a point w_1 to another point w_2 in the complex plane! So we define integration using a curve γ which is piecewise smooth: if γ is parametrized via $z : [a, b] \to \Omega$, then

$$\int_\gamma f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

We finished last class by introducing the idea of a closed curve (where $z(a) = z(b)$).

Definition 37

A curve is **simple** if there are no self-intersections: z is injective on the open interval (a, b). A **simple closed curve** is closed and simple.

Example 38

Circles are useful simple closed curves: a circle C centered at z_0 with radius r satisfies

$$z = z_0 + Re^{it}, \quad 0 \le t \le 2\pi.$$

(This is the positive orientation of C: we can also trace it in reverse, giving us the negative orientation, by using e^{-it} instead of e^{it}.)

Just like in the one-variable case, our integral has some basic properties:

$$\int_\gamma (\lambda f(z) + \mu g(z))dz = \lambda \int_\gamma f(z)dz + \mu \int_\gamma g(z)dz.$$

Also, we can reverse the orientation and travel backwards: just like $\int_a^b f(x)dx = -\int_b^a f(x)dx$,

$$\int_\gamma f(z)dz = -\int_{\gamma^-} f(z)dz.$$

Definition 39

Let the **length** of a parameterized curve γ on $[a, b]$ be defined as

$$\text{length}(\gamma) = \int_a^b |z'(t)| dt.$$

This last notion gives us a particularly useful fact:

Proposition 40 (ML inequality)

$$\left| \int_\gamma f(z) dz \right| \leq \sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma).$$

Proof. The left hand side

$$\left| \int_\gamma f(z) dz \right| = \left| \int_a^b f(z(t)) z'(t) dt \right| \leq \left| \int_a^b |\sup(f(z(t)))| z'(t) dt \right|,$$

and now we can pull the supremum term out of the integral. □

Recall that last time, we formulated something that looks like the Fundamental Theorem of Calculus:

Theorem 41

If F is a holomorphic function and $F' = f$, and γ is a parametrized curve,

$$\int_\gamma f(z) dz = F(b) - F(a).$$

This immediately gives us a few results if we look at special examples of curves:

Corollary 42

Under the same assumptions, $\int_\gamma f(z) dz = 0$ for any closed curve $\gamma \subseteq \Omega$. Equivalently, for any curves γ_1 and γ_2 **with the same start and end points**,

$$\int_{\gamma_1} f(z) dz = \int_{\gamma_2} f(z) dz.$$

Example 43

Let $f(z) = z^n$, where $n \in \mathbb{Z}$. (f is holomorphic on $\Omega = \mathbb{C} - \{0\}$.) Let C be the unit circle: what is $\int_C z^n dz$?

Since

$$\left(\frac{z^{n+1}}{n+1} \right)' = z^n \quad \forall n \neq -1,$$

this integral is 0 for $n \neq -1$ (because f has a primitive). On the other hand, the integral is $2\pi i$ for $n = -1$, so this function $\frac{1}{z}$ does not have a primitive on Ω: there is no holomorphic F on Ω such that $F' = \frac{1}{z}$.

Next, let's consider a problem which is classic in one-variable calculus:

Example 44

Compute
$$\int_0^{2\pi} \cos^n x dx.$$

We can do this by integrating by parts, but there's a faster way by integrating using a complex variable! If we parametrize via $z = e^{ix}$, notice that
$$\cos x = \frac{e^{ix} + e^{-ix}}{2} = \frac{z}{2} + \frac{1}{2z}.$$
So if we want $\cos^n x$, the right thing for us to consider is the integral
$$\boxed{\int_C \left(\frac{z}{2} + \frac{1}{2z}\right)^n \frac{dz}{iz} = \frac{1}{2^n i} \int_C \left(z + \frac{1}{z}\right)^n \frac{dz}{z}}.$$
This is because
$$z = e^{ix} \implies dz = ie^{ix} dx,$$
and we can check that this reduces to the original problem. Now we can directly work out the calculations: our integrand is a linear combination of z^ns, which all integrate to 0 over the unit circle except for the $\frac{1}{z}$ term. So it suffices to find the constant term of $\left(z + \frac{1}{z}\right)$, and that is only nonzero when n is even!

So when n is odd, our integral is 0 (because of cancellation by parity). Otherwise, our integral can be written as
$$\frac{1}{2^n i} \int_C \left(\cdots + \binom{n}{n/2} + \cdots\right) \frac{dz}{z},$$
which simplifies to
$$\binom{n}{n/2} \cdot \frac{2\pi i}{2^n i} = \boxed{\binom{n}{n/2} \frac{\pi}{2^{n-1}}}.$$

This is one of the motivations for studying complex holomorphic functions: real-valued integrals can be turned into complex-valued problems.

So when does a function f have a primitive? As we mentioned before, we know that we must have $\int_\gamma f(z) dz = 0$ for any γ. We claim that if $f : \Omega \to \mathbb{C}$ is a holomorphic function, and Ω is "nice," then
$$\int_\gamma f(z) dz = 0.$$
In other words, any sufficiently nice holomorphic function does have a primitive:

Theorem 45 (Goursat, simple case)

If f is holomorphic, and $\gamma = T$ is any triangle such that the interior of T is contained in Ω (so our domain can't have a hole), then
$$\int_T f(z) dz = 0.$$

This might seem crude, but because we can draw triangles anywhere inside circles (to connect any two points), we can use it to prove the following result:

Theorem 46

Any function f that is holomorphic on an open disk has a primitive.

Proof. First, let's make sure we can construct F in a well-defined way. Let z_0 be the center of our disk. Define the unique curve γ_z which first goes horizontally and then vertically from z to z_0, and now define

$$F(z) = \int_{\gamma_z} f(z)dz.$$

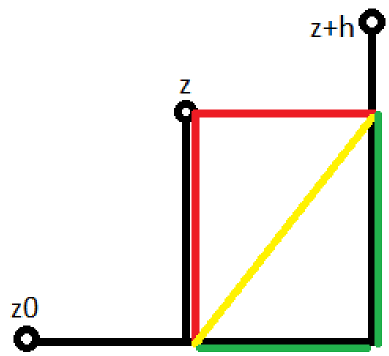
Now let's show that F is holomorphic, and that $F' = f$. **We're trying to show that**

$$F(z+h) - F(z) = f(z)h + \phi(z,h)h,$$

where $\phi(z,h) \to 0$ as $h \to 0$: this is because we can rewrite this as

$$f(z) = \frac{F(z+h) - F(z)}{h} - \phi(h),$$

and taking $h \to 0$ means that the derivative exists and $F'(z) = f(z)$. Let's draw a picture:



The black paths from z_0 to z and $z+h$ are the curves along which we're integrating to define $F(z)$, but we claim that the integral along the green part is the same as the integral along the red part. This is because (CCW means counterclockwise and CW means clockwise)

$$\int_{\text{red+green CCW}} f(z)dz = \int_{\text{red+yellow CCW}} f(z)dz + \int_{\text{yellow+green CCW}} f(z)dz = 0 + 0 = 0$$

by Goursat's theorem! So now we can just work with the curve η that goes from z to $z+h$ horizontally and then vertically, and now we're trying to show that

$$F(z+h) - F(z) = \int_\eta f(w)dw$$

approaches the right hand side $f(z)h + \phi(z,h)h$. Since f is holomorphic, we can use the linear approximation to write

$$f(w) = f(z) + (w-z)f'(z) + (w-z)\psi(w-z),$$

where $|\psi|$ goes to 0 as $w - z$ goes to 0. Integrating this expression,

$$F(z_h) - F(z) = \int_\eta f(w)dw = \int_\eta \left(f(z) + (w-z)f'(z) + \psi(w)(w-z)\right) dw.$$

$f(z)$ is a constant, so the first term here gives us $f(z)h$. The second term is $f'(z) \cdot \frac{h^2}{2}$, which goes to zero faster than $f(z)h$, and the third term is even smaller than that, so we're done: we've showed that $F(z+h) - F(z)$ approaches $f(z)h$, which means that $F' = f$. $\square$

Corollary 47

If f is holomorphic on an open disk Ω, then $\int_\gamma f(z)dz = 0$ for any closed curve $\gamma \subseteq \Omega$.

4 September 17, 2019

Professor Zhang has no preference about whether we turn our problem sets in online or in person, so we can do either one! (The TA may have a preference, though.)

Last time, we proved the following result:

Theorem 48 (Cauchy's theorem for open disks)

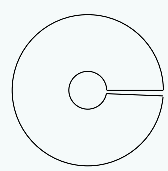
If a function $f : D \to \mathbb{C}$ is holomorphic on the whole disk, then it has a primitive on the disk. In particular, $\int_\gamma f(z)dz = 0$ for any closed curve γ contained in D.

We proved this using Goursat's theorem by connecting any point in the disk to the center of the disk in a unique way (first horizontal, then vertical). This is nice because we can use the Fundamental Theorem of Calculus: if F is the primitive of f, then any integral $\int_\gamma f(z)dz$ is just the difference of F between the endpoints.

Let's now generalize this beyond just open disks! Somehow, we're not using that many properties of our disk right now: all we need to do is to be able to connect any two points in a well-defined way.

Example 49 (Keyhole contour)

Consider the following simple closed curve: notice that we can still connect any two points inside the closed regionwith horizontal and vertical line segments.



So a **toy contour** is one whose interior domain allows us to connect any two points with a simple "zig-zag" curve of vertical and horizontal line segments. Other examples of toy contours include semicircles, rectangles, and so on.

Remark 50. *It's hard to come up with an example of something that isn't a toy contour, but pathological examples can be constructed (consider the space-filling curve).*

Later on, we'll prove something that helps us work with these constructions:

Theorem 51

If f is holomorphic on the interior of a toy contour, then $\int_\gamma f(z)dz = 0$ for any closed curve γ.

These kind of contour integrals came up initially because people were interested in computing the value of definite integrals. Here's an example:

Problem 52

Show that

$$\int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x y}\,dx = e^{-\pi y^2}$$

for any $y \in \mathbb{R}$.

Note that this is related to the **Fourier transform** of a function $f : \mathbb{R} \to \mathbb{R}$:

$$\hat{f}(y) = \int f(x) e^{-2\pi i x y}\,dx,$$

is the **Fourier transform** of f, provided that we have some nice convergence properties. So the equality in our example tells us that $e^{-\pi x^2}$ is its own Fourier transform! Also, notice that plugging in $y = 0$ here yields the famous result

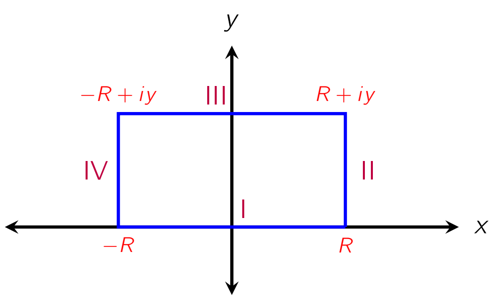
$$\int_{-\infty}^{\infty} e^{-\pi x^2} = 1.$$

We can also compute this integral with a double integral over polar coordinates, but somehow the result we're showing here is more general.

Solution. Our hope is that there's a way for us to break up a closed curve into an "easy" and a "hard" contour integral: this will be a general motivation for these kinds of problems! First of all, this integral is indeed absolutely convergent because it goes to 0 very quickly. Multiplying both sides by $e^{\pi y^2}$, notice the problem is equivalent to showing that

$$\int_{-\infty}^{\infty} e^{-\pi (x+iy)^2}\,dx = 1.$$

We'll apply Cauchy's theorem on the function $f(z) = e^{-\pi z^2}$, which is holomorphic on all of $\mathbb{C}$. Consider the following contour γ_R:



We'll traverse this rectangle **counterclockwise.** Since $f(z)$ is holomorphic on the plane, $\int_{\gamma_R} f(z)dz = 0$. Notice that by definition,

$$\int_{-\infty}^{\infty} e^{-\pi (x+iy)^2}\,dx = \lim_{R\to\infty} \int_{-R}^{R} e^{-\pi (x+iy)^2}\,dx,$$

which should explain the bounds that we define in our rectangle. Let's write out the contour integral along the path itself: along the **horizontal parts**, we have

$$\int_I + \int_{III} = \int_{-R}^{R} e^{-\pi x^2}\,dx - \int_{-R}^{R} e^{-\pi (x+iy)^2}\,dx$$

(where the negative sign comes from the fact that we're traversing in the $-x$-direction). The **vertical parts** are basically "unwanted," but they're pretty small as we go to infinity. In particular, by the ML inequality, the total value of those integrals is at most the length of the contour times the supremum of $|e^{-\pi z^2}|$ on the segment:

$$\int_{II} + \int_{IV} \leq 2|y| \cdot \sup_y |e^{-\pi(R+iy)^2}| = 2|y|e^{-\pi R^2},$$

which goes to 0 as $R \to \infty$ (since y is fixed).

So now we're almost done: by Cauchy's theorem,

$$0 = \int_I + \int_{III} + \int_{II} + \int_{IV},$$

and taking the limit as $R \to \infty$, the last two terms here disappear. We're therefore left with

$$0 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx - \int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} dx.$$

The first integral is equal to 1 (exercise in 18.02), which means $\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} dx = 1$ as well, as desired! $\square$

Corollary 53

The result of Problem 52 is actually true for any $y \in \mathbb{C}$, because the equality in the form

$$\int_{-\infty}^{\infty} e^{-\pi(x+iy)^2} dx = 1$$

allows us to "absorb" complex parts of y into x.

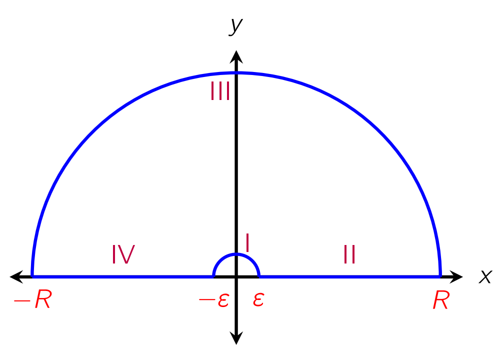
This was one of the early motivations for Cauchy's work in complex analysis: taking contours lets us evaluate definite integrals more easily!

Problem 54

Show that

$$\int_0^{\infty} \frac{1-\cos x}{x^2} = \frac{\pi}{2}.$$

Solution. Whenever we see a trig function in complex analysis, we should try replacing it with an exponential function. Define $f(z) = \frac{1-e^{iz}}{z^2}$: there's a denominator of z^2 here, so f is not holomorphic at $z = 0$. So we need to avoid that point, and we do this by drawing the following contour $\gamma_{R,\varepsilon}$:



Again, traverse this counterclockwise: we'll eventually take $\varepsilon \to 0$ and $R \to \infty$. First of all, on II and IV of the contour,

$$\int_{II} + \int_{IV} = \int_{\varepsilon}^{R} \frac{1-e^{ix}}{x^2} dx + \int_{-R}^{-\varepsilon} \frac{1-e^{ix}}{x^2} dx,$$

since we're only traversing the real line. Since the denominator is an even function, we can combine the two integrals: since $e^{ix} + e^{-ix} = 2\cos x$,

$$= 2\int_{\varepsilon}^{R} \frac{1-\cos x}{x^2}dx.$$

(Notice that taking $\varepsilon \to 0$ and $R \to \infty$ gives us twice the integral that we're after!) So now we just need to get rid of the other two semicircles.

Notice that we're always working on the upper half-plane: if $z = x + iy$, and $y > 0$, then

$$\left|\frac{1-e^{iz}}{z^2}\right| \le \frac{2}{|z|^2},$$

because $|e^{iz}| = |e^{ix-y}| = e^{-y} < 1$ for all positive y. This means that our integral is bounded: again by the ML inequality,

$$\left|\int_{III}\right| \le \frac{2}{R^2}\cdot\frac{\pi}{2}R,$$

which is proportional to $\frac{1}{R}$. Taking $R \to \infty$, we find that this quantity must vanish, so the contribution from III in the integral goes to 0.

Finally, let's look at the small arc I. We're now looking at

$$\int_{IV} = \int_{IV} \frac{1-e^{iz}}{z^2}dz.$$

Here's a key idea: **as $\varepsilon \to 0$, the length of this curve goes to 0, so it's okay if we replace this function by adding a bounded term.** Since $e^{iz} = 1 + iz + O(z^2)$,

$$\frac{1-e^{iz}}{z^2} = \frac{1-1-iz-O(z^2)}{z^2} = \frac{-i}{z} + \phi(z),$$

where $\phi(z)$ is bounded. So now

$$\int_{IV} = \int_{IV}\frac{-idz}{z} + \int_{IV}\phi(z)dz,$$

and we can ignore the second term because the length of the integral goes to 0. We've done the first integral before: $\frac{dz}{z}$ can be parametrized, and so this just gives us (flipped sign because orientation is clockwise instead of counterclockwise)

$$= -i\int_{IV}\frac{dz}{z} = -i\cdot -\pi = \pi.$$

Thus

$$\lim_{R\to\infty}\lim_{\varepsilon\to 0} 2\int_{\varepsilon}^{R}\frac{1-\cos x}{x^2}dx = \pi,$$

which gives what we want! □

Notice that a large contribution here came from the circle around $z = 0$. This will generalize:

Theorem 55 (Cauchy integral formula)

Given a holomorphic function $f : D \to \mathbb{C}$, for any $\zeta \in D$,

$$f(\zeta) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}dz,$$

where we integrate once counterclockwise around the circle C enclosing D.

This is a generalization of the fact we found earlier: we know that if f is the constant function 1,

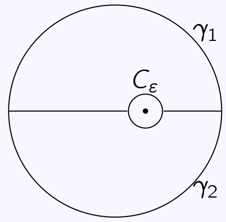
$$\frac{1}{2\pi i}\int_C \frac{dz}{z} = 1.$$

Notice that this allows us to find (for example) $f(0)$ for any holomorphic function, if we can integrate $\frac{f(z)}{z}$ easily around the disk.

Proof. We'll start by doing a construction that makes our problem a lot easier:

Lemma 56

In the Cauchy integral formula, we can also integrate around an arbitrarily small disk C_ε that contains ζ:



This is because we can integrate around the top and bottom toy contours: by the Cauchy formula, those integrals are 0, but their sum gives us the integral around C once counterclockwise, minus the integral around C_ε once clockwise:

$$0 = \int_{\gamma_1}\frac{f(z)}{z-\zeta} + \int_{\gamma_2}\frac{f(z)}{z-\zeta} = \int_C \frac{f(z)}{z-\zeta} - \int_{C_\varepsilon}\frac{f(z)}{z-\zeta} \implies \boxed{\int_C \frac{f(z)}{z-\zeta} = \int_{C_\varepsilon}\frac{f(z)}{z-\zeta}}.$$

This is nice, because we can now do the same trick as we did in the above problem: add a bounded function. Because f is holomorphic,

$$f(z) = f(\zeta) + (z-\zeta)g(z),$$

where $g(z)$ is bounded inside of some sufficiently small disk (because it's continuous and approximately the value of the derivative $f'(\zeta)$), and thus

$$\int_{C_\varepsilon}\frac{f(z)}{z-\zeta} = f(\zeta)\int_{C_\varepsilon}\frac{1}{z-\zeta} + \int_{C_\varepsilon} g(z)dz.$$

Taking $\varepsilon \to 0$, the second term disappears, while the first integral is something we've already computed before:

$$\int_{C_\varepsilon}\frac{f(z)}{z-\zeta} = f(\zeta)\cdot 2\pi i.$$

Putting everything together,

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}dz = \frac{1}{2\pi i}\int_{C_\varepsilon}\frac{f(z)}{z-\zeta}dz = \frac{1}{2\pi i}\cdot 2\pi i\cdot f(\zeta) = f(\zeta),$$

as desired. □

This can be generalized, too: we can actually calculate all derivatives of a point with a similar contour integral.

Theorem 57

For all n,

$$f^{(n)}(\zeta) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-\zeta)^{n+1}}dz.$$

5 September 19, 2019

Today, we'll start to gain a better understanding of holomorphic functions, and then we'll move on to some applications!

Recall that a function $f : \Omega \to \mathbb{C}$ defined on an open connected region is **holomorphic** if it is complex differentiable at every point in the region: this definition only requires the existence of the first derivative. Recently, we've been looking at integration: Cauchy's theorem tells us that on a **toy contour** γ contained in our holomorphic region, we always have $\int_\gamma f(z)dz = 0$. This led us to the following result:

Theorem 58 (Cauchy's integral formula)
Let f be holomorphic on an open region Ω, and let circle $C \subseteq \Omega$. For all points z in the interior of C,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta.$$

(This looks slightly different from last time, because we've flipped the roles of z and ζ.)

What's powerful about this is that the right side only uses the value of f on the circle C: Cauchy's integral formula allows us to find the value of f anywhere inside it!

Corollary 59
Consider the special case where our circle C is centered around $z_0 = 0$. Parametrize our circle with $\zeta = Re^{it}$, where $0 \le t \le 2\pi$: then $\frac{d\zeta}{\zeta} = idt$, so

$$f(0) = \frac{1}{2\pi}\int_C f(Re^{it})dt.$$

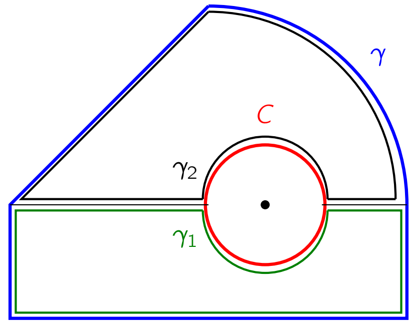
In other words, **the value of the function f at the center of the circle is the average of the value of f around the circle!** This is in stark contrast to the real-valued case: if we just have a one-variable function $f : \mathbb{R} \to \mathbb{R}$, a "circle" around any point x is only two points ($x - \varepsilon$ and $x + \varepsilon$), and if a continuous function's value $f(x)$ at any point is the average of $f(x + \varepsilon)$ and $f(x - \varepsilon)$, then f is just a linear function.

Things are a bit more interesting if we allow a larger domain $u : \mathbb{R}^2 \to \mathbb{R}$ though: if we try to impose the same condition that the average value of u on C is the value at the center, these end up being the **harmonic functions**

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Remember that the main jump in going from real-valued to complex-valued functions is that we don't just have a real-valued function u: instead, we have $f = u + iv$. Indeed, it was a homework problem to show that u and v are in fact harmonic functions.

By the way, we don't need to have a circle in the formulation of the Cauchy integral formula! If f is holomorphic, $\frac{f(\zeta)}{\zeta - z}$ is holomorphic everywhere except at z, so we can use Cauchy's theorem in the form $\int_{\gamma'} \frac{f(\zeta)}{\zeta - z} = 0$ to "switch" between curves:



Basically, if we integrate around all curves counterclockwise,

$$\int_C \frac{f(\zeta)}{\zeta - z} = \int_C \frac{f(\zeta)}{\zeta - z} + \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z} + \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z} = \int_\gamma \frac{f(\zeta)}{\zeta - z},$$

since the line and circle inside γ cancel out in orientation.

However, we do need to be careful to not let our curve self-intersect: the idea is that we might go around counterclockwise more than once. We'll discuss this more later, though.

Earlier in this class, we also discussed representing functions with power series:

Definition 60

A function f defined on an open region Ω is **analytic** if

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

for $z \in \Omega$. That is, f is equal to its power series everywhere.

We know that analytic functions on a domain Ω must be holomorphic, because derivatives are defined if the power series is defined. What we're going to show is that the converse is true too: all holomorphic functions are actually analytic!

Theorem 61 (Regularity Theorem)

If f is holomorphic, then f is infinitely complex differentiable (in other words, $f', f'', \cdots$ are also holomorphic). In addition, for any $z \in D$ and $n \geq 0$,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

This is even more remarkable than our first result: we can recover **all higher-order derivatives** from the value of f on a single circle as well! We'll prove this using the following fundamental result:

Theorem 62

All holomorphic functions are analytic:

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$$

for all $z \in \Omega$. (In particular, this power series is convergent.)

Proof. **If we can prove that all holomorphic functions are analytic, then the regularity theorem follows**. This is because we've already shown that

$$\int_C \zeta^n = \begin{cases} 0 & n \neq -1 \\ 2\pi i & n = -1. \end{cases}$$

Thus, if we consider for example $z = 0$, then

$$\boxed{\frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}} = \frac{n!}{2\pi i}\int_C \frac{\sum a_i \zeta^i}{\zeta^{n+1}} d\zeta$$

will only pull out the term in the numerator where $i = n$, and that gives us

$$\frac{n!}{2\pi i}\int_C a_n \zeta^{-1} d\zeta = \frac{n!}{2\pi i}\cdot 2\pi i a_n.$$

Then we know that in a power series expansion, $a_n = \frac{f^{(n)}(0)}{n!}$, so this is equal to

$$n! \cdot \frac{f^{(n)}(0)}{n!} = \boxed{f^{(n)}(0)},$$

as desired.

So let's show that holomorphic functions are analytic. We're supposed to show that for any point z', there exists a small circle around z' such that f is equal to its power series. First of all, we can change coordinates so that $z' = 0$, and let's say that the circle around 0 has radius R (pick R small enough so that the circle is contained within the region Ω).

Let D be the open disk of radius R centered at 0. By Cauchy's theorem, for all $z \in D$,

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta.$$

We know the power series expansion

$$\frac{1}{1-z} = \sum_{n \geq 0} z^n$$

holds for all $|z| < 1$, so let's try to apply this to the denominator of our expression. We have

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\cdot\frac{1}{1 - \frac{z}{\zeta}} = \frac{1}{\zeta}\sum_{n \geq 0}^{\infty}\left(\frac{z}{\zeta}\right)^n$$

for all $|z| < |\zeta|$. Remember that z is inside the disk of radius R and ζ is on the boundary, so $|z| < |\zeta|$ does indeed hold: thus we can indeed go ahead and write (for all $z \in D$)

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i}\int_C \sum_{n \geq 0}^{\infty} f(\zeta)\frac{z^n}{\zeta^{n+1}} d\zeta.$$

Now because of convergence** (see below), we can **swap the order of the sum and integral** to get

$$= \frac{1}{2\pi i}\sum_{n \geq 0}^{\infty} z^n \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta,$$

and now the integrals are constants with respect to z, so this is just a power series that holds for all $|z| < R$. Thus, f is indeed equal to its power series on a neighborhood (specifically D), meaning f is analytic.

Thus f is infinitely complex differentiable, and comparing coefficients as above gives the regularity theorem, as desired. □

To clear up the convergence issue, we use the following result:

Proposition 63

Let C be compact. If

$$f(\zeta) = \sum_{n \geq 0}^{\infty} A_n(\zeta)$$

converges **uniformly**, meaning that for any ε, there exists **a fixed** N such that for all $\zeta \in C$, we have

$$\left| \sum_{n=N}^{M} A_n(\zeta) \right| < \varepsilon,$$

then we can swap the sum and integral:

$$\int_C \sum_{n \geq 0}^{\infty} A_n(\zeta) = \sum_{n \geq 0}^{\infty} \int_C A_n(\zeta).$$

Basically, we have to pick our N independent of the ζ that we're using. Let's show that the function we were using in our proof is indeed uniformly convergent:

$$\left| \sum_{n=N}^{M} \frac{z^n}{\zeta^{n+1}} f(\zeta) \right| \leq \sum_{n=N}^{M} \frac{|z|^n}{|\zeta|^{n+1}} |f(\zeta)| \leq \sum_{n=N}^{M} \frac{|z|^n}{R^{n+1}} \sup_C |f(\zeta)|.$$

The supremum exists and is finite, because C is compact, and that means our sum is absolutely convergent as long as $|z| < R$. Since the right hand side here doesn't even depend on ζ, we can pick an N independent of ζ, and we've verified the necessary conditions.

In fact, this generalizes to the following criterion:

Proposition 64

If $\sup_{\zeta \in C} |A_n(\zeta)| \leq a_n$ and $\sum_{n \geq 0} a_n$ converges, then we can use Proposition 63.

The reason all of this is powerful can be seen when we look at it in contrast to real-valued functions:

Example 65

Consider the function

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

This function is continuous, and it turns out to actually be infinitely differentiable. But all derivatives are equal to 0, so f is **not equal to its power series at** $x = 0$.

With this, we can start proving more high-powered theorems about our holomorphic functions.

Definition 66

A holomorphic function f is **entire** if it is holomorphic on all of $\mathbb{C}$.

Theorem 67 (Liouville)

If f is entire and bounded, then it is constant.

Again, this is contrast to things like $f(x) = \sin x$ in the real case!

Proof. We'll prove that the derivative $f'(z) = 0$ for all z, which implies that f is constant (as long the region it's defined on is connected). Notice that by the Regularity Theorem,

$$f'(z) = \frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

Since f is bounded, we can bound the magnitudes here:

$$|f'(z)| \le \frac{1}{2\pi} \int_C \left| \frac{f(\zeta)}{(\zeta - z)^2} \right|,$$

and if C is a circle centered at z with radius R, this can be bounded by

$$\frac{1}{2\pi} \cdot \sup_{\zeta \in C} |f(\zeta)| \frac{1}{R^2} 2\pi R = \frac{\sup_{\zeta \in C} |f(\zeta)|}{R}.$$

The numerator is bounded by a fixed constant, so taking $R \to \infty$ makes this 0. Thus $|f'(z)| \le 0$, so $f'(z) = 0$ and we're done. $\square$

Now we're getting to the fun part of the class: we'll get to more applications next time.

6 September 24, 2019

The average score on the problem set is about 54 out of 60, which is pretty good. Next Thursday, we have the first midterm exam, but it will only cover content from the first two problem sets. Solutions will be posted either tonight or tomorrow, so that we have a week to study for the exam.

Today, we'll continue with applications of the results we've shown so far. Recall that we have the Cauchy integral formula

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta,$$

which can alternatively just be thought of informally as

$$f(z) = \int_C \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

and then differentiating both sides with respect to z repeatedly. (Interchanging the order of the integral and derivative is okay because of uniform convergence.) Here C is a circle of radius R centered at the point z, so we can say the following:

Theorem 68 (Cauchy's inequality)

For all nonnegative integers n,

$$|f^{(n)}(z)| \le \sup_{\zeta \in C} \frac{n!}{2\pi} \frac{|f(\zeta)|}{|\zeta - z|^{n+1}} \cdot 2\pi R = \frac{n!}{R^n} \sup_{\zeta \in C} |f(\zeta)|.$$

Last time, we showed the **regularity theorem** along the way to showing that any holomorphic f is infinitely complex differentiable. We also showed that holomorphic functions are all analytic: it has a power series expansion at every point which is equal to f in some disk. If we combine this with Cauchy's inequality, we get something stronger.

Consider a region (open and connected) Ω, and pick a point z close to the boundary. If we consider the largest disk that we can possibly draw while still being contained in Ω, this will indeed still be convergent by Hadamard's formula (because $|f^{(n)}(z)| \lesssim \frac{1}{R^n}$ from the above inequality). **So this tells us something powerful:** we can draw a pretty large disk around each point, as long as we stay inside Ω!

Corollary 69

If $f : \mathbb{C} \to \mathbb{C}$ is entire (holomorphic on all of $\mathbb{C}$), then the power series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

is convergent for all $z \in \mathbb{C}$.

Let's now dive into some of the applications. Last time, we proved Liouville's theorem, which says that any **entire bounded function is constant**. We did this by proving that the first derivative is zero, but here's another proof: we just need to show that all coefficients in the power series are equal to 0.

Proof of Liouville's theorem. By Cauchy's inequality, if we integrate around a circle γ_R centered at 0,

$$|f^{(n)}(0)| \le \frac{n!}{R^n} \sup_{\zeta \in \mathbb{C}} |f(\zeta)|.$$

Since $|f(\zeta)| \le C$ for all ζ (by assumption),

$$|f^{(n)}(0)| \le \frac{n!C}{R^n},$$

and taking $R \to \infty$ means that $|f^{(n)}(0)| = 0$, so the nth derivative must be equal to 0.

Since this works for all $n > 0$, the power series for f must only have a constant term, and this means f is constant. □

We can actually use this in a clever way:

Theorem 70 (Fundamental Theorem of Algebra)

Let $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree n (so all $a_i \in \mathbb{C}$, and $a_n \neq 0$). Then P has n roots, counting multiplicity: this means that we can write

$$P(z) = a_n \prod_{i=1}^{n} (z - z_i)$$

for some (not necessarily distinct) $z_i \in \mathbb{C}$.

We can write this explicitly for low degrees, but it's hard to do the same for higher degrees!

Proof. We proceed by induction: it's enough to show that every polynomial with degree at least 1 has at least one root z_1. (The base case is easy: for $n = 1$, $a_0 + a_1 z$ has the root $z = -\frac{a_0}{a_1}$, which exists because $a_1 \neq 0$.) Then by long division, we can write $P = (z - z_1)Q(z)$, and now $Q(z) = \frac{P(z)}{z - z_1}$ is a polynomial of degree $n - 1$. But we can then use the inductive hypothesis to show that $Q(z) = (z - z_2) \cdots (z - z_n)$, and then we're done!

We may want to argue similarly to how we show that a cubic has at least one real root: since $P(x) \to \pm\infty$ as $x \to \pm\infty$ (respectively), there exists a value where $P = 0$ by continuity. But we can't quite do this for even-degree polynomials, so we'll need to be more clever.

So here's what we do: assume for the sake of contradiction $P(z) \neq 0$ for all $z \in \mathbb{C}$. Define $f(z) = \frac{1}{P(z)}$: this is holomorphic because we've made the assumption that the denominator is never zero. Thus, P is entire, so f is entire as well: it's of the form

$$\frac{1}{a_n} \cdot \frac{1}{z^n + O(z^{n-1})}.$$

As $|z| \to \infty$, f approaches 0, because the z^n term will dominate. So f is uniformly bounded over all of $\mathbb{C}$, because drawing a circle of (sufficiently large) radius R, the function is bounded inside the circle because we have a compact domain, and it's also bounded outside (because f goes to 0 as $|z| \to \infty$). Thus f is constant, which is a contradiction!

Thus, $P(z) = 0$ for some $z \in \mathbb{C}$, and we're done. □

Let's now move on to yet another powerful fact about complex-valued functions. Let's say that someone erases part of a domain Ω: it turns out that holomorphic functions can essentially "recover what is erased." This is called **analytic continuation**.

Theorem 71

Let Ω be a connected open region, and say that $f : \Omega \to \mathbb{C}$ is holomorphic. Say that we have a sequence of points $\{z_n\}$ such that $f(z_n) = 0$, and $\lim_n z_n = z \in \Omega$. Then $f = 0$ everywhere in Ω.

In other words, if the set of roots $f^{-1}(0)$ contains a limit point in Ω, then f is identically zero. Here, a **limit point** is one where there exists a sequence of points that approach it.)

In contrast, this is not true if f is not holomorphic: it's good to think of a real-valued example!

Proof. **Step 1**: First, we show that f is zero on a neighborhood containing z. Without loss of generality, let's say the limit point is $z_\infty = 0$.

Suppose for the sake of contradiction that this is false: then f's power series around 0 is nonzero and looks like

$$f(z) = \sum_{n \ge 0}^{\infty} a_n z^n = a_{n_0} z^{n_0} + O(z^{n_1}) = a_{n_0} z^{n_0}(1 + O(z)),$$

where a_{n_0} is the first nonzero coefficient. As we take $z \to 0$, the $O(z)$ contribution goes away, and thus $f(z)$ approaches $a_{n_0} z^{n_0}$. This means that there exists a small disk around z such that $f(z) \neq 0$ for all $z \neq 0$ in the disk.

But this contradicts the definition of a limit point: for any disk around 0, there should be a z_n such that $f(z_n) = 0$. Thus f must indeed be zero on the neighborhood.

Step 2: Now define U to be the interior of $f^{-1}(0)$: this is an open subset of the Ω, and we're trying to show that $U = \Omega$. (Note that $f^{-1}(0)$ is a closed set because we map closed sets to closed sets.) But the first step also implies that U is closed! This is because we've shown in step 1 that given any sequence $\{z_n\} \to z_\infty$ where $z_n \in U$, we have $z_\infty \in U$ as well, so U contains all of its limit points.

Therefore U is both open and closed, and it is not empty (because it contains a small disk around z). Since Ω is connected, it cannot be decomposed into the disjoint union $U \sqcup (\Omega - U)$ unless one of those is empty. Thus $U = \Omega$, as desired. □

In general, call $f^{-1}(0)$ the **zero set** of f. It's closed, and we've shown that unless f is the zero function, the zero set cannot have any limit points. This has the following consequence:

Corollary 72

If f, g are holomorphic functions on a connected open region Ω, and $f = g$ on any disk D, then $f = g$ on Ω.

This motivates us to consider the following idea:

Definition 73

Let $f : \Omega \to \mathbb{C}$ and $F : \Omega' \to \mathbb{C}$ be holomorphic, where $\Omega \subseteq \Omega'$. If $f = F$ on all of Ω, then F is an **analytic continuation** of f.

Note that such an analytic continuation must be unique, but it doesn't necessarily need to exist!

Example 74

We know that

$$f(z) = 1 + z + z^2 + \cdots + z^n$$

is convergent for all $|z| < 1$ by Hadamard's formula. We know that this agrees with $F(z) = \frac{1}{1-z}$, which is holomorphic on $\mathbb{C} \setminus \{1\}$. Thus, F is an analytic continuation of f.

Next, we present the converse of Cauchy's theorem:

Theorem 75 (Morera)

If f is continuous on an open disk D, and $\int_T f(z)dz = 0$ for any triangle $T \subseteq D$, then f is holomorphic.

This is more suitable if we know something about integrating f without having to compute its derivative!

Proof. This is basically a consequence of regularity. Recall the proof of Cauchy's theorem that used Goursat's theorem: all we really used was that $\int_T f(z)dz = 0$ for any closed triangle. So we can repeat basically the same construction to find a function F that is holomorphic such that $F' = f$, and now by regularity f is holomorphic as well (in particular, $f' = F''$). $\square$

We'll discuss applications of this next time!

7 September 26, 2019

We'll start with a quick note about one of the homework problems from last time. The problem asks to evaluate the definite integral

$$\int_0^\infty \frac{\sin x}{x} dx,$$

and the standard thing to do here is to instead evaluate the contour integral of function $f(z) = \frac{e^{iz}-1}{z}$, which is holomorphic even at $z = 0$ (if we define $f(0) = 1$).

Remark 76. *We can see this by expanding out the power series:*

$$\frac{e^{iz}-1}{z} = \frac{1 + iz + O(z^2) - 1}{z} = i + O(z),$$

which is indeed a valid power series, and it is an entire function.

The reason we're bringing this up is that finding the integral over the larger arc is a bit tricky: how can we show that it converges to the value we want? We parametrize via polar coordinates: letting $z = Re^{it}$ and doing the relevant simplifications, we need to show that

$$\int_0^\pi e^{-R\sin t} dt$$

goes to 0 as $R \to \infty$. This is more difficult than the example we did in class, because we could use the ML inequality to show that the value was bounded by $\frac{c}{R^2}$. This time, we'd get $\frac{2}{R}$, which is not sufficient! If we tried to naively use the ML inequality again, the expression doesn't go to 0:

$$\sup_{z \in \gamma} |f(z)| \cdot \text{length}(\gamma) = \frac{2}{R} \cdot 2\pi R = 4\pi.$$

So how exactly do we bound the integral? The idea is that saying something like

$$\left| \frac{e^{iz}}{z} \right| \le \frac{1}{R}$$

is a really bad approximation over most of the curve. The only large contributions are in the parts where z is very close to R or $-R$: we basically need to just extinguish those cases.

First, note that it's sufficient to only consider the integral from 0 to $\frac{\pi}{2}$. Notice that $\sin t$ is concave on the interval $(0, \frac{\pi}{2})$, so

$$\sin t \ge \frac{2}{\pi} t \quad \forall 0 \le t \le \frac{\pi}{2}.$$

We can then say that

$$\int_0^{\pi/2} e^{-R \sin t} \le \int_0^{\pi/2} e^{-cRt}$$

for some positive constant $c > 0$. And now we can evaluate directly: this is equal to

$$\frac{1}{Rc}\left(1 - e^{-Rc\pi/2}\right)$$

which goes to 0 as $R \to \infty$.

Now we'll go back into the material of the class! Last time, we stated Morera's theorem: if $f : \Omega \to \mathbb{C}$ is continuous, and $\int f(z)dz = 0$ for any triangle inside Ω, then f is holomorphic.

Recall that we define our power series within the disk $D(R)$ with R given by Hadamard's formula via

$$\sum_{n=0}^{\infty} a_n z^n = \lim_{N \to \infty} \sum_{n=0}^{N} a_n z^n.$$

Definition 77

For any compact set $C \subseteq \Omega$, a set of functions $\{f_n(z)\}$ **uniformly converges** to f on C if for every $\varepsilon > 0$, there exists an N (depending on ε) such that for all $n > N$,

$$|f_n(z) - f(z)| < \varepsilon$$

for all $z \in C$.

Pointwise convergence says that we **fix** z **first**, while uniform convergence says that we can find an N such that this is true for all z simultaneously. So uniform convergence is stronger!

It turns out that in the power series expressions above, we do actually have uniform convergence

$$\sum_{n=0}^{N} a_n z^n \to \sum_{n=0}^{\infty} a_n z^n$$

for every closed disk $\overline{D} \subseteq \Omega$. We want to show that the limit of the holomorphic functions $\sum_{n=0}^{N} a_n z^n$ is holomorphic as well, and we can state that as part of a more general result:

Theorem 78

Let $\{f_n\}_{n=0}^{\infty}$ be a sequence of holomorphic functions on Ω such that $f_n(z) \to f$ uniformly on every compact subset of Ω. Then the limit $f(z) = \lim_{n\to\infty} f_n(z)$ is holomorphic on Ω.

Note that it's actually pretty hard to have uniform convergence everywhere on Ω: for example, the geometric series

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

is not uniformly convergent on the entire unit disk! So the theorem only asking for compact subsets of Ω is nice.

Proof. It's enough to prove that $f(z)$ is holomorphic in a neighborhood N around every point z_0, and by Morera's theorem, it's sufficient to show that

$$\int_T f(z)dz = 0$$

for all triangles T contained in N. Notice that applying the uniform convergence criterion,

$$\left|\int_T (f - f_n)(z)dz\right| \le \int_T |(f - f_n)(z)| \le \varepsilon \cdot \text{length}(T)$$

for all $n \ge N(\varepsilon)$ for any fixed $\varepsilon > 0$. So if we take $\varepsilon \to 0$ and $N(\varepsilon) \to \infty$ appropriately, this expression goes to 0. This means that

$$\lim_{n\to\infty} \int_T (f - f_n)(z)dz = \lim_{n\to\infty} \int_T f(z)dz - \int_T f_n(z)dz = 0,$$

and now we're done by Morera's theorem. $\square$

In particular, this means we've now actually shown that power series are holomorphic!

Definition 79

The **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is defined for $s \in \Omega = \{s \in C : \text{Re}(s) > 1\}$.

Recall that the magnitude of each term here only depends on the real part:

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\text{Re}(s)}},$$

so $\zeta(s)$ as we've defined it here is convergent as long as $\text{Re}(s) > 1$ (by the integral test). Thus, if we consider the half-plane $\overline{\Omega_{1+\varepsilon}} = \{z : \text{Re}(z) \ge 1 + \varepsilon\}$ for any $\varepsilon > 0$, we can say that

$$\zeta_N(s) = \sum_{n=1}^{N} \frac{1}{n^s}$$

is uniformly convergent to $\zeta(s)$ on any compact set $C \subseteq \overline{\Omega_{1+\varepsilon}}$ by the same reasoning as in the proof of Theorem 78. Also, notice that any compact set C is contained within one of these half-planes $\overline{\Omega_{1+\varepsilon}}$, because the compact set cannot "touch" $\text{Re}(z) = 1$. Formally, since $\text{Re}(z) > 1$ for every $z \in C$,

$$\inf_{z\in C} \text{Re}(z) > 1,$$

because C is closed and therefore contains all of its limit points. Thus applying Theorem 78, $\zeta(s)$ is indeed holomorphic on the half-plane $\{z : \text{Re}(z) > 1\}$.

Remark 80. *It's harder to analyze the analytic continuation of $\zeta(s)$ past this half-plane, but we'll talk about that later!*

Here's another nice convergence property:

Theorem 81

Let holomorphic functions $f_n \to f$ converge uniformly on a compact set C. Then $f_n' \to f'$ also converges uniformly on C.

Proof sketch. The idea is that by Cauchy's integral formula, we can write

$$f' = \int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

and prove uniform convergence from here. $\square$

Example 82

We would like to consider the function

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{dt}{t}$$

and show that it is convergent (at least for $\text{Re}(s) > 0$). But this is sort of hard for us to deal with right now, so let's just look at something simpler] like

$$\gamma(s) = \int_1^2 e^{-t} t^s dt.$$

Can we say that this is holomorphic?

Theorem 83

Suppose we have a function $F(z, t)$ defined on $(z, t) \in \Omega \times [0, 1]$. Say that F is holomorphic in z for every $t \in [0, 1]$, and it's also continuous on $\Omega \times [0, 1]$. Then

$$f(z) = \int_0^1 F(z, t) dt$$

is holomorphic on Ω.

Notice that this is not required to be holomorphic in the t-variable, just the z-variable.

Proof sketch. Remember that integrals of a continuous function can be defined by Riemann sums of the form

$$\int_0^1 F(z, t) dt = \lim_{n \to \infty} \left(\sum_{i=1}^n F(z, \tfrac{i}{n}) \right) \cdot \frac{1}{n},$$

and for any fixed n, $\left(F(z, \frac{i}{n})\right) \cdot \frac{1}{n}$ is indeed holomorphic (because it's a finite sum of holomorphic functions), so we just need to prove uniform convergence on $C \times [0, 1]$ for any compact set $C \in \Omega$ to apply Theorem 78. $\square$

We'll return to the gamma function later in this class – it's a bit more subtle.

Let's move on by taking another look at the fact that

$$\int_C \frac{1}{z^n} = \begin{cases} 2\pi i & n = 1 \\ 0 & n \neq 1. \end{cases}$$

If we want to for example study rational functions $\frac{p(z)}{q(z)}$, we need to know a little more about the **singularities** when the denominators are 0: there's mild-ish ones that look like $\frac{1}{z^n}$ as $z \to 0$. These are at least okay because they always approach ∞ as z approaches the problem point.

But there's also bad ones like $e^{1/z}$: notice that as $z \to 0$, if we approach from the positive real line, we get ∞, but if we approach from the negative real line, we get 0. So that limit is undefined, and there's some bad behavior going on.

Lemma 84

Let $f \neq 0$ be holomorphic on Ω, and let $z_0 \in \Omega$. Then there exists an open disk D centered around z_0 and a unique function g and integer $n \geq 0$ such that for all $z \in D$,

$$f(z) = (z - z_0)^n \cdot g(z),$$

g is holomorphic, and $g(z) \neq 0$ on the **whole disk**.

In particular, zeros of f are isolated!

Proof. For simplicity, say that $z_0 = 0$: then we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n = a_{n_0} z^{n_0} (1 + O(z))$$

uniquely for some n_0 corresponding to the first nonzero term. Then just pick our neighborhood small enough so that the $O(z)$ term has magnitude less than 1.

This construction is unique, because if we had

$$z^{n_0} g_0(z) = z^{n_1} g_1(z),$$

then we must have $n_0 = n_1$ (or else one would vanish when the other doesn't), and this implies that $g_0(z) = g_1(z)$. $\square$

So in some sense, holomorphic functions behave locally like polynomials.

Definition 85

The **multiplicity** of a root z_0 is the value of n_0 in the unique representation

$$P(z) = (z - z_0)^{n_0} Q(z)$$

where $Q(z_0) \neq 0$.

Can we define something similar for $\frac{1}{z}$-like things as well?

Definition 86

Let a **deleted neighborhood** of z_0 be defined as $D - \{z_0\}$ for some disk D centered around z_0.

Definition 87

If $f(z)$ is holomorphic on a deleted neighborhood of z_0, then f has a **pole** at z_0 if $\frac{1}{f}$ is holomorphic at z_0. The **order** of the pole is the multiplicity of z_0 at $\frac{1}{f}$.

We'll talk a bit more about this kind of object more next time!

8 October 1, 2019

This lecture is being given by Yongyi Chen.

Last class, we started talking about zeros and poles of a function f, and we proved the following theorem:

Theorem 88

If f has a pole of order n at z_0, then we can write

$$f(z) = (z - z_0)^{-n} h(z),$$

where h is holomorphic near z_0 and $h(z_0) \neq 0$.

This allows us to write out a more explicit form, known as a **Laurent series**:

Corollary 89

If f has a pole of order n at z_0, then

$$f(z) = \left(\frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0} \right) + G(z),$$

where G is holomorphic near z_0.

Proof. By Theorem 88, we can directly write this out:

$$f(z) = (z - z_0)^{-n} h(z) = (z - z_0)^{-n}(A_0 + A_1(z - z_0) + \cdots) = \frac{A_0}{(z - z_0)^n} + \cdots + \frac{A_{n-1}}{z - z_0} + (A_n + \cdots).$$

$\square$

Definition 90

In the above expression for f, $\left(\frac{a_{-n}}{(z-z_0)^n} + \frac{a_{-n+1}}{(z-z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z-z_0} \right)$ is known as the **principal part** of f, and a_{-1} is the **residue** of f at z_0, denoted $\mathrm{res}_{z_0}(f)$.

Remark 91. *If f has a* **simple pole** *at z_0, meaning that $f = \frac{a_{-1}}{z-z_0} + G(z)$, then we can compute*

$$\mathrm{res}_{z_0}(f) = \lim_{z \to z_0} (z - z_0) f(z),$$

since $G(z)$ is finite and therefore $(z - z_0)G(z)$ goes to zero. Furthermore, if

$$f = \frac{a_{-n}}{(z - z_0)^n} + \frac{a_{-n+1}}{(z - z_0)^{n-1}} + \cdots + \frac{a_{-1}}{z - z_0},$$

then we can clear denominators via

$$(z - z_0)^n f = a_{-n} + a_{-n+1}(z - z_0) + \cdots + a_{-1}(z - z_0)^{n-1} + (z - z_0)^n G(z),$$

and then take the $(n-1)$*th derivative and let* $z \to z_0$*: this yields*

$$\text{res}_{z_0} f = \lim_{z \to z_0} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z - z_0)^n f(z)\right].$$

With this in mind, we're going to talk about the residue formula today.

Theorem 92 (Residue Theorem)
If we have a holomorphic function f in an open set containing a curve C and its interior, except for a point z_0, then integrating once counterclockwise around C,

$$\int_C f(z)dz = 2\pi i \, \text{res}_{z_0}(f).$$

Proof sketch. First, we need to shrink our circle γ so that it is contained in the region where the Laurent expansion for z_0 is valid. We can do this with a similar trick as what we've been doing in this class: Cauchy's theorem tells us that integrals around closed curves with no enclosed poles are 0.

Then we know that

$$f(z) = \frac{a_n}{(z - z_0)^n} + \cdots + \frac{a_{-1}}{z - z_0} + G(z)$$

on the whole circle γ (and its interior excluding z_0), so by Cauchy's formula, integrating this term by term makes everything go away except for

$$\int_\gamma \frac{a_{-1}}{z - z_0} dz = 2\pi i a_{-1}.$$

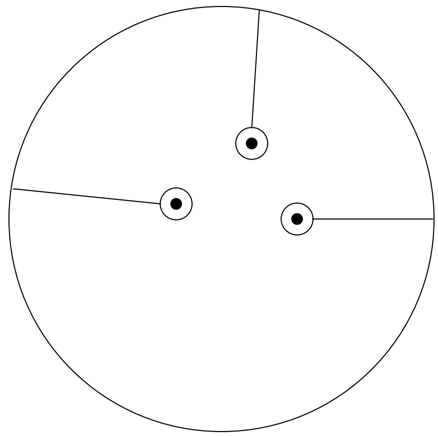
□

As a small generalization, we can have functions with multiple poles:

Corollary 93
If f has poles $z_1, \cdots, z_n$ inside a circle γ, then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \text{res}_{z_j} f.$$

Proof. The idea again is that we can replace the contour around the big circle γ with small circles around each pole, and then we can apply the residue theorem. We may also see this referred to as a "keyhole contour:"



∎

Note also that we don't necessarily need to have circles: any closed curves will work.

In the rest of this lecture, we'll do three example applications of the residue formula.

Problem 94

Compute the value of

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\, dx.$$

Solution. Normally, we'd just say that this is (by regular calculus)

$$\arctan x|_{-\infty}^{\infty} = \pi.$$

But here's a different way of approaching this problem: consider the function $f(z) = \frac{1}{1+z^2} = \frac{1}{(z+i)(z-i)}$. This has poles when the denominator is 0, which occurs at $z = \pm i$. At $z = i$, this function looks like $\frac{1}{2i} \cdot \frac{1}{z-i}$, so the residue is $\frac{1}{2i}$. Similarly, at $-i$, the residue is $\frac{-1}{2i}$.

Consider the following contour integral: integrate $f(z) = \frac{1}{1+z^2}$ along a semicircle of radius R. As $R \to \infty$, the total integral along the arc part goes to zero by the ML inequality, so what we're left with is the integral just on the real line! More explicitly, when $|z| = R$,

$$\left|\frac{1}{1+z^2}\right| \le \frac{1}{R^2-1}$$

by the triangle inequality, and thus the integral along the arc is at most

$$\pi R \cdot \frac{1}{R^2-1},$$

which goes to 0 as $R \to \infty$.

So now by the residue theorem, the integral along the real line as $R \to \infty$ (which is what we want) is equal to $2\pi i$ times the residue at i, which is

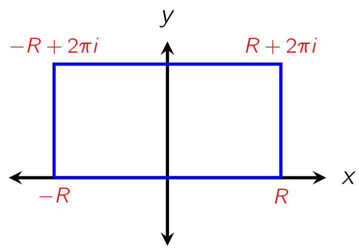
$$2\pi i \cdot \frac{1}{2i} = \pi,$$

as desired. ∎

Problem 95

Show that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin \pi a}$$
for all $0 < a < 1$.

Solution. Consider the contour integral of $f(z) = \frac{e^{az}}{1+e^z}$ around the rectangle γ_R with vertices at $R, R+2\pi i, -R+2\pi i$, and $-R$ for a real number R.



First, we show that the left and right parts go to zero as $R \to \infty$:
$$\lim_{R\to\infty} \int_{\pm R}^{\pm R+2\pi i} \frac{e^{az}}{1+e^z}\,dz = 0.$$
This is true because the real part of z is R along this whole contour integral, and this means the numerator satisfies
$$|e^{az}| = e^{aR}$$
while
$$|1+e^z| \geq e^R - 1$$
by the triangle inequality, so
$$\left|\frac{e^{az}}{1+e^z}\right| \leq \frac{e^{aR}}{e^R - 1},$$
which goes to 0 as long as $0 < a < 1$. Since we're only integrating along a curve of finite length (2π), the integral over this curve goes to zero as well.

So now we're left with only the integrals along the top and bottom. Define the bottom integral
$$A_R = \int_{-R}^{R} \frac{e^{az}}{1+e^z}\,dz :$$
this is what we will eventually want. Similarly, we can find the top by substituting in the shift: notice that we have a negative sign because we have to integrate from right to left:
$$-\int_{-R+2\pi i}^{R+2\pi i} \frac{e^{az}}{1+e^z}\,dz = -\int_{-R}^{R} \frac{e^{a(z+2\pi i)}}{1+e^z+2\pi i} = -\int_{-R}^{R} \frac{e^{2\pi i a}\cdot e^{az}}{1+e^z} = -e^{2\pi i a}\cdot A_R.$$
So now putting everything together, we add up the whole contour integral:
$$\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \lim_{R\to\infty} A_R(1-e^{2\pi i a}).$$
Remember that $A = \lim_{R\to\infty} A_R$ is exactly the integral that we're trying to find!

But on the other hand, we can also evaluate this integral with the residue theorem. Inside this rectangle, the only pole is where $e^z = -1$, which only occurs when $z = \pi i$. So the residue formula tells us that

$$A(1 - e^{2\pi i a}) = 2\pi i \operatorname{res}_{\pi i} f,$$

and now we just need to compute this residue. As mentioned earlier in class, the easiest way to do this to take

$$\lim_{z \to \pi i} (z - \pi i) \frac{e^{az}}{1 + e^z}.$$

This can also be written as

$$\lim_{z \to \pi i} e^{az} \frac{z - \pi i}{e^z - e^{\pi i}} = e^{a\pi i} \lim_{z \to \pi i} \frac{1}{\left(\frac{e^z - e^{\pi i}}{z - \pi i}\right)}.$$

Then the denominator is the definition of the derivative of e^z at πi, which is just $e^{\pi i}$. This gives us a residue of $-e^{a\pi i}$, which gives an answer of

$$A(1 - e^{2\pi i a}) = 2\pi i \cdot -e^{a\pi i} \implies A = \frac{-2\pi i e^{a\pi i}}{1 - e^{2a\pi i}}.$$

Simplifying this some more, we get

$$-2\pi i \frac{1}{e^{-a\pi i} - e^{a\pi i}} = \frac{2\pi i}{2i \sin(a\pi)} = \frac{\pi}{\sin(a\pi)},$$

as desired. $\square$

It's important to note that we did assume these are simple poles, and we should check this for ourselves as an exercise.

For our final example, we'll compute a Fourier transform:

Problem 96

Show that for all $\xi \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{\cosh \pi x} = \frac{1}{\cosh(\pi \xi)}.$$

Note that we define

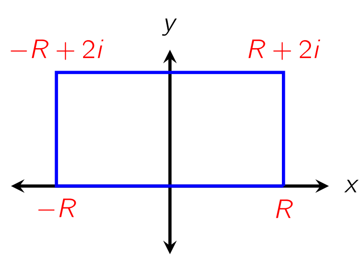
$$\cosh x = \frac{e^x + e^{-x}}{2},$$

which is the definition of $\cos x$ but with x instead of ix. So what this problem tells us is that

$$\mathcal{F}\left(\frac{1}{\cosh(\pi x)}\right)(\xi) = \frac{1}{\cosh(\pi \xi)}:$$

the **Fourier transform** of this function is itself!

Solution. We'll again use the residue theorem: we'll integrate $f(z) = \frac{e^{-2\pi i z \xi}}{\cosh(\pi z)}$ over a rectangle of height $2i$.



As in the last example, let's verify that the left and right parts of this integral go to 0:

$$\int_{\pm R}^{\pm R+2i} \frac{e^{-2\pi i z\xi}}{\cosh(\pi z)} dz = 0.$$

We can again just bound: the integrand first, we expand out the definition of cosh to get

$$\left|\frac{e^{-2\pi i z\xi}}{\cosh(\pi z)}\right| = \left|\frac{2e^{-2\pi i z\xi}}{e^{\pi z} + e^{-\pi z}}\right|.$$

Let $z = R + ti$, where $0 < t < 2$: then the imaginary parts of the exponent go away when looking at magnitude, and

$$|e^{-2\pi i z\xi}| \le e^{2\pi t\xi} = e^{4\pi|\xi|}$$

is bounded by a constant. On the other hand, the denominator

$$\left|e^{\pi z} + e^{-\pi z}\right| = \left|e^{\pi R}e^{\pi i t} + e^{-\pi R}e^{-\pi i t}\right| \ge |e^{\pi R} - e^{-\pi R}|$$

by the triangle inequality. Notably, this goes to infinity, so the value of $f(z)$ must go to 0 on the left and right sides of the rectangle, and therefore the integrals there are indeed zero (because the length of those sides is a constant).

So now we just need to look at the top and bottom. Again, the bottom is the answer that we want, so we denote

$$A_R = \int_{-R}^{R} f(z)dz.$$

Meanwhile, the top integral (again, remember that we have to integrate from right to left, so we pick up a negative sign)

$$-\int_{-R}^{R} f(z+2i)dz = -\int_{-R}^{R} \frac{2e^{-2\pi i(z+2i)\xi}}{e^{\pi(z+2i)} + e^{-\pi(z+2i)}}.$$

We've cleverly picked our bounds so that the denominator looks exactly the same as in A_R, and the numerator is just a constant away from what we have in A_R: thus, this integral is equal to

$$-e^{4\pi\xi}A_R.$$

So now putting everything together, the integral around the whole contour is just

$$\lim_{R\to\infty} \int_{\gamma_R} f(z)dz = \lim_{R\to\infty} A_R(1 - e^{4\pi\xi}).$$

It remains to look at the residues: we have a pole when $f(z) = \cosh \pi z = 0$, which happens when $e^{\pi z} = -e^{-\pi z}$, which happens at $z = \frac{i}{2}$ or $z = \frac{3i}{2}$. What are the residues at these points? Again, these are simple poles, so

$$\text{res}_{i/2} f = \lim_{z\to\frac{i}{2}} \left(z - \frac{i}{2}\right) f(z) = e^{-2\pi i z\xi} \cdot \frac{2(z-\frac{i}{2})}{e^{\pi z} + e^{-\pi z}}.$$

Manipulating this in a similar way, we find that this is

$$\lim_{z\to\frac{i}{2}} 2e^{-2\pi i z\xi} \cdot e^{\pi z} \cdot \frac{z - i/2}{e^{2\pi z} - e^{2\pi \cdot i/2}},$$

where the last term is just the inverse of the derivative of $e^{2\pi z}$ at $z = \frac{i}{2}$, which yields $\frac{-1}{2\pi}$. Then plugging in $z = \frac{i}{2}$ everywhere else, we have that the residue at $\frac{i}{2}$ is

$$2e^{\pi\xi}e^{\pi i/2} \cdot \frac{-1}{2\pi} = -\frac{ie^{\pi\xi}}{\pi}.$$

We find through a similar calculation that

$$\operatorname{res}_{3i/2} f = +\frac{ie^{3\pi\xi}}{\pi},$$

and now we can put everything together: by the residue theorem,

$$\lim_{R\to\infty}(1-e^{4\pi\xi})A_R = 2\pi i\left(-\frac{ie^{\pi\xi}}{\pi}+\frac{ie^{3\pi\xi}}{\pi}\right) = 2(e^{\pi\xi}-e^{3\pi\xi}) = -2e^{2\pi\xi}(e^{\pi\xi}-e^{-\pi\xi}).$$

Thus, the answer A that we want is

$$A = \frac{-2e^{2\pi\xi}(e^{\pi\xi}-e^{-\pi\xi})}{1-e^{4\pi\xi}} = -2e^{2\pi\xi}\cdot -e^{-2\pi\xi}\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{2\pi\xi}-e^{-2\pi\xi}},$$

and now the first two exponentials cancel, leaving us with a difference of squares

$$= \frac{2}{e^{\pi\xi}+e^{-\pi\xi}} = \frac{1}{\cosh(\pi\xi)},$$

as desired. $\square$

9 October 8, 2019

The class average on the midterm is a 91 out of 100: this is higher than Professor Zhang expected. There was a small oversight: one of the problems on the test had us compute

$$\int_0^\infty e^x \cos x.$$

But some of us didn't even do the contour integral, because this can be solved with more elementary methods (like integration by parts)! Also, even if we do the contour integral, we end up with

$$R\int_0^{\pi/4} e^{-R\cos t}dt,$$

which is pretty easy to bound. It would be a bit more difficult to bound if we had $\sin t$ instead of $\cos t$, but this did not end up happening! For the next exam, the questions will be more difficult.

Let's also discuss the last problem on the exam: we wanted to show that if an entire function satisfy $|f(z)| < |z|^n$ for all $|z| > 1$, then f is a polynomial. One thing to do is to say that because f is entire, we can write it as its power series expansion

$$f(z) = \sum_{i=0}^\infty \frac{f^{(i)}(0)}{i!}z^i,$$

and then we just need to show that the derivative vanishes for all $i > n$ using Cauchy's inequality with a circle of radius R:

$$|f^{(i)}(0)| \le \frac{i!}{R^i}R^n,$$

and then take $R \to \infty$.

It's important to note that we've only shown the higher-order derivatives are 0 at $z = 0$, and to finish, we can use the fact that f is entire, so it's equal to its power series at 0 everywhere. But the second way to finish is to show that the $n+1$th derivative (**as a function**) is equal to 0. Basically, pick a circle of radius R around some arbitrary point $z \in \mathbb{C}$: applying the same Cauchy inequality,

$$|f^{(n+1)}(z)| \le \frac{(n+1)!}{R^{n+1}}\sup_{\zeta\in C_R}|f(\zeta)|.$$

But then ζ is at most $z + R$ away from the origin, so by the problem statement,

$$|f^{(n+1)}(z)| \le \frac{(n+1)!}{R^{n+1}}(z+R)^n,$$

and again taking $R \to \infty$ gives the result: $\frac{z+R}{R}$ goes to 1 as $R \to \infty$. We could also use N instead of n, which tells us that $f^{(n)}$ is bounded by a constant $n!$, meaning that it's constant by Liouville's theorem.

Remark 97. *If we had trouble with the first two problems on the exam, we should make sure to review the relevant concepts more.*

So let's restate where we are again. We're starting to move away from strictly holomorphic functions a little bit: now we allow some mild, isolated singularities in our function. Let's go over some definitions again:

Definition 98

An **isolated singularity** of a function f is a point z_0 such that f is holomorphic in $\Omega - \{z_0\}$. Such a singularity is a **pole** if $\frac{1}{f(z)}$ is holomorphic in a neighborhood of z_0.

The main idea is that we can add, subtract, and multiply two holomorphic functions, but we can't divide without potentially introducing poles.

Definition 99

The **order** of a pole z_0 of f is the order of the zero of $\frac{1}{f}$ at z_0.

For example, $\frac{1}{z}$ has a pole of order 1. In general, we can write $f(z)$ as a (Laurent) series around z_0 as

$$f(z) = \sum_{i=1}^{r} a_{-i}\frac{1}{(z-z_0)^i} + g(z),$$

where g is holomorphic. This means that the bad part of f just looks like a rational function (quotient of polynomials), and once we subtract the **principal part** (everything except $g(z)$), our function is holomorphic.

Definition 100

Let the **residue** of the pole be the coefficient a_{-1} of $\frac{1}{z-z_0}$ in our series: denote this $\text{Res}_{z_0} f$.

This helps us generalize the Cauchy integral formula:

Theorem 101 (Residue formula)

Given a closed curve γ,

$$\int_\gamma f(z)dz = 2\pi i \sum_{\text{poles } z_i} \text{Res}_{z_i} f(z).$$

With this, we can do some more hard integrals!

Example 102

If we want to compute

$$\int_{-\infty}^{\infty} \frac{1}{1+x^6} dx,$$

we can integrate this around a semicircle of radius R: the arc part goes away, so the integral along the real line is just $2\pi i$ times the sum of the residues.

We no longer have to do the annoying keyhole contour anymore!

So now, let's move on. Why do we care about these singularities, other than the value of those residues? This is the "Riemann principle:" roughly, if we know the zeros and poles of our function, then it's basically determined. We know that a small piece of a holomorphic function will give us the function everywhere: now we have even more flexibility! Let's study some more properties of these singularities.

Definition 103

An isolated singularity z_0 of a function $f : \Omega - \{z_0\} \to \mathbb{C}$ is called **removable** if f extends to a holomorphic function on Ω (we can fill in $f(z_0)$ with some value).

Example 104

We know that $\lim_{x\to 0} \frac{\sin x}{x} = 1$, and indeed the singularity of $f(z) = \frac{\sin z}{z}$ is removable at $z = 0$.

Theorem 105 (Riemann)

A singularity z_0 of $f : \Omega - \{z_0\} \to \mathbb{C}$ is removable if and only if $f(z)$ is bounded in a neighborhood around z_0 (not containing z_0 itself).

Notice that this also implies that the limit $\lim_{z\to z_0} f(z)$ exists: for example, we can't have f approach two different values from two directions (like it does in the real-valued case for $f(x) = \frac{1}{x}$).

One idea is that if f is continuous on Ω, and it's holomorphic on $\Omega - \{z_0\}$, it's easy to show that f is holomorphic on Ω. (Use Morera's theorem, which says that this is equivalent to $\int_T f(z)dz = 0$ for any triangle, on an increasingly small triangle: by continuity, the value of $|f(z)|$ is bounded.)

But there's another way to prove this theorem:

Proof. It's easy to show that if the singularity is removable, then f is bounded. The other direction is the tricky part.

We claim that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

for any circle C around our singularity z_0. This two-variable function $F(\zeta, z) = \frac{f(\zeta)}{\zeta - z}$ is defined on $C \times D$, where D is the disk enclosed by C, and this function is holomorphic in z for any given ζ (because $\zeta \neq z$). Thus, if we have the above equality, then we show that $f(z)$ is holomorphic on D (by Theorem 83, parameterizing the curve C on $[0, 1]$). That would mean we can indeed extend f to a holomorphic function, and we'd be done.

So to prove our claim, make a keyhole contour which excludes z and z_0, which tells us that

$$\int_C f(z)dz = \int_{\gamma_{z_0}} f(z)dz + \int_{\gamma_z} f(z)dz,$$

where γ_{z_0} and γ_z are small circles around z_0 and z. But notice that the part around z gives us the familiar Cauchy integral formula term, and the part around z_0 goes to 0 (because f is bounded around z_0 and the length goes to zero). $\square$

Corollary 106

Let $f : \Omega - \{z_0\} \to \mathbb{C}$ be a holomorphic function. Then z_0 is a pole if and only if $|f(z)|$ goes to ∞ when $z \to z_0$: in other words, $\frac{1}{|f(z)|} \to 0$.

Proof. It is clear that if z_0 is a pole, $|f(z)|$ goes to ∞. Looking at the other direction, if $\frac{1}{f}$ goes to 0 as $z \to z_0$, then $\frac{1}{f}$ is bounded near z_0, so by the Riemann theorem, $\frac{1}{f}$ has a removable singularity at z_0, so $\frac{1}{f}(z_0) = 0$. This is the definition of z_0 being a pole of f. $\square$

So is there a way for us to classify all isolated singularities at this point? We have removable singularities that can be fixed, poles that have some finite order, and then the rest are called **essential singularities** (basically, we can't do anything about them). The first kind tells us that $f(z)$ is bounded, and the second that $f(z)$ is unbounded. So our last case is weird: it means f does not have a limit as $z \to z_0$.

Example 107

Consider $f(z) = e^{1/z}$. There is an isolated singularity at $z_0 = 0$: as we approach 0 from the real line, f approaches ∞, but as we approach 0 from the negative real line, f approaches 0. So 0 is an essential singularity.

Next time, we'll look a bit more at these essential singularities.

10 October 10, 2019

Hopefully we should now have access to the grades! There's always some problem with Stellar...

Today, we're going to continue discussing functions with isolated singularities. Last time, we classified our singularities:

- Removable, meaning f is bounded near the singularity,
- Pole, meaning that f at least goes to ∞ (it has a limit),
- Essential, which means there is no consistent behavior near our problem point.

Theorem 108 (Casorati–Weierstrass)

Let f be holomorphic on $\Omega - \{z_0\}$, and say that f has an essential singularity at z_0. Then the image of f is dense in $\mathbb{C}$.

In other words, f takes on values arbitrarily close to any complex number! (It might not be surjective: for example, $e^{1/z}$ is never zero, but it gets arbitrarily close to 0.) Specifically, in every small neighborhood of w, f takes on at least one point in that neighborhood.

In particular, the image of f is dense in $\mathbb{C}$ even if we shrink Ω to a smaller neighborhood of z_0. We won't prove this, but more generally it is possible to say that (Picard's great theorem) f takes on every complex value infinitely often, with at most one exception (again, $z = 0$ in this case).

Proof. Suppose for the sake of contradiction that the image is not dense in $\mathbb{C}$. Then $\text{Im}(f)$ avoids some open ball $D_R(w)$ of radius R centered around w. In other words, for all $z \in \Omega - \{z_0\}$,

$$|f(z) - w| \geq R > 0.$$

Consider the function $g(z) = \frac{1}{f(z)-w}$: since the denominator is never zero, this is a new holomorphic function on $\Omega - \{z_0\}$ with an isolated singularity at z_0. This time, $|g(z)|$ is bounded from above by $\frac{1}{R}$, so g is bounded near the singularity. Thus by Riemann's theorem, z_0 is now a removable singularity. But that implies $f = w + \frac{1}{g}$ also has a removable singularity or pole at z_0, which is a contradiction with our assumption. Thus the image of f is indeed dense, as desired. $\square$

So now we have a nice description for all three categories of singularities about the behavior of f near that singularity. With this, we now want to be able to "divide" holomorphic functions:

Definition 109

A function f is **meromorphic** if we can write $f(z) = \frac{g(z)}{h(z)}$ on any local neighborhood of z, where g, h are holomorphic. More formally, f is defined on $\Omega - \{z_1, z_2, \cdots\} \to \mathbb{C}$ such that f is holomorphic on its domain, $z_1, z_2, \cdots$ are poles, and all z_i are isolated: $\{z_1, z_2, \cdots\}$ has no limit points in Ω.

Example 110

Let g, h be two holomorphic functions of Ω. Then $f = \frac{g}{h}$ is indeed meromorphic as long as h isn't identically zero (because zeros of h are isolated, or else we'd have a limit point of zeros, making h identically zero). This also tells us that rational functions are meromorphic.

Often, we care about the behavior of a complex-valued meromorphic function $f(z)$ as $z \to \infty$ (for example, when we're computing a contour integral). Often we do this by replacing z with $\frac{1}{z}$, so we just have a limit as our input goes to 0. But what if we treat ∞ equally as any other point on the complex plane?

Definition 111

The **extended complex plane** is the union $\mathbb{C} \cup \{\infty\}$, and say that $f(z)$ is holomorphic (resp. has a pole) at ∞ if and only if $f\left(\frac{1}{z}\right)$ has a removable singularity (resp. has a pole) at $z = 0$. (An essential singularity at ∞ is defined similarly.)

Note that there's no ambiguity about which "direction" of ∞ we're using here, because saying this in terms of $\frac{1}{z}$ going to 0 is well-defined.

Example 112

If $f(z) = z^n$ for any $n \in \mathbb{Z}$, then $f\left(\frac{1}{z}\right) = \frac{1}{z^n}$ has a removable singularity when n is nonpositive and a pole when n is positive, so f is holomorphic at ∞ for $n \leq 0$, and it has a pole at ∞ for $n > 0$. More generally, rational functions are meromorphic on $\mathbb{C} \cup \{\infty\}$, because plugging in $\frac{1}{z}$ in for z still gives us a ratio of polynomials, and such a function either has a removable singularity or pole at 0, depending on the factors in that polynomial.

Example 113

Consider $f(z) = e^z$: changing coordinates yields an essential singularity at $z = \infty$.

What's the motivation for doing something like this (adding a new plane to the complex plane)? The idea is that this gives us a much nicer characterization:

Theorem 114

The only possible meromorphic functions on $\mathbb{C} \cup \{\infty\}$ are the rational functions $\frac{P}{Q}$, where P and Q are polynomials and Q is nonzero.

Of course, this is a strict subset of all meromorphic functions: adding ∞ to our domain in the theorem statement removes functions like e^z. Recall Riemann's principle, which tells us loosely in this case that meromorphic functions on $\mathbb{C} \cup \{\infty\}$ are basically determined by the locations of their zeros and poles (up to a constant). Here, this makes sense, because we can write rational functions (by the Fundamental Theorem of Algebra) as a product of linear factors!

Proof. If $f(z)$ is holomorphic or has a pole at ∞, then either $f(z)$ is bounded or goes to ∞ as $z \to \infty$.

Define $F(z) = f\left(\frac{1}{z}\right)$: this is meromorphic at $z = 0$. Recall from earlier that a function with a pole can be written as

$$f = \text{principal part} + \text{holomorphic function},$$

and around $z = 0$, the principal part is of the form

$$\frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots + \frac{a_{-r}}{z^r},$$

if the pole at $z = 0$ has order r. Thus, if we define

$$f_\infty(z) = \sum_{i=1}^{r} a_{-i} z^i,$$

this has a pole at ∞.

Remember that our zeros and poles are isolated, and in particular the pole at ∞ is isolated. Thus, we can draw a neighborhood of radius R around $z = 0$ for $F(z) = f\left(\frac{1}{z}\right)$ that doesn't contain any other zeros and poles, which means **there is a disk of radius $\frac{1}{R}$ for our original function $f(z)$ for which there are no zeros or poles outside that disk other than at ∞.** So there are a **finite number of zeros and poles** (otherwise we'd have a limit point), and thus for each pole, we can define the principal part corresponding to it:

$$f_{z_i} = \sum_{j=1}^{r_i} \frac{a^i_{-j}}{(z - z_i)^j}.$$

Now for each one, collect the principal parts, so that we can define the function

$$g = f - \left(\sum_{\{z_i\}} f_{z_i} \right).$$

(Remember that the set of $\{z_i\}$ contains ∞ if it's a pole, too.) This is now holomorphic on $\mathbb{C} \cup \{\infty\}$: we claim that this means g must be constant. This is because a function g being holomorphic at ∞ means $g(z)$ is bounded as $z \to \infty$. So g is bounded for large values of $|z|$, and then g is bounded on any disk of the form $z < R$ (because g is holomorphic).

Thus by Liouville's theorem, g is constant. So indeed f is the sum of a constant and all the principal parts, which is a finite sum of rational functions, so f is indeed a rational function. □

The central idea is that **we can cover $\mathbb{C}$ with two bounded disks** once we start using the extended complex plane:

$$\mathbb{C} \cup \{\infty\} = \{|z| \le R+1\} \cup \left\{ \left| \frac{1}{z} \right| \le \frac{1}{R} \right\}.$$

This almost makes $\mathbb{C} \cup \{\infty\}$ "compact" (see **one-point compactification**). This makes ∞ quite powerful, and it motivates a better study of the extended complex numbers. Here's a geometric story:

Definition 115

The **Riemann sphere** is defined via

$$S^2 = \{(x, y, z) | x^2 + y^2 + z^2 = 1\} \subseteq \mathbb{R}^3,$$

where we associate a point $a + bi$ in the complex plane with the **second intersection of the sphere with the line connecting** $(0, 0, 1)$ and $(a, b, 0)$, and we associate ∞ with $(0, 0, 1)$.

This is called the **stereographic projection**, and it bijectively identifies any point in $\mathbb{C}$ with a corresponding point S^2 except for the point $(0, 0, 1)$ (and the edge cases are accounted for, too). So considering the behavior of $|z| > R$ can be thought of as considering a small neighborhood around the north pole $(0, 0, 1)$ on the Riemann sphere. This is particularly nice because the sphere is "symmetric:" ∞ doesn't look particularly distinguished, which makes the extended complex plane more natural.

Remark 116. *The Riemann sphere is (the simplest example of) a 1-dimensional* **complex manifold** *which is compact. (It's a bounded, closed subset inside $\mathbb{R}^3$.)*

11 October 17, 2019

Recently, we've been discovering some properties of meromorphic functions (which have isolated singularities). Let's start looking at some applications now.

Say that $f : \Omega \to \mathbb{C}$ is meromorphic (implicitly, there is a countable set of poles in Ω, without any limit points). One rough form of Riemann's principle says that "knowing the zeros and poles tells us the function f." We considered this last time using extended complex plane $\mathbb{C} \cup \{\infty\}$: is there anything else we can say?

It turns out this is related to the question of defining the log function in the complex plane. When we define $\log x$ in the real-valued case, we define it as a function $u : \mathbb{R}_{>0} \to \mathbb{R}$ via

$$u'(x) = \frac{1}{x}, \quad u(1) = 0.$$

So it seems to make sense to define the complex log function similarly: perhaps we can define a function log such that

$$\frac{d}{dz} \log f(z) = \frac{f'(z)}{f(z)}$$

for all f, z. The right hand side here is a well-defined meromorphic function, so the derivative here is well-defined.

Well, what does this have to do with zeros and poles?

Theorem 117 (Argument principle)
Say $\gamma \subset \Omega$ be a simple closed curve such that f has no zeros or poles on γ. Then

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \text{number of zeros} - \text{number of poles}$$

(with multiplicity), where we're counting the number of poles and zeros **inside** γ.

In most cases, we take γ to be a circle. Intuitively, the integrand on the left hand side is really "$d(\log f)$," so if $\log f$ were holomorphic on Ω, the integral would have to be zero. But defining log is pretty hard: it seems to make sense to say that

$$\log(Re^{i\theta}) \stackrel{?}{=} \log R + (i\theta).$$

Unfortunately, the argument θ is only defined up to multiples of 2π, so this gives us a bit of trouble. And intuitively, those changes in θ are exactly what's causing the contribution on the right hand side.

Proof. Note that $\frac{f'}{f}$ is a meromorphic function, so by the Residue formula,

$$\int_\gamma \frac{f'(z)}{f(z)}dz = 2\pi i \sum \text{Res}_{z_0}\frac{f'}{f}$$

Notice that the poles of $\frac{f'}{f}$ come from the zeros of f and the poles of f', but the poles of f' are all poles of f (of order one larger). So we're only looking at points z_0 that are poles or zeros of f, and locally around each point, we can always write

$$f(z) = (z - z_0)^{r_{z_0}} g(z),$$

so that $g(z_0) \neq 0$, g is holomorphic, and r is some integer (positive if we have a zero and negative if we have a pole, with multiplicity). So that means that

$$\frac{f'}{f} = \frac{g'}{g} + \frac{r_{z_0}}{z - z_0}.$$

Here, $\frac{g'}{g}$ is holomorphic around z_0 because g doesn't vanish there, and thus the residue at z_0 of $\frac{f'}{f}$ is just r_{z_0}. Plugging this back into the residue formula, we indeed find that

$$\int_\gamma \frac{f'}{f}dz = 2\pi i \sum_{\text{zeros and poles}} r,$$

which is exactly what we want. $\square$

So this "log derivative" of f has nice residues: they're always integers, corresponding to the order of the zeros and poles. Notably, this means that if we know the function $\frac{f'}{f}$, we can approximate the integral to find the number of zeros with a computer. So we don't even need an exact answer: just a good enough approximation to the nearest integer! (This could technically disprove the Riemann hypothesis if we try hard enough and got lucky...)

Let's now use this to look at some applications: the idea is that a small perturbation of our function shouldn't change the number of zeros too much.

Theorem 118 (Rouché)
Let f, g be holomorphic on a region Ω containing a circle γ. If $|f| > |g|$ everywhere on γ, then the number of zeros inside γ is the same for f and $f + g$ (counting multiplicity).

Notice that the theorem assumption is only about the behavior of f and $f + g$ **on** the circle, not inside.

Proof. It suffices to show that f and $f + tg$ have the same number of zeros for all $t \in [0, 1]$ (because $|tg| < |g|$). By the argument principle,

$$\frac{1}{2\pi i} \int_\gamma \frac{(f + tg)'}{(f + tg)} dz = \eta(t)$$

yields the number of zeros of $f + tg$ inside the disk for any $0 \le t \le 1$ ($f + tg$ is holomorphic, so it has no poles). Our goal is to show that this function is constant: that is, it doesn't change in terms of t.

$\eta(t)$ always takes on integer values, so to prove it's constant, we just need to show that it's continuous. The denominator $f + tg$ is always nonzero because $|f| > |tg|$, and $\frac{(f+tg)'}{(f+tg)}$ is a function $\gamma \times [0, 1] \to \mathbb{C}$ taking (z, t) to $\frac{f'(z)+tg'(z)}{f(z)+tg(z)}$. This is continuous with respect to both z and t (because the numerator and denominator are), so when we integrate over γ, we'll still have a continuous function in t by Theorem 83. $\square$

With this, we can get another result:

Definition 119

A function $f : \Omega \to \mathbb{C}$ is an **open map** if the image of every open subset of Ω is open.

Example 120 (Not an open map)

Consider the function $f : \mathbb{R} \to \mathbb{R}$ sending x to x^2. This is not an open map, because the image of $(-1, 1)$ is $[0, 1)$.

As we've seen time and time again, though, holomorphic functions behave much better than smooth real-valued functions:

Corollary 121 (Open Mapping Theorem)

Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Then f is an open map.

Intuitively, this just tells us that the image of f cannot be isolated, unlike in the real-valued case above where there's a sharp cutoff at 0.

Proof. It's sufficient to prove that if $f(z_0) = w_0$, every point in a small neighborhood of w_0 is also in the image of f. We'll take $w_0 = 0$ for simplicity (we can just shift the constant in f), and we'll also assume $z_0 = 0$ (that's another shift).

We just need to show that $f(z) = w$ if $|w|$ is small. Choose a circle γ around 0 in our domain space, such that $z = 0$ is the only zero inside or on γ (we can do this because there are no limit points of zeros).

Now we can apply Rouché's theorem. Notice that γ is compact, and $f \ne 0$ on γ, so the infimum of $f(z)$ is nonzero. Thus, for any complex number w, as long as $|w|$ is less than $M = \inf |f(z)|$ on γ, we can apply Rouche with $g(z) = w$. Then f has one zero, so $f + g = f(z) - w$ has one zero inside γ as well, and thus the image of f also includes w. Thus, we've found a neighborhood of radius M around 0 which is contained in the image of f. $\square$

This actually gives us a very cool result:

Corollary 122 (Maximum Modulus Principle)

Let a holomorphic function f be defined on an open subset Ω. If f is nonconstant, then it does not attain its maximum value on Ω.

Again, this is in stark contrast to the real-valued case.

Proof. Since f is nonconstant, assume that f does attain its maximal value M at z_0. Because f is open by the Open Mapping Theorem, the image also contains a small disk around M, and thus there is a point in the disk with larger magnitude, which is a contradiction. $\square$

Another proof. We can think of this as the **mean-value property** of a holomorphic function: given any disk centered at z_0 and a circle of radius r around it,

$$f(z_0) = \int_0^{2\pi} f(z_0 + re^{i\theta})d\theta.$$

Since $f(z_0)$ is the average value of $f(z_0 + re^{i\theta})$, then f must be constant on the whole circle, and that makes f constant because it contains a limit point. $\square$

In contrast, if we have a closed disk $\overline{D}$, we know that f does attain its maximum value (because the disk is compact). Thus, the maximum value of f is always attained on its boundary, not its interior!

So let's go back to the logarithm that we discussed from the beginning of class, and we'll have to do this by "deforming" our function. To study f and $f + g$ in Rouché's theorem, we instead looked at the family of functions $F_t(z) = f + tg$. We can think of this as a "curve" connecting the functions $F_0(z) = f$ and $F_1(z) = f + g$. We want to think about when it's okay for us to "switch" between two different curves in an integration.

Definition 123

Say we have two (piecewise smooth) curves in the complex plane γ_0, γ_1 parametrized from $[a, b] \to \Omega$, such that $\gamma_0(a) = \gamma_1(a)$ and $\gamma_0(b) = \gamma_1(b)$. Then γ_0, γ_1 are **homotopic** or **homotopically equivalent**, denoted $\gamma_0 \sim \gamma_1$, if we can continuously deform γ_0 into γ_1: there exists a family of curves

$$F(s, t) : [0, 1] \times [a, b] \to \Omega$$

such that $F(0, t) = \gamma_0(t), F(1, t) = \gamma_1(t)$, and we also have $F(s, a) = \gamma_0(a), F(s, b) = \gamma_0(b)$ for all $s \in [0, 1]$ (that is, we have the same beginning and starting points for all curves).

Notably, this means that any two curves in our family F are homotopic.

Theorem 124

Let f be holomorphic on Ω. Then if $\gamma_0 \sim \gamma_1$ are homotopic, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz.$$

Remember that Goursat's theorem is a very special case of this: if we pick P and Q to be two vertices of a triangle, then the triangle forms two different paths from P to Q with equal contour integral.

We won't go through the proof here, but the idea is that we can cover the homotopy by a bunch of small disks.

Definition 125

A connected open region Ω is **simply connected** if any two curves with the same starting and ending points are homotopic.

Basically, if we have two paths from a point A to B, we can reverse the orientation of one of them to get a closed curve. So in a simply connected region, we're saying that **any closed curve can be deformed to a point**. It's a bit hard to check this definition, though!

Example 126

A region Ω is **convex** if the line segment between two points $P, Q \in \Omega$ is always contained in Ω. Convex regions are simply connected, because we can just "connect" the corresponding points along the curve: given $\gamma_0(t), \gamma_1(t)$, just define

$$F(s,t) = s\gamma_0(t) + (1-s)\gamma_1(t).$$

Example 127

A region Ω is **star-shaped** if there exists a point $z_0 \in \Omega$ such that the line between z_0 and z is always contained in Ω for any $z \in \Omega$. Notably, $\mathbb{C} - (-\infty, 0]$ is star-shaped with center $z = 1$. It's a bit more annoying to show, but these are also simply connected.

In general, toy contours are simply connected. The whole point is that any properties we've been stating for disks (which are simply connected, because they're convex) can be extended to simply connected regions in general. That's what we'll talk about next time for finding primitives of holomorphic functions!

12 October 22, 2019

Last time, we started talking about **simply connected** domains Ω: recall from last time that convex and star-shaped regions are simply-connected. The idea is that in these domains (which we already assume to be connected), we have a nice theory of integration for holomorphic functions:

Theorem 128

Let Ω be simply connected, and let $f : \Omega \to \mathbb{C}$ be holomorphic. Then there exists a holomorphic function F, unique up to a constant, such that $F' = f$ on Ω.

We previously proved this for an open disk $\Omega = D$: any open disk is simply connected, because it is convex, meaning any two curves with the same start and end points are homotopic. So this is a generalization of Cauchy's theorem: the proof is basically to set $F(z_0) = 0$ for some $z_0 \in \Omega$ and define $F(z)$ as an integral of f from z_0 to z, and this is well-defined by the definition of homotopy.

Corollary 129

If $\gamma \subset \Omega$ is a closed curve inside a simply connected domain, then

$$\int_\gamma f(z)dz = 0.$$

Example 130 (Non-example)

Consider the annulus formed by deleting a smaller disk from a larger disk. Then the remaining region is connected, but it's not simply connected, because we cannot shrink a curve that goes once around the inner disk to a point: equivalently, we can't deform half of the curve to the other half without going across the removed disk.

Alternatively, we can also show that the annulus is not simply connected by using Corollary 129: just let $f(z) = \frac{1}{z}$ and make the inner disk contain $z = 0$. Then f is holomorphic on the annulus, but $\int_C f(z)dz \neq 0$ for any circle once around the origin. So Ω **can't be simply connected if we remove any small disk around the origin**: in general, Corollary 129 is a good way to test if our domain can be simply connected.

As we mentioned at the end of class last time,

$$\Omega = \mathbb{C} - (-\infty, 0]$$

is a star-shaped region, because we can connect any point in Ω to the point $z_0 = 1$ with a straight line while staying entirely within Ω. (In particular, we can delete any half-line from $\mathbb{C}$ and this works.) So that means that because $\frac{1}{z}$ is defined on all of Ω, it also has a primitive on Ω.

Let's introduce a result that is helpful:

Fact 131 (Jordan Curve Theorem)

Say γ is a simple closed curve (it does not have self-intersections like the figure-eight). Then $\mathbb{C} - \gamma$ can be written as a disjoint union of two open sets $\Omega \sqcup U$, where Ω, called the **interior part**, is bounded, and U is unbounded. Furthermore, Ω is simply connected.

This is, as our intuition tells us, proving that curves have an inside and an outside. Thus, this tells us how to produce simply connected domains in an easy way, but it's only a special class of domains (because any Ω created this way is bounded).

Remark 132. *The concept of "simply connected" domains will lead us to algebraic topology if we continue to study in that direction.*

So now let's try to define the **complex logarithm** for a complex number $z = re^{i\theta} \neq 0, \theta = \arg(z) \in [0, 2\pi)$ via

$$\log z = \log r + i\theta.$$

Does this work? The idea is that we can always define log in a small disk around any z_0 not containing 0. Then we can draw a bunch of small disks around the origin that overlap, which basically gives us an annulus on which log is defined. We see, then, that if we define log on the unit circle, we start off with $\log z = 0$, but the value increases to $2i\pi$ by the time we've wrapped once around. (That's not so good for continuity reasons.)

So instead, we'll delete the negative real line, and we'll only define log on a simply connected domain!

Theorem 133

Let Ω be any simply connected domain which does not contain 0, and say it contains the point $z = 1$ (for simplicity). Then there exists a unique holomorphic function $\log : \Omega \to \mathbb{C}$ such that

$$(\log z)' = \frac{1}{z}, \quad \log(1) = 0.$$

Proof. Apply Theorem 128 to $f = \frac{1}{z}$. □

Theorem 134

For all $z \in \Omega$, we have

$$e^{\log z} = \log e^z = z.$$

Proof. To prove these two functions are equal on a connected domain, we just need to prove that they have the same derivative and the same value at any specific point. Indeed, $z = 1$ yields $e^0 = 1$, so we just need to show that the derivatives are equal.

It suffices to show two things: first of all, $ze^{-F} = 1$ (which just requires it having derivative 0, because we've already verified the value at one particular point). Then the derivative here is

$$(ze^{-F})' = (1 - zF')e^{-F} = 0.$$

Similarly,

$$[\log(e^z)]' = \frac{1}{e^z} \cdot (e^z)' = 1,$$

and now we've shown both directions. $\square$

By extension, this also tells us that the function $\log z$ is equal to the real-valued $\log r$ if $z = r$ is real.

Definition 135

The **principal branch** of $\log z$ is defined on $\Omega = \mathbb{C} - (-\infty, 0]$: concretely, for any $z = re^{i\theta}$ and $\theta \in (-\pi, \pi)$,

$$\log z = \log r + i\theta.$$

So the range of $\log z$ is an infinite rectangle of the form

$$\{(a + bi) : a \in \mathbb{R}, b \in (-\pi, \pi)\}.$$

We can define a power series expansion around $z = 1$ for the principal branch of the holomorphic function $f(z) = \log z$ as well:

$$\log(1 - z) = -\sum_{n=1}^{\infty} \frac{z^n}{n}.$$

for all $|z| < 1$, which agrees with the real-valued power series.

One more consequence of this:

Proposition 136

Say we have a holomorphic function $f : \Omega \to \mathbb{C} - \{0\}$ defined on a simply connected domain. Then there exists a function $g : \Omega \to \mathbb{C}$ such that

$$f(z) = e^{g(z)}.$$

Intuitively, this is saying that $g = \log(f(z))$. Note that we can just add $2\pi i$ to the value of g and still have a valid equality, so it's not exactly unique.

Proof. We can't apply the results earlier directly, because the image of f doesn't necessarily lie in the star-shaped region $\mathbb{C} - (-\infty, 0]$. But we can use the same kind of strategy here: we're trying to show that

$$g = \log(f(z)) \implies g' = \frac{f'}{f},$$

so we define (fixing some point $z_0 \in \Omega$)

$$g(z) = \int_{z_0}^{z} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

Since Ω is simply connected, this is well-defined! The rest is just verification that $f(z) = e^{g(z)}$ by taking derivatives. $\square$

So this completes the foundation of holomorphic and meromorphic functions that we'll talk about in this class: poles, residues, the argument principle, and the concept of simple connectivity. For the rest of the semester, we'll discuss the geometric nature of these functions (chapter 8), and then we'll move on to selected topics in chapter 5, 6, and 7 – notably, entire functions, so that we can study the Γ and ζ functions. Hopefully, we'll have some time to discuss the application of these ideas to the prime number theorem.

We've shown just now that the two regions

$$\{(a+bi) : a \in \mathbb{R}, b \in (-\pi, \pi)\} = \mathbb{R} \times (-\pi, \pi), \quad \mathbb{C} - (-\infty, 0]$$

are connected by a pair of bijective functions, so they're in some sense equivalent. This motivates the general definition:

Definition 137

Let $U, V \subseteq \mathbb{C}$ be open. A map $f : U \to V$ is called **conformal** or **biholomorphic** if it is holomorphic and bijective.

Lemma 138

Let $f : U \to V$ be a holomorphic, injective map. Then $f'(z) \neq 0$ for all $z \in U$. In addition, if f is actually bijective, then $g = f^{-1}$ is also holomorphic.

This isn't true in the real-valued case: consider the function

$$f(x) = x^3,$$

which is bijective and smooth, but its inverse $g(x) = \sqrt[3]{x}$ is not differentiable at $x = 0$. (Notably, x^3 is not bijective as a complex-valued function, because every nonzero z has three cube roots.) This also explains why conformal maps are called biholomorphic (we have a pair of holomorphic functions)! Remember that the open mapping theorem tells us that the image is always an open set, because f can't be constant if it's injective.

Example 139

Consider entire functions $f : \mathbb{C} \to \mathbb{C}$: if they are holomorphic and injective, they must be of the form $f(z) = az + b$ with $a \neq 0$ (homework problem). So we can't actually come up with any other conformal maps with $U = V = \mathbb{C}$.

Example 140

Let $\mathcal{H}$ be the upper half-plane

$$\{z = x + iy, y > 0\}.$$

Then $\mathcal{H}$ is conformally equivalent to the unit disk $D = \{|z| < 1\}$ via the map f sending

$$f(z) = \frac{i - z}{i + z}.$$

This has image equal to the unit disk, because $|i - z| < |i + z|$ whenever $\text{Im}(z) > 0$. And it's holomorphic, because $z = -i$ is not contained in the domain $\mathcal{H}$.

This means that these two regions are actually equivalent in our study, because **given any function on one of our domains, we can just compose them with the conformal map** to get a function on the other domain. So the holomorphic function theory on the two regions is basically the same.

More usefully, if we plug in a real number into $f(z) = \frac{i-z}{i+z}$, notice that we end up with a complex number of norm 1. So the boundary of $\mathcal{H}$ is basically mapped to the unit circle, except with the one distinguished point at -1, and to fix this, we just add a point at infinity.So basically

$$\mathcal{H} \cup (\mathbb{R} \cup \{\infty\}) \text{ is conformal to } D \cup C = \overline{D}.$$

Next time, we'll start with an explanation of the lemma above.

13 October 24, 2019

Last time, we introduced the concept of a **conformal** or **biholomorphic** map, which just requires our function to be bijective and holomorphic. This is a priori much weaker than being biholomorphic: why is there any reason the inverse function has to be holomorphic? Let's return to a lemma from last time:

Theorem 141
If a holomorphic function $f : U \to V$ is injective, then $f'(z_0) \neq 0$ for every point $z_0 \in U$. Then if f is bijective, the inverse map $g = f^{-1}$ is holomorphic as well.

So the injectivity argument tells us something strong about the derivative! Notice that the first condition is a local question, because we only care about the function in a neighborhood of z_0.

Proof of the first point. We can assume without loss of generality that $z_0 = 0$ and $f(z_0) = 0$, so we have a holomorphic function f that maps the origin to itself. Near 0, we can pick a small disk $D_r(0)$ contained in U such that the power series expansion near 0 is valid: this can be uniquely written as

$$f(z) = az^n + h(z) = z^n g(z),$$

where $g(0) = a \neq 0$. We know that g is also holomorphic on $D_r(0)$, and by the open mapping theorem, the image of $D_r(0)$ is some open map containing 0. We claim that for sufficiently small r and sufficiently small w,

$$\text{number of solutions}\{z \in D_r(0), f(z) = w\} = n$$

if we count roots with multiplicity. This is because we can make the higher-order terms arbitrarily small by picking r such that everywhere in D_r,

$$|h(z)| < |az^n|,$$

and thus by Rouché's theorem, the number of solutions to az^n and $f(z) = az^n + (h(z) - w)$ are the same, because $h(z) - w$ is a small modification of f (and we can now pick w small enough so that $|h(z) - w| \le |h(z)| + |w| < |az^n|$ everywhere on the circle, because the circle is compact). Since $az^n = 0$ has n solutions inside the disk with multiplicity, this means $f(z) - w$ also has n solutions inside the disk.

But this contradicts injectivity when $n > 1$: we should only have one solution to the equation $f(z) - w = 0$. So we must have $n = 1$. The only problem we need to worry about is the case where we have an n-fold repeated root for every single w, but this is something we can try checking ourselves: this would mean the derivative vanishes on every point of the small disk, meaning that it is a constant. $\square$

So the main point here is that if we shrink the neighborhood small enough, our function f always looks like $f(z) = cz^n$ if we shrink enough. And then we can basically use the argument principle to finish.

This second part is now essentially using the **implicit function theorem**:

Proof of the second part. To show that $g = f^{-1}$ is holomorphic, we can say that if $f(z) = w$ and $f(z_0) = w_0$ (by bijectivity)

$$g'(w_0) = \frac{g(w) - g(w_0)}{w - w_0} = \frac{z - z_0}{f(z) - z_0} = \frac{1}{\frac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{f'(z_0)}$$

because f is holomorphic. And since f' is nonzero everywhere by assumption, this means g has a well-defined derivative everywhere, as desired. $\square$

So we've now justified calling these bijective maps biholomorphic! In general, showing that a map is conformal is pretty easy: we just need to show bijectivity (injectivity is enough here, because we can restrict the image) and that the derivative is always nonzero.

Let's see some more examples:

Example 142

If we take the principal branch $\log z$, we map $\mathbb{C} - (-\infty, 0]$ to the rectangular strip

$$\{a + bi : a \in \mathbb{R}, b \in (-\pi, \pi)\}.$$

Notably, if we look at the negative real line (which we've restricted from the domain of $\log z$), we can say that the "lower part" of that line corresponds to the boundary line $r - i\pi, r \in \mathbb{R}$, and the "upper part" of that line corresponds to the boundary line $r + i\pi, r \in \mathbb{R}$.

Notice, though, that if we remove the entire real line and only consider the upper half-plane of $\mathbb{C}$ for $\log z$, the image is now restricted to the upper half of the rectangular strip

$$\{a + bi : a \in \mathbb{R}, b \in (0, \pi)\}.$$

But somehow 0 is now in the middle of the real line, and it still doesn't appear on our boundary: there's some interesting thing that's going on with points at infinity here.

Example 143

As another example of this, again consider the map

$$f(z) = \frac{i - z}{i + z}.$$

Recall that this sends the upper half plane to the unit disk.

If we plug in a real number $r \in \mathbb{R}$ into this, $\frac{i-r}{i+r}$ has magnitude 1, so we always end up on the unit circle. That means the real line maps to the unit circle, except it misses the point -1, because that would correspond to plugging in ∞ in for z. Here is a good opportunity to explain the name **conformal**: **conformal maps preserve angles between curves**, as long as we remember orientation. In particular, if we have a curve with a certain orientation, the orientation should be preserved! So because the real line is going counterclockwise around the half-plane, we should expect that we should go counterclockwise around the unit disk as well, and this is indeed correct.

By the way, the inverse map here is

$$z = i\frac{1 - w}{1 + w},$$

and we can start to play around with the behavior of this in certain regions. For example, if we restrict ourselves to the first quadrant

$$z \in \{a + bi : a, b > 0\},$$

then the image is restricted to the upper semicircle. And we can keep playing this game to get some interesting results: go from a semicircle to a quadrant, and then compose a log with that to get a rectangular stripe, and so on. The easiest way to do all of this is to keep track of the boundary and orientation of each region we work with.

So why do we care about any of this? We mentioned earlier that if two regions are conformally equivalent to each other, then defining functions on the two regions are essentially equivalent: we can compose with the conformal map to get a function on the other region. Here's another motivation:

Problem 144 (Dirichlet problem)

Consider the partial differential equation on some region $\Omega \subseteq \mathbb{C}$

$$\Delta u = 0$$

where Δ is the Laplacian operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Say that we know that $u = \phi$ is a given real-valued continuous function on the boundary: can we find u everywhere?

u is called a **harmonic function**. Here's an example of such a problem:

Example 145

Let $\Omega = D$, and this means that the boundary $\partial\Omega = C$ is a circle. Say we're given a real-valued $u = \phi$ on C: can we find u everywhere in D?

Well, this case is just the Cauchy integral formula. We know that holomorphic functions are related to harmonic functions, because (as a homework exercise) we can always complete a harmonic function to a holomorphic one. Such a function has to satisfy the **mean-value property**

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} u(e^{i\theta}) d\theta,$$

and then (as a homework problem we just did) we can find a conformal equivalence inside the disk sending 0 to an arbitrary point α. So this means we can actually find the value of $u(\alpha)$ everywhere inside the disk. The key idea here is that the mean value property looks special, but we can apply a biholomorphic map from the disk to itself so that any point α takes the role of 0, and then we can derive a new formula of the form

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) P(z, \theta) d\theta$$

by using the chain rule (again, as we did in the homework).

That only solves the Dirichlet problem for circles, though. In general, what's our strategy?

Lemma 146

Say that $\Delta u = 0$ on Ω, and $\Phi : V \to \Omega$ is holomorphic. Then $u \circ \Phi$ is harmonic as well.

Proof. We can write $u = \text{Re}(f)$ for a holomorphic function f, as detailed above. Now $f \circ \Phi$ is holomorphic, and indeed $u \circ \Phi$ is the real part of $f \circ \phi$, so it must be harmonic. $\square$

Here, this tells us that the Dirichlet equation cares only about the **conformal equivalence class**, not the general shape of Ω. So now things become a bit easier: if we need to solve an equation on a specific region, we can try to transform it into a different region.

Well, let $f : U \to V$ be a conformal map. Then if U is simply connected, V is also simply connected. Notably, $\mathbb{C}$ can't be conformally equivalent to D, because that would be a bounded function that is entire (meaning it must be constant). But it turns out this is the only exception:

Theorem 147 (Riemann mapping theorem)
Let $U \subsetneq \mathbb{C}$ be simply connected. Then U is conformal to D.

This is the start of a lot of modern geometry and topology, and it's probably the most nontrivial result we've encountered so far! This result might be surprising when we see it at first, because it's so easy to come up with simply connected domains – in some sense, they are all equivalent here. And of course, this also justifies why we study the unit disk so much.

14 October 29, 2019

We've been motivating the importance of conformal maps: one key result is the **Riemann mapping theorem**, which tells us that if we have a simply connected $\Omega \subsetneq \mathbb{C}$, then it is conformal to the unit disk. So all functions on such Ω are essentially equivalent to functions on D, and thus it makes sense to only consider functions that are defined on the unit disk.

Today, our focus will be to start classifying these conformal maps: we'll try to classify all self-conformal maps from D to itself. (These are also called **automorphisms**, and they form a group under function composition.) This is particularly useful, because if we can find any map $F : \Omega \to D$, then all conformal maps from $\Omega \to D$ are of the form $\phi \circ F$, where ϕ is an automorphism of D.

Example 148
Any rotation $f(z) = e^{i\theta}z$ is an automorphism of the unit disk.

Theorem 149 (Schwarz lemma)
Let $f : D \to D$ be holomorphic but not necessarily conformal, and say that $f(0) = 0$. Then

- $|f(z)| \le |z|$.
- If the equality holds for any point z_0, then $f(z)$ is a rotation.
- $|f'(0)| \le 1$, and equality only holds when f is a rotation.

Proof. Because $f(0) = 0$, we can consider the function $g(z) = \frac{f(z)}{z}$, which has a removable singularity at $z = 0$ which we can replace (because f has a power series expansion with constant term 0). This goes from $D \to \mathbb{C}$ a priori, and we want to show that the image also lands inside the unit disk.

So now by the Maximum Modulus Principle, the maximum value of g cannot be attained on the interior. We can't use the maximum modulus principle on the whole disk D, because we don't necessarily have nice behavior there, but we can apply it to a disk D_r for any $r < 1$. Then

$$\sup_{z \in D_r} |g(z)| \le \sup_{|z|=r} |g(z)| \le \frac{1}{r}$$

because we know that $|f| \le 1$ everywhere on its domain (because it's a map to the unit disk). This is true for all r, so take $r \to 1$ to get the result. Thus $|g(z)| \le 1$, and thus

$$\frac{|f(z)|}{|z|} \le 1 \implies |f(z)| \le |z|.$$

This is the first bullet point, and notice that if equality holds, then $|g(z)| = 1$ attains its maximum value inside the disk, so g is constant. This means that $g(z) = e^{i\theta}$ everywhere for some θ, and thus $f(z) = e^{i\theta} z$, proving the second point.

Finally, notice that $|f'(0)| = |g(0)| \le 1$ (by power series expansion). If $g(0)$ has magnitude 1, then g is constant everywhere, and again we have $f(z) = e^{i\theta} z$, and we're done. $\square$

So now we can apply the Schwarz lemma to classify automorphisms of D. Rotations are indeed automorphisms, but we've also seen a few other examples before:

Example 150

Let $\alpha \in D$. Then we can apply the **fractional linear transform** to get a holomorphic map

$$\phi_\alpha(z) = \frac{\alpha - z}{1 - \overline{\alpha} z}.$$

Let's show that this is indeed an automorphism. To do that, we need to (1) show that $|\phi_\alpha(z)| < 1$ for all z, and (2) show that the map is bijective.

For (1), note that ϕ_α is a holomorphic map on a domain slightly larger than the unit disk, because $\overline{\alpha}$ in the denominator has norm less than 1. So we can define ϕ_α on a disk of radius $1 + \varepsilon$ instead of just 1; now by the maximum modulus principle, we can just verify that $|\phi_\alpha(z)| < 1$ for all $|z| = 1$. And this is true because we can pull out a $-z$ from the numerator, and then the numerator and denominator are conjugates! Since ϕ_α is clearly not a constant map, we can't have equality anywhere inside, and thus the image is contained in D.

And now for (2), bijectivity is easy: if $w = \frac{\alpha - z}{1 - \overline{\alpha} z}$, we can solve for z in terms of w, and it turns out that

$$z = \frac{\alpha - w}{1 - \overline{\alpha} w} = \phi_\alpha(w),$$

meaning that ϕ_α is actually an involution on the unit disk! So it's definitely a bijection.

Theorem 151

All automorphisms of D are of the form

$$f(z) = e^{i\theta} \phi_\alpha(z)$$

for some $\theta \in \mathbb{R}, \alpha \in D$.

Proof. We've already shown that all of these maps are conformal (they're compositions of conformal maps), so it remains to show that all conformal maps are of this form.

Notice that ϕ_α sends 0 to α and vice versa, so given any conformal map F that sends α to β, we can compose it with ϕ_β and also pre-compose it with ϕ_α. Then

$$G(0) = \phi_\beta \circ F \circ \phi_\alpha(0) = 0,$$

so this gives us one fixed point! In other words, any conformal map can be expressed via

$$F = \phi_\beta^{-1} \circ G \circ \phi_\alpha^{-1},$$

so if we can show that all automorphisms G that send 0 to themselves are of the form in the theorem statement, we'll also be done.

Lemma 152

If G (an automorphism of the disk) fixes 0, then it must be a rotation of the form $e^{i\theta}z$.

Proof of lemma. We know that G and G^{-1} are both maps from D to themselves, so by the Schwarz lemma, $|G'(0)| \le 1, |{G^{-1}}'(0)| \le 1$. But these two numbers are reciprocals:

$${G^{-1}}'(0) = \frac{1}{G'(G^{-1}(0))} = \frac{1}{G'(0)},$$

so both derivatives must have magnitude 1, meaning that G is indeed a rotation (by the equality case of the Schwarz lemma). $\square$

So now because G is a rotation,

$$F = \phi_\beta^{-1} \circ (\text{rotation}) \circ \phi_\alpha,$$

and we're now almost done. To get this in the form (rotation) $\circ\, \phi_\alpha$, we can verify that the composition of two maps $e^{i\theta}\phi_\alpha$ is another one of that form. But actually, we can just set $\beta = 0$: ϕ_β is then the identity, which gives us what we want. $\square$

Basically, knowing the example ϕ_α gave us a lot of control here. The other key to this proof is the Schwarz lemma, which tells us about the strong rigidity of holomorphic functions.

Let's reformulate this a bit with linear algebra: it turns out that we can represent such automorphisms of the unit disk with matrices in $SU(1,1)$, the special unitary group! Basically, consider the hermitian norm on $\mathbb{C}^2$ defined by

$$||(w_1, w_2)||^2 = w_1\overline{w_1} - w_2\overline{w_2}.$$

We can also represent this in matrix form as

$$||(w_1, w_2)||^2 = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Definition 153

Let $J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (this characterizes the inner product). The **unitary group $U(1,1)$** is the set of matrices that preserve the inner product:

$$g \in M_2(\mathbb{C}) : gJg^* = J,$$

where g^* is the conjugate transpose of g. The **special unitary group $SU(1,1)$** are those matrices in $U(1,1)$ with determinant 1.

Consider the set of $\mathbf{w} = (w_1, w_2)$ such that

$$||\mathbf{w}||^2 = ||(w_1, w_2)||^2 < 0 \implies |w_1| < |w_2|.$$

This can be thought of as the set of points $z = \frac{w_1}{w_2}$ in the open unit disk! Then if we have a matrix $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SU(1,1)$, then

$$g \cdot \mathbf{w} = \begin{bmatrix} aw_1 + bw_2 \\ cw_1 + dw_2 \end{bmatrix}$$

should also correspond to a point inside the unit disk (because the norm $||g\mathbf{w}||^2 = ||\mathbf{w}||^2$ is preserved), and thus we can define our map to be

$$f(z) = \frac{aw_1 + bw_2}{cw_1 + dw_2} = \frac{az + b}{cz + d},$$

which is exactly the fractional linear transform that we've been talking about! And if we choose g to be of the form

$$g = \lambda \begin{bmatrix} 1 & -\alpha \\ -\overline{\alpha} & 1 \end{bmatrix},$$

that indeed gives us $\phi_\alpha(z) = \frac{z-\alpha}{1-\overline{\alpha}z}$, and if we pick

$$g = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix},$$

we get our rotation map (just by an angle 2θ). So the whole point is that everything we've been doing can indeed be written as an element of the special unitary group: we would just need to show that these do actually generate all of $SU(1,1)$, and that takes a bit of work.

So let's transfer everything back to the upper half-plane! If we want a conformal map $\mathbb{H} \to D$ that sends $i \to 0$, one example is $w = \psi(z) = \frac{i-z}{i+z}$, which has inverse $z = i\frac{1-w}{1+w}$.

Theorem 154

All automorphisms of $\mathbb{H}$, the upper-half plane, are of the form

$$\phi_g(z) = \frac{az + b}{cz + d}$$

where $a, b, c, d \in \mathbb{R}$ and $ad - bc = 1$ (that is, $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is an element of the special linear group $SL_2(\mathbb{R})$).

One direct proof is to just do our some calculations: consider the following commutative diagram, which corresponds automorphisms of $\mathbb{H}$ with automorphisms of D:

$$\begin{tikzcd} \mathbb{H} \arrow[r, "\psi"] \arrow[d, "F"] & D \arrow[d, "G"] \\ \mathbb{H} \arrow[r, "\psi"] & D \end{tikzcd}$$

Thus we can write any map F as

$$F = \psi^{-1} \circ G \circ \psi,$$

where G is an automorphism of the unit disk, and ψ is the map from $\mathbb{H}$ to D that we defined above. But this isn't really telling us that much about the structure of the group: instead, what's really going on here is that

$$SU(1,1)/(\text{scalar matrices}) \cong SL_2(\mathbb{R})/(\pm 1),$$

where the modding out happens because we can multiply all components by certain scalars and still get the same matrix. And notably, we can look at subgroups of the two matrix groups as well: automorphisms of the half-plane that send i to themselves correspond exactly to the rotation matrices

$$g = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

in the orthogonal group! So there's some interesting theory here that we should explore on our own if we're interested.

15 October 31, 2019

First of all, we should remember the second midterm exam is next Tuesday, November 5 (during class). The next homework assignment will be posted soon, but the submission deadline will be the week after the exam (on November 14).

We're moving to a slightly more technical study of complex-valued functions now. To put things into context, we started with the easiest functions, polynomials: they're particularly nice because a degree-d polynomial always has d roots. We then slightly extended this by considering rational functions, which are the meromorphic functions on the extended complex plane $\mathbb{C} \cup \infty$: these are just ratios of our polynomials. (Specifically, polynomials are the subset of the rational functions which are entire, meaning they have no poles.)

In general, if we have entire functions that don't necessarily need to be meromorphic at ∞, this gives us a larger class of functions that can be represented as power series, such as e^{az} and $\sin z$. We'll focus on this particular class of functions this week, and we'll try to **classify these entire functions** (with some constraints).

To do this, let's first think about some important properties of polynomials: any degree-d polynomial can be written in the form

$$P(z) = a \prod_{i=1}^{d} (z - z_i).$$

So knowing the set of all zeros $z_1, \cdots, z_d$ determines our polynomial P up to a constant! Notably, it's okay to have a zero counted twice, because that just means that root has some multiplicity greater than 1.

Problem 155

Let's try to make a similar characterization for entire functions: if we know the locations of all zeros, maybe we can figure out what form our function f takes.

In general, though, this seems like it can be difficult, because just taking $\prod(z - z_i)$ might not give us anything convergent! So we need to do a bit of preliminary work first.

Theorem 156 (Jensen)

Say that $f : D_{R+\varepsilon} \to \mathbb{C}$ is holomorphic with $f(0) \neq 0$, such that f has no zeros on the boundary ∂D_R (we can pick such an R, because otherwise we'd have uncountably many zeros for f). Suppose that we have a set of zeros inside D_R (with multiplicity) $\{z_1, \cdots z_n\}$ (this is a finite set because the closed disk D_R is compact, so any countably infinite set of points would have a subsequence containing a limit point, meaning f is identically zero). Then

$$\log|f(0)| = \sum_{i=1}^{n} \log \frac{|z_i|}{R} + \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta.$$

Why do we care about this? We know that the **argument principle** calculates the number of zeros inside D_R:

$$\frac{1}{2\pi}\int_{\partial D_R}\frac{f'(z)}{f(z)}dz,$$

but it's not always good enough for the current purposes. The argument principle tells us how many zeros we have, but Jensen also gives us, in some sense, how large the magnitude of these zeros z_i are! In addition, the argument principle involves derivatives of f, so if we only have asymptotic bounds for f, it's easier to use Jensen.

Proof. Say we have functions f, g that both satisfy the properties (not equal to 0 at 0, no zeros on the boundary ∂D_R). Then notice that fg also satisfies this property, because

$$\log|fg| = \log|f| + \log|g|,$$

and the first term on the right side is "split" between the zeros of f and the zeros of g. Specifically, it's useful to think about the function

$$g = \frac{f}{\prod_{i=1}^{n}(z - z_i)}.$$

Here, g **has no zeros**, because we've divided them all out from f. In particular, g is holomorphic inside D_R with no poles. so it suffices to prove the result for functions g that are holomorphic with no zeros inside D_R, and also for linear factors $(z - z_i)$. Taking the product of these will give us any function f.

We'll need to use the complex log in both cases: recall that $\text{Re}(\log a) = \log|a|$) for any complex number a, because the argument only contributes to the imaginary part of the log.

- First, let's show Jensen for the case where g has no zeros. Then we want to show that

  $$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi}\log|g(Re^{i\theta})|d\theta.$$

  This looks a lot like the mean-value property for holomorphic functions, which tells us that

  $$f(0) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\theta})d\theta.$$

  We just have an extra log term, and we need to make sure this is well-defined when we use complex numbers. Well, since $D_{R+\varepsilon}$ is simply connected, and f never takes the value 0, the image of $D_{R+\varepsilon}$ is a simply connected domain that does not contain 0. Thus, we can define a version of log, call it g, such that we can write $f = e^g \implies g = \log f$ for all points in $D_{R+\varepsilon}$. Then the mean-value property on g tells us that

  $$g(0) = \frac{1}{2\pi}\int_0^{2\pi} g(Re^{i\theta})d\theta,$$

  and then take the real part of this to get the result. (This is essentially just using the mean value property for harmonic functions.) **Notice here that we can't always define** log **as simply as usual**: the simply connected image of f might look like a spiral, so we can't cut out a half-line from the origin directly.

- Finally, let's do the case where $f = z - z_i$ is a linear function. We just need to show that

  $$\log|z_i| = \log\frac{|z_i|}{R} + \frac{1}{2\pi}\int_0^{2\pi}\log|Re^{i\theta} - z_i|d\theta.$$

  Subtract $\log|z_i|$ from both sides (to cancel), and cancel the $\log R$ from the fraction with a $\log R$ from the integral: it then suffices to show that

$$0 = \int_0^{2\pi} \log|(1 - ae^{i\theta})| d\theta,$$

where $a = \frac{z_i}{R}$ has magnitude less than 1 by definition. But now $g = \log(1 - az)$ is holomorphic on some disk $D_{1+\varepsilon}$, because $1 - az$ is always nonzero. So now we can use the mean value theorem again: the integral is equal to $\text{Re}(g(0)) = 1$.

$\square$

With this, let's start to classify our entire functions. The number of zeros for a polynomial gives its degree, and this degree measures how fast this function grows. Precisely, notice that if a polynomial has degree d,

$$\sup_{|z| \in D_R} |f(z)|^d \sim R^d.$$

So let's try to be more precise with this order of growth, but also allow us to quantify the growth of things like exponential functions:

Definition 157

Say that f is entire, and $\rho \in \mathbb{R}_{\geq 0}$. Then f has an **order of growth at most ρ** if for all $z \in \mathbb{C}$,

$$|f(z)| \leq Ae^{B|z|^\rho}$$

for some real numbers $A, B > 0$. In particular, the **order of growth** of f is defined via

$$\rho_f = \inf\{\rho : f \text{ has order at most } \rho\}.$$

Example 158

e^z has order of growth at most 1 (but this doesn't work for any constant less than 1), so $\rho_f = 1$. In general, e^{z^d} has order of growth at most d for any $d \in \mathbb{Z}$, and indeed $\rho_f = d$ for these functions.

But there's a small complication: suppose that we know that ρ_f is the order of growth of f: **this does not imply that** f has an order of growth at most ρ_f, because of the infimum definition! For example, $f(z) = z^d$ has order of growth at least ε for all $\varepsilon > 0$ (because any exponential grows faster than polynomial), so $\rho_f = 0$. But this doesn't mean that f has an order of growth at most 0...

Intuitively, the idea is that polynomials have order of growth 0 because they only have finitely many zeros. So if we want something bigger, we must allow infinitely many zeros for our entire functions.

Definition 159

For a function f, define $n(r) = \text{zero}_{D_r}(f)$ to be the number of zeros inside a disk $|z_i| < r$, with multiplicity.

Theorem 160

Say that f has an order at most $\rho < \infty$. Then

- $n(r) \le Cr^\rho$ for some constant C and all sufficiently large r: the number of zeros can't grow too fast.
- Say that our zeros of f are $\{z_1, z_2, \cdots\}$. Then for any $s > \rho$,

$$\sum_{z_i \ne 0, i \ge 1} \frac{1}{|z_i|^s} < \infty$$

converges to a finite value.

Example 161

If we take $f = \sin(\pi z)$, the zeros occur precisely at the integers (this was a homework problem). Let's check that the conditions of the theorem hold.

Indeed, because f has order at most 1 (sin is basically like the exponential),

$$n(r) = 2r + 1 \le Cr$$

is true. In addition, it's definitely true that

$$\sum_{i=1}^{\infty} \frac{1}{|z_i|^s} < \infty$$

for all $s > 1$. Notice that both of our conditions are tight in this case! (In general, we can take $\cos(z^d)$ to get a function with $O(r^d)$ zeros inside a disk of radius r.)

Start of the proof. First, let's show that the first result implies the second. We can break up into different annuli, depending on how far away we are from the origin, to get a good enough bound. Fix some r_0, and now consider the zeros between r_0^n and r_0^{n+1} for each n. This is a good bound, because each zero in each annulus contributes at most the smallest magnitude in that annulus:

$$\sum_{i=1}^{\infty} \frac{1}{|z_i|^s} = \sum_n \sum_{r_0^n < |z_i| < r_0^{n+1}} \frac{1}{|z_i|^s} \le \sum_n C r_0^{(n+1)\rho} \cdot \frac{1}{(r_0^n)^s} \le c r_0^\rho \sum_n \frac{1}{r_0^n (s-p)}.$$

And now the term inside the sum is a geometric series, so this is a finite value for any $s > \rho$, as desired. □

We'll finish the rest of the proof next time.

16 November 7, 2019

The midterm exams will be graded by next class.

We've been studying entire functions with a finite order of growth: basically, we want to understand more than just polynomial functions, but we don't want to look at all general holomorphic functions. The intuition is that a function with many zeros must grow very fast!

Quantitatively, we stated the following reuslt last time:

Theorem 162
Say that f is entire and of order at most ρ. Then

- The number of zeros $n(r)$ inside the $|z| < r$ (counting multiplicity) is at most cr^ρ for sufficiently large r.
- Summing over all zeros (that are nonzero),
$$\sum_{\text{nonzero zeros}} \frac{1}{|z_i|^s} < \infty \quad \forall s > \rho.$$

Note that $n(r) : \mathbb{R}_{>0} \to \mathbb{Z}$, which counts the number of zeros, is well-defined, because we can only have finitely many zeros inside a disk of any finite radius. (This is because the closed disk is compact, and we can't have our set of zeros contain a limit point.) However, notice that $n(r)$ is obviously not continuous, because it only takes on integer values.

We showed last time that the first point proves the second: we split up the sum in annuli of $r^k \le |z| \le r^{k+1}$, and we showed that this gives a convergent sum.

How do we proceed from here? Last time, we proved Jensen's formula, which tells us that
$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})d\theta| = -\sum \frac{\log|z_i|}{R} + \log|f(0)|,$$
where we're summing over all zeros inside a disk of radius R. This can be reformulated in terms of our counting zero function $n(r)$: note that
$$\log\frac{A}{B} = \int_B^A \frac{dr}{r},$$
so our sum over zeros can be written as
$$\sum_i \int_{|z_i|}^R \frac{dr}{r} = \sum_{z_i}\int_0^R 1_{[|z_i|,R]}\frac{dr}{r},$$
where $1_{[|z_i|,R]}$ is the **indicator function** for our closed interval $[|z_i, R]$, meaning it takes on 1 inside the interval and 0 everywhere else. We can then swap the integral and the sum (because the integrand is positive), and this means we're essentially counting how many zeros are inside a disk of some radius: thus,
$$\frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})d\theta| = \int_0^R n(r)\frac{dr}{r} + \log|f(0)|,$$
which is a much cleaner way of writing the formula! (And even though we have discontinuities, they're mild enough to still let us Riemann integrate over $\frac{n(r)}{r}$, because they're simple jumps of a monotone function.)

By the way, here's a slightly easier way to remember the formula:

Sketchy proof using the argument principle. Recall the **argument principle**, which tells us that at **all continuous points** of $n(r)$ (meaning we have no zeros on the boundary),
$$n(r) = \frac{1}{2\pi i}\int_{|z|=r} d\log f(z),$$
where $d\log f(z) = \frac{f'(z)}{f(z)}dz$. We can then rewrite this (by parameterizing $z = re^{i\theta}$) as
$$n(r) = \frac{r}{2\pi}\int_0^{2\pi} d\log|f(re^{i\theta})|$$

(because $z = re^{i\theta} \implies dz = rde^{i\theta}$, which gives us a constant r that we can pull out of the integral). And now we have a good way to write $\frac{n(r)}{r}$:

$$\frac{n(r)}{r} = \frac{1}{2\pi}\int_0^{2\pi} d\log|f(re^{i\theta})| \implies \boxed{\int_0^R \frac{n(r)}{r}dr} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^R d\log|f(re^{i\theta})|dr$$

and now the inside integral can be calculated with the Fundamental Theorem of Calculus:

$$= \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})| - \log|f(0)|d\theta,$$

which is exactly

$$= \boxed{-\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta}.$$

□

Another way to think about this is the following:

Proposition 163

We can write

$$\frac{n(r)}{r} = \frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\right).$$

Then integrating means we can just evaluate the parenthetical term at R and 0.

Proof. We have

$$\frac{d}{dr}\left(\frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\right) = \frac{d}{dr}\left(\frac{1}{2\pi i}\int_{|z|=1} \log f(rz)\frac{dz}{z}\right),$$

and now we can swap the integral and the derivative: since

$$\frac{d}{dr}\log f(rz) = \frac{f'(rz)}{f(rz)} \cdot z,$$

we end up with

$$\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(rz)}{f(rz)}z \cdot \frac{dz}{z} = \frac{1}{r}\frac{1}{2\pi i}\int_{|z|=1} \frac{f'(rz)}{f(rz)}d(rz),$$

which is exactly $\frac{n(r)}{r}$ by the argument principle. □

The key idea here is that

$$\frac{d}{dr}\log f(rz) = \frac{z}{r}\frac{d}{dz}\log f(rz),$$

which might seem kind of silly. But we're integrating over z versus over r, which are two very different ways of looking at the problem. So now we can prove the first point of Theorem 162:

Proof. We know that $n(r)$ is increasing, so we can estimate via

$$\int_0^{2R} \frac{n(r)}{r}dr \ge \int_R^{2R} \frac{n(r)}{r}dr \ge n(R)\log 2.$$

And now by our modified Jensen, we can rewrite the left hand side:

$$n(R)\log 2 \le -\log|f(0)| + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta.$$

Since f has order at most ρ, $f \le Ae^{B(2R)^\rho}$ for some constants A, B, so we know that the magnitude of $\log|f|$ is at most a constant plus $B(2R)^\rho$ everywhere that we're integrating it, which yields

$$n(R)\log 2 \le -\log|f(0)| + A' + B'R^\rho,$$

which shows that $n(R) \le cR^\rho$ as desired. $\square$

We showed last time that we could get equality (both conditions of this theorem are tight). Let's try to go back to another question from last week, then: if we know all the zeros of a function f, can we recover what it is? If we have finitely many zeros, the function $f(z) = \prod_{i=1}^n (z - z_i)$ works, but it's hard to justify directly that

$$\prod_{i=1}^{\infty}(z - z_i) = \lim_{N\to\infty}\prod_{i=1}^{N}(z - z_i)$$

makes sense.

Well, we know that if we have an infinite product that converges, then $a_i \to 1$ (just like in the infinite sum case, where $a_i \to 0$). The whole point is that we can use the complex log to make everything look like an infinite sum.

Proposition 164

Say that $\prod_{i=1}^\infty a_i$ converges, which means that a_i eventually lies in the disk $|z-1| < \frac{1}{2}$. (We can drop the first finitely many terms.) Then we can define log on the disk, and we have

$$\sum_{i=1}^{\infty}\log a_i = \log\prod_{i=1}^{\infty} a_i.$$

(Basically, we can just define both sides to be the partial sum / product of the first N terms, and take $N \to \infty$.)

Lemma 165

Suppose we're in this situation, where $|b_i| = |a_i - 1| < \frac{1}{2}$. Then if $\sum_{i=1}^\infty |b_i|$ converges, then $\sum_{i=1}^\infty \log(1+b_i)$ absolutely converges.

Proof. We know that for all $|z| < \frac{1}{2}$,

$$|\log(1+z)| < cz$$

for some constant c, because $\frac{\log(1+z)}{z}$ is holomorphic on the small disk and is therefore bounded, and this shows the result. $\square$

One subtle point is that convergence is only implied in one direction: **if** $\sum \log a_i$ converges, then $\lim_{N\to\infty}\prod_{i=1}^N a_i$ converges, but not necessarily the other way raound (for example, consider $a_i = 1 - \frac{1}{i}$ for all $i \ge 2$). So we can say that **if** $\sum \log a_i$ converges, then $\prod_{i=1}^\infty a_i$ converges and is nonzero. In addition, we can say that if $\sum_i |b_i| < \infty$ converges to a finite value, then $\prod a_i$ converges to a nonzero value. And in this case, the only way for the infinite product to be zero is for one of the individual terms to be zero.

Corollary 166

Suppose that $b_i \in \mathbb{R}_{>0}$ satisfy $\sum_{i=1}^{\infty} b_i < \infty$ and $b_i < 1$ for all i. Say that we have holomorphic functions $F_i(z)$ from $\Omega \to \mathbb{C}$, such that $|F_i(z) - 1| < b_i \quad \forall i$. Then

- $\prod_{i=1}^{\infty} F_i(z)$ converges uniformly on every compact subset of Ω to a holomorphic function $F : \Omega \to \mathbb{C}$.
- We have for any $z \in \Omega$ that
$$\frac{F'(z)}{F(z)} = \sum_{i=1}^{\infty} \frac{F_i'(z)}{F_i(z)}$$
(this is basically the derivative of $\log F = \sum \log F_i$).

Proof. Choose a compact subset K inside Ω. Then $\log F_i$ is well-defined, because its value lies in a small disk near 1, so its absolute value is bounded by a fixed, absolute constant times b_i. Thus $\log F_i$ is holomorphic for all i, and then the infinite sum $\sum_{i=1}^{\infty} F_i$ is uniformly convergent to $\log F$. This is because one sufficient condition for uniform convergence for a sum $\sum_{i=1}^{\infty} f_i$ is if all $|f_i| \le c_i$ and $\sum c_i < \infty$, which is indeed the case here on our compact subset K! And this uniform convergence means that $f_{i'}$ converges to f' as well, so we can indeed take the term-by-term derivative, giving us the desired result. $\square$

Example 167

Consider the function
$$f(z) = \frac{\sin(\pi z)}{\pi},$$
which has zeros precisely at the integers. Thus, we claim that
$$f(z) = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

First of all, the right hand side is convergent, because the sum of the distances from 1 of the individual terms,
$$\sum_{n=1}^{\infty} \frac{|z|^2}{n^2} < \infty$$
is convergent for all z in a compact (in particular bounded) subset of $\mathbb{C}$. So the above corollary gives us uniform convergence to a holomorphic function. It takes a bit more work to show that it's actually equal to $\frac{\sin(\pi z)}{\pi}$, though. Also, note that we paired up the roots z_i and $-z_i$, because Theorem 162 only tells us that $\frac{1}{|z_i|^s}$ is convergent for any $s > 1$, not $s = 1$, so it's currently a bit hard for us to study
$$\sum_{z_i} \left(1 - \frac{z}{z_i}\right).$$
We'll work a bit harder next time to get there.

17 November 12, 2019

The average score on exam 2 was an 82 out of 100. The first two problems are more standard, so we won't talk about them, but the others are worth spending some time on:

Problem 168

Say a function $f(z)$ has a pole at $z_0 \neq 0$ and is holomorphic everywhere else in $|z| < 2$. Show that $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$, where $f = \sum a_n z^n$ is the power series expansion of f.

Solution. We first isolate the pole by writing

$$f(z) = \frac{a_{-m}}{(z-z_0)^m} + \cdots + \frac{a_{-1}}{z-z_0} + g(z),$$

where the order of the pole is $m \geq 1$ and $g(z)$ is holomorphic in $|z| < 2$. But that means g has a large radius of holomorphicity – in particular, by Hadamard's formula, the coefficients of the power series for g satisfy $|b_n| < R^{-n}$ for some $|z_0| < R < 2$, which we will see goes to zero much faster than the terms from the principal parts. So we can ignore the contributions from those coefficients of $g(z)$.

Thus, the result follows if we can show that the coefficients coming from the other terms have the correct ratio. Indeed, for a simple pole we can write

$$\frac{1}{z-z_0} = -\frac{1}{z_0}\sum_{n\geq 0}\frac{z^n}{z_0^n},$$

and indeed this power series' coefficients do have this ratio of z_0. Then for the general case with order m poles, just take the derivative of the expression $m-1$ times – this will give us a polynomial factor, but the limiting ratio is still determined by the z_0 exponential. □

The crucial concept here is that the pole contributes a principal part to our function, but we can isolate it pretty easily.

Problem 169

Suppose $f : \mathbb{D} - \{0\} \to \mathbb{C}$ is holomorphic and has an **isolated** singularity at 0. If $\operatorname{Re} f < 2019$ (that is, the real part of f is bounded from above), then show that we have a **removable** singularity at 0.

Solution. The easiest way to do this is to use a conformal map. We know that a half-plane can be mapped to a unit disk, so we can compose f with a biholomorphic map g such that the composition is bounded, so the composition has a removable singularity. And that means that f is bounded near 0, and we're done.

But another way to do this is to consider the classification of poles: by Casorati-Weierstrass, the image of f must be dense if f has an essential singularity at 0, which doesn't work here because $\operatorname{Re} f < 2019$. So we just need to show it's not a pole: if it were, we could write

$$f(z) = z^{-n}g(z)$$

for some integer n and holomorphic function g with $g(0) \neq a$. So now we can make f a large real number by taking $z = ra^{1/m}$ for a small $r \in \mathbb{R}$. Then taking $r \to 0$ yields a contradiction with $\operatorname{Re} f < 2019$. So the singularity is not a pole or essential singularity, and thus it must be removable. □

The difficulty of the final exam will probably be between the two difficulties of the midterms.

We'll resume the example from last time: we were trying to show that

$$\frac{\sin \pi z}{\pi} \stackrel{?}{=} z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right),$$

where the right hand side converges for any given z because $\sum \frac{z^2}{n^2}$ is finite for any z. (This comes from the fact that $\prod_n (1 + c_n)$ converges if $\sum |c_n|$ converges, since $\log(1+z) = z + O(z^2)$.) We do have to be a bit careful: we need

$|c_n| < 1$ for this condition to hold, but we can just define our product on $|z| < R$ and consider a partial sum

$$\sum_{n \geq R} \left| \frac{z^2}{n^2} \right| < \infty.$$

So we basically define (for any R)

$$\phi_R(z) = \prod_{n<R} \left(1 - \frac{z^2}{n^2}\right) \cdot \prod_{n \geq R} \left(1 - \frac{z^2}{n^2}\right)$$

to be a holomorphic function on $|z| < R$. We know the infinite product converges and is nonzero inside $|z| < R$, so this is a good definition! And thus this product converges to an entire function, because we can pick an arbitrarily large R.

Remark 170. *This function ϕ has zeros exactly at the integers, but we actually have to work a little bit to show that: we just know that for any R, ϕ_R has zeros at the integers for all $|z| < R$. But we can't actually say that it has zeros for integers greater than R, because ϕ_R isn't actually defined there!*

We can also check for ourselves that all zeros are simple (multiplicity one). So the zeroset of this product is the same as $\frac{\sin \pi z}{\pi}$: how do we show that these two functions are actually equal? Let's try taking the log derivative $d \log$ of both sides, because an infinite sum is easier to deal with than an infinite product: the right hand side becomes

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{-\frac{2z}{n^2}}{1 - \frac{z^2}{n^2}} = \boxed{\frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}},$$

and the left hand side becomes

$$g(z) = \boxed{\pi \frac{\cos \pi z}{\sin \pi z}}.$$

Can we now prove that these two boxed expressions are equal? Doing a partial fraction decomposition,

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{1}{z-n} + \frac{1}{z+n} = \text{''} \sum_{n \in \mathbb{Z}} \frac{1}{z-n}.\text{''}$$

This last equality is a little harder to justify, because it isn't absolutely convergent: instead, we have to take the partial sums where $|n| < N$ and take $N \to \infty$. But the point is that it has some important structural properties: our function $\sum_{n \in \mathbb{Z}} \frac{1}{z-n}$ is periodic with period 1 in the real direction and it also has a simple pole at 0 with residue 1 (the only poles are at these integers). And obviously, $g(z) = \pi \frac{\cos \pi z}{\sin \pi z}$ satisfies all of these properties as well (we can verify the residue calculation).

So if we take the function $h(z) = f(z) - g(z)$, our goal is to show that $h = 0$. We know that the singularities cancel out, so h is actually an entire function. Both f and g are also odd, so h is also an odd function: it now suffices to show that h is bounded, which means it must be a constant.

And to show that it's bounded, we know that $h(z) = h(a + bi)$, where $0 < a \leq 1$ (we can translate in the real direction by the periodic condition). Since h is entire, if we take any rectangle with imaginary part from $-R$ to R, h is bounded. That means it's enough to show that as the imaginary part goes to ∞ (and the real part is contained in $(0, 1]$), h is still bounded.

This is just a calculation: we'll actually show that both f and g are bounded as long as the magnitude of the imaginary part is at least R. We have

$$f = \pi \cot \pi z = \pi i \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}},$$

and $|e^{i\pi z}| = e^{-\pi y}$, which goes to 0 as $y \to \infty$. So f approaches some constant as $y \to \infty$, meaning it is indeed bounded. On the other hand,

$$g(z) = \frac{1}{z} + \sum \frac{2z}{z^2 - n^2},$$

and we can use the triangle inequality to say that

$$|g(z)| \le \frac{1}{|z|} + \sum \frac{2|z|}{|z^2 - n^2|}$$

and here

$$|z^2 - n^2| = |x^2 - y^2 + 2xyi - n^2| \ge n^2 + y^2 - 1$$

because $0 < x \le 1$. And thus

$$|g(z)| \le \frac{1}{|z|} + |z| \sum_{n \ge 1}^{\infty} \frac{1}{n^2 + y^2 - 1},$$

which is bounded because (the first term goes to zero and) $|z|$ is roughly y, and then the infinite sum

$$y \sum_{n=1}^{\infty} \frac{1}{n^2 + y^2 - 1} \sim y \int_0^{\infty} \frac{1}{x^2 + y^2} dx.$$

Changing variables, this becomes

$$= \int_0^{\infty} \frac{dx}{x^2 + 1},$$

which is a finite constant! So we've now shown f and g are both bounded, so h is bounded, and therefore it must be zero.

So $f = g$, and it remains to show that the things before the log derivative are also equal:

Proposition 171

If $\frac{\phi'}{\phi} = 0$, then ϕ is constant.

Proof. This means ϕ' is zero. $\square$

So now we finally have the equality we're looking for! We know that

$$\pi \cot(\pi z) = \frac{1}{z} + z \sum_{n=1}^{\infty} \frac{2}{z^2 - n^2},$$

and now let's use this to find the value of $\sum \frac{1}{n^2}$. Plugging in $z = 0$ might seem a little sketchy, but what we're basically saying is

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = \frac{\pi \cot \pi z - \frac{1}{z}}{z} = \frac{(\pi z)\cot(\pi z) - 1}{z^2}$$

has a removable singularity at $z = 0$ (because the left hand side is holomorphic there). And we can use Mathematica or grind out L'Hopital's rule to find that indeed $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. We'll see if we can get a nicer proof next time! The point is that we can also use this to find things like $\sum \frac{1}{n^4}$ and so on: that's for another time.

18 November 14, 2019

Let's start by polishing up the calculation from last class. Recall that we can consider the function

$$f(z) = \frac{\pi z \cot(\pi z) - 1}{z^2} = \sum_{n=1}^{\infty} \frac{1}{z^2 - n^2},$$

and here both sides of the equation are holomorphic at $z = 0$ (because they are equal and the right side is holomorphic). So let's try to write down the power series expansions on both sides. First of all, the right hand side has

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = \sum_{n=1}^{\infty} \frac{-1/n^2}{1 - \frac{z^2}{n^2}} = -\sum_{i=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{n^{2i}} z^{2i}$$

where we expand out the power series term by term, and we know that this converges absolutely given any z so we can swap the order of summation. And then notice that this inner term is exactly the Riemann zeta function:

$$\sum_{n=1}^{\infty} \frac{1}{z^2 - n^2} = -\sum_{i=1}^{\infty} \zeta(2i) z^{2i},$$

where $\zeta(2i) = \sum_{n=1}^{\infty} \frac{1}{n^{2i}}$. Last time, we computed the first term of both sides: the $i = 1$ term gives us $\zeta(2) = \frac{\pi^2}{6}$, but we can actually expand the power series of $\cot(\pi z)$ to find $\zeta(4), \zeta(6)$, and so on! The key idea in getting this done is that we need to find a power series expansion for $\frac{1}{e^{2\pi i z} - 1}$, and if we're careful enough, we can actually prove that

$$\frac{\zeta(2i)}{\pi^{2i}} \in \mathbb{Q}$$

for all positive integers i, because $\frac{1}{e^w - 1}$ actually has all rational terms in its power series and then we can do a change of variable $w = 2\pi i z$.

So now let's go back to our initial question of studying entire functions. Our goal is to classify functions based on their zero set alone: how much information can we find? Say our zeros are $\{z_1, z_2, \cdots\}$, and say that $|a_n| \to \infty$. (This is a necessary condition, because otherwise we have infinitely many zeros in a bounded region, which means we'd have a limit point and the function is identically zero. Also, note that we need to have countably many zeros: uncountably many zeros would also give us a limit point.) Our goal is then to find an entire function f with $f(z_i) = 0$.

With a polynomial, we can just take the product $(z - z_i)$, and in the case last time where the zeros were at the integers only (but we do have an infinite number of zeros), we found that we could take $\frac{\sin \pi z}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$ (we had to combine the positive and negative zeros to ensure convergence). The intuition here is that instead of taking a product $(z - z_i)$, we want to take a product of $1 - \frac{z}{z_i}$ instead, because that at least gives us a chance of having a convergent product! Indeed, let's try defining a function

$$f(z) = \prod_{i=1}^{\infty} \left(1 - \frac{z}{z_i}\right).$$

What are the conditions for this to be convergent? Recall the criterion for product convergence:

Proposition 172

Say $|c_i| < 1$ for all i. If $\sum |c_i|$ is convergent, then $\prod (1 + c_i)$ is convergent.

The main idea is that $\log(1 + z) \le cz$ as long as z is sufficiently small. This criterion was **not** satisfied when we tried to define sin as a product of $1 - \frac{z}{n}$, because the sum of $\frac{z}{n}$ diverges, but it **was** satisfied when we defined it as a product of $1 - \frac{z^2}{n^2}$.

So this already tells us something about our attempted definition of f:

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|} < \infty \implies f \text{ is well-defined with the correct zeroset.}$$

Specifically, we can define f to be the product of $1 - \frac{z}{a_i}$ over all zeros with $|a_i| < R$, and then take $R \to \infty$: this gives us uniform convergence on any disk, so f defined as an infinite product is indeed meromorphic.

Unfortunately, this is a very strong condition, and it doesn't even apply to a simple function like sin. Remember that by definition, f has order of growth at most ρ if $|f(z)| \le Ae^{B|z|^\rho}$: say that ρ is some finite number. We know under this condition that we have a precise result for the number of zeros:

$$n(r) \le Cr^\rho,$$

where this function counts the number of zeros (with multiplicity) inside a disk $|z| < r$. We did a computation a few lectures ago that showed that

$$\sum \frac{1}{|a_i|^s} < \infty \quad \forall s > \rho.$$

So this already gives us a first approximation: we hope that if f is an entire function with growth order $\rho < 1$, then

$$f(z) = e^{g(z)} z^m \prod_{i=1}^{\infty} \left(1 - \frac{z}{a_i}\right)$$

for some holomorphic function $g(z)$. We already know the convergence of the infinite product holds, but we just need to show that f is actually this infinite product.

The trick here is that whenever we have a factor like $1 - \frac{z}{a}$, we can improve the convergence if we replace with $1 - \left(\frac{z}{a}\right)^n$. Unfortunately, this changes the zero set: the hope is that we can instead multiply by an exponential function instead to have similar improved convergence without changing what the zeros of $f(z)$ look like. Basically, can we find a function g such that

$$(1 - z)e^{g(z)} \to 1 - z^{n+1}$$

as $z \to 0$? Well, taking logs of both sides,

$$\log(1-z) + g(z) \sim \log(1 - z^{n+1}) \implies -z - \frac{z^2}{2} - \cdots + g(z) \approx -z^{n+1} \implies \boxed{g(z) = z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots + \frac{z^n}{n} - z^{n+1}}.$$

We can ignore the last term because it's much smaller as $z \to 0$, and this heuristic becomes an important object:

Definition 173

The **Weierstrass canonical factor of degree n** is

$$E_n(z) = (1 - z) \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}\right)$$

The last term z^{n+1} is essentially an error term here:

Lemma 174

If $|z| \le \frac{1}{2}$, then $|E_n(z) - 1| < cz^{n+1}$ for some absolute constant c independent of n.

So the main idea here is that

$$\sum_{i=1}^{\infty} \frac{1}{|a_i|^{n+1}} < \infty \implies z^m \prod_{i=1}^{\infty} E_n\left(\frac{z}{a_i}\right) \text{ entire with desired zeros.}$$

The key is that now we no longer need to worry about the order of f being smaller than 1: in general, we can just pick a large enough n in the Weierstrass canonical factor.

Proof. Note that by the definition of log, $1 - z = e^{\log(1-z)}$, and

$$E_n(z) = e^{\log(1-z)+z+\frac{z^2}{2}+\cdots+\frac{z^2}{n}} = e^w$$

where $w = -\sum_{k=n+1}^{\infty} \frac{z^k}{k}$. But because $|z| \le \frac{1}{2}$,

$$|w| \le |z|^{n+1} \sum_{k=n+1}^{\infty} \frac{|z|^{k-n-1}}{n} \le |z|^{k+1} \sum_{j=0}^{\infty} 2^{-j} \le 2|z|^{k+1},$$

and thus

$$|E_n(z) - 1| = |e^w - 1| \le c|z|^{k+1}$$

for some constant c, as desired. (We can take the absolute constant $2e$.) □

Theorem 175 (Weierstrass)

Let $\{a_1, a_2, \cdots\}$ be a sequence of complex numbers with $|a_n| \to \infty$. Then there exists an entire function f vanishing at all $z = a_n$, with multiplicity, and nowhere else:

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right).$$

Any other such entire function is of the form $f(z)e^{g(z)}$ for an entire function g.

Proof. We can show that this function is holomorphic and has the appropriate zeros inside any disk of radius R (and take $R \to \infty$). First of all, there are only finitely many zeros inside the disk of radius $2R$, and indeed the product $z^m \prod_{n=1}^{N} E_n\left(\frac{z}{a_n}\right)$ is a polynomial with exactly the zeroset that we want.

Now for all other points, $|E_k(z/a_k) - 1| \le c|z/a_k|^{k+1}$ for some absolute constant c, and because $z < R$ and $a_k > 2R$, this is at most $c\frac{1}{2^{k+1}}$. Thus, the sum $\sum_{k=1}^{\infty} |E_k(z/a_k) - 1|$ is convergent, and thus the infinite product is convergent and has the appropriate zero set, as desired.

The last part of this can be proved by the following argument: if f_1, f_2 are two functions that satisfy the theorem statement, then $\frac{f_1}{f_2}$ is entire, holomorphic, and has no zeros. Thus, there exists a function g such that $e^{g(z)} = \frac{f_1}{f_2}$ (since the range of $\frac{f_1}{f_2}$ is simply connected and does not contain 0); this is the $g(z)$ in the problem statement. □

Weierstrass' theorem can actually be strengthened to a stronger result related to the order of growth:

Theorem 176 (Hadamard)

Suppose that f is entire with order $n \le \rho < n+1$ (this integer n is unique). Then

$$f(z) = e^{P(z)} z^m \prod_{i=1}^{\infty} E_n\left(\frac{z}{a_i}\right),$$

where P is a polynomial of degree at most n.

Here, we're basically choosing a uniform degree for our canonical factor!

Example 177

Let's verify that Hadamard's formula recovers our expansion for $\frac{\sin \pi z}{\pi}$.

Since this function has order $\rho = 1$, we take $n = 1$, and this tells us that we can take

$$\frac{\sin \pi z}{\pi} = e^{g(z)} z \prod_{k \in \mathbb{Z} \neq 0} \left(1 - \frac{z}{k}\right) e^{z/k} = e^{A+Bz} z \prod_{k=1}^{\infty} \left(1 - \frac{z^2}{k^2}\right)$$

for some A, B (the $e^{z/k}$ cancels out for positive and negative). And now if we look at the leading terms, $A = 0$ for the linear factors to work out, and $B = 0$ because the left hand side is odd. So we've verified our infinite product in another way!

Theorem 178

If ρ, the order of f, is not an integer, then $f(z) = a$ has infinitely many solutions.

This is in stark contrast to something like $f = e^{p(z)}$, which has no zeros at all.

Proof. It's equivalent to show that f has infinitely many zeros. Suppose that f had finitely many zeros: then by Hadamard's formula,

$$f = e^{g(z)} \cdot h(z)$$

for some polynomials g and h. But then f has growth order equal to the degree of g, which is an integer: contradiction. $\square$

19 November 19, 2019

Last time, we stated Hadamard's formula, which is a strengthening of Weierstrass' theorem. Basically, Weierstrass tells us that if we have a set of zeros with $|a_n| \to \infty$ (where repeats are allowed, corresponding to multiplicity), then there exists an entire function

$$z^m \prod_{n=1}^{\infty} E_n \left(\frac{z}{a_n}\right)$$

where E_n is the canonical Weierstrass factor

$$E_n(z) = (1 - z) e^{z + \frac{z^2}{2} + \cdots + \frac{z^n}{n}},$$

which has much better convergence properties (because it behaves like $1 + O(z^{n+1})$ as $z \to 0$). So larger n make this very close to 1, which makes the product more likely to converge to a finite value.

The point is that the degree of the canonical factor gets larger and larger to ensure that we have convergence:

$$\sum_{n=1}^{\infty} \frac{1}{|a_n|^n} < \infty$$

for any sequence with $|a_n| \to \infty$. But it turns out that we don't need to change the factor n: Hadamard says that if f is entire with order $k \le \rho < k + 1$, then we can write

$$f(z) = e^{g(z)} z^m \prod_{i=1}^{\infty} E_k \left(\frac{z}{a_i}\right),$$

where g is a polynomial of degree at most k. (Note, though, that this doesn't go the other way around: even if we have finitely many zeros, our function can still have arbitrarily high order because of the factor $e^{g(z)}$.)

Intuitively, this works because $\sum_n \frac{1}{|a_n|^{k+1}}$ is convergent if $\rho < k+1$: this makes the Weierstrass canonical factor of degree k strong enough for our purposes. So one thing useful to consider is the set

$$\left\{ s \in \mathbb{R}_{\geq 0} | \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty \right\},$$

which is either empty, $[\rho, \infty)$, or (ρ, ∞) (because a larger exponent only increases the convergence properties). We'll formalize this:

Definition 179

Given a sequence of nonzero complex numbers $\{a_n \neq 0\}$ with $|a_n| \to \infty$, define

$$\lambda = \inf\{s \in \mathbb{R}_{\geq 0} | \sum_{n=1}^{\infty} \frac{1}{|a_n|^s} < \infty\}.$$

Example 180

$a_n = n$ has $\lambda = 1$ (because $\sum \frac{1}{n^s}$ converges for all $s \in (1, \infty)$). Similarly, $a_n = \sqrt{n}$ has $\lambda = 2$.

Example 181

$a_n = 2^n$ has $\lambda = 0$, because $\sum \frac{1}{2^{sn}}$ converges as long as s is positive.

So sequences that have smaller λ have zeros that go to ∞ faster (and also correspond generally to functions with lower order, except for those $e^{g(z)}$ terms).

Let's loosen the condition $k \leq \rho < k+1$ to just $\rho < k+1$: we lose uniqueness, but often we can only get upper bounds on the growth order of a function. So then if $\rho = \frac{1}{2}$ for example, we can pick any nonnegative integer k, and then

$$\prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right) = \prod \left(1 - \frac{z}{a_n}\right) e^{\frac{z}{a_n} + \cdots + \frac{z^k}{k a_n^k}}.$$

If we remove the highest order term from the exponent, we get a product of two infinite products:

$$= \prod_{n=1}^{\infty} e^{\frac{z^k}{k a_n^k}} \cdot \prod_{i=1}^{\infty} E_{k-1}\left(\frac{z}{a_n}\right)$$

But the first term here is convergent as long as $\rho < k$ (which is slightly stronger than $\rho < k+1$). So there's a bit of consistency between different possibilities for our factor k: the reason that we need to constrain $k \leq \rho < k+1$ is so that if we take out that highest order term, we won't have something that's convergent anymore.

Last time, we showed that Hadamard's formula recovers the correct form for $\frac{\sin \pi z}{\pi}$. Basically, if we know the order and the zeros for a function, that's enough to write it down. But how do we go in reverse: what's the growth order of the function

$$f(z) = z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)?$$

Well, we know that this is convergent if $\lambda < k+1$, because this means by definition that $\sum \frac{1}{|a_n|^{k+1}}$ is convergent. There's a small window here:

Theorem 182

If $\prod E_k(\frac{z}{a_n})$ has growth order ρ, then

$$\lambda \le \rho \le k+1.$$

Note that we usually only care about bounds, and since λ and $k+1$ are separated by at most 1. $\lambda \le \rho$ is easy, because a function of order ρ must have $\sum \frac{1}{|a_n|}^{\rho+\varepsilon}$ convergent, and λ is the infimum of all such $\rho + \varepsilon$. Proving that $\rho \le k+1$ is showing that the product has order at most $k+1$, and that's more substantial.

Example 183

Remembering that $a_n = 2^n$ has $\lambda = 0$, we can choose $k = 0$ to yield the convergent product

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{2^n}\right).$$

The above theorem tells us that $0 \le \rho \le 1$: what's the actual order?

This is a special example: note that

$$f(z^m) = \prod_{n=1}^{\infty}\left(1 - \left(\frac{z}{2^{n/m}}\right)^m\right)$$

gives us m roots for every root in our original polynomial, and this can be rewritten as

$$= \prod_{n=1}^{\infty}\prod_{i=1}^{m}\left(1 - \zeta_m^i \frac{z}{2^{n/m}}\right).$$

The new function $f(z^m)$ has order between 0 and 1 as well, because it's also written with Weierstrass canonical factors of degree 0. But this is true for any $m \ge 1$, which means that

$$|f(z^m)| \le Ae^{B|z|} \implies |f(z)| \le Ae^{B|z|^{1/m}}.$$

Taking $m \to \infty$ implies the order of f itself must indeed be 0. (Notice that we didn't really use any properties here except that $\lambda = 0$.)

The central point here is that we can now construct functions with essentially arbitrary zero sets: if we want to find a function with zeroset $\mathbb{Z}_{\ge 0}$, then

$$\prod_{n=1}^{\infty}\left(1 - \frac{z}{n}\right)e^{z/n}$$

is a function which has order between 1 and 2 (because $\lambda = 1 < k+1$ and $k = 1$ here).

Definition 184

The **Gamma function**

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^{s-1}dt = \lim_{\varepsilon \to 0}\int_{\varepsilon}^{1/\varepsilon}\phi(t)dt$$

is defined (a priori only) on the positive real numbers $s \in \mathbb{R}_{>0}$.

We should be a bit careful about the singularities here: as long as $s > 0$,

$$\int_0^{\varepsilon} t^{s-1}dt < \infty,$$

and also e^{-t} dominates t^{s-1} as $t \to \infty$ for all real numbers s. So the main trouble is the value of $e^{-t}t^{s-1}$ near 0.

Let's first think about how to relate this function to the infinite product that we've been talking about. First of all, we'll need some preliminary results:

Theorem 185

The gamma function

$$\int_0^\infty e^{-t}t^{s-1}dt$$

converges and is holomorphic for any s with $\operatorname{Re} s > 0$. In addition, it satisfies the functional equation

$$\Gamma(s+1) = s\Gamma(s), \Gamma(1) = 1.$$

(We'll prove this next time.) This actually implies that there is a **meromorphic continuation to** $\mathbb{C}$! We have the function defined for all $\operatorname{Re} s > 0$, so then we can define

$$\Gamma(s) = \frac{\Gamma(s+1)}{s}$$

for all $-1 < \operatorname{Re} s \le 0$, and then iteratively do this for the strip $(-2, 1]$, then $(-3, 2]$ and so on. And the functional equation tells us that this definition is indeed consistent, and it tells us Γ except at the points $0, -1, -2, \cdots$. But we just have a simple pole near each of these points: specifically, if we define the Γ function up to real part $-n$, then

$$\Gamma(s) = \frac{\Gamma(s+n+1)}{s(s+1)(s+2)\cdots(s+n)},$$

and the poles occur at $s = 0, -1, -2, \cdots, -n$ (now just take $n \to \infty$). In particular,

$$\operatorname{Res}_0\Gamma(s) = \Gamma(1) = 1,$$

and then the residue at -1 is just $\frac{1}{-1} = -1$, at -2 is $\frac{-1}{-2} = \frac{1}{2}$, and so on: basically, Γ is meromorphic with a pole at $-n$ of $(-1)^n\frac{1}{n!}$ for all $n \ge 0$.

Let's finish today by justifying that this definition of the gamma function is well-defined for all complex numbers with $\operatorname{Re} s > 0$. If we define

$$F_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} e^{-t}t^{s-1}dt,$$

this is an entire function because we're integrating over a bounded domain for any fixed ε. Our goal is to show that F_ε converges to a holomorphic function $F = \Gamma(s)$. To do that, we just need to show uniform convergence for s in a **compact** domain, and we can do this by just looking at some region $\delta < \sigma < M$ with bounded real part (because showing absolute convergence can just be done by taking σ instead of s). And that's just showing that the error terms $\int_0^\varepsilon e^{-t}t^{\sigma-1}dt$ go to 0 uniformly with respect to s in this region, and similarly show this for $\int_{1/\varepsilon}^\infty e^{-t}t^{\sigma-1}dt$. And that's just basically bounding based on the boundary values δ and M.

20 November 21, 2019

We're going to continue discussing the Gamma function today, which is the first nontrivial example of meromorphic continuation that we've discussed. Recall that we start by defining

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

for all $\text{Re}(s) > 0$, but we want to also define it for $\text{Re}\, s < 0$. To do this, notice that $\Gamma(s)$ is holomorphic on its current domain, so if there exists a meromorphic continuation, it must be unique: if f and g are meromorphic on Ω', and $f = g$ on some nonempty region $\Omega \subseteq \Omega'$, then $f = g$ on Ω'. (We technically only proved this for holomorphic functions, but we can just remove the poles from our region and it's still open.)

But as we saw on the homework, it's possible to have a function that cannot be continued beyond its domain. So we need to construct an example explicitly: we use the fact that

$$\Gamma(s+1) = s\Gamma(s) \quad \forall\, \text{Re}(s) > 0.$$

Then we can define Γ iteratively for $\text{Re}\, s \in (-1, 0], (-2, -1]$, and so on, to find that Γ has poles at $0, 1, -2, \cdots$ and is holomorphic everywhere else! Explicitly, if $\text{Re}(s) \in (-1, \infty)$, we define

$$\tilde{\Gamma}(s) = \frac{1}{s}\Gamma(s+1),$$

and we check that we do have $\Gamma(s) = \tilde{\Gamma}(s)$ for all $\text{Re}(s) > 0$ (so our new function is consistent with our current definition). Then $\tilde{\Gamma}$ still satisfies our functional equation everywhere, so we can define this to be Γ, and define a new $\tilde{\Gamma}$ on $\text{Re}(s) \in (-2, \infty)$, and so on.

Proof of the functional equation. Note that

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1-1} dt = -\int_0^\infty t^s d(e^{-t})$$

and by integration by parts, this is

$$= -t^s e^{-t}|_0^\infty + \int e^{-t} d(t^s).$$

The first term is 0 at both endpoints, so the boundary terms go away, and we're left with

$$\int e^{-t} d(t^s) = \int s t^{s-1} e^{-t} dt = s\Gamma(s),$$

as desired. And we should also verify that $\Gamma(1) = 1 = \int_0^\infty e^{-t} dt$, which is true. □

Corollary 186

$\Gamma(n) = (n-1)!$ for all positive integers n. In other words, the gamma function is an interpolation of the factorial function!

We'll also go through an alternate proof of the meromorphic continuation of $\Gamma(s)$ here, which will help us with our study of the zeta function:

Proof. Note that we can write

$$\Gamma(s) = \int_0^1 e^{-t} t^{s-1} dt + \int_1^\infty e^{-t} t^{s-1} dt.$$

The second part here is already convergent (and holomorphic) for all s, because the singularity at ∞ is well-behaved (the function just decays to 0). Meanwhile, if we write out the power series $e^{-t} = \sum_{n=0}^\infty \frac{(-1)^n t^n}{n}$ and integrate the first part term by term (we're still working with $\text{Re}(s) > 0$),

$$\int_0^1 e^{-t} t^{s-1} dt = \sum_n \frac{(-1)^n}{n!} \int t^{n+s-1} dt = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+s)}.$$

And now notice that for any $s \in \mathbb{C}$, at most one of these terms can be bad. Each term is a meromorphic function of s, and each one has a different pole, so we can write

$$\int_0^1 e^{-t}t^{s-1}dt = \frac{1}{s} + \sum_{n\geq 1}^{\infty}(-1)^n \frac{1}{n!(n+s)}.$$

Notice that this sum now has no poles for $\text{Re}(s) > -(1-\varepsilon)$ except for at $s = 0$, and we have uniform convergence of the infinite sum, so it is holomorphic! So isolating the first term from our infinite sum gives us a meromorphic function on $\text{Re}(s) > -(1-\varepsilon)$.

And in general, we can separate the first N terms from this infinite sum to show that we indeed have a meromorphic function on the half-plane $\text{Re}\, s \in (-N+\varepsilon, \infty)$ with poles at $0, -1, \cdots, -N+1$. $\square$

This strategy works slowly, but it doesn't require a functional equation! So this is a good thing to do when we care about showing existence but not necessarily the exact functional form.

We mentioned earlier on in the class that $\sin \pi z$ is an entire function with order at most 1, and zeros at all of the integers. Let's relate this to Γ, which we know has poles at all nonpositive integers:

Theorem 187

For all $s \in \mathbb{C}$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}.$$

Intuitively, $\Gamma(s)$ catches all zeros that are 0 or smaller, and $\Gamma(1-s)$ catches all zeros that are 1 or larger! So we get a disjoint union which yields all of the integers.

Proof sketch. First of all, if we replace $s \to s+2$, both sides stay constant (by the functional equation for Γ). Thus, it suffices to show this equality for $\text{Re}(s) \in [0, 2]$. But we can do this by showing that the ratio of the two sides (which is entire, because both sides have the same set of zeros) is bounded as the imaginary part goes to ∞, so by Liouville the ratio must be constant! And doing this is just some routine bounding. $\square$

But instead of diving into the details, we'll instead move on, because we have a much better understanding of the function Γ:

Theorem 188

$\frac{1}{\Gamma(s)}$ is an entire function of order $\rho = 1$. In fact,

$$\left|\frac{1}{\Gamma(s)}\right| \leq Ae^{B|s|\log|s|}$$

for some constants A, B.

(Here, notice that $\log|s|$ grows slower than any $|s|^\varepsilon$ for $\varepsilon > 0$.) This proof is actually pretty technical, so it's not so great for us to discuss during class, even though the conclusion is very important.

Well, given this knowledge, we can apply Hadamard's theorem:

Theorem 189

We can write

$$\frac{1}{\Gamma(s)} = e^{\gamma s} s \prod \left(1 + \frac{s}{n}\right) e^{-s/n},$$

where γ is the Euler constant

$$\gamma = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} - \log n.$$

(This Euler constant $\gamma \approx 0.577$ exists because $\log n$ is the integral $\int_1^n \frac{1}{x} dx$.) Note that we have

Proof. Everything except the $e^{\gamma s}$ term comes from Hadamard's factorization theorem: we know that

$$\frac{1}{\Gamma(s)} = e^{A+Bs} s \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n}.$$

Remark 190 (Asymptotic notation)**.** *If we have two functions f, g, then as $z \to \infty$, then $f = O(g) \implies |f| \le C|g|$ and $f = o(g) \implies \lim \frac{f}{g} \to 0$. $f \sim g$ means that $\lim \frac{f}{g} = 1$.*

Because Γ has a pole of residue 1 at $s = 0$, we know that the left hand side should be $s + O(s^2)$ as $s \to 0$, while the right hand side is $e^A s + O(s^2)$ (leading term comes from taking only the constant factors of everything, times the lone s). This already tells us that $e^A = 1$, or $A = 0$.

Thus, it remains to find the value of B: as $s \to 1$, we find that

$$1 = e^B \cdot \prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}.$$

Turning the product into an infinite sum,

$$1 = e^B e^{\sum_n (\log(1+1/n) - 1/n)}.$$

But now

$$\sum_{n=1}^{N} \log\left(1 + \frac{1}{n}\right) - \frac{1}{n} = \sum_{n=1}^{N} \log \frac{n+1}{n} - \sum_{n=1}^{N} \frac{1}{n} = \log(N+1) - \sum_{n=1}^{N} \frac{1}{n}.$$

As we take $N \to \infty$, this expression indeed converges to the negative Euler constant! So $e^B = e^\gamma$, which means $B = \gamma + 2\pi i k$ for some integer k. And now $k = 0$, because plugging in any real number $s > 0$ must make the right hand side real: this is only possible if the exponent for e^{Bs} is always real-valued. $\square$

So we know that $\frac{1}{\Gamma(s)}$ is entire of order $\rho = 1$, and it has zeros $\{0, -1, -2, \cdots\}$. We also know that $\Gamma(1) = 1$, $\frac{1}{\Gamma}(\mathbb{R}) \in \mathbb{R}$. **This is enough to determine the entire function**, without even using the integral definition! So we should generally remember the characteristic properties instead of the numerical definition.

Problem 191

We know the double-angle identity

$$\sin(2\theta) = 2 \sin\theta \sin\left(\frac{\pi}{2} - \theta\right).$$

Can we do something similar for the Γ function?

Perhaps we can consider

$$\frac{1}{\Gamma(s)} \frac{1}{\Gamma\left(s+\frac{1}{2}\right)}.$$

This has zeros at all of the negative half-integers, so it's natural to imagine that the right hand side is related to $\frac{1}{\Gamma(2s)}$. Plugging in $s = \frac{1}{2}$ shows that the equality should be

$$\frac{1}{\Gamma(s)} \frac{1}{\Gamma\left(s+\frac{1}{2}\right)} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{1}{\Gamma(2s)}.$$

But we can recover the exact value of $\Gamma(\frac{1}{2})$:

$$\frac{1}{\Gamma(s)\Gamma(1-s)} = \frac{\sin \pi s}{\pi} \implies \Gamma\left(\frac{1}{2}\right)^2 = \pi \implies \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

And we can confirm that this is true by plugging in both sides into the product factorization of the Γ function in Theorem 189.

We'll see some of these strategies come up again for the ζ function!

21 November 26, 2019

We're going to continue looking at sophisticated ways to do a meromorphic continuation today.

Definition 192

The **Riemann zeta function**

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$$

is defined and absolutely convergent for all real numbers $s \in \mathbb{R}_{>1}$.

Just like in the Gamma function, we first extend from real numbers to complex numbers with an appropriate real part:

Theorem 193

$\zeta(s)$ defines a holomorphic function on the region $\text{Re}(s) > 1$.

Proof. Consider the region $\text{Re}(s) > \delta$, where δ is any real number larger than 1. Each individual term of the zeta function is holomorphic on this region, so we just need to argue that there is uniform convergence. Indeed,

$$\sum_{n \geq 1} \left| \frac{1}{n^s} \right| = \sum_{n \geq 1} \left| \frac{1}{n^\sigma} \right| < \sum_{n \geq 1} \frac{1}{n^\delta} < \infty,$$

where σ denotes the real part of s, so it is indeed uniformly convergent! (Here, we're using the fact that $\sum_{n \geq 1} F_n(z)$ is uniformly convergent on a region Ω if we can write $|F_n(z)| \leq a_n$ for all $z \in \Omega$, where the a_n don't depend on z, and we have $\sum_n a_n < \infty$. So if we can bound our function termwise, that is good enough.) And now move $\delta \to 1$, and we've indeed shown that we have a holomorphic function on the half-plane $\text{Re}(s) > 1$. $\square$

If we want to extend the zeta function beyond this half-plane (a priori this may not even exist at all), we need to prove some functional properties. In the Gamma function, we just used the fact that $\Gamma(s+1) = s\Gamma(s)$: can we do something similar over here?

Definition 194

The **theta function**

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau} = \sum_{n\in\mathbb{Z}} q^{n^2},$$

(where $q = e^{\pi i \tau}$) is defined for all τ with $\text{Im}(\tau) > 0$.

Basically, we need $|q| < 1$ for this last power series to be well-defined, which happens as long as the imaginary part of τ is positive.

Example 195

The theta function is interesting on its own: for example,

$$\theta(q)^4 = \sum_{N\geq 0} r_4(N) q^N,$$

where $r_4(N)$ is the number of ways to write N as the sum of four squares $N = a^2 + b^2 + c^2 + d^2$, $a, b, c, d \in \mathbb{Z}$. So there's a very strong number theoretic flavor here: for example, Lagrange's four squares theorem tells us that the coefficients of θ^4 are all nonzero!

Let's establish a few properties of the theta function: let's first study this only on the **imaginary axis**. Define

$$u(t) = \theta(it), \quad t \in \mathbb{R}_{>0} :$$

plugging this into the definition gives us a convergent sum

$$u(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 t}.$$

These positive exponents decay very fast, so it's easy to see that $|u(t) - 1| \leq Ce^{-\pi t}$ for some absolute constant C as $t \to \infty$. The trouble point, though, is near $t = 0$: then each individual term approaches 1, so our sum does diverge near the origin. In other words, we don't have a great estimate of the θ function when $t \to 0$. This is fixed by the following functional equation:

Theorem 196

We have the "inversion property"

$$u(t) = \sqrt{\frac{1}{t}}\, u\left(\frac{1}{t}\right).$$

If we assume that this is true, we get an asymptotic bound

$$|u(t)| \leq C'\sqrt{\frac{1}{t}}$$

as $t \to 0$, because u goes to 1 as $t \to \infty$. (This is a hard result to get without the inversion formula.)

Proof. We'll use something more general, known as the **Poisson summation formula**: given a **nice** function $f : \mathbb{R} \to \mathbb{C}$,

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n),$$

where $\hat{f}(x) = \int_{-\infty}^{\infty} f(y)e^{2\pi ixy}dy$ is the Fourier transform of f. (Being **nice** means that $\hat{f}$ exists, and that both sums are convergent.)

One particularly nice class of functions (sufficient for the Poisson summation) is the **Schwartz functions**, which are those f which are

- smooth and
- for every N, $|f(x)x^N|$ is bounded over all $\mathbb{R}$.

(This class has the nice property that the Fourier transform is an operator that takes Schwartz functions to Schwartz functions!) For example, we can consider the function

$$f(x) = e^{-\pi x^2} \implies \hat{f}(x) = f(x)$$

(we showed this with the Cauchy integral formula). Obviously, we don't want to directly plug this into the Poisson summation formula because it'll be "true by plagiarism," but it's more interesting to think about a small tweak: for any $t > 0$, we can take

$$f_t(x) = e^{-\pi tx^2} = e^{-\pi(\sqrt{t}x)^2} \implies \hat{f}_t(x) = \frac{1}{\sqrt{t}}e^{-\pi x^2/t}.$$

So now applying the Poisson summation formula to f_t,

$$\sum_{n\in\mathbb{Z}} e^{-\pi tn^2} = \frac{1}{\sqrt{t}}\sum_{n\in\mathbb{Z}} e^{-\pi n^2/t}$$

and we're done. $\square$

Remark 197. *This formula almost works for complex numbers too – we have to be a bit careful with the random constants and square roots, but this has some kind of connection to* **modular forms**.

So now let's try to connect our zeta function $\zeta(t)$ to the function $u(t)$ by considering

$$\int_0^\infty (u(t) - 1)t^{s-1}dt.$$

for all $\mathrm{Re}(s) > \frac{1}{2}$ – this is convergent because of the behavior of u as $t \to 0$ and $t \to \infty$. We can rewrite this as

$$= \int_0^\infty 2\sum_{n=1}^\infty e^{-\pi n^2 t}t^{s-1}dt,$$

and by absolute convergence, we can interchange the order of summation:

$$= 2\sum_{n\geq 1}^\infty \int e^{-\pi n^2 t}t^{s-1}dt.$$

Making the change of variables $t \to t/(\pi n^2)$, the integral now becomes familiar:

$$2\cdot\pi^{-s}\left(\sum_{n\geq 1}^\infty \frac{1}{n^{2s}}\right)\Gamma(s).$$

This gives us the following result if we rename $2s$ as s:

Lemma 198

We have for all $\text{Re}(s) > 1$ that

$$\pi^{-s/2}\zeta(s)\Gamma\left(\frac{s}{2}\right) = \frac{1}{2}\int_0^\infty (u(t)-1)t^{s/2-1}dt.$$

The intuition from here is that u satisfies a functional equation, so if we change t to $\frac{1}{t}$, that will give us something useful:

$$\int_0^\infty (u(t)-1)t^{s/2-1} = \int_0^\infty \left(u\left(\frac{1}{t}\right)-1\right)\frac{1}{\sqrt{t}}t^{s/2}\frac{dt}{t}.$$

Changing variables so $t \to \frac{1}{t}$,

$$= \int_0^\infty (u(t)-1)\, t^{(1-s)/2}\frac{dt}{t}.$$

(Note that the calculus trick here is that under this transformation, $\frac{dt}{t}$ remains the same but just picks up a negative sign.) So the idea is that there's some relation between s and $1-s$:

$$s : \int_0^\infty (u(t)-1)t^{s/2}\frac{dt}{t} \implies 1-s : \int_0^\infty (u(t)-1)t^{(1-s)/2}\frac{dt}{t}.$$

We haven't fully justified this, so let's just work out the analysis details:

Theorem 199

If we define

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s),$$

then ξ satisfies the functional equation $\xi(s) = \xi(1-s)$, and ξ is meromorphic on all of $\mathbb{C}$.

Proof. The proof ideas are the same as before: we take the integral from 0 to ∞ and break it up into a part from 0 to 1 and a part from 1 to ∞. Then ξ looks like

$$\xi(s) = \frac{1}{2}\int_0^\infty (u(t)-1)t^{s/2}\frac{dt}{t},$$

and our second property tells us that the latter integral looks like

$$\int_1^\infty (u(t)-1)t^{s/2}\frac{dt}{t},$$

which is an entire function of s (because there's no singularities anywhere). So it's only important to look at the first piece: we just want to do a change of variables $t \to \frac{1}{t}$, which changes the bounds $[0,1]$ to $[1,\infty]$! This gives us

$$\int_1^\infty \left(u\left(\frac{1}{t}\right)-1\right)t^{-s/2}\frac{dt}{t},$$

and now using the functional equation makes this

$$= \int_1^\infty \left(u(t)\sqrt{t}-1\right)t^{-s/2}\frac{dt}{t}.$$

This isn't exactly what we want: we want to show decay of this integral, but we only have decay of $|u(t)-1|$. So we can separate this out:

$$= \int_1^\infty (u(t)-1)\sqrt{t}\cdot t^{-s/2}\frac{dt}{t} + \int_1^\infty \left(-\frac{1}{\sqrt{t}}+1\right)t^{(1-s)/2}\frac{dt}{t}.$$

This extra term is easy to compute directly, so plugging everything back in, we find that

$$\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \xi(s) = \frac{1}{2}\int_1^\infty (u(t)-1)t^{s/2}\frac{dt}{t} + \frac{1}{2}\int_1^\infty (u(t)-1)t^{(1-s)/2}\frac{dt}{t} + \frac{1}{s} + \frac{1}{1-s}.$$

Each term of the integral is an entire function because of the decay of $u(t)-1$, and we're adding that to two meromorphic functions. And indeed $\xi(s) = \xi(1-s)$ by direct substitution! $\square$

So we know that $\xi(s)$ has a simple pole at $s = 0, 1$ and it's holomorphic everywhere else. So now we can write down

$$\zeta(s) = \pi^{s/2}\frac{1}{\Gamma(s/2)}\xi(s).$$

Because $\frac{1}{\Gamma}$ is holomorphic, ξ only has poles at 0 and 1, but the pole at 0 cancels out with the zero of $\frac{1}{\Gamma}$ at 0. Thus, we have the following conclusion:

Corollary 200

The zeta function is meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ with residue 1. It also has zeros at $-2, -4, \cdots$, corresponding to the poles of Γ at the negative integers. (These are known as the **trivial zeros** of the zeta function.)

The Riemann hypothesis says that these are all of the zeros except when $\text{Re}(s) = \frac{1}{2}$: the place where we're not so sure is everywhere else in the critical strip where $0 < \text{Re}(s) < 1$.

We'll try to make use of this in the next few classes to prove the prime number theorem.

22 December 3, 2019

Today and Thursday will be devoted to a proof of the Prime Number Theorem:

Definition 201

A **prime number** is a positive integer whose only divisors are 1 and itself. The **prime counting function**

$$\pi(x) = \#\{p \le x : p \text{ prime}\}.$$

It's reasonable to view this as a function which is defined for all real numbers $x \in \mathbb{R}_{\ge 0}$: it is then piecewise smooth with jumps at some selected integers. We can alternatively write this function simply as

$$\pi(x) = \sum_{p \le x} 1$$

(for the next few lectures, summing over p means summing over primes). We'll be working towards the following result:

Theorem 202 (Prime Number Theorem)

Asymptotically, we have

$$\pi(x) \sim \frac{x}{\log x},$$

meaning that $\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1$.

This essentially tells us that the density of prime numbers is roughly proportional to $\frac{1}{\log x}$: this is a more quantitative description of the sparsity of primes!

We'll divide up this proof into a few steps: first, we'll show that $\zeta(s)$ does not vanish on $\text{Re}(s) = 1$, and then we'll reduce the prime number theorem to a question about $\zeta(s)$.

Theorem 203 (Zero-free region of $\zeta(s)$)

The function $\zeta(s) = \sum_{n\geq 1} \frac{1}{n^s}$ is nonzero for all $s = 1 + it$, $t \in \mathbb{R}$.

(This is technically a baby version of the Riemann hypothesis, which says that $\zeta(\sigma + it) \neq 0$ for all $\sigma > \frac{1}{2}$.) This result is powerful and interesting on its own, though!

Proof. First, we'll talk about the **Euler product**. Every positive integer can be uniquely written as a product

$$n = \prod_{p_i} p_i^{a_i},$$

which implies that we can rewrite ζ as

$$\zeta(s) = \sum \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \prod_p \frac{1}{1 - p^{-s}}$$

for all $\text{Re}(s) > 1$ (we pick one of the factors p^{as} for each prime p). Now if we take $s \to 1$, we know that ζ has a meromorphic continuation to all of $\mathbb{C}$ and has a simple pole at $s = 1$, so ζ should be divergent as $s \to 1$. Thus,

$$\lim_{s\to 1} \prod_p \frac{1}{1 - p^{-s}}$$

should be divergent, so we must have infinitely many prime numbers, and we at least have $\pi(x) \to \infty$. So this gives us some hope that we can study the value of ζ a bit more carefully: maybe we can look at other points of ζ to get more information!

So now let's return to the thing we're trying to prove. Suppose $1 + it$ is a zero for ζ: consider the points $1, 1 + it, 1 + 2it$. We can't plug these into the Euler product, but because our function is meromorphic, we can approach those points: given a real number $\sigma > 1$, consider the points $\sigma, \sigma + it$, and $\sigma + 2it$, and look at the expression

$$\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it).$$

As $\sigma \to 1$, we get a pole of order 3 from $\zeta(1)$, but we get a zero of order 4 (or $4m$ for some positive integer m) from $\zeta(1 + it)$. Thus, this function approaches 0 as $\sigma \to 1$. (If we want to be more explicit, we can write out the poewr series here and see that the leading term is indeed higher than constant order.

Now we can consider

$$\log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| :$$

this should go to $-\infty$ as $\sigma \to 1$, because the expression inside the log goes to 0. But that's actually not true:

Lemma 204

For all $\sigma > 1$ and $t \in \mathbb{R}$,

$$\log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| \geq 0.$$

Proof of lemma. To show this, we need to estimate the log. Note that ζ does not vanish for any $\text{Re}(s) > 1$ (from the Euler product), because $\sum_p |p^{-s}|$ is absolutely convergent for any $\text{Re}(s) > 1$. We can therefore use the power series expansion and the Euler product here:

$$\log \zeta(s) = -\sum_p \log(1 - p^{-s}) = \sum_p \left(\sum_{n \ge 1} \frac{1}{np^{ns}} \right),$$

and now taking the absolute value of the ζ means we just care about the real part of each term: if $s = \sigma + it$, then

$$= \sum_p \sum_n \frac{1}{n} \operatorname{Re}\left(e^{-n \log ps}\right) = \sum_p \sum_n \frac{1}{np^{n\sigma}} \cos(n \log(p)t).$$

So now to deal with the expression in the lemma,

$$\log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)| = 3 \log |\zeta(\sigma)| + 4 \log |\zeta(\sigma + it)| + \log |\zeta(\sigma + 2it)|,$$

and now all three have the same σ term: plugging everything in and defining $\theta = (n \log p)t$,

$$= \sum_p \sum_n \frac{1}{npn^{\sigma}} (3 + 4\cos\theta + \cos(2\theta)).$$

Now $3 + 4\cos\theta + \cos(2\theta) = 2\cos^2\theta + 4\cos\theta + 2 = 2(\cos\theta + 1)^2 \ge 0$ for all θ, so each term is nonnegative and thus the whole sum must be nonnegative as well. $\square$

Thus, if $\log |\zeta(\sigma)^3 \zeta(\sigma + it)^4 \zeta(\sigma + 2it)|$ is nonnegative for all $\sigma > 1$, it can't go to $-\infty$ as $\sigma \to 1$, and we've found a contradiction. So no zero on the line $\operatorname{Re}(s) + 1$ can exist. $\square$

With this, we've proved there's no zeros on a closed subset of $\mathbb{C}$ (the line $\operatorname{Re}(s) = 1$), so there is an open neighborhood around any $s = 1 + it$ with no zeros. It turns out that we can show with a bit more work (and the Hadamard product) that

$$\zeta(\sigma + it) \ne 0 \quad \forall \sigma > 1 - \frac{c}{\log(2 + |t|)},$$

but this is about as much as we know about the zero-free region: it's not yet known that there are no zeros for any $\sigma = 1 - \varepsilon$, $\varepsilon > 0$.

Remark 205. *We can prove that there are many zeros on the line* $\operatorname{Re}(s) = \frac{1}{2}$ *(for example, using the argument principle or more sophisticated methods). We currently know that at least 40 percent of all zeros are on this line!*

So with this, we'll move on to a more routine part of the proof: how do we connect this zero-free region to the prime number theorem? It turns out it's easier to approximate something related:

Definition 206

The **Chebyshev function** is defined via

$$\vartheta(x) = \sum_{p \le x} \log p.$$

Now not all prime numbers are created equal: since $\log p$ is proportional to the number of digits of p, this would be (for example) the amount of time it takes for a computer to write down this prime. We may want to guess that this is proportional to x (to cancel out with the $\frac{1}{\log x}$ factor), and this actually turns out to be correct.

Also, maybe we should consider powers of prime numbers as well:

Definition 207

Let

$$\psi(x) = \sum_{n \le x} \Lambda(n),$$

where $\Lambda(n) = \log p$ if n is a power of p, and 0 otherwise.

Since there are $\lfloor \log_p(n) \rfloor$ powers of p less than n, we can rewrite

$$\psi(x) = \sum_{p} \lfloor \log_p(n) \rfloor \log_p,$$

and crudely this is the size

$$\approx \sum_{p<x} \log x \approx \log x \pi(x).$$

For now, the distinction between these three different counting functions may not be clear: they are all useful, and we'll try to prove something about them now.

Lemma 208

The prime number theorem ($\pi(x) \sim \frac{x}{\log x}$) is equivalent to showing that $\vartheta(x) \sim x$.

Proof. We have

$$\vartheta(x) = \sum_{p \le x} \log p \le \sum_{p \le x} \log x = \log x \pi(x),$$

and now we can prove the other direction by showing that $\theta(x) \ge (1-\varepsilon)\log(x)\pi(x)$ for any $\varepsilon > 0$. We can split up the sum for θ:

$$\vartheta(x) = \sum_{p<\sqrt{x}} \log p + \sum_{\sqrt{x} \le p < x} \log p.$$

In general, we can say for any δ that

$$\vartheta(x) \ge \sum_{x^\delta \le p \le x} \log p \ge \log x^\delta \sum_{x^\delta \le p \le x} 1 = \delta \log x(\pi(x) - \pi(x^\delta)).$$

Now if $\pi(x) \sim \frac{x}{\log x}$, we know that (taking $\delta = 1 - \varepsilon$)

$$\vartheta(x) \gtrsim \delta \log x \left(\frac{x}{\log x} - \frac{x^\delta}{\log x^\delta} \right)$$

and the second term becomes much smaller as $x \to \infty$, which shows that $\vartheta(x) > \delta x$ as $x \to \infty$ for all $\delta = 1 - \varepsilon$, and also we know in this case that $\vartheta(x) \le x$, so indeed $\vartheta(x) \sim x$. The other direction follows similarly. $\square$

We can also prove similarly that the third function we defined is also related here:

Lemma 209

$\vartheta(x) \sim x$ is also equivalent to $\psi(x) \sim x$.

It turns out ψ will be the easiest function to relate to ζ:

Lemma 210

We have

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log(\zeta(s)) = \sum_{n\geq 1}\Lambda(n)n^{-s},$$

where λ is defined the same way as in $\psi(x)$.

So if we know enough about $\frac{\zeta'}{\zeta}$, we may be able to deduce information about the coefficients $\Lambda(n)$.

Proof. Going back to the Euler product form

$$\log\zeta(s) = \sum_p \sum_{n\geq 1}^{\infty} \frac{1}{np^{ns}},$$

and now taking a derivative on both sides,

$$\frac{d}{ds}\left(\frac{1}{m}p^{-ms}\right) = -p^{-ms}\log p,$$

so

$$-\frac{d}{ds}\log\zeta(s) = \sum_p \sum_m p^{-ms}\log p = \sum_{n\geq 1}\Lambda(n)n^{-s},$$

because only the powers of p will show up and we get an extra $\log p$ from each one, $\square$

Next time, we'll see how the nonvanishing of ζ is related and finish up the proof.

23 December 5, 2019

Today, we'll finish the proof of the Prime Number Theorem, and then do a bit of a review session on Tuesday. (A list of topics for the final will be posted soon so we can review at home.) Our final is Monday morning from 9–12.

Recall that we're trying to prove that the asymptotic behavior of the prime counting function

$$\lim_{x\to\infty}\frac{\pi(x)}{x/\log x} = 1.$$

We showed last time that this is equivalent to examining the behavior of a new weighted counting function,

$$\psi(x) = \sum_{n\leq x}\Lambda(n), \quad \Lambda(n) = \begin{cases}\log p & n = p^a \\ 0 & \text{otherwise},\end{cases}$$

or the Chebyshev function $\vartheta(x) = \sum_{p\leq x}\log p$ (which only sums over the primes). Then we just need to show that $\psi(x) \sim x$ (this is the same as $\vartheta(x)\sim x$, because notice that the only differences between these two weighted counting functions come from the primes less than $\sqrt{n}$, which become negligible).

At the end of last class, we noted that Λ is a good way to connect the prime counting function to the zeta function:

$$\boxed{-\frac{\zeta'(s)}{\zeta(s)} = \sum_{n=1}^{\infty}\Lambda(n)n^{-s}}.$$

If we define $\lambda(n)$ (with a lowercase λ) to be $\log p$ when n is prime and 0 otherwise (so not counting prime powers), it is actually harder to find $\sum_{n=1}^{\infty}\lambda(n)n^{-s}$: this means we're only summing over primes $\sum_p p^{-s}\log p$, which we can write as a derivative of the function $f(s) = -\sum_p p^{-s}$. This function is a bit farther away from the Riemann zeta function: we can study its properties, but it's less clear how to make the connections. So we'll stick with $\Lambda(n)$ and the $\psi(x)$ function here.

Well, the above boxed expression is some kind of generating function for Λ. How can we turn it into something that is useful?

Definition 211

The **Laplace transform** of a bounded piecewise continuous function $f : [0, \infty) \to \mathbb{R}$ is a complex-valued function

$$\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t) dt$$

defined for all $\text{Re}(s) > 0$. More generally, if $|f(t)| \le Ae^{t\sigma}$, then this function is convergent for all $\text{Re}(s) > \sigma$.

Because we're assuming our function f is bounded, and the exponential term e^{-ts} decays, this is a holomorphic function on the half-complex plane. This is very similar to the Fourier transform: if we write $s = x + iy$, then

$$\mathcal{L}(f)(s) = \int_0^\infty e^{tx+tiy} f(t) dt = \int_0^\infty e^{-ity}(e^{-tx} f(t)) dt.$$

So the Laplace transform is taking the Fourier transform of $e^{-tx} f(t)$, plus some kind of "normalizing factor."

Proposition 212

We have

$$\mathcal{L}(\psi(e^t))(s) = -\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)}.$$

First, we need to show an intermediate result:

Lemma 213

$\psi(x) \le cx$.

From this, we can deduce that $\psi(e^t) \le ce^t$, and then $\sigma = 1$ so the Laplace transform can exist for all $\text{Re}(s) > 1$.

Proof. We can show that $\vartheta(x) \le c'x$ for some constant C' first: this is easier to prove. Note that $\theta(x) = \log \prod_{p \le x} p$, so we need a way to estimate the product of primes not exceeding x. We can consider

$$\binom{2n}{n} = \frac{(2n)!}{n!^2} :$$

this contains factors of all primes $n \le p \le 2n$, so

$$\prod_{n \le p \le 2n} p \le \binom{2n}{n} \le 2^{2n},$$

where the last inequality comes because $2^{2n} = \sum \binom{2n}{i}$ by the binomial expansion. So taking the log of both sides,

$$\theta(2n) - \theta(n) \le 2n \log 2,$$

and this shows the result for all powers of 2: this means that $\vartheta(2^n) \le c'2^n$, and ϑ is monotonic, so we indeed have $\vartheta(x) \le 2c'x$ for all x, as desired.

And now ϑ and ψ are within a constant multiple of each other, so we've shown the result. $\square$

Now we can move on to the proof of Proposition 212:

Proof. We have

$$\mathcal{L}(\psi(e^t)) = \int e^{-ts}\psi(e^t)dt.$$

Change variables so that $x = e^t$ to get

$$= \int_1^\infty x^{-s}\psi(x)\frac{dx}{x}.$$

Since ψ is piecewise constant and only changes at integers, we can sum over integers:

$$= \sum_n \psi(n)\int_n^{n+1} x^{-s}\frac{dx}{x} = \sum_n \psi(n)\frac{1}{s}(n^{-s} - (n+1)^{-s}).$$

So now we can collect the n^{-s} terms from adjacent terms to get

$$= \frac{1}{s}\sum_n \psi(n)(n^{-s} - (n+1)^{-s}) = \frac{1}{s}\sum_n \Lambda(n)n^{-s},$$

and now apply the above equality to get the result. □

With this, we now have the Laplace transform of our function ψ, and we want to discover some properties.

Fact 214

One thing we can consider is the inverse Laplace transform: for all piecewise functions f,

$$f(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \mathcal{L}(f)(s)e^{ts}ds.$$

where C is a fixed constant greater than σ.

This convergence is **not absolute**: it's the limit as we take an integral from $C - iT$ to $c + iT$ and take $T \to \infty$. The advantage of this is that we can possibly do a meromorphic continuation beyond $c > \sigma$ by drawing a contour integral!

Remember that a $\frac{f'(z)}{f(z)}$ has simple poles exactly where f has zeros or poles, with residues equal to the multiplicities. The point, then, is that because $\zeta(s)$ has a simple pole at $s = 1$, $\mathcal{L}(\psi(e^t)) = -\frac{1}{s}\frac{\zeta'(s)}{\zeta(s)}$ has a pole at 1 as well, and we can write

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + g(s),$$

where g is holomorphic on $\text{Re}(s) = 1$. **Here, we've used the fact that ζ does not have any poles on** $\text{Re}(s) = 1$ (this is the nonvanishing theorem of last time), and in particular it's holomorphic on an open subset containing $\text{Re}(s) = 1$. So we can shift things by a bit:

$$\mathcal{L}\left(\frac{\psi(e^t)}{e^t} - 1\right) = -\frac{1}{s+1}\left(\frac{\zeta'(s+1)}{\zeta(s+1)} + \frac{1}{s}\right),$$

and this new function is holomorphic on $\text{Re}(s) = 0$. So to summarize, this is the holomorphic function we care about:

$$f(t) = \frac{\psi(e^t)}{e^t} - 1.$$

This has an analytic continuation to $\text{Re}(s) = 0$, and this uses the nonvanishing of ζ on $\text{Re}(s) = 1$. In addition, from our lemma earlier, this is uniformly bounded.

Theorem 215

Given a piecewise bounded continuous function $f : [0, \infty) \to \mathbb{R}$, **assume that** $g(s) = \mathcal{L}(f)(s)$ **has a holomorphic continuation to** $\text{Re}(s) = 0$. Then $\int_0^\infty f(t)dt$ converges to $g(0)$ (as if we just plugged in $s = 0$).

(The asumptions are nontrivial: a priori, we only know this continuation is possible for $\text{Re}(s) > 0$.) Basically, if we can move our domain of definition to $s = 0$, we can plug in $s = 0$ as well. This is called a **Tauberian-type theorem**.

This proof is within the techniques we have studied, but it is tedious, so we'll skip it here. How can we use this to finish our proof? Since $\mathcal{L}(f)$ is holomorphic, we know that the integral

$$\int_0^\infty f(t)dt = \lim_{T\to\infty} f(t)dt$$

exists. Let's use this to finish:

Lemma 216

If $\int_0^\infty \left(\frac{\psi(e^t)}{e^t} - 1\right) dt$ exists, and $\psi(x)$ is nondecreasing, then $\psi(x) \sim x$.

Proof. Suppose otherwise: then there exists some ε such that there are either infinitely many a such that $\frac{\psi(a)}{a} > (1+\varepsilon)a$ or infinitely many such that $\frac{\psi(a)}{a} < (1-\varepsilon)a$. Without loss of generality, we'll do the first case and let $\lambda + 1 + \varepsilon$. Now notice that the problem statement is equivalent to having

$$\int_1^\infty \left(\frac{\psi(x)}{x} - 1\right) \frac{dx}{x}$$

converge (by making a change of variables). Then

$$\int_a^{\lambda a} \left(\frac{\psi(x)}{x} - 1\right) \frac{dx}{x} \geq \int_a^{\lambda a} \left(\frac{\lambda a}{x} - 1\right) \frac{dx}{x} = \int_1^{\lambda} \left(\frac{\lambda}{x} - 1\right) \frac{dx}{x} > 0.$$

This is some positive constant independent of a. But this is a contradiction, because for the integral to be convergent, we must not have any interval with bounds approaching infinity which integrates to a positive constant. $\square$

The only thing we didn't do is the Tauberian-type result here. To summarize, the key idea is that the Laplace transform connecting ψ to ζ allows us to use analytic continuation results!

We'll finish this class with something which is not a proof but gives some more intuition for the prime number theorem.

Theorem 217 (Explicit formula for $\psi(x)$)

We have for all $x > 2$ (**for a slightly modified** ψ **function**) that

$$\psi(x) = x - \sum_{\zeta(\rho)=0} \frac{x^\rho}{\rho} + \frac{\zeta'(0)}{\zeta(0)}.$$

(This includes the trivial zeros $\rho = -2, -4, \cdots$, which contribute a $\frac{1}{2}\log(1 - x^{-2})$ term.)

Here, this is not an absolute convergence either: we have to take a limit where ρ has bounded imaginary part and take the limit. If the Riemann hypothesis holds, $\text{Re}(\rho) = \frac{1}{2}$ for all nontrivial zeros, so each of the terms here is proportional to $\sqrt{x}$. So that would show that

$$\psi(x) - x = O\left(x^{1/2+\varepsilon}\right)$$

for an arbitrarily small ε. And from this, we'd also be able to show a particularly good approximation to $\pi(x)$:

$$\pi(x) - \int_2^x \frac{dt}{\log t} = O\left(x^{1/2+\varepsilon}\right).$$

(It takes more time to show those asymptotic bounds, but this is just for reference.) And after all, if we have the inverse Laplace transform

$$\psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} -\frac{\zeta'(s)}{\zeta(s)} x^s \frac{ds}{s},$$

the idea is that we can move our contour from some $c > 1$ towards the negative real line: every time we go past a zero of ζ, we pick up the residue there! We just need to justify the convergence issues.

24 December 10, 2019

We'll go over many of the topics which will be covered on the final exam (which is next Monday from 9–12 in room 2-190). A list of topics has been uploaded on Stellar: the exam will cover chapters 2, 3, 8.1-8.2, 5, and 6.1 (this is the order they were covered in class). We should expect about 70 percent of the exam to be on the first three of these chapters, and the remaining 30 percent on the last two.

Chapter 2 is about some basic theory of complex analysis, primarily Cauchy's integral formula, so we'll skip it for now. Chapter 3 is about meromorphic functions, and we'll start by **classifying all isolated singularities**:

- removable singularities (bounded in a neighborhood),
- poles (separate the function into a principal part and a holomorphic part near the pole),
- and essential singularities (Casorati-Weierstrass: the image is dense in a small neighborhood).

We should know how to compute the residue of these poles, and we should be able to use the residue formula to compute integrals:

$$\frac{1}{2\pi i}\int_C f(z)dz \sum_{z_i} \text{Res}_{z_i} f,$$

where we're adding over all poles z_i inside C. This is one of the major motivations to study complex analysis: we can compute certain contour integrals this way, which gives us results that we can't necessarily find with elementary calculus.

Example 218

Consider the definite integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2},$$

and pretend we never learned trigonometry.

We can integrate $f(z) = \frac{1}{1+z^2}$ counterclockwise around a semicircle of radius R in the upper half-plane, and take $R \to \infty$. For large enough R, the only pole inside this semicircle is at $z = i$: note that the residue here is

$$\lim_{z\to i}(z-i)\frac{1}{1+z^2} = \lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i}.$$

Since the contribution from the arc part of the semicircle goes to 0 by the ML inequality,

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{dx}{1+x^2} = 2\pi i \text{Res}_{z=i} f = 2\pi i \frac{1}{2i} = \pi,$$

which is correct if we remember trigonometry again! (We should expect at least one problem where we need to compute integrals.)

One related idea is the **argument principle**, which concerns the log derivative

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \#\text{ zeros} - \#\text{ poles}.$$

Example 219

Let's say we want to find the number of zeros of $z^3 + 3z + 1$ whose absolute value is at most 1.

Applying the formula directly is a bit difficult, but we can consider Rouché's theorem, which tells us that f and $f+g$ have the same number of zeros inside the region $|z| = 1$ if $|f| > |g|$ everywhere on the boundary. So use $f(z) = 3z$ and $g(z) = z^3 + 1$: since $|3z| = 3$ on the boundary, but $|g| \le |z|^3 + 1 = 2$ by the triangle inequality, $|f| = 3z$ and $|g| = z^3 + 3z + 1$ have the same number of zeros inside the disk, so $z^3 + 3z + 1$ has exactly one zero with absolute value at most 1.

On a related note, we have the open mapping theorem, which tells us that any nonconstant holomorphic function is an open map: it sends open sets to open sets. This directly gives us the maximum modulus principle: we can never attain a maximal value of a function f on its interior.

The next topic we discussed is that of a simply connected domain, which helped us define the logarithm. For example, the star-shaped region $\mathbb{C} - (-\infty, 0]$ is simply connected (we should know some examples, though we won't be asked to prove they're simply connected), so we can define the **principal branch** $\log z$ on this domain. On a related note, we should know the statement of the Riemann mapping theorem, which says that proper, simply connected, open subsets of $\mathbb{C}$ are conformal (or biholomorphic) with the unit disk D. One key example of this is the upper half-plane: even if this is unbounded and looks very different from D, we should think of them as very similar to each other.

A bit more about conformal maps: it's good for us to know some examples (from chapter 8.1). $\log z$, for example, defines a conformal map from $\mathbb{C} - (-\infty, 0]$ to the infinite strip $\mathbb{R} \times (-\pi, \pi) \subseteq \mathbb{C}$ (we have bounded imaginary part). More important ones include the ones that keep coming up in the study of the unit disk: for example, the Schwarz lemma helps us determine the automorphism group of D (and therefore the upper half-plane $\mathbb{H}$ by extension). We should definitely know how to write down these automorphisms, as well as the maps between D and $\mathbb{H}$. For example, there is a map $\phi : D \to D$ that sends $\alpha \to 0$, and the way to remember this is in terms of the fractional linear map

$$\phi_\alpha = \frac{\alpha - z}{1 - z\overline{\alpha}}.$$

(How do we memorize this? This map should send 0 and α to each other, and it should be holomorphic, so there shouldn't be conjugates of z. It also sends the boundary δD to itself.) Then ϕ_α can be composed with a rotation, but that's all the automorphisms of D: any such map can be written as $e^{i\theta}\phi_\alpha$ for some $\alpha \in D, \theta \in [0, 2\pi)$. Doing some calculations, we can find that all automorphisms of $\mathbb{H}$ can be written in the form $\frac{az+b}{cz+d}$, where $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$ has determinant 1.

The remainder of the exam will cover the last two chapters. When looking at entire functions, recall that we used Jensen's formula to connect zeros of an entire function f to the value of $\log |f|$ integrated on a circle. We don't need to memorize the formula, but it is useful to understand the relationship here: this helps us discuss infinite products of the form $\prod_i (1 + c_i)$, where the convergence can be reduced to considering the convergence of $\sum |c_i|$. Here, each c_i can be a function: we only need to know how to apply this criterion when each function is bounded by a constant.

Example 220

Consider the product (as we've discussed before)

$$\prod_{n=1}^{\infty}\left(1-\frac{z^2}{n^2}\right).$$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is constant, $\sum_{n\geq 1}\left|\frac{z^2}{n^2}\right|$ is uniformly bounded for any compact set in the complex plane, so the product will be holomorphic everywhere. This leads us to a more general construction of the Weierstrass canonical factor: we should understand rather than memorize that

$$E_n(z) = (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^n}{n}}$$

comes from the first n terms of the Taylor series expansion of $\log(1-z)$. In our discussion of entire functions, we said that a function f has an **order of growth** at most ρ if $|f(z)| \leq Ae^{B|z|^\rho}$ for sufficiently large $|z|$, and the order of growth of a function is the infimum of all such ρ. This gives us the (extremely powerful) Hadamard product factorization: if a function has order $k \leq \rho < k+1$ for some integer k, then we can write

$$f = e^{g(z)}z^m \prod_{a_i \neq 0} E_k\left(\frac{z}{a_i}\right).$$

(Here, unlike in the Weierstrass product, we can use the same canonical factor for all roots.)

Example 221

Consider (again) the function $f(z) = \frac{\sin \pi z}{\pi}$.

We know this function has (simple) zeros at the integers, and the order is equal to 1 (because it's a linear combination of exponential functions). So $k = 1$, and

$$\frac{\sin \pi z}{\pi} = z\prod_{n\in\mathbb{Z}_{\neq 0}}\left(1-\frac{z}{n}\right)e^{z/n}.$$

We're not required to know the proof (it's very technical), but it's important for us to know how to write out the product expansion.

The final topic will be the Gamma function: we should know how it's defined (initially just for positive real s) and how to make the meromorphic continuation to all of $\mathbb{C}$. It's important for us to be able to know the locations and residues of all the poles, as well as the connection to the factorial function.