8.03: Vibrations and Waves

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My recitations for this class were taught by **Professor Wolfgang Ketterle**.

1 September 6, 2018

Professor Lee is not here today; Professor Comin will be teaching the lecture instead. This class is going to help us master mathematical formulism surrounding a lot of "wave" topics: this includes things like heat, radiation, and even gravity (sort of).

1.1 Overview

There are three textbooks that are required for this class; we can get two of them through the library and the last is given to us in PDF format already.

There's an 8.03 website we can access, and there's also a Piazza class forum that we should sign up for. The class has 2 quizzes and a final that are obviously required. Homework-wise, there are ten problem sets; they are usually due at **4pm** on Friday (no late submissions at all!), but the lowest pset grade will be dropped. We should submit these in the drop boxes between buildings 16 and 8 on the third floor.

There are office hours spread throughout the week, and we should try to go to some of them. Also, these lectures are being recorded on video! They will likely be posted on MIT OCW after this semester.

1.2 Topics of the class

Many concepts in this class will explain natural phenomena. This ranges from sound and brain waves, to water and electromagnetic waves, to probability density waves and gravitational waves in modern or quantum mechanics. About the first third of the lectures will be mechanical waves, and then the next third will cover electromagnetism (Maxwell, wave equation, propagation, radiation). There will be some optics near the end and finally some quantum stuff to close everything out.

The whole point here is to **translate physics into math** and **assemble models from fundamental motion**.

1.3 Simple harmonic oscillator

We'll start with a simple model. Consider a rigid body, where the only thing we care about is the mass of that object (think of it as a point at the center of mass). Then, we add a mechanical element: a spring, which we'll also only look at in terms of one number: the stiffness of that spring. Basically, we want to think of a good mathematical model to describe our system. Then, we will try to see if it works physically, and once it does, we can predict how it will act.

Example 1

We have a body of mass m lying on a flat surface attached to one end of a spring with stiffness k. This spring is attached to a fixed end (such as a wall).

We know this will only move in one direction – the direction in which the spring compresses (by symmetry), so we'll write the mass's position as $x(t)$, where $x < 0$ if it compresses the spring and $x > 0$ if it stretches it. Then we can call $x(0) = 0$ the **equilibrium position**.

We can make some observations first:

Fact 2

First of all, if we start at the **equilibrium** position, where $x(0) = 0$ and $\dot{x}(0) = 0$, then $x(t) = 0$. (Nothing happens if everything is at equilibrium).

But we can change this by starting with some displacement $x(0) = x_0$ while keeping the initial velocity $\dot{x}(0) = 0$.

Fact 3

We observe that there is some kind of periodic oscillation through time, and that there is some notion of a frequency.

Let's try to start from these initial conditions. We want to predict where our body is at some arbitrary time t; that is, we want to solve for $x(t)$.

In physics, it is always good to start from the free-body diagram. The normal force and force of gravity (which act in the vertical direction) cancel out by our constraints, so the net force is always in the x-direction, and it is provided entirely by the spring. We know (intuitively) that the spring will act opposite the direction of displacement; call this force F_s.

Newton and Hooke come to the rescue here: Newton's second law says that the total force on an object can be written as $\sum \vec{F} = F_s = m \sum \vec{a}$, and since we're in one dimension, we can write this as $F_x = ma$ or $F_x = m\ddot{x}$ (dots refer to time derivatives). In addition, Hooke's law tells us that the force from a spring is proportional to the displacement; in particular, $F_s = -kx(t)$. (Notice that the negative sign causes the restoring force to point towards the equilibrium position at all times.) Thus, setting these equal,

$$m\ddot{x} = -kx \implies \boxed{\ddot{x} + \frac{k}{m}x = 0}.$$

There's no magical reason why Hooke's law should be true here; for all we know, the force equation could be $F_s = -kx^3$. But Hooke's law holds up pretty well empirically, especially with simple springs. So with this equation, we've now linked our **dynamical variables** (the position of our mass, as well as its derivatives) to create an **equation of motion.** This will help us describe the behavior of the mass-spring system!

Definition 4

To simplify our notation, let $\omega^2 = \frac{k}{m}$. (ω is sometimes known as the **natural frequency** of the mass-spring system).

Sometimes, we use numerical methods to solve differential equations, but that won't be necessary here. We want a function $f(t)$ with second time derivative proportional to itself; two such functions are sine and cosine! In our particular case, the solutions turn out to be $x_1(t) = \cos(\omega t)$ and $x_2(t) = \sin(\omega t)$; it is easily verified that these are linearly independent and satisfy our equation of motion.

But we have two functions, and we want a unique solution. The idea is that we can take **any** linear combination of these solutions $c_1x_1 + c_2x_2$ will also work, since we have a **linear homogeneous differential equation.**

So our general solution is (for some real numbers a, b)

$$x(t) = a\cos\omega t + b\sin\omega t$$

and to get the specific solution we want, we need to use our **initial conditions**. Let's say that $x(0) = x_0$; then plugging this into our general solution, we find that $a = x_0$. Similarly, if $\dot{x}(0) = 0$, we can differentiate the above equation to find that $b = 0$. In general, if $\dot{x}(0) = v_0$, then $b = \frac{v_0}{\omega}$ will work.

Fact 5

In an nth order differential equation, we will have n linearly independent solutions, and n initial conditions will help us find the exact solution. We've just seen an example of this in action with $n = 2$: math works!

Regardless of these initial conditions, the frequency $\omega = \sqrt{\frac{k}{m}}$ is some constant of motion, and the period of oscillation is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}}.$$

Let's test our model. We will measure the period of two different mass-spring models. If our model works, the period will scale as the square root of the mass.

Example 6

There are two masses hanging (vertically) by identical springs from a fixed end at the front of the classroom. Mass 1 is 427 grams, and mass 2 is 713 grams. This means that $\frac{T_2}{T_1}$ is about $\sqrt{1.70} \approx 1.3$.

To make the reaction time contribute less to error, we will measure 5 periods of each. (This is a good experimental technique in general.) On two different runs, the smaller mass completed five periods in 3.64 seconds and 3.96 seconds, for an estimated period of around 0.76 seconds. The larger mass (which is actually just two balls attached together) did this in 5.63 seconds, which is a period of 1.12 seconds. $\frac{1.12}{0.76} = 1.5$, which is... sort of close to 1.3. (Hmm.) Basically, physical situations can't be perfectly modeled because of friction and so on.

But here's a question: are properties like the frequency of the system different because we did this experiment vertically? It turns out the answer is **no**: the equilibrium is just translated downward by $y_0 = \frac{mg}{k}$, while the actual oscillation stays the same. This is because the new differential equation is

$$m\ddot{y} = -ky + mg = -k(y - y_0),$$

and $\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 (y - y_0)}{\partial t^2}$, meaning our equation still holds with a translated variable.

Remark 7. *This leads to an interesting thought. It seems that if we know the position and velocity of every atom in the universe, we will know all future (and even all past) values for those positions and velocities, just by solving a bunch of differential equations. (This is known as* **Laplace's Demon***.) That was what classical physics was all about: determinism let us measure reality and make good, verifiable predictions. But then quantum mechanics came along, and now we're all sad.*

Fact 8
The series of steps of observation (measurement), abstraction (model), and prediction (testing) are why science is important and useful for describing the world.

Back to the failure of our experiment above (and therefore of our model). Reality is always more complex than the models that we come up with; if Hooke's law is ideal, **why does it work really well** for applications like heat, sound, and radiation?

Well, we know that if our force $F(x)$ is conservative, it can be written as $-\frac{dV(x)}{dx}$ for some potential function $V(x)$. Specifically, in Hooke's case, our potential looks like $V(x) = V_0 + \frac{1}{2}kx^2$, which is a perfect parabola. Many potentials, like the Lennard-Jones potential between two atoms, do not look like parabolas. But no matter what $V(x)$ looks like, the equilibrium position has to be at a **local minimum** for the potential function. At a local minimum, the first derivative is 0, so the **Taylor series approximation** has no linear term! As $x \to x_0$, we approach Hooke's law, since terms that are $O(x^3)$ are much smaller than the x^2 term itself. Formally, our Taylor series will look like

$$V(x) = V_0 + \frac{1}{2}V''(x_0)(x - x_0)^2 + \frac{1}{6}V'''(x_0)(x - x_0)^3 + O((x - x_0)^4) \cdots$$

Fact 9
Often, we'll study situations in this class with small oscillations, so we're close enough to equilibrium conditions to make such approximations.

Specifically, we'll make our small oscillation condition such that the third term is much smaller than the second term; that is,

$$(x - x_0) \ll \frac{3V''(x_0)}{V'''(x_0)}.$$

This is very important, because we can assume the force is linear and the potential is quadratic for small perturbations from equilibrium. And this is one of the core concepts that justifies why simple harmonic motion is a good approximation.

We can actually write our general solution for the harmonic oscillator in a more compact form:

$$x(t) = a\cos\omega t + b\sin\omega t = A\cos(\omega t + \phi)$$

A is here the **amplitude**, and ϕ is the **phase** of the oscillation at $t = 0$ And these are equivalent if we set $A = \sqrt{a^2 + b^2}$ and $\tan\phi = \frac{b}{a}$.

1.4 Complex numbers

We use complex numbers as a generalization of real numbers. If $i = \sqrt{-1}$, then any number $x + iy$ can be written as $Re^{i\phi}$, which is also $R(\cos\phi + i\sin\phi)$. This might look very similar to what we had earlier! This means that we can just take the real or imaginary part of a complex exponential to give us a real solution to the equation of motion, and this is nice because **equations of motion are easier to express in complex numbers**.

Realistically, we're missing **drag force** in this model. We'll talk about some other factors in the next lecture!

2 September 10, 2018 (Recitation)

Professor Ketterle studies **ultracold atomic matter**. Cooling down particles makes it easier to be precise in experiments, and a lot of physical phenomena only happens at very low temperatures!

There are three kinds of waves that are dealt with in physics (broadly):

- Mechanical waves can be observed and even demonstrated – they are described by some position function $x(t)$.
- Electromagnetic waves become more abstract; we can't see them directly, and it was weird in the 19th century to think that something could oscillate even in a vacuum, since there is no mechanical equivalent to an EM wave. These are described by some functions $E(t), B(t)$ that are related by Maxwell's equations. But electromagnetic waves do follow basically the same concepts and equations as sound waves, and we use similar differential equations to work with them.
- When we get to quantum mechanics, we have a wavefunction $\psi(t)$ which even has a complex component. There is a unifying component here: we do write mechanical and EM waves as the real part of a complex function for convenience and mathematical tricks. But the difference is that quantum mechanical waves are actually complex-valued!

Example 10

When Professor Ketterle received the Nobel Prize, it was for working on a Bose-Einstein condensate, a very cold collection of atoms. It turns out that all of the atoms in such a collection follow the same wavefunction.

One interesting way we often observe light is by "combining it with itself!" We have destructive interference at certain points if we, for example, create a **double-slit interference pattern**. So sometimes, motion and motion adds up to nothing (standing waves in the water), light and light adds to darkness, and so on.

Fact 11

Take two Bose-Einstein condensates, and we get an interference pattern. This means that sometimes atoms and atoms give nothing! (Note: I had no idea what this meant, but this is explained much later in the class, particularly in the second-to-last recitation.)

One question to think about: how fast and how slow can waves vibrate and still be considered as oscillations? A heartbeat oscillates at 1 Hertz, or once per second. Earthquake aftershocks happen on the order of hours, and "helioseismology" happens on the order of a few minutes. (The frequency is a few millihertz.) And orbits of planets happen on the order of years! But we can maybe still think of all of these as oscillations or waves.

On the other side of things, music is played, creating sound waves at thousands of hertz. Looking at light waves, visible light oscillates at 10^{14} hertz. But it is electron oscillation that gives off light, and electrons can oscillate up to 10^{16} hertz if they are bound very tightly to a heavy metal. Getting smaller, nuclei themselves vibrate too due to the strong nuclear force, which gives a frequency on the order of 10^{22} hertz.

Vibrations and waves are important because of the concept of equilibrium. At any stable point of equilibrium, we're at a local minimum, so small perturbations gives us a quadratic potential.

Fact 12

One way to think of this is that "Hooke's law is valid as long as it is valid." It's an approximation, not a physical law.

Example 13
The double pendulum is cool because it is not very predictable! (Changing the initial position of the two pendulums changes the motion dramatically.) In other words, the behavior of oscillatory systems is not always as simple when the angle or magnitude of displacement is large.

Monday recitations will be usually very general: giving help, asking for hints, presenting cool ideas. Wednesday recitations will talk more about details (like how to work through calculations), and we'll have time to work by ourselves as well.

3 September 11, 2018

Unfortunately, the recitation I'm in happens to be very full, so it is advised that some people switch to recitation section 4. Also, one of the problem set problems is bad because the units don't work out – we should use the revised version online!

Professor Lee creates a "quark soup" from matter in his research. Basically, if we strip off all of the electrons from water and then compress the protons, we get a new medium called a "quark gluon plasma." To do this, physicists collide lead ions to create a large amount of energy in a small space using the Large Hadron Collider.

3.1 Review

Last time, we found that the **complex exponential** is a really good way to represent solutions to a harmonic oscillator. The idea is that many different systems will give equations of the form

$$M\ddot{x} = -kx,$$

where M represents a "generalized mass" and k a "generalized spring constant." In such a system, we can calculate the kinetic energy $KE = \frac{1}{2}M\left(\frac{dx}{dt}\right)^2$ and potential energy $PE = \frac{1}{2}kx^2$, and our total energy

$$E = \frac{1}{2}M\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

must be constant. Letting $\omega_0 = \sqrt{\frac{k}{m}}$, which simplifies our equation to $\ddot{x} + \omega_0^2 x = 0$, we know our solutions are of the form $x(t) = A\cos(\omega_0 t + \phi)$ for some parameters A, ϕ. Plugging this into the equation for energy, since $\frac{dx}{dt} = -A\omega_0 \sin(\omega_0 t + \phi)$, we will end up with

$$E = \frac{1}{2}M\left(-A\omega_0 \sin(\omega_0 t + \phi)\right)^2 + \frac{1}{2}k\left(A\cos(\omega_0 t + \phi)\right)^2 = \frac{1}{2}kA^2$$

which is constant! So in a simple harmonic oscillator, **kinetic energy and potential energy are being dynamically converted back and forth**.

Fact 14 (Other SHM examples)
Vertical springs (with gravity), pendulums at small angles, and LC circuits can all be basically described using simple harmonic motion.

3.2 New examples of simple harmonic motion

Example 15

Instead of a having simple mass, now let's say that we have a rod fixed at one end, hanging vertically. (Assume that its motion is confined to a plane.)

We will use Newton's law in the rotational form $\tau = I\alpha$. First of all, let's establish a coordinate system: let θ be the angle from the vertical for the rod, where counterclockwise is positive and clockwise is negative Additionally, let's say we have initial conditions $\theta(0) = \theta_i, \theta'(0) = 0$. Then if the rod has length ℓ, the vector from the fixed end to the center of mass has length $\frac{\ell}{2}$.

From here, we just need to consider the free-body diagram. The only forces on the rod are a force $F_g = mg$ downward and a tension force pointed in the radial direction, which contributes no torque. Thus

$$\tau = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin\theta = -\frac{\ell}{2} mg \sin\theta,$$

which means that Newton's law gives us the equation

$$\tau = I\ddot{\theta} = \frac{1}{3}m\ell^2\ddot{\theta} = -\frac{mg\ell}{2}\sin\theta(t) \implies \boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0},$$

and as before, if we define the natural frequency $\omega_0 = \sqrt{\frac{3g}{2l}}$ and use a **small angle approximation**, we get a similar equation of the form $\ddot{\theta} = \omega_0^2\theta$.

Fact 16

How good is the small angle approximation? If $\theta = 1^\circ$, $\frac{\sin\theta}{\theta} = 99.99\%$. Similarly, 5° gives 99.95%, and 10° gives 99.5%, so the approximation is honestly pretty good.

Once we make that approximation, we can just write our solution as $\theta(t) = A\cos(\omega_0 t + \phi)$, where $\omega_0^2 = \frac{3g}{2l}$. Plugging in our initial conditions, we find that $\phi = 0, A = \theta_0$. Thus $\boxed{\theta(t) = \theta_0\cos(\omega_0 t)}$ is the solution we want.

But if we try this in the real world, the rod doesn't oscillate forever – eventually it stops moving! So we need to introduce a new idea here.

3.3 Drag force

We'll continue with the example above. We'll add a **drag force** which is proportional to $\dot{\theta}$:

$$\tau_{\text{drag}}(t) = -b\dot{\theta}.$$

This is a good model for **slow propagating particles** – in those situations, having drag proportional to velocity works well. (However, for very high-velocity particles, drag ends up being more proportional to the velocity squared).

So let's solve our equation: the same free-body setup tells us that

$$\tau = I\ddot{\theta} = -mg\frac{l}{2}\sin\theta - b\dot{\theta},$$

and with the small angle approximation, we end up with

$$\ddot{\theta} + \frac{3b}{ml^2}\dot{\theta} + \frac{3g}{2l}\theta = 0$$

Define $\omega_0^2 = \frac{3b}{ml^2}$ and $\Gamma = \frac{3b}{ml^2}$; our equation has now been written in the generalized form

$$\boxed{\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = 0.}$$

Fact 17

Does drag force slow down, speed up, or do nothing to the frequency of motion? 20 people say it will do nothing, 12 say it will slow down, and 1 says it will speed up.

To solve this, we use mathematics in a pretty way. Let our trial solution be of the form $z(t) = e^{i\alpha t}$, where α is some complex number. Then $\dot{z} = i\alpha z, \ddot{z} = -\alpha^2 z$, so

$$\ddot{z} + \Gamma\dot{z} + \omega_0^2 z = 0 \implies z(-\alpha^2 + i\Gamma\alpha - \omega_0^2) = 0.$$

Now by the quadratic formula, we get

$$\alpha = \frac{i\Gamma \pm \sqrt{4\omega_0^2 - \Gamma^2}}{2}.$$

Now we break into cases – remember that ω_0 is the natural frequency with no drag force. Our central question is really what happens to the **sign of the expression under the square root**.

1. (Underdamped motion) If $\omega_0^2 > \frac{\Gamma^2}{4}$; that is, the drag force is quite small, we define $\omega^2 = \omega_0^2 - \frac{\Gamma^2}{4}$, and now we find that $\alpha = \frac{i\Gamma}{2} \pm \omega$. Then plugging in α_+ and α_-, we have our solutions

$$z_+(t) = e^{-\frac{\Gamma}{2}t}e^{i\omega t}, \quad z_-(t) = e^{-\frac{\Gamma}{2}t}e^{-i\omega t}.$$

   If we take the average of these, we get $\boxed{\theta_1(t) = e^{-\frac{\Gamma}{2}t}\cos(\omega t)}$, and if we take their difference over $2i$, we get $\boxed{\theta_2(t) = e^{-\frac{\Gamma}{2}t}\sin(\omega t)}$. Thus, the general solution is going to be $\boxed{e^{-\frac{\Gamma}{2}t}\left(a\cos(\omega t) + b\sin(\omega t)\right)}$ for some parameters a, b from the initial conditions. And like before, we can rewrite the sinusoidal part as $A\cos(\omega t + \phi)$ if we'd like.

2. (Critically damped) In this next case, we have $\omega_0^2 = \frac{\Gamma^2}{4}$. Then we have the solution $\alpha = \frac{i\Gamma}{2}$, so $\omega = 0$. So one way to interpret the solutions here is to take the θ_1 and θ_2 solutions from the above case and **send ω to zero**. We get $e^{-\frac{\Gamma}{2}t}$ from θ_1, and a Taylor approximation for θ_2 yields an additional linear term $te^{-\frac{\Gamma}{2}t}$. (There are ways to formalize this as well, but we can indeed plug these two solutions into the differential equation and see that everything works out.) So our general solution in this case will be

$$\boxed{\theta(t) = e^{-\frac{\Gamma}{2}}(A + Bt)}.$$

   Some kind of magic happens once we hit this critical value, and the oscillations stop! Here, the system behavior changes, and our motion can only cross the equilibrium point **at most once**. (This happens only if $A + Bt = 0$, which intuitively occurs when we throw our mass across the equilibrium point really fast.)

3. (Overdamped) If $\omega_0^2 < \frac{\Gamma^2}{4}$, then we find that $\alpha = i\left(\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right)$, and the square root term is now real. Thus, α is pure imaginary: our solutions will be

$$\boxed{\theta(t) = A_+e^{-\Gamma_+ t} + A_-e^{-\Gamma_- t}},$$

   where A_+, A_- are free parameters. So in this case, we have a linear combination of two exponential functions, and again, the mass can only cross the equilibrium position at most once.

4 September 12, 2018 (Recitation)

We'll spend some time today trying to think about behavior of not-quite-SHM. Simple harmonic motion is periodic, and the period of that motion is independent of the amplitude. Are there motions that are periodic but not harmonic, and so on?

Fact 18

Planetary motion is periodic and sort of harmonic-looking, but the period does depend on the amplitude, so this is not as simple of a system as the one we've been discussing.

A bouncing ball is also periodic. But the graph of the potential for an ordinary bouncing ball is the union of the lines $x = 0$ (an "infinitely steep potential" because the ball cannot go below the ground) and $y = kx$ (from gravitational potential energy). So the **potential function's shape is not parabolic** – this is a useful way of thinking about our system!

In particular, we can construct a potential function that is parabolic for $x > 0$ and infinite at $x = 0$ (think of half a cereal bowl). This gives a motion that is periodic, with period half that of the ordinary case, which is still not harmonic.

Example 19

If our potential is shaped as $|x|$ instead of x^2, we can think of this as a bouncing ball under gravity, except when it hits 0, it gets inverted, and so does gravity. Since the potential is still not quadratic, this periodic motion **is** dependent on amplitude.

(One way to understand this situation is that the cusp at $x = 0$ has infinite curvature, but that's hard for us to really visualize right now.)

Problem 20

Suppose we want to compute the period of a pendulum, but we do not use the small angle approximation. Then is the period faster or slower for a larger amplitude?

We're thinking of our potential as quadratic, but it's actually a cosine function! Thus, the potential is less steep, so the motion will actually have **less restoring force** than in the ideal (quadratic) case. Thus, the period will be **slower** for larger amplitudes.

5 September 13, 2018

Pset 1 is due tomorrow, and pset 2 will be posted online today.

5.1 Review

Last time, we found that energy is conserved in a simple harmonic oscillator and is constantly being converted between kinetic and potential energy. We were also able to distinguish between different behaviors based on the damping force magnitude (in which case energy is being dissipated due to drag).

Definition 21

Let ω_0 be the undamped frequency of an oscillator, and let Γ be the drag constant. Define the **quality factor** $Q = \frac{\omega_0}{\Gamma}$.

Then critical damping occurs at $Q = 0.5$, with overdamped motion for all $Q < 0.5$. Simple harmonic motion assumes $Q = \infty$ (and $\Gamma \to 0$, meaning there is no damping).

5.2 Adding the driving force

We'll now add another element to our system. Consider the rod from last time with the additional drag force, and now add an **additional torque**

$$\tau_{\text{Drive}} = d_0 \cos \omega_d t.$$

Then our total torque becomes

$$\tau(t) = \tau_g(t) + \tau_{\text{Drag}}(t) + \tau_{\text{Drive}}(t),$$

so if we plug this into $\tau = I\alpha$, we end up with the equation of motion

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = \frac{d_0}{I}\cos\omega_d t,$$

where we should recall that we have actual values for Γ, ω_0: $\Gamma = \frac{3b}{ml^2}$ and $\omega_0 = \sqrt{\frac{3g}{2l}}$. Define $f_0 = \frac{d_0}{I}$ for simplicity, which gives us the following:

Definition 22

The **standard form** for a (damped, driven) harmonic oscillator is

$$\boxed{\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = f_0 \cos \omega_d t}.$$

Fact 23

A poll: will the resulting frequency of the damped oscillator be ω, ω_d, or neither of them? 2 say ω, 8 say ω_d, and 30 say neither.

To figure this out, let's solve the differential equation, again using our complex exponential function. Let $z(t)$ be the "exponential version" of our solution $\theta(t)$: then the differential equation that we need to solve becomes

$$\ddot{z} + \Gamma\dot{z} + \omega_0^2 z = f_0 e^{i\omega_d t}.$$

We guess that our solution is of the form $Ae^{i(\omega_d t - \delta)}$. (Just kidding, this isn't a guess – Professor Lee already knows the correct form of the solution.) The idea is that we'll get some nice cancellation of the exponential term, since the equation becomes

$$Ae^{i(\omega_d t-\delta)}(-\omega^2 + i\omega_d\Gamma + \omega_0^2) = f_0 e^{i\omega_d t} \implies \boxed{A(-\omega^2 + i\omega_d\Gamma + \omega_0^2) = f_0 e^{i\delta}}.$$

This is a complex-valued equation, which is secretly two real-valued equations in two variables! We can now solve for A, δ, first by splitting both sides into the real and imaginary parts: we have

$$A(\omega_0^2 - \omega_d^2) + i(A\omega_d\Gamma) = f_0 \cos\delta + i(f_0 \sin\delta),$$

so we know that $\boxed{A(\omega_0^2 - \omega_d)^2 = f_0 \cos\delta}$ and $\boxed{A\omega_d\Gamma = f_0 \sin\delta}$. Squaring both and adding, we find that

$$A^2((\omega_0^2 - \omega_d^2)^2 + (\omega_d\Gamma)^2) = f_0^2,$$

and thus $\boxed{A(\omega_d) = \frac{f_0}{\sqrt{(\omega_0 - \omega_d)^2 + \omega_d^2\Gamma^2}}}$. (Just to make this clear, this is the amplitude of the motion as a function of the driving frequency ω_d.) We can also divide the real-part and imaginary-part equations to find that

$$\tan\delta = \frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2},$$

so $\boxed{\delta = \tan^{-1}\left(\frac{\Gamma\omega_d}{\omega_0^2 - \omega_d^2}\right)}$. We've now determined the values of δ and A, and to finish, just take the real part of our exponential solution, which is $A(\omega_d)\cos(\omega_d t - \delta(\omega_d))$. In other words, the **amplitude** and **phase lag** of this system are fixed, as long as we specify ω_d, the driving frequency.

But wait – there's no free parameters, so this can't be the actual answer to our question. (We need a way to deal with initial conditions.) This is because what we've found is just a **particular solution** θ_p of the **inhomogeneous** differential equation. The **general solution** of this differential equation will be of the form

$$\theta(t) = \theta_p + c_1\theta_1 + c_2\theta_2$$

where θ_1, θ_2 are solutions of the same harmonic oscillator equation, except with **no driving force.** This is known as the **complementary solution**.

Fact 24

The particular solution θ_p is also called the **steady-state** solution, since the complementary solution $c_1\theta_1 + c_2\theta_2$ will die out as $t \to \infty$. In other words, the frequency of the driving force will usually win out!

There's some even more interesting behavior going on here:

Proposition 25

A driving force with small oscillations can increase the amplitude of a pendulum dramatically, and a very large amplitude or high-frequency oscillation can do very little to the amplitude.

To understand this, we can look more carefully at the expressions for $A(\omega_d)$ and $\tan\delta$. As $\omega_d \to 0$, $A \to \frac{f_0}{\omega_0^2}$, and there is no phase difference as $\tan\delta = 0$. On the other hand, if $\omega_d \to \infty$, then $A(\omega_d) \to 0$, and $\tan\delta \to 0$ as well. Intuitively, the integral of force over any period is 0, so nothing can actually happen consistently enough to influence the motion.

The most interesting case comes when $\omega_d \to \omega_0$. Now $A(\omega_d) = \frac{f_0}{\omega_0\Gamma}$ – notice that if Γ is small, this amplitude can be really large! This is called **resonance behavior**, and there is actually a local maximum for the amplitude if we plot A versus ω_d – not exactly at ω_0, but close. And if we plot the phase $\delta(\omega_d)$, ω_d starts at 0 and goes to π.

Fact 26

80 hertz is the resonance frequency of the human eye.

Example 27

All objects have a resonance frequency in some sense. This means that if we produce a sound wave at the frequency of a glass, the glass may shatter.

A loud sound was played at the resonant frequency of a glass at the front of the room. Unfortunately, that glass did not actually break. It's okay – we can wait for the last class, where we'll try again!

5.3 Summary

In a damped driven oscillator, the transient (damped) behavior dies out, and the driving force will win at the end. This driven behavior is often called the steady state oscillation (or solution). There are also many situations in which **resonance** can occur, and those can have very important applications.

Next time, we'll start dealing with more complicated dynamics: we'll have multiple oscillators talk to each other.

6 September 17, 2018 (Recitation)

We'll start by discussing an idea from the problem set. Buildings often vibrate, and that puts forces on masses inside the building. But there is a concept known as **vibration isolation**, where we can put objects on springs to cancel out the vibration to some degree.

One question that was asked: how do we determine the signs in an RLC circuit? Notice that we're comparing our RLC circuit equation

$$L\ddot{Q} + R\dot{Q} + \frac{1}{C}Q = \varepsilon(t)$$

to the spring-mass system

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = F(t).$$

we know that the left hand side of the spring-mass equation has plus signs, because the restoring forces should both be in the **opposite** direction to the displacement. So the signs are correct in the spring-mass version. And since RLC circuits are harmonic oscillators, we know the left-hand side of the RLC equation also has the right signs. And finally, how do we figure out if there is a positive or negative sign for $\varepsilon(t)$? We start off with a free-body diagram and calculate $m\ddot{x} = F$, so we can start with $L\ddot{Q} = \varepsilon$ and figure it out.

There's a better way to go through the derivation, though! In an LC circuit, the energy in a capacitor, plus the energy in an inductor, comes out to

$$\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}L\dot{Q}^2,$$

which should be constant. Adding in dissipation from the resistor, we're losing some energy, so

$$\frac{d}{dt}\left(\frac{1}{2}\frac{Q^2}{C} + \frac{1}{2}L\dot{Q}^2\right) = -\dot{Q}^2 R.$$

Indeed, evaluating these derivatives yields the correct signs on the right hand side! And if we put in the driving ε term into the equation, that also pops out in the correct way.

We'll finish by studying some more properties of the damped harmonic oscillator. In the equation $\ddot{x} + \Gamma\dot{x} = 0$ (**viscous damping** without a Hooke spring term), we have exponential decay of the form $e^{-\Gamma t}$. But when we add in the Hooke spring term in the case of **underdamped motion**, the exponential factor is $e^{-\frac{\Gamma}{2}}$ instead.

Problem 28

Where does the factor of 2 come from (between the viscous damping and the damped harmonic oscillator)?

If we plot the kinetic energy for the viscous case (where our equation is just $\ddot{x} + \Gamma\dot{x} = 0$) as a function of time, $\frac{1}{2}m\dot{x}^2$ decays exponentially with factor 2Γ. Similarly, this means that the loss of energy $\dot{E}$ will also decay by 2Γ. But things are different for the harmonic oscillator – the loss of energy is dictated by $-F\dot{v} = -\Gamma\dot{x}^2$. Here, the **frictional loss is only half of the maximum**, because the average value of $\sin^2$ is $\frac{1}{2}$ over its period. Thus the energy will decay at half the rate it normally does – this means we get a decay of Γ for the energy, and therefore there is a decay rate of $\frac{\Gamma}{2}$ for underdamped motion. (Another way to think about this is that "half of our system's energy is stored in potential energy.)

Problem 29

In contrast, with the overdamped harmonic oscillator, we have a superposition of two exponentials $e^{-\Gamma_+ t}$ and $e^{-\Gamma_- t}$. Why are there two different time constants of decay?

Take the limit as $\Gamma \gg \omega_0$. Then the two constants Γ_+ and Γ_- approach Γ and $\frac{\omega_0^2}{\Gamma}$ (by binomial expansion).

Let's analyze these two constants separately. Notice that Γ is the viscous damping decay rate for velocity; it's what we get if we take **only the first two terms** in the spring-mass equation (without the Hooke term). Basically, in this case the $\omega_0^2 x$ term can be ignored, and we have $\dot{v} = -\Gamma v$ (viscous damping for the **velocity**).

But if the velocity is quickly reduced because Γ is large, the acceleration term $\ddot{x}$ can be neglected. (Now the friction force and the restoring force are almost equal, since the acceleration is so small.) So now we have $\Gamma\dot{x} + \omega_0^2 x = 0$, and this is viscous damping for the **position**! Notice too that the decay rate here, $\frac{\omega_0^2}{\Gamma}$, is very small, so it takes very long to change the velocity.

And this is why critically damped motion is important: we often **want to damp motion as fast as possible**. Now we understand that as Γ gets larger and larger, velocity is damped quickly, but position damping suffers as a result. So the optimal damping occurs when the velocity damping and position damping are on the same order, and that's a physical way to interpret critical damping.

7 September 18, 2018

7.1 Review

So far, we have learned how to solve **damped driven oscillators**: we separate the motion into transient behavior (exponentially dying away with time) and a steady-state solution (often sinusoidal). In particular, the steady-state solution is the harmonic oscillation which comes directly from the driving force, and it has frequency ω_d rather than the natural frequency ω_0. There is also a **resonance** frequency for most oscillating systems: if the natural frequency lines up with the driven motion, amplitude reaches a near-maximum. This kind of behavior can be seen in many different situations, from RLC circuits to particle physics.

7.2 Coupled oscillators

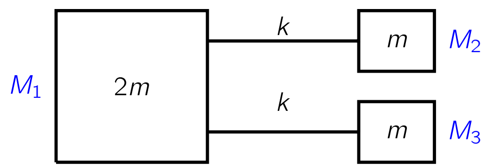
Now that we're making our system more complicated, we're going to go back to doing a simple case: let's increase the number of oscillators and remove drag and driving force from the picture. There's a lot of different situations in which this can happen: we can connect two springs in parallel, two pendulums with a spring or a rod, or we can use two degrees of freedom, and so on.

In general, coupled oscillators are very hard to understand! Instead, we'll assume that we have **very small** oscillations so that approximations are easier to make.

Example 30

Three masses are moving horizontally. One mass, M_1, has mass $2m$ and starts at position $x = 0$. Two other masses, M_2 and M_3, of mass m are at $x = \ell$, each connected by springs with constant k to the first mass, where ℓ is the relaxed length of the springs.

Here's a schematic diagram:



To analyze what's going on here, it's best to look at some simple cases:

Definition 31

A **normal mode** of a physical system is a solution where every part of the system is oscillating at the **same phase** and **same frequency**.

These are our "fundamental" situations, and it turns out that we can take linear combinations of them to get a general solution. Let x_1, x_2, x_3 be the positions of masses M_1, M_2, M_3: we can find the normal modes by direct inspection.

- **Mode A**: Say that the large mass M_1 is fixed, and let M_2 and M_3 move in opposite directions. Then the forces will always cancel out on M_1. We can solve for the positions x_2 and x_3 separately and easily, because M_1 is fixed; it's as if we have a spring-mass single oscillator with spring constant k and mass m, so the frequency of motion is $\omega_A = \sqrt{\frac{k}{m}}$.

  We might ask: aren't the two masses out of phase by 180 degrees, and isn't M_1 not oscillating? It's okay: all masses are in phase if we think of x_1 as having zero amplitude and x_2, x_3 as having opposite signs.

- **Mode B**: We can have M_2 and M_3 move together in phase, so we basically have two masses of $2m$ (M_1 is the one on the left, and $M_2 + M_3$ is the one on the right) connected by two springs of spring constant k for a total spring constant of $2k$. (Of course, in this normal mode, M_1 oscillates opposite to M_2 and M_3.)

  If each mass is displaced by Δx in opposite directions, each mass of $2m$ experiences a force of $-2k \cdot 2\Delta x$. Thus, this case gives a frequency of $\omega_B = \sqrt{\frac{2k}{m}}$ – notice that this is different from ω_A!

- **Mode C:** We can also have all masses moving at the same constant velocity. The springs will not change length, so there is no back-and-forth motion, and $\omega_C = 0$.

  Is this really oscillation, though? We can consider sinuosidal motion $x(t) = A\cos\omega t + B\sin\omega t$, and let $\omega \to 0$ to first order. Then this becomes $x(t) \approx A + B\omega t$, which is actually a **linear motion**! So constant velocity is actually just "very very slow oscillation."

Notice that we have three masses, each of which has a second-order differential equation from Newton's second law. Thus, we need six independent parameters to describe a general solution of our motion. Luckily, each of our three modes gives us two parameters, so our general solution can be found if we just add them together in a linear combination:

$$\begin{aligned} x_1 &= 0 & &+B\cos(\omega_B t+\phi_B)+(C+vt), \\ x_2 &= A\cos(\omega_A t+\phi_A) & &-B\cos(\omega_B t+\phi_B)+(C+vt), \\ x_3 &= -A\cos(\omega_A t+\phi_A) & &-B\cos(\omega_B t+\phi_B)+(C+vt) \end{aligned}$$

will be the general solution to this system by the uniqueness theorem. And we didn't even need to find or use the equations of motion from the free-body diagram!

But if we replace the mass on the left with $3m$ (instead of $2m$), the problem can't be solved so explicitly. But it turns out there's a nice, well-defined way to solve these in general using mathematics!

Example 32

Let's solve Example 30 again, but this time, we'll actually find the equations of motion and try to do the problem analytically.

Again, let x_1, x_2, x_3 be the displacements from the three masses' equilibrium position. There are two spring forces acting on M_1:

$$2m\ddot{x}_1 = F_{\text{net}} = k(x_2 - x_1) + k(x_3 - x_1) \implies \boxed{2m\ddot{x}_1 = -2kx_1 + kx_2 + kx_3}$$

and similarly, we can find equations for the other two masses:

$$m\ddot{x}_2 = k(x_1 - x_2) \implies \boxed{m\ddot{x}_2 = kx_1 - kx_2}$$

$$m\ddot{x}_3 = k(x_1 - x_3) \implies \boxed{m\ddot{x}_2 = kx_1 - kx_3}$$

The key insight is to **write these equations together as a matrix** – specifically, they can be written as $M\ddot{X} = -KX$, where M, K are now **matrices** and X is now a **column vector**. Then we can verify that

$$M = \begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}, K = \begin{bmatrix} 2k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix}$$

satisfies the equation $M\ddot{X} = -KX$ for $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

But we still need to be a bit clever to solve this, and we'll do this by again looking at the normal modes. Let's write our matrix in complex form: $X_j = \text{Re}(Z_j)$, and let's guess (but not really) our solution will be of the (normal mode) form

$$Z = e^{i(\omega t+\phi)} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}.$$

Then by definition, all three masses will oscillate at the same frequency ω and phase ϕ, just with amplitudes A_1, A_2, A_3. Since we have an exponential, $\ddot{Z} = -\omega^2 Z$, so we just want to make sure that $M\ddot{Z} = M\omega^2 Z = KZ$ (lthis part should look familiar). This means that (taking the real part, so Z is replaced with A again)

$$\omega^2 A = M^{-1}KA \implies \boxed{\left(M^{-1}K - \omega^2 I\right) A = 0}.$$

Notice that $M^{-1}K$ is a kind of interaction matrix: if it is not a diagonal matrix, then there is coupled behavior, and the oscillators are talking to each other.

So normal modes actually have to do with **eigenvectors**! We want ω^2 to be an eigenvector of $M^{-1}K$, which means we want

$$\det\left[M^{-1}K - \omega^2 I\right] = 0$$

for our matrices M, K. Luckily, M^{-1} is easy to calculate: M is diagonal, so

$$M^{-1}K = \begin{bmatrix} \frac{1}{2m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} 2k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix}$$

and after a lot of calculation (expanding out the determinant explicitly), letting $\frac{k}{m} = \omega_0^2$, we want

$$\det(M^{-1}K - \omega^2 I) = 0 \implies \left(\omega_0^2 - \omega^2\right)\left(\omega_0^4 - 2\omega_0^2\omega^2 + \omega^4 - \omega_0^4\right) = 0 \implies \omega^2(\omega_0^2 - \omega^2)(\omega^2 - 2\omega_0^2) = 0,$$

and this gives us our eigenvalues: $\boxed{\omega = 0, \omega_0, \sqrt{2}\omega_0}$. Those correspond exactly to the normal modes that we found earlier, and now we have a general method for solving coupled oscillator problems!

Next time, we will identify special forms of motion in this model, and we'll take a look at the beat phenomenon and driven oscillation.

8 September 19, 2018 (Recitation)

We'll start with an idea that was briefly mentioned in class:

Proposition 33

Suppose there are N masses in d dimensions in a coupled system. Then we have Nd degrees of freedom, and each of those will give us a second order differential equation.

For example, given N masses connected by a network of springs, assume that there exists an equilibrium position for everything simultaneously, and we only care about small pertubations from the equilibrium. Then all forces will be linear; for example, in three-dimensions, we have equations of the form

$$m\ddot{x} = \sum_{i=1}^{N} a_i x_i + b_i y_i + c_i z_i.$$

In general, these can be written in matrix form

$$M\ddot{X} = KX \implies \ddot{X} = \left(M^{-1}K\right)X,$$

where X is a $3N$-dimensional vector and $M^{-1}K$ is a $3N$ by $3N$ matrix. Just to review, here's where the normal modes come in! If we have all components of X oscillating at the same phase and same frequency ω, then

$$\ddot{X} = \boxed{\omega^2 X = M^{-1}KX}$$

becomes an eigenvalue problem. And we care about these normal modes because **we can write any solution as a linear combination of those normal modes** (by the existence theorem)!

There is one normal mode that is important: sometimes we can translate the whole system at a constant velocity (center-of-mass motion). This has frequency $\omega = 0$, and here's one way to think about why that's true: a "very soft" rubber band with very low spring constant has almost no force, and this corresponds to a very small ω. Alternatively, $A \sin \omega t \to A\omega t$ as ω gets smaller. This means that it is important to distinguish the case where we have **free space** (conservation of linear and angular momentum) from the other case!

We'll close with a graphical way to think about the damped driven harmonic oscillator. For a motion of the form $x(t) = x_0 \cos(\omega t + \phi)$, represent that motion as a (rotating) vector of unit length (called a **phasor**) $e^{i\omega t + \phi}$. Then the velocity vector $v(t)$ is ω times as long as $x(t)$, and it is positioned 90 degrees counterclockwise of x. Finally, acceleration will have magnitude ω^2 times as long as x, and will be 180 degrees out of phase.

Then if we're trying to solve

$$\ddot{x} + \Gamma \dot{x} + \omega_0^2 x = \frac{F}{m} e^{i\omega t} \implies x = Ae^{i(\omega t - \delta)},$$

in the complex plane, we can imagine rotating x around until the two vectors on the left and right side coincide!

9 September 20, 2018

Tomorrow is a student holiday (career fair), so the second problem set is now due next Monday instead of Friday. Also, all lecture notes and slides are posted on the 8.03 website, and some additional links and extra resources are also posted there.

9.1 Review

Last time, we learned how to write down the equation of motion for a system of coupled oscillators in matrix form. After doing this, we can find the relative amplitudes of the eigenvectors, which correspond to the normal modes. In fact, all general solutions are linear combinations of normal modes – the key point is that there's always simple harmonic motion in a coupled system.

9.2 A slightly harder example

Example 34

Consider two pendulums of mass m with massless strings of length ℓ, connected by a spring with constant k at relaxed length ℓ_0. This system is placed on earth, and we displace the right hand side mass by some small amount x_0. There is no initial velocity for either spring.

If we run the experiment, we see that one pendulum rocks back and forth, but it slowly stops as the other starts to rock back and forth. In other words, kinetic energy is propagating back and forth between the two oscillators. Let's translate this to math!

Remember that we should always **define our coordinates relative to the equilibrium position**. Let x_1, x_2 be the displacements of the two masses – our initial conditions are that $x_1(0) = 0$, $\dot{x}_1(0) = 0$, $x_2(0) = x_0$, and $\dot{x}_2(0) = 0$.

As always, we use a force (free-body) diagram to start writing equations. Define the x-direction to be horizontal and the y-direction to be positive as we go up (against gravity). On each spring, there are three forces: the tension force from the pendulum of magnitude T, the gravitational force $-mg\hat{\mathbf{j}}$, and the spring force $F_s = k(x_2 - x_1)\hat{\mathbf{i}}$. Let θ_1, θ_2 be angle displacements of the pendulum from the vertical (positive towards the right). Thus, in the x and y-direction on mass 1,

$$m\ddot{x} = k(x_2 - x_1) - T_1 \sin\theta_1, m\ddot{y} = T_1 \cos\theta_1 - mg,$$

and by the small angle approximation for cosine,

$$m\ddot{x} = k(x_2 - x_1) - T_1 \sin\theta_1, m\ddot{y} = T_1 - mg.$$

But with this small-angle approximation, we're essentially ignoring the y-direction of motion, so $m\ddot{y} = 0 \implies T_1 = mg$, and also $\sin\theta_1 = \frac{x_1}{L}$. Thus,

$$m\ddot{x}_1 = -mg\frac{x_1}{L} + k(x_2 - x_1) \implies \boxed{m\ddot{x}_1 = -\left(k + \frac{mg}{l}\right)x_1 + kx_2}.$$

Similarly, we can get an equation for x_2: it turns out to be

$$\boxed{m\ddot{x}_2 = kx_1 - \left(k + \frac{mg}{l}\right)x_2}.$$

So writing these equations in our matrix form, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $K = \begin{bmatrix} k + \frac{mg}{l} & -k \\ -k & k + \frac{mg}{l} \end{bmatrix}$ (remember to take negative signs)!, and $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, so we now have $M\ddot{X} = -KX$. We now skip ahead to the eigenvalue problem: recall that our trial solution tells us that for a normal mode A, $\ddot{A} = -\omega^2 A$, so we want to find the eigenvalues that satisfy

$$\omega^2 A = M^{-1}KA = \begin{bmatrix} \frac{k}{m} + \frac{g}{l} & -\frac{k}{m} \\ -\frac{k}{m} & \frac{k}{m} + \frac{g}{l} \end{bmatrix} A.$$

If we let $\omega_p^2 = \frac{g}{l}$ and $\omega_s^2 = \frac{k}{m}$, we want

$$\det \begin{bmatrix} \omega_s^2 + \omega_p^2 - \omega^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 + \omega_p^2 - \omega^2 \end{bmatrix} == 0.$$

This happens when (the determinant is a difference of squares) $|\omega_s^2 + \omega_p^2 - \omega^2| = \omega_s^2$, so either $\omega = \omega_p$ or $\omega = 2\omega_s + \omega_p$.

We have the eigenvalues, so now it is time to find the eigenvectors. When $\omega^2 = \frac{g}{l}$, we want

$$(M^{-1}K - \omega^2 I)A = \begin{bmatrix} \omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & \omega_s^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

so the eigenvector is any multiple of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and in the other case, we want

$$(M^{-1}K - \omega^2 I)A = \begin{bmatrix} -\omega_s^2 & -\omega_s^2 \\ -\omega_s^2 & -\omega_s^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0$$

so the eigenvector is any multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

And in this case, the normal modes are simple enough that we can describe them explicitly. $\omega = \omega_p$ is the normal mode where the two pendulums are exactly side by side in phase – it is as if the two are moving independently, since the spring is never stretched! The other case, $\omega = \sqrt{\omega_p^2 + 2\omega_s^2}$, corresponds to the situation where the masses are moving in opposite directions, and the frequency is larger because the restoring force is stronger.

So let's write down our general solution. Any solution $x(t)$ is of the form

$$X = \text{Re}(Z) = \text{Re}(Ae^{i(\omega t + \phi)}).$$

Thus, we can write the first normal mode in the vector form

$$X^{(1)} = \cos(\omega_1 t + \phi_1) \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(where $\omega_1 = \sqrt{\frac{g}{l}}$), and we can write the second normal mode as

$$X^{(2)} = \cos(\omega_2 t + \phi_2) \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(where $\omega_2 = \sqrt{\frac{g}{l} + \frac{2k}{m}}$). Then the general form will be

$$X = c_1 x^{(1)} + c_2 x^{(2)}.$$

In non-vector form, this is

$$\begin{aligned} x_1 &= \alpha \cos(\omega_1 t + \phi_1) + \beta \cos(\omega_2 t + \phi_2) \\ x_2 &= \alpha \cos(\omega_1 t + \phi_1) - \beta \cos(\omega_2 t + \phi_2) \end{aligned}$$

and we have our four free parameters: $\alpha, \beta, \phi_1, \phi_2$. Indeed, having four parameters exactly matches with us having four initial conditions, and this is because we have two second-order differential equations! With our initial conditions, it turns out $\alpha = \frac{x_0}{2}, \beta = -\frac{x_0}{2}, \phi_1 = \phi_2 = 0$.

What's amazing is that the matrix $M^{-1}K$ tells us how the individual components of the system interact with each other. And the eigenvalues give the frequencies where the system behaves as nicely as possible!

If we plug in all our free parameters, we have that

$$\boxed{x_1(t) = \frac{x_0}{2}(\cos(\omega_1 t) - \cos(\omega_2 t))}, \boxed{x_2(t) = \frac{x_0}{2}(\cos(\omega_1 t) + \cos(\omega_2 t))}$$

and using the sum to product formula,

$$\boxed{x_1(t) = -x_0 \sin\left(\frac{\omega_1 + \omega_2}{2} t\right) \sin\left(\frac{\omega_1 - \omega_2}{2} t\right)}, \boxed{x_2(t) = x_0 \cos\left(\frac{\omega_1 + \omega_2}{2} t\right) \cos\left(\frac{\omega_1 - \omega_2}{2} t\right)}.$$

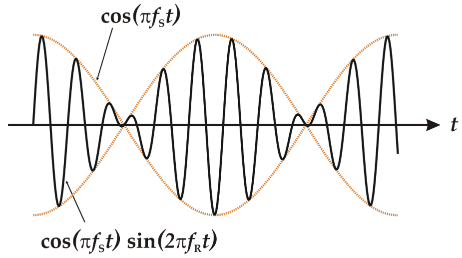
9.3 The beat phenomenon

Now suppose we take $\omega_1 \approx \omega_2$ (for example, make the masses large, so the force from the springs are small compared to the force from the pendulums). Then tracing out the path $x(t)$ for one of the two masses, we essentially get a grouped cosine wave: fast oscillations are "filling out" the curve from the smaller frequency wave! It seems that a slower motion is modulating the oscillation.

Definition 35 (Beat phenomenon)

When we add two sinusoidal waves together, we get a product of two sine waves, where the **envelope** (slower frequency) modulates the **carrier** (faster frequency).

Here's a picture (from Google) of what this might look like: the outside sine wave is the envelope.



Specifically, we can find the period of these two shapes directly: $T_{\text{carrier}} = \frac{2\pi}{\left|\frac{\omega_1+\omega_2}{2}\right|}$, and $T_{\text{beat}} = \frac{2\pi}{|\omega_1 - \omega_2|}$ (half the normal period, since the envelope looks symmetric on the positive and negative parts).

Fact 36

We may have heard of "beats" as a concept in music, but we can notice that the two-pendulum system is also exhibiting an envelope-carrier behavior. This means the beat phenomenon occurs in mechanical waves too, not just in sound waves!

It turns out that if we follow the motion traced out by path $(x_1(t), x_2(t))$ along a **normal axis** (along the direction of an eigenvector), we can see simple harmonic motion. This is because eigenvectors are a way to diagonalize our matrix, and viewing motion along only those directions will **decouple** the motion (so there isn't interaction between them)!

Next time, we will **excite** normal modes with a driving force, and we'll deal with an **infinite** number of oscillators as well.

10 September 24, 2018 (Recitation)

This recitation is being taught by **Pearson Miller**, the graduate TA for this class.

Let's talk through the main ideas of this class so far. We've been looking at damped, driven harmonic oscillators of the form

$$m\ddot{x} + b\dot{x} + kx = F_0\cos(\omega_d t)$$

which will always have a solution of the form

$$x(t) = x_{\text{trans}}(t) + x_{\text{steady}}(t)$$

where the steady state solution is sinusoidal (we can write $x_{\text{steady}}(t) = A\cos(\omega_d t + \phi)$) and the transient solution will decay to 0 as long as there is some damping ($b \neq 0$). This transient motion can be of different forms depending on whether the oscillator is overdamped (exponential decay), critically damped (exponential decay times a linear factor), or underdamped (exponential decay with oscillation).

Next, we have normal modes, where we have to bash a lot more (hooray). The main idea is that we'll have two or more systems (coupled) in the form

$$m\ddot{x}_i = -\sum_j b_{i,j}\dot{x}_j - \sum_j k_{i,j}x_j + f_i\cos(\omega_d t)$$

In a lot of problems we'll solve (and a lot of problems that come up), many of these terms disappear and we have a simpler mathematical equation. (For example, we could remove the driving and drag force.) But no matter what, we can rewrite this set of equations as a single vector equation

$$M\ddot{\vec{X}} = -B\dot{\vec{X}} - K\vec{X} + \vec{F}\cos(\omega_d t),$$

where M, B, K are now matrices and X, F are vectors. Multiplying both sides by M^{-1}, which is always a diagonal matrix (because it represents the masses of individual objects in our system),

$$\ddot{\vec{X}} = -M^{-1}B\dot{\vec{X}} - M^{-1}K\vec{X} + M^{-1}\vec{F}\cos(\omega_d t).$$

Now a normal mode must be of the form

$$\vec{n}_i = \vec{A}_i\cos(\omega t),$$

(basically, we have eigenvectors), so then $\ddot{X} = -\omega^2 X$ gives us an eigenvalue problem.

Example 37

A mass is free to move on a horizontal track, and another mass is hanging from this first mass via a pendulum of length L. How can we describe the motion of this system?

We wish to find two normal modes. Parametrize the top mass by its displacement X and the bottom mass by its angle θ from the vertical. So one normal mode is

$$\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where the whole system is just translating (and the pendulum is not moving). The other one can be found in a similar style of inspection, but let's just do it mathematically. Looking at the top mass,

$$F_{\text{net}} = m\ddot{x} = T\sin\theta,$$

where T is the tension force, and

$$\tau_{\text{net}} = mL^2\ddot{\theta} = -mgL\sin\theta,$$

since the tension force does not contribute any torque. Since we're dealing with small oscillations, we can make the small angle approximation $T = mg$, and we get the two equations

$$m\ddot{x} = mg\theta$$
$$\ddot{\theta} = -\frac{g}{L}\theta.$$

Writing $\vec{X} = \begin{bmatrix} x \\ \theta \end{bmatrix}$, we have that

$$\ddot{\vec{X}} = \begin{bmatrix} 0 & g \\ 0 & \frac{g}{L} \end{bmatrix}\vec{X},$$

and to find the eigenvalues of this interaction matrix, we need to find the values of ω^2 that give eigenvalues for our matrix, so

$$\det \begin{bmatrix} 0 & g \\ 0 & \frac{g}{L} \end{bmatrix} = 0 \implies -\omega^2 \left(\frac{g}{L} - \omega^2 \right) = 0.$$

This tells us that the two normal modes are the one of the **natural pendulum** and of **translational velocity**. One has $\omega = 0$ with eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and the other has $\omega = \sqrt{\frac{g}{L}}$ for an eigenvector of (remember to plug in ω^2 for our eigenvalue) $\begin{bmatrix} L \\ -1 \end{bmatrix}$. Both of these make sense, because the center-of-mass position should be conserved throughout the motion, and indeed that is the case here!

11 September 25, 2018

Today, we'll start by applying a driving force to the coupled oscillators from last lecture.

11.1 Summary of coupled oscillators

There are many different situations where we can have oscillators talking to each other: LC circuits that are coupled together, pendulums attached by a spring, spring-mass systems with multiple masses, and so on.

If we arbitrarily excite the system, the motion will not necessarily be harmonic. Energy will migrate, and the amplitudes of the motion will vary. However, the motion is a linear combination of the normal modes, each of which leads to harmonic motion. In those cases, the amplitudes will stay at a constant ratio, and all energy will stay in the individual components.

To solve for normal modes, first write our system as a single matrix equation of motion in the form $M\ddot{X} = -KX$. The normal modes will satisfy $\ddot{X} = -\omega^2 X$, which turns the problem into an eigenvalue problem.

Finally, when we add two harmonic waves together, where the two frequencies are similar but not identical, we will occasionally see a **beat phenomenon**, which consists of an envelope and a carrier.

11.2 A coupled spring-mass system

Example 38

Consider a spring-mass system with two masses of mass m connected to each other and to walls on either sides by springs with spring constant k.

As always, we define our coordinates with respect to the equilibrium position. Let x_1 be the displacement of the left mass and x_2 be the displacement of the right mass. Then

$$F_1 = m\ddot{x}_1 = -kx_1 + k(x_2 - x_1),$$
$$F_2 = m\ddot{x}_2 = -kx_2 + k(x_1 - x_2),$$

so writing this in matrix form,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Writing $\omega_0^2 = \frac{k}{m}$, we now have

$$\ddot{X} = -\begin{bmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{bmatrix} X.$$

As always, we guess that X is of the form $Ae^{i\omega t + \phi}$. Then $\ddot{X} = -\omega^2 X$, so we have

$$-\omega^2 A = -\begin{bmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 \end{bmatrix} A,$$

and eigenvalues ω^2 of this matrix come up when

$$\det \begin{bmatrix} 2\omega_0^2 - \omega^2 & -\omega_0^2 \\ -\omega_0^2 & 2\omega_0^2 - \omega^2 \end{bmatrix} = 0.$$

Expanding, we want $\omega^4 - 4\omega_0^2\omega^2 + 3\omega_0^4 = 0$, which means $\omega = \omega_0$ or $\omega = 3\omega_0$, corresponding to the eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ respectively.

As always, it's good to ask "what do these normal modes mean?". In one case, the two masses move in phase, and the middle spring is never stretched. In the other, the two masses move in opposite directions with an effective spring constant of $3k$. This gives the full solution with four parameters:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_0 t + \phi_1) + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(3\omega_0 t + \phi_2).}$$

11.3 Adding a driving force

Example 39

Take the setup from above, but change the right wall to have an oscillating position of $\Delta \cos(\omega_d t)$. What happens to our equation of motion?

The only thing that changes is the second equation of motion: we now have an additional term

$$F_1 = m\ddot{x}_1 = -kx_1 + k(x_2 - x_1),$$
$$F_2 = m\ddot{x}_2 = -kx_2 + k(x_1 - x_2) + k\Delta \cos(\omega_d t).$$

Writing these equations above in the matrix form $M\ddot{X} = -KX + F\cos(\omega_d t)$,

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ k\Delta \end{bmatrix} \cos(\omega_d t)$$

Again, we can multiply both sides by M^{-1} to get an equation of the form $\boxed{\ddot{X} = -M^{-1}KX + M^{-1}F\cos(\omega_d t)}$. We want to find a particular solution, so we use complex notation again: write $X = \operatorname{Re} Z$. Then we want to solve the equation

$$\ddot{Z} + M^{-1}KZ = M^{-1}Fe^{i\omega_d t},$$

and we guess the solution to be of the form $Z = Be^{i\omega_d t}$ (to fit the form of the particular solution) for some real-valued vector B. (There is no phase difference, because there is no dissipation in this system!) Then

$$Be^{i\omega_d t}\left(-\omega_d^2 I + M^{-1}K\right) = M^{-1}Fe^{I\omega_d t}.$$

The exponentials cancel as expected, and we're left with the vector equation

$$\boxed{\left(M^{-1}K - \omega_d^2 I\right)B = M^{-1}F}.$$

Let's plug in the numbers we have: we want to solve for B_1, B_2 such that

$$\begin{bmatrix} \frac{2k}{m} - \omega_d^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega_d^2 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{k\Delta}{m} \end{bmatrix}.$$

We can solve this by Cramer's rule! Recall that $\frac{k}{m} = \omega_0^2$, so

$$B_1 = \frac{\det \begin{bmatrix} 0 & -\frac{k}{m} \\ \frac{k\Delta}{m} & \frac{2k}{m} - \omega_d^2 \end{bmatrix}}{\det \begin{bmatrix} \frac{2k}{m} - \omega_d^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega_d^2 \end{bmatrix}} = \frac{\frac{k^2\Delta}{m^2}}{(\omega_d^2 - \omega_0^2)(\omega_d^2 - 3\omega_0^2)},$$

and similarly

$$B_2 = \frac{\det \begin{bmatrix} \frac{2k}{m} - \omega_d^2 & 0 \\ -\frac{k}{m} & \frac{k\Delta}{m} \end{bmatrix}}{\det \begin{bmatrix} \frac{2k}{m} - \omega_d^2 & -\frac{k}{m} \\ -\frac{k}{m} & \frac{2k}{m} - \omega_d^2 \end{bmatrix}} = \frac{\frac{2k^2\Delta}{m^2} - \frac{k\Delta\omega_d^2}{m}}{(\omega_d^2 - \omega_0^2)(\omega_d^2 - 3\omega_0^2)}.$$

Notice that this time because the equation is inhomogenous, the magnitude of our vector B does matter, not just the ratio. But we can still consider

$$\frac{B_1}{B_2} = \frac{\frac{k}{m}}{\frac{2k}{m} - \omega_d^2}.$$

If $\omega_d^2 = 3\omega_0^2$, so we're exciting the first normal mode, then Cramer's rule tells us that B_1 and B_2 both go off to infinity, since the denominator is 0 (this is just **resonance behavior**). However, still notice that $\frac{B_1}{B_2} = -1$; this amplitude ratio matches the normal mode ratio! Similarly, if $\omega_d^2 = \omega_0^2$, $\frac{B_2}{B_1} = 1$.

And our general solution for this system with the driving force is the same solution as last time, but we add on another term:

$$\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\omega_0 t + \phi_1) + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(3\omega_0 t + \phi_2) + \begin{bmatrix} B_1(\omega_d) \\ B_2(\omega_d) \end{bmatrix} \cos(\omega_d t).}$$

We've found that our normal modes occur at $\omega = \omega_0$ and $\omega_0\sqrt{3}$. But here's another interesting fact: at $\omega_0\sqrt{2}$, we have $B_2 = 0$, so we can actually tune our frequency such that one of the masses will not move! (The right wall and the left mass will both oscillate, but the forces on the right mass always cancels out completely.)

It turns out this has a useful application:

Example 40

Taipei 101 is in an unfortunate place called Taiwan, which has 2200 earthquakes a year, 200 of which we can feel. How does Taipei 101 prevent earthquakes and oscillation from making it fall over? There is a 660 metric ton ball in the middle of the building, which acts as a **tuned mass damper**. This way, during an earthquake, the ball will oscillate, not the building!

Next time, we will talk about symmetries of systems and eigenvectors, as well as how we can use them to help solve problems.

12 September 26, 2018 (Recitation)

Let's first talk about mutual inductance, which is the electromagnetic system analogous to coupled oscillators. If two loops (or solenoid-like components) are near each other, then the voltage induced is proportional to the change in magnetic field, which is the change in current. Thus we can write

$$u_j = M_{ji}\dot{I}_i,$$

where M_{ji} is referred to as the **mutual inductance**. This is probably more familiar in the form where $i = j$; then $M_{ii} = L$ is referred to as the **self-inductance**.

So in our homework problem, we have (for the coupled LC circuits)

$$\frac{Q_1}{C_1} + L_1\dot{I}_1 + M_{12}\dot{I}_2 = 0$$

(How do we know the signs work out? It's hard to justify and probably not worth talking about right now.) One interesting question: is $M_{12} = M_{21}$? To answer this, we try to relate mechanics to electromagnetism. A general harmonic oscillator vector equation where $M\ddot{X} = -KX$ has a **symmetric matrix** K, because energy is generally in the form

$$E = \sum \frac{1}{2}K_{ij}x_i x_j,$$

where an entry K_{ij} in our matrix will be related to the derivatives with respect to i and j. Since the order of mixed partials doesn't matter, $K_{ij} = K_{ji}$.

Perhaps we can make an analogous argument here: now looking at magnetic field energy, we know the energy density is proportional to B^2. Well, by Biot-Savart, current is generated by some combination I_1 and I_2; as a result, B^2 is going to be in the form $\sum M_{ij}I_iI_j$. Thus the total magnetic energy is a bilinear expression in currents as well, so we do indeed have $M_{ij} = M_{ji}$, and our matrix M **is symmetric**.

So what's the equivalent of potential and kinetic energy in a general system? Usually, they are related to the **coordinates and derivatives**, respectively. Since the energy in a capacitor is proportional to Q^2, and the energy in an inductor is proportional to $\dot{Q}^2$, the capacitor is the potential energy, and the inductor is the kinetic energy.

So the terms like M_{ij} in front of second derivatives generally have to do with "masses," and in some cases (such as with mutual inductance), the "mass matrix" may not be diagonal, but it is still symmetric!

13 September 27, 2018

We are getting an extension on the problem set; the next one will also be shorter and only contain 3 problems.

13.1 Review

We learned last time that driving force can **excite** certain normal modes if the driving frequency matches the normal mode frequency. This is not that surprising; it's like the resonance case with a single oscillator. In general, though, the solution will have a particular solution (related to the driven force) plus a homogeneous solution (which have unknown coefficients and depend on initial conditions).

Example 41

Tuning forks can excite other tuning forks of the same frequency. This actually makes them vibrate and produce sound even if they aren't hit!

Today, we will find a way to solve for normal modes without actually knowing the details of $M^{-1}K$. Here's some motivation for what we're doing: consider the coupled pendulum, as well as the coupled spring-mass system, which we studied in previous classes. In both cases, we can place a mirror in the middle, reflect the whole system, and the result will look the same! We'll soon see why this is useful.

13.2 Symmetry

Example 42

Let's say we have two pendulums attached by a spring, with coordinates x_1, x_2. Notice that if we reflect the system – that is, $x_1 \to -x_2, x_2 \to -x_1$ – the system looks the same, and all normal modes look the same.

The symmetry matrix is therefore a transformation

$$X \to SX, S = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

So we know that for our vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, $SX = \tilde{X}(t) = \begin{bmatrix} -x_2(t) \\ -x_1(t) \end{bmatrix}$.

Fact 43

Suppose a symmetry sends X to $\tilde{X}$. If X is a solution, then $\tilde{X}$ is a solution as well.

Definition 44

The **commutator** of two operators (matrices) A and B is $[A, B] = AB - BA$. If $[A, B] = 0$, then A and B **commute**.

Theorem 45

Suppose that a symmetry matrix S has **all different eigenvalues**. If $[M^{-1}K, S] = 0$, then the eigenvectors of S are also the eigenvectors of $M^{-1}K$. (However, they can have different eigenvalues.)

Proof. Let's assume that X is a solution to the equation $\boxed{\ddot{X}(t) = -M^{-1}KX(t)}$, and so is $\tilde{X}(t) = SX(t)$. Then

$$\ddot{\tilde{X}}(t) = -M^{-1}K\tilde{X}(t)$$

because $\tilde{X}$ is also a solution. Substituting in $\tilde{X} = SX$, we find that

$$S\ddot{X}(t) = -M^{-1}KSX(t),$$

but now taking the boxed equation and multiplying both sides by S,

$$S\ddot{X}(t) = -SM^{-1}KX(t).$$

Thus, equating the right hand sides, we know that $M^{-1}KS = SM^{-1}K$ (since $x(t)$ is not always zero), and therefore $[S, M^{-1}K] = 0$.

Now, we show that the two operators share eigenvectors. Let $X(t) = A\cos(\omega t + \phi)$, and plug this back into the initial equation: we know that (just like with any other solution)

$$\omega^2 A = M^{-1}KA.$$

Let's say that A is an eigenvector of S with eigenvalue β, meaning that $SA = \beta A$ for some β. We're assuming here that $[S, M^{-1}K] = 0$, and all eigenvalues of our matrix S are different. Then

$$SM^{-1}KA = M^{-1}KSA = M^{-1}K\beta A = \beta M^{-1}KA,$$

so $M^{-1}KA$ is an eigenvector of S with eigenvalue β. But if all β are distinct, $M^{-1}KA$ must be proportional to A (they must be scalar multiples of each other). Thus,

$$M^{-1}KA = \omega_n^2 A$$

for some ω_n. And since S and $M^{-1}K$ have the same size, the eigenvectors of $M^{-1}K$ are just the eigenvectors of S if we repeat this argument for each eigenvector! □

13.3 A concrete example: infinite coupled oscillators

Example 46 (Hard to solve analytically)
Consider an infinite system of masses in a horizontal line. They are all on pendulums, and they are all connected by springs. All masses have the same mass, and all springs have the same spring constant.

The matrix $M^{-1}K$ will have some (positive) entries on the diagonal, as well as a different (negative) entry on all off-by-one entries. So it's hard to solve that eigenvalue problem on its own.

Instead, let's do a simpler version.

Example 47
Consider infinitely many masses in the horizontal direction, indexed by the integers. Each mass has mass m, and all masses with labels $\{j, j+1\}$ are connected by a spring with spring constant k and relaxed length a.

Label the displacement of masses $\cdots, x_{j-1}, x_j, x_{j+1}, \cdots$. Focus on mass x_j; the free-body calculations tell us that

$$m\ddot{x}_j = k(x_{j-1} - 2x_j + x_{j+1}).$$

Let's take some normal mode of the form $x_j = A_j \cos(\omega t + \phi) = \text{Re}\, A_j e^{i(\omega t + \phi)}$. If we write out our entries for the

$M^{-1}K$ matrix, we will have

$$M^{-1}K = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & ⋰ \\ \cdots & \frac{2k}{m} & -\frac{k}{m} & 0 & 0 & \cdots \\ \cdots & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & 0 & \cdots \\ \cdots & 0 & -\frac{k}{m} & \frac{2k}{m} & -\frac{k}{m} & \cdots \\ ⋰ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We can now solve this using **space translation symmetry**. Move the entire system by a units (one relaxed spring length) to the right; now everything still looks the same, and any solution X is taken to a solution $\tilde{X}$! So this is a valid symmetry, and our space translation $A' = SA$ takes the form

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

To find the normal modes of our system, let's find the eigenvectors of S. Let $SA = \beta A$. Then

$$A = \begin{bmatrix} \vdots \\ A_j \\ A_{j+1} \\ \vdots \end{bmatrix}, \quad SA = \begin{bmatrix} \vdots \\ A_{j+1} \\ A_{j+2} \\ \vdots \end{bmatrix},$$

so for an eigenvector A, $A_{j+1} = \beta A_j$ for all β. So now if $A_0 = 1, A_1 = \beta, A_2 = \beta^2$, and so on. The magnitudes in a normal mode are only determined by this β, and there seem to be an infinite number of possible values for β!

Recall that our symmetry argument tells us that $[S, M^{-1}K] = 0$, so the eigenvectors are the same for S and $M^{-1}K$. All that is left is to evaluate $M^{-1}KA = \omega^2 A$ to find the eigenvalues (and therefore the frequencies of our normal modes).

Fact 48

The important thing is that we've solved for the eigenvectors for **every system** with space-invariant symmetry! We just need the eigenvalues for this specific case, and here's the first time we actually use our equation of motion.

Focus on the jth term of both sides. Defining $\omega_0^2 = \frac{k}{m}$, we have the left side equal to

$$-\frac{K}{M}A_{j-1} + \frac{2k}{m}A_j - \frac{k}{m}A_{j+1} = \omega_0^2(-A_{j-1} + 2A_j - A_{j+1})$$

and setting this equal to $\omega^2 A = \omega^2 \beta^j$, we find that

$$\omega^2 = \omega_0^2 \left(-\frac{1}{\beta} + 2 - \beta \right)$$

and thus, every β that we pick gives us a specific value of ω.

But notice that $\beta = b, \frac{1}{b}$ give the same eigenvalue, and if $|\beta| \neq 1$, things will explode in either the ∞ or $-\infty$ direction. So to have a **physically feasible system**, we actually need $\beta = e^{i\theta}$ for some $\theta \in \mathbb{R}$. So let $\beta = e^{ika}$ (where

a is still the relaxed length of the spring). Plugging this in,

$$\begin{aligned}\omega^2 &= \omega_0^2\left(-e^{ika} + 2 + e^{ika}\right)\\ &= \omega_0^2(2 - 2\cos(ka))\end{aligned}$$

and this gives us many different possible frequencies: we can have $\omega = \omega_0\sqrt{2(1-\cos(ka))}$ for any $k \in \mathbb{R}$. But notice that $k, -k$ give the same ω; that is, $A_j = \beta^j = e^{ijka}$ and e^{-ijka} have the same value of ω^2. Adding these together, we get $\boxed{A_j = 2\cos(jka)}$. And at the end of the day, what we've found is that **space translation actually produces sinusoidal waves**! If we look at our system as a whole, the amplitudes of the masses trace out a cosine shape.

Next time, we will look at more examples of infinite systems, and we'll find applications of this to smaller, more finite systems. Finally, we'll look at a **continuous infinite system**.

14 October 1, 2018 (Recitation)

We'll start by summarizing some of the material from lecture. Recall that in our matrix $M^{-1}K$, the eigenvalues ω^2 correspond to the squares of the frequencies of our normal mode oscillation, and the eigenvectors give us the amplitude ratios. But there is an interesting fact: **two matrices that commute can be simultaneously diagonalized by an eigenvector basis.** The idea is that we make a basis transformation (to the eigenvector basis), and this decouples the motion in both cases.

Well, S and $M^{-1}K$ often commute. The eigenvalues of a symmetry matrix often have magnitude 1 (because symmetries like reflection or rotation can't change the length of a vector), while the eigenvalues of $M^{-1}K$ can be something else. But remember that the simultaneous diagonalization tells us about **eigenvectors**, not eigenvalues; it's generally easy to find the eigenvectors of S and then find the corresponding eigenvalues in $M^{-1}K$!

Example 49

Suppose we have two masses m_1 and m_2 related by mirror symmetry (which in one dimension is the same as rotating by 180 degrees).

Then S is really an operator in space, which sends $x_1 \to -x_2$ and $x_2 \to -x_1$. Since we have an explicit way to describe our symmetry, that tells us about the eigenvectors (normal modes) of such a physical system without needing to do much more work.

Next question: what is a **dispersion relation**? (This is some vocabulary on our problem set.) In a vacuum, light of all frequencies travel at the same time. But if we put light in a fiber or some other medium with an index of refraction, that index of refraction depends on the frequency! So we will often see red light before blue light, since red light travels faster in glass (for example) than blue light.

Specifically, we know that if $n(\omega)$ is an index of fraction depend on our frequency ω, we can write down the equation

$$c_{\text{fiber}} = \frac{c_{\text{vacuum}}}{n(\omega)}.$$

In a vacuum, we can say that $\omega = ck$, where $k = \frac{2\pi}{\lambda}$ is the wave number. This is the case where we have no dispersion; with dispersion, the equation becomes $\omega = \frac{c}{n(\omega)}k$. (And the function $\omega(k)$ can depend on the vector $\vec{k}$ as well.)

But in quantum mechanics, dispersion relations have to do with energy and momentum as well. The de Broglie relation tells us the equation $p = \hbar k$, and the energy of a photon is $E = \hbar\omega$. (Here, $\hbar$ is just some constant.)

Since $E = \frac{p^2}{2m}$, this is actually a **quadratic dispersion relation** between k and ω! And there is something else about band structures in solids (which I don't understand, sorry). We'll talk more about dispersion relations in the following lectures though.

Example 50

Consider a system consisting of (wall)-(spring)-(mass 1)-(spring-mass 2)-(spring)-(oscillating wall).

Let's try to look at general physical descriptions and answer questions like (1) can one mass be stationary? and (2) what do we know about frequencies? One way to think about this question physically is just look at mass 2 and track its energy flow! If it never moves, then the rest of the system cannot ever move as well (since no energy moves through mass 2).

15 October 2, 2018

Professor Comin is teaching this class. We have our first exam on October 11, at the same time as usual lecture. Material covered will come from the first eight lectures.

15.1 Review

Last week, we talked about the infinite system of coupled oscillators. Symmetry is important here for finding normal modes; in fact, they are very helpful since finding the eigenvalues of $M^{-1}K$ doesn't actually make as much sense with infinitely many dimensions. In that case, we could shift the whole system forward by one fundamental unit, which gave us a symmetry matrix which we could then solve for eigenvalues and eigenvectors.

We also mentioned that if we have a symmetry matrix S and a dynamical matrix $M^{-1}K$, if those two commute (which they do in our case), $SM^{-1}K = M^{-1}KS \implies$ all eigenvectors for S are eigenvectors for $M^{-1}K$, and vice versa, if the eigenvalues of S are all different. We can also reverse this argument: if the eigenvalues of $M^{-1}K$ are all different, eigenvectors will also be the eigenvectors for S. This is a purely mathematical argument!

The reason this is so useful is that $M^{-1}K$ is a lot harder to solve for once the system has lots of dynamical variables. Writing this out mathematically, if $SA = \beta A$, we know that $SM^{-1}KA = \beta M^{-1}KA \implies M^{-1}KA$ is an eigenvector for S with eigenvalue β, so $M^{-1}KA = \alpha A$ for some α. Thus we have found that A is also an eigenvector for $M^{-1}K$!

See notes above from September 27 for how to solve the infinite spring oscillator. Here are the important points:

- The symmetry matrix S has 1s on the superdiagonal (the entries above the diagonal) and 0s everywhere else. This is known as a space-translation symmetry.
- If $A' = SA$, then $A' = \beta A$ for an eigenvector for some complex number β. Since β is the same for each component, this means that $A_j = A_0\beta^j$ for all j. We can set $A_0 = 1$ since eigenvectors can be scaled arbitrarily.
- To make sure the amplitude stays bounded in both directions, we need $|\beta| = 1$, so write it as $\beta = e^{ika}$ for some real number k. Here k is the **wave number**, which has to do with the number of waves per distance (sort of like frequency but for time).
- This means $A_j = e^{ijka}$ for some $k \in \mathbb{R}$ will produce an eigenvector or normal mode $A^{(k)}$ (where (k) is a label). We should remember to multiply by the time-component as well if we're writing out the equation explicitly!

15.2 Another infinite system of masses

Example 51

Consider an infinite number of masses attached on a string that can only oscillate vertically. There is a constant tension force T throughout the string (assume the horizontal force is negligible).

Let each mass m_j have y-coordinate y_j. If the angle of the string on the left and right of the mass are θ_L and θ_R respectively, the force in the y-direction is approximately $T(\theta_R + \theta_L)$ (where $\sin\theta \approx \theta$). But since we're using the small angle approximation, we can also use $\theta \approx \tan\theta$, which gives us a nice form for our equations of motion:

$$\boxed{m\ddot{y}_j = \frac{T}{a}\left(y_{j-1} - 2y_j + y_{j+1}\right)}.$$

Since this holds for any j, we can write the matrix equation as

$$\ddot{Y} = \frac{T}{ma}\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \iddots \\ \cdots & -2 & 1 & 0 & 0 & 0 & \cdots \\ \cdots & 1 & -2 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & -2 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & -2 & 1 & \cdots \\ \cdots & 0 & 0 & 0 & 1 & -2 & \cdots \\ \iddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} Y.$$

This is a huge mess, and trying to solve for eigenvalues directly by taking determinants isn't going to work. But remember that we already found our eigenvectors during the previous lecture! Translational symmetry tells us that our eigenvectors **have to** be of the form

$$y_j(t) = e^{ikja}e^{i\omega t},$$

so we can actually ignore this complicated matrix altogether and just solve one equation at a time. Plugging this into our boxed equation of motion,

$$-m\omega^2 e^{ikja}e^{i\omega t} = \frac{T}{a}\left(e^{ik(j-1)a}e^{i\omega t} - 2e^{ikja}e^{i\omega t} + e^{ik(j+1)a}e^{i\omega t}\right).$$

All the exponentials cancel out, and we're left with

$$-m\omega^2 = \frac{2T}{a}(\cos(ka) - 1)$$

and now we've found our eigenvalues for each k: $\boxed{\omega^2 = \frac{2T}{ma}(1 - \cos(ka))}$, and we are done with the problem!

Definition 52

The function that relates ω to k is called the **dispersion relation**.

Writing $1 - \cos(ka) = 2\sin^2\left(\frac{ka}{2}\right)$, we now have

$$\omega = \sqrt{\frac{4T}{ma}}\left|\sin\left(\frac{ka}{2}\right)\right|$$

as our relation between ω and k. Plotting this shows a cusp at $t = 0$. This function is even, so two wave numbers with the same magnitude have the same frequency ω.

We've thus found a continuous function $\omega(k)$ to describe frequencies for the normal modes. So in summary, we start with a normal mode "label" k, and we get eigenvectors $A^{(k)}$ with corresponding frequencies $\omega^{(k)}$.

The actual solutions come from the real part of our complex normal modes, plus a phase term (since the amplitude can be complex). This is the real part of $e^{ijka}e^{i\omega t}$, plus a phase, which gives $\boxed{A_j^{(k)} = A\cos(kja + \omega(k)t + \phi^{(k)}}$. (This is a traveling wave!) So our final solution can be written as some superposition of our normal modes

$$y_j(t) = \sum_k A_k \cos\left(kja + \omega(k)t + \phi^{(k)}\right)$$

(where we really should be using an integral instead of a sum), and where we impose initial conditions to say more specific things about our coefficients A_k. The maximum amplitude is attained at $kx - \omega t + \phi = 0$, so this wave travels at a **linear rate** if we fix our attention on a wave crest.

15.3 Boundary conditions

We'll put a bit more structure on our system again:

Example 53

Suppose we now have $N + 1$ equal masses in a row (so like last time, but with a finite number of masses), and the masses $y_0(t) = y_N(t) = 0$ are fixed. What can we say about our motion?

A lot of the key players in this problem look the same as the one in the previous one we were working on. So we can use results from the infinite system of masses, but we just have to impose specific boundary conditions.

Notice that $\cos(kaj + \omega t)$ and $\cos(-kaj + \omega t)$ have the same frequency, since they have opposite wave numbers. Thus, we can take a linear combination of these solutions. One of these travels to the left, and one travels to the right; this gives us a standing wave! When we add these two traveling waves together, some spots will stay still: those where the waves are oscillating completely out of phase. (These are called **nodes**.) In particular, let's try adding these opposite wave number solutions together:

$$\text{Re}(e^{i\omega t}(e^{i(kaj)} + e^{-ikaj})) = \text{Re}(e^{i\omega t}) \cdot 2\cos(kaj) = 2\cos(\omega t)\cos(kaj),$$

and we've decoupled the time and space components of our wave. At any point where $\cos(kaj) = 0$, we will have a node (which stays fixed) at all times t. We couldn't do this when we had just one normal mode for a specific frequency ω! So our trial solution here is of the form

$$y_j^{(k)} = A\cos(kaj + \phi)\cos(\omega t + \phi'),$$

and now it's time to impose our boundary conditions. For $y_0^{(k)} = 0$ to be true, we need $\cos(\phi) = 0 \implies \phi = \frac{\pi}{2}$. Then for $y_N^{(k)} = 0$ to also be true, we need $\cos\left(kaN + \frac{\pi}{2}\right) = 0 \implies kaN = m\pi \implies k = \frac{m\pi}{aN}$. **Not all** k **work now**: we can only pick integer m, and in fact picking larger and larger m will actually give redundant solutions since we have a finite number of masses.

In this problem, we worked with what we call a **fixed boundary** condition, but we can also consider the case where we have **free boundaries** – that is, we require there to be is no force at the boundary point. Then we can just impose the conditions $y_{-1}(t) = y_0(t)$ and $y_{N+1}(t) = y_N(t)$, which will give essentially the effect that we want.

16 October 3, 2018 (Recitation)

This recitation is mostly a review of what we've been covering in lecture. We have been discussing chains of coupled oscillators, which are a step towards discussing the continuous wave equation. Just like with other coupled systems, our chain can be described by a matrix $M^{-1}K$, as well as by a symmetry matrix S. When $[M^{-1}K, S] = 0$, the two matrices have the same eigenvectors. And we can usually find the eigenvectors of S more easily than $M^{-1}K$, either by guessing or solving the characteristic equation.

However, symmetry matrices usually don't change magnitudes of vectors (that is, amplitudes of motion). Thus, the magnitude of almost all eigenvalues will be 1. And in this particular case, writing this eigenvalue as $e^{i\alpha}$ (which are the only such eigenvalues that make the eigenvectors normalizable), we can let $\alpha = ka$ for some k. Now we have an eigenvalue of e^{ika}, where k is essentially the inverse wavelength (times 2π) – this concept is related to the wave number.

For the chains we've been talking about, we have a normal mode vector A (describing all of our amplitudes) such that the jth component $A_j = e^{ik(ja)}$. Well, ja is the x-position of mass j, relative to mass 0. Thus, the masses trace out the graph of e^{ikx}, where x is the x-coordinate! Now once we have our eigenvectors A, we can plug them into $M^{-1}K$ to get us our eigenvalues $-\omega^2$, and $\omega(k)$ is now the dispersion relation.

This is still not the question we care about, though. We're solving the infinite chain problem here - how do we turn this into a finite chain problem? It turns out we can maintain the same eigenmodes because of all the symmetry! If we want to impose some boundary conditions – that is, take a subset of the chain with an open or fixed end – we just need to select the relevant wave numbers k so that the traced out e^{ikx} is valid.

In particular, notice that $\omega(k) = \omega(-k)$ was always true in the infinite chain (the dispersion relation was even). As a result, we can let $\alpha e^{ikja} + \beta e^{-ikja}$ be a normal mode as well. We can now vary both k and $\frac{\alpha}{\beta}$ to satisfy the boundary conditions that we have! Notice that e^{ikja} and e^{-ikja}, when multiplied by $e^{i\omega t}$, create traveling waves in different directions. In particular, when we have $|\alpha| = |\beta|$, we can add the two traveling waves together to create $A_j = \cos(kja + \phi)$, which gives us a **standing wave**. Now k adjusts the wavelength of that standing wave: pictorally, we want the nodes to match up, so we stretch the wave to make sure both endpoints are at nodes.

17 October 4, 2018

Today, there is a guest professor giving the lecture.

17.1 Review

Understanding physics is often about describing a system and getting insight out of it. And usually, nature cannot be described just using English, or Chinese, or any other language – we can only explain it using mathematics.

Physics and mathematics are clearly not the same; one is physical and one is abstract. However, they are parallel ways of describing the same situation. In other words, we can use nature as an "analog computer" to solve mathematical equations, or we can use the solutions to mathematical equations to describe physical situations. When we study physics, we go through a sequence: certain phenomena give insights or practical consequences that were not realized before, and thus physics is "selecting specific phenomena" to study. For example, the problem of describing a system's **slight disturbance from equilibrium** is very common, and that's what we've been doing so far.

For example, any situation in one degree of freedom can be described by a certain equation. Each time we make the problem more complicated, we get **more and more insight**. The equation of motion for a simple harmonic motion in one variable is described by

$$\frac{d^2\psi(t)}{dt^2} = -\omega^2\psi(t),$$

and this describes a wide variety of motions (so it has practical significance). Thus, we solve the system mathematically: we find that the solution takes the form $\psi(t) = \alpha e^{i\omega t}$.

This doesn't make a lot of sense at face value – what does i, the square root of -1, actually mean? – but we can extend our model: describe the motion as $\alpha e^{i\omega t} + \beta e^{-i\omega t}$, and with appropriate choices of α, β, we can get purely real solutions which also satisfy the initial conditions.

The next thing we do is to think about having coupled systems: with N degrees of freedom, we get N differential equations. We get very interesting solutions called **normal modes**: the whole system oscillates in a very simple way. This can be figured out using an eigenvalue problem: in those simple cases, we have an eigenvector basis, and in those new coordinates, we have decoupled the system and created simple harmonic motion.

Then we moved on to a symmetry matrix: there, something new comes up. It turns out the normal modes are related to the symmetry matrix! This becomes useful: consider a coupled system with hundreds and hundreds of pendulums lined up next to each other. Even such a complicated system can have a simple solution where everything oscillates at the same frequency, and the displacements of the pendulums trace out a sinusoidal curve. (This is remarkably simple!)

One of the main ways in which we can describe a complicated system is using matrices: we often represent a system as $\ddot{Y} = -M^{-1}KY$. Then examining the system at normal modes, we have an eigenvalue/eigenvector situation: if the normal mode is $Ae^{i\omega t}$, then we want $(M^{-1}K - \omega^2)A = 0$. This can give many eigenvectors and corresponding eigenvalues.

Last time, we reduced the problem to an eigenvector Y with components

$$Y_j = \sum_{\text{all } k \in \mathbb{R}} c e^{\pm i(\omega t + \phi)} e^{\pm ijka}.$$

To proceed, we use the initial conditions, which often eliminate many terms and tell us the values for those coefficients c. But mathematically, ω and k are related, and there is a **dispersion relation** $\omega(k)$; in this case, we have $\omega^2 = \frac{4T}{ma}\left(\sin^2\left(\frac{ka}{2}\right)\right)$. If the system is finite, we also have further constraints that restrict the possible values of ω, k for us (these come from boundary conditions).

Keep in mind that ω, k are just arbitrary constants until we solve the system! Afterwards, we can say that ω is the angular frequency and k is the wave number, but those aren't obvious a priori just from the mathematics

17.2 Deriving the wave equation

If we are trying to solve the vector equation $\omega^2 A = M^{-1}KA$, it's best to do it term by term. After all, if A has a thousand terms, we don't want to deal with them all at once!

Basically any system where mass j only depends on its neighbors will be pretty simple. For example, consider the masses on a string spaced a apart with tension T: we have the equation

$$\omega^2 A_j = \frac{T}{ma}(-A_{j-1} + 2A_j - A_{j+1})$$

for all j. One key idea: let's **label all of these masses with their position instead of a subscript**. Since the masses are spaced out by a, this equation becomes

$$\omega^2 A(x) = \frac{T}{ma}[-A(x-a) + 2A(x) - A(x+a)]$$

But now, let's use the Taylor expansion (that is, we'll make a linear approximation). We know that $f(x+\Delta x) = f(x) + \Delta x \frac{df}{dx} + \frac{1}{2}\Delta x^2 \frac{d^2 f}{dx^2} + \cdots$, and using this here,

$$A(x-a) \approx A(x) - aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$
$$A(x-a) \approx A(x) + aA'(x) + \frac{1}{2}a^2 A''(x) + \cdots$$

and plugging these in, we get that

$$\omega^2 A(x) = \frac{T}{ma}\left[2A(x) - 2A(x) - a^2 A''(x) + O(a^3) + \cdots\right] \implies \boxed{\omega^2 A(x) = -\frac{Ta}{m}A''(x) + \cdots}.$$

So now if a is small, we have managed to write all the equations of motion with just one equation! But $\omega^2 A = M^{-1}KA$, and the operator $\frac{d^2}{dt^2}$ and $M^{-1}K$ are negatives of each other (since our initial equation was $\ddot{X} = -M^{-1}KX$). Thus,

$$\boxed{\frac{d^2\psi(x,t)}{dt^2}} = M^{-1}K\psi(x,t) = \omega^2\psi(x,t) = \boxed{\frac{T}{\rho}\frac{d^2\psi(x,t)}{dx^2}}$$

for some constant ρ which is the mass of the string per unit length. This is the **continuous wave equation**, and it describes systems as long as a, spacing between masses, is small!

This is beautiful in two ways: it gives interesting solutions, and it provides new insight. Recall that in the discrete case, we had our dispersion relation $\omega^2 = \frac{4T}{ma}\sin^2\frac{ka}{2}$. Now let $a \ll \frac{2\pi}{k}$; since $\sin x \approx x$, this now becomes $\boxed{\omega^2 = \frac{4T}{ma}\cdot\left(\frac{ka}{2}\right)^2 = \frac{T}{\rho}k^2}$, and we have a **linear relation** between ω and k. And in fact, this constant $\frac{T}{\rho}$ turns out to be related to the propagation velocity of the waves!

So if we define $\frac{\omega}{k} = v_p = \sqrt{\frac{T}{\rho}}$, we have the nice form

$$\boxed{\frac{d^2\psi(x,t)}{dt^2} = v_p^2\frac{d^2\psi(x,t)}{dx^2}}.$$

Now that we know what we're working towards, we will derive this equation in a simpler and more insightful way. Consider an infinite string with tension T and mass per unit length ρ; zoom in on a piece from x to $x+\Delta x$. Using Newton's second law and writing $\psi = \psi(x,t)$, since the net force on this piece of string is the difference in forces on the two ends,

$$F = ma \implies T\frac{d\psi}{dx}(x+\Delta x) - T\frac{d\psi}{dx}(x) = \rho\Delta x\frac{d^2\psi}{dt^2}$$

and by the Taylor expansion,

$$T\frac{d^2\psi}{dx^2}\Delta x = \rho\Delta x\frac{d^2\psi}{dt^2}$$

which reduces to the same wave equation. This only assumes $\frac{d\psi}{dx}$ is small, and it doesn't require us to have a notion of spacing (like the variable a above).

18 October 10, 2018: Recitation

Tomorrow is the exam. Here's today's first question: when do we include a phase shift when solving for the general solution? Consider the damped driven harmonic oscillator with equation of motion

$$\ddot{x} + \Gamma\dot{x} + \omega_0^2 x = f\cos(\omega t).$$

If there is no damping, we do not need any phase shift. This is because we can just take $x = A\cos(\omega t)$, and substituting this in makes every term a multiple of $\cos(\omega t)$. (In fact, if there isn't a driving force, we just have the simple harmonic oscillator.) We **do** end up with a phase shift in the general solution: it will be $C\cos(\omega t + \alpha)$ for some α, but this is just a free parameter.

The other sense in which a phase shift can happen is when we have an actual solution $x = A\cos(\omega t + \phi)$ with phase relative to the driving force. What does that mean? Well, $\phi = 0°$ or $180°$ if there is no damping (or all terms have an even number of derivatives). But once we introduce damping, we have odd derivatives of $\cos(\omega t)$. Then cosines and sines get mixed up, and we will have an actual phase shift.

Let's simplify this four-term equation into two terms by throwing away two of them: then we have the three different equations

$$\ddot{x} = f\cos(\omega t),$$
$$\Gamma\dot{x} = f\cos(\omega t),$$
$$\omega_0^2 x = f\cos(\omega t).$$

There are three regimes of frequency, which correspond to these three different equations. In the low frequency case, where $\omega \to 0$, the **third equation** is most relevant (because derivatives add a factor of ω, making them negligible). Similarly, the first equation is valid at high ω, and the second is valid at a near-resonant case: where $\omega \approx \omega_0$.

- At low frequency (third equation), the solution to the driven oscillator looks a lot like $x = \frac{f}{\omega^2}\cos(\omega t)$ – everything is in phase.
- At high frequency (first equation), we find that $x = -\frac{f}{\omega^2}\cos(\omega t)$; this is almost identical to the previous case, but now the negative sign means we are 180 degrees out of phase.
- Finally, in the resonant case (second equation), we have $x = \frac{f}{\Gamma\omega}\sin(\omega t)$, where again $\omega = \omega_0$. Notice here that $\sin(\omega t) = \cos(\omega t - 90°)$, so the phase is 90 degrees.

In the damped case, the resonant frequency is shifted a bit from the normal ω: it is $\omega_{\text{res}} = \sqrt{\omega_0^2 - \Gamma^2/4}$. Notice, though, that the phase shift still happens at ω_0, not the resonance frequency.

But we're losing a level of understanding by having the amplitude x related to a mass (because f represents a force). Let's compare apples to apples!

Example 54

Consider an undamped harmonic oscillator mass-spring system. However, instead of having a driving force, move the other end of the spring by a distance $\Delta\cos(\omega t)$.

Now the force equation becomes
$$m\ddot{x} = -k(x - x_s)$$
where x_s is the position of the support. This means that we can rewrite our equation as
$$\ddot{x} + \frac{k}{m}x = \frac{k}{m}\Delta\cos(\omega t),$$
and indeed it's true that a displacement Δ is **directly related** to the displacement x. So that is why it is possibly a good idea to replace f, the force amplitude, with $\Delta\omega_0^2$.

But now $\frac{f}{\omega_0^2} = \Delta$, and that explains why we can use soft springs for vibration damping! For high frequencies ω above the resonance frequency, the displacement x will vibrate at near-180-degree phase.

Now let's apply this idea to a chain. Recall that in the infinite case of oscillators, we had the equation

$$A_{j+1} = \beta A_j \implies \beta = e^{ika}.$$

But mathematically, β could have been arbitrary; we just said $|\beta| = 1$ to avoid amplitudes blowing up. What does it mean if we're not in that case? Notice there is a maximum possible frequency for which normal modes exist:

$$\omega^2 = c(1 - \cos(ka)) \leq 2c.$$

What happens when we drive the system at a frequency faster than when the dispersion relation allows? For one thing, the cosine term $\cos(ka) = \frac{e^{ika}+e^{-ika}}{2}$ will now have an imaginary k, and this will lead to exponentially decaying behavior. (This has to do with **evanescent waves**.)

But recall that we had our solution of the form $x(t) = -\frac{\omega_0^2}{\omega^2}\Delta$ for large ω in the damped, driven harmonic oscillator. So if $\omega > \omega_0$, each mass in a system drives the next one, but the motion is smaller by a factor of $\frac{\omega_0^2}{\omega}$. And indeed, this is the exponential decay that we see mathematically.

19 October 15, 2018 (Recitation)

Let's talk about the normal modes from one of the problems on our exam, with a slightly modified setup:

Example 55

We have a one-dimensional system. Three masses of $M_1 = m$, $M_2 = am$, and $M_3 = m$ appear in that order, pairwise connected by relaxed springs of length k. Find the normal modes of this system.

This requires solving a cubic equation, which is simple in the case of $a = 2$ and complicated in general. But there's another way we can approach this problem!

Let's think about the symmetry matrix S. In this case, unfortunately, our eigenvalues are degenerate. If we flip the picture around the middle mass, our symmetry matrix is

$$\begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix},$$

and the eigenvalues must be ± 1, since $S^2 = I$. So one of the eigenvalues appears twice; in this case it is -1.

Remember that we care about the eigenvectors, not the eigenvalues, of S. What are the eigenvectors in this case? The two simple ones are $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ for the eigenvalue $\lambda = 1$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda = -1$. But there's another dimension to this: $\begin{bmatrix} 1 \\ b \\ 1 \end{bmatrix}$ is always an eigenvector of eigenvalue -1 for all real numbers b, and we have a **degeneracy**.

Normally, we like having all distinct eigenvalues, because it means that our eigenvectors are orthogonal. But in this case, there is a two-dimensional subspace of eigenvectors for $\lambda = -1$. It's very good physically to pick $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ because that corresponds to translational motion, and then it makes sense to pick the other one to keep the center of mass constant (so that we're decoupling translational and oscillatory motion). So we want the eigenvector to be $\begin{bmatrix} 1 \\ -2/a \\ 1 \end{bmatrix}$, and now the dot product of two eigenvectors isn't always going to be 0 (We can check that directly).

Recall that the K matrix is symmetric, and M^{-1} is diagonal. Unfortunately, $M^{-1}K$ is not symmetric, so we can't always guarantee that its eigenvectors are orthogonal in the normal sense. But we can redefine the scalar product in terms of the M matrix as a metric!

Fact 56

Let the scalar (dot) product between two (amplitude) vectors $\vec{a}$ and $\vec{b}$ be

$$\vec{a} \cdot \vec{b} = \vec{a}^T M \vec{b}.$$

Basically, this means that heavier masses have more "weight" towards the product. And now, the scalar product of any two eigenvectors does actually become 0. Everything about symmetric matrices applies - we've just changed our metric! And this kind of mathematical trick works as long as both M and K are symmetric.

So in this problem, how do we find eigenvalues nicely? In the problem from the quiz with $a = 2$, we know that we have center-of-mass motion, which corresponds to $\omega^2 = 0$. There is also a motion in which the middle mass doesn't move: physically, this means the outside masses are connected to a "wall" (the stationary mass) with springs of constant k and $2k$, for a frequency of $\omega^2 = \frac{3k}{m}$. Finally, we can use the **trace of the matrix** to find the last one: since the trace (sum of the diagonal entries) is the sum of the eigenvalues, ω^2/ω_0^2 must be 5, so we have a frequency of $\frac{2k}{m}$ for the last one. And notice that if we change the middle mass from $2m$ to am, we know that the final eigenvalue yields a frequency of

$$\omega = \sqrt{\frac{\left(1 + \frac{2}{a}\right) k}{m}}.$$

This is the most interesting normal mode, and let's look at the two limiting cases. If $a = \infty$, the middle mass never moves, so the two normal modes are just parallel and antiparallel motion of the outside masses. And on the other hand, if $a = 0$, then the outer masses are basically walls.

Example 57

For our next topic, let's return to a damped harmonic oscillator again and look at a few different situations.

In an overdamped system, a mass will only cross the x-axis at most one time: 0 if it starts at or near rest, 1 if we give it a hard kick.

Meanwhile, in an underdamped system, consider an underdamped situation that is driven at a frequency ω, and say that we drive the system (starting at rest) close to resonance.

Question 58. *How long does it take for the oscillator to reach at least* 50% *of its final energy?*

The idea is that for small Γ, and therefore a large time-constant, we will start off with a **beat phenomenon**. Our general solution takes the form

$$X(t) = A\cos(\omega_d t - \delta) + Be^{-\Gamma/2t}\cos(\omega t - \alpha).$$

(Adding together two cosine waves with similar frequencies creates a beat pattern.) When we start at rest, our initial conditions require these two terms to destructively interfere. This means that $|A| = |B|$, and at the beginning of our motion (where the damping exponent is small and therefore negligible), we have a maximum amplitude of about $2A$ (constructive interference). And getting from destructive interference at the beginning to constructive interference takes a time **proportional to** $\frac{1}{\omega-\omega_d}$ (the cosine terms' arguments differ by a constant times $(\omega-\omega_d)t$).

But if we increase our damping and make Γ get larger, we don't see this beat phenomenon. Instead the motion will damp out to the steady-state solution too quickly. So the behavior of this system depends on a comparison of Γ^{-1}, the time constant term, to $\frac{1}{\omega-\omega_d}$, the term constituting the beat frequency!

For the final part of this recitation, let's prepare for next lecture. In the coupled oscillator system, we have equations of the form

$$-\ddot{y}_j = \frac{T}{ma}\left(-y_{j-1} + 2y_j - y_{j+1}\right),$$

where j is discrete and we have some total number of masses N (thus this is valid for $j = 1, \cdots, N$). But then, we can also have a wave equation with a continuous variable x: we showed in class that we get the equation

$$\boxed{\frac{\partial^2}{\partial t^2}\psi(x,t) = v_p^2 \frac{\partial^2}{\partial x^2}\psi(x,t)}.$$

It turns out that we can go from the discrete case to the continuous case by setting $a \to 0$ while keeping $Na = L$ constant. Basically, distribute the masses over a constant length L while adding more and more of them! And analyzing the frequency can come from analyzing the eigenvalues

$$\omega^2 = c\sin^2\left(\frac{ka}{2}\right)$$

for some constant c, and this approaches $\omega^2 = v_p^2 k^2$ as a gets smaller (by the small angle approximation).

One last question – why does the wave equation relate **second derivatives** of ψ? The left hand side is (after multiplying by a mass) essentially a force, but Hooke's law gives a linear stretch as a function of distance, so the second derivative $\frac{\partial^2\psi}{\partial x}$ seems a bit out of place. The reason this makes sense is because stretching the string in the same direction would give an equal linear stretch in both directions, so it's the **deviation from linearity** that we care about. In other words, the wave equation considers differential force!

20 October 16, 2018

Exam 1 had an average of 80 and standard deviation of 12. However, those numbers should mean nothing to us, since they will not be used for grade curves. We're only competing with ourselves! And if we need a tutor, we should message the instructors.

20.1 Review: wave equation from coupled oscillators

Recall the **wave equation** from last time:

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = v_p^2 \frac{\partial^2\psi(x,t)}{\partial x^2},$$

where $\psi(x,t)$ describes the displacement of the wave at position x and time t. As we've been saying, we can describe this as a continuous limit of the infinite coupled oscillators! Recall the dispersion relation

$$\omega^2 = 4\frac{T}{ma}\sin^2\frac{ka}{2}$$

Now, let $a \ll \frac{2\pi}{k}$. Then the small angle approximation tells us

$$\omega^2 = 4\frac{T}{ma}\left(\frac{ka}{2}\right)^2 = \frac{T}{\rho_L}k^2,$$

where ρ_L is the mass per unit length. So now we can define the quantity

$$v_p = \frac{\omega}{k} = \sqrt{\frac{T}{\rho_L}}.$$

This is actually called the **phase velocity** of our wave, and it's interesting that it's always a constant regardless of the frequency of our waves!

(Mathematically, what is changing in this limit? Our $M^{-1}K$ matrix becomes an operator $-\frac{T}{\rho_L}\frac{\partial^2}{\partial x^2}$, and our equations $\psi_j \to \psi(x, t)$ go from being discrete to continuous. And this limit yields possibly the most important equation in physics.)

Fact 59

All coupled systems obey the one-dimensional wave equation if space-translation symmetry holds, as long as we ignore higher order terms.

20.2 Solving the wave equation

Partial differential equations are difficult to solve, so let's simplify by assuming we have the separable form

$$\psi(x, t) = A(x)B(t).$$

Plugging this in, we have

$$A(x)\frac{\partial^2 B(t)}{\partial t^2} = v_p^2 B(t)\frac{\partial^2 A(x)}{\partial x^2},$$

and now collecting all terms of x on the right side and all terms of t on the left,

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2}.$$

But if we change x, the left hand side stays the same, so the right hand side must be constant for all x. Similarly, if we try changing t, the left hand side must stay constant. Thus, we can set this equal to $-k_m^2$ for some real k_m. (Yes, we will eventually worry about the positive case too.) This means

$$\frac{1}{v_p^2 B(t)}\frac{\partial^2 B(t)}{\partial t^2} = \frac{1}{A(x)}\frac{\partial^2 A(x)}{\partial x^2} = -k_m^2,$$

which yield the one-dimensional equations

$$\boxed{\frac{\partial^2 B(t)}{\partial t^2} + k_m^2 v_p^2 B(t) = 0}, \quad \boxed{\frac{\partial^2 A(x)}{\partial x^2} + k_m^2 A(x) = 0}.$$

And now **this is just simple harmonic motion**! So we have our solution

$$A(x) = A_m \sin(k_m x + \alpha_m), \quad B(t) = B_m \sin(\omega_m t + \beta_m)$$

where $\omega_m = k_m v_p$. This explains why we wanted $-k_m^2$ to be negative – the other case gives us exponential functions!

Fact 60

And remember that this is what we expected: this system has space-translation symmetry, so it should be a linear combination of e^{ikx}.

So now let's put this all together, defining $C_m = A_m B_m$: we have

$$\boxed{\psi(x, t) = A(x)B(t) = C_m \sin(k_m x + \alpha_m)\sin(\omega_m t + \beta_m)},$$

and recall that ω_m and k_m are related by the dispersion relation $\omega_m = k_m v_p$.

Example 61

If we have a string that we vibrate up and down, we can see these normal modes! But it is hard to excite higher normal modes, because our arm has to move much faster.

Okay, but what is actually going on? It turns out this is the **standing wave** solution: these solutions are the "normal modes" for the wave equation, and the full solution will be a **superposition** of the infinite number of normal modes. And because these normal modes are sinusoidal, we can use **Fourier series** to decompose the shape of the wave!

20.3 Fourier decomposition

To understand what's going on, we'll do a specific example.

Example 62

A string has length L. It is attached at one end to a wall (at $x = 0$) and at the other end to a massless, frictionless ring that can freely move up and down. Let T be the tension, and let ρ_L be the mass density of the string.

What are the boundary conditions here? We know that $\psi(0, t) = 0$ for all t, since the point at $x = 0$ is fixed on the wall. Similarly, at $x = L$, the boundary condition is actually that

$$\frac{\partial \psi}{\partial x}(L, t) = 0.$$

This is because the only vertical force on the ring comes from the string tension, so if the slope of ψ is not zero there, we have a nonzero force but zero mass, and thus this creates (unphysical) infinite acceleration! This is a bit contrived, but we will try and work with it.

Okay, so we have our two boundary conditions: let's **plug them in to our normal modes**. Since $\psi_m(0, t) = 0$, we have

$$\psi_m(0, t) = A(0)B(t) = A_m \sin(\alpha_m)\sin(\omega_m t + \beta_m) = 0,$$

so we must have $\sin(\alpha_m) = 0$ to have a nontrivial solution, meaning $\alpha_m = 0$.

Now, let's look at the second boundary condition: taking the partial derivative with respect to x at L, noting that $\alpha_m = 0$,

$$\frac{\partial \psi_m(L, t)}{\partial t} = A'(L)B(t) = A_m k_m \cos(k_m L)\sin(\omega_m t + \beta_m) = 0.$$

If this is valid for all t and the amplitude is nonzero, we must have $\cos(k_m L) = 0$, so $k_m L = \frac{(2m-1)\pi}{2}$ for some positive integer m. (That's what the subscript m is for: it indexes our normal modes!)

So now k_m is not arbitrary: we can always write $k_m = \frac{(2m-1)\pi}{2L}$. The next step is to calculate our wavelengths:

$$\lambda_m = \frac{2\pi}{k_m} = \frac{4L}{2m-1}.$$

So the standing waves that correspond to valid solutions are sine waves with wavelength $4L$, $\frac{4L}{3}$, $\frac{4L}{5}$, and so on.

And the general solution with these boundary conditions must look like

$$\psi(x,t) = \sum_{m=1}^{\infty} A_m \sin(k_m x)\sin(\omega_m t + \beta),$$

where $k_m = \frac{(2m-1)\pi}{2L}$ and $\omega_m = k_m v_p$. (And frequencies and wavelengths are inversely related, since v_p, their product, is constant for this wave!)

So we can use boundary conditions to find k_m and α_m. But to describe the whole system, we need to find a linear combination of the normal modes. How are we supposed to find A_m?

Fact 63

Fourier decomposition is really useful! Given any (continuous) shape, we can approximate it by "picking out the normal modes."

Example 64

Consider the wave

$$x(t) = \begin{cases} 0 & 0 \le x < \frac{L}{2} \\ h & \frac{L}{2} < x \le L \end{cases}$$

(the middle point $\frac{L}{2}$ can really take on any value), and distort the system such that this is the initial position of the string $\psi(x,0)$. Also, we make sure initial velocity is zero for everything, and the entire string is not moving initially: $\dot{\psi}(x,0) = 0$.

Now looking at our general solution $\psi(x,t)$, we find that the time-derivative must satisfy

$$\dot{\psi}(x,0) = \sum_{m=1}^{\infty} A_m \omega_m \sin(k_m x)\cos(\beta_m) = 0$$

for all possible values of x. Thus we must have $\cos(\beta_m) = 0 \implies \beta_m = \frac{\pi}{2}$, and we have now eliminated all parameters except for A_m.

We now use a useful result:

Theorem 65 (Orthogonality of the sine function (in this case))

Define k_m, k_n as above. Then

$$\int_0^L \sin(k_m x)\sin(k_n x)dx = \begin{cases} \frac{L}{2} & m = n \\ 0 & m \neq n. \end{cases}$$

So actually this tells us that

$$\boxed{A_m = \frac{2}{L}\int_0^L \psi(x,0)\sin(k_m x)dx},$$

because the $\sin(\omega_m t + \beta_m)$ term evaluates to 1 at $t = 0$, and only one term (the $\sin(k_m x)$ term) will survive when we write $\psi(x, 0)$ as a linear combination of $\sin(k_n x)$s. In our case, this simplifies to

$$A_m = \frac{2}{L}\int_{L/2}^{L} h\sin(k_m x)dx = \frac{2}{L}\frac{-h}{k_m}\left(\cos(k_\alpha L) - \cos(k_\alpha L/2)\right).$$

The bottom line is that this Fourier decomposition lets us pick out the values of A_m with a simple integral! So in general, we find α_m and k_m by our boundary conditions (ends of strings and so on). Then we can find β_m and A_m using the initial conditions ($\psi(x, 0)$)!

But we haven't actually gotten to the **traveling wave** solution yet: one that involves $x \pm v_p t$ or $kx \pm \omega t$. It turns out that such functions are also special kinds of solutions, and those will also tell us important properties related to wave propagation.

21 October 17, 2018 (Recitation)

Let's compare the continuous string system to a discrete case with N masses spaced out. They seem to actually be very similar; in particular, both trace out a sinusoidal pattern, but in the discrete case, the masses are spread out at equal intervals. More explicitly, the normal modes in the discrete case follow are proportional to sine functions

$$A_{j,n} \propto \sin(kja + \alpha_n),$$

and if we let $ja = x$ (since the masses are spread out by a), this turns into

$$\psi_n(x) \propto \sin(k_n x + \alpha_n).$$

And as we've said before, the discrete version has a dispersion relation $\omega^2 = c\sin^2\frac{ka}{2}$, while the continuous version has has $\omega^2 = ck^2$. In other words, we are just setting a to be small and using the small angle approximation to go from the discrete to the continuous case.

So now, we find α_n using the left boundary condition, and we find the k_n that work using the right boundary condition. This means the k_ns have to be discrete.

Example 66
What do the normal modes of a continuous string of length L look like if both ends are fixed?

Knowing that our normal modes are sine waves, we know that we must have $k_n = \frac{2\pi}{\lambda}$, where $L = n\frac{\lambda}{2}$ (there have to be an integer number of half-wavelengths in the string), and this tells us that we need $\boxed{k_n = \frac{n\pi}{L}}$.

But what are the differences between the normal modes of a string and discrete masses? If we have a discrete number of masses N, there is a shortest wavelength we can have: after a certain point, the **normal modes start to repeat**. In particular, adding a constant of c or $2\pi + c$ from one mass to another does exactly the same to the sin function! So there really are only N normal modes in the discrete case.

Remark 67. *But once we go to the continuous limit, we get ∞ number of masses, and there is no cutoff for the shortest wavelength, and there are infinitely many normal modes.*

Let's talk about the general solution for both systems in more detail. In the discrete system, normal modes generate a general solution where the jth mass follows

$$y_j(t) = \sum_m A_{j,m} y_m(t),$$

and $y_m(t) = \sin(\omega_m t + \beta_m)$. Here, $A_{j,m}$, the amplitude of the mass, is of the form $A_m \sin(jk_m a + \alpha_m)$. Boundary conditions then tell us about k_m and α_m (fixed, open end, and so on), and the initial conditions tell us the rest about A_m and β_m.

Example 68

If one end is fixed and the other end is attached to a massless rod, then the possible wavelengths are $\frac{\lambda}{4}$, $\frac{3\lambda}{4}$, and so on. Then since $\lambda_n = \frac{4L}{2n+1}$, $k_n = \frac{\pi}{2L}(2n+1)$.

But when we switch over to the continuous case, nothing really changes! We now have x, the position of the mass element j, at position ja, so we replace $ja \to x$. Thus, $A_{j,m} = A_m \sin(jk_m a + \alpha_m) \implies \psi(x) = A_m \sin(k_m x + \alpha_m)$. This is really all the same! The only change is that we have an infinite number of modes, since we can detect arbitrarily small wavelengths.

Let's talk a bit more about the mathematics of this Fourier decomposition: how do we match coefficients to their initial conditions? In the initial condition, we usually are given $\psi(x, 0)$ and $\dot{\psi}(x, 0)$, and we need to find the values of A_m, β_m. Recall that the time components of our normal modes are of the form $A_m \sin(\omega_m t + \beta_m)$, and we want to split this up into something that is easier. Cosine functions have zero derivative at $t = 0$, while sine functions have zero value at $t = 0$, so it makes sense to instead write

$$A_m \sin(\omega_m t + \beta_m) = A_m^s \sin(\omega_m t) + A_m^c \cos(\omega_m t).$$

Now our general solution is of the form

$$\psi(x, t) = \sum_{m=1}^{\infty} (A_m^s \sin(\omega_m t) + A_m^c \cos(\omega_m t)) \sin(k_m x + \alpha_m).$$

But now, $\psi(x, 0)$ removes all the sines:

$$\psi(x, 0) = \sum_{m=1}^{\infty} A_m^c \sin(k_m x + \alpha_m),$$

and $\dot{\psi}(x, 0)$ removes all the cosines:

$$\dot{\psi}(x, 0) = \sum_{m=1}^{\infty} A_m^s \omega_m \sin(k_m x + \alpha_m).$$

Now we can do our Fourier "filtering" separately for each case! Integrating and normalizing will give us each coefficient that we're looking for:

$$A_m^c = \frac{2}{L} \int_0^L \psi(x, 0) \sin(k_m x + \alpha_m) dx,$$

$$\omega_m A_m^s = \frac{2}{L} \int_0^L \dot{\psi}(x, 0) \sin(k_M x + \alpha_m) dx.$$

So we're using the normal modes as basis vectors, and we're projecting down onto one at a time with this Fourier integration trick.

22 October 18, 2018

The next pset comes with a questionnaire; it is anonymous, so we shouldn't turn it in with the rest of the pset. Also, the exam date for Exam 2 has been changed to November 15.

22.1 Review

Last class, we found the normal modes of the wave equation in the standing wave form

$$\psi(x, t) = \sum_m A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m).$$

We decide k_m and α_m using boundary conditions, and then we find β_m and A_m using the initial conditions. To do this, we perform a Fourier decomposition using the $\sin(k_m x + \alpha_m)$s as basis vectors.

22.2 Traveling waves

This time, we'll look at a different type of solution: **progressing** or **traveling waves**. It turns out we can write solutions in the form $F(x \pm v_p t)$ or $G(kx \pm \omega t)$ (where $\frac{\omega}{k} = v_p$).

Indeed, let's try plugging in a trial solution $\psi(x, t) = f(x - v_p(t))$. Define $\tau = x - v_p t$; then, by the chain rule,

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial x} = f'(\tau) \implies \frac{\partial^2 f}{\partial x^2} = f''(\tau).$$

On the other hand,

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial \tau}\frac{\partial \tau}{\partial t} = -v_p f'(\tau) \implies \frac{\partial^2 f}{\partial t^2} = v_p^2 f''(\tau).$$

But with just this amount of work, we've verified that this trial solution works in the wave equation! So this works for any function f which is twice differentiable.

Fact 69

$f(\tau)$ tells us the shape of the progressing wave. In particular, $f(x - v_p t)$ is a wave packet that travels to the right with "velocity" v_p.

However, it's important to note that the string itself only moves vertically! Every particle on the string moves up and down, but it is possible for a wave shape to move horizontally.

Example 70

If two waves of opposite amplitude travel towards each other in other directions, what happens when they collide?

Destructive interference will make the wave shape cancel out, but how does that make sense in terms of conservation of energy? The answer is that even when the string is all at its equilibrum position, **it's moving at some velocity**! This motivates us to calculate the **energy** that is stored in the string. It comes in two types:

- Kinetic energy: $\frac{1}{2}\Delta m v^2$ for a small unit of mass Δm. Specifically, given a small segment dx, $\Delta m = \rho_L dx \implies \boxed{KE = \int \frac{1}{2}\rho_L \left(\frac{\partial}{\partial t}\psi(x, t)\right)^2 dx}$.

- Potential energy: recall that work can be calculated with the equation $dW = F \cdot ds$. If we stretch the string of length dx so that the two ends are a vertical distance $d\psi$ apart (we can assume it's a line), the new length is $ds = \sqrt{dx^2 + d\psi^2}$, and we care about the differential

$$F \cdot (ds - dx) = T\left(\sqrt{dx^2 + d\psi^2} - dx\right) = T\,dx\left(\sqrt{1 + \left(\frac{\partial \psi}{\partial x}\right)^2} - 1\right)$$

  Now, by the Taylor series approximation, this is approximately

$$dW = T\,dx\left(1 + \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 - 1\right) = \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 dx$$

  which gives us our formula for potential energy:

$$\boxed{PE = \int \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 dx}.$$

22.3 The power of traveling waves

Problem 71

How do we predict what happens to the shape of a wave at $t = T$ given the wave shape at $t = 0$?

We can use brute force – the previous lecture tells us that we can find the values of A_m for each normal mode, so we can just plug in $t = T$ into our explicit formula.

But we can also write solutions as the **superposition of two traveling waves**! Consider the function $g(x, t) = f(x + vt) + f(x - vt)$ for some function f. We can verify that

$$\frac{\partial g}{\partial t} = vf'(x + vt) - vf'(x - vt),$$

and evaluating this at $t = 0$ yields $vf'(x) - vf'(x) = 0$. So such a function will always have zero derivative at $t = 0$, which means this is good for systems which are **initially at rest**.

In particular, notice that there is a connection to normal modes here – for example, the first normal mode of a string with fixed ends is also just a superposition of two traveling waves. For a general strategy, we **decompose a stationary initial condition** $y(x, 0)$ into two copies of itself, and then just let that be $f(\tau)$! It is now easy to predict $t = T$, because we know that the shape of $f(\tau)$ stays constant, and we just have a copy of the wave shape that has translated by v_pT to the left, plus a copy that has translated v_pT to the right:

$$\boxed{y(x, T) = f(x - vT) + f(x + vT)}.$$

And indeed, if we deform a string with some pattern and let it evolve from rest, we'll see this kind of behavior experimentally as well.

22.4 Reflection and transmission

We are interested in how waves evolve, and in particular, we're interested in what happens to waves when they hit something different from what they've been propagating through.

Example 72

Connect two Bell wave machines (which we've been using to demonstrate wave motion) together, one with a higher ρ_L than the other. Both have string tension T, but the right one has ρ_L four times as big than the left one. Assume a massless junction – how does a wave propagate through this system?

We know that the phase velocity (speed of the wave) $v_p = \sqrt{\frac{T}{\rho_L}}$ will be twice as large for the left one as for the right one. To rephrase our question more precisely, if we have a traveling wave with amplitude A propagating through the left Bell wave machine, what happens when it hits the junction? It could create a reflected wave (traveling to the left), continue on (creating a wave through the right Bell wave machine), or do both.

Secretly, this is a boundary condition problem! We'll define the position of the left string via a function $\psi_L(x, t)$ and the right one similarly via $\psi_R(x, t)$. Suppose that the boundary occurs at the coordinate $x = 0$.

- The string must be continuous at the junction, so $\boxed{\psi_L(0^-, t) = \psi_R(0^+, t)}$.
- The slope of the string is also continuous. To justify this, we can imagine a massless ring at the junction; the vertical components of the force must cancel out. Thus $\boxed{\left.\frac{\partial \psi_L}{\partial x}\right|_{x=0} = \left.\frac{\partial \psi_R}{\partial x}\right|_{x=0}}$.

With this, we'll now solve the problem analytically: let f_i be the incoming (initial) wave, and let f_r be the reflected wave on the left string. Then

$$\psi_L(x, t) = f_i(-k_1 x + \omega t) + f_r(k_1 x + \omega t),$$

where $\frac{\omega}{k_1} = v_p$. Similarly, we will find that

$$\psi_R(x, t) = f_t(-k_2 x + \omega t).$$

Plugging these into the boundary conditions,

$$\boxed{f_i(\omega t) + f_r(\omega t) = f_t(\omega t)}$$

(from the first condition) and

$$-k_1 f_i'(\omega t) + k_1 f_r'(\omega t) = -k_2 f_t'(\omega t)$$

(from the second condition), and now we can solve for the relation between f_i, f_r, f_t! Integrating the last equation on both sides with respect to time, we have $-\frac{k_1}{\omega} f_i(\omega t) + \frac{k_1}{\omega} f_r(\omega t) = -\frac{k_2}{\omega} f_t(\omega t)$ (we ignore the integration constant for now). Since $\frac{\omega}{k_1} = v_1$ and $\frac{\omega}{k_2} = v_2$, we clear denominators and find that

$$\boxed{v_2(f_r(\omega t) - f_i(\omega t)) = -v_1 f_t(\omega t)}.$$

Now we can solve the two boxed equations! A bit of algebra shows that

$$\boxed{f_r(\omega t) = \left(\frac{v_2 - v_1}{v_1 + v_2}\right) f_i(\omega t)}, \quad \boxed{f_t(\omega t) = \left(\frac{2v_2}{v_1 + v_2}\right) f_i(\omega t)}.$$

And this motivates us to define the **reflection** and **transmission coefficients**

$$R = \frac{v_2 - v_1}{v_1 + v_2}, T = \frac{2v_2}{v_1 + v_2},$$

which tell us how much of the wave passes through the boundary. For instance, in the example we proposed earlier, $v_1 < v_2$, so $R, T > 0$. Both the reflected and the transmitted wave are going to have a positive amplitude!

To finish, we'll look at the two extremes when we vary the density ρ_L for the right side. If we have a wave pulse approaching a wall, ρ_L of the wall is basically infinite, and the velocity $v_2 \approx 0$. And this tells us that

$$R = \frac{0 - v_1}{v_1 + 0} = -1, \quad T = \frac{2 \cdot 0}{v_1 + 0} = 0.$$

In other words, **when a wave on a string hits a fixed end, it gets flipped upside down.**

On the other hand, if we have a wave pulse connected to air (a free end), $\rho_L \to 0$, so $v_2 \to \infty$. Plugging in a large value of v_2 sends $R \to 1$ and $T \to 2$ – this means that the wave is reflected back right-side-up with equal intensity. (Unfortunately, the mass of the air is zero, so no energy is created from the transmission coefficient T.)

23 October 22, 2018 (Recitation)

(This recitation was taught by Pearson, the graduate TA.) Recall that the wave equation takes the form

$$v^2 \partial_{xx} u = \partial_{tt} u,$$

where v is the speed of propagation. (There's no way to get away from this equation no matter what field we're looking at!) We will take a look at formulas for transmission and reflection. We know that there are two ways of writing the general solutions: we've been primarily discussing and working with the normal modes of the form

$$u_n(x, t) = A_n \cos(k_n x + \alpha_n) \cos(\omega_n t + \beta_n),$$

where $\omega_n = k_n v$. But today, we will talk more about the traveling wave solutions

$$u(x, t) = f(x - vt) + g(x + vt).$$

This strategy is pretty powerful: notice that f and g are actually functions of one variable. And this is going to keep coming up, because this kind of decomposition can be used pretty often.

Let's talk about reflections. We've mostly been dealing with waves propagating in one specific medium (a vacuum, or air, or a fixed tension, and so on). But waves don't always act that nicely:

> **Fact 73**
> Connecting two different media will give a discontinuity in our wave function $u(x, t)$.

This is because the two media will have different wave speeds (for instance, from different tension forces or other properties of the system), so the velocity of a wave will not be continuous. Also, we have a local boundary condition by connecting the two systems with some force. (To visualize this, we can think of quantum mechanics: we have the same kind of discontinuity in a delta potential function, where the slope changes abruptly.)

To solve the reflection problem, we define an **incident wave** $f(x - vt)$. As mentioned in class, we will also get back a reflected wave with multiplicative factor R and a transmitted wave multiplicative factor T, where all amplitudes are relative to A. In other words, if $f(x - vt) = Ae^{ikx}e^{i\omega t}$, we can write the reflected wave as made of $RAe^{ikx}e^{i\omega t}$ and the transmitted wave $TAe^{ikx}e^{i\omega t}$.

Let's look at some simple cases:

- If we have a wall (fixed end), then $R = -1, T = 0$, and the wave is entirely reflected.
- If the medium is the same on both sides of the boundary (that is, $v_1 = v_2$), $R = 0, T = 1$, and the wave is entirely transmitted.

- Finally, if the medium for the transmitted wave has zero density, we have $R = 1, T = 2$. (This can also be interpreted as a free end for a string.)

Notice that in all three cases, the difference between the reflection and transmission coefficient is 1. In particular, we can take the limit $\lim_{x \to 0^-} u(x, t) = \lim_{x \to 0^+} u(x, t)$, which gives $1 + R = T$.

How did we find the actual values of R and T from here? We needed another equation to solve, and this last equation depends on further initial conditions. Well, the wave and its derivative have to be continuous at the point of transmission, so the frequency of the wave must be the same on the two sides. (Thus both the reflected and transmitted wave have a component proportional to $f(\omega t)$.) The wave numbers k will be different, and so will the speed of propagation, but this does give a constraint that relates the two different media. That's how we found the equations for R and T during lecture!

Now let's look at another case:

Example 74

Suppose we have some force on a point mass, and we want to look at the force balance at that discontinuity. (For example, maybe we have a bead hanging on a string, affected by gravity.)

Now the actual position $u(x, t)$ must be continuous, but the velocity no longer needs to be continuous. The way to approach this problem is to look at the forces on the mass in the middle: if we have a tension force T on both sides, Newton's law imposes the boundary condition

$$T(\partial_x u^+ - \partial_x u^-) = m u_{tt}(x)|_{x=0}.$$

In other words, having a force on this point mass causes a **discontinuity in the first derivative** from one side of the string to the other.

24 October 23, 2018

Today's lecture is being given by Professor Comin. First, a few quick announcements: remember that the pset has a questionnaire and is due Friday, the exam is now on November 15th, and Professor Lee's office hours are moved.

24.1 Review

We've talked about the wave equation in the past few lectures: basically, we can represent solutions in standing wave **normal modes**, just like in the discrete case, or we can look at them as traveling waves written in terms of $x \pm v_p t$. Notice that our normal modes are almost in the form $\cos(kx \pm \omega t + \phi)$, and the progressing or traveling waves are just like this, but replacing cosine with a more general function.

Notably, the traveling wave solution is a much nicer method to solve these kinds of problems, because we can take initial conditions that are hard to represent as cosines and sines (like a square wave) and avoid the Fourier series method by representing that initial function as $f(x + vt) + f(x - vt)$.

It's finally time to extend our model and stop doing mechanics with only mechanical transverse waves.

Definition 75

A **transverse** wave is one in which displacement is orthogonal to the direction of propagation. Meanwhile, a **longitudinal** wave is one in which displacement is parallel, like in sound waves.

For example, if we have a discrete set of masses along a line, they could be connected by strings that provide a restoring force in the perpendicular direction, or they could be connected by springs that provide a restoring force in the parallel direction (if we constrain the masses to one dimension). In fact, these have very similar equations: recall that

$$m\ddot{\psi}_j = \frac{T}{a}\left(\psi_{j+1} + \psi_{j-1} - 2\psi_j\right)$$

is the equation in the transverse case. Then simply replacing $\frac{T}{a}$ with k gives the equation in the longitudinal case! **The physics is all the same**, because all restoring forces are all proportional to displacement.

So **how do longitudinal waves behave** in the continuous model? Consider a (not massless) spring that can be stretched and compressed at different locations. Then we can have regions of closer and farther loops, which produces a wave (imagine a slinky). It turns out that the dispersion relation for such a spring wave is

$$\omega^2 = \frac{K\ell}{\rho_\ell}k^2,$$

where K is the stiffness of the spring and ℓ is its length; this is analogous to the dispersion relation $\omega^2 = \frac{T}{\rho}k^2$ for the string wave.

24.2 Sound waves

Sound waves are created based on differences of **effective pressure** in a medium. Air molecules move in a somewhat-harmonic motion, and in regions of low density, pressure is lower. In short, sound waves propagate because particles like to move to areas of lower pressure, and this leads us to something that looks a lot like the wave equation!

To understand this, let's first recall the alternate derivation of the wave equation for a string with tension T. Consider a small element of the string from x to $x + \Delta x$. This element has mass $\Delta m = \rho\Delta x$, and the forces on this mass element are just the tension forces from the left and right of the small string. The tension force is then

$$F = ma = \boxed{\rho\Delta x\frac{\partial^2\psi}{\partial t^2}} = F_R + F_L = \boxed{T\frac{\partial\psi}{\partial x}(x+\Delta x, t) - T\frac{\partial\psi}{\partial x}(x, t)},$$

and dividing through by Δx and taking Δx to 0, we get that $\rho\frac{\partial^2\psi}{\partial t^2} = T\frac{\partial^2\psi}{\partial x^2}$, which is the usual wave equation as desired. We'll now extend this to fluids and sound:

Example 76

Consider a cylinder with cross-sectional area A, filled with a fluid or gas, and let the x-coordinate be perpendicular to this cross-sectional area.

Pick a section of the cylinder from x to $x + \Delta x$. In the equilibrium case, the pressure at both x and $x + \Delta x$ is p_0. Then the force on any volume element $v_0 = A\Delta x$ is zero, since $F = PA$ and the cross-sectional area is the same on both sides. The initial volume here is then $V_0 = A\Delta x$, and there is some initial density ρ_0.

So now let's displace the molecules by a small amount relative to our width Δx. If the molecules at x move by some amount $\psi(x, t)$, while the molecules at $x + \Delta x$ move by $\psi(x + \Delta x, t)$, then our new volume is

$$V = A\left(\Delta x + \psi(x+\Delta x, t) - \psi(x, t)\right) = A\Delta x + A(\psi(x+\Delta x, t) - \psi(x, t)) = V_0 + \boxed{A(\psi(x+\Delta x, t) - \psi(x, t))},$$

where the boxed part is our change in volume ΔV. Also, note that the total force acting on the fluid is related to the pressure on both sides (we can assume the pressure on the left side is still the pressure at position x if we have small oscillations, and similarly, the pressure on the right side is the pressure at $x + \Delta x$). This yields a total force

$$\sum F = A\left(p(x,t) - p(x+\Delta x, t)\right) = -A\Delta p = -A\frac{\partial p}{\partial x}\Delta x$$

for small Δx. Meanwhile, the mass is conserved throughout this process, so $\rho_0 V_0 = \rho' V'$ (where ρ represents the density). By Newton's second law, then,

$$ma = F \implies \boxed{\rho_0 V_0 \frac{\partial^2 \psi}{\partial t^2} = -A\frac{\partial p}{\partial x}\Delta x}$$

and since $A\Delta x = V_0$, we get the simplified equation of motion

$$\boxed{\rho_0 \frac{\partial^2 \psi}{\partial t^2} = -\frac{\partial p}{\partial x}}.$$

Unfortunately, this is not a closed equation: we need some other relationship between p and ψ.

Proposition 77 (Equation of state)

At a fixed temperature, for an **ideal gas**, $pV = Nk_bT = C$ for some constant (since N, the number of particles, is fixed). Meanwhile, a **real gas** has $pV^\gamma = C$. Here, γ is the adiabatic constant, and it's related to properties of the air molecules themselves.

Thus, $\frac{\Delta p}{p} = -\gamma\frac{\Delta V}{v}$ (this comes from isolating p and taking a derivative). We know that $V_0 = A\Delta x$, and $\Delta V = A(\psi(x+\Delta x, t) - \psi(x,t))$ from our calculation earlier. But since $\Delta V = A\frac{\partial \psi}{\partial x}\Delta x$, if we plug everything back in,

$$\frac{\Delta p}{p} = -\gamma\frac{A\frac{\partial \psi}{\partial x}\Delta x}{A\Delta x} = -\gamma\frac{\partial \psi}{\partial x} \implies \Delta p = -p_0\gamma\frac{\partial \psi}{\partial x}.$$

Now, because the pressure $p(x) = p_0 + \Delta p(x)$ deviates only based on Δp, we can now calculate

$$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(-p_0\gamma\frac{\partial \psi}{\partial x}\right) = -p_0\gamma\frac{\partial^2 \psi}{\partial x^2}.$$

Plugging this back in to our equation of motion will yield our **wave equation for sound:**

$$\boxed{\rho_0\frac{\partial^2 \psi}{\partial t^2}} = -\frac{\partial p}{\partial x} = \boxed{p_0\gamma\frac{\partial^2 \psi}{\partial x^2}}.$$

Compare this to the wave equation $\rho\frac{\partial^2 \psi}{\partial t^2} = T\frac{\partial^2 \psi}{\partial x^2}$. Density ρ_0 plays the role of the mass density ρ, and $p_0\gamma$ plays the role of the constant tension of the string! And now for sound, the wave velocity satisfies

$$v^2 = \frac{p_0\gamma}{\rho_0}$$

γ is dimensionless, p_0 has dimensions force per area, and ρ has dimensions mass per volume. This gives meters squared per second squared, which is indeed the correct set of units to use.

24.3 Do we have an ideal gas?

This equation of state PV^γ was debated pretty wildly hundreds of years ago. Newton claimed that we should treat PV as constant: as the wave propagates, heat is conducted quickly, so the temperature cannot rise or fall, and this means PV is constant. Laplace says that heat flow is negligible, so PV^γ is constant.

To figure out the actual answer experimentally, we can measure the speed of sound by setting up a standing wave and using $\lambda = \frac{v_s}{\nu}$. Newton's equation gives 289 meters per second, and Laplace gives 342 meters per second. Derham's experiment in 1708 gives a correct answer of 348 **meters per second**, so Laplace was correct!

Black-boxing a lot of chemistry, γ is the adiabatic index, which is related to α, the number of degrees of freedom divided by 2. In particular, $\gamma = \frac{1+\alpha}{\alpha}$. A monoatomic gas has 3 translational degrees of freedom, so $\alpha = \frac{3}{2}$ and $\gamma = \frac{5}{3}$. Diatomic molecules have 3 translational degrees of freedom (of the center of mass) and 2 rotational degrees of freedom (we can't count the rotation along the rigid axis). Thus $\alpha = \frac{5}{2}$ and $\gamma = \frac{7}{5}$.

And finally, how do we deal with boundary conditions for sound waves? If we have a tube filled with gas, a closed boundary forces $\psi(0, t) = 0$ – the air is not allowed to be displaced. On the other hand, an open boundary or atmosphere means the pressure is p_0 right outside the tube. Continuity of pressure means that the pressure $p(x)$ is p_0 there, and therefore $\Delta p(x) = 0$ **at the end of the tube**, which gives a boundary condition that the **derivative** $\frac{\partial \psi}{\partial x} = 0$.

25 October 24, 2018 (Recitation)

When we study the wave equation, there are two ways to approach it. One way is to look at it from the **discrete coupled oscillator** point of view, where we separate our temporal and spatial components into a normal mode $A(x)B(t)$. This allows us to deal with **restricted geometry** easily: if we have boundary conditions that we'd like to describe mathematically, it's often easy to translate those conditions into restrictions on our normal modes. However, we can also look at pulses of the form $G(x \pm vt)$, and this approach helps us look at propagation, reflections, and transmissions.

But can we get propagating pulses out of normal modes? The answer is yes – all solutions can be described by (linear combinations of) normal modes. It may seem odd that standing waves, which only move up and down, can actually propagate. However, the key insight is that **different normal modes have different frequencies**: if a wave is propagating to the right, then the left part of a packet is moving down, while the right part is moving up. Everything can be described by studying the different phases and frequencies of normal modes!

Remark 78. *Normal modes have one thing built in: they can automatically describe a reflection, because we can deal with boundary conditions easily. So if a wave pulse hits a wall, the normal modes can explain how and why an opposite-traveling negative amplitude wave is formed.*

One thing to notice, though: we can never have an infinitely-oscillating traveling wave that fulfills, for example, a fixed-end boundary condition. This is because there are no nodes! Specifically, if we're forced to have a node, then $y(0) = 0$ at all times t, so $\dot{y}(0) = 0$. But an infinite traveling wave is always moving when its displacement is 0. (Similarly, a single wave pulse would never work as long as it's constrained to our limits.)

So what is happening when we have a reflection of a wave against a wall? The constraint that $y(L) = 0$ just means that we actually extend our function so that $y(L - k) = y(L + k)$! So there is an opposite traveling wave pulse outside the wall, which "enters the picture" just as our initial wave pulse exits.

We should emphasize here that we have such solutions only when our wave equation is non-dispersive. Recall that

$$\omega^2 = \frac{4T}{ma} \sin^2 \frac{ka}{2}$$

is the **actual** dispersion relation $\omega(k)$ for our coupled system, and we get to a linear dispersion relation $\omega = v_p k$ **only when** ka is small. In general, there are actually **two different velocities** at play here: $v_p = \frac{\omega}{k}$ and $v_g = \frac{\partial \omega}{\partial k}$, called the phase and group velocity respectively. (And for $\omega = \tilde{v}k$, we find that $v_p = v_g = \tilde{v}$ do not depend on k at all and are always equal.)

Well, we can superimpose traveling waves to create a phase packet, and we can do a similar Fourier decomposition to figure out "how much" of each fundamental traveling wave we want. We know that each traveling wave moves, and as long as they all move at the same velocity, the sum will also move at that same velocity – the shape will not change. But if our wave equation is dispersive, this does not work! It's possible that the wave will broaden as it moves, and that's another explanation for why we call this a **dispersion relation**.

Proposition 79

Electromagnetic waves are non-dispersive.

Here's a way to check: we know of a supernova explosion billions of light-years away, and we can measure it in the infrared, visible, and X-ray spectrum. It turns out that the X-ray pulse arrives exactly at the same time as the visible parts, even though the two different spectra correspond to different frequencies.

Let's move on and discuss **energy and power** for traveling waves. Specifically, **where is potential energy maximized** for a traveling wave? Remember from earlier calculations that potential energy in a traveling wave comes from **extra tension** in the string: in a given length element dx, it is $\frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2 dx$. (Displacement itself does not matter, only the difference in stretch.)

But where is kinetic energy maximized? It's at the same place! The wave equation tells us that the kinetic energy can be written as $\frac{\rho_L}{L}\left(\frac{\partial\psi}{\partial t}\right)^2 dx$, and remember that a traveling wave is only in terms of one single parameter $x - vt$. So the partial derivatives for time and position are directly related to each other, and this leads to an interesting idea:

Proposition 80

Any traveling wave has equal potential and kinetic energy, and maximums occur at the **same** location. (This is in stark contrast to a standing wave $\psi(x, t) = A(x)B(t)$, in which kinetic and potential energy are being exchanged back in forth.)

Finally, let's talk about **power** (that is, the flow of energy in a traveling wave). It turns out that we have an explicit formula for the transmitted power:

$$P = -T\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t}.$$

In particular, single out a specific length element dx. The energy of this element can change because we provide more power on the left side or from the right side. So we can write that the change in energy of this length element is

$$\dot{E} = P_L + P_R.$$

Power here is force times velocity, which is $T\frac{dy}{dx}$ times $\frac{dy}{dt}$ – plugging this in gives us the above formula for P. (And there is a dot product, but we have already constrained everything to the y-direction.)

And now as an exercise, we can evaluate this expression for P at two positions x and $x + dx$ to find the **net power flow**. Subtract those two values, and we can find (from the subsequent derivation) the exchange of potential and kinetic energy in a wave.

26 October 25, 2018

Today's lecture is being given by Professor Ketterle. We're going to start talking about light today; one of the main ideas here is that **light is just like any other wave**. Remember that waves propagate, and a sound packet in an organ pipe moves at the speed of sound through that pipe. It turns out that light does the same thing, only at a different speed.

Fact 81

Light can travel 30 centimeters in about a nanosecond – the speed of light is about 3×10^8 m/s.

So in one clock cycle of a computer, light travels about 30 centimeters. As another example, it takes about a second for light to travel to the moon. This year's Nobel Prize was given for **short light pulses**: now the light pulse that propagates has width in small fractions of a millimeter.

Fact 82

A trillion-frame-per-second camera captured a light pulse moving through a Coke bottle. This camera took "pictures in x, t dimensions" instead of x, y dimensions – specifically, the frame-by-frame pictures gave us a sense of how light actually propagates.

26.1 The EM wave equation

Let's go back to our wave equation. We know that we can have mechanical and sound waves; today, we will talk about electromagnetic waves. This is a bigger step than we might initially imagine: until now, the waves have been motions of particles, like displacement of a string, spring, or gas. But now, **EM waves are just fields**. This is more abstract than mechanical waves, but this is the first step towards understanding quantum wavefunctions. EM waves do describe photons, but we don't need to use those photons to describe the waves – all of the theory we need is described by Maxwell's equations.

Fact 83

Here are Maxwell's equations for reference; they describe electric and magnetic fields and how they interplay.

- Gauss' law: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$; this tells us how charge is created.
- Gauss' Law for magnetism: $\vec{\nabla} \cdot \vec{B} = 0$; this explains us that there are no magnetic charges or monopoles.
- Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$; this says an electric field can be created by a changing magnetic field, and
- Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$; this describes how a current is surrounded by magnetic fields, and a changing electric field (called a displacement current) also contributes to this magnetic field.

If we're working in a vacuum, there are no charges or currents, so our equations reduce dramatically:

- Gauss' law: $\vec{\nabla} \cdot \vec{E} = 0$.
- Gauss' Law for magnetism: $\vec{\nabla} \cdot \vec{B} = 0$.
- Faraday's Law: $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$.
- Ampere's Law: $\vec{\nabla} \times \vec{B} = \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$.

These just seem like a bunch of vector relations for now, but where are the waves? The first step is to **eliminate the magnetic field**:

Proposition 84
For any vector $\vec{A}$, we have
$$\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A},$$
where $\vec{\nabla} \cdot \vec{\nabla} = \vec{\nabla}^2$ is the Laplace operator.

(This can be proved by expanding the expressions out explicitly.) Applying this to the electric field $\vec{A} = \vec{E}$, we find that
$$\boxed{\vec{\nabla} \times \left(\vec{\nabla} \times \vec{E}\right)} = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = \vec{\nabla}(0) - (\vec{\nabla} \cdot \vec{\nabla})\vec{E} = \boxed{-\vec{\nabla}^2 \vec{E}}.$$
(Here, in three dimensions, the Laplacian can be written in coordinate form as $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.) But now Faraday's law tells us that the left hand side is the curl of $-\frac{\partial \vec{B}}{\partial t}$, which is (exchanging the derivative and curl) $-\frac{\partial}{\partial t}\left(\vec{\nabla} \times B\right) = \boxed{-\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}$. Setting the boxed things equal, this gives the wave equation for electric fields
$$\boxed{\vec{\nabla}^2 \vec{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}}.$$
And in fact, an identical equation can be found for the magnetic field $\vec{B}$, so **both the electric and magnetic components** fulfill this 3D wave equation!

And $\frac{1}{\mu_0 \varepsilon_0}$ takes the role of the (squared) speed of light in our wave equation, so we know that $v_p = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} = 3 \times 10^8$ m/s.

26.2 A bit of history

Light initially wasn't something that was thought to propagate, and the first person to propose such an idea was Galileo Galilei. He did an experiment, and he found that light was at least 10 times the speed of sound.

Ole Romer in 1675 was the first person to find a limit on the speed of light – he used very large distances by looking at the rotation of the moon Io around Jupiter. This moon rotates every 4 hours or so, but when the Earth is on the opposite side of the earth (half a year later), there was a 20 minute time delay! In other words, there must be a time lag for light to travel a distance equal to the diameter of the orbit of the earth, and this let Romer estimate the speed of light to be about 2×10^8 meters per second.

James Bradley used the aberration of starlight in 1728 to get a much closer estimate of 3×10^8 meter per second. It's a triangulation argument: if the earth is moving tangentially, the light will hit the surface at an angle depending on which way it moves, and we can use this to compare the ratio of light speed to Earth speed.

And recall that when a flashlight sends out all colors of light, all colors will arrive at the same time, since the speed of light c does not depend on k, ω, or the frequency. This was found to be true to 20 digits precision about 20 years ago.

26.3 Back to solutions of the wave equation

Let's reduce this equation to the one dimensional case. Suppose that we have the trial solution
$$\vec{E} = E_0 e^{i(kz - \omega t)} \hat{x},$$

which means the electric field is only a displacement (that is, only has amplitude) in the x-direction and propagates in the positive z-direction. (Notice that we can always **choose** our coordinate system to satisfy these constraints.)

Then let's verify that this works in our wave equation. The left hand side, the Laplacian, is only dependent on z, and we have

$$\vec{\nabla}^2 \vec{E} = \frac{\partial^2 E_x}{\partial z^2}\hat{x}$$

while the right hand side is

$$\mu_0\varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0\varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}\hat{x}.$$

Cancelling $\hat{x}$ from the right hand sides and looking at real parts of both sides now, we have $\boxed{E_x = E\cos(kz - \omega t)}$, so the two sides simplify to

$$-E_0k^2\cos(kz - \omega t) = -\mu_0\varepsilon_0\omega^2 E_0\cos(kz - \omega t) \implies \boxed{\frac{\omega}{k} = \frac{1}{\sqrt{\mu_0\varepsilon_0}} = c}.$$

So as expected, the well-known one-dimensional wave solution satisfies the EM wave equation, as long as we have this linear relation $\frac{\omega}{k} = c$. This means that electormagnetic waves are indeed **non-dispersive**, because the speed of the wave does not depend on k.

But what about the associated magnetic field for this choice of $\vec{E}$? We know from Faraday's law that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t},$$

so we can plug our $\vec{E}$ in and integrate. The only non-vanishing part of the left side of this equation, the curl of $\vec{E}$, is

$$\frac{\partial E_x}{\partial z}\hat{y} = -kE_0\sin(kx - \omega t)\hat{y}.$$

Set this equal to $-\frac{\partial \vec{B}}{\partial t}$, and we find after integration that $|\vec{B}| = \frac{E_0}{c}\cos(kz - \omega t)\hat{y}$. So the **magnitudes have a direct relation**, and notice that $\vec{E}$ and $\vec{B}$ are **orthogonal** and **both perpendicular** to the direction of propagation! Here's another way to write this: defining the vector $\vec{k} = k\hat{e}_z$, we have

$$\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}.$$

where $\hat{k}$ is the unit vector in the direction of propagation.

Fact 85

Notice that $\cos(kz - \omega t)$ is a common term between $\vec{E}$ and $\vec{B}$, so the two are in phase, both in time and space!

Let's now more formally show why $\vec{E}, \vec{B}, \hat{k}$ are all always orthogonal for linearly polarized plane waves. Let's say we have an electric field of the form

$$\vec{E} = \vec{E}_0 e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}.$$

Taking the partial x-derivative gives us back $\vec{E}$ by ik_x, and a similar thing happens for the y- and z-derivatives. But this means that the del operator is easy to write down: we simply have

$$\vec{\nabla} \cdot \vec{E} = i\vec{k} \cdot \vec{E}.$$

Since the divergence must be 0 by Maxwell's equations, so $\vec{E}$ and $\vec{k}$ are indeed orthogonal. Next, we take the curl of the electric field: we'll find that

$$\boxed{i\vec{k} \times \vec{E}} = \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = \boxed{-i\omega\vec{B}}.$$

Equating the directions of the left and right expressions indeed shows that $\hat{k} \times \hat{E} = \hat{B}$. (And this is an easier calculation than bashing everything out with coordinates!)

With this, we can think about how these electromagnetic waves propagate. At the moment, nothing really happens: they translate in a particular direction $\hat{k}$, just like our progressing waves $f(x - vt)$ did in the mechanical case.

Example 86

Let's add a sheet of matter that reflects our electromagnetic wave back. What will happen to the $\vec{E}$ and $\vec{B}$ fields?

Recall that when we have reflection of a mechanical wave at a fixed end, the reflected wave is upside down, and when we have reflection at a free end, the reflected wave is right-side up. It turns out that $\vec{E}$ and $\vec{B}$ are **reflected in different ways**.

To explain this, note that metal sheets have charges, and charges shield electric fields. Thus, a good conductor with infinite conductivity should impose the boundary condition $\vec{E} = 0$. Well, if we have an incident wave of the form $\vec{E}_I = \frac{E_0}{2}\cos(kz - \omega t)\hat{x}$, and this hits a conducting sheet at $z = 0$, then we must have the reflected wave propagate backwards: $\vec{E}_R = c\frac{E_0}{2}\cos(-kz - \omega t)\hat{x}$ for some reflection coefficient c. And because we want $\vec{E}_I + \vec{E}_R = 0$ at $z = 0$ to satisfy the equation, we indeed need $c = -1$. And this means that **electric fields are flipped upside down**. On the other hand, by Maxwell's equations, the associated B field here is $\vec{B}_I = \frac{E_0}{2c}\cos(kz - \omega t)\hat{x}$, and the right-hand rule shows that $\vec{B}_R$ is still positive: **magnetic fields stay right-side up**.

Now adding up the incident and reflected fields,

$$\vec{E} = \vec{E}_I + \vec{E}_R = \frac{E_0}{2}\left(\cos(kz - \omega t) - \cos(-kz - \omega t)\right) = \boxed{E_0 \sin(\omega t)\sin(kz)\hat{x}},$$

and similarly

$$\vec{B} = \vec{B}_I + \vec{B}_R = \boxed{\frac{E_0}{c}\cos(\omega t)\cos(kz)\hat{y}}.$$

So standing waves are different from traveling waves: the electric and magnetic waves are **out of phase by 90 degrees, both in time and in space**.

Here's a fun application of what we've been talking about:

Example 87

The frequency of the microwaves in a microwave oven is about $\nu = 2.45$ GHz. Microwaves heat up **water molecules**, which are polar – the water dipole basically gets sped up, and it hits other molecules to produce heat. We therefore cannot heat up a bunch of perfectly neutral, nonpolar molecules.

The wavelength of these microwaves is about 12 centimeters, and we can see the standing waves that the microwave oven creates with our own eyes if we don't put our food on a rotating plate!

27 October 29, 2018 (Recitation)

Today, we will study both the energy and the flow of energy in mechanical and electromagnetic systems. Hopefully we will be able to connect the similarities and differences.

First of all, what's the difference between waves on a string and waves that propagate in a vacuum? Electromagnetic waves depend on both $\vec{E}$ and $\vec{B}$, and it turns out that the two wave equations for $\vec{E}$ and $\vec{B}$ must be satisfied at the same time - we can't have one without the other! But we'll see that despite this idea, there are lots of similarities.

- Consider the energy density in our waves. At a given point, the electric field energy density is $\frac{1}{2}\epsilon_0 E^2$, and the magnetic field density is $\frac{1}{2\mu_0}B^2$. We find that these are always the same in a propagating wave, since $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E}$. But it turns out we also have two different forms of energy in mechanical waves too: we have $\frac{\rho_L}{2}\left(\frac{\partial\psi}{\partial t}\right)^2$, the kinetic energy of the wave, and $\frac{T}{2}\left(\frac{\partial\psi}{\partial x}\right)^2$, the potential energy. (And these are equal on average.)

- Can we also find similarities in **boundary conditions**? If an electromagnetic wave hits metal with infinite conductivity ($\sigma = \infty$), we have a boundary condition $E = 0$ (like a fixed end), while having a free-end boundary condition for B. Well, consider a standing mechanical wave that is fixed at $x = 0$. Then $\psi(x, t) = \dot{\psi}(x, t) = 0$ is indeed a fixed end, but there is no restriction on $\frac{\partial\psi}{\partial x}$: it's a free end for the derivative. (And there is indeed a "fixed end" boundary condition for $\frac{\partial\psi}{\partial x}$ when we have an free end.) So at a boundary condition for mechanical waves, we have two parameters, $\frac{\partial\psi}{\partial t}$ and $\frac{\partial\psi}{\partial x}$, which have opposite boundary conditions as well!

- In a traveling wave, $\vec{E}$ and $\vec{B}$ are always in phase, both temporally and spatially. Well, in a traveling mechanical wave, the kinetic and potential energy are also in phase: this is because $f(x - vt)$ only depends on one function, so it can only depend on the derivatives of f. So all energy is in phase, both in space and in time, for a traveling wave in both cases.

- What about standing waves? Mechanical waves transfer energy continuously between potential and kinetic energies in a way that makes them always behave out of phase. More intuitively, we can think of standing waves of the form
$$\psi(x, t) = \sin(\omega t + \phi)\cos(kx + \delta),$$
and notice that taking the time and space derivatives will give two cosine terms and two sine terms, respectively. So standing waves have energy out of phase both in time and in space! Maximum potential and kinetic occur at different times, and they also occur at nodes and antinodes respectively. There's a small difference: notice that the oscillation of energy in the mechanical has frequency 2ω instead of ω (because of the product-to-sum trig identity). But the main point is that the same thing happens with EM waves: a standing wave has $\vec{E}$ and $\vec{B}$ out of phase, both in space and in time.

Remark 88. *A good physical picture we can use to explain this last point is to look at boundary conditions. For a fixed end, we always meet boundary conditions by adding an incident and reflected wave together, such that they are in such a phase to destructively interfere where we want. On the other hand, if we use a loose end, we make them constructively interfere as best we can. So the fixed end has a node at the end, and the loose end has an antinode at the end.* **Because $\vec{E}$ and $\vec{B}$ react differently to boundary conditions, they must be out of phase.**

- Next, let's look at how $\vec{E}$ and $\vec{B}$ are related. Notice that curl of $\vec{E}$ relates to the time dependence to B by Faraday's law:
$$\partial_x E \to \partial_t B.$$
On the other hand, Ampere's law tells us that the curl of B is related to the displacement current:
$$\partial_x B \to \partial_t E.$$
But velocity and strain in the mechanical case are related in much the same way, using $\frac{\partial\psi}{\partial t}$ and $\frac{\partial\psi}{\partial x}$!

- Now consider the flow of energy through traveling waves. We know that we can calculate the power in mechanical waves via
$$P = F \cdot v = -T\frac{\partial\psi}{\partial x}\frac{\partial\psi}{\partial t},$$

With electromagnetic waves, the Poynting vector $\vec{S}$ is a measure of the energy per surface area per time. This is similar to having mass flow (recall that both mass and energy are conserved), and the expression is given by

$$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$$

In one dimension, this is really a product of $\vec{E}$ and $\vec{B}$, just like the power in mechanical waves is a product of velocity and strain.

- So how do we go from energy flow to actual energy? If we are given a current density ρ, this number also gives us a charge per unit area per time. Conservation laws of mass, energy, charge all tell us that the change in a given volume must flow out through the surface. So what we have is a continuity equation

$$\frac{d}{dt} \int \rho \, dV = -\int_S \vec{j} \, dA,$$

and we can use the divergence theorem to replace the right hand side with $-\int \text{div}\, \vec{j} \, dV$. So we can delete the integrals, since they are over identical, arbitrary surfaces, and this tells us that $\boxed{\dot{\rho} = -\text{div}\, j}$.

Well, something analogous works for EM waves! We have an energy density

$$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2,$$

and using an analogous argument, we must have $\boxed{\dot{u} = -\vec{\nabla} S}$. We can show this by finding the divergence of $E \times B$, which we'll probably do next time.

Fact 89

All in all, we can draw nice connections between one-dimensional plane EM waves and traveling waves on a string.

So back to the mechanical system: we have an energy density described by

$$u = \frac{\rho_L}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 + \frac{T}{2}\left(\frac{\partial \psi}{\partial x}\right)^2,$$

and we want to relate this density to our power P. We can integrate to find the change in energy between x_1 and x_2:

$$\frac{d}{dt} \int_{x_1}^{x_2} u \, dx = P(x_1) - P(x_2),$$

and now set $x_2 = x_1 + dx$ to get

$$\frac{d}{dt} u \, dx = P(x_1) - P(x_1 + dx).$$

Dividing by dx, $\boxed{\frac{\partial}{\partial t} u = -\frac{\partial}{\partial x} P}$. (And in multiple dimensions, the right hand side just becomes the divergence of P.) So let's calculate this by plugging in what we have for P: we'll get

$$T \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial t}\right)$$

and using the product rule, this indeed expands to the same expression as $\frac{\partial}{\partial t} u$, after using the wave equation. Physics works!

28 October 30, 2018

28.1 Review

Recall that massive strings, sound waves, and EM waves all satisfy the wave equation. It's important that Maxwell's equations in a vacuum give us the wave equations for the $\vec{E}$ field (and analogously B field)

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t^2};$$

it's remarkable that c can be found in terms of μ_0, ε_0, which are both constants that can be calculated without looking at light speed itself.

Last time, we looked at boundary conditions: our progressing EM waves (which have $\vec{E}$ and $\vec{B}$ in phase both in space and time) can be bounced back by a perfect conductor, which converts them to standing waves (where $\vec{E}$ and $\vec{B}$ are out of phase, both in space and time). Note that in all progressing waves so far, we have been dealing with **plane waves**! In other words, because our $\vec{E}$ and $\vec{B}$ only depend on t and one spatial variable (the direction of propagation), **each cross-sectional plane** will give the same values of $\vec{E}$ and $\vec{B}$.

One point to note: $\vec{E}$ and $\vec{B}$ can't both be 0 at any point in a standing wave, since energy will not be conserved. If we add up the energy in a standing wave, we will notice that the total energy is conserved, but the curve is not static (the amount of energy at each point isn't constant)! There is energy transfer, and at all points except nodes, there will be power transfer.

28.2 Phase and group velocity

We're going to introduce a new idea now: **how do we transfer information using our waves**? Infinite plane waves are nice, but unfortunately, a harmonic progressing wave like $\vec{E} = E_0 \cos(kz - \omega t)\hat{x}$ fills up the whole universe, so it can't actually give any information. But we can cut off the harmonic wave at a starting point and ending point to create a message! To analyze systems like this, let's start even simpler and go back to the one-dimensional string.

Suppose we send a **pulse** that can be detected, so that a nonzero displacement (or analogously, a nonzero electric field) gives a 1 and the default state (zero displacement) gives a 0. The propagation of the pulse can be described by a wave equation $f(x - vt)$, where v is the speed of propagation.

If we send a square pulse, the pattern isn't sinusoidal, but it seems like we can still Fourier decompose it. But our situation has been nice so far, because the dispersion relation $\omega(k) = v_p k$ is linear. This is idealized: a real life piano string has stiffness, so if we bend it, it will want to bend back.

Example 90

Suppose our string (or medium) satisfies the modified wave equation

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \left[\frac{\partial^2 \psi}{\partial x^2} - \alpha \frac{\partial^4 \psi}{\partial x^4} \right]$$

for some positive α. (There are no $\frac{\partial^3 \psi}{\partial x^3}$ terms, because symmetry tells us we can't have odd terms in this equation.)

Fact 91

What will happen to the speed of propagation? 12 people say it will increase, 16 say it will decrease, and 8 say it will stay the same.

To find out the answer, we can plug in a test function $A\cos(kx - \omega t)$, and our goal is to find the dispersion relation $\omega(k)$. The equation becomes

$$-\omega^2 A\cos(kx - \omega t) = v^2\left[-k^2 - \alpha k^4\right] A\cos(kx - \omega t),$$

and cancelling out common terms yields

$$\omega^2 = v^2(k^2 + \alpha k^4).$$

So $v_p = \frac{\omega}{k} = v\sqrt{1 + \alpha k^2}$, which is **larger** than our initial v. So the speed is larger than the natural v, and this makes sense because stiffness increases the restoring force!

So now if we graph ω vs k, the graph $\omega(k)$ will curve up instead of remaining linear – our initial wave equation is idealized. But this is really bad if we're sending a message. In particular, if we send our square pulse on a (non-ideal) string and let it propagate, some components will move faster than others (those with higher frequency), and the **wave will generally spread out**.

Example 92

Let's say we want to add the two progressing waves

$$\psi_1(x, t) = A\sin(k_1 x - \omega_1 t), \quad \psi_2(x, t) = A\sin(k_2 x - \omega_2 t).$$

Then $v_1 = \frac{\omega_1}{k_1}$ is the phase velocity of wave 1, and similarly $v_2 = \frac{\omega_2}{k_2}$. The sum-to-product formula tells us that

$$\psi = \psi_1 + \psi_2 = 2\sin\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)\cos\left(\frac{k_1 - k_2}{2}x - \frac{\omega_1 - \omega_2}{2}t\right)$$

If we assume $k_1 \approx k_2 \approx k$, $\omega_1 \approx \omega_2 \approx \omega$, then we get the **beat phenomenon** with a fast-oscillating **carrier** and outside **envelope.**

But the carrier and envelope aren't necessarily traveling at the speed! The carrier wave is moving at a **phase velocity**

$$v_p = \frac{\omega_1 + \omega_2}{k_1 + k_2},$$

while the speed of the envelope is the **group velocity**

$$v_g = \frac{\omega_1 - \omega_2}{k_1 - k_2} = \frac{\Delta\omega}{\Delta k}.$$

And assuming that the differences in ω and k are small, this gives us $\boxed{v_g = \frac{d\omega}{dk}}$.

So let's return to the dispersion relation we had for our stiff string, which was

$$\omega = kv\sqrt{1 + \alpha k^2}.$$

Fact 93

If we plot a point (k, ω) on the dispersion curve, the group velocity is the slope at the point, while the slope of the line connecting $(0, 0)$ to (k, ω) is the phase velocity.

Since $\omega(k)$ is concave up, the group velocity is faster than the phase velocity, which means the envelope moves faster than the wave itself.

Definition 94

A medium is **non-dispersive** if $\frac{\omega}{k}$ is constant and **dispersive** otherwise.

It turns out it's even possible to have a negative group velocity while having a positive phase velocity. (As a silly example, in a moonwalk, the body is moving backward, while the hands are moving forward.) And we can search up the concept of **negative stiffness** if we want to see physical applications of this.

With this more complicated dispersion relation, let's return to our wave on a string again, this time with boundary conditions. If a string has tension T, mass density ρ_L, and stiffness α, and $\psi(0, t) = \psi(L, t) = 0$, we can again solve for the allowed $k_m = \frac{m\pi}{L}, \alpha_m = 0$, just like before. It turns out that α does not affect the wave numbers of the normal modes at all, because we still have the same requirements on wavelength! So each normal mode is of the form

$$\psi_m(x, t) = \sin(k_m x)\sin(\omega_m t + \beta_m),$$

and this time, $\omega_m = v k_m \sqrt{1 + \alpha k^2}$. In other words, **the dispersion does not affect the wave numbers**, but it does affect the frequencies and phase velocities of the individual normal modes.

If we graph ω_m against k_m, the wave numbers k_ms are still equally spaced (apart by $\frac{m\pi}{L}$), but the ω_ms get wider and wider! So if we try to superimpose the first two normal modes, it would take a very long time to get the periods to line up.

But what's a good way to understand the phase velocity intuitively?

Example 95

Consider a plane wavefront that hits a wall at an angle. Then the wavefront will move faster than the actual speed of the propagating wave (for example, imagine a wave crashing on the shore at a very slight angle).

In particular, if we just choose a good enough angle, a detector at the wall can see the peaks and troughs move faster than the speed of light! The reason this doesn't break rules about lightspeed is because **phase velocity doesn't actually carry any information**. This means it's just the shape that is traveling "faster than the speed of light," not any actual information being carried by the wave.

29 October 31, 2018 (Recitation)

Two quick tips. Recall that light in a vacuum travels with the dispersion relation $\omega = ck$, which means the frequency ν satisfies $\omega = \nu 2\pi$. (This is just a definition – traveling 2π radians per second and 1 revolution per second are the same thing.) Also, when we have an exponential $e^{i(\text{blah})}$ as a solution, **both** the real and imaginary parts must solve the differential equation by themselves. So both cosines and sines will work!

Let's go back to EM waves and derive the continuity equation for energy – we will show that $-\frac{\partial u}{\partial t} = \text{div}\vec{S}$. (Here, we're defining the **Poynting vector** $S = \frac{1}{\mu_0}\vec{E} \times \vec{B}$.) So in other words, we are trying to show that

$$\frac{1}{\mu_0}\text{div}(\vec{E} \times \vec{B}) = -\frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0}B^2\right).$$

So the first thing we need to do is evaluate the left-hand side $\vec{\nabla} \cdot (\vec{E} \times \vec{B})$. Note that this is a **triple scalar product**, and it finds the area of a parallelepiped, so we can usually permute the entries however we'd like. Thus, it seems like we can almost write

$$\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{E}) \cdots,$$

but **not quite**! We have to think of it as a product rule in which the first term "acts" on $\vec{E}$ and the second "acts" on B. This is because when we write out this triple scalar product as a determinant, we can treat $\vec{\nabla}$ as a vector on its own $(\partial_x, \partial_y, \partial z)$, as long as ∂xy still becomes $x\partial y + y\partial x$. This means that the actual expression we want is

$$\boxed{\vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \vec{B}(\vec{\nabla} \times \vec{E}) + \vec{E}(\vec{B} \times \vec{\nabla})} = \vec{B}(\vec{\nabla} \times \vec{E}) - \vec{E}(\vec{\nabla} \times \vec{B}).$$

Plugging in Maxwell's equations, this tells us that the left hand side is

$$\frac{1}{\mu_0}\left(\vec{B} \cdot -\frac{\partial \vec{B}}{\partial t} - \vec{E}\mu_0\varepsilon_0\frac{\partial \vec{E}}{\partial t}\right).$$

Indeed, evaluating the derivative on the right hand side and using Maxwell's equation also gives us this expression, and we've proved the formula.

Next, let's talk about the concepts of **phase and group velocity** from last lecture. In class, we took our expression

$$A\sin(k_1 x - \omega_1 t) + A\sin(k_2 x - \omega_2 t)$$

and wrote it as a product. One term in the product corresponds to the average $\sin\left(\frac{k_1+k_2}{2} - \frac{\omega_1+\omega_2}{2}\right)$, and the other corresponds to the difference $\cos\left(\frac{k_1-k_2}{2} - \frac{\omega_1-\omega_2}{2}\right)$. The first term gives the average ω over the average k, while the second term gives the differential $\Delta\omega$ in ratio to Δk. And the latter is the group velocity: we can make the envelope (by Fourier synthesis) a sharp pulse or some other shape by controlling this derivative $\frac{d\omega}{dk}$.

Fact 96

We can even use something called **Fourier integrals** to make a single pulse and avoid periodicity completely, but this isn't too relevant to the class.

So let's ask some questions about our velocities v_p and v_g. If the dispersion relation is "curved" one way or the other, we can have $v_g > v_p$ or vice versa, and it's also possible to have v_p and v_g to have opposite signs.

But **can v_p and v_g be faster than the speed of light**? It turns out the answer is yes in both cases. We know information cannot be transferred faster than the speed of light, but wavefronts are not particles, so nothing is violated if $v_p > c$! (Phase velocity tells us about maxima and minima: no information could be sent with just that data.)

The situation is more subtle with the group velocity: why can we have $v_g > c$? The basic answer is that the group velocity is defined as the velocity of the **maximum height** of the envelope. If a pulse hits a new medium, it's possible the first part of pulse gets transmitted, but not the rest. But the maximum point of the pulse is still before the medium, so the key idea is that the **leading wavefront** is transmitted, rather than the part which tells us the group velocity. In other words, the geometric maximum does not necessarily correspond to the location of the energy.

30 November 1, 2018

30.1 Review

Recall from last time that pulses will deform and spread out in a dispersive medium, because different Fourier coefficients travel at different speeds. We quantified this using the different concepts of a **phase velocity** $v_p = \frac{\omega}{k}$ and **group velocity** $v_g = \frac{\partial \omega}{\partial k}$, corresponding to the motion of the carrier and envelope.

So we'll ask two questions today: what is group velocity, and how do we solve the problem of dispersion?

30.2 The Fourier transform

Recall that we can always describe the displacement in a medium by a single wave function $\psi(x, t)$. In a non-dispersive medium, everything travels at the same speed $v = v_p = v_g$, so we can just describe our wave as a propagating function

$$\psi(x, t) = f\left(t - \frac{x}{v}\right).$$

But in a dispersive medium, we need to be more careful: one way is to divide the wave into individual components and then sum everything back together.

Theorem 97 (Fourier transform)

Given a function $f(t)$, we can decompose it into harmonic oscillations:

$$f(t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega t} d\omega,$$

where $C(\omega)$ is a complex amplitude (which can encode a phase) that we'll derive soon.

Notably, f does not have to be periodic! This is an important distinction from the more discrete Fourier series that we discussed earlier in this class.

So, let's say we've written our function $f(t)$ this way, and this is the signal that we want to transmit. (We can think of this as $\psi(0, t)$.) To find the actual transmitted wave $\psi(x, t)$, we just need to add in the space component:

$$\boxed{\psi(x, t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega t + ik(\omega)x} d\omega}.$$

Here, each component propagates at its own propagating speed: notice that the exponential looks a lot like our simpler expression $kx - \omega t$, and we can get k for any value of ω by inverting the dispersion relation $\omega(k)$.

Example 98

Let's do this for a non-dispersive medium: say we have $k(\omega) = \frac{\omega}{v}$.

So our function becomes

$$\psi(x, t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\left(\omega t - \frac{\omega}{v}x\right)} d\omega = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega\left(t - \frac{x}{v}\right)} d\omega = f\left(t - \frac{x}{v}\right)$$

as expected, because k is always a constant times ω and thus we can factor out the common terms.

And finding the coefficients $C(\omega)$ looks a lot like the Fourier decomposition problem: we'll use our "mode picker" again.

Definition 99

The **Dirac delta function** $\delta(x)$ is defined as

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

with the property that $\int_{-\infty}^{\infty} \delta(x)dx = 1$.

An important corollary is that we can integrate the delta function against another function to "pick out" a particular value:

$$\int_{-\infty}^{\infty} \delta(x-\alpha)f(\alpha)d\alpha = f(x).$$

Proposition 100 (Orthogonality of the Exponential Function)

For all $\omega, \omega' \in \mathbb{R}$,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt = \delta(\omega-\omega')$$

This is essentially because whenever $\omega \neq \omega'$, the integrand is even, so the integral is zero when we integrate from $-\infty$ to ∞. So we can actually evaluate the following expression as a double integral:

$$\begin{aligned}
&\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} C(\omega')e^{-i\omega' t}d\omega'\right)e^{i\omega t}dt \\
&= \boxed{\frac{1}{2\pi}}\int_{-\infty}^{\infty} C(\omega')d\omega'\left(\int_{-\infty}^{\infty} e^{i(\omega-\omega')t}dt\right) \\
&= \int_{-\infty}^{\infty} C(\omega')\delta(\omega-\omega')d\omega = \boxed{C(\omega)},
\end{aligned}$$

and we've **found an expression for our coefficients** $C(\omega)$! Notice that this is basically just the Fourier series idea in the continuous case. But we still haven't answered our problem: how do we deal with dispersion of (for example) EM waves in non-vacuum material?

30.3 A cool application: amplitude modulation

We'll use the ideas that lead to **AM (Amplitude Modulation)** radio! Let $f_s(t)$ be a signal I want to send, and let's say that sound has a frequency around 1 kilohertz (for instance, high-pitched music). Then we can put it in a **carrier** $\cos(\omega_0 t) = e^{i\omega_0 t}$, so we actually send the wave

$$f(t) = f_s(t)\cos(\omega_0 t).$$

And we're going to make the ω_0 be a much higher frequency than the rest: ω_0 is often anywhere from 0.1 to 30 megahertz.

Why do we want to do this? Our new signal has a very interesting property that $C(\omega)$ is nonzero around $\omega \approx \omega_0$, but all other contributions become very, very small. This is because we're forcing a very fast oscillation! Imagine we have a contribution from the $\cos(\omega_s t)$ frequency that we want to send. After being modulated,

$$\cos(\omega_s t)\cos(\omega_0 t) = \frac{1}{2}\left[\cos((\omega_0+\omega_s)t) + \cos((\omega_0-\omega_s)t)\right]$$

Since $\omega_0 \gg \omega_s$, this is now only a small adjustment relative to the carrier frequency. And remember that we have a beat phenomenon going on, so we care about the shape of the **envelope**, not the carrier: this travels at the **group velocity** of the wave.

So now if we consider our dispersion relation $\omega(k)$, instead of looking at the contribution from $\omega = 0$ to $\omega = \omega_s$, we're now looking at a very narrow wavenumber slice k around $k = k_0$ and frequency around $\omega = \omega_0$:

$$\omega(k) = \omega_0 + (k-k_0)\frac{\partial \omega}{\partial k} + \cdots \implies \boxed{\omega \approx \omega_0 + (k-k_0)v_g}.$$

Proposition 101

Suppose we make a linear approximation for the dispersion relation $\omega(k)$ at a point (ω_0, k_0). Then the signal $f(t) = \mathrm{Re}\left(f_s(t)e^{-i\omega_0 t}\right)$ will turn into the a signal traveling at the group velocity, modulated by $e^{i(\omega_0 t - k_0 x}$.

In other words, we will have a wave function

$$\psi(x,t) = \mathrm{Re}\left(f_s\left(t - \tfrac{x}{v_g}\right)e^{-i(\omega_0 t - k_0 x)}\right),$$

where the envelope is moving at the group velocity v_g and the carrier is moving at the phase velocity $\frac{\omega_0}{k_0}$. Let's show the calculations for this:

Proof. Let

$$f_s(t) = \int_{-\infty}^{\infty} C(\omega)e^{-i\omega t}d\omega.$$

Make our "AM radio," modulating by a frequency ω_0, so

$$f(t) = f_s(t)e^{-i\omega_0 t} = \int_{-\infty}^{\infty} C(\omega)e^{-i(\omega+\omega_0)t}d\omega.$$

Change the variables so that $\omega \to \omega - \omega_0$: this yields

$$f(t) = \int_{-\infty}^{\infty} C(\omega-\omega_0)e^{-i\omega t}d\omega.$$

Now, we wish to prepare our $\psi(x,t)$, and recall that we prepare our individual components based on frequency. Since we're making the linear approximation $\omega = \omega_0 + (k-k_0)v_g$ from above, we now know that $k = \frac{\omega-\omega_0}{v_g} + k_0 = \frac{\omega}{v_g} + b$ (with our new variable change), so

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} C(\omega-\omega_0)e^{-i\omega t + ik(\omega)x}d\omega \\ &= \int_{-\infty}^{\infty} C(\omega-\omega_0)e^{-i\omega t + i\left(\frac{\omega}{v_g}+b\right)x}d\omega.\end{aligned}$$

Collecting all of the ω terms together,

$$\psi(x,t) = \int_{-\infty}^{\infty} C(\omega-\omega_0)e^{-i\omega\left(t-\frac{x}{v_g}\right)}e^{ibx}d\omega,$$

and now changing our variables back with $\omega \to \omega + \omega_0$, we find that

$$\begin{aligned}\psi(x,t) &= \int_{-\infty}^{\infty} C(\omega) e^{-i(\omega+\omega_0)\left(t-\frac{x}{v_g}\right)} e^{ibx} d\omega \\ &= \boxed{\int_{-\infty}^{\infty} C(\omega) e^{-i\omega\left(t-\frac{x}{v_g}\right)}} e^{-i\omega_0 t} e^{i\left(\frac{\omega_0}{v_g}+b\right)x} d\omega.\end{aligned}$$

But the first boxed term is $f_s\left(t - \frac{x}{v_g}\right)$, our original signal, and since $\frac{\omega_0}{v_g} + b$ was defined to be k_0, the rest modulates our signal f_s as $e^{(ik_0 x - i\omega_0)t}$, as desired. $\square$

So the consequence is that as long as we can make that approximation to first order, the dispersive medium will not matter: **we're using such a narrow band of frequencies that they will move almost at the same speed**! So we pick our ω_0 (and corresponding k_0), and the group velocity – that is, the speed at which things are actually transmitted – will be the value of $\frac{\partial \omega}{\partial k}$ in this range, which will be basically constant.

Fact 102

It's possible that the first-order approximation isn't very good, and this does happen if we somehow have a very unfortunate medium where we can't get a nice linear fit anywhere. But in practice, usually they do, and that's why AM radios are still around! (And as a sidenote, radio waves are refracted by the ionosphere so that they can be transmitted around the world.)

With this, we've now learned how to send material through a dispersive medium! Next time, we will talk about the uncertainty principle.

31 November 5, 2018 (Recitation)

A lot of people talk about slow light: we can make the group velocity much slower than c, and Professor Ketterle's lab managed to make the group velocity approximately 1 meter per second.

But let's demystify what's going on here. The velocity in a medium can be found with the equation

$$c_{\text{med}} = c_{\text{vacuum}}/n,$$

where n is the index of refraction. Glass has an index n between 1.5 and 2, so we can slow down light by having it pass through glass. But this is a lot less of a change than a factor of 3×10^8.

Well, everything goes back to Maxwell's equation. Recall that we calculated the speed of light to be

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}},$$

so in different mediums, there are different values for each of these constants. For example, if magnetic fields want to move, they have to create both a displacement current and a real current. So let's say a photon is at resonance with an atom. This means that an electron is raised to a higher energy level, but because this is unstable, it will move back down and emit another photon.

And indeed that's how slow light works. A photon hits a Bose-Einstein condensate, and there is some average time τ before another photon is emitted and the light continues propagating. Then the subsequent photon continues moving until hits another atom, and so on! This gives the illusion that a single photon is moving at a much slower speed than c.

On the other hand, we can have frozen light: make $\tau \to \infty$, and we can make light infinitely slow. So light will enter a medium, and by controlling resonance, we can "capture" a light pulse. And that just means the light has been permanently absorbed, which is a kind of **photography**.

Fact 103

The whole point of photography is to record the intensity distribution, but there's a phase distribution as well which we measure via **holography**! This is still classical light and relates to Maxwell's equations. But light also has some quantum properties, and this is something I don't understand yet.

Next, let's talk about the Fourier and inverse Fourier transform. Another way to phrase what we said during lecture is that we can write any function in terms of exponential functions using the form

$$f(x) = \int dk\, a(k) e^{-ikx}.$$

Think of e^{ikx} as a basis and f as a vector in this vector space. So to extract a specific element, we just project onto that specific direction:

$$a_k = \hat{e}_k \cdot \vec{A} \implies a(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{ikx} dx$$

So **integrals act as inner products** in a normal vector space. And the reason the extraction e^{ikx} has a positive exponent while the original Fourier transform has a negative exponent is that the inner product of two complex-valued scalar functions is defined as

$$\langle f, g \rangle = \int f(x) g^*(x) dx,$$

where g^* denotes the conjugate of g. We do this so that $\langle f, g \rangle$ can define a **norm** – we need to make sure the scalar product of f with itself is a positive real number! (And notice that $a(k) = a(-k)$ implies that f will always be real, since the complex parts will always cancel out.)

With that, let's compare the discrete and continuous Fourier transforms. If we have a function defined from $x = 0$ to $x = L$ with the boundary conditions $f(0) = f(L) = 0$, we know that we can write f as a sum of normal modes:

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(k_n x), \quad k_n = \frac{n\pi}{L}.$$

Well, this function is defined for all real numbers, not just the range $[0, L]$: it's periodic with period $2L$, and the extension is an odd function. To find the coefficients a_n, we know that

$$a_n = \frac{2}{L} \int_0^L f(x) \sin(k_n x) dx.$$

If we let $L \to \infty$, our Fourier series turns from a sum $f(x) = \sum a_n \sin(k_n x)$ into an integral $\int a_n \sin(k_n x) dn$. But $k_n = \frac{n\pi}{L}$, so we can make a u-substitution of $dn = dk \frac{L}{\pi}$. This gives

$$f(x) = \frac{L}{\pi} \int a_n \sin(k_n x) dk,$$

and since $a_k = \frac{2}{L} \int_0^L f(x) \sin(k_n x) dx$, we can normalize: let $a_k = a_n \frac{L}{\pi}$ and take $L \to \infty$, and we recover the familiar equations

$$a_k = \frac{2}{\pi} \int_0^{\infty} f(x) \sin k_n x dx, \quad f(x) = \int_0^{\infty} a_k \sin(kx) dk.$$

In other words, we've replaced sums by integrals and gone from the discrete to the continuous Fourier transform.

32 November 6, 2018

32.1 Clearing up some loose ends

Recall that last time, we had a delta function which we didn't define too rigorously. We'll be a bit more careful now:

Definition 104

Define

$$\delta_n(\omega) = \frac{1}{2\pi}\int_{-n}^{n} e^{i\omega t} dt = \frac{\sin(\omega n)}{\omega n}.$$

The integral from $-\infty$ to ∞ of δ_n is 1 for any n.

Now we take $n \to \infty$; the function itself does not exist, but we can say that the definite integral

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \delta_n(x) f(x) dx = f(0)$$

does converge, because since this limit does exist. If we do this, we can also shift our variables to say that (using this limit definition)

$$\delta(\omega - \omega') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(\omega-\omega')t} dt.$$

32.2 Review

Exam 2 will be in Walker (50-340) during normal lecture time. It will cover lectures 9 to 17 (so it will go up to the material covered on Thursday).

Recall from last time that we can start with a function $f(t)$ (which is the displacement or wavefunction that we measure at $x = 0$) and then write it in the Fourier form

$$f(t) = \int_{-\infty}^{\infty} C(\omega) e^{-i\omega t} d\omega.$$

Using the inverse Fourier transform, we can also find our coefficients

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t) e^{i\omega t} dt.$$

And now, even with a dispersive medium, we can actually write out $\psi(x, t)$ by propagating each frequency separately:

$$\psi(x, t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{i\omega t} e^{ik(\omega)x},$$

where $k(\omega)$ is determined from the dispersion relation. We then extended this to the idea of AM radio: since most dispersion relations become **more linear** as ω increases, we can take our signal and make it travel using a fast carrier. Then the envelope will travel at the group velocity $v_g = \frac{d\omega}{dk}$ (evaluated at ω_0) and the carrier will travel at the phase velocity, $v_p = \frac{\omega_0}{k_0}$.

32.3 The uncertainty principle

Let's start with a sample problem:

Example 105

Let $f(t) = e^{-\Gamma|t|}$. What are the Fourier coefficients $C(\omega)$?

We can easily integrate:

$$\begin{aligned} C(\omega) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}e^{\Gamma|t|}dt \\ &= \frac{1}{2\pi}\left[\int_{-\infty}^{0} e^{\Gamma t}e^{i\omega t}dt + \int_{0}^{\infty} e^{-\Gamma t}e^{i\omega t}\right] \\ &= \frac{1}{2\pi}\left[\frac{1}{\Gamma + i\omega} + \frac{1}{\Gamma - i\omega}\right] \\ &= \frac{1}{2\pi}\frac{2\Gamma}{\Gamma^2+\omega^2} = \boxed{\frac{\Gamma}{\pi(\Gamma^2+\omega^2)}}. \end{aligned}$$

If we plot $C(\omega)$, which gives the "wave population" in frequency space, we get a narrow peak. And as we increase Γ, the frequency space gets wider – basically, we need larger values of ω to describe a narrow pulse at some fixed time.

And this intuition is related to the uncertainty principle. **If we want to have a very precise arrival time of our signal $f(t)$, we don't hava precise measurement for the frequency of the signal**. To make this more precise, we'll need to be more mathematical:

Definition 106

The **intensity** of a pulse is a function proportional to $|f(t)|^2$.

We know the energy of the electric field is proportional to E^2, and potential and kinetic energy in a mechanical system also have this quadratic factor.

Definition 107

The **average value** of a function $g(t)$ be

$$\langle g(t)\rangle = \frac{\int_{-\infty}^{\infty} g(t)|f(t)|^2}{\int_{-\infty}^{\infty}|f(t)|^2}$$

This should look a lot like the center of mass calculation that we have seen before, for example, when calculating the moment of inertia. It's basically a way to find the average of a continuous distribution!

Next, we want a measure of spread: a lot of people like to talk about it when looking at exam grades.

Definition 108

For a function $f(t)$, let the **spread in time** be the average of the squared deviations:

$$\Delta t^2 = \langle (t - \langle t\rangle)^2\rangle = \frac{\int_{-\infty}^{\infty} dt\,|(t-\langle t\rangle)f(t)|^2}{\int_{-\infty}^{\infty} dt\,|f(t)|^2}.$$

Similarly, let the **spread in frequency** for a function $C(\omega)$ be

$$\Delta\omega^2 = \langle(\omega - \langle\omega\rangle)^2\rangle.$$

If we've done some probability before, these are the **variances** of the distributions for t and ω (in time and frequency space, respectively). A small spread means we have a good idea what the actual value of our function is.

Then **uncertainty** says Δt and $\Delta\omega$ can't be small at the same time!

> **Theorem 109** (Uncertainty principle)
> For any function $f(t)$ with Fourier coefficients $C(\omega)$,
> $$\Delta\omega\Delta t \geq \frac{1}{2}.$$

Proof. First of all, here is one tool that will be useful for us: notice that

$$\int_{-\infty}^{\infty} \omega C(\omega)e^{-i\omega t}d\omega = i\frac{\partial}{\partial t}\int_{-\infty}^{\infty} C(\omega)e^{-i\omega t}d\omega = i\frac{\partial}{\partial t}f(t).$$

So we can "generate an ω in an integral" by tacking on a derivative $i\frac{\partial}{\partial t}$. (This is a more common trick in quantum mechanics.) First, let's try to figure out what our $\langle\omega\rangle$ is: it's hard to find an average of ω directly using the equation, so let's see if we can write out ω in terms of t.

Specifically, we're going to replace ω with the operator $i\frac{\partial}{\partial t}$. We find that

$$\langle\omega\rangle = \frac{\int_{-\infty}^{\infty}\omega|f(t)|^2dt}{\int_{-\infty}^{\infty}|f(t)|^2dt} = \frac{\int_{-\infty}^{\infty}\omega f^*(t)f(t)dt}{\int_{-\infty}^{\infty}|f(t)|^2dt},$$

where f^* is the complex conjugate, and now substituting with the above tool (note that the location of the derivative does matter), we have that

$$\boxed{\langle\omega\rangle = \frac{\int_{-\infty}^{\infty} f^*(t)i\frac{\partial}{\partial t}f(t)dt}{\int_{-\infty}^{\infty}|f(t)|^2}dt}.$$

With this, we can start working towards a formula for our variance:

$$\Delta\omega^2 = \langle(\omega-\langle\omega\rangle)^2\rangle = \frac{\int_{-\infty}^{\infty}\left|\boxed{\left(i\frac{\partial}{\partial t}-\langle\omega\rangle\right)}f(t)\right|^2dt}{\int_{-\infty}^{\infty}|f(t)|^2dt},$$

where the boxed part is $\omega - \langle\omega\rangle$.

Now we define a function

$$r(\kappa, t) = \left([t-\langle t\rangle] - i\kappa[i\tfrac{\partial}{\partial t} - \langle\omega\rangle]\right)f(t)$$

and define the auxiliary variables $T = t - \langle t\rangle$, $\Omega = i\frac{\partial}{\partial t} - \langle\omega\rangle$. Then our function can be written

$$r(\kappa, t) = (T - i\kappa\Omega)f(t).$$

Also define the function

$$R(\kappa) = \frac{\int_{-\infty}^{\infty}dt|r(\kappa,t)|^2}{\int_{-\infty}^{\infty}dt|f(t)|^2}.$$

This is greater than or equal to 0, since the integrals in the numerator and denominator are both positive. **Let's first evaluate the numerator**: we know that

$$|r(\kappa,t)|^2 = r(\kappa,t)r(\kappa,t)^* = (T-i\kappa\Omega)f\cdot(T+i\kappa\Omega^*)f^* = |Tf|^2 + |\kappa\Omega f|^2 + i\kappa\left[Tf\Omega^*f^* - \Omega fTf^*\right].$$

Let's look at the three terms separately. The third term becomes

$$i\kappa\left[Tf\Omega^*f^* - \Omega fTf^*\right] = i\kappa\left[Tf(-i\tfrac{\partial}{\partial t} - \langle\omega\rangle)f^* - \left(i\tfrac{\partial}{\partial t} - \langle\omega\rangle\right)fTf^*\right].$$

Luckily, we can cancel the $\langle\omega\rangle T f f^*$ terms, and we'll end up with

$$\kappa T\left(f\frac{\partial f^*}{\partial t}+\frac{\partial f}{\partial t}f^*\right)=\kappa T\frac{\partial}{\partial t}(ff^*).$$

Integrating this from $-\infty$ to ∞ (as we do in the numerator of $R(\kappa)$, we get

$$\int_{-\infty}^{\infty}\kappa T\frac{\partial}{\partial t}(ff^*)dt,$$

and integration by parts simplifies this to

$$\kappa T f f^*|_{-\infty}^{\infty}-\kappa\int_{-\infty}^{\infty}\frac{\partial T}{\partial t}|f|^2dt.$$

If f is localized (that is, it goes to 0 on the ends), the first part here goes to 0. For the second part, $\frac{\partial T}{\partial t}$ is the derivative of $t-\langle t\rangle$ with respect to t, which is just 1. So what's left is just $-\kappa\int_{-\infty}^{\infty}|f|^2dt$! Remembering that this is the numerator of $R(\kappa)$, **this third term simply turns out to be $-\kappa$.**

But the other two terms are simpler: the first term is the definition of Δt^2, and the second term turns out to be $\kappa^2\Delta\omega^2$. So we've found that

$$R=\Delta t^2+\kappa^2\Delta\omega^2-\kappa>0.$$

Now choose the optimal κ: we want to use the value such that $\frac{dR}{d\kappa}=0$, and it turns out $\kappa=\frac{1}{2\Delta\omega^2}$. Plugging this back in,

$$\Delta t^2-\frac{1}{4\Delta\omega^2}\geq 0,$$

and clearing denominators yields

$$4\Delta t^2\Delta\omega^2-1\geq 0\implies\Delta t\Delta\omega\geq\frac{1}{2},$$

as desired. □

So in summary, we found the intensity, average, and spread of t and ω. We used some random stuff from quantum mechanics and found that a weird function turns out to be R. But conveniently, this gives us that the product of the standard deviations of ω and t are bounded from below!

So this means we can't have very narrow ω and t at the same time. If we wanted to make an argument to turn this into the familiar quantum mechanics uncertainty principle, we can rewrite as

$$v\Delta t\frac{\Delta\omega}{v}\geq\frac{1}{2}.$$

The first term is Δx and the second term is Δk, so the position and wavenumber have some uncertainty relation too. Well, we have $p=\hbar k$ from the de Broglie wavelength, so we plug this in to get

$$\Delta x\Delta p\geq\frac{\hbar}{2},$$

which is Heisenberg's uncertainty principle. But notice that the idea really comes from the Fourier transforms and properties of waves! **Uncertainty does not come from quantum mechanics, but actually from wave mechanics.**

Here's one more interesting consequence. Take these equations (from quantum mechanics and relativity) for granted:

$$E=\hbar\omega,\ p=\hbar k,\ E^2=p^2c^2+m^2c^4.$$

We can use these to find the **dispersion relation**

$$\omega^2 = c^2k^2 + \omega_0^2, \quad \omega_0^2 = \frac{mc^2}{\hbar}.$$

So if we look at $\frac{\omega}{k}$, we have a nonlinear dispersion relation when ω_0 is nonzero or if photons have mass! It's useful now to use some data from a **pulsar**, which is a rapidly rotating neutron star. It emits pulses of radiation, and we can use it to measure the arrival time of light at different angular frequencies.

It turns out that ω_0 was experimentally found to be nonzero – we found that the mass of a photon is about 1.3×10^{-49} grams, which is definitely not true (photons are massless). And the explanation for this result is that the free electrons in space actually distorted the EM field.

33 November 7, 2018 (Recitation)

We'll start today with a concrete example. Let's say we're trying to find the Fourier coefficients for the function

$$f(t) = e^{-\frac{t^2}{2\sigma^2}}.$$

As always, we can find our Fourier coefficients with the equations

$$f(t) = \int_{-\infty}^{\infty} C(\omega)e^{-i\omega t}d\omega,$$

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt.$$

There are three examples of important curves we're likely to see often: Gaussian curve, exponential decay, and the box function.

Whenever we have a resonance curve with peaks, they are usually Gaussian or Lorentzian. Gaussian distributions are proportional to e^{-t^2}, and these come up because of the Central Limit Theorem. On the other hand, Lorentzian distributions with only one "lifetime" or other parameter are often characterized by a distribution of $\frac{1}{(\omega-\omega_0)^2+(\Gamma/2)^2}$. Notice that this falls off as $\frac{1}{\omega^2}$, which is much slower than the subexponential decay rate for the Gaussian.

Well, it turns out the Fourier transform connects a lot of these common functions! Gaussians are sent to other gaussians, exponential decay and Lorentzian distributions are sent to each other, and box distributions are related to a $\sin(x)/x$ function.

Anyway, let's integrate to answer our initial question:

$$C(\omega) = \frac{1}{2\pi}\int e^{-\frac{t^2}{2\sigma^2}+i\omega t}dt.$$

Since we're integrating with complex numbers, we technically need to be a bit more mathematically precise: we can complete the square and then do a **complex contour integral**. Integrating around the real axis can be completed in the complex plane by drawing a semicircle, and as the radius of that semicircle goes to infinity, the contribution from the rest of the semicircle goes to 0. (But we don't need to really understand this.)

Next, we're going to look at the uncertainty principle: if $f(t)$ is a pulse, we can find its approximate or effective width Δt. Similarly, we can find an effective width in frequency $\Delta\omega$.

In particular, we use a quadratic weight: we say that the average value of the variable t is

$$\langle t\rangle = \frac{\int tf(t)^2dt}{\int f(t)^2dt}$$

Usually, f is normalized, so the denominator is 1, and we have a probability weighting in the numerator. Similarly, we can find $\langle t^2 \rangle$ (by replacing the t in the numerator with t^2), and now we can find the variance

$$\Delta t^2 = \langle (t - \langle t \rangle)^2 \rangle = \langle t^2 \rangle - 2\langle t \rangle \langle t \rangle = \boxed{\langle t^2 \rangle - \langle t \rangle^2},$$

so we can think of variance in terms of the averages of t^2 and t.

By the way, a shorter pulse has a wider Fourier transform, and vice versa. That's what the uncertainty relation tells us! Intuitively, we can draw a phasor diagram. At time $t = 0$, all components are equal to 1, so everything lines up to give the maximum amplitude for our wave. However, after that, things stop lining up, and having different frequencies ω give us a spread in angle. Well, the spread in angles of phasors is related to $\Delta\omega \cdot t$, and so that's another way in which the uncertainty relation comes up!

Let's do a concrete example: let's say we have a function that is β from $-\frac{\sigma}{2}$ to $\frac{\sigma}{2}$ and 0 everywhere else. Then we can find the Fourier transform easily:

$$C(\omega) = \frac{1}{2\pi} \int_{-\sigma/2}^{\sigma/2} \beta e^{i\omega t} dt = \frac{\beta}{2\pi} \left[\frac{1}{i\omega} e^{i\omega t} \right]_{-\sigma/2}^{\sigma/2} = \frac{\beta \sin(\sigma\omega/2)}{\pi\omega}$$

Can we do anything with uncertainty here? The amplitude decays as $\frac{1}{\omega}$ here, so we can say that the uncertainty in frequency is on the order of the size between two zeros. So $\Delta\omega = \frac{2\pi}{\sigma}$, and notice that

$$\Delta\omega\Delta t \approx \frac{2\pi}{\sigma}\sigma = \text{constant}$$

as expected.

Remark 110. *When computing Fourier transforms, we often have $C(\omega) = C(-\omega)$. So often, we can write our function as*

$$f(t) = \int_0^\infty C(\omega)(e^{i\omega t} + e^{-i\omega t}) d\omega,$$

which means we can represent our function f simply as

$$f(t) = \frac{2\beta}{\pi} \int_0^\infty \frac{\sin(\sigma\omega/2)}{\omega} \cos(\omega t) d\omega.$$

In this case, we don't even need to use complex numbers!

In the remainder of this recitation, we'll preview some of the upcoming material and talk about **multiple dimensions**. Let's say we have a set of $N = 3$ coupled oscillators in one dimension; this gives us 3 normal modes. Then all waves are normal modes of the form

$$\psi_n = A \sin(k_n j a + \alpha) \sin(\omega_n t + \phi).$$

Let's review where all of these terms come from: A_n and ϕ_n are determined by initial conditions, but k_n, α_n are determined by boundary conditions. (For example, a fixed end gives $\alpha = 0$ and discrete values for k_n.) Also, these 3 normal modes give 6 free parameters to fulfill 6 initial conditions: 3 displacements and 3 initial velocities.

But let's extend this now. Look at a 3 by 3 grid of masses connected by springs in the natural way, and we now have a 2-dimensional system. But here's what's important about this: **any pattern in the x-direction which is constant along y will be a valid solution**, since the net vertical force is 0. But this means we have a **separable** system! So our normal modes are now of the form

$$\psi_{n,m} = A_{n,m} \sin(k_n j a + \alpha_n) \sin(k_m i b + \alpha_m) \sin(\omega_n t + \phi_n),$$

where we just combine an x-normal mode and a y-normal mode together! So that's why we have 3×3 normal modes, and we'll see some more detail in the upcoming lectures.

34 November 8, 2018

34.1 Review

Let's recap the past few topics: we have been looking at waves in media that are dispersive versus nondispersive. At a constant velocity, $\omega = v|k|$, and the waveform is preserved, which means information can be transmitted. On the other hand, if we have a nonlinear medium with multiple ω (in other words, not a cosine function), the waveform will be distorted, so information will be lost.

One way we've found to fix this is to add a faster carrier, since the dispersion relation is much more linear for large frequencies ω. We used this to define $\Delta\omega$ and Δt, and this gave us an uncertainty principle: we cannot have both Δt and $\Delta\omega$ small at the same time.

34.2 Generalizing to more dimensions

Next, let's look at 2D and 3D waves. We're only going to look at a small subset of those examples: those that are analytically solvable, which usually means we have a symmetric system. Also, small-angle approximations are important for us, so we'll ignore higher-order terms.

Recall that in one dimension, we looked at a linear chain of coupled oscillators. They were separated by a distance a and all had mass m, and there was a constant tension T. Oscillations were transverse, and we tried to find $y(t)$ for all masses. First, we considered the discrete case and solved for the positions of individual masses $y_j(t)$, using our equations of motion of the form

$$y_j = \frac{T}{ma}(y_{j-1} - 2y_j + y_{j+1}).$$

Example 111

Consider a 2-dimensional mesh of beads in the xy-plane, where motion is only in the z-direction. The spacing horizontally is a_H, while the spacing vertically is a_V. Index the beads as (j_x, j_y). Vertical masses are connected by strings of tension T_V, while horizontal masses are connected by strings of tension T_H. Edge beads are also connected to a rigid frame. What's $z_{(j_x,j_y)}(t)$?

Drawing a free-body diagram, we can add the x and y-direction forces together. For any mass, we only get contributions from the four masses next to it. This yields the equation

$$m\ddot{z}_{(j_x,j_y)}(t) = \frac{T_H}{a_H}\left(z_{(j_x+1,j_y)} + z_{(j_x-1,j_y)} - 2z_{(j_x,j_y)}\right) + \frac{T_V}{a_V}\left(z_{(j_x,j_y+1)} + z_{(j_x,j_y-1)} - 2z_{(j_x,j_y)}\right),$$

where we're just adding the forces in each of the 1D cases.

How do we proceed? Recall that in the one-dimensional case, we found our normal modes of the form

$$y_j(t) = Ae^{i(kaj-\omega t)},$$

with a dispersion relation

$$\omega = \sqrt{\frac{4T}{ma}}\left|\sin\frac{ka}{2}\right|.$$

Well, now we can use those to find our solutions! This time, we'll be trying to find normal modes of the form

$$z_{(j_x,j_y)}(t) = Ae^{i(k_x a_H j_x + k_y a_v j_y - \omega t)}.$$

Notice that the real part of this can be written as

$$\cos(k_x a_H j_x + k_y a_v j_y - \omega t) = \cos(k_x x_{(j_x,j_y)} + k_y y_{(j_x,j_y)} - \omega t) = \cos(k_x x + k_y y - \omega t),$$

and this is a generalization of a traveling wave! It can be written as $\cos(\vec{k} \cdot \vec{r} - \omega t)$, where $\vec{E} = (k_x, k_y)$ and $vecr = (x, y)$. So just like in the 1-D case, we can turn this problem into the continuous case now.

To find the dispersion relation in two dimension, we substitue everything back into the equation of motion, and we can show that we will have (in a discrete case)

$$\omega^2 = \frac{4T_H}{ma_H} \sin^2\left(\frac{k_x \cdot a_H}{2}\right) + \frac{4T_V}{ma_v} \sin^2\left(\frac{k_y \cdot a_v}{2}\right),$$

which is a vaguely Pythagorean relation. We can now label any normal mode not just as a number but as an ordered pair (k_x, k_y), and now our dispersion relation can be plotted as ω versus a k_x, k_y plane!

And now we can think about boundary conditions as well. If the displacement must go to zero at the walls, this just gives us conditions of the form

$$\psi(0, y, t) = \psi(L_H, y, t) = \psi(x, 0, t) = \psi(x, L_V, t) = 0$$

However, remember that our normal modes were traveling waves, so we need to find analogous **standing waves** in the 2D case. Remember that in the one-dimensional case, we wanted things of the form $\cos(kx)\cos(\omega t)$, since we could control the position x and time t separately. Well, we'll similarly convert into the form

$$\cos(k_x x + k_y y - \omega t) \to \cos(k_x x)\cos(k_y y)\cos(\omega t)$$

(where each cos term can also be a sin).

With this, we can talk about **Chladni figures**. Basically, we vibrate a mechanical plate with sand on it at certain frequencies, and the grains of sand will end up in the nodes, which are the most stable spots. In other words, **resonance will excite a normal mode**! Our normal modes are of the form

$$\psi_{(n_x,n_y)}(x, y, t) = a_{n_x,n_y} \sin\left(\frac{\pi n_x x}{L_H}\right) \sin\left(\frac{\pi n_y \cdot y}{L_v}\right) \cos(\omega t + \varphi').$$

And finding the continuum limit of the mesh is not too bad: we can take $a_H, a_v \to 0$ and let $T_H = T_v = T$ for simplicity. Then our dispersion relation becomes

$$\omega^2 = \frac{T}{a}\frac{a^2}{m}\left(k_x^2 + k_y^2\right) \implies \boxed{\omega^2 = v^2|\vec{k}|^2}$$

in the small angle approximation. Here, $\frac{T}{a} = T_s$ corresponds to the surface tension, while $\rho_s = \frac{m}{a^2}$ corresponds to mass density. **So we've recovered a "linear" dispersion relation**! And the main difference is that because we're in two dimensions, we have an infinite number of wavevectors for a given frequency.

34.3 Boundary conditions and Snell's law

So now let's consider **an interface between two media** in two dimensions. Remember that we considered the problem of a propagating wave in the x-direction which hit an interface at $x = 0$; this time, let's **divide the xy-plane** with

$x = 0$ as an interface. We have two different media: say that we have the parameters $\rho, T, v = \sqrt{T/\rho}$ on the left side and $\rho', T', v' = \sqrt{T'/\rho'}$ on the right side. **Then what can we say about the generalized displacements** $\psi_L(\vec{r}, t)$ and $\psi_L(\vec{r}, t)$?

Let's assume we have an incident plane wave propagating with wave vector $\vec{k_i} = (k_x, k_y)$. Assuming it's a cosine function, we can write

$$\psi_i(\vec{r}, t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}.$$

Here, the wavefronts are perpendicular lines to $\vec{k}$, and the function will have maxima along those wavefronts. Before we do any calculations, the main idea is that

$$|\vec{k}| = \frac{2\pi}{\lambda} \implies \frac{2\pi}{\lambda} = \frac{\omega}{v} \implies \lambda = \frac{v2\pi}{\omega},$$

so λ, the wavelength, should be proportional to the velocity. This means that when we change wavefronts, the spacing between the wavefronts must change.

So what are our boundary conditions at $x = 0$? Continuity tells us that

$$\psi_L(0^-, y, t) = \psi_R(0^+, y, t)$$

We'll make an ansatz: $\psi_L = \psi_i + \psi_r$ (the sum of the incident and reflected waves), and $\psi_R = \psi_t$, the transmitted wave. The reflected wave and transmitted wave **must have the same frequency** as the incident wave, so that continuity is satisfied at all times t.

Because we're working in two dimensions, we don't know the directions or magnitudes of the wave vectors $\vec{k}$ for the reflected or transmitted wave. The only thing we know (a priori) is that

$$|\vec{k_\alpha}| = \frac{\omega}{v}, \quad |\vec{k_\beta}| = \frac{\omega}{v'},$$

where $\vec{k_\alpha}$ and $\vec{k_\beta}$ are the wave vectors (directions of propagation) for the reflected and transmitted waves, respectively. And so it seems that the problem is very hard: we could have the transmitted wave sum over a bunch of different β, and similarly the reflected wave could sum over a bunch of different α. **But now we plug in** $x = 0$: our general boundary condition is now that **at all times** t, we must have

$$Ae^{i(k_y\cdot y-\omega t)} + \sum_\alpha AR_\alpha e^{i(k_{\alpha,y}\cdot y-\omega t)} = \sum_\beta AT_\beta e^{i(k_{\beta,y}\cdot y-\omega t)}$$

and this can only be satisfied if $k_{\alpha,y} = k_{\beta,y} = k_y$. This means the y-components are constrained with a pretty simple expression: we must have the reflected component be a reflection with angle of incidence equal to angle of reflection. On the other hand, k_β must have the same y-projection while having a different length: this gives us Snell's law

$$v \sin\theta_t = v' \sin\theta.$$

Rewriting this expression in terms of the **index of refraction** $n = \frac{c}{v}$, we have the familiar form

$$\boxed{n_1 \sin(\theta_1) = n_2 \sin(\theta_2)}.$$

35 November 13, 2018

Exam 2 will be on Thursday in Walker, just like last time. It will cover lectures 9 to 17, so it does not include material from this class. (And there will be a review session tomorrow at 4pm in E25-111, like last time.)

We've been talking about extending the wave equation to more dimensions; today, we're going to talk about a change in the direction of a field in 2 or 3 dimensions.

35.1 Review

Recall that we talked about Chladni figures, which trace out normal modes with sand on a metal plate. In practice, they look different from the simulated version, but that's because of the stiffness of the plate (we don't have a linear dispersion relation). Those kinds of higher order terms allow the different dimensions to talk to each other, which is bad for our separable conditions. (Also, we can search up Bessel functions for the cylindrical boundary condition case, but that's mostly a sidenote.)

We also talked for a while about geometrical optics and the proof of Snell's law. The key idea was that **at the boundary, the sum of the waves have to be equal on both sides**. Notice that this law doesn't just apply for light - we never used Maxwell's equations! So the derivation works for any plane wave hitting an interface.

35.2 A cool application of Snell's law

We can solve for $\sin\theta_2$ in our equation: this yields

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1.$$

Notice that if $n_1 > n_2$, we get a problem with this equation as $\sin\theta_1 \to 1$, since no value of θ_2 works.

Proposition 112 (Total internal reflection)
If $n_1 > n_2$ and $\sin\theta_1 > \frac{n_2}{n_1}$, then the refracted wave disappears – the whole wave is reflected back into the initial medium.

This is used in fiber cables to transfer information! Basically, light will keep bouncing inside the cable and will not escape into the air if the index of refraction of the inner material is high enough.

35.3 Polarization

The next physical phenomenon we'll try to explain here is that of **rainbows**. The first point we'll make is about the direction of the electric field in our electromagnetic wave – how do we know that EM waves are transverse, and what is a good way to test this?

The key idea is that there are different ways to **polarize** light – in other words, an EM wave propagating in the z-direction can have an electric field in the x- or y-direction. Let's start with the theoretical calculations: we know that we can write our electric field as

$$\vec{E}(z,t) = \text{Re}\left[\vec{\psi}_0 e^{i(kz-\omega t)}\right]$$

where $\vec{\psi}_0$ must be in only perpendicular directions to the direction of propagation: $\vec{\psi}_0 = \psi_1\hat{x} + \psi_2\hat{y}$ for some $\psi_1 = A_1 e^{i\phi_1}, \psi_2 = A_2 e^{i\psi_2}$. So then we can write our electric field in matrix form:

$$\vec{E} = \text{Re}\left(Ze^{i(kz-\omega t)}\right), \quad Z = \begin{bmatrix}\psi_1 \\ \psi_2\end{bmatrix}.$$

In other words, we can understand $\vec{E}$ as the superposition of two different plane waves!

Example 113

Consider the example where $\vec{E}$ is the sum of $\vec{E}_1 = E_0 \cos(kz - \omega t)\hat{x}$ and $\vec{E}_2 = E_0 \cos(kz - \omega t)\hat{y}$. What is the locus of $\vec{E}$-fields?

Since the x- and y-components are always equal, $\vec{E}$ will oscillate along the line $y = x$. This means $\vec{E} = \vec{E}_1 + \vec{E}_2$ is going to be **linearly polarized**, since the locus is a line. Writing out our field explicitly,

$$\vec{E} = \text{Re}\left((E_0\hat{x} + E_0\hat{y})e^{i(kz-\omega t)}\right),$$

and here $Z = E_0 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. If we replace $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with other vectors, we will still have linearly polarized light as long as both entries are **real-valued constants**.

Example 114

What if $\vec{E}_1 = E_0 \cos(kz - \omega t)\hat{x}$, but $\vec{E}_2 = E_0 \sin(kz - \omega t)\hat{y}$ (so we have different trigonometric functions)

Adding these together,

$$\begin{aligned} \vec{E}_2 &= E_0 \cos(kz - \omega t)\hat{x} + E_0 \cos\left(kz - \omega t - \frac{\pi}{2}\right)\hat{y} \\ &= \text{Re}\left((E_0\hat{x} + E_0 e^{-i\pi/2}\hat{y})e^{i(kz-\omega t)}\right). \end{aligned}$$

We know that $e^{-i\pi/2} = -i$, so we now have a defining matrix $Z = E_0 \begin{bmatrix} 1 \\ -i \end{bmatrix}$. So phase differences between the x- and y-direction show up in Z as well! If we plot $\vec{E}$ on the xy-plane, we find a **circle** (traced out clockwise, since $kz - \omega t$ decreases as t increases). So this is **circularly polarized** light! With $Z = \begin{bmatrix} 1 \\ i \end{bmatrix}$, we can similarly get a counter-clockwise circularly polarized wave.

We can even construct examples where the electric field is changing in both phase and amplitude! This happens when our superposition $\vec{E}_1 + \vec{E}_2$ has different coefficients in front of the two components.

Example 115

Suppose the two plane waves we're superimposing are $\vec{E}_1 = \frac{E_0}{2} \cos(kz - \omega t)\hat{x}$ and $\vec{E}_2 = E_0 \sin(kz - \omega t)\hat{y}$.

Then $Z = E_0 \begin{bmatrix} \frac{1}{2} \\ -i \end{bmatrix}$, and if we plot the locus, we get **elliptically polarized** light.

Example 116

People might be getting bored, so we'll do one more. Suppose we have $\vec{E}_1 = E_0 \cos(kz - \omega t)\hat{x}$ and $\vec{E}_2 = E_0 \cos(kz - \omega t + \Delta\phi)\hat{y}$ for some phase $\Delta\phi$.

If this phase difference $\Delta\phi$ is equal to either $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, we get circularly polarized light, and if $\Delta\phi = 0, \pi$, we have linearly polarized light. **But in any other case, it turns out we will have elliptically polarized light!**

To explain this, imagine starting with the locus $y = x$, traced out for $\Delta\phi = 0$. To get to a circle, we need some kind of smooth transformation, and we can imagine splitting the line open and growing an ellipse inside it! This information is also encoded in our vector $Z = E_0 \begin{bmatrix} 1 \\ \cos\Delta\phi + i \sin\Delta\phi \end{bmatrix}$.

But something else is more common in daily life: **unpolarized light**. This isn't saying that we have complete symmetry – if we put many electromagnetic waves together with different phases such that all of the $\vec{E}$-fields add up to 0, then there is no light at all! Instead, we have **differently polarized light** at every moment, such that the direction of polarization is almost random.

35.4 How do we filter light to be polarized?

Definition 117

Unpolarized light is the sum of a bunch of EM waves that are produced independently by a large number of uncorrelated emitters. They can emit at different times and polarizations.

Now, we can meet the **polarizer**. Suppose we have a bunch of vertical metal rods that are perfect conductors (assume that the rods are very close to each other), and the electric field component is only vertical. Then the electric field must cancel out to 0, so electrons in the conductors will be driven, and it will cancel all the electric field (and cause a reflection). Specifically, this means that no vertical electric field will pass through.

But now if we rotate the polarizer by 90 degrees, the electrons in the conductors can only move horizontally. So an electric field in the vertical direction will not be affected! So we can filter out the light in a specific direction: define the **easy axis** to be the direction in which the electric field can easily pass through.

When drawing a diagram, we often represent a linearly polarized polarizer as a circle, where an arrow inside the circle points in the direction of the easy axis. If the easy axis is in the x-direction, the polarizer can be described as a matrix:

$$P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \implies Z' = P_0 Z,$$

where Z' is the final electric field vector after going through the polarizer. Similarly, if the easy axis is in the y-direction, it is at an angle of $\frac{\pi}{2}$ from the x-direction, so we write it as

$$P_{\pi/2} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

In general, we can find that

$$P_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}.$$

Calculating the electric field after polarization is a pretty straightforward calculation: say we have light that is oscillating in the x-direction with intensity I_0, and let it pass through a polarizer with easy axis θ off the x-axis. Well, we can compose the light into an I_{parallel} and I_{perp}; only the parallel component will pass through.

We have to go back to the electric field to do this projection: decompose $\vec{E}_0$ into a parallel and perpendicular component. Since the magnitude of $\vec{E}_{\text{parallel}}$ is $|\vec{E}_f| = |\vec{E}_0|\cos\theta$, the intensity (proportional to the electric field squared) will be $\boxed{I_f = I_0\cos^2\theta}$.

Fact 118

Computers emit polarized light – if we wear polarized sunglasses and tilt them 90 degrees, we'll see a noticeable difference.

So now take an unpolarized light source and have it pass through a polarizer. Then the intensity is halved to $\frac{I_0}{2}$, and we can add in another polarizer in the perpendicular direction after it, so that **no light passes through the system of two polarizers**.

Now say that we emit light from our source one photon at a time – each photon comes out at random, and we can observe that each one either passes through or doesn't. But we have a bit of a problem – if there is a single photon, how can it be halfway along the easy axis and halfway along the perpendicular axis? (After all, we can't split a photon.) So let's rotate one of the two polarizers so they're off by $\frac{\pi}{4}$. It seems reasonable that because the intensity of the light is $\frac{I_0}{2}$ after the individual photons have passed through the first polarizer, the intensity of the light with this $\frac{\pi}{4}$ angle should be either $\frac{I_0}{2}$ or 0.

But experimentally, the intensity is $\frac{I_0}{4}$, and that means there is probability involved: each photon has some random chance to pass through or not! And that's why we need quantum mechanics to describe the world.

Next time, we'll talk a bit about the quarter wave plate and how we can generate EM waves.

36 November 14, 2018 (Recitation)

This recitation is being taught by Pearson (the graduate TA). Let's quickly go over the topics that will be on tomorrow's midterm.

- The wave equation in one dimension is $\partial_{tt}u = c^2\partial_{xx}u$. We should know how to solve this for standing waves of the form $A\sin\omega t\sin kx$, as well as traveling waves of the form $f(\omega t \pm kx)$.
- Boundary conditions (fixed, massless ring, free, forced); we should understand how to deal with these in the infinite, semi-infinite, and bounded domains.
- We should be able to identify normal modes of a given system – this means being able to find Fourier series and coefficients.
- It's important for us to understand the difference between a dispersive and non-dispersive medium – the latter is of the form $\omega = vk$. On a related note, we should understand the difference between a group velocity $v_g = \frac{\partial\omega}{\partial k}$ and a phase velocity $v_p = \frac{\omega}{k}$.
- We should be able to work with air, sound, and pressure waves.
- The two-dimensional wave equation: we should know that we can separate solutions to $\partial_{tt}u = c^2\nabla^2 u$ into the form $A\sin(\omega t)\sin(k_x x)\sin(k_y y)$. This also gives us Snell's law as a boundary condition.
- We'll be given Maxwell's equations, but we should be able to work with electromagnetic waves. It is important to note that $\vec{E}, \vec{B}, \vec{K}$ are perpendicular, and we should be able to work with the Poynting vector $\vec{S}$, energy density U, and so on.
- We should (at least qualitatively) understand the uncertainty principle?

Here's the selected points that were discussed:

- Let's take a closer look at sound waves. It turns out air pressure, p, and air density, ρ, are both proportional to $\sin kx$, the displacement at x. An open end corresponds to a node (or fixed end) for pressure, and a closed end corresponds to an antinode. As always, fixed ends at the origin correspond to sine functions and open ends correspond to cosine functions.
- In general, for an equation of motion, to generate the dispersion relation, we plug in the ansatz back in. Again, it's only a non-dispersive medium if ω is linear in k, and this is the only case where v_g and v_p are always equal.

- Finally, imagine we fix a boundary condition for some function of the form $y(L, t) = f(t)$. Solving this is a bit tricky; the best way is to Fourier decompose.

37 November 19, 2018 (Recitation)

(Notes from this recitation are slightly incomplete.)

We'll start by talking about **pressure boundary conditions**. Normally, a closed end corresponds to the function (which defines the wave) having a value of zero, while an open end corresponds to zero derivative. However, in this case, notice that the difference in pressure $\Delta p = \frac{\partial\psi}{\partial x}$ actually corresponds to the derivative of the function instead of its actual value! Thus, we actually have zero **value** for the change in pressure Δp at an open end, while we have zero **derivative** for Δp at a closed end. To explain this intuitively, the pressure must stay continuous at an open end. Thus, the pressure must be equal to the atmospheric pressure, which means $\Delta p = 0$.

Next, let's look some more at two and three-dimensional wave equations. Usually our equations of motion have to do with the Laplacian rather than a second derivative, but because of the Pythagorean theorem, this will still give us $\omega = v_s|k|$ in a multi-dimensional non-dispersive medium. (This is because $x^2 + y^2 + z^2 = |k|^2$.)

One important note here: whether or not a normal mode can contribute a nonzero amount to the wave solution **depends on the boundary conditions**. Sometimes, we have to be careful not to let the wave number k_x be equal to 0, because if we have a sine wave $\sin(k_x x)$, the normal mode is then identically zero. Thus in those cases, we must take the smallest nonzero k. However, if our boundary conditions give us cosine waves $\cos(k_x x)$, it is okay to have $k_x = 0$, because the normal mode is identically 1 from the x-direction contribution.

Finally, let's look at how rainbows are formed: once light enters a water droplet, we can track the number of reflections inside a droplet before it is refracted again. For each number of reflections, we can calculate the overall angle of reflection α with respect to θ, the angle of entry. Plot this graph, and there is a maximum value for α (as a function of θ). And this is the point where we have maximum brightness in a rainbow.

38 November 20, 2018

This is a general reminder that grade cutoffs are all public. IN this class, the exams are worth 20, 20, and 35 percent, and the problem sets are worth 25 percent. The typical cutoffs for A, B, C, and passing are 75-80, 65-70, 52-57, and 47-52 respectively.

38.1 Review

Last time, we asked the question "how do we know that EM waves are transverse?" This is predicted from Maxwell's equations, but having the notion of **polarization** gives another justification for this fact. We talked about linearly polarized light, where the electric field is always pointed along a specific axis, and also about circularly polarized light, where we add two differently-polarized light sources (generally perpendicular) with different phases (off by $\pi/2$).

Fact 119
Linearly polarized light is the sum of counter-clockwise and clockwise polarized light, just like a stationary wave can be decomposed into a left and right progressing waves!

We also discussed polarizers, which have an **easy axis** that allows a light wave with that orientation to pass through, but not light in the perpendicular orientation.

38.2 More editing of polarization

Today, we will talk about waveplates, which have different indices of refraction n_x and n_y in different directions! Consider an electromagnetic wave propagating in the z-direction for a total distance ℓ. The wave number for light polarized in the x-direction is

$$k_x = \frac{n_x}{c}\omega = \frac{2\pi}{\lambda_x},$$

and similar for $k_y = \frac{2\pi}{\lambda_y}$, so the phase difference between light polarized in the two directions is

$$\Delta\phi = \frac{2\pi\ell}{\lambda_x} - \frac{2\pi\ell}{\lambda y} = \frac{(n_x - n_y)}{c}\omega\ell$$

So we can now tune ℓ so that we can create any kind of phase difference.

Definition 120

In a **quarter wave plate**, a medium has different indices of refraction in the two axes so that the phase difference between them is $\frac{\pi}{2}$. Let the axis with a smaller phase difference be the **fast axis**, and let the one with larger phase be the **slow axis**.

We're curious what will happen to a light wave that is passed through this quarter wave plate: let's examine a few possible scenarios.

Example 121

Say that the incident wave is linearly polarized along the fast axis.

Since all of the light in this particular case is passing through the fast axis, we will still have an exiting linearly polarized light along the fast axis.

Example 122

Now say that the light is polarized at a 45° degree angle from the fast axis, so that the two components (fast and slow) are equal.

Then the slow axis will be delayed by $\frac{\pi}{2}$, and remember from last lecture that this corresponds to **circularly polarized light**. So this quarter wave plate can turn linearly polarized light into something different!

Example 123

Finally, say that the light is polarized at an angle θ from the fast axis.

Now, we have a $\cos\theta$ projection onto the fast axis and a $\sin\theta$ projection onto the slow axis. This produces **elliptically polarized light**, and we can describe the transformation in terms of a matrix

$$Q_\theta = \begin{bmatrix} \cos^2\theta + i\sin^2\theta & (1-i)\sin\theta\cos\theta \\ (1-i)\sin\theta\cos\theta & \sin^2\theta + i\cos^2\theta \end{bmatrix},$$

which encodes both the magnitude and phase for the exiting light.

Fact 124

There are two different demonstrations we can do here. First of all, sugar is a chiral molecule and has a preferred orientation – it will actually rotate polarized light clockwise or counterclockwise. Secondly, we can put a quarter wave plate between two perpendicular polarizers, which will allow light to pass through!

38.3 Radiation from accelerating charges

With this, we move on to our next question: **how do we create electromagnetic waves**? There is radiation coming from faraway stars, even though the surface area is increasing on the order of r^2. And our goal is to create this radiation, which can **propagate energy through space**.

To be more quantitative about that, suppose we have a light source at some point in the universe. Draw a circle that is some distance away from the light source – the Poynting vector is $\frac{1}{\mu_0}\vec{E} \times \vec{B}$, and we can use this to find the radiation power across a surface area of a sphere at some radius. But we can draw another surface that is farther away, and the total power is (in both cases) the length of the Poynting vector times the area.

Proposition 125

Since the power should be the same at all radii, the Poynting vector should be proportional to $\frac{1}{A} = \frac{1}{r^2}$. Therefore, $\vec{E}$ and $\vec{B}$ both fall off as $\frac{1}{r}$.

Our goal is therefore to set up a system where **the electric and magnetic field are inversely proportional to the radius**.

Example 126

Let's look at a stationary charge.

By Gauss's law, the electric field $\vec{E}$ will have magnitude $\frac{q}{4\pi\epsilon_0 r^2}\hat{r} \propto \frac{1}{r^2}$. However, no magnetic field is created – $\vec{B} = 0$ – so there is nothing radiating. Thus, just having a stationary charge does not create the radiation we're looking for (the Poynting vector is zero).

Example 127

Now let's say we have a charge moving at a constant speed u, and let $\beta = \frac{u}{c}$.

The calculations here are a bit messier, but we ultimately find that

$$\vec{E} = \frac{q}{4\pi\varepsilon_0 e^2}\frac{1-\beta^2}{(1-\beta^2\sin^2\theta)^{3/2}}\hat{r} \propto \frac{1}{r^2}.$$

This means that the magnetic field created has the same order of magnitude:

$$\vec{B} = \frac{\vec{u}\times\vec{E}}{c^2} \propto \frac{1}{r^2}.$$

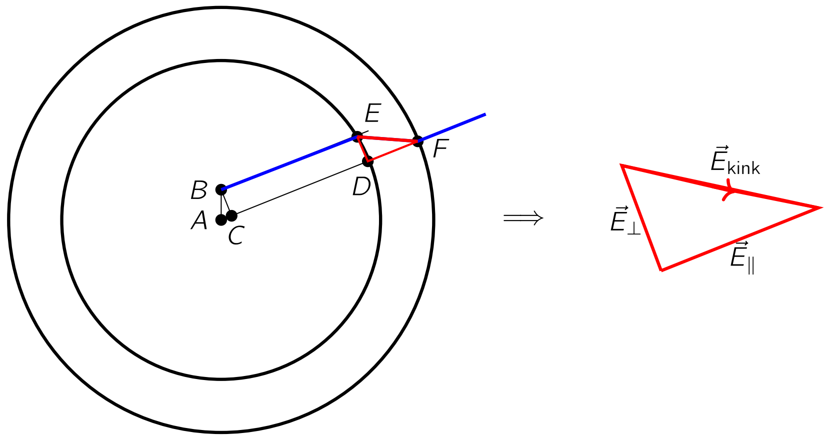
Unfortunately, the energy is proportional to $\frac{1}{r^4}$ in this case! Thus the power goes to 0 as the radius r goes to infinity, and we still don't have radiation.

Ultimately, the answer is that we need to **accelerate the charge** to get the correct radiation:

Example 128

Let's say we have a charge that is initially at rest at a point A, accelerates at an acceleration a (as a function of time) to the point A' in Δt, and then travels at speed $u = a\Delta t$ to the point B at time t. Assume $\Delta t \ll t$ and $u \ll c$.

Here's a schematic diagram: the colored lines from B to E to F represent the **direction of electromagnetic wave observed in this given direction** $\vec{AF}$. Notably, there will be a kink in the perpendicular direction of the electric field between points E and F, which is what we're most interested in.



It's helpful to give a bit more description of what's going on here:

- **The point A' is approximately at the point A in our diagram**. This is because the distance traveled from A to B is ut (where u is the final speed of the charge), and this is much larger than the distance traveled during the acceleration period Δt.
- The two circles represent the propagation of electromagnetic waves, moving outward at the speed of light c. The outer circle is the propagation from point A (at the beginning of our motion), and the inner circle is the propagation from the constant-velocity portion of the movement. Although the circles should have different centers, the distance between A, A', and B is much smaller than the radius (because the speed $u \ll c$).

Our goal is to calculate the perpendicular electric field component $\vec{E}_\perp$, which is ultimately what will give us our radiation. First of all, notice that $DF = c\Delta t$, since the acceleration takes place between the two circles. Also, we can assume $BCDE$ is a rectangle because the short sides are much smaller than the longer sides.

The red triangle on the right formed by E_{kink} is similar to triangle DEF (both represent the direction of the electric field at that point), so we can calculate the ratio **at a radius r away from the source**:

$$\frac{E_\perp}{E_\parallel} = \frac{DE}{DF} = \frac{-u_\perp t}{c\Delta t} = -\frac{(a_\perp \Delta t)\cdot t}{c\Delta t} = -\frac{a_\perp r}{c^2},$$

where $u_\perp$ and $a_\perp$ are the components of u, a along the direction BC, and **where we used $r = ct$ in the last equality**. But by Gauss' law, we know that $\vec{E}_\parallel = \frac{q}{4\pi\varepsilon_0 r^2}$, so

$$\vec{E}_\perp = \vec{E}_\parallel \cdot \frac{-a_\perp r}{c^2} = \frac{-q \cdot a_\perp\left(t - \frac{r}{c}\right)}{4\pi\varepsilon_0 c^2 r} \propto \frac{1}{r}.$$

Here, $a_\perp$, labeled in blue, is a **function of retarded time**. Basically, it takes time for the information to propagate, so the measured kink depends on the acceleration of the particle from $\frac{r}{c}$ seconds ago! And now that our perpendicular electric field has the right proportionality constant, we've finally found the source of **electromagnetic radiation**.

39 November 21, 2018 (Recitation)

We'll start by talking about polarizers – how can we write out the matrix transformations? Generally, it's not very helpful to use matrices except for large computations, because the whole point of having matrices is to do **rotations**. After all, if we start with linearly polarized light of the form $\vec{E} = E_0 \cos(kx - \omega t)\hat{y}$, passing through polarizers and wave plates, we are just decomposing into a new coordinate frame each time – we can call it the fast and slow axis, or the easy and not-easy axis. So our projections are basically multiplying by $\cos\theta$ and $\sin\theta$ whenever necessary! In other words, if we rotate our coordinates so that

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta,$$

this corresponds exactly to the matrix

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

On the other hand, throwing away a component and turning a vector $\begin{bmatrix} x \\ y \end{bmatrix}$ into $\begin{bmatrix} x \\ 0 \end{bmatrix}$ is just

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, if we are changing a phase angle of one axis by a phase angle θ, that is just

$$M' = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & 1 \end{bmatrix}.$$

The whole point is that only relative angles matter: we can throw away components perpendicular to the easy axis of a polarizer, and it's pretty easy for us to understand each of these components on their own.

So why use matrices? For example, if we have a polarizer, a half wave plate, and a third polarizer at angles α, β, γ, we can just compose the transformations without thinking:

$$A = R(\alpha) \to M \to R(\alpha - \beta) \to M' \to R(\beta - \gamma) \to M.$$

The whole point is that we rotate our axis a bunch of times, and we can go through matrix computation without needing to think about too much of the physics!

Next, let's talk a bit about the Fourier transform of a Gaussian (which is something we discussed a few recitations ago). Recall that we wanted to compute the Fourier coefficient

$$F(k) = \int_{-\infty}^{\infty} e^{-x^2} e^{ikx}\, dx,$$

and by completing the square, we find that $F(k)$ is just a constant times $\int_{-\infty}^{\infty} e^{-\left(x + \frac{ik}{2}\right)^2}$. And we can compare this to the ordinary Gaussian integral $\int_{-\infty}^{\infty} e^{-x^2}$:

Proposition 129

Even though we're integrating along a different line in the complex plane, we actually have

$$\int_{-\infty}^{\infty} e^{-(x+ci)^2} = \int_{-\infty}^{\infty} e^{-x^2}$$

for any real number c.

Fact 130

This next part is really not necessary for understanding of 8.03, but it's some extra mathematical justification. (Feel free to take a look at 18.112 for more explanation.)

Proof. This has to do with complex integration. In the ordinary case, we integrate along the real line, and in the new case, we integrate with a line parallel to it. Let's pick a contour to be a counterclockwise rectangle with horizontal bounds $-R$ to R and vertical boundaries 0 and $\frac{ik}{2}$. e^{-x^2} is analytic; there are no poles, so the contour integral around the whole rectangle is 0 by Cauchy's theorem.

We know the bottom part of the rectangle, which is the standard Gaussian integral, evaluates to $\sqrt{\pi}$ as $R \to \infty$. Also, as $R \to \infty$, the side parts of the rectangle go to 0, since e^{-x^2} exponentially vanishes. So if we integrate from right to left on the top part of the rectangle, it must be $-\sqrt{\pi}$. Therefore, the integral from $-\infty$ to ∞ in the proper way along that new axis is also $\sqrt{\pi}$, as desired. $\square$

Now let's take a look at an overhead projector, which has polarized light.

Problem 131

If we have light polarized in the y-direction, can we use polarizers (which just filter out light) to get it polarized in the x-direction?

We know that light comes in quanta (such as the photon). If photons have an x-polarization, it would initially seem like we can never "filter out light" and turn this into a y-polarization. But photons' polarization is a lot like the spin of a particle, because we need to use wave or probabilistic descriptions. So there are some extra tricks we can pull!

Example 132

Start with y-polarized light. If we use a 45° polarizer, we only get half of the light through (the photon goes through with 50% probability), but now all the light is polarized in that direction. So now we use another polarizer in the x-direction, and we now have $\frac{1}{4}$ of the initial light polarized in the x-direction!

In particular, if we insert a polarizer between two perpendicular polarizers, **we can actually increase the amount of light** that comes out. This might seem counterintuitive, especially since polarizers are supposed to only filter light.

Example 133

Place a half wave plate between two polarizers – let's say the first polarizer makes the light y-polarized. (Half wave plates flip the electric field along one axis by 180 degrees and fixes the field along the other one.)

If the wave plate is **parallel** to either polarizer, **no light passes through**. This is because the half wave plate just flips around one of the two directions by 180 degrees, and this does not allow anything to flip through.

However, if the wave plate is at a 45 **degree angle**, we can decompose the light into the diagonal directions: for example, if the flipped axis is along $y = x$, we can think of the relevant vector as being decomposed as

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}.$$

After passing through the half wave plate, the first term is negated and the second term stays the same; our vector becomes

$$\begin{bmatrix} -1/2 \\ -1/2 \end{bmatrix} + \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

This actually changes the net electric field by a 90 degree rotation, and this **allows the light to pass through** the second polarizer!

Finally, let's talk a little about the physics of a half wave plate. The idea is that there is a different index of refraction in one direction than the other, which causes a phase shift for the electric fields in the two directions. Professor Ketterle brought some Saran wrap to class today – let's change the material by stretching it in one direction! The molecules are being aligned in a way that changes the direction of $\vec{E}$-field propagation.

So let's put the Saran wrap at a 45 degrees angle under the (polarized) overhead projector – indeed, light does change its polarization slightly! In fact, we can sometimes stretch the wrap in a way to create circularly polarized light, which would always allow light to pass through the second polarizer, regardless of orientation. And as a bonus fact, the phase shift may be different for different colors, so we will be able to see more of some colors than others with this trick, creating a beautiful color spectrum.

40 November 26, 2018 (Recitation)

Recall that we can write a propagating light wave polarized at 45° to the x-axis as

$$\vec{E} = E_0 \cos(\omega t) \left(\frac{1}{\sqrt{2}} \hat{e}_x + \frac{1}{\sqrt{2}} \hat{e}_y \right),$$

and the whole point is that we can always write the electric field as a linear combination of two fixed orthonormal vectors. Each time after putting light through a polarizer at any angle θ, we can do a change of basis with a fast and slow axis.

(Question: do the axes need to be orthogonal? For all practical purposes, yes.)

Also recall that when we put light through a wave plate, we add a phase ϕ to $\cos(\omega t)$. For example, a phase shift of $\frac{\pi}{2}$ turns a sine wave into a cosine wave and vice versa – the whole point is that after putting linearly polarized light at an angle through a quarter wave plate, we now have circular or elliptically polarized light.

So what was the sugar solution from lecture doing to the electric field when it rotated the polarization? Mathematically, we're switching to a **circular polarization basis**

$$\frac{1}{\sqrt{2}} \left(\hat{e}_x \pm i \hat{e} y \right),$$

which correspond to right-hand and light-hand circular polarization. The sugar is like the wave plate: it changes the phase along one of the two basis vectors when we write our electric field in the form

$$\vec{E} = E_{\text{clockwise}} \frac{\hat{e}_x + i\hat{e}_y}{\sqrt{2}} + E_{\text{counterclockwise}} \frac{\hat{e}_x - i\hat{e}_y}{\sqrt{2}}.$$

So if we start with only clockwise or only counterclockwise circularly polarized light, nothing really happens to it (just like with fast axis oriented light in quarter wave plates). But if we have a linear combination of the two circular directions at equal intensity, we actually end up with linear polarization. Then when we use the sugar as a half wave plate, we still get linear polarization, but rotated 90 degrees!

Fact 134

In general, if one of the two directions (clockwise and counterclockwise) is delayed by ϕ, the sugar outputs a linear polarization that has been rotated by $\phi/2$.

We'll finish by discussing how to create **electromagnetic radiation** (the topic of last lecture). Stationary charges or pieces of wire give $E, B \propto r^{-2}$, which doesn't quite work because energy decays to 0, meaning that energy is not leaving the source. Instead, the idea is to take a charge and **accelerate it through space**. Then the Coulomb field has a kind of kink: it is transverse, and we find that

$$\vec{E}_\perp = -\frac{q}{4\pi\varepsilon_0} \frac{1}{r^2} \frac{\vec{a}_\perp r}{c^2} \propto \frac{1}{r},$$

which is what we wanted. But the acceleration $\vec{a}$ has two features: (1) it is the **transverse component** that matters, and (2) it is the acceleration at the retarded time $\left(t - \frac{r}{c}\right)$ that affects the current electric field $\vec{E}(t)$, due to the time needed for light to travel. Another way to rephrase point (1) is that longitudinal motion does not affect the radiation – for example, rod antennas radiate electromagnetic radiation in the perpendicular direction, not the direction along the antenna itself! (And consider radiation created not by a linear rod but by **orbiting charges** such as electrons. We can see circularly polarized or linearly polarized light based on the plane in which we're observing the orbiting electron.)

41 November 27, 2018

Problem set 9 is due today, and the last pset is due next Friday.

41.1 Review

We've been talking about electromagnetic waves in a vacuum. Last time, we talked about accelerating charges: without this, the Poynting vector decays too quickly as a function of the radius r, so the energy is not transmitted properly. However, if we accelerate the charge over a short Δt, we create a kink in the electric field. The formula we derived was that

$$\vec{E}_{\text{rad}}(\vec{r}, t) = -\frac{q\vec{a}_\perp(t')}{4\pi\varepsilon_0 c^2 r},$$

where $t' = t - \frac{r}{c}$ is a retarded time that accounts for how long it takes for the information to be transmitted. And also note that the **negative sign** comes up because the kink is pointed in the opposite direction to the acceleration of the charge (look back at the diagram if this is not clear).

In particular, the magnetic field $\vec{B} = \frac{1}{c}\hat{r} \times \vec{E}_{\text{rad}}(t')$ is also proportional to $\frac{1}{r}$, so the Poynting vector is proportional to $\frac{1}{r^2}$, and the total energy through any sphere is constant. So this is how we propagate energy "to the edge of the universe!"

41.2 More applications: Maxwell's equation in matter

Today, we're going to talk about how to take **good photos**. Polarization can change a lot about the light in a photograph – for example, consider unpolarized sunlight that is traveling in some direction. When it hits a molecule in the air, it will change direction by Snell's law, but the refracted and reflected components will both be polarized in some way!

> **Example 135**
> Suppose sunlight is coming in horizontally and hits some object in the sky, and we are directly below the molecule. Then the light that hits us will be extremely polarized.

So when we put a polarizer on our camera, this often eliminates most of the light coming from the sky! And in fact, we can put polarization filters to filter out light reflecting off windows or the surface of the water. We'll find out why in this lecture.

To progress, we need to introduce the idea of having electromagnetic fields in matter. In a perfect conductor, we have infinite charge, and it costs no energy to move them around. But in a dielectric, things are different: charges are attached to specific atoms. This means that when we write down Maxwell's equations, there are two different kinds of charges. There are some charges that are **free**, and some others that are **bound** to specific atoms.

Let $\rho = \rho_f + \rho_b$ be the charge density. We can define the **electric displacement field** as

$$\vec{D} = \varepsilon_0 \vec{E} + \vec{p},$$

where $\vec{p}$ is the electric dipole moment. We can think of this as a kind of flux, and we can also describe the bound charges via

$$-\vec{\nabla} \cdot \vec{p} = \rho_b.$$

Then we know that

$$\varepsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{p},$$

so rearranging yields

$$\boxed{\vec{\nabla} \cdot \vec{D}} = \vec{\nabla}(\varepsilon_0 \vec{E} + \vec{p}) = \boxed{\rho_f}.$$

This is very nice, because the displacement field is now only related to free charges, making this as close to a vacuum as possible.

Our next step is to look at electric current, which we can break up into a components of free, bound, and dipole (polarization) currents:

$$\vec{J} = \vec{J_f} + \vec{J_b} + \vec{J_p}.$$

We can also define a magnetic field similarly called the **demagnetizing field**, and it takes the form

$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

where $\vec{M}$ is the magnetic dipole which satisfies

$$\vec{\nabla} \times \vec{M} = \vec{J_b}.$$

So looking back at the original Maxwell's equations, we can plug in our new forms of current to find that

$$\boxed{\frac{1}{\mu_0}(\vec{\nabla} \times \vec{B})} = \vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \boxed{\vec{J_f} + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{p}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}.$$

Here, $\vec{J_f}$ is the free charge current, $\vec{\nabla} \times \vec{M}$ is the contribution from bound current, and $\frac{\partial \vec{p}}{\partial t}$ is the polarization current. So now

$$\vec{\nabla} \times \left(\frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J_f} + \frac{\partial}{\partial t} \left(\varepsilon_0 \vec{E} + \vec{p} \right),$$

which means we've successfully factored out the contribution from bound current:

$$\vec{\nabla} \times \vec{H} = \vec{J_f} + \frac{\partial D}{\partial t}.$$

We should keep in mind that this all becomes more complicated if we take more physics classes. But for now, **we can pretend we don't have bound current**! So now with our definitions, we have some modified equations.

Theorem 136 (Maxwell's equations in matter)

The following modified equations hold for electric and magnetic fields in matter:

- $\vec{\nabla} \cdot \vec{D} = \rho_f$,
- $\vec{\nabla} \cdot \vec{B} = 0$,
- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,
- $\vec{\nabla} \times \vec{H} = \vec{J_f} + \frac{\partial D}{\partial t}$.

It turns out that when we have very small electric fields $\vec{E}$, we often have $\vec{p} \propto \vec{E}$, and we can write $\vec{D} = \varepsilon \vec{E}$. Similarly, $\vec{M}$ can be proportional to $\vec{B}$, which gives $\vec{H} = \frac{\vec{B}}{\mu}$. Then ε is a way to quantify the resistance of forming an electric field, and μ does the analogous thing for forming a magnetic field. (Usually, μ is approximately equal to μ_0, except for superconductors which have $\mu \approx 0$.)

So if we have no free charge or current ($\rho_f = \vec{J_f} = 0$), the equations simplify to

- $\vec{\nabla} \cdot \vec{D} = 0$,
- $\vec{\nabla} \cdot \vec{B} = 0$,
- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,
- $\vec{\nabla} \times \vec{H} = \frac{\partial D}{\partial t}$.

We'll now make a further simplification and consider a **linear homogeneous isotropic material**, where we have the proportionality $\vec{D} = \varepsilon \vec{E}$ and $\vec{H} = \frac{\vec{B}}{\mu}$ as mentioned above. Then our equations simplify even more to

- $\vec{\nabla} \cdot \vec{E} = 0$,
- $\vec{\nabla} \cdot \vec{B} = 0$,
- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$,
- $\vec{\nabla} \times \vec{B} = \mu\varepsilon \frac{\partial E}{\partial t}$.

This is identical to the normal Maxwell's equation except with different constants, so the wave equation should look very similar! Indeed, the speed of light is now $\frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$, which means we can also define the index of refraction to be

$$\boxed{n = \frac{\sqrt{\mu\varepsilon}}{\sqrt{\mu_0 \varepsilon_0}}}.$$

In most materials, $\mu \approx \mu_0$, so if $\varepsilon > \varepsilon_0$, the resistance of forming an electric field is larger than in vacuum, meaning $n > 1$. It's also possible to have $n < 1$, but nothing propagates in such materials, so causality is not violated (no information is propagating faster than the speed of light).

41.3 More applications: boundary conditions, reflection, and transmission

Now, let's go back to the car window and Snell's law. Imagine an interface with two materials of indices n_1 and n_2, where the materials have speeds (of propagation) v_1 and v_2 and permeabilities μ_1, ε_1 and μ_2, ε_2 respectively. Let's say a polarized incident plane wave is of the form

$$\vec{E}_I(\vec{r}, t) = \vec{E}_{0I} \cos(\vec{k}_I \cdot \vec{r} - \omega t).$$

The magnetic field corresponding to this wave is

$$\vec{B}_I(\vec{r}, t) = \frac{1}{v_1}(\hat{k}_i \times \vec{E}_I).$$

When the wave comes in contact with the medium, we also have a reflected and transmitted wave $\vec{E}_R$ and $\vec{E}_T$, which are of the form

$$\vec{R}_I(\vec{r}, t) = \vec{E}_{0R} \cos(\vec{k_R} \cdot \vec{r} - \omega t)$$

and

$$\vec{T}_I(\vec{r}, t) = \vec{E}_{0T} \cos(\vec{k_T} \cdot \vec{r} - \omega t).$$

We found last time that the frequencies ω must be the same to satisfy boundary conditions at $z = 0$. In addition, we must have

$$\vec{K}_I \cdot \vec{r} = \vec{K}_R \cdot \vec{r} = \vec{K}_T \cdot \vec{r}$$

as well as some relations between the angles from Snell's law:

$$\theta_I = \theta_R, n_1 \sin\theta_I = n_2 \sin\theta_T.$$

Now, let's say the light polarization is in a given plane – the xz-plane. We know that $\vec{E}^{(1)} = \vec{E}_{0I} + \vec{E}_{0R}$ and $\vec{E}^{(2)} = \vec{E}_{0T}$, but now it seems we're stuck. So now it's time to use specific boundary conditions for **electromagnetic waves**! We know that $\vec{\nabla} \cdot \vec{D} = 0$, which means that if we take a pillbox including the plane $z = 0$ and shrink the thickness to 0, the nonzero faces have $\vec{D}$ equal to $\varepsilon_1 E_\perp^{(1)}$ and $\varepsilon_2 E_\perp^{(2)}$. Therefore, Gauss's law tells us that

$$\boxed{\varepsilon_1 E_\perp^{(1)} = \varepsilon_2 E_\perp^{(2)}}.$$

Also, by Faraday's law, we have

$$\oint \vec{E} \cdot d\ell = -\frac{d}{dt}\int \vec{B} \cdot d\vec{a},$$

so draw a rectangular loop perpendicular to $z = 0$ and take the thickness (in the perpendicular direction) to 0 – the right hand side must become 0, because the area shrinks to zero and thus the time-derivative also goes to zero. This tells us that the line integral is zero, which means that

$$\boxed{E_\parallel^{(1)} = E_\parallel^{(2)}}.$$

So now we have some equations to work with. Looking in the perpendicular direction, since $\theta_R = \theta_I$ (reflection),

$$\varepsilon_1(-E_{0I} \sin\theta_I + E_{0R} \sin\theta_I) = -\varepsilon_2 E_{0T} \sin\theta_T,$$

and in the parallel direction,

$$E_{0I}\cos\theta_I + E_{0R}\cos\theta_I = E_{0T}\cos\theta_T.$$

Now we simplify: the first equation here becomes (by Snell's law)

$$(E_{0I} - E_{0R}) = \frac{\varepsilon_2\sin(\theta_T)}{\varepsilon_1\sin(\theta_I)} = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}E_{0T} = \beta E_{0T}$$

(where we define the new variable $\beta = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2}$). Similarly, we can rewrite the second equation as

$$(E_{0I} + E_{0R}) = \frac{\cos(\theta_T)}{\cos(\theta_I)}E_{0T} = \alpha E_{0T}$$

(where $\alpha = \frac{\cos\theta_T}{\cos\theta_I}$). Since we started off with E_{0I}, so we can now solve for the other variables. This tells us that

$$\vec{E_{0R}} = \frac{\alpha - \beta}{\alpha + \beta}E_{0I}$$

and

$$\vec{E_{0T}} = \frac{2}{\alpha + \beta}E_{0I}.$$

This means that in the plane case (where the incident plane wave is polarized), we have our reflection and transmission coefficients

$$\boxed{R = \frac{\alpha - \beta}{\alpha + \beta}}, \quad T = \boxed{\frac{2}{\alpha + \beta}}.$$

Again, let's look at some extreme cases.

Example 137

Suppose we have **normal incidence**: the light enters at a 90 degree angle to the plane.

Then $\alpha = \frac{\cos\theta_T}{\cos\theta_I} = 1$. If the material has $\mu_1 \approx \mu_2 \approx \mu_0$, then $n_1 = \frac{\sqrt{\mu_1\varepsilon_1}}{\sqrt{\mu_0\varepsilon_0}} \approx \sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$. So $\varepsilon_1 \propto n_1^2$, so

$$\beta = \frac{\varepsilon_2 n_1}{\varepsilon_1 n_2} = \frac{n_2}{n_1}$$

which gives reflection and transmission coefficients of $R = \frac{n_1 - n_2}{n_1 + n_2}, T = \frac{2n_1}{n_1 + n_2}$. (These should look fairly familiar from the one-dimensional case!)

Example 138

When $\alpha = \beta$, there is no reflection from this polarized light in this plane! This is called **Brewster's angle**, and this occurs when the reflected and transmitted light are at 90° to each other.

We'll discuss a final question next lecture about soap bubbles, and then we'll move on to wave interference.

42 November 28, 2018

(My computer crashed, and I lost most of the information from this recitation. Here's the most notable topics.)

The Larmor frequency is the cyclotron frequency of an electron moving at speed v in a magnetic field B – it is given by the equation $\omega = \frac{v}{r} = \frac{eB}{m}$. There is another Larmor frequency, which comes from angular momentum of electrons in an electron cloud experiencing a torque from their magnetic moment $\vec{\tau} = \vec{M} \times \vec{B}$. This number is only off by a factor of 2, but it is completely different from the first one.

Our next question: why is there no electromagnetic radiation from an atom, which has electrons around a nucleus? The central idea is that we should not think about electrons as orbiting but instead as a **cloud**. Then if the distribution is radially symmetric, there is no moving charge, so no radiation can be emitted.

Well, in an atom, there are normal modes constrained to certain radii, because we have circular boundary conditions. In those cases, the atom is stable and there is no radiation. But if the electron is between two normal modes, we get a beat frequency, which does cause radiation. This is unstable, and eventually the electron must go back to one of the stable states.

With this, let's derive the Larmor formula. We have an equation for the radiating electric field:

$$\vec{E}_{\text{rad}}(t) = \frac{q}{4\pi\varepsilon_0}\frac{1}{r}\frac{\vec{a}}{c^2},$$

where the acceleration $\vec{a}$ is evaluated by a retarded time of $\frac{r}{c}$.

Theorem 139 (Larmor formula)
The power radiated away by such a charge is

$$P = \frac{q^2|a|^2}{6\pi\varepsilon_0 c^3}.$$

Proof. Note that energy flux can be written as

$$\frac{1}{2}\varepsilon_0 E^2 \cdot 2 \cdot A \cdot c,$$

where the factor of 2 comes from the energy is the same for electric and magnetic field, A is the surface area of a sphere at some radius r, and the factor of c comes from the speed of radiation. We can plug in our electric field and write A in terms of r, and we find that

$$\varepsilon_0 \frac{q^2|a|^2}{16\pi^2\varepsilon_0^2 r^2 c^4} \cdot 4\pi r^2 = \frac{q^2|a|^2}{4\pi\varepsilon_0 c^3},$$

and now we just want to get to a weird factor of 6 instead of 4. To correct for this factor of $\frac{3}{2}$, $\vec{a}$ is a vector, so we only care about the **perpendicular part**. This means we have to integrate $\sin^2\theta$ over the sphere, and that indeed corrects the constant in the desired way. □

43 November 29, 2018

43.1 Review

Remember that in matter, we replace $\vec{E}$ and $\vec{B}$ with some modified terms $\vec{D}$ and $\vec{H}$, which lead us to some modified Maxwell's equations. We also talked about Brewster's angle: we can determine the magnitude of the reflected and transmitted wave, and sometimes we can polarize the light that is reflected.

43.2 Soap and interference of EM waves

Our next topic of study will be that of electromagnetic wave interference, which will help us understand why soap has color.

Fact 140

How thick is the film in a soap bubble? Is it 1 millimeter (pin head), 100 micron (human hair), or 100 nanometer (virus)? Poll says 1, 30, 13.

We're going to need to do a fairly involved calculation to get the answer. Note that the intensity of an EM wave in matter for $\mu \approx \mu_0$ can be writen as

$$I = |\vec{S}| = \frac{1}{\mu_0}|\vec{E} \times \vec{B}| = \frac{n}{\mu_0 c}|\vec{E}^2| = cn\varepsilon_0|\vec{E}^2|.$$

Also (from the last lecture), an EM wave that passes from one transparent medium to another at normal incidence has reflection and transmission coefficients

$$R = \frac{n_1 - n_2}{n_1 + n_2}, \quad T = \frac{2n_1}{n_1 + n_2}.$$

Remember that we can have interference between two EM waves: they can enhance or cancel each other. In particular, if we can add together two waves of different phase

$$\vec{E} = \vec{E_1} + \vec{E_2} = A_1 \cos(\omega t - kz + \phi_1)\hat{x} + A_2 \cos(\omega t - kz + \phi_2)\hat{x},$$

then we get a cross term when we take $\vec{E}^2$, which can be converted by product to sum to give

$$|\vec{E}|^2 = A_1^2 \cos^2(\omega t - kz + \phi_1) + A_2^2 \cos^2(\omega t - kz + \phi_2) 2A_1A_2\left(\frac{1}{2}\cos(2\omega t - 2kz + \phi_1 + \phi_2) + \frac{1}{2}\cos(\phi_1 - \phi_2)\right).$$

From here, we can find the (time-averaged) intensity by integrating over a period:

$$\langle I \rangle = \frac{1}{t}\int_0^T I\,dt = cn\varepsilon_0\left[\frac{A_1^2}{2} + \frac{A_2^2}{2} + A_1A_2\cos(\phi_1 - \phi_2)\right]$$

Letting $\delta = \phi_1 - \phi_2$ be the phase difference, we can plot $\langle I \rangle$ as a function of δ. the maxima are reached at $0, 2\pi, 4\pi$, which are constructive interference, and the minima are reached at $\pi, 3\pi$, and so on, which are destructive interference. These give $\langle I \rangle$ proportional to $\frac{1}{2}(A_1 + A_2)^2$ and $\frac{1}{2}(A_1 - A_2)^2$, respectively.

So how do we use all of this calculate the width of the soap bubble? Notice that the surface of this bubble is the **boundary between two media**. Suppose we have an incident wave of amplitude A, which produces a reflected wave with amplitude RA and transmitted wave with amplitude TA. If $n_1 > n_2$, which means $R > 0$ (soap to air, for example), and there is no flip in sign for amplitude for the reflected wave. But if we go in reverse, $n_1 < n_2$ and $R < 0$, which gives a flip in amplitude, which is equivalent to introducing a phase difference $\delta = \pi$.

So now given $n_1 = 1, n_2 = 1.5$ (air to soap), we find that $R = -0.2, T = 0.8$. So the intensity of the reflected light is

$$I_R = 0.2^2 I_0 = 0.04 I_0,$$

while the intensity of the transmitted light is

$$I_T = n_2 0.8^2 I_0 = 0.96 I_0$$

(indeed conservation of energy works!). This means that 4 percent of the light will go backward, and 96 percent of the light will pass into the bubble – note that the backwards light has had its amplitude flipped as well.

But later, we hit the inner boundary of the soap bubble. Since we're going from soap to air, we have $R = +0.2$ this time, and this means a total of $4\% \cdot 96\%$ of our initial light will be reflected (with no change in sign). As a result, the magnitude of the light that comes out the other side is $96\% \cdot 96\%$.

But this process can repeat many times! (We assume that the soap film walls are some distance d apart, which is small enough that we have parallel walls.) There are many different optical passes through the bubble that lead to the **light reflecting back at us**: we can get the 4% of the light that is reflected immediately, or we can get the $4\% \cdot 96\% \cdot 96\%$ of the light which is reflected off the second wall. We'll ignore the terms after that, because only 4% of the light is reflected each time (and thus the probabilities get small very quickly).

So now consider the interference between those two waves – we'll assume they are roughly the same amplitude. The phase difference between the waves comes from the different lengths that the paths travel, as well as from the **change in sign from the reflection**. Thus our total phase is

$$\delta = \frac{2d}{\lambda/n_2} \cdot 2\pi + \pi,$$

where the dividing by n_2 comes because the wavelength of the light is decreased to accommodate the slower speed. So to have constructive interference, we must have $\Delta = 2N\pi$, and to have destructive interference, we have $(2N+1)\pi$. But taking $d \to 0$, δ always approaches π, meaning we always have destructive interference! So the soap bubble will always look more and more transparent.

On the other hand, how do we set up constructive interference? We need $d = \frac{(2N-1)\lambda}{4n_2}$ for some N (and in general, we want $d = \frac{N\lambda}{2n_2}$ to get destructive interference). Thus, to find the wavelengths λ that create constructive interference if we fix d, we just need to solve for possible values λ:

$$\lambda_{\text{max}} = \frac{4dn_2}{2N-1}.$$

Now we can play the game: if $d \approx 100$ nm, then λ_{max} takes on the values $600, 200, 120$ nm. Well, red is about 650 nm, and violet is about 400 nm, so the constructive interference will lead us to see a **red color** in the soap bubble! In contrast, if we try making $d \approx 100$ microns, this won't work: there's too many different peaks at once that all contribute, and we'll just see white light! Since we can actually see color in a bubble, its width must be closer to 100 nanometers, and we've finally answered our question.

43.3 Propagation of waves: diffraction and interference

We're moving on to our final topic of the semester:

Proposition 141 (Huygen's principle)
All points on a wavefront become a new source of spherical waves.

For example, a plane wave just keeps propagating as a plane wave, because drawing circles from points on a straight-line wavefront creates a new straight-line wavefront. The proof of this result is a bit involved, and it turns out it only holds for odd values of $n \geq 3$, so we're pretty lucky!

With this, it's time to get into the double-slit experiment.

Example 142
Suppose we have a wall with 2 narrow slits, one above the other, which are infinitely wide but very narrow. Call them A and B, and let the vertical distance between them be d. Shine incident light onto the wall (and let it shine through the slits) – we want to know the pattern on the screen L units away from the wall.

To understand what's going on here, pick a point P on our screen. By Huygen's principle, the light at P is a combination of the light from A to P and the light from B to P. If $L \gg d$, it doesn't really matter whether we measure the angle of inclination from A or from B, so we can say that the point P is at an angle θ above the horizontal from both points A and B. Then the difference in path length is

$$r_{BP} - r_{AP} \approx d \sin \theta,$$

and the phase difference between the waves from the two light sources is

$$\delta = \frac{d \sin \theta}{\lambda} \cdot 2\pi = kd \sin \theta.$$

Suppose that the polarization of the light is in the z-direction, for simplicity. Then the intensity of the light depends on $\vec{E_A} + \vec{E_B}$, and we can write this out as

$$(E_0 e^{i(\omega t - kr_{AP})} + E_0 e^{i(\omega t - kr_{BP})})\hat{z} = E_0 e^{i(\omega t - kr_{AP})} e^{-i\delta/2} \left[e^{i\delta/2} + e^{-i\delta/2} \right] \hat{z}$$

But the bracketed term is $2 \cos\left(\frac{\delta}{2}\right)$, so the intensity is proportional to $|\vec{E}^2|$, which is proportional to $\cos^2\left(\frac{\delta}{2}\right)$. **This means that the intensity oscillates sinusoidally based on vertical distance**! We'll study this interference pattern a bit more next time.

44 December 3, 2018 (Recitation)

Today, we'll talk about Maxwell's equations in a medium: how does this affect n, the index of refraction, how do reflection and transmission work, and finally, what happens when waves add up?

First, let's address a problem on the homeowrk about an accelerated charge. We know that the electric field has an $\frac{1}{r}$ dependence

$$\vec{E} = \frac{-q}{4\pi\varepsilon_0} \frac{1}{r} \frac{\vec{a}_{perp}}{c^2},$$

which allows the Poynting vector to decay as $\frac{1}{r^2}$ and creates actual energy propagation. There is a total power emitted here: we integrate the Poynting vector over the sphere to get **Larmor's formula**

$$P = \frac{q^2 |a|^2 \mu_0}{6\pi\varepsilon_0 c}.$$

Note that if energy is lost, we get **radiation damping**, and the charge must decelerate. We'll use this to talk about the **Abraham-Lorentz force**. The easiest derivation of the result is to consider a somewhat periodic system. Integrate the power over a period: we have

$$\int_{\tau_1}^{\tau_2} -P dt = \int_{\tau_1}^{\tau_2} \vec{F}_{\text{rad}} \cdot \vec{v} dt,$$

and our goal is to find the vector $\vec{F}_{\text{rad}}$. The trick now is that this is equal to (by integration by parts)

$$\int_{\tau_1}^{\tau_2} \frac{\mu_0 q^2}{6\pi c} \frac{d\vec{v}}{dt} \frac{d\vec{v}}{dt} = \int_{\tau_1}^{\tau_2} \frac{\mu_0 q^2}{6\pi c} \frac{d^2 v}{dt^2} v dt,$$

and that gives us the formula

$$\vec{F}_{\text{rad}} = \frac{\mu_0 q^2}{6\pi c} \dddot{x}.$$

So what we call the **viscous radiation force** comes from the jerk (the derivative of the acceleration) of our charge's motion!

(By the way, there's also a boundary term we've ignored, which vanishes because it's at the beginning and end of a cycle.)

So now let's go back to Maxwell's equations in a medium. They largely differ from Maxwell's equations in vacuum because materials have a polarization $\vec{P}$ and a magnetization $\vec{M}$ when exposed to electric and magnetic fields:

$$\vec{P} = \int \rho(r)(\vec{r_0} - \vec{r})dV, \quad \vec{M} = \frac{1}{2}\int(\vec{r} \times \vec{J})dV.$$

Some additional features here: the magnetic moment contributes to the torque via $\vec{\tau} = \vec{\mu} \times \vec{B}$. Also, we can write the charge density in terms of the polarization with the equation $\rho = \text{div}\vec{P}$. (One way to think about this last statement is that a bunch of dipoles lined up in the same direction will cancel out to zero charge density.)

What about bound charges? As discussed during lecture, the bound charge density also relates to the polarization via $\rho_b = -\vec{\nabla}\vec{P}$, and now, the total current generated (from the magnetic response in Ampere's law) gets a term from the bound charge, $\vec{\nabla} \times \vec{M}$, as well as a contribution from polarization via $\frac{\partial \vec{P}}{\partial t}$.

Luckily, often in many materials, $\vec{P} \propto \vec{E}$, and $\vec{M} \propto \vec{B}$. So in such a linear medium, things are usually pretty simple: let α be the **polarizability** constant such that that $\vec{P} = \alpha\vec{E}$, which means $\varepsilon = \varepsilon_0 + \alpha$, and write $\vec{M} = \kappa\vec{B}$, defining $\frac{1}{\mu} = \frac{1}{\mu_0} - \kappa$. Now if we take another look at the Maxwell equations, we have that

$$\vec{\nabla} \cdot \vec{E} = 0$$

since $\vec{P}$ is proportional to $\vec{E}$, which has divergence 0. In fact, similar arguments show that everything stays the same as the vacuum case, except that μ_0, ε_0 become μ and ε! So we can return to our familiar equation and find the speed of light in matter to be

$$v^2 = \frac{1}{\mu\varepsilon}.$$

With this idea of polarization in mind, let's now take another look at the interface between two materials. We know the electric field in the parallel direction must be the same on both sides of the interface, and then in the perpendicular direction, Gauss' law tells us that

$$\varepsilon_1\vec{E}_\perp^{(1)} = \varepsilon_2\vec{E}_\perp^{(2)}.$$

Let's review why is this true. Stokes' law tells us that

$$\oint \vec{E} \cdot \vec{dl} = \int \vec{\nabla} \times \vec{E} dA,$$

and if we make a loop which is a rectangle with long ends parallel to the interface and infinitesimally short in the perpendicular direction, we can make the integral evaluate to

$$\ell(E_\parallel^{(1)} - E_\parallel^{(2)}) = 0,$$

since the area integral is over an infinitesimally small area and therefore must approach 0. To derive the equation for the perpendicular direction, use a pillbox in Gauss' law

$$\int \vec{E} d\vec{A} = \int \text{div}\vec{E} dV.$$

Again, with infinitesimal thickness, we're left with

$$A(\vec{E}_\perp^{(1)} - \vec{E}_\perp^{(2)}) = \frac{\rho}{\varepsilon_0}A\delta.$$

But this time, we can't just take δ to zero: we have a singularity because the charge density is discontinuous! Here's where the polarization contribution comes in, and that's why we get an extra factor of ε_1 and ε_2 instead of reducing as simply as in the parallel case.

45 December 4, 2018

Today we're going to talk some more about interference: we will see the effects with lasers (in the double-slit experiment), water ripples, and with phased radar.

45.1 Review

Consider a wall with two (horizontal) slits A and B separated vertically by a distance d. By Huygen's principle, we can add the spherical wave sources from A to B when we shine light into the wall; then, a second distant wall will have intensity $A\cos^2\frac{\delta}{2}$, where $\delta = \frac{d\sin\theta}{\lambda}\cdot 2\pi = kd\sin\theta$ is the phase difference between the two incident electric fields, and θ is the angle from the horizontal. If we plot this intensity, we see that in some points, the light will constructively interfere, and in others, they will cancel each other out. But there are still questions to be answered – what if we have more than N slits? And why is it that there is some kind of larger pattern of nodes and antinodes beyond what is described with our $A\cos^2\frac{\delta}{2}$ pattern?

45.2 The N-slit interference pattern

In real life, **radar** works by having a focused wave sent out at some frequency and having it reflect off some object (such as an airplane). Unfortunately, when we accelerate a charge in some direction, the wave propagates in many different directions, so we'll need to find a way to distinguish them.

It turns out the N-slit interference pattern does a good job:

Example 143

Let's say we have N slits, each separated by some vertical distance d on our initial wall of light sources. What is the electric field at some point P on the other wall, which is an angle θ above the lowest slit?

We want to add up all of the contributions from the individual light sources; let's say R is the distance between the lowest slit and the point P. If $\delta = d\sin\theta$, we can approximate the total electric field via

$$E = E_0\left[e^{i(\omega t-kR)} + e^{i(\omega t-kR-\delta)} + e^{i(\omega t-kR-2\delta)} + \cdots + e^{i(\omega t-kR-(N-1)\delta)}\right].$$

This is an infinite geometric series, so we can rewrite it as

$$E_0e^{i(\omega t-kR)}\left[1+e^{-i\delta}+\cdots+e^{-(N-1)\delta}\right] = E_0e^{i(\omega t-kR)}\left(\frac{1-e^{i\delta N}}{1-e^{-i\delta}}\right).$$

We can write this fraction more nicely by doing a little bit of algebraic manipulation:

$$= E_0e^{i(\omega t-kR)}\frac{e^{-i\delta N/2}}{e^{-i\delta/2}}\left[\frac{e^{i\delta N/2}-e^{-i\delta N/2}}{e^{i\delta/2}-e^{-i\delta/2}}\right]$$

$$= E_0e^{i(\omega t-kR)}e^{-i\delta(N-1)/2}\frac{\sin(N\delta/2)}{\sin(\delta/2)}.$$

So $\langle I \rangle$ is proportional to $|\vec{E}|^2$, and we have our equation

$$\boxed{\langle I \rangle = I_0 \left[\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right]^2}.$$

To understand this equation, let's try varying δ. At $\delta \approx 0$, which means $\theta \approx 0$, all of our light sources go straight towards the wall, and there is no phase. This means that all N vectors constructively interfere, and thus we get N^2 times the intensity. Now increase δ; the vectors start to curve more until $\boxed{\delta = \frac{2\pi}{N}}$, at which point the sum of the vectors exactly cancels out to 0.

But recall that $\delta \propto \sin\theta$ by definition, and the boxed equation above says that position δ of the first destructive interference is decreasing as a function of N. This means that **the central maxima becomes smaller and smaller as we add more slits**, which allows us to have focused light in specific and narrow directions.

This allows us to create a focused beam of light at $\theta = 0$ (horizontally across), but it can also be useful to have the light be focused at a different point. To do this, note that if we add a phase of $\Delta\phi$ between adjacent sources, this makes its way into the geometric series and we get

$$\delta = \frac{2\pi}{\lambda} d \sin\theta - \Delta\phi.$$

This means the central maxima is moved to $\sin\theta = \frac{\Delta\phi\lambda}{2\pi d}$, and now we can sweep our radar at arbitrary angles!

45.3 Single-slit diffraction

Our next topic is actually very similar to interference, but the difference is that there is only **one slit**. The key point is that if our slit is wide enough, we can think of different points in the same slit as acting as spherical wave sources. One way to do the mathematics here is to take the N-slit experiment and take $d \to 0$ and $N \to \infty$ in such a way that Nd stays constant and equal to some D, the width of the slit.

But we can also write this down as a **Fourier transform problem**! Let $C(k_x, k_y)$ be a Fourier coefficient proportional to the total electric field (it lives in the wavenumber space). We can calculate it via the inverse Fourier transform in two dimensions:

$$C(k_x, k_y) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(x, y) e^{-i(k\cdot r)},$$

where $f(x, y)$ tells us about the shape of the source – f is 1 if there is light and 0 if there isn't! (And note that we can ignore the ωt term because it is a common factor.)

Let's apply this to the single-slit diffraction problem: **we're trying to pick out the normal modes that contribute to the interference pattern.** say it is infinitely long along the y-direction and has a width of D along the x-direction. Then the function which dictates our light source looks like

$$f(x, y) = \begin{cases} 1, & -D/2 \le x \le D/2 \\ 0, & |x| > D/2. \end{cases}$$

So we can calculate the Fourier coefficient

$$C(k_x, k_y) = \frac{1}{4\pi^2} \int_{-D/2}^{D/2} dx e^{-ik_x x} \int_{-\infty}^{\infty} dy e^{-ik_y y}.$$

This can be directly evaluated – the inner integral is just a delta function, and we find that

$$C(k_x, k_y) = \frac{1}{2\pi}\delta(k_y) \cdot \frac{e^{-ik_xD/2} - e^{ik_xD/2}}{-ik_x},$$

which simplifies to

$$C(k_x, k_y) = \frac{\delta(k_y)}{2\pi}\frac{2\sin\frac{k_xD}{2}}{k_x}.$$

This means $\vec{E}$ is proportional to $\frac{\sin(k_xD/2)}{k_x}$. The delta function tells us that $k_y = 0$ to have a nonzero contribution from the wavenumber (k_x, k_y); in other words, the light must go straight in the y-direction. So we can now calculate the intensity based on the wave number: we have

$$I \propto |C|^2 \propto \frac{\sin^2(k_xD/2)}{k_x^2}.$$

This lives in frequency space, and we want to get back to our original problem. If x is the horizontal displacement between the two walls, and $\vec{r}$ is the vector from the light source to the point P, $\frac{x}{r} = \frac{k_x}{k}$ (since k_x is along the x-direction and k is along the direction of propagation), and therefore we know that

$$k_x = \frac{2\pi\sin\theta}{\lambda}, \quad k_y = 0.$$

Plugging this in, we have our final formula for the intensity of the light:

$$I \propto \frac{\sin^2\left(\frac{\pi D\sin\theta}{\lambda}\right)}{\left(\frac{2\pi\sin\theta}{\lambda}\right)^2}.$$

Defining $\beta = \frac{\pi D\sin\theta}{\lambda}$, we finally get a good description of the intensity distribution:

$$\boxed{I \propto \frac{\sin^2(\beta)}{\beta^2}}.$$

If we plot this, we find that our minima happen occur (with intensity $I = 0$) for $\sin\theta = \frac{k\lambda}{D}$, and the light decays as $\frac{1}{\beta^2}$. (And notice this only works when $\lambda \ll D$ so that we can make the appropriate approximations.)

So the width of the central maxima is proportional to $\frac{1}{D}$. And an interesting fact comes out of this: as we decrease the width of the slit, the central maxima width gets larger!

45.4 Resolution and the size of the maxima

So now let's look at a slightly different geometry: consider a circle with diameter D instead of a narrow slit. If θ now represents the angle from the center of the circle, it turns out that the intensity distribution looks like

$$I(\theta) = I_0\left(\frac{J_1(\beta)}{\beta}\right)^2$$

where J_1 is a Bessel function of the first kind. This results in a central intensity distribution, and to measure how narrow this distribution is, we can look at the distance before the first minima is reached. If we set $J_1(x) = 0$, we have $x \approx 3.83$m and substituting in our β above yields that the position of the first minima is at

$$\sin\theta \approx \frac{1.22\lambda}{D}.$$

This brings us to the idea of **resolution** – for instance, a 240 cm telescope gets a much better picture of a nebula than a 40 cm telescope, because the width of the central picture is proportional to $\frac{1}{D}$.

From there, let's also consider this resolution idea in modern technology. The pupil is 5 millimeters in diameter, and the angular resolution (the distance between maxima and minima) is $\frac{1.22\lambda}{D}$, which is about $1.22 \cdot 10^{-4}$ with a typical wavelength that we can see. Well, the iPhone X has 459 ppi, so the Δx between two adjacent pixels is $\frac{2.54cm}{459} = 55$ micrometers. Then the angle between those pixels, $\Delta\theta$, is about $\frac{\Delta x}{20cm} = 2.8 \times 10^{-4}$. So this is nearing the limit of what the human eye is physically able to see!

46 December 5, 2018 (Recitation)

Today, we will talk about the **interference of matter waves**. In particular, there are interesting similarities and differences between having two sources of electromagnetic waves versus two matter waves.

First of all, if we have two waves of any material originating from two sources, we (at some distance away from those sources) will observe an intensity of (in the case of two sources of light)

$$I = I_0|e^{i\phi_1} + e^{i\phi_2}|^2,$$

where ϕ_1 and ϕ_2 are the phases of the two waves. Since we take the magnitude when finding intensity, we can just use the phase **difference**, and we get

$$I = I_0|1 + e^{ik\Delta r}|^2$$

as long as all of the light originates from the point sources with the same phase. This is a generic expression, but now we can add some bells and whistles. If we are close to the sources of the light, the intensity from the two light sources will be unequal! If we're distances r_1, r_2 away from our two sources, we instead get an intensity of

$$I = I_0\left|\frac{1}{r_1} + \frac{e^{ik\Delta r}}{r_2}\right|^2.$$

Remark 144. *One way to create "two light sources" is to have a light source near a mirror – light can either bounce off the mirror or travel straight. Then we will have a difference in path length again (from traveling the extra distance to and from the mirror), which contributes a phase difference. But it's also possible that the light goes out of phase from the mirror, which would contribute an additional $e^{i\Delta}$ term.*

And to make the problem more complicated, let's add a piece of glass between the light source and mirror. If there is a spherical glass shell of thickness d, we can calculate the phase shift: if k_g is the wavenumber in glass and k_v is the wavenumber in vacuum, we have

$$\Delta\phi = k_g d - k_v d = d(k_g - k_v) = d(n-1)k_v,$$

where n is the index of refraction of the glass. This gives us a final phase shift of $\boxed{2\frac{d(n-1)2\pi}{\lambda}}$, since we have to go through the glass twice for one of the light sources.

With that, let's move on to something else and relate matter and light intereference together:

Problem 145

What is the locus of all points that form a fixed maxima in the interference pattern for light?

Solution. Remember that a maximum occurs when we have constructive interference between our two light sources. This means that we must keep the **difference in path length** constant, so this locus looks like a **hyperbola**. ☐

With that, let's talk about the interference of two Bose-Einstein condensates. Because these condensates are so cold, we get coherent wave sources from all of the individual atoms! (An analogy is using a laser instead of a lightbulb.)

Problem 146

Take a condensate and break it into two parts; what happens now with the interference pattern?

If this condensate behaved like a gas and we let it expand, we could take a snapshot of the gas t seconds after it starts expanding – then the speed of a particle that is moved by a distance of x is $\frac{x}{t}$. But in quantum mechanics, we have a wavelength $\lambda = \frac{h}{mv}$, which means that **wavelengths are longer near the center**. But now things are more interesting:

Solution. To find the phases for our Bose-Einstein condensates, we integrate $\int k\,dr = \int kr\,dr = \frac{kr^2}{2}$. Then the locus of all points that satisfy $r_2^2 - r_1^2 = c$ is **a vertical line**, rather than a hyperbola. ☐

47 December 6, 2018

The final exam will be on December 21st from 9 to 12. It will be in Walker, covering all material from lectures 1 to 23 and all the psets. (Next Tuesday will be a review session in class.) The last pset is due on Friday, and one problem set is dropped for this class.

47.1 Review

Last time, we discussed the concepts of diffraction and resolution. We found that if we interfere light patterns from two narrow slits, we get a diffraction pattern that is very close to sinusoidal. But the single-slit diffraction adds a larger modulating factor! In particular, if we have two slits that are wide (length D) and separated by a distance d, we will find that the intensity follows a distribution

$$I = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \left(\frac{\sin(N\delta/2)}{\sin(\delta/2)}\right)^2$$

where the first term comes from diffraction (with $\beta = \frac{\pi D}{\lambda}\sin\theta$) and the second term comes from multi-slit interferenc (with $\delta = kd\sin\theta$).

So far in 8.03, we have talked about mechanical and electromagnetic waves. Now we'll take a quick detour into probability density waves!

47.2 Electrons - waves or particles?

Throughout our study of light in this class, it has looked a lot like a wave. But in 1887, Hertz discovered the **photoelectric effect**. The basic idea behind this effect is that shining light with some specific energy $E = h\nu$ onto metal excites electrons. Scientists found then that the maximum energy of the electron could be written as $K = h\nu - \phi$, where ϕ is some potential of the material.

Here, ϕ can be interpreted as the amount of energy that is needed to release the electron. But the experiments didn't seem to make sense: if we shined low-frequency but high-intensity light onto the medal, no electrons were excited. The ultimate conclusion was that **it is the frequency** – not the intensity – of the light that matters when we try to get the energy above ϕ. But now it sounds like the light is like a particle! In other words, the photoelectric effect leads to a photon description of light, and it seems to break the wave description of light.

But the problems don't stop there. Consider a scenario where we shoot billiard balls through a double slit: the distribution should look like two Gaussians around the slits, because the balls pass through one slit or the other. Similarly, if we shoot individual electrons one at a time, they should behave like particles as well, and the wall can act as a detector.

Fact 147

But when we run the double-slit experiment experimentally, even when we shoot a single electron at a time, we still see an interference pattern. So electrons must be "waves" in some sense!

Let's look at the math behind this more carefully: suppose the billiard balls can pass through slit 1, which yields a probability distribution of P_1, or through P_2, which yields probability distribution P_2. So balls follow some joint probability distribution P_{12} which is a combination of P_1 and P_2.

Now if we switch to the electron problem, we can shoot a beam of lots and lots of electrons, and we can do things like block one of the two slits (to force electrons to go through the other lsit). This results in intensity distributions of $I_1 = |\psi_1|^2$ when blocking slit 2 and $I_2 = |\psi_2|^2$ when blocking slit 1. But it turns out the superposition gives

$$I_{12} = |\psi_1 + \psi_2|^2 = I_1 + I_2 + 2\sqrt{I_1 I_2}\cos\delta,$$

which is not $I_1 + I_2$! So in the classical case, we add probability distributions, but in the (quantum) case, we add the underlying **wavefunctions** which we square later. And this introduces interference cross-terms.

This seems like a paradox, so let's add an additional light source near slit 1. Basically, having this will allow us to know which slit the electron went through, so that we can analyze the probability distributions more carefully.

Fact 148

This doesn't quite work, either. Under this kind of setup, the distribution becomes two Gaussians again – the electron returns to being a "classical billiard ball!" So **the interference disappears once we begin to observe it**.

What's curious is that when we reduce the intensity of the light, so that eventually the light is only emitting a photon every few seconds, we get a probability distribution **in between** the two extremes of "classical billiard ball" and "full intereference pattern!" So if we know which slit the electron passed through, we get no interference, and if we don't know, we get interference.

Finally, recall that we can reduce the energy carried by each photon by making $E = h\nu = \frac{hc}{\lambda}$ smaller. At a certain point, we can make λ large enough to be comparable to D. Then those photons still produce a (single-slit) interference pattern, because we've lost the ability to figure out which part of the slit it went through!

Remark 149. *By the way, this all relates to the uncertainty principle. At the beginning, we knew that the momentum of the particle in the parallel direction to the wall was almost zero. But once it passes through the slit, we lose a lot of that information! So the uncertainty principle explains why we get a narrower pattern when we have a larger uncertainty of Δx (where the particle went through the diffraction slit).*

47.3 Particles as waves

We'll close by formally tying everything to quantum mechanics.

Proposition 150
We can describe a particle with a **wavefunction** ψ such that the probability of finding a particle in a spot is proportional to $|\psi|^2$.

This is pretty weird from a classical point of view; if we have probability, we can never predict the exact outcome of an experiment.

Example 151
Consider a particle constrained to a box of length L: this is controlled by a potential function which is ∞ for $x < 0$ and $x > L$, and suppose that our wavefunction satisfies $\psi(0) = \psi(L) = 0$.

We've solved many wave equations like this before with closed ends: we know that the normal modes take the form

$$\psi_m(x) = A_m \sin(k_m x),$$

so adding in the time dependence, we have a wavefunction which evolves via

$$\psi_m(x, t) = A_m \sin(k_M x) e^{i\omega_m t}$$

where m is a natural number. We're still missing the actual wave equation, though:

Theorem 152 (Schrodinger's equation)
Wavefunctions evolve via the differential equation

$$i\hbar \frac{d}{dt}\psi(x, t) = -\left[\frac{\hbar^2}{2m}\frac{\partial}{\partial x^2} + V(x, t)\right]\psi(x, t).$$

From this, let's try to find the **dispersion relation**. Plugging in the mth normal mode, we find that when $V = 0$ (for a free particle),

$$\hbar\omega_m \psi_m = \left[\frac{\hbar^2 k_m^2}{2m}\right]\psi_m,$$

and this means $\omega = \frac{\hbar k^2}{2m}$. If we put this together with de Broglie's relation $p = \hbar k$, notice that the group velocity of this dispersion relation is

$$v_g = \frac{d\omega}{dk} = \frac{\hbar k}{m} = \frac{p}{m} = v.$$

So this is pretty remarkable – a wave-particle is a wave packet whose velocity is the group velocity (classically) and the phase velocity (at small scales)! As a final comment, there's an extension of this, known as the **standard model**, and it currently describes everything we know except gravity. And gravity, according to general relativity, produces waves that are distortions of space time – that field of study also has normal modes.

48 December 10, 2018 (Recitation)

We'll start today by talking some more about electromagnetic waves in a medium. We know that in a vacuum, the dispersion relation is $\omega = ck$. However, in a medium, the relation changes to some function $\omega(k)$. To explain this, note that most materials contain electrons, and the response from the material is mostly electric (and not magnetic).

In other words, $\varepsilon \neq \varepsilon_0$, but it's okay to say that $\mu \approx \mu_0$. The critical idea is to consider the fourth Maxwell's equation

$$\nabla \times B = \mu_0 \left(J + \varepsilon_0 \frac{\partial E}{\partial t} \right).$$

In a real material, we have real current J (which can be categorized into bound and free current), but the important part is that we have an extra current. To analyze this, we can think of electrons as being attached to nuclei using a spring, and using the Newtonian model, we find that the term J is proportional to E.

And here, remember that there is a **phase shift** in the harmonic oscillator! If our frequency is large, and $\omega \gg \omega_0$, we get a phase shift of π; on the other hand, when $\omega \ll \omega_0$, we get a phase shift of around 0. So that gives us a contribution to J, and our dispersion relation ends up just being $\omega = v_p k$, where $v_p = \frac{c}{n}$ is the speed of light in the medium.

In other words, a medium having "no response" to light means that we have $n - 1 = 0 \implies n = 1$. And it makes sense to think about both $n > 1$ and $n < 1$ as well: we can study each of these cases further to understand properties of the materials.

With the remainder of the time in this recitation, we're going to do a **lab tour** of a Bose-Einstein condensate. How do we make such a condensate? Start by considering a setup of hot atoms – these are created by heating up metal (or other solid material) and taking the resulting evaporated beam and tightening it into something that can propagate. In a new vacuum chamber, we can bring the temperature to almost $T = 0$ (usually measured in nanoKelvin). This is done by either laser cooling or evaporative cooling, and one way to describe what exactly is happening is to imagine a stream of sand hitting a ping-pong ball until it slows down. Laser cooling has a limit, known as the recoil limit. But Doppler shifts actually help us out here, since the lasers essentially provide a **viscous force**! And finally, we want a spatial restoring force as well to make sure our whole setup stays in place: we place a magnetic field around where we want the atom to be.

A natural question to ask is why the lasers don't heat up the atom. With normal materials, lasers heat up a material because of molecular vibrations. But in a single atom, no such thing can happen! We can only have an atom's electron excited, and then the Doppler shifted light will have higher frequency and therefore higher energy, cooling the atom further.

49 December 11, 2018

We should fill out the course evaluations – they are important and anonymous, and they were most of the reason why last lecture made some connections to quantum mechanics. The final exam is going to be on December 21 in Walker – Professor Comin will run another review session on Thursday; see Piazza!

So what will we be talking about in our final exam? We started with a single oscillator, talking about the equation of motion, damping, driving force, and unknown coefficients using initial conditions. We then moved on to coupled systems (normal modes, eigenvalue problems, driven systems, and resonance). From there, we discussed the infinite coupled system, using symmetry, and finally applied to the continuous case for the wave equation. These can be described using both progressive and standing waves, and depending on whether we have a bound or infinite system, we use Fourier decompositions and think about the uncertainty principle. Overall, our question is often around the dispersion relation $\omega(k)$.

After this, we moved on to less mechanical behavior. We thought about group and phase velocity, as well as signal transmission for various waves. Moving on to 2 and 3-dimensional systems, we looked at special normal modes and specific topics in optics (reflection, Snell's law, and so on). Polarization was also an important idea: how can we make light linearly, circularly, or elliptically polarized, and in general, how do we generate electromagnetic charges using accelerated charges? Moving away from EM waves that are just in a vacuum, we talked about rainbows, Brewster's angle, and other boundary conditions. Finally, we looked at interference and diffraction as a connection to QM.

With that, let's start the review! A single particle with force proportional to its displacement is in **harmonic motion**. All such systems can be described in the functional form $\ddot{x} + \omega_0^2 x = 0$, which dictates the natural frequency of this system.

When we introduce more complications such as a drag force, it's easier to use complex numbers! This is because exponential functions are so easy to differentiate (while keeping the functional form intact). Once we add drag, it's very interesting that there are (up to) four different scenarios: $\Gamma = 0, \Gamma < 2\omega_0, \Gamma = 2\omega_0, \Gamma > 2\omega_0$, which have very different solutions and properties.

Next, we add a driving force of the form $f_0 \cos(\omega_d t)$. Generally this gives a steady-state solution (that is sinusoidal), plus a (generally decaying) homogeneous or transient solution which slowly decays out. The end frequency will always be ω_d.

After we've exhausted the complication of a single oscillator, we move on to talk about coupled oscillators. We try to deal with these problems by thinking of the whole system as a "harmonic oscillator" with normal modes! If $M\ddot{X} = -KX$ for matrices M, K and a vector of displacements X, we find that we get an eigenvalue problem of the form

$$\omega^2 A = M^{-1} K A,$$

where ω is the angular frequency and A gives us the normal mode! By the way, we can usually diagonalize our system to an eigenvector basis, in which case we do see harmonic oscillation.

Near resonance behavior, if $\omega_d \approx \omega_0$, the amplitude $A(\omega)$ is very large. Remember that we defined the **quality** $Q = \frac{\omega_0}{\Gamma}$, which is larger for a system that is less damped. Generally, the larger the quality, the more the system will respond to resonance.

We then move on to infinite systems; translation symmetry is important here! **Symmetry matrices** S have the same eigenvectors as the original system matrix $M^{-1}K$, as long as the two commute and the eigenvalues of S are all different (which they do). This makes it easier for us to solve the eigenvalue problem: we just need to find the eigenvectors A such that $SA = \beta A$, and we plug those specific eigenvalues A back into $M^{-1}KA = \omega^2 A$ to find the value of ω^2 and thus the frequency for our normal modes.

Specifically, consider an infinite system of masses indexed by j. Then by symmetry, eigenvectors satisfy $A_j \propto e^{ijka}$, where a is the spacing between masses and k is the wavenumber. The important idea here is that this only happens if we have space-translation symmetry. As we decrease spacing, we can replace our $M^{-1}K$ matrix with a differential operator, and in the limit change the vector of ψ_j, individual displacements, into a single wave function $\psi(x, t)$. This gives us the **wave equation**, which sets up a distinction between the second partial derivative with respect to time and space!

To study solutions of this new wave equation, we often use normal modes of the form $A_m \sin(k_m x + \alpha_m) \sin(\omega_m t + \beta_m)$. Here, k_m and α_m are decided by the boundary conditions: for example, if there is a fixed end at $x = 0$, we must have $\alpha_m = 0$. After this, we decide A_m and β_m using initial conditions; importantly, ω_m is related to k_m by the dispersion relation, which depends on the system mechanics! Alternatively, for a more intuitive understanding of the solutions, we can use progressing waves of the form $f(x \pm vt)$.

The wave equation can explain lots of things, including the behavior of pressure and sound waves. One point that was mentioned briefly during the semester: Newton thought PV must be constant, because heat is conducted fast enough to avoid temperature from rising or falling. Meanwhile, Laplace thought that heat flow is negligible, so PV^γ is constant. (Laplace ends up being correct here, but we should wait until 8.044 for an explanation of this.)

If we change our system a bit, we can get a nonlinear dispersion relation $\omega(k)$. Then we can track the speed of the wave in two ways: the **carrier** moves at the phase velocity, $v_p = \frac{\omega}{k}$, but the **envelope** is moving at the group velocity $v_g = \frac{d\omega}{dk}$. We can create phase and group velocities that are arbitrarily high, but no information is being transmitted.

Such systems are more complicated than the ideal wave equation, but we can solve for a dispersive medium's equation of motion using a **Fourier transform**! Writing $f(t) = \int_{-\infty}^{\infty} d\omega C(\omega) e^{-i\omega t}$, we decompose our function into different frequencies and propagate them separately using the dispersion relation. This gives rise to the uncertainty relation: we can't send a very brief pulse without getting a wide range of frequencies, and vice versa.

Moving on to higher dimensions, we can often treat the x-, and y-directions separately and multiply the normal modes together. Reflection and transmission work differently here: if we try to model a wave traveling between two media with different speeds of propagation, we get Snell's law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$. A nice application of this is that we get total internal reflection, which leads to technology like optical fibers.

Next, let's talk about **polarization**. If we're given a direction of propagation for an electromagnetic wave, there are many possible directions for the direction of the electric field. It can be linear, circular, or elliptical as a function of time, and we can build polarizers with an easy axis (where the E field can go through). This allows us to produce interesting results with tools like wave plates and polarizers. Polarization can be filtered out in a camera, and if we have an incident ray at an interesting (Brewster's) angle, we can create a reflected ray that is polarized.

Producing EM waves is hard: we have to actually accelerate a charge to get a perpendicular direction for the electric field! This creates a kink and yields an electric field proportional to $\frac{1}{r}$, which allows for **electromagnetic radiation** to propagate.

Finally, we talked about superposition of waves. In particular, interference patterns are important for determining the intensity of light patterns, and this can be used in looking at radar and other ways to **constructively and destructively interfere** different pulses together. Intensity plots are interesting: they are sinusoidal for a double-slit interference, with an additional modulation by a factor that depends on single-slit diffraction. We can further complicate the problem and create N-slit interference patterns to create large central maxima with "children" peaks as well. This gets pretty hard to work with, but an interesting application of all of this gives a limit on the resolution of our eye.

Finally, all of this is connected to quantum mechanics: in that model, we don't have determinancy anymore, and the distribution of interference patterns depends on whether we are "observing" the experiment itself. Quantum waves are probability density waves, but they still have similar normal modes, and we get a dispersion relation that is very interesting. And a final important idea: energy is now quantized because we have restricted our normal modes! Basically, 8.04 is going to be a weird class.

50 December 12, 2018 (Recitation)

This is a review session by Pearson (the graduate TA).

Let's start with mechanics. A harmonic oscillator follows the Newton's second law equation

$$\ddot{x} = \beta\dot{x} + \omega_0^2 x + F(t),$$

where F can be an impulse or a periodic function. We can break the solution $x(t)$ into a transient and a steady-state solution, and we can also use the idea of resonance to increase the steady-state amplitude. (Remember that the resonance frequency is typically around the natural frequency ω_0, but it depends on the natural frequency as well as the damping term.) We should also know how to arrive at this equation by looking at a physical system and using Newton's law: this type of harmonic oscillator behavior occurs in springs, LC circuits, and pendulums, as long as we use the small-angle Taylor approximation. (And we should know how to find the range of values where making such an approximation is valid!)

Next, when we couple oscillators together, we get our normal modes. We should know how to find the normal frequencies, as well as the amplitudes at which these normal modes oscillate. We can use the symmetry matrix to make our life easier, but this sometimes gives degeneracies in eigenvalues and therefore inaccurate eigenvectors for the $M^{-1}K$ matrix. So often, we should just use our own common sense instead of trying to calculate too much!

Moving on to continuous media, it's good for us to remember the derivation of the wave equation

$$\partial_{tt}u = c^2\nabla^2 u.$$

There's two ways to get a solution: progressing and standing, and understand how to calculate transmission and reflection coefficients. Here, it's particularly important for us to understand the dispersion relation and use it to understand propagation of the waves. Fourier modes and transforms are generally going to be reasonably easy to calculate: either a step function or a Gaussian. It's important to know that the transform of a Gaussian is a Gaussian!

Moving on to electromagnetic waves, we should know Maxwell's equations and solutions in a vacuum. It's good to understand the Poynting vector and how power works, how to get $\vec{E}$ and $\vec{B}$ fields from each other, and things like that. Doing the same thing in a homogeneous medium just replaces ε_0 with ε and μ_0 with μ.

Boundary conditions are a bit different for EM waves: we should understand waveguides, as well as how the parallel and perpendicular components interact. Finally, we should understand how radiation is produced from an accelerated charge and perhaps connect this with interference and/or diffraction.

In general for optics problems, we should understand principles of reflection and Snell's law. There's lots of questions about polarization we can ask: how does reflection change polarization, particularly if it's at an angle? What does Brewster's angle mean? And finally, we should be able to extract key properties of n-slit interference and diffraction.

The syllabus technically lists quantum and gravitational waves, but we won't be asked directly about them.