8.033: Relativity

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Introduction

We'll first go through the main points from the syllabus. Office hours will be right after class, in this classroom if possible. Today, since there isn't enough material, there won't be formal office hours. Everyone besides Professor Vitale also has office hours, listed on the syllabus.

The purpose of this class is to study relativity, both special and some general. Special relativity deals with velocities larger than what we deal with in our daily lives: we'll see that Newtonian equations start to fail us dramatically. Later in the class, we'll start to deal with gravity and acceleration.

We will hopefully see that everything makes sense. That's what's great about physics: we may need to use math, but everything will make sense. This can't necessarily be said about quantum mechanics – the "why?" is a hard question there.

18.02, 8.01, and 8.02 are prereqs. 18.06 as a coreq is recommended for those of us who enjoy thinking about the geometry of what's going on.

There's one required textbook, listed on the syllabus, as well as other recommended books. Problem sets and mock exams will be on the LMOD website, too. Homework is due weekly on Friday at 5:30pm in the usual boxes, and midterm dates will be confirmed by the end of the week. Note that November 6, the date of the second midterm, is chosen so that we'll get grades back by drop date.

Every teacher says the same thing: don't wait until the night before to do your psets! We'll see especially from 8.033 that adiabatic learning will be much better than step-by-step. From a long tradition, we drop the worst pset grade, but use this for real emergencies. Late homework will not be graded unless we arrange specially.

Also, this is a physics class, not a philosophy class, so we should keep problem sets as concise as possible. Verbosity is not particularly encouraged.

Remark 1. *For some reason, relativity brings the literacy out of students...*

On a more serious note, conciseness is important because at some point, we'll want to write scientific articles which also need to be short and to the point.

Also, we will not use Piazza in this class – questions will be dealt with by asking directly in person or by email. Professor Vitale doesn't like Piazza very much, because he thinks it's important for us to know how to express our doubts and questions ourselves. (We can yell; no need to raise our hands.) It's also great to have questions during class because typically questions are shared – probably three or four people also in the class have the same question.

Finally, there is a tutoring service starting week 6 of the class, aimed for students who are struggling to pass. This shouldn't be our first option if we want to ask for help, though.

The 8.033 team all actively works with relativity in their work, so this will be a fun class!

Fact 2
Grades are not curved. The strictest grade curves will be A/B = 90, B/C = 80, and so on, and there are many things that all factor into our final grade.

Lecture notes will be posted after each class. If they're not there by the end of the day, we should send Professor Vitale an email. (And these notes exist, too!)

1 September 4, 2019

We're going to start with a slow introduction to special relativity: we'll review some definitions and results from 8.01 in a way that will be relevant for the future, and we'll talk about why we need to study beyond Newtonian mechanics.

Definition 3
A **reference frame** or **frame** is a three-dimensional coordinate system (e.g. (x, y, z)), where every point has a **clock** that records time, and all clocks are in sync.

It's important to stress here that we shouldn't imagine a reference frame as having an observer! Instead, there is an observer at every point with a clock, so whenever an event happens, we can quantify where we are (x, y, z), as well as when we are (t).

Definition 4
An **event** is something that happens somewhere at some time.

For example, "Betelgeuse blows up" is an event. All observers will agree that this event happened, so we don't necessarily need a frame to quantify whether this happens. However, depending on which coordinates we're using, we may assign different numbers to label the same event! For example, we can say that Betelgeuse blew up at a spot (x, y, z) at time t, and we know this because our coordinate frame gives us a clock at this specific point. This is a bit cumbersome to write down, so let's come up with some shorthand: if S is the standard Cartesian frame, we'll write

$$B \underset{S}{\rightarrow} (t, x, y, z).$$

Notice that if we change frames, for example to spherical coordinates in a frame S', we still have the same event B. However, we now have some different labels

$$B \underset{S'}{\longrightarrow} (t, r, \theta, \phi).$$

In this class, we're going to use **four-vectors** to make our notation more concise. we've already seen this in the examples above, but we can be a bit more abstract: we'll define $\mathbf{x} = (t, x, y, z)$ to be a four-vector (in bold), where the entries of the vector depend on our frame.

In this class, we'll grab coordinates from our vector via

$$\mathbf{x}^0 = t, \mathbf{x}^1 = x, \mathbf{x}^2 = y, \mathbf{x}^3 = z.$$

Concisely, this will be denoted x^μ. **By convention, all Greek letters in this class take on the values** $0, 1, 2, 3$**.** If we only want to refer to the space part of the four-vector (so not 0), we will use a Latin letter like i, j, k.

Let's now return to physics by going back to 8.01. Most of that class is about particles that move: something we see often is a particle moving in a path. At any time t, we can (or want to) figure out what point $\vec{r}(t)$ the particle is currently at. Remember that these coordinates depend on our frame, and we're using the standard frame S here. We'll write this as

$$\vec{r}(t) = (x(t), y(t), z(t)) = x(t)\hat{e}_x + y(t)\hat{e}_y + z(t)\hat{e}_z,$$

where the $\hat{e}$ vectors are unit basis vectors. From this, we can find the velocity

$$\vec{u}(t) = \frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right).$$

We've been lucky so far in our physics experience, because the derivative of the vector is just the term-by-term derivative of the individual components. We can then find the acceleration in a similar way:

$$\vec{a}(t) = \frac{d\vec{u}}{dt} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right).$$

This is as much as we need to do physics with a particle in 8.01: let's reframe this model for the purposes of 8.033. Let's think of the particle's trajectory as a **sequence of events**, writing it as a path of the form

$$P : \text{the particle is at a specific spot now.}$$

This stresses that we don't need a frame to think of a particle's trajectory, but eventually we do need one to do calculations for it! If we add a coordinate frame S to it, we can then turn each event

$$P_i \to (t_i, x_i, y_i, z_i)$$

into a specific four-vector. There's an uncountable number of events, so we usually index our events by a real number τ. This gives us a parameterization

$$P(\tau) = (t(\tau), x(\tau), y(\tau), z(\tau)).$$

There's a few frames that have played important roles in our physics journey so far:

Definition 5

An **inertial frame** is one where $\vec{F} = m\vec{a}$ is valid.

Remark 6. *Note that this is a vector equation, so it's really three equations: we can technically write it out as* $\begin{cases} F^x = ma^x \\ F^y = ma^t \\ F^z = ma^z \end{cases}$ *or as $F^i = ma^i$. This is one reason why we like to use indices in the way introduced!*

In 8.01, if we are given an inertial frame, we can find infinitely many other inertial frames by starting from the initial frame and using what are called **Galilean transformations**:

1. rigid translations (move the origin), which can be described by $\vec{x} \to \vec{x}' = \vec{x} + \vec{a}$ and $t \to t' = t + T'$, or more simply just as $x^\mu \to x^{\mu'} = x^\mu + a^\mu$,
2. rigid rotations (in three dimensions), described as $\vec{x} \to \vec{x}' = R\vec{x}$ for some rotation matrix R (with t held constant),
3. or boosts, where $\vec{x} \to \vec{x}' = \vec{x} - \vec{v}t$ and t is held constant.

Physically, this just means that our axes are parallel to the original ones, but the origin is moving at some constant velocity.

Notably, there's no "correct" reference frame: $\vec{F} = m\vec{a}$ holds in all such inertial frames, so there's no experiment that can help us distinguish one from another as being correct. In other words, doing physics in inertial frame S versus inertial frame S' is equivalent, but we can still consider S' relevant to S. Just remember that it's perfectly valid to switch between any inertial frames!

Let's verify that these boosts indeed preserve the laws of mechanics:

Proposition 7
If $\vec{F} = m\vec{a}$, then $\vec{F}' = m\vec{a}'$ in our boosted frame.

Proof. To make things simpler, whenever we do this kind of argument, **let's assume $\vec{v}$ is parallel to the x-axis.** (We can do the problem separately for all three axes and compose the transformations together.) Consider an event P that happens: if the four-vector in frame S is (t, x, y, z), then the four-vector in S' is $(t, x', y', z') = (t, x - vt, y, z)$.

Let's also assume that the event happens at $t = t' = 0$: we can use a time-translation transformation to make sure this happens, so that the frames coincide at the time of the event.

Since we're trying to verify Newton's laws, consider a particle following path $\vec{u} = \frac{d\vec{r}}{dt}$. Then the velocity $\vec{u}'$ in our boosted frame satisfies

$$u^{x'} = \frac{dx'}{dt'} = \frac{d(x - vt)}{dt} = \frac{dx}{dt} - v = u^x - v$$

(by substituting in the definition of our boost). This makes sense: if we are moving our frame at a velocity v, we see all particles shifted by a velocity v. Also, notice that $u^{y'} = \frac{dy'}{dt'} = \frac{dy}{dt} = u^y$, and similarly $u^{z'} = u^z$. So the velocity of our particle in our new frame is

$$\vec{u}' = (u^x - v, u^y, u^z).$$

Now, the acceleration $\vec{a}'$ satisfies

$$a^{x'} = \frac{du^{x'}}{dt'} = \frac{d(u^x - v)}{dt} = \frac{du^x}{dt} - \frac{dv}{dt} = a^x$$

(since v is constant), and similarly $a^{y'} = a^y, a^{z'} = a^z$. This means that the accelerations $\vec{a}' = \vec{a}$: in both frames, we describe the acceleration of the particle equivalently, which is great!

What about the force $\vec{F}$, then? Let's consider this for a specific force: Newtonian gravity. if we have two particles of mass M, m along the x-axis, at positions x_1, x_2, we know that

$$\vec{F} = -\frac{mMG}{r^3}\vec{r} = -\frac{mMG}{(x_2 - x_1)^2}\hat{e}_x.$$

But we can transform the old coordinates into our new coordinates in frame S: along the x-direction,

$$F^{x'} = -MmG(x_2' - x_1')^{-2} = -MmG((x_2 - vt) - (x_1 - vt))^{-2} = -MmG(x_2 - x_1)^{-2} = F^x,$$

and thus the forces are numerically equivalent as desired. This works in all three directions, so $\vec{F}' = \vec{F}$, and therefore $\vec{F} = m\vec{a} \iff \vec{F}' = m\vec{a}'$ holds in both frames. $\square$

This does assume that nothing happens to the mass m under a transformation like this, but we have to believe that for now!

On the other hand, do we need a lot of work for inverse transformations? No, because we're exploiting symmetry here! If we want to change frames in reverse, we change v to $-v$, and we have to replace primes with no primes (because we're renaming our transformation). That means the transformation $S' \to S$ is of the form

$$\begin{cases} t = t' \\ x = x' + vt' \\ y = y' \\ z = z'. \end{cases}$$

This model for physics worked for 150 years, but then in 1870, Maxwell came up with Maxwell's equations, which implied that light waves travel at a constant speed $\approx 3 \times 10^8 \frac{m}{s}$. But this is independent of our reference frame, and the boosts we've been talking about are supposed to transform our velocity via $u' = u - v$. So this statement about the speed of light can seemingly only be true in one frame: we'll figure out what's going on here tomorrow!

2 September 5, 2019 (Recitation)

My recitations are being taught by Professor Netta Engelhardt. (Office hours are Thursday 5-6pm in room 6-403.)

Professor Engelhardt studies quantum gravity, so she's very excited to introduce this theory to us! Recitations are meant to help us understand the content better, so if we want to see something being covered, we should let Professor Engelhardt know.

An important theme throughout this class is that relativity describes our universe, even if we can't see it in our daily life. In other words, we shouldn't count something as sci-fi just because it sounds like it!

We'll start with a bit of review. During lecture, we introduced the concept of a **reference frame**, which is basically just a coordinate system (t, x, y, z). The standard Cartesian coordinates (hopefully the notation won't be a hinderance for too long) have basis vectors

$$\mathbf{e}_0 = (1, 0, 0, 0), \mathbf{e}_1 = (0, 1, 0, 0), \mathbf{e}_2 = (0, 0, 1, 0), \mathbf{e}_3 = (0, 0, 0, 1).$$

We can also use polar coordinates (r, θ, ϕ): the following vectors correspond to t, r, θ, ϕ respectively:

$$\mathbf{e}_0' = (1, 0, 0, 0), \mathbf{e}_1' = (0, 1, 0, 0), \mathbf{e}_2' = \left(0, 0, \frac{1}{r}, 0\right), \mathbf{e}_3' = \left(0, 0, 0, \frac{1}{r \sin\theta}\right),$$

where we should notice that we no longer have constants because spherical coordinates "scale" based on r. It's important to note that both reference frames here are equally valid!

Out of our reference frames, some of them are inertial: basically, we like to work with frames where we don't have acceleration. Galileo observed that the earth is more or less at an inertial frame: the acceleration is so small that we don't have to worry about it, so that's why this early field is called "Galilean relativity." Things like a rock in space would be in an inertial frame if there were literally no forces on it at all.

What are some examples of Galilean relativity?

Example 8 (Length invariance)
A train is moving at a constant velocity $\vec{v}$ with respect to an observer on the ground. The ground observer measures a rod on the ground to have length ℓ: what would an observer on the train measure the length of the rod to be?

We have two frames in this problem: S, the ground frame, and S' the train frame. For simplicity, let's say that the velocity and rod are both along the x-direction.

Pick a coordinate system for the ground frame Ssuch that the endpoints of the rod are at x_A and x_B: then the rod has length $x_B - x_A$. Similarly, the length of the rod in S' is $\ell' = x'_B - x'_A$. But we know that $x' = x - vt$, so

$$\ell' = (x_B - vt) - (x_A - vt) = x_B - x_A = \ell,$$

and we indeed have length invariance. (In a few weeks, we're going to find that we have a lot more problems because the times might be different, and there's tons of other confusion too.)

Let's look at some combined transformations now:

Example 9

Consider a block which is moving with a velocity $\vec{v}_0$ attached to a pulley on an inclined ramp of angle θ, and say that we have a ball on the topmost spot of the block. At time t_0, the ball is dropped off the block at position $\vec{x}_0$. What is the equation dictating the motion of the ball $\vec{x}(t)$?

This is an example where an intermediate reference frame may be useful! We want to undo two confusing parts of this: the boost of $\vec{v}$ from the block, and a rotation so that we can have more reasonable coordinate axes.

So let's start from the reference frame S of the box with shifted axes: then the ball is just in free-fall from rest, so

$$\vec{x}(t) = \frac{1}{2}\vec{g}(t - t_0)^2 + \vec{x}_0.$$

Meanwhile, if we are looking in frame S', so we have normal axes and are stationary relative to the ramp and ground, we can write a separate equation

$$\vec{x}'(t) = \frac{1}{2}\vec{g}'(t - t_0)^2 + \vec{v}_0(t - t_0) + \vec{x}'_0.$$

As of right now, these are not related yet, so now it's time for us to use our transformations! How do we find $\vec{x}'_0$? Notice that the actual transformation being done to $\vec{x}$ and $\vec{g}$ is a combination of the rotation and a boost, and thus

$$\vec{x}'(t) = [R(-\theta)] \cdot \vec{x} + \vec{v}_0 t, \quad \vec{g}' = [R(-\theta)]\vec{g},$$

where R is a rotation matrix. Plugging in what we know,

$$\vec{x}' = [R(-\theta)] \cdot \vec{x} + \vec{v}_0 t = [R(-\theta)] \cdot (\frac{1}{2}\vec{g}(t - t_0)^2 + \vec{x}_0) + \vec{v}_0 t$$

and this now simplifies to

$$\frac{1}{2}\vec{g}'(t - t_0)^2 + [R(-\theta)] \cdot \vec{x}_0 + \vec{v}_0 t.$$

Equating this to the $\vec{x}'(t) = \frac{1}{2}\vec{g}'(t - t_0)^2 + \vec{v}_0(t - t_0) + \vec{x}'_0$ we had before, we find that

$$\vec{v}_0(t - t_0) + \vec{x}'_0 = [R(-\theta)] \cdot \vec{x}_0 + \vec{v}_0 t,$$

or that $\boxed{\vec{x}'_0 = R(-\theta)\vec{x}_0 + \vec{v}_0 t_0}$. Plugging in our $\vec{x}'_0$ and $\vec{g}'$ gives us a closed expression for $\vec{x}'(t)$.

Let's move on to thinking about index notation! In this class, we'll describe events using four-vectors $\mathbf{v}$. Four-vectors don't need to be expressed as coordinates, but we can write them down as

$$\sum_\mu v^\mu \mathbf{e}_\mu,$$

where μ ranges from 0 to 3. In index notation, **we omit the summation notation**: we write something like

$$v^0\mathbf{e}_0 + v^1\mathbf{e}_1 + v^2\mathbf{e}_2 + v^3\mathbf{e}_3 = v^\mu \mathbf{e}_\mu,$$

which saves a lot of clutter. This is useful when we want to do things like matrix multiplication! Matrices are described via two indices M_{ij}: specifically, the matrix looks something like

$$M = \sum_{\mu,\nu} M_\mu{}^\nu e^\mu e_\nu.$$

If we want to multiply a matrix by a vector, we can write it out in component form, and we'll see that the rows and columns we're multiplying across will have their indices cancel out! By the way, the upper and lower indices will make more sense soon, but they basically correspond to column and row vectors, respectively.

3 September 9, 2019

Let's start with a quick reminder of where we are. Galilean relativity helps us switch between reference frames: for example, if we have two Cartesian coordinate systems, where one of them (S') is moving $\vec{v} = (v, 0, 0)$ relative to the other (S), and they are coincident at $t = 0$, we can make the transformation

$$\begin{cases} t' = t \\ x' = x - vt \\ y' = y \\ z' = z \end{cases} \implies \begin{cases} u^{x'} = u^x - v \\ u^{y'} = u^y \\ u^{z'} = u^z \end{cases},$$

and then the accelerations that we measure in the two frames are the same. So regardless of which inertial frame we use, the laws of physics look the same, and therefore all such frames are equivalent.

But then electromagnetism was fully developed, and problems start to arise. By the end of today, we're going to start discussing the solutions!

Almost all of the theory of electromagnetism comes from Maxwell's equations:

- $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$
- $\vec{\nabla} \cdot \vec{B} = 0$
- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\vec{\nabla} \times \vec{B} = \mu_0(\vec{J} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t})$

where ρ is the charge density and J is the current density. We also need to know that a particle experiences a force

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$$

in an electric and magnetic field, and now we can basically do everything we need to solve problems.

One solution to Maxwell's equations can be found in a vacuum, where $\rho = \vec{J} = 0$: then we get wave solutions, known as **electromagnetic waves**, of the form

$$\vec{E} = E_0 \cos(kx - \omega t)\hat{e}_y, \quad \vec{B} = B_0 \cos(kx - \omega t)\hat{e}_z$$

(only along the x-direction for simplicity), where the speed of propagation is

$$c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} \approx 3 \times 10^8 \text{ m/s}.$$

Indeed, if we shoot a laser beam in the lab, this is what we measure. But something is wrong here: if we believe Galilean relativity, the velocity of anything is supposed to depend on the frame of reference we're using. But Maxwell's equations don't require a special frame for anything: **in which frame is the speed of light equal to what we measure?** In fact, it would also be weird if we found that Maxwell's equations depend on the frame that we're working with, because this would imply some lack of symmetry.

So people did realize that there was this discrepancy between Galilean relativity and Maxwell's equations, and they realized there are only three possibilities for explaining this:

- Relativity only applies to mechanics and not electromagnetism; electromagnetism only has the simple form dictated by Maxwell's equations in a specific frame.
- Relativity applies to all laws, and Maxwell's equations are wrong.
- Relativity applies to all laws, but Galilean transformations are wrong.

This third option was discarded pretty quickly because it seemed nonsensical, and the second was also discarded because Maxwell's equations were consistent with all lab experiments. So everyone mostly decided on possibility 1: there exists one frame that is special, called the **ether**, and in all other frames, $\vec{u}' = \vec{c} - \vec{v}$.

Fact 10

This is something that can be tested, so scientists tried and failed systematically. See the Michelson-Morley experiment.

Around 20 years after Maxwell's equations were discovered, Einstein proposed in 1905 the following:

- Maxwell's equations are correct as stated.
- The speed of light c is the same in all frames; that's why all experiments couldn't find any deviation.
- Relativity does hold, but Galilean transformations are wrong!

It turns out what's going wrong is the most innocuous-looking equation in our transformation: $t' = t$.

Proposition 11

It's not possible to synchronize the clocks of two different frames S, S' so that $t = t'$ and the measured time between events $\Delta t_{AB} = \Delta t'_{AB}$ is always consistent.

Let's discuss this. Suppose person X is sitting at the origin of frame S, and individuals $Y_1, Y_2, \cdots$ are sitting somewhere else: every person has a clock, and we want to synchronize X's clock with everyone else.

Remark 12. *(Joke) We're already doing a bad job, because Professor Vitale has 3pm on his watch, but the clock in the back says 3:01pm.*

But synchronizing is hard, because it takes time to get information from one place to another. So we should use the speed of light to synchronize clocks, because that's something that doesn't change based on our reference frame. So this gives us a procedure: if person Y_i is a distance $r_{(i)}$ away from X, then person X releases light from the origin at $t = 0$, and person Y_i should program their clock to a time

$$t = \frac{r_{(i)}}{c},$$

the time it takes for the light to travel from X to Y_i. This seems straightforward for now.

But the consequences of this are not so straightforward. We have a new question now: how do we decide if two events are simultaneous? For example, if we have two events A and B (dropping y and z-coordinates):

$$A: \text{ lightbulb turned on at } (t_A, x_A), \quad B: \text{ lightbulb turned on at } (t_B, x_B),$$

how can we decide if $t_A = t_B$? One thing to do is to put an observer O at the midpoint of x_A and x_B. If $t_A = t_B$, because the speed of light is the same, the observer will see the two flashes of light at the same time. (This is equivalent to the first example where we are trying to synchronize clocks.) On the other hand, if B happens after A, the light from A will arrive first.

But the problems start when we introduce another frame S'. We have the same events A and B, but let's say another observer O' starts where O is and moves with velocity v relative to O. If the light beams leave A and B simultaneously at $t = 0$, by the time the light beams have gotten to O, the observer in S' has already moved, so they won't observe the two beams at the same time. So simultaneity is a relative statement: $\Delta t_{AB} = 0 \not\Rightarrow \Delta t'_{AB} = 0$!

In fact, time intervals also depend on a frame (they're relative as well).

Example 13

Let S be a platform at rest, and let S' be the frame of a train moving with velocity v. Inside the train, passengers have a clock: the way it works is that a beam of light comes from the floor of the train, bounces off a mirror on the ceiling, and comes back to where it starts. (This is one tick.)

There are three events here:

- Light leaves the bottom laser
- Light hits the mirror
- Light arrives back at the laser.

How long does this process take? In our S' frame, the three events have coordinates

$$A \underset{S'}{\to} (t_{A'}, x_{A'}, 0), \quad B \underset{S'}{\to} (t_{B'}, x_{A'}, h), \quad C \underset{S}{\to} (t_{C'}, x_{A'}, 0).$$

The round trip takes time

$$\Delta t' = t_{C'} - t_{A'} = \frac{2h}{c},$$

just because light travels a total distance of $2h$. This is easy!

But in our S frame, those same events look different: since the train is moving, we see the light at different x-positions for A, B, and C. Explicitly,

$$A \underset{S}{\to} (t_A, x_A, 0), \quad B \underset{S}{\to} (t_B, x_B, h), \quad C \underset{S}{\to} (t_C, x_C, 0)$$

where notably $x_A \neq x_B$. In Δt seconds, the train travels a horizontal distance of $v\Delta t$, and (by symmetry, half of this is traversed between A and B), so the light actually travels a distance of

$$\sqrt{h^2 + \left(\frac{v\Delta t}{2}\right)^2}$$

between position A and B, and therefore the time that this event takes is

$$\Delta t = \frac{2\sqrt{h^2 + \left(\frac{v\Delta t}{2}\right)^2}}{c}.$$

Solving for Δt, we find that

$$\Delta t = \frac{2h}{c\sqrt{1 - \frac{v^2}{c^2}}} = \Delta t' \cdot \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then for all velocities $0 < v < c$, $\Delta t > \Delta t'$: this is what is called **time dilation**. The event takes longer to happen in the S frame than it does in the S' frame!

Definition 14

For simplicity of notation, we will denote $\beta = \frac{v}{c}$ and

$$\gamma(|\vec{v}|) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}.$$

(γ depends only on the magnitude and not the direction of the velocity.)

So this is an example where we can experimentally verify our hypothesis: run an experiment in two different frames, and calculate whether the times are different. Notably, though, this requires synchronizing clocks in different positions, which is a deeper question than we'd initially thought!

Definition 15

The **proper time interval** τ between two events is the time interval measured in the frame where they are co-located.

So S' does measure the proper time interval because events A and C occur at the same x-position, but S does not. So our time dilation result can be rewritten in another way:

$$\boxed{\Delta t = \Delta\tau \cdot \gamma(v)}.$$

This discrepancy is only relevant with γ is significantly different from 1. $\Delta t = \Delta \tau$ if $v = 0$, and as v approaches c, the difference becomes more and more relevant, but we need to work with velocities that are significantly different from those in our daily life to get anything noticeable!

There are larger implications here, too: if we try to measure the length of a moving object, we want to measure the position of the two extreme ends at the same time. But remember that simultaneity is relative now! As an illustrative example, let's say that our laser from above is now moving horizontally instead of velocity, and let's say we're trying to calculate Δx, the length of the train.

In the S' frame, $\Delta x' = \frac{c\Delta t'}{2}$, because the light is traveling a total distance of $2\Delta x'$ in $\Delta t'$ seconds. But S describes the situation differently: as light goes to the right, the train is also moving to the right, so light has to catch up with the train. The first leg (left to right) takes a time $\frac{\Delta x + v\Delta t_1}{c}$, which yields $\Delta t_1 = \frac{\Delta x}{c\left(1-\frac{v}{c}\right)}$. The second leg (right to left) is similar: $t_2 = \frac{\Delta x - v\Delta t_2}{c}$, and thus $\Delta t_2 = \frac{\Delta x}{c\left(1+\frac{v}{c}\right)}$. So the total time

$$\Delta t = \Delta t_1 + \Delta t_2 = \frac{2\Delta x}{c}\gamma(v)^2.$$

But now remember that Δt and $\Delta t'$ are related: plugging in $\Delta t = \gamma \Delta t'$,

$$\Delta t' = \frac{2\Delta x \gamma}{c}, \quad \Delta x' = \frac{c\Delta t'}{2},$$

and thus

$$\frac{2\Delta x'}{c} = \frac{2\Delta x\gamma}{c} \implies \boxed{\Delta x = \frac{\Delta x'}{\gamma(v)} = \sqrt{1 - \frac{v^2}{c^2}}\Delta x'} < \Delta x'.$$

So the two frames disagree by a factor of $\frac{1}{\gamma}$ about the length of the train: that means that if we do a measurement of a length in a non-rest frame, we get a smaller length. This is **length contraction**!

Does something similar happen along the other space dimensions, though? The answer is no! Imagine if something happened to the length of the train along the z-coordinate (height). Then if it passes through a tunnel of the same height as itself, we can't have the tunnel or train change height, or experimental results would look different. So length contraction must only happen along the direction of movement.

So is there anything that our events agree upon, if they don't agree about either time or space intervals? It turns out that our entire transformation will need to be modified, and we'll do that next time!

4 September 10, 2019 (Recitation)

The Michelson-Morley experiment was meant to test the existence of ether, and we'll start this class by explaining where the concept came from. The lesson is that physicists will go very far to avoid something that they think is a pillar of physics!

There's three ideas at work here:

- All laws are the same in all inertial frames (relativity).
- Maxwell's equations for electromagnetism; notably, we think this is a law of nature.
- Galilean transformations, including Galilean addition of velocities.

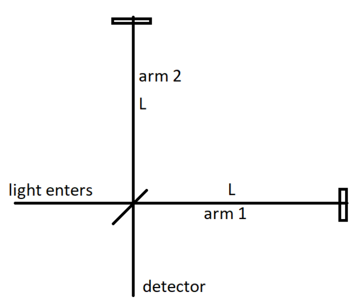
These three seem to contradict each other: Maxwell's equations predict a particular speed of light, which contradicts the workings of Galilean transformations. So physicists chose to remove the wrong assumptions at the time: they assumed that Maxwell's equations only hold in a particular reference frame, and in this special frame, the speed of light is c. This wasn't crazy at the time, because we do know that the speed of light changes in a medium: for example,

$$c_{\text{water}} = \frac{1}{\sqrt{\varepsilon_{\text{water}}\mu_{\text{water}}}}.$$

This is only the speed of light in the specific reference frame of running water, so the logic was that a preferred reference frame for c in vacuum would make sense, too! It seemed odd that there could be anything in vacuum, so the natural answer was to try to find this fluid-like substance, called the **luminiferous ether**. This made sense for explaining electromagnetic waves too: maybe EM waves just "wiggle through the ether."

So Michelson and Morley tried to find this experimentally in 1887. We know that $\vec{u}_{\text{ether}} = \vec{c}$, and if the Earth is moving relative to that ether, $\vec{u}_{\text{earth}} = \vec{c} - \vec{v}$: we should be able to measure this discrepancy noticeably.

So they set up a contraption where light enters from the left, gets split into two beams, which bounce off and recombine:



If S is the ether reference frame, and S' is the earth reference frame, and the horizontal arm is along the direction of the earth's movement, by Galilean addition of velocities, the light will take slightly different amounts of time to finish bouncing off the two mirrors, so the two light beams will be out of phase! Numerically, if it takes t_1 seconds for light to travel through arm 1, and it takes t_2 seconds for light to travel through arm 2, then the difference in time

$\Delta t = t_2 - t_1$ gives us different results:

$$\frac{c\Delta t}{\lambda} = \begin{cases} 0, \pm 1, \pm 2 & \text{constructive interference,} \\ \pm\frac{1}{2}, \pm\frac{3}{2} & \text{destructive interference,} \\ \text{others} & \text{something in between.} \end{cases}$$

If the earth is moving at a velocity of $\vec{v} = v\vec{e}_x$ relative to the ether, the time it takes (in reference frame S) for one light beam is

$$t_1 = \frac{L}{c-v} + \frac{L}{c+v} = \frac{2L}{c(1-\frac{v^2}{c^2})} = \frac{2L}{c}\gamma(v)^2$$

(because the ether helps in one direction and hurts in the other). Meanwhile, to compute t_2, we first compute it in the earth's reference frame. Much like in the example during lecture with the clock on the train, the total distance that the light travels follows

$$\left(\frac{ct_2}{2}\right)^2 = L^2 + \left(\frac{vt_2}{2}\right)^2 \implies t_2 = \frac{2L}{c}\gamma(v).$$

So the difference between them

$$\Delta t = t_2 - t_1 = \frac{2L}{c}(\gamma - \gamma^2),$$

and now since we expect $v \ll c$, we can Taylor expand in $\beta = \frac{v}{c}$: since $(1+nx)^\alpha \approx 1 + \alpha nx + O(x^2)$, we can expand out $\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \approx 1 + \frac{v^2}{2c^2}$ to find that

$$\Delta t \approx \frac{Lv^2}{c^3}.$$

So the quantity that we really want is

$$\frac{c\Delta t}{\lambda} = \frac{Lv^2}{\lambda c^2}.$$

Plugging in $L \approx 5$ meters, $\lambda \approx 10^{-7}$ meters, and $\left(\frac{v}{c}\right)^2 \approx 10^{-8}$, we should see a phase difference of order 10^{-1}. But Michelson and Morley found nothing noticeable to the order of 10^{-2}: in fact, they rotated it in all angles, and they also waited six months so that the Earth was moving in the other direction. Still nothing!

Fact 16

Physicists like to hold on to assumptions like this. Don't take any particular principle as the holy grail!

So the inevitable conclusion came: **light travels at the same speed in all reference frames!** This has survived a lot of different tests, and it has been tested extensively, so we can assume it's true.

So what does that tell us about time dilation?

Example 17

Suppose we have some clock A moving with respect to a clock B at a velocity $\vec{v} = v\vec{e}_x$. If clock A is going half as fast as clock B from the reference of B, find v.

The first step is always to fix some reference frames: let's say that clock A is at rest in frame S', and clock B is at rest in frame S. We can use the Lorentz transformation

$$t = \gamma(v)\left(t' - \frac{vx'}{c^2}\right),$$

but there's a significant simplification we can make here: since S' is the reference frame of clock A, the second term disappears (as $\Delta x' = 0$ in this frame)! In this frame, if we have an event that we measure, then

$$\Delta t = \gamma \Delta t'.$$

Since $\gamma = 2$, we can invert to solve for v, which turns out to be around $0.866c$.

There are a lot of different kinds of time delay that exist, coming from different physics equations. It's a good idea for us to try to keep track of them as much as possible.

Similarly, we can solve problems with length contraction:

Example 18

How quickly must a meter stick move for an observer at rest to perceive its length as half a meter?

Remark 19. *Last time, the first question asked was "what is the length of the meter stick?"...*

Let S be the reference frame of the meter stick, and let S' be the observer's frame. Then the length in the S' frame is

$$\Delta x' = \frac{\Delta x}{\gamma},$$

and $\gamma = 2$ again and we have the same speed of $0.866c$.

5 September 11, 2019

First of all, a point of clarification: if a train is moving relative to a station, and there is a clock on both the station and the train, it is indeed true that an observer on the train will see the clock on the station is moving slower, and the observer on the station will see that the clock on the train is moving slower. There's no contradiction here, notably because measured time depends on positions in different reference frames, but we should ask during office hours if we're still confused!

Last time, we discussed that Galilean relativity is incorrect: as a consequence of the speed of light always being some constant c, simultaneity is no longer a concept. In fact, events will take longer if they are not at rest (by a factor of γ), and measured lengths will seem shorter (also by a factor of γ).

Today, we'll follow up logically: we'll replace our Galilean transformations, and we'll ask whether S and S' actually have anything in common at all (if we don't have length or time conservation). Finally, we'll talk a little about the concept of causality.

Remember that our Galilean transformations are currently incorrect: if a frame S' is moving relative to a frame S with velocity v,

$$t' \neq t, \quad x' \neq x - vt.$$

(We always assume that we're moving along the x-axis.) Nothing happens along the y- or z-axis, but we still need to remove two of our four equations. Remember that if $v \ll c$, we should still recover the Galilean transformation in the limit of our Lorentz transformation.

Let's say we have an event E described in two frames

$$E \underset{S}{\rightarrow} (t, x, y, z), \quad E \underset{S'}{\rightarrow} (t', x', y', z').$$

We may think that t' can be some generic function $f(t, x, y, z)$, and so can $x' = g(t, x, y, z)$. But t' and x' shouldn't depend on y and z, because they're staying constant. In addition, t' and x' should be linear in x and t:

$$\boxed{t' = At + Bx, \quad x' = Dt + Fx.}$$

This is the correct form for our equations because of homogeneity: if there were an x^2 term, for example, lengths would depend on the exact coordinates rather than just differences. Our goal here is to find A, B, D, F.

Example 20

Let's say that event E_0 is the event where the origins of S and S' are overlapping.

This can be described via

$$E_0 \underset{S}{\rightarrow} (0, 0), E_0 \underset{S'}{\rightarrow} (0, 0)$$

because we've synchronized our clocks in the two frames. Unfortunately, plugging these into $t' = At+Bx, x' = Dt+Fx$ teaches us nothing beyond $0 = 0$: this was already built in, so it can't give us any more information.

So let's do something more instructive:

Example 21

Say that event E_0 occurs again (just for synchronization). Let's say that A is a person at rest in frame S and B is a person at rest in frame S'. Event E_1 is where something happens to B, and event E_2 is where something happens to A.

This time, we know that

$$E_1 \underset{S}{\rightarrow} (t, vt), E_1 \underset{S'}{\rightarrow} (t', 0)$$

(because B traveled vt meters in t seconds from A's point of view). Similarly, E_2 happens at S's origin, so

$$E_2 \underset{S}{\rightarrow} (T, 0), E_2 \underset{S'}{\rightarrow} (T', -vT').$$

So let's plug event E_1 into our boxed equations:

$$x' = Dt + Fx \implies 0 = Dt + Fvt \implies D = -Fv,$$

and thus we can write $x' = -Fvt + Fx = F(x - vt)$. Similarly, plugging event E_2 in gives

$$-vT' = -FvT + 0 \implies T' = FT,$$

and since $t' = At + Bx \implies T' = AT$, we now know that $A = F$: we have simplified our equations to the form

$$t' = At + Bx, \quad x' = A(x - vt).$$

Example 22

Now let's say that a laser beam is shot horizontally by B when the two frames overlap at $t = t' = 0$: this is event ε_0. At some later time, the laser beam hits a wall, and this is event ε_1.

We're now using the fact that the speed of light c is constant: this means that $x = ct$ in both reference frames! Plugging this in,

$$t' = At + Bx = At + Bct \implies t' = (A + Bc)t,$$

and

$$x' = A(x - vt) \implies ct' = A(x - vt) = At(c - v).$$

Plugging in $t' = (A + Bc)t$ from above,

$$c(At + Bct) = At(c - v) \implies Bc^2 = -Av \implies B = -\frac{v}{c^2}A.$$

We now only have one unknown variable! To find the value of A, we're going to need to use a different setup:

Example 23

Now, the laser beam is shot at an angle from the horizontal for event ε_0, where it eventually hits the ceiling for event ε_1.

Because distance is speed times time,

$$ct = \sqrt{x^2 + y^2 + z^2}, \quad ct' = \sqrt{x'^2 + y'^2 + z'^2}.$$

Squaring both sides, because $y' = y$ and $z' = z$.

$$y^2 + z^2 = c^2t^2 - x^2 = c^2t'^2 - x'^2 \implies \boxed{c^2(t^2 - t'^2) = x^2 - x'^2}.$$

Now we just substitute in what we know: substitute in $x' = A(x - vt)$ and $t' = A\left(t - \frac{v}{c^2}x\right)$ to find

$$c^2\left(t^2 - \left(A\left(t - \frac{v}{c^2}x\right)\right)^2\right) = x^2 - A^2(x - vt)^2.$$

It turns out that all of the ts and xs go away, and we find that

$$A^2 = \frac{1}{1 - \frac{v^2}{c^2}} = \gamma^2.$$

This means we're done:

Theorem 24 (Lorentz transformation)

The transformed coordinates for a reference frame S' moving at velocity v relative to S are

$$t' = \gamma\left(t - \frac{v}{c^2}x\right), \quad x' = \gamma(x - vt), \quad y' = y, \quad z' = z.$$

One way to make this easier is to work with the variable ct instead of t: because c is a universal constant, everything still works out! This then gives us the equations in the symmetrical form

$$ct' = \gamma\left(ct - \frac{v}{c}x\right), x' = \gamma\left(x - \frac{v}{c}t\right).$$

Remark 25. *Lorentz found these first by trying to explain Michelson-Morley, and he showed first that if coordinates were connected this way, Michelson-Morley should not yield a positive result.*

It's also straightforward to show that if $v \ll c$, this reduces to the Galilean transformations! (Notably, $\gamma \to 1$.)

Because these transformations are linear, we can represent them nicely as

$$\begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix}.$$

A few other remarks: there's nothing special about the velocity being parallel to the x-axis, or the axes being parallel. We can still write everything in terms of boosts, translations, and rotations, but the math is just more ugly!

By the way, the inverse transformation is not too difficult: just replace primes with non-primes, and replace v with v'. Because γ is still γ and β becomes $-\beta$, this yields

$$\begin{bmatrix} ct \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ct' \\ x' \\ y' \\ z' \end{bmatrix}.$$

It can be checked that these 4×4 matrices are inverses of each other, which is what we want! By the way, if we want to calculate a time interval, we just put Δs everywhere:

$$c\Delta t = \gamma(c\Delta t' + \beta\Delta x'), \quad \Delta x = \gamma(\Delta x' + \beta c\Delta t').$$

With this, notice that there's actually something we maintain universally between different reference frames:

Definition 26

The **spacetime invariant**, or **ST** for short, is defined to be

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

This turns out to be **conserved under Lorentz transformations!** So even if different frames disagree about an event's coordinates, they will agree on this specific quantity.

Note that Δs^2 can be negative or positive. How can we interpret this?

Example 27

Consider frame S, where an archer A is shooting an arrow at a target: this is event T_1. Event T_2 is where the arrow hits the target. Call the distance between the archer and the target Δx.

Obviously, these two events have a causal connection: event T_2 has been caused by event T_1. The coordinates

$$T_1 \underset{S}{\rightarrow} (0, 0), \quad T_2 \underset{S}{\rightarrow} (\Delta t, \Delta x)$$

yield a spacetime invariant of

$$\Delta s^2 = -c^2\Delta t^2 + \Delta x^2.$$

Since this can be rewritten as

$$-c^2\Delta t^2 \left(1 - \left(\frac{1}{c}\frac{\Delta x}{\Delta t}\right)^2\right) = -c^2\Delta t^2 \left(1 - \frac{u^2}{c^2}\right),$$

our constant is always negative because $u < c$. So event T_2 being caused by T_1 has a Δs^2 that is negative.

Since every observer agrees about the value of Δs^2, they will agree about the sign!

Definition 28

Events with $\Delta s^2 < 0$ are called **timelike**, events with $\Delta s^2 > 0$ are called **spacelike**, and events with $\Delta s^2 = 0$ are called **lightlike**.

There are a few important facts about timelike separated events:

- One is allowed to be the cause of the other.
- There exists a frame S' in which $\Delta x' = 0$.

In this instance, we take the frame of the arrow: we claim the relative velocity here is $v = \frac{\Delta x}{\Delta t}$. Indeed, our original coordinates satisfy

$$\Delta x' = 0 = \gamma(\Delta x - \beta c \Delta t) \implies \Delta x = \beta c \Delta t \implies v = \frac{\Delta x}{\Delta t}.$$

As an exercise, we can also calculate $\Delta t'$ and verify explicitly that the calculated value of $\Delta s'^2$ is the same as Δs^2.

However, is it possible to find a frame where event E_2 actually occurred before E_1? This would be bad, because causality would be violated. The answer is no: if $\Delta t < 0$, then $\Delta t' < 0$. If we want to try this on our own, assume that a frame contradicts this, so $\Delta t < 0$ and $\Delta t' > 0$. Then we'll find that the relative speed between frames must be larger than c, which can't happen (as we'll see in the next few lectures).

Also, just to review, $\Delta t' = \Delta \tau$ is the **proper time** for this event (because the arrow is at rest in frame S'), and $\Delta t = \gamma \Delta \tau$ by time dilation. (We can even verify this by plugging back directly into the Lorentz transformation: $\Delta x' = 0$.)

Remark 29. *It's important to stress that the simple relations like*

$$\Delta t = \gamma \Delta t'$$

only occur when the events happen at the same position! Otherwise, we need to use more generalized transformations.

So what about the situation where $\Delta s^2 > 0$?

Example 30

Let's say that event S_1 is the event "8.033 starts at 2:30pm," and event S_2 is that "A star blows up in Andromeda 2.2×10^{22} meters away 1 million years after today."

We can calculate

$$\Delta s^2 \approx 4 \times 10^{44} m^2 > 0,$$

and for events like this, it turns out we can find a frame where S_2 happens **before** S_1! So if Δt is negative, we can find a frame where $\Delta t'$ is positive.

Why is this not a problem? The minimum time it takes for anything to propagate from here to Andromeda is $\frac{\Delta x}{c} = 3.2 \times 10^6$ years, which is larger than 1×10^6 years. So if something happened 1 million years from now, it could not have been influenced by anything that happened here! So the two events must be related. In addition, there's no frame S' where $\Delta x' = 0$: again, we can see that we would need the speed of our frame to be larger than c.

Finally, what if $\Delta s^2 = 0$?

Example 31

Consider the events $L_1 = (0, 0)$ and $L_2 = (\Delta t, \Delta x)$, where $\Delta s^2 = 0 = -c^2 \Delta t^2 + \Delta x^2$. Then $\Delta x = c \Delta t$.

So the events can be caused if the information is sent exactly at the speed of light. This is an edge case, and we can confirm that the Lorentz transformation works out here.

So we've shown that there's something we can all agree upon, regardless of frame, which has significance about causality. There will be a lot of paradoxes in relativity that can come up, which we can resolve by having a good understanding of what's going on here!

6 September 12, 2019 (Recitation)

First of all, how do we do matrix multiplication with index notation? Having indices on the bottom or the top is actually important: it tells us how the components transform. Objects v^μ and v_μ are actually different! So if we have two vectors v^μ and u^ν, we can't actually multiply these together: we have to talk about a lower index somehow.

How do we go from a higher index to a lower index? We'll talk about this next lecture, but for now, we should just know that all matrices should have one upper index and one lower index. Also, does it make sense to multiply

$$V^{\mu'}U_\mu?$$

The answer is no, because we're talking about components in different reference frames: we should only try things like

$$V^{\mu'}U_{\mu'} = \sum_{\mu'} V^{\mu'}U_{\mu'}.$$

Let's do some practice problems!

Example 32

Consider the invariant intervals $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. (Convention in high-energy physics is to set $c = 1$ to simplify equations, but we should remember to remind instructors if they forget them.) This is invariant under Lorentz transformations: write ds^2 in spherical coordinates.

It's tempting to just say that because

$$dx^2 + dy^2 + dz^2 = d(x^2 + y^2 + z^2) = dr^2,$$

this just reduces to $dr^2 - c^2(dt)^2$. But we should make sure we know what differentials mean: this says that we're actually taking the derivative with respect to some parameter λ! So we should be writing out things like

$$y = r\sin\theta \implies \frac{dy}{d\lambda} = \frac{dr}{d\lambda}\sin\theta + r\cos\theta\frac{d\theta}{d\lambda} \implies dy = \sin\theta dr + r\cos\theta d\theta,$$

and so on. So then we square and add these up! It's therefore important to mention that x and dx are different: we should always think that we're taking derivatives. The actual expression looks like

$$ds^2 = -(cdt)^2 + (dr)^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.$$

Notably, the form of ds^2 changes, but the actual value does not! Intuitively, we should think of ds^2 as the length of a curve (sort of like a worldline).

Example 33
Let (x, t) be a coordinate system of an inertial reference frame. Alice "sees" a flash of red light at $x = x_0, t = t_0$, and she sees a flash of green light at $x = x_1, t = t_1$. Meanwhile, Bob records both flashes at the same position. How fast is Alice moving relative to Bob, and which color occurs first according to Bob? What is Δs^2 in Bob's reference frame?

Let $\Delta x = x_1 - x_0$ and $\Delta t = t_1 - t_0$. Because Bob sees the two events at the same position in his rest frame, Alice must be moving at a speed

$$v = \frac{\Delta x}{\Delta t}.$$

In particular, notice that events can only be observed at the position that they occur, so Bob's reference frame is actually the inertial reference frame!

The idea for the second question is that if $t_0 < t_1$, and because the speed of the frames relative to each other is less than c, there is potentially causal dependence, so $t_0' < t_1'$ for Bob as well.

Example 34 (Relativistic torpedo)
Two submarines A and B are traveling towards each other, and the relative speed between them is v (measured from the point of view of one of the submarines). Each submarine has proper length L_0. A fires a torpedo from its tail up towards B when B's tail passes A's head: what will happen?

Basically, B sees A as being much shorter than it, so B thinks it will be hit. But A sees B as being much shorter, so A doesn't think B will be hit. What's going on?

Define some events:

1. The heads of the submarines line up (remember that the front points right for A and left for B).
2. The head of A lines up with the tail of B.
3. The torpedo is fired from A's reference frame.
4. The head of B aligns with the tail of A.
5. The two tails align.

Let's put coordinates on all of these events! In A's frame, define $x_A = 0$ to be its head: then event 1 occurs at $(x_A, t_A) = (0, 0)$. For event 2, B's length is $\frac{L_0}{\gamma}$ and the relative velocity is v, so $(x_A, t_A) = (0, \frac{L}{\gamma v})$. For event 3, in A's reference frame, the torpedo is fired at $(-L_0, \frac{L_0}{\gamma v})$ (same time). Here's what's important: event 4 occurs at $(x_A, t_A) = (-L_0, \frac{L_0}{v})$, because A is not contracted in our example. Notice that this means $\frac{L_0}{v} - \frac{L_0}{\gamma v} > 0$: event 4 does occur after event 3.

So now doing a Lorentz transformation, we can check that this also holds for event B, because $v < c$!

7 September 16, 2019

Let's start by summarizing where we are so far in this class: last time, we found the Lorentz transformations for motion along the x-direction:

$$\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \\ y' = y \\ z' = z \end{cases}$$

where we're defining

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad \beta = \frac{v}{c}.$$

Remember that we can rewrite these using index notation: if

$$x^{\mu'} = (ct', x', y', z'), \quad x^\mu = (ct, x, y, z),$$

then the two sets of coordinates can be linked via

$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu.$$

This uses the **repeated index convention**: if we see the same index on the same side of an equation, we are summing over it. Notably, we know from above that

$$\Lambda^{\mu'}{}_\nu = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This tell us how to convert from S to S', and for now we should just think of the index notation by saying that the bottom index is our starting frame and the top index is the one we're going to. If we want to convert from $S' \to S$, we don't need to derive anything complicated: we just have

$$x^\mu = \Lambda^\mu{}_{\nu'} x^{\nu'}, \quad \Lambda^\mu{}_{\nu'} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We also noted that if we have two events

$$A \underset{S}{\to} (ct, x, y, z), \quad B \underset{S}{\to} (c(t + dt), x + dx, y + dy, z + dz),$$

and we have similar descriptions of the events A and B in the S' frame, we can define a spacetime invariant

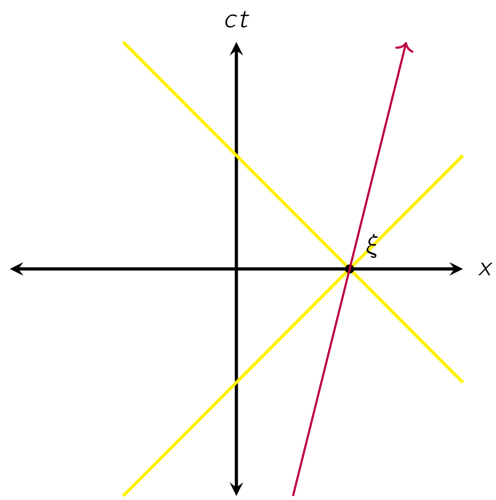
$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2,$$

which is the same for both reference frames! As a general note, this teaches us that it's better to use invariants rather than coordinates when possible.

The sign of ds^2 has significance, too: events with $ds^2 = 0$ are light-like, events with $ds^2 > 0$ are space-like (so they can't be linked, because they're not causally connected), and events with $ds^2 < 0$ are time-like (so there is potential for causality). In this last case, if $dt > 0, dt' > 0$: this means that no matter which frame we're in, everyone agrees about their order.

Today, we're going to start figuring out how to visualize relativity! Since objects move in time, too, we want to include this on our coordinate axes as well. To do this, because we usually focus on events that move along the x-axis, we'll show x and ct on the two primary axes. (Using ct instead of t ensures that units are homogeneous.) These are called **space-time diagrams**, or **ST** for short.

Remember that events are just things that happen in space and time, so events are going to be some point in our diagram. Here's an example of a particle moving towards the positive x-direction with constant velocity:

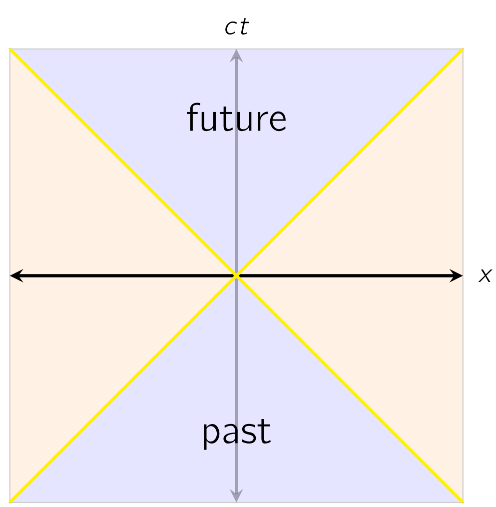


In particular, a particle that does not move will correspond to a vertical line, because $\Delta x = 0$. The nice thing about these particles is that $\frac{\Delta x}{\Delta t}$ corresponds to a slope: if a particle is moving at an angle ξ from the horizontal,

$$\tan \xi = \frac{c\Delta x}{\Delta x} = \frac{c}{u} = \frac{1}{\beta}.$$

So the trajectory of an object moving at a constant velocity will be a straight line! Notably, a particle can only move as fast as the speed of light, so $\beta = 1 \implies \xi = 45^\circ$. So **light moves at 45 degrees in the spacetime diagram if it's moving in the x-direction.**

This is nice, because it allows us to describe causal relationships! Let's say that we have an event at the origin. It's always a good idea to draw light beams through that point, because it breaks our region into the following areas:



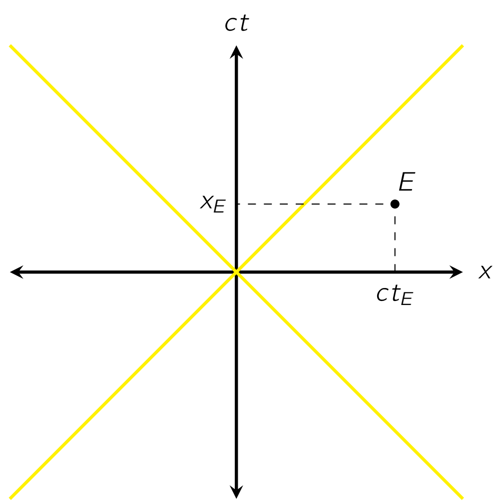
- All events on the light lines (yellow) have $ds^2 = 0$.
- All events in the top and bottom regions (blue) have $ds^2 < 0$.
- All events in the left and right regions (orange) have $ds^2 > 0$.

This is because $ds^2 = -c^2dt^2 + dx^2$ (in the case where we only care about the x-direction), and $|cdt| > |dx|$ for the top and bottom regions (and vice versa on the left and right). These regions generalize to points that aren't just the origin, either: just draw light lines starting from those points instead.

Physicists have a "poetic" nature, so these have more specific names. The top region is called the **future** (because they happen later), the bottom region is called the **past**, and the left and right regions are called **elsewhere** (because... they are elsewhere?). **This partition is nice because all observers divide up spacetime into the same partitions!**

Remark 35. *Note that the light lines are not actually lines, because we have other space dimensions as well: if we reinstated other space dimension, the light lines become cones.*

So now let's start thinking about how to translate between reference frames! Suppose that we have some event in one reference frame: usually, we just draw it and find the value of x and ct by seeing its coordinates along each axis:

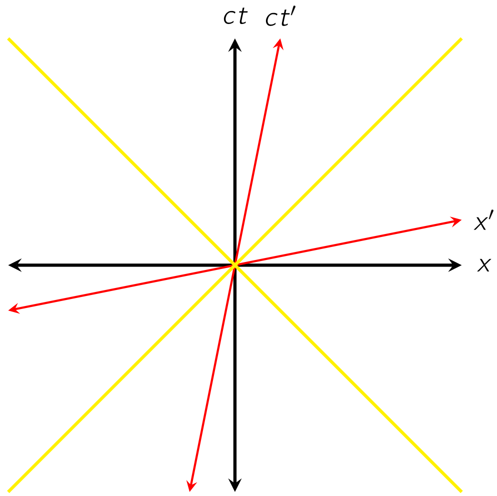


But in a different reference frame, if we want to find the value of ct' and x', we need to know how to draw our axes, and we also need to know how to draw tick marks! Let's see how we can formally figure out where the x' and ct' axis look on our diagram.

By definition, the ct' axis is the set of points where $x' = 0$, and the x' axis is the set of points where $ct' = 0$. Remember that we have our Lorentz transformations:

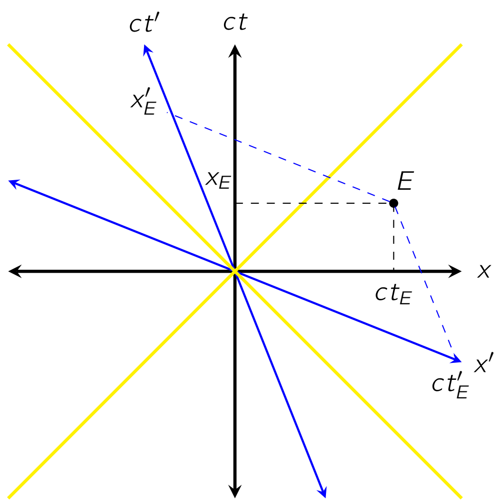
$$x' = \gamma(x - \beta ct), \quad ct' = \gamma(ct - \beta x).$$

So now we have equations to work with! For the ct' axis, we must have $x' = 0 \implies \boxed{x = \beta ct}$. Similarly, for the x' axis, we must have $ct = \beta x$. So the new axes are **straight lines with slope** β, $\frac{1}{\beta}$.



Notice that the angle between the ct and ct' axis is the same as the angle between the x and the x' axis here. We can also sanity check that if $\beta \to 0$ (so our new reference frame gets slower and slower), our ct'-axis approaches the ct-axis, and the x'-axis approaches the x-axis. Meanwhile, as S' gets faster, the red lines move closer and closer to the light lines!

One final note about this situation: if S' is moving to the left instead of to the right, the axes will move in the opposite direction, as we can see below.

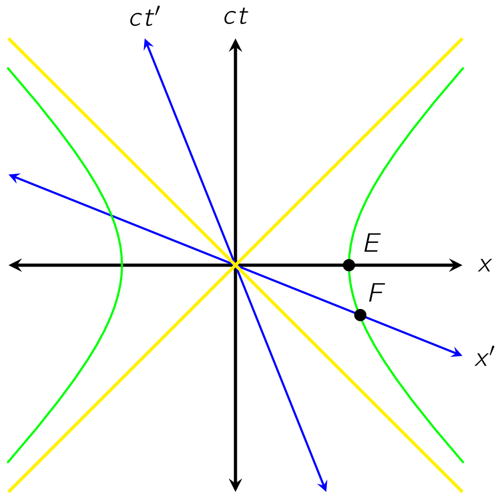


So how do we assign coordinates to an event E? We do the same projection that we did in the normal case, but we do it with a parallelogram instead of a rectangle! Notice that this immediately shows us that S and S' cannot always agree on simultaneity: two events that have the same t in reference frame S are along the same horizontal line, but those are not going to have the same value of t' in our frame S', and vice versa. This is also true for colocation: two events that have the same x in reference frame S are along the same vertical line, but that vertical line doesn't correspond to a line of constant x' in our frame S'.

Our next question: how do we draw tick marks? We need a way to think about equal displacement from the origin in different frames. Luckily, we have our spacetime invariant ds^2 to do this! Consider a situation where $ds^2 = 1$. Then

$$1 = -c^2 dt^2 + dx^2 = -c^2 dt'^2 + dx'^2,$$

so if an event occurs at $E \underset{S}{\to} (0, 1)$ (which clearly satisfies our equation), it must lie on specific hyperbolas:



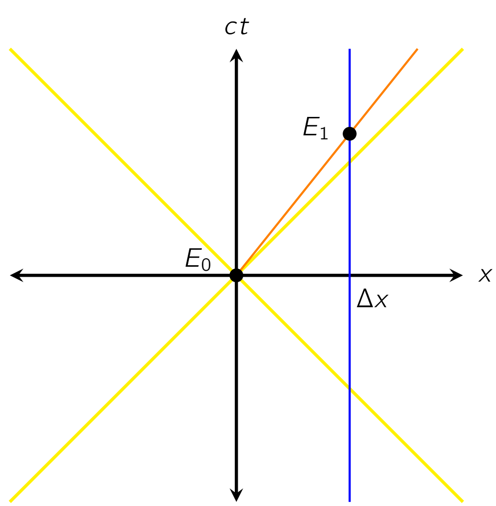
But don't forget that an event $F \underset{S'}{\rightarrow} (0,1)$ must also lie on these hyperbolas, because ds^2 is an invariant! This means that the **intersection of our x' axis** with the hyperbola is the point $(0,1)$ in our new frame. We can generalize this, too: just find the hyperbola with $ds^2 = 4$ to find $(t,x) = (0,2)$, and intersect this with the x' axis to find $(0,2)$ in the S' frame. Similarly, the hyperbola with $ds^2 = -1$ gives us $(1,0)$, and now we're done! We've now found a way to associate steps in one frame with steps in the other frame.

Remark 36. *This is similar to Euclidean geometry, where we see that the set of vectors $\vec{r}$ such that $||\vec{r}|| = x^2 + y^2$ is constant traces out a circle. So if we rotate our reference frame, we find our new basis vectors by intersecting them with our unit circle. The only difference is that the Minkowskian geometry we work with just has slightly different characteristics!*

Example 37

Say a clock is moving at a constant velocity $u\hat{e}_x$ at $t = 0$ from the origin, described via $E_0 \underset{S}{\rightarrow} (0,0)$. Some time later, the clock is still moving, and at some final point, the clock hits a wall and stops, so $E_1 \underset{S}{\rightarrow} \left(\frac{\Delta x}{u}, \Delta x\right)$.

Let's try drawing the wall and the clock in our spacetime diagram:



So how long does it take for this to happen (aka what does the clock read when it hits the wall), given that the S-frame measures a total time of Δt? In the rest frame of the clock S', the clock stays stationary, and the wall moves towards it at $-u\hat{e}_x$. One way to calculate this is that the proper time is

$$\Delta\tau = \Delta t' = t'_{E_1} - t'_{E_0},$$

and in any other frame, $\Delta t = \gamma(v)\Delta\tau$. But a more general way to approach this kind of problem is to use the invariant

$$ds^2 = -c^2dt^2 + dx^2 = -c^2dt'^2 + dx'^2.$$

This can be simplified to

$$-c^2dt^2 + \Delta x^2 = -c^2d\tau^2,$$

and we can take an integral

$$\Delta\tau = c\int_{E_0}^{E_1} d\tau = c\int dt\left(1 - \frac{dx^2}{c^2dt^2}\right)^{1/2}.$$

Since the velocity of the watch is constant, we can pull it out of the integral: since $\frac{dx}{dt} = v$,

$$\Delta\tau = c\left(1 - \frac{v^2}{c^2}\right)\left(\int_{E_0}^{E_1} d\tau\right),$$

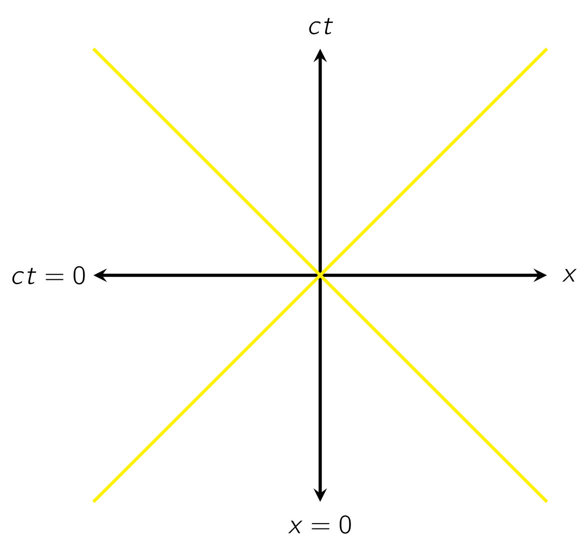
and now we're done:

$$c\Delta\tau = c\left(1 - \frac{v^2}{c^2}\right)d\tau.$$

This will be important when we aren't just traveling through straight lines in our spacetime diagram!

8 September 17, 2019 (Recitation)

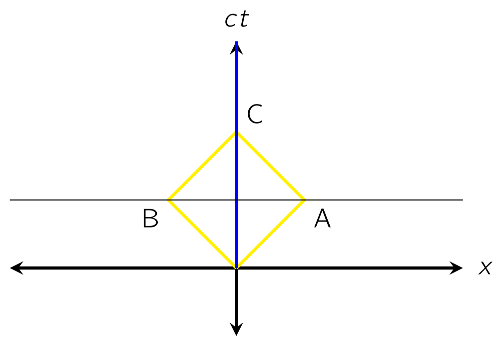
We'll be doing a lot of work with spacetime diagrams today. Remember that our basic image looks like this: light is always at a 45-degree angle to the horizontal, because it travels exactly 1 light-second in one second.



Example 38

An observer moves in an inertial reference frame, and they emit rays of light to the left and right at $t = 0$. The light is then reflected off of mirrors at some x-position, and then the light comes back to the observer (in an exactly synchronized way). How do we describe this in a spacetime diagram?

In the observer's point of view, they are stationary, so they stay at $x = 0$ for the whole time. Let's label events A, B, and C:



The line connecting A and B is called a **line of simultaneity**: events A and B happen at the same time in this frame.

Example 39

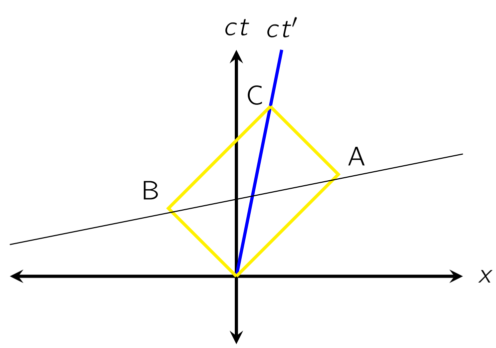
But now suppose that we are in a different inertial reference frame: now we see the observer moving away from us at a speed v. What happens to our line of simultaneity?

Notice that

$$\frac{c}{v} = \frac{\text{rise}}{\text{run}} = \text{slope of the line}$$

is constant if our frame is moving at a constant velocity v, so we will see the observer moving along some line in our spacetime diagram. (Note that the slope of that line must always be at more than 45 degrees from the horizontal, because $v < c$.)

So what will happen to events A, B, and C? Light **always moves at a 45 degree angle** to the horizontal in all reference frames, so our picture will look something like this:



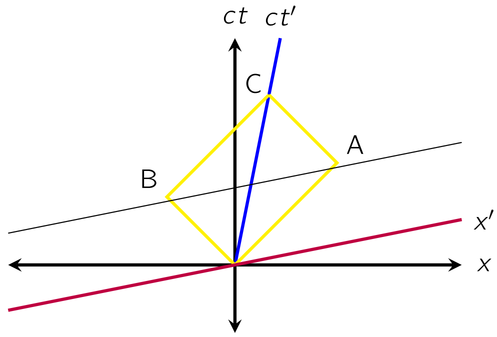
So simultaneity is indeed relative: the line connecting A and B is no longer a line of simultaneity in our reference frame.

By the way, note that if we have a third observer who is shifted relative to the first observer but is **not moving**, they will perceive events A and B as simultaneous. To verify this in a spacetime diagram, we can just draw light cones starting from the midpoint of A and B in space, so that those cones hit A and B at the same time.

Problem 40

In our frame, how would we draw the first observer's line of simultaneity through the origin?

Parallel lines need to stay parallel (because we have a linear transformation here), so the line will stay parallel to line AB in our new frame. This line of simultaneity corresponds to a line where $t' = 0$, so it's also a transformed axis in the spacetime diagram:



Here, A and B are **space-like separated**: A and B do not live in each other's light cone, so it is possible to change frames so that either happens before the other.

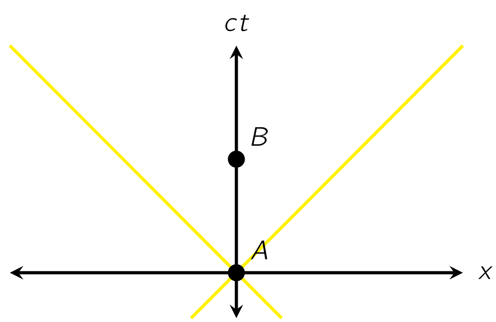
On the other hand, what about two events that are **time-like separated?** If we have two events $\tilde{A}$ and $\tilde{B}$ such that one lies in the lightcone of the other, this will be true regardless of which reference frame we're using. (We may say that $\tilde{A}$ is **to the past** of $\tilde{B}$ if $\tilde{B}$ is in the light cone of $\tilde{A}$.)

Now that we have our spacetime diagrams, let's revisit a problem from earlier:

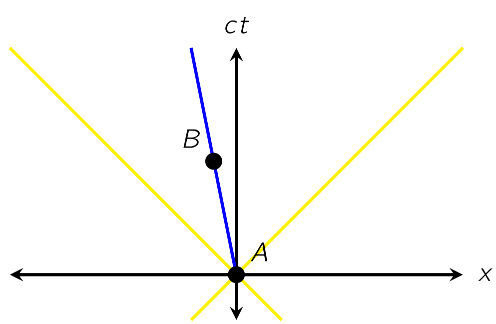
Problem 41

Consider reference frame 1, where we have a flash of red light at $(x_1, ct_1) = (0, 0)$ (event A) and a flash of green light at $(x_1, ct_1) = (X, cT)$. We also have another reference frame 2, where event A occurs at $(x_2, ct_2) = (0, 0)$ and event B occurs at $(0, c\tilde{T})$. What is the relative velocity between the two frames?

Solution. In reference frame 2, event B is directly above event A:

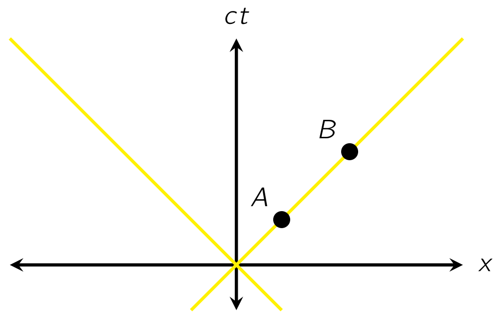


Everything's pretty simple so far. Meanwhile, in reference frame 1, our axes are shifted by a bit:



Now event B occurs at (X, cT), so we can immediately calculate the relative velocity: $\frac{c}{v} = \frac{cT}{X}$, as we expect. $\square$

Finally, let's review the idea of light-like separation: two events A and B are **null-separated** if A lies on the boundary of B or vice versa.



It's a bit weird to also say that B is in the future of A for this case, so we often say that B is in the **causal future** of A (a term borrowed from general relativity). So here's the central idea of today's recitation: the light cones are what we care about, because they give us relations between different events!

9 September 18, 2019

First, an announcement: it's up to instructors to decide deadlines for the problem set, and the 8.033 team has decided that problem set 2 will be due on Monday instead of Friday because of the Career Fair. Of course, the downside is that grades will take longer to come in.

We've been showing that space and time intervals between events are not agreed upon by different frames, but there is a quantity $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$ which is invariant under Lorentz transformations.

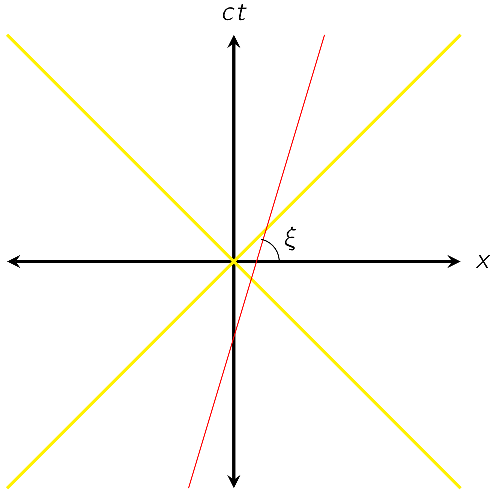
Remark 42. *This spacetime element may be written differently in different books: for example, some use $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$. The central idea is the same, though, and it's just down to convention.*

Just like we can think of the norm of a vector $||d\vec{x}||^2$, we'll come back to the notion of a "norm" for our four-vectors later. We like to work with invariants like this, but occasionally we do need to convert coordinates: we've found that the way to do this is through the Lorentz transformation

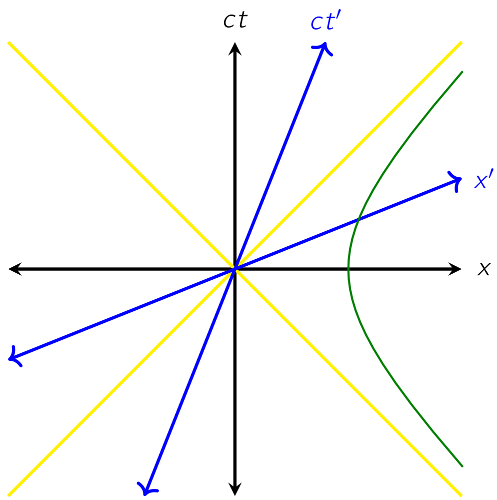
$$dx^{\mu'} = \Lambda^{\mu'}{}_{\nu} dx^{\nu},$$

where we're summing over ν on the right side, and where Λ is the Lorentz transformation matrix.

Last time, we introduced the **space-time diagram**: this is a pictoral way to represent what happens in space and time for individual events (points in our diagram), or trajectories of objects (curves or lines). Then light always travels at a 45-degree angle, and other objects traveling at a speed v travel through lines at an angle ξ such that $\tan \xi = \frac{c}{v}$:



For any event E, we can draw light cones originating from that event in our diagram: then the region above E is the **future**, and the region below E is the **past**. Light lines are important because they stay constant even when we change frames:



If we want to try the tick marks, we consider the green hyperbola: points along one axis correspond to points along the other axis!

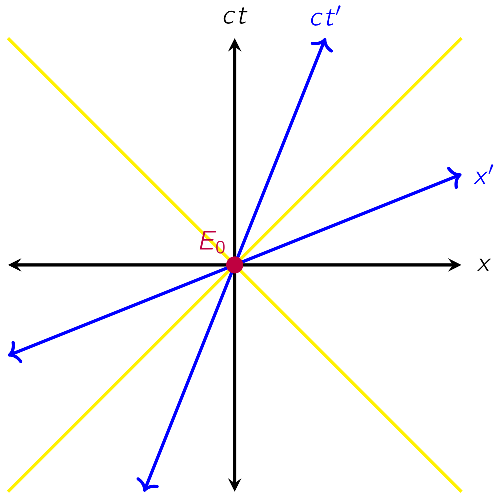
Today, we'll first see some more practical uses of spacetime diagrams, primarily to visualize time dilation and length contraction. Then, we'll talk about a few "paradoxes" in special relativity, and we may spend some time on figuring out how to work with vectors.

Let's start with **time dilation.** Recall that the **proper time** of an event $\Delta\tau$ is the time interval as measured in a frame where they happen in the same position: in any other frame, we know that $\Delta t = \gamma\Delta\tau$, where $\gamma = \gamma(v)$ depends on the relative velocity of the two frames we're considering. Let's show this pictorally!

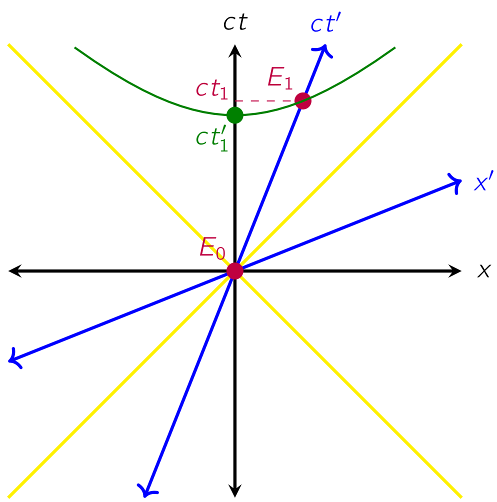
Example 43

Consider a clock moving in the S' frame. At $t' = t = 0$, the clock is at the origin of both frames, and S' is moving at a velocity $\vec{v} = v\hat{e}_x$. Some time later, the clock and its frame will have moved by some amount.

Let the two events be E_0 and E_1. We can write the coordinates fairly easily: $E_0 \underset{S'}{\rightarrow} (0, 0)$, $E_0 \underset{S}{\rightarrow} (0, 0)$, $E_1 \underset{S'}{\rightarrow} (\Delta\tau, 0)$, $E_1 \underset{S}{\rightarrow} (\Delta t, v\Delta t)$. Here's the diagram with both frames' axes, as well as the position of our first event:



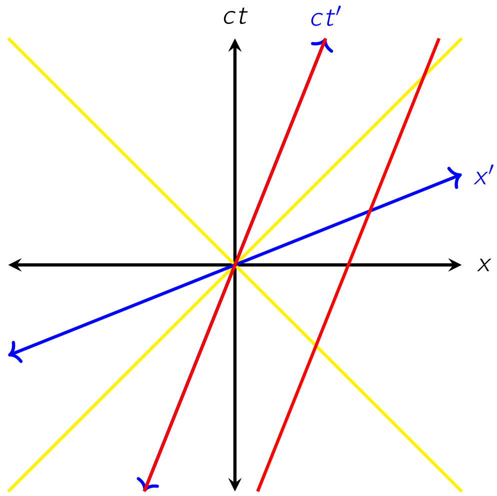
We know that the second event E_1 must occur at $x' = 0$, so it's along the ct'-axis. But we can draw a hyperbola through that event:



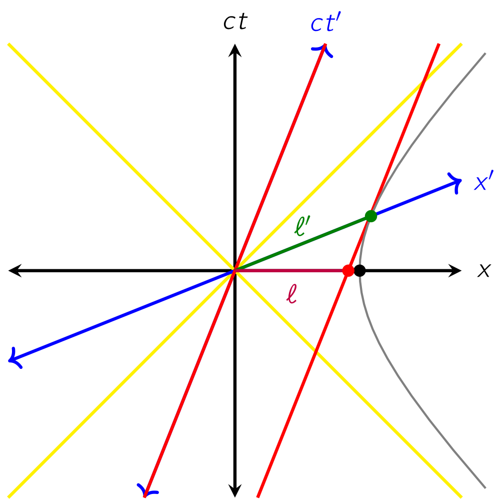
This proves that $ct' < ct$, meaning the proper time is indeed smaller than any measured time in any other reference frame!

Similarly, we can do something like this for **length contraction.** Say that we have an object of length ℓ' in its rest frame: if we have another frame moving at a velocity v, then the measured length from that frame is $\ell = \frac{\ell'}{\gamma} < \ell'$.

Again, we'll draw a spacetime diagram for this, but the picture will look slightly different! The key of measuring length is that we need to measure **both edges** of the object at the same time, but since different frames represent time differently, they won't measure with the same events. Let's call the left and right sides of the ruler A and B. In our current frame, A and B just move at constant speed, parallel in our spacetime diagram:



So if we're trying to measure in a given frame, we need to measure **along a line of simultaneity**. But notice that lines of simulaneity look different in our different frames:



Again, notice that $\ell < \ell'$ because of where our hyperbola lands on the x-axis, and thus length contraction behaves as we expect

It's now time for us to move on: let's start to think about some **paradoxes** in special relativity! Many people tried to think of situations where relativity yielded paradoxical results, but unconsciously the problem was that people applied intuition from Newtonian mechanics.

Problem 44 (Professor V and the guillotines)

Consider an object of length d, moving relative to a frame S with a relativistic velocity v. Let the S' frame be the frame where that object is not moving.

There are two blades at rest in frame S, at a distance d apart from each other. As soon as the object enters the spot between the guillotines, the blades come down.

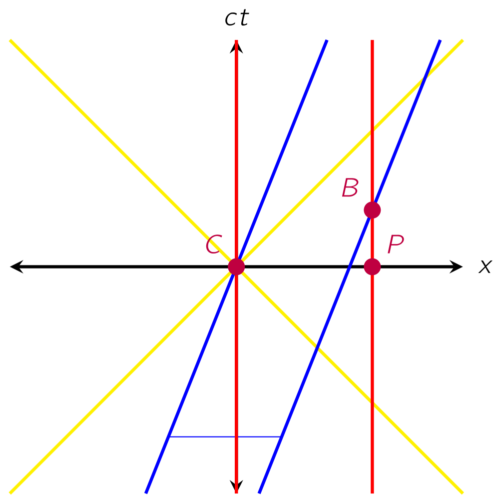
From the S' point of view, the separation between the guillotine blades becomes $\frac{d}{\gamma}$. Since this distance is less than d, the object will definitely get cut. But from the S point of view, it seems like the object gets shrunk to a distance of $\frac{d}{\gamma}$, so it seems like the object won't be cut. Which is true?

Since Professor Vitale is still here to give the lecture, it's pretty clear which happened!

Solution. We can introduce a bunch of events, Lorentz transform, and see what happens. But we'll do the spacetime diagram way, because it's more instructive and harder for us to do at home on our own!

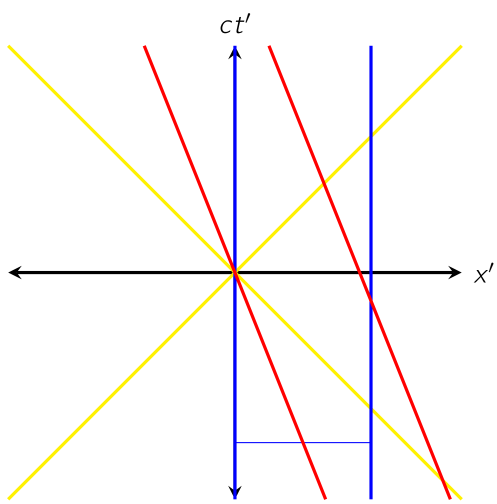
Let event A be the one where the front end of the object gets to the x-coordinate of the first blade ($x = 0$), event B when the front end of the object gets to the x-coordinate of the second blade ($x = d$), event C when the back end of the object gets to the x-coordinate of the first blade ($x = 0$), **which is also when the first blade goes down**, and event D when the back end of the object gets to the x-coordinate of the second blade ($x = d$). We'll also consider event P (for pain), where the second guillotine is released. (From the S-frame, this event happens at the same time as event C).

We can assign t and x-coordinates to all of these events and carry out the Lorentz transformations. But instead, let's do this with a diagram. Here's what happens in the S-frame (or blade frame):

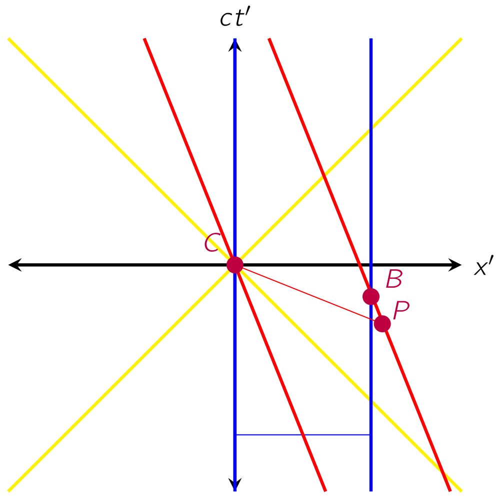


In this frame, the object is completely fine: the total length of the object is less than the length between the guillotines, and event B happens after event P.

But something different happens in the S'-axis. Now, the object is not moving, but the blades are! This time, notice that the distance between the blades as measured in S' is less than d:



At first glance, it now seems like the object will get cut, but adding in the events B, C, P tells the full story. We have transformed axes now! Since events C and P happen at the same time in the S-frame, **they won't happen at the same time** in the S'-frame:



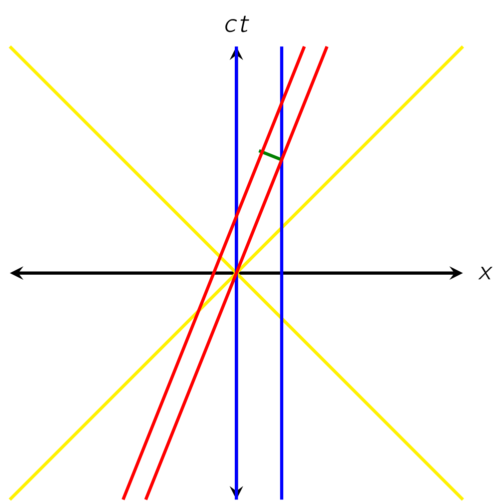
So the second guillotine goes down **before** the first one, and thus we are safe! □

Next, let's talk about the concept of **rigidity**. A rigid body in mechanics is one where the distance between any two points is constant, and we claim that this can no longer be the case!

Problem 45

Consider a rod of length d moving into a barn of length d. When the right edge of the rod hits the wall of the barn, the door closes. From the point of view of the barn, the rod has length less than d, so the door can be closed. But from the point of view of the rod, the barn has shrunk, so the door can't be closed without closing the rod! Who is right?

The problem here is that we've **assumed the rod stops instantaneously**. In reality, when the right side of the rod hits the barn, the left side doesn't know about this yet and will continue moving! Here's a diagram: blue represents the building, and red represents the rod.



Basically, the rod stops at the **green line**, rather than all at once (and in particular, the length of the rod does not look constant in all frames).

We'll finish today by talking about the geometry of vectors. Our goal is to appreciate that the "length" (in this case, ds^2) of a vector in space time isn't just the sum of the squares of the components! Let's start by talking about important points in Euclidean geometry.

Every vector in Euclidean space can be written as

$$d\vec{x} = dx^1 \hat{e}_x + dx^2 \hat{e}_y + dx^3 \hat{e}_z = dx^i \hat{e}_i.$$

If we change coordinates, the values of dx^i and $\hat{e}_i$ may change, but the value of

$$||d\hat{x}||^2 = 1(dx^1)^2 + 1(dx^2)^2 + 1(dx^3)^2$$

does not change! In addition, this norm can be written in a slightly different way: letting 1 denote the identity matrix,

$$= d\vec{x}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} d\vec{x} = d\vec{x}^T \cdot 1 \cdot d\vec{x}.$$

This is closely related to the normal dot product

$$\vec{v} \cdot \vec{w} = v^1 w^1 + v^2 w^2 + v^3 w^3 = \vec{v}^T 1 \vec{w}.$$

That 1 in the middle corresponds to the **metric** in our space: in Euclidean space, we have $\eta = 1$. But in general, we'll be working with a metric η which we can think of as a map

$$\eta : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R},$$

taking two vectors to their "dot product:"

$$\vec{A}, \vec{B} \to \eta(\vec{A}, \vec{B}) = \eta(\vec{B}, \vec{A}) = \vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}.$$

In general, this "dot product" is going to become important once we start working with our time coordinate! The following statements are thus all equivalent in Euclidean geometry:

$$\vec{A} \cdot \vec{B} = \vec{A}^T 1 \vec{B} = \vec{A} \eta \vec{B} = A^i \eta_{ij} B^j = A^i \delta_{ij} B^j = A^1 B^1 + A^2 B^2 + A^3 C^3.$$

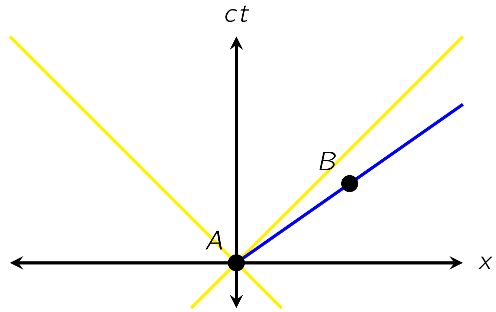
Next time, we'll start seeing how things go from the simple Euclidean geometry to spacetime geometry!

10 September 19, 2019 (Recitation)

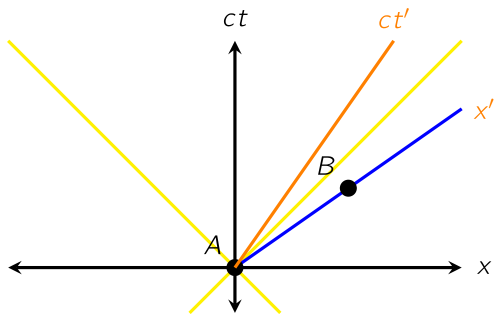
Recall that between any two events, we can compute the spacetime invariant ds^2 and organize information as follows:

- Spacelike, $ds^2 > 0$, events are not in each others' future lightcones,
- Timelike, $ds^2 < 0$, one event is in another's future light cone,
- Null or lightlike, $ds^2 = 0$, one event lies on the boundary of the other's future lightcone.

Let's talk a bit about faster-than-light travel and see what paradoxes come up! Say an athlete can travel at a speed $v > c$. Then in our reference frame, that athlete's **worldline** has slope less than 1.



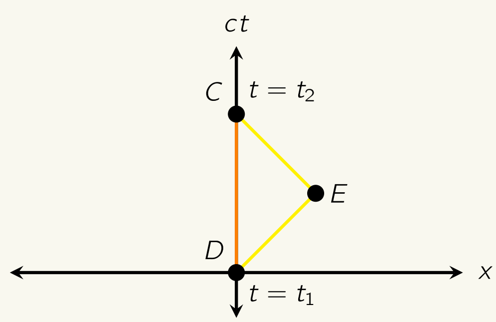
Notice that events A and B on this worldline are spacelike-separated. That is already a bit strange, but now it gets worse: suppose we have another friend who is not as fast, in such a way that their worldline is the reflection of the athlete's over the light line: now, in the slow friend's point of view, A and B can actually happen simultaneously at the same time (because the blue line **is** the red line's line of simultaneity)! In fact, if the slow friend travels a bit faster, they will actually see B happen before A, which is absurd: it allows us to do backwards time travel, which has many paradoxes of its own.



So the resolution is that indeed, **nothing can travel faster than the speed of light!**

Problem 46

We've been saying that ds^2 is an invariant interval, so let's try something. Consider two events C and D, timelike-separated.



Notice that the spacetime interval between C and D satisfies $ds^2 < 0$, but the spacetime interval between C and E and between D and E are both 0. So ds^2 seems like it should be $0 + 0 = 0$: what's going on here?

Solution. The idea is that the geometry changes between reference frames. If we tried to measure ds^2 in the reference frame from D to E to C, **this is not inertial**, because we have acceleration at point E. This solves the twin paradox, too: the idea is that ds^2 is not actually invariant in noninertial reference frames. ☐

Problem 47

Let's do some index notation practice! As always, $\mathbf{v}$ denotes a four-vector.

In this problem, ∇ denotes $\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z - c\hat{t}\partial_t$.

- $(\mathbf{v} \cdot \nabla)\mathbf{v}$:
$$\mathbf{v}^\nu \partial_\nu \mathbf{v}^\mu \hat{e}_\mu, \text{ or more sloppily, } \mathbf{v}^\nu \partial_\nu \mathbf{v}^\mu.$$
- $\text{Tr}[M^T M]$, where M is a matrix. If M has indices $M_\mu{}^\nu$, then M^T has indices $M^\sigma{}_\rho$. Since we're doing matrix multiplication here, we have
$$\text{Tr}[M_\mu{}^\nu M_\nu{}^\sigma] = M_\mu{}^\nu M_\nu{}^\mu,$$
since we have to sum through the ones where $\mu = \sigma$.
- $\nabla \cdot \nabla = \nabla^\mu \nabla_\mu$.
- $\nabla \cdot \nabla f$, where f is a function: this just becomes
$$\nabla^\mu \nabla_\mu f.$$
- $(\nabla \cdot \nabla)\mathbf{v}$ then just becomes
$$\nabla^\mu \nabla_\mu v^\nu.$$

If this is something we're confused about, we should review it because it'll become important later!

11 September 23, 2019

We got psets back in class today if we didn't pick them up during recitation: we should email TAs if we think there is a grading error.

Today, we're going to finish the discussion about geometrical properties of Euclidean spacetime. We'll review some aspects of Euclidean geometry, particularly in Cartesian coordinates, and then we'll figure out how to introduce time into this geometrical picture.

Recall that any vector $\vec{x}$ in three-dimensional space can be written as
$$d\vec{x} = dx^1 \hat{e}_1 + dx^2 \hat{e}_2 + dx^3 \hat{e}_3$$
(decomposing along the basis vectors). In index notation, this can be written as $dx^i \hat{e}_i$ (remember that Latin indices range over only $1, 2, 3$). When we change coordinates, the components of the vector, as well as the basis vectors of that vector, will change. But the vector itself isn't changing, just its description!

There is a quantity that doesn't change under rotations (as long as we stick with an orthonormal basis):
$$||d\vec{x}||^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2.$$
This is a special case of the dot product $d\vec{x} \cdot d\vec{x} = ||d\vec{x}||^2$. That dot product can be rewritten as the **bilinear form** (terminology doesn't really matter here)
$$d\vec{x}^T \mathbf{1} d\vec{y} = \begin{bmatrix} dx^1 & dx^2 & dx^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dy^1 \\ dy^2 \\ dy^3 \end{bmatrix}.$$

This middle identity matrix is really not that simple in general, though. In fact, this matrix is where our general **metric**, denoted η or g, will come into the picture soon. We can think of our metric as a function

$$\eta(\vec{A}, \vec{B}) = \vec{A} \cdot \vec{B} \in \mathbb{R}$$

taking two vectors to their dot product. In index notation, this can be written as

$$\vec{A} \cdot \vec{B} = A^i B^j \eta_{ij} :$$

in the case where η is the identity matrix, this reduces to

$$A^i B^j \delta_{ij} = A^i B^i.$$

Remark 48. *The order of expressions in $A^i B^j \eta_{ij}$ doesn't matter, because all of those are coordinates and therefore numbers. However, we cannot write $\vec{A}^T \eta \vec{B}$ in a different order – that would give us a different object!*

Rewriting this in another way,

$$\vec{A} \cdot \vec{B} = (A^i \hat{e}_i) \cdot (B^j \hat{E}_j) = A^i B^j \hat{e}_i \cdot \hat{e}_j,$$

and since $\hat{e}_i \cdot \hat{e}_j = \delta_{ij}$, this also gives exactly the same expression as what we have above. So the lesson is that the metric takes in two vectors and returns a number, and its existence does not depend on the frame. To assign the coordinates, the right way to do this is to let

$$\eta_{ij} = \hat{e}_i \cdot \hat{e}_j.$$

(This last detail will become helpful when we add time into the mix.)

So how do we think about a change of coordinates? Suppose we have new coordinates of the form

$$x^{i'} = f^{i'}(x^j),$$

where f' represents, for example, a Lorentz transformation. This equation in differential form looks like

$$dx^{i'} = \frac{\partial x^{i'}}{\partial x^j} dx^j = \Lambda^{i'}{}_j dx^j$$

for some matrix Λ. The index at the bottom, j, represents the frame we start from, and the index at the top, i', represents the frame we're going to.

In problem set 1, we asked the question "how do the unit vectors change?" We saw that they follow

$$\hat{e}'_i = \Lambda^j_{i'} \hat{e}_j,$$

where $\Lambda^j_{i'}$ is now the inverse matrix of what we initially had.

Definition 49

A **covariant vector**, also known as a 1-form, is an object of the form A_i (one index down), such that coordinates change the same way as the basis vectors:

$$A_{i'} = \Lambda^j{}_{i'} A_j.$$

Meanwhile, a **contravariant vector** is of the form A^i and changes in the same way as the differentials:

$$A^{i'} = \Lambda^{i'}{}_j A^j.$$

Most of the quantities we're used to working with are contravariant vectors (for example, the position of a particle), but an example of a covariant vector is the **gradient**

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

In index notation, this is often denoted $\frac{\partial}{\partial x^i}$ or ∂_i. Notice that by the chain rule,

$$\partial_i = \frac{\partial}{\partial x^i} = \frac{\partial x^j}{\partial x^{i'}} \cdot \frac{\partial}{\partial x^j} = \Lambda^j{}_{i'}\partial_j,$$

and this indeed varies in the same way as the basis vectors!

Definition 50

When we sum over a repeated index (one up, one down), call this a **contraction**. For example,

$$A^i\eta_{ij} = A^1\eta_{1j} + A^2\eta_{2j} + A^3\eta_{3j}.$$

Well, what happens to a matrix, like η, when we change coordinates? Remember that we defined η to be the dot product of our basis vectors, so

$$\eta_{i'j'} = \hat{e}_{i'} \cdot \hat{e}_{j'} = (\Lambda^\ell{}_{i'}\hat{e}_\ell) \cdot (\Lambda^m{}_{j'}\hat{e}_m) = \Lambda^\ell{}_{i'}\Lambda^m{}_{j'}\eta_{\ell m}.$$

(last step because we just have coordinates instead of vectors or matrices). So this object η does not behave like a covariant or contravariant vector: it requires two Λs to be converted! That's why there are two indices on the bottom: this is called a 2-form or bilinear form. Specifically, this is actually a more general example of a **tensor**.

So here's the bottom line: under a transformation,

$$A^{i'} = \Lambda^{i'}{}_j A^j, \quad B_{i'} = \Lambda^j{}_{i'}B_j, \quad \eta_{i'j'} = \Lambda^\ell{}_{i'}\Lambda^m{}_{j'}\eta_{\ell m}.$$

The way to remember what to do is just to pair things up!

Definition 51

Define the inverse of a matrix η_{ij} via

$$\eta^{-1} = \eta^{ij}$$

such that

$$\eta^{ij}\eta_{j\ell} = \delta^i{}_\ell.$$

Matrices are also able to do things like "lower" and "raise" an index:

$$\eta^{ij}B_j = B^i, \eta_{ij}B^j = B_i.$$

Remark 52. *All of this has deep mathematical justification, but we should take differential geometry if we're interested in these details.*

So let's say we have a vector

$$\vec{A} = (A^1, A^2, A^3) \to A^i.$$

Then

$$A_i = \eta_{ij}A^j = \eta_{i1}A^1 + \eta_{i2}A^2 + \eta_{i3}A^3.$$

If we write this out for all i, and because η_{ij} is the Kronecker delta, this yields

$$A_1 = A^1, A_2 = A^2, A_3 = A^3.$$

That's exactly why no one ever bothers telling us that there's two types of geometrical opposites: in the conditions of Euclidean geometry, they happen to work out perfectly! But the point is that we're about to introduce time, and remember that ds^2 has a minus sign in front of the time. So things aren't going to be perfect anymore.

We now have our four-vector (remember that in these notes, bold refers to four-vectors)

$$\mathbf{dx} = (cdt, dx, dy, dz) = dx^\mu,$$

which lives in **Minkowski spacetime**. This time, we don't agree about the norm $||\vec{x}||^2$ anymore, but rather the **spacetime invariant**

$$ds^2 = -(cdt)^2 + dx^2 + dy^2 + dz^2.$$

We can write this in the same way as in Euclidean space:

$$= dx^\mu dx^\nu \eta_{\mu\nu},$$

where the metric is no longer just the identity:

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

And now we can take our previous discussion and carry over most of the properties. For example, it's still true that

$$\eta_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu,$$

if we define the dot product relative to our metric

$$\mathbf{A} \cdot \mathbf{B} = \eta(\mathbf{A}, \mathbf{B}) = \eta_{\mu\nu} A^\mu B^\nu.$$

So we can basically sum everything up in the statement

$$ds^2 = \mathbf{dx} \cdot \mathbf{dx}.$$

The ideas about covariant and contravariant vectors still hold! In particular, we can still say that

$$A^{\mu'} = \Lambda^{\mu'}{}_\nu A^\nu, \quad A^\mu \eta_{\mu\nu} = A_\nu,$$

and so on. But this latter statement is a little different when we write it down: expanding out for μ, ν, notice that we now have

$$A_0 = -A^0, A_1 = A^1, A_2 = A^2, A_3 = A^3.$$

Now η, which we've been calling a **tensor**, has two indices down, because it behaves in a particular way when we change frame.

Definition 53

A **tensor** of rank $\begin{bmatrix} U \\ L \end{bmatrix}$ is a geometrical object R such that we can represent it in the form

$$R^{\mu_1,\mu_2,\cdots,\mu_U}{}_{\nu_1,\cdots,\nu_L}.$$

Contravariant vectors are type $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, covariant vectors are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$-tensors, and so on. In particular, when we change frame,

$$R^{\mu_1',\mu_2',\cdots,\mu_U'}{}_{\nu_1',\cdots,\nu_L'} = \Lambda^{\mu_1'}{}_{\alpha_1}\Lambda^{\mu_2'}{}_{\alpha_2}\cdots\Lambda^{\mu_U'}{}_{\alpha_U}\Lambda^{\beta_1}{}_{\nu_1'}\Lambda^{\beta_2}{}_{\nu_2'}\cdots\Lambda^{\beta_L}{}_{\nu_L'}R^{\mu_1,\mu_2,\cdots,\mu_U}{}_{\nu_1,\cdots,\nu_L}$$

requires $U+L$ matrices to change to our new coordinates.

The ideas of contraction still hold:

$$A^\mu B_\mu = A^0B_0 + A^1B_1 + A^2B_2 + A^3B_3.$$

(Note that this is no longer the dot product! Contraction is more generic, because it doesn't require a metric.) Contraction can even be done within the same tensor:

$$R^\alpha{}_{\mu\alpha\beta} = R^0{}_{\mu0\beta} + R^1{}_{\mu1\beta} + R^2{}_{\mu2\beta} + R^3{}_{\mu3\beta}.$$

This is indeed actually useful for physics, both for this class and in general. The starting point of this class is that if we perform Galilean transformations, Newton's laws stay the same, but Maxwell's equations do not stay the same. What we'll see in the rest of this class is that if we can write the laws of physics in a way that equates two tensors with the same rank, such as

$$R^{\mu\nu} = Kg^{\mu\nu} + a^{\mu\nu},$$

then we can be sure that changing frames will keep the laws the same. This is because **all tensors of the same rank behave the same way under transformations!**

Example 54

If $R^{\mu\nu} = Kg^{\mu\nu}$ in frame S, then we can compute the components in S' via

$$R^{\mu'\nu'} = \Lambda^{\mu'}{}_\alpha\Lambda^{\nu'}{}_\beta R^{\alpha\beta}.$$

Similarly,

$$g^{\mu'\nu'} = \Lambda^{\mu'}{}_\alpha\Lambda^{\nu'}{}_\beta g^{\alpha\beta}.$$

So both sides are transforming exactly the same way, and thus the equality will still hold correctly!

Next time, we'll start seeing how dynamics, as well as electromagnetism, can be written in tensorial form.

12 September 24, 2019 (Recitation)

Today, we're going to talk about many of the ideas from lecture in more detail. We'll talk about the metric, because this is where we start to see how curved spacetime might make a difference!

Fact 55

One way to think about a 4-vector is to treat it as a tangent vector to a curve γ in 4-dimensional space.

To be more specific, we can parametrize γ with respect to some parameter λ. Then $\mathbf{x}^\mu(\lambda) = (t(\lambda), x(\lambda), y(\lambda), z(\lambda))$ are the specific coordinates: thus, then the tangent vector

$$\mathbf{V}^\mu = \left.\frac{dx^\mu(\lambda)}{d\lambda}\right|_{\lambda=\lambda_0}$$

gives us the coordinates of our four-vector. Note here that x^μ has an upper index, so V^μ does as well.

We care about this object $\mathbf{V}$ when we look at it in different reference frames: under a transformation, we want them to behave via

$$V^{\mu'} = V^\mu \Lambda^{\nu'}{}_\mu,$$

where Λ is a matrix. (We can check that these 4-vectors form a vector space.) Notice that this is in contrast to **one-forms**, which can be defined as follows:

Definition 56

A linear map from the set of 4-vectors to the reals

$$\omega : \mathbf{V} \to \omega(V) = \omega_\mu \mathbf{V}^\mu \in \mathbb{R}$$

is known as a **one-form**.

The space of 1-forms is known as the **dual space** or **Hom space** (it's dual to the vector space of four-vectors). We know that Lorentz transformations are implemented with a Lorentz matrix, and notably

$$\omega_{\mu'} = \Lambda^?{}_? \omega_?$$

From here, we know that we must have one free index, and we must contract any other indices that show up, so we can fill in the rest accordingly:

$$\omega_{\mu'} = \Lambda^\nu{}_{\mu'} \omega_\nu$$

Note that Λ is a matrix, and expressions like $\Lambda^\nu{}_{\mu'}\omega_\nu$ correspond to multiplying a matrix by a column vector, while expressions like $V^\mu \Lambda^{\nu'}{}_\mu$ correspond to multiplying that same matrix by a row vector! (Upper indices refer to rows, and lower indices refer to columns.)

Remark 57. *Note that there's a symmetry here, though. While one-forms map vectors to reals, we can also think of vectors as mapping one-forms to reals:*

$$V : \omega \to v(\omega) = \omega(V).$$

Well, we can generalize this idea: we want objects that take some number of vectors and one-forms and return a real number.

Definition 58

A **multilinear map** T (to the reals) which takes k one-forms and ℓ vectors

$$T : \{\omega^{(1)}, \cdots, \omega^{(k)}, V^{(1)}, \cdots, V^{(\ell)}\} \to \mathbb{R}$$

is a **rank (k, ℓ) tensor**.

For example, for a rank $(1,1)$ tensor, we can check that the sum of two one-forms is a one-form, and the sum of two vectors is a vector, so we follow the rule

$$T(a\omega + b\eta, cU + dV) = acT(\omega, U) + adT(\omega, V) + bcT(\eta, U) + bdT(\eta, V).$$

Remark 59. *The "rank" of a tensor has nothing to do with the "rank" of a matrix. Rank here basically denotes dimension. Also, note that only a rank (1, 1) tensor is actually a matrix: we're often sloppy, like with the metric, in denoting (0, 2) or (2, 0) tensors in matrix form.*

Let's write down the components here. For T to take in k one-forms in its argument, it must be able to contract with k lower indices. Similarly, it must be able to take ℓ upper indices, so the coordinates can be written as

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell}.$$

How does this behave under a transformation? We've talked about this before: each of the $k+\ell$ indices is changed by a matrix Λ with one primed and one un-primed coordinate. To get this to match up, we have

$$T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_{\ell'}} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_\ell} \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k} \cdot \Lambda^{\nu'}{}_{\nu_1'} \cdots \Lambda^{\ell'}{}_{\nu_\ell'},$$

where each Λ is a Lorentz transformation!

Question 60. *Does the order here matter?*

The answer is generally yes. For example, say we have a $(0,2)$ tensor which can be written as two one-forms: $T_{\mu\nu} = \omega_\mu \eta_\nu$. Then $T_{10} = \omega_1 \eta_0$, but $T_{01} = \omega_0 \eta_1$, so in general, $T_{\mu\nu}$ and $T_{\nu\mu}$ will not be the same!

Problem 61

A rank (1, 1) tensor takes a vector and a one-form, and it outputs a scalar. What do you get if you only feed it a vector? Only a one-form?

Given any v, the tensor $T(v, \cdot)$ takes in a one-form and outputs a scalar, so it is a vector. Similarly, the tensor $T(\cdot, \omega)$ takes in a vector and outputs a scalar, so it is a one-form. Explicitly,

$$T^\mu{}_\nu v^\nu = u^\mu, \quad T^\mu{}_\nu \omega_u = \omega_\nu.$$

Definition 62

A **Lorentzian metric** is a rank $(0,2)$ tensor which is symmetric and has one negative and three positive eigenvalues.

(In contrast, a **Riemannian metric** has four positive eigenvalues. For example, the identity matrix is the standard Riemannian metric.)

Let's break down the definition here. g is symmetric if

$$g(\mathbf{u}, \mathbf{v}) = g(\mathbf{v}, \mathbf{u}) \implies g_{\alpha\beta} u^\alpha v^\beta = g_{\alpha\beta} v^\alpha u^\beta = g_{\beta\alpha} v^\beta u^\alpha$$

(last step just by relabeling). So this means that $\boxed{g_{\alpha\beta} = g_{\beta\alpha}}$: g looks like a symmetric matrix!

We want to be able to take a dot product of a vector with itself. We know that we can't multiply a column vector by a column vector, so we have to somehow send a column vector into a row. That's what the metric is for: for a vector $\mathbf{v}$, we define

$$\mathbf{v}_\mu = \eta_{\mu\nu} \mathbf{v}^\nu.$$

In some sense, $\mathbf{v}_\mu$ is the **dual vector** to $\mathbf{v}$:

$$\omega_\mu v^\nu = \omega^\mu v^\nu \eta_{\mu\nu} = \omega^\mu v_\nu.$$

That's the power of having the metric be symmetric! This is particularly significant because general relativity is a study of metrics, which actually encode the geometry of spacetime. Consider the invariant interval

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -(cdt)^2 + dx^2 + dy^2 + dz^2,$$

measuring the infinitesimal length of a curve as given to us by the object η. Suppose that instead of η, we had something else: maybe we want to divide $dx^2 + dy^2 + dz^2$ by t^2. Then the curve is being looked at in a different geometry! So that's what η will encode: this one in particular corresponds to inflation.

13 September 25, 2019

Today, we're going to go back to physics: we'll start using four-vectors and matrices to solve physics problems. First, let's summarize the main important points of last time. We defined a **contravariant** four-vector to transform like $\mathbf{dx}$: any four-vector $\mathbf{A}$ transforms via

$$A^{\mu'} = \Lambda^{\mu'}{}_\nu A^\nu.$$

In contrast, a **covariant** vector, transforms like the unit basis vectors $\mathbf{e}_\mu$: these are also called one-forms, and we'll denote them here on out as **A**. The coordinates transform via

$$A_{\mu'} = \Lambda^\nu{}_{\mu'} A_\nu.$$

In general, a **rank** (U, L) tensor has U upper indices and L lower indices, and they require $U + L$ basechange matrices whenever we change frames. The main tensor we care about so far is the **metric**, which is a $(0, 2)$ tensor: it defines the dot product of two four-vectors via

$$\eta_{\mu\nu} : \mathbf{A} \cdot \mathbf{B} = A^\mu B^\nu \eta_{\mu\nu}.$$

In our Minkowski space, we defined

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

One reason we care about the metric is that helps us raise or lower an index, which secretly switches us between the vector space of four-vectors and its dual space:

$$A_\nu \eta^{\mu\nu} = A^\mu, \quad B^\mu \eta_{\mu\alpha} = B_\alpha.$$

Today, we're going to start by generalizing the idea of a velocity (and then by extension to define the momentum). Why do we want to do this? We've shown that transformations in velocity of the form $u' = u - v$ don't really work, so we need to make some updates.

Recall that in 8.01, we defined our velocity

$$\vec{u} = \frac{d\vec{x}}{dt}.$$

This is not a four-vector, so it's not clear how this changes under frames. There are nice interpretations of $\vec{u}$, such as the geometric approach of having it be a **tangent vector** to the curve that we're moving through in spacetime.

That's how we'll be approaching it here today!

A big difference here is that we need to track the position of a particle in both space and time: this is some **world line**. What if we try to define the velocity as

$$\mathbf{u} = \frac{d\mathbf{x}}{dt}?$$

The problem here is that the denominator here, t, depends on the frame we're in, so $\frac{d}{dt}$ is not quite a scalar. Instead, we'll use something that's actually well-defined: **we use $d\tau$, the proper time, instead of dt**:

Definition 63

The **four-velocity U** of a four-vector is defined via

$$\mathbf{U} = \frac{d\mathbf{x}}{d\tau}.$$

So is it possible to relate this to the three-dimensional velocity $\vec{u}$?

Example 64

Consider the simple case where an object is moving in some frame S with a velocity $\vec{u} = u^x \hat{e}_x$.

The idea is to solve this problem in a frame where it's particularly easy to solve. This is a four-vector now: if we manage to find the four components in some frame, we can just use Λ, our Lorentz transformation matrix, to finish!

Consider S', the rest frame of that object, which is moving right relative to S with a velocity $u^x \hat{e}_x$. Since the particle is not moving,

$$S' : dt' = d\tau, dx = dy = dz = 0.$$

Therefore,

$$\mathbf{U} \underset{S'}{\rightarrow} \left(\frac{d(ct')}{dt}, 0, 0, 0 \right) = (c, 0, 0, 0).$$

Since $\mathbf{U}$ is a four-vector,

$$U^\mu = \Lambda^\mu{}_{\nu'} U^{\nu'},$$

and since S is moving relative to S' with velocity $-u^x$, that negative sign cancels out with the one in $-\beta\gamma$, and

$$\Lambda^\mu{}_{\nu'} = \begin{bmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Plugging this in, most of our terms will cancel. Looking component-wise,

$$U^0 = \Lambda^0{}_{0'} U^{0'} = \gamma(\vec{u})c, \quad U^1 = \Lambda^1{}_{0'} U^{0'} = \beta(\vec{u})\gamma(\vec{u})c,$$

and we can see clearly that $U^2 = U^3 = 0$. Thus,

$$\mathbf{U} \underset{S}{\rightarrow} \gamma(\vec{u}) \left(c, u^x, 0, 0 \right).$$

In the most generic case where the particle has all three components of its three-velocity in the S-frame,

$$\mathbf{U} = \boxed{\gamma(\vec{u}) \left(c, u^x, u^y, u^z \right)}.$$

Remark 65. *Do be careful: $U^i \neq (\vec{u})^i$, because there's an extra factor of γ in each term.*

Here's another way to do the problem: say that we're working in some arbitrary frame S. Then $dt = \gamma d\tau$, where γ is the speed between the two frames. We can then write

$$\mathbf{U} = \frac{d\mathbf{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\mathbf{x}}{dt} = \gamma(\vec{u})\frac{d\mathbf{x}}{dt},$$

and this also gives us the correct expression for the four-velocity.

How do we generalize the Galilean transformations to relativity? We're going to derive the velocity addition formulas! Consider two frames

$$\mathbf{U} \underset{S}{\rightarrow} \gamma(\vec{u})(c, \vec{u}), \quad \mathbf{U} \underset{S'}{\rightarrow} \gamma(\vec{u'})(c, \vec{u'}).$$

(Here, this shorthand notation means $(\gamma(\vec{u})c, \gamma(\vec{u})u^x, \gamma(\vec{u})u^y, \gamma(\vec{u})e^z)$.) We can write the transformation in tensorial form as

$$U^{\mu'} = \Lambda^{\mu'}{}_\nu U^\nu,$$

and writing out the Lorentz transformation from S to S' for $\mu' = 0$ yields (notice that the only nonzero components of $\Lambda^{0'}{}_\mu$ are for $\mu = 0, 1$)

$$\gamma(\vec{u'})c = U^{0'} = \Lambda^{0'}{}_0 U^0 + \Lambda^{0'}{}_1 U^1 = \gamma(\vec{v})\gamma(\vec{u})c + \left(-\frac{v}{c}\gamma(\vec{v})\right)\gamma(\vec{u})u^x,$$

so dividing through by c,

$$\boxed{\gamma(\vec{u'}) = \gamma(\vec{v})\gamma(\vec{u})\left(1 - \frac{vu^x}{c^2}\right)}.$$

Next, let's look at the next term, where $\mu' = 1$:

$$\gamma(\vec{u'})u^{x'} = U^{1'} = \Lambda^{1'}{}_0 U^0 + \Lambda^{1'}{}_1 U^1 = \left(-\frac{v}{c}\gamma(\vec{v})\right)\gamma(\vec{u})c + \gamma(\vec{v})\gamma(\vec{u})u^x.$$

Plugging in the value of $\gamma(\vec{u'})$ from above yields

$$\gamma(\vec{v})\gamma(\vec{u})\left(1 - \frac{vu^x}{c^2}\right)u^{x'} = \gamma(\vec{v})\gamma(\vec{u})(-v + u^x) \implies \boxed{u^{x'} = \frac{u^x - v}{1 - \frac{vu^x}{c^2}}}.$$

But this tells us how the ordinary velocities in the old and new frame are related! Notice that as $u, v \ll c$, this approaches the Galilean transformation $u^{x'} = u^x - v$. In addition, if $u^x = c$, any v keeps c constant: now we've actually gotten a result consistent with Michelson-Morley.

We might think that nothing happens along the y and z frames, but this is no longer true! In particular, $u^y = \frac{dy}{dt}$, while $u^{y'} = \frac{dy'}{dt'}$: nothing happens to dy, but something does happen to dt. Specifically,

$$U^{2'} = \Lambda^{2'}{}_\mu U^\mu = \Lambda^{2'}{}_2 U^2,$$

(all other terms cancel out), which means that

$$\gamma(\vec{u'})u^{y'} = \gamma(\vec{u})u^y.$$

Thus, we can plug in the boxed impression from above to find that

$$u^{y'} = \frac{\gamma(\vec{u})}{\gamma(\vec{u'})}u^y = \boxed{\frac{u^y}{\gamma(\vec{v})\left(1 - \frac{vu^x}{c^2}\right)}}.$$

This is how the y-component of our velocity changes: it's not constant anymore! (The z-component behaves exactly the same way.)

Recall that we found earlier that $\mathbf{dx} \cdot \mathbf{dx}$ is a scalar whose value is independent of the frame. Well, what's the value of $\mathbf{U} \cdot \mathbf{U}$? Here's an example where working on specific frame is the best: since we have a scalar, it's invariant, and so we can calculate it in the rest frame. Then we just have

$$\mathbf{U} \cdot \mathbf{U} = U^0 U^0 \eta_{00} = -c^2.$$

This is an invariant, because c is an invariant!

Now we're ready to move on to the concept of a **generalized momentum**. In 8.01, we just define $\vec{p} = m\vec{v}$, and this quantity is nice because this is a conserved quantity. We want to again come up with a four-vector, and it turns out that

$$\mathbf{p} = m\mathbf{U}$$

actually does the job here! $\mathbf{p}$ is called the **four-momentum**, and since it's a scalar times a vector, it's indeed a four-vector.

Expanding out, we have

$$\mathbf{p} \underset{S}{\rightarrow} m\gamma(\vec{u})(c, u^x, u^y, u^z).$$

Again, let's look at the dot-product of this vector with itself:

$$\mathbf{p} \cdot \mathbf{p} = (m\mathbf{U}) \cdot (m\mathbf{U}) = -m^2 c^2.$$

This means that the **mass is conserved**, because $m^2 = -\frac{\mathbf{p} \cdot \mathbf{p}}{c^2}$ is conserved. (This is fairly handwavy, but this isn't the main focus here.)

Next, looking at the specific components of p, let's first consider the space components:

$$p^i = m\gamma(\vec{u}) u^i,$$

which can be Taylor expanded to

$$mu^i \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right),$$

which looks a lot like the classical momentum of a particle when $|u| \ll c$.

Definition 66

Thus, in special relativity, we define the three-dimensional **momentum**

$$\vec{p} = m\gamma(\vec{u})\vec{u}.$$

Let's look a bit more specifically at the time component now: notice that

$$cp^0 = m\gamma(\vec{u})c^2 \rightarrow mc^2 \left(1 + \frac{1}{2}\frac{u^2}{c^2} + \cdots\right) = mc^2 + \frac{1}{2}mu^2 + \cdots.$$

So the $\frac{1}{2}mu^2$ looks like the classical kinetic energy of a particle, and this new mc^2 term seems like it's an energy as well! This is called the **rest energy** of a particle. The idea is that even if a particle is not moving, $\gamma \rightarrow 1$, and $cp^0 \rightarrow mc^2$. So the four-momentum doesn't go to zero: there's some kind of "energy" associated with each amount of mass!

Definition 67

We therefore define the **energy** in terms of our four-momentum via $E = cp^0$.

Putting these two definitions together,

$$\mathbf{p} \xrightarrow{S} \left(\frac{E}{c}, \vec{p}\right):$$

then

$$\mathbf{p} \cdot \mathbf{p} = p^\mu p^\nu \eta_{\mu\nu} = -\frac{E^2}{c^2} + p^{x2} + p^{y2} + p^{z2} = \frac{E'^2}{c^2} = p^{x'2} + p^{y'2} + p^{z'2} = -m^2c^2$$

is conserved in all frames! In particular, the energy and momentum depend on the frame that we're in, and we can "convert" between the two.

Fact 68

This means that if we have some N particles that go into a collision and M particles come out, we have conservation of four-momentum

$$p^\mu_{\text{total}} = \sum_{i=1}^{N} p^\mu_i = \sum_{j=1}^{M} p^\mu_j.$$

The first component tells us that the total energy is conserved, and the other tells us that the three-dimensional momenta are the same.

We'll talk more about generalized energy and generalized momentum next time!

14 September 26, 2019 (Recitation)

First, we'll take a look at some of the homework problems.

The **Levi-Civita symbol** or **antisymmetric symbol** is how we write a cross-product in index notation:

$$\varepsilon_{ijk} = \begin{cases} +1 & ijk \text{ is an even permutation of } (1,\ 2,\ 3) \\ -1 & ijk \text{ is an odd permutation} of (1, 2, 3) \\ 0 & \text{otherwise (some two entries are equal)} \end{cases}$$

A permutation is even if it takes an even number of swaps from $(1, 2, 3)$ to get to it, and odd if it takes an odd number of swaps.

Recall that last time, we defined a rank $(2,\ 0)$ **symmetric tensor** to be one where $T^{ij} = T^{ji}$. Contracting this with ε_{ijk},

$$\varepsilon_{ijk}T^{ij} = \varepsilon_{ijk}T^{ji},$$

and now relabeling so that $(i, j) \to (j, i)$,

$$= \varepsilon_{jik}T^{ij} = -\varepsilon_{ijk}T^{ij}.$$

Thus, $\varepsilon_{ijk}T^{ij}$ is its own negative, and thus ε_{ijk} **annihilates all symmetric tensors**! We'll get to work a lot with this in the problem set.

For the rest of this class, let's do some review and problems. We talked about four-vectors last recitation as curves through four-dimensional space, parametrized by some λ. Then the **four-velocity** is just a special version of this: given two events A and B that are timelike-separated, parametrize this via the proper time τ, and define

$$U^\mu = \frac{dx^\mu}{d\tau}.$$

We derived last time that this takes the form

$$\mathbf{U} \to \gamma(c, \vec{v}),$$

and we found that the squared norm of this vector is

$$\eta_{\mu\nu} U^\mu U^\nu = -c^2$$

in one reference frame, and thus it's equal to $-c^2$ in all reference frames (because it's a Lorentz scalar, which is invariant under transformations). Note that this is a negative quantity because it is timelike! We can do something similar with proper length instead to get positive quantity for spacelike events, but this is generally less interesting.

After this, we introduced the **four-momentum** $\mathbf{p} = m\mathbf{U} \implies p^\mu = mU^\mu$. Importantly, we also found that

$$\mathbf{p} \cdot \mathbf{p} = -m^2 c^2$$

is constant across all frames. Consistently with the four-velocity, the four-momentum takes the form $\gamma \left(\frac{mc^2}{c}, \vec{p}\right)$, and we often denote the first component of this vector to be $\frac{E}{c}$. This means that we actually define the energy

$$E = \gamma mc^2,$$

(which is unfortunately one symbol away from the famous equation that we know). Note that being **conserved** is different from being **invariant**. Four-momentum is conserved before and after a process, but it isn't necessarily invariant in all situations!

Problem 69

Consider a particle with rest momentum m, so its rest energy is mc^2. Suppose a particle is moving relative to our reference frame at a speed v away from, aligned along our x-axis. In our reference frame, the particle's energy is γmc^2: what is the particle's four-momentum in our reference frame?

Solution. The y- and z-components of four-momentum will be zero, because the momentum in those directions is zero. In the x-direction,

$$p^x = \gamma mv = \gamma m \left(c\sqrt{1 - \frac{1}{\gamma^2}} \right) = \boxed{mc\sqrt{\gamma^2 - 1}}.$$

Alternatively, we can bash out the invariant $\mathbf{p} \cdot \mathbf{p}$: if we write the four-vector as $\left(\frac{E}{c}, p^x, 0, 0\right)$ (where $p^x = \gamma \vec{p}^x$),

$$\mathbf{p} \cdot \mathbf{p} = -m^2 c^2,$$

but we can also expand this out as

$$p^\mu p^\nu \eta_{\mu\nu} = -(p^0)^2 + (p^1)^2 = -\gamma^2 m^2 c^2 + (p^x)^2.$$

Thus, again we have

$$(p^x)^2 = (\gamma^2 - 1)m^2 c^2 \implies p^x = mc\sqrt{\gamma^2 - 1},$$

and we can confirm that setting this equal to $p^x = \gamma mv$ makes everything consistent with the Lorentz factor. □

Next, let's consider the norm of the four-momentum of a photon,

$$\mathbf{p}^2 = \mathbf{p} \cdot \mathbf{p} = 0.$$

If we were in Euclidean space, that would mean the momentum is zero. In general, if $g_{\mu\nu}$ had four positive eigenvalues, then we can diagonalize our matrix and say that in some frame,

$$p^\mu p^\nu g_{\mu\nu} = (p^t)^2 g_{tt} + (p^x)^2 g_{xx} + (p^y)^2 g_{yy} + (p^z)^2 g_{zz}.$$

Since each term here is nonnegative, we must have all terms zero for the sum to be zero. Since all $g_{\mu\mu}$s are positive, this means all p^μs are zero.

But in relativity, $g_{\mu\nu}$ will have a negative eigenvalue, so the four-momentum of a photon is not just the zero vector!

> **Problem 70**
> Is it possible for us to have an electron and a photon interact and output an electron, given that $\mathbf{p} \cdot \mathbf{p} = 0$?

Say that the starting electron has momentum $\mathbf{p}_b$, the photon has momentum $\mathbf{q}_b$, and the final electron has momentum $\mathbf{p}_a$. Then it's true that

$$p_b^\mu + q_b^\mu = p_a^\mu,$$

and also it's true that

$$(p_b)_\mu + (q_b)_\mu = (p_a)_\mu.$$

(Remember that b and a are just subscripts, NOT indices!) Multiplying everything together,

$$\left(p_b^\mu + q_b^\mu\right)\left((p_b)_\mu + (q_b)_\mu\right) = p_a^\mu p_{a\mu} = -m^2c^2,$$

where m is the mass of an electron. Expanding the left side, we have

$$p_b^\mu p_{b\mu} + q_b^\mu q_{b\mu} + p_b^\mu q_{b\mu} + q_b^\mu p_{b\mu} = -m^2c^2.$$

But we know the first two terms – they are $-m^2c^2$ and 0, respectively – so we can cancel those terms with the $-m^2c^2$ and write out the other two terms with the metric:

$$\eta_{\mu\nu} p_b^\mu q_b^\nu + \eta_{\mu\nu} p_b^\nu q_b^\mu$$

and now because η is a symmetric tensor, we can write this as

$$\eta_{\mu\nu} p_b^\mu q_b^\nu + \eta_{\nu\mu} p_b^\nu q_b^\mu$$

and because these are all dummy indices, this is just equal to $2p_b^\mu q_{b\mu}$. So

$$p_b^\mu q_{b\mu} = 0,$$

This is a constant, so let's pick a convenient frame: the initial electron's rest frame! Then $\mathbf{p}_b = (mc, 0, 0, 0)$ and $\mathbf{q}_b = \left(\frac{E_q}{c}, q^x, q^y, q^z\right)$, so

$$p_b^\mu q_{b\mu} = -mE_q = 0.$$

Since electrons are not massless, the energy of the photon must be zero. But that means the particle doesn't exist, so the electron and photon cannot interact in this way!

15 September 30, 2019

Today, we'll finish our transition between classical and special relativistic mechanics: we'll see how to generalize the idea of acceleration and forces, which completes our study of velocity, energy, and momentum.

Recall that we've recently defined the four-velocity in terms of the regular velocity $\vec{u}$ via

$$\mathbf{U} \underset{S}{\rightarrow} \gamma(u) \cdot (c, u^x, u^y, u^z) = \gamma(u)(c, \vec{u}).$$

We found this by defining the four-velocity as the derivative $\frac{d\mathbf{x}}{d\tau}$: it was mentioned that the space component of the four-velocity doesn't just correspond to the regular three-velocity:

$$\frac{\vec{u}^i}{c} = \frac{U^i}{U^0}.$$

If we want to change frames, we can just use a Lorentz matrix to find the new components:

$$U^{\mu'} = \Lambda^{\mu'}{}_{\nu} U^{\nu}.$$

Regardless of what frame we're in, dotting the four-velocity with itself (under the metric) gives a constant $-c^2$.

We then defined the four-momentum $\mathbf{p} = m\mathbf{U}$: this just yields

$$\mathbf{p} \underset{S}{\rightarrow} \gamma(u)(mc, m\vec{u}) = \left(\frac{E}{c}, \vec{p}\right)$$

where $E = \gamma mc^2$ is the total energy and $\vec{p} = m\gamma(u)\vec{u}$. These are both conserved quantities in the same ways as in classical mechanics, but there's an extra factor of γ here! To make use of these conservation laws, note that we can define a **total momentum** for a system of particles

$$\mathbf{p}_{\text{tot}} = \sum_{\ell} \mathbf{p}_{\ell}.$$

This is useful if we have a particle collision: the total momentum before and after the collision must be equal, and so must the energies. It's possible that some particles are destroyed or created – there's (at least for now) no restriction on the actual number of particles here!

Remark 71. *Conservation and invariance are two different concepts. For example, energy is conserved, but the numerical value of energy can be different in different reference frames.*

Today, we'll talk a bit more about momentum and energy, and then we'll move to forces and acceleration. Note that if we write out the dot product for the four-momentum,

$$p^{\mu} p_{\mu} = \eta_{\mu\nu} p^{\mu} p^{\nu} = -m^2 c^2 = -\frac{E^2}{c^2} + \vec{p} \cdot \vec{p}.$$

(We often denote $p^2 = |\vec{p}|^2$ for convenience.) This gives us a very nice expression when we simplify the equality between the last two:

$$\boxed{E^2 = m^2 c^4 + p^2 c^2}.$$

Remember that we defined the kinetic energy

$$K = E - mc^2 = (\gamma - 1)mc^2.$$

The following quantities are then equivalent:

$$E = \gamma mc^2 = KE + mc^2 = p^0 c = \sqrt{p^2 c^2 + m^2 c^4}.$$

If we change frames so that $p^{\mu'} = \Lambda^{\mu'}{}_{\nu} p^{\nu}$, notice that we still have

$$E'^2 = m^2 c^4 + p'^2 c^2$$

because $\mathbf{p} \cdot \mathbf{p}$ is a scalar, which is constant. Similarly, if we repeat this logic for $\mathbf{p}_{\text{tot}}$ (as defined above), we can define

$$p^{\mu}_{\text{tot}} p_{\text{tot}\ \mu} = -\frac{\varepsilon_0^2}{c^2} = -(p_{\text{tot}})^2 + \vec{p}_{\text{tot}} \cdot \vec{p}_{\text{tot}}$$

in accordance with the one-particle definition. This result needs to be constant, because it's the dot-product of a four-vector with itself. We call ε_0 the **center-of-mass energy**: this is in some way a measure of the total energy of the system! Note that because it's the dot-product of two four-vectors, **it is both conserved and invariant**.

So we can calculate ε_0 at any time and in any frame, and its value is always the same!

Definition 72

Define the **center-of-mass frame** to be the reference frame where $\vec{p}_{\text{tot}} = 0$.

This means that the space components of the total four-momentum are all zero, but then this gives us

$$-\frac{\varepsilon_0^2}{c^2} = -(p^0_{\text{tot}})^2.$$

This total energy available can be "spent" in a lot of ways, though! This is not exactly like the "proper time" or "proper distance," where we talk about a frame where the particle is at rest. It's true that the system is overall at rest, but the particles don't necessarily have to be at rest now.

Remember when we first defined a four-vector, we chose the first component to be $x^0 = ct$ to make the units work out – we usually measure time in seconds, and we usually measure space in meters (this is called using **international units**). **But if we define both space and time in meters**, choosing appropriate units so that light travels 1 meter in 1 seconds, velocity is unitless, and $c = 1$. This is known as using **natural units**, and it makes a lot of our expressions look much simpler:

$$ds^2 = -dt^2 + dx^2, U^{\mu} U_{\mu} = 1.$$

If we ever need to convert back, it's not too hard: just match all of the dimensions and replace t with ct. So from this point on, we should expect the convention of $c = 1$!

Problem 73 (Creating a π^+ particle)

Consider the collision

$$p + p \to p + n + \pi^+$$

which turns two protons into a proton, a neutron, and a pion. How fast do the protons need to move to make this happen (how much kinetic energy must be given to one of the protons for it to smash into the other one, which is at rest)?

We're using a system of units used in particle physics: the mass of a proton is 938 MeV (notice that even though this is units of energy, that's fine because $c = 1$! If we want, it's really divided by c^2), the mass of a neutron is 940 MeV, and the mass of a pion is 140 MeV.

Solution. If we use the same lab frame S before and after, the momentum needs to be conserved, so the whole system after must be moving to the right. Note that the kinetic energy that comes from the first proton is really contributing to the extra mass after the collision!

But what is the minimum kinetic energy here? We claim that we shouldn't have one particle at rest and the other smash against it, but instead two particles that smash into each other headfirst. Let's consider S', the **the center-of-mass frame**. In this new frame, two protons approach each other with four-momentum $\mathbf{p}_1$ and $\mathbf{p}_2$. Then by conservation, we can say that the system is at rest at the end, so we don't waste any energy to make the products of the collision move.

Before the collision, the components of our momentum are

$$p_1^0 = \gamma(u')m_p, \quad p_2^0 = \gamma(u')m_p, \vec{p_1} = \gamma(u')\vec{u'}, \vec{p}_2 = \gamma(\vec{u'})(-\vec{u}'),$$

and after the collision, the "most energy-efficient end result" is one where all $\vec{p} = 0$, $p_p^0 = m_p$, $p_m^0 = m_m$, and $p_{\pi^+} m_\pi^+$. So now looking at conservation of energy,

$$2\gamma(u')m_p = m_p + m_m + m_{\pi^+},$$

and now we're almost done: we need

$$\gamma(u') = \frac{m_p + m_m + m_\pi}{2m_p} \approx 1.08,$$

which corresponds to a velocity of about $0.37c$.

So how do we go back to the answer in our lab frame? We just do a vector transformation:

$$p^\mu = \Lambda^\mu{}_{\nu'} p^{\nu'}$$

where Λ is the basechange matrix. We only need the energy, so setting $\mu = 0$, we find that

$$p^0 = \Lambda^0{}_{\nu'} p^{\nu'} = \gamma(u')^2 m_p(1 + u'^2).$$

Then the kinetic energy

$$K_1 = E_1 - m_p \approx \boxed{305 \text{ MeV}}.$$

□

Note that this is twice as much as the rest mass of the pion. The idea is that some of the kinetic energy is wasted in making the final products move to the right, so that we have conservation of momentum! Also, this explains why many particle accelerators are built so that two particles start off with equal and opposite momenta. Notice for example that the kinetic energy in this better frame

$$K_{1'} = (\gamma(u') - 1)m_p \approx 75MeV \implies K_{\text{tot}} \approx \boxed{150 \text{ MeV}},$$

which is much smaller than what we have to pay in the S-frame (in fact, it's exactly equal to the extra mass we need)!

Remark 74. *Basically, the important point here is that energy can become both "things" and the kinetic energy of those things. This is how we create things like the Higgs boson, too!*

Problem 75

What happens if the mass m of our particle goes to 0? For example, can we consider photons?

It may be tempting to say that $E = \gamma mc^2$ means that photons have no energy, but this isn't true. Notably,

$$p^\mu p_\mu = -m^2 = 0,$$

but this doesn't imply that $\mathbf{p}$ is zero: it just tells us that $-E^2 + p^2$ is zero! So a particle with zero mass is one where $E = |\vec{p}|$. If we pick our x-axis to be in the direction of motion,

$$p^\mu \underset{S}{\to} (E, E, 0, 0).$$

Now note that it's always true that

$$\vec{p} = \gamma m \vec{u}, \quad E = \gamma m.$$

Since the magnitude $-E^2 + p^2$ is zero, we must have $\vec{u} = 1$: that is, the particle **must move at the speed of light!**

Remark 76. *Notice that even if m goes to 0, γ goes to ∞, so in the limit E can stay finite. As an exercise, we can check that there is no frame S' where a massless particle is at rest, meaning that $\vec{u} = 0$.*

We'll now introduce the ideas of acceleration and forces in our theory here, and we'll also introduce the twin paradox soon. Acceleration in special relativity is not trivial, and this is because we've been dealing with inertial frames this whole time. If we have accelerations, our velocities will not be constant!

Definition 77

Define the **four-acceleration** to be

$$\mathbf{A} = \frac{d\mathbf{U}}{d\tau} = \frac{d^2\mathbf{U}}{d\tau^2}.$$

Proposition 78

We always have

$$\mathbf{A} \cdot \mathbf{U} = 0.$$

Proof. Take the derivative of

$$U^\mu U_\mu = 1$$

with respect to τ: then the right hand side vanishes, and we're left with

$$\frac{dU^\mu}{d\tau} U_\mu + U^\mu \frac{dU_\mu}{d\tau} = 2A^\mu U_\mu = 0.$$

□

An immediately nice consequence of this can be defined if we try to relate inertial frames to accelerating particles:

Definition 79

For an accelerating particle, let a **momentarily comoving rest frame** (or **MCRF** for short) to be one where the particle is not moving at a particular instant in time.

Then we know that if $\vec{a}$ denotes the three-acceleration, we can say in the MCRF

$$\mathbf{U} = (1, 0, 0, 0) \implies \mathbf{A} = (0, a^x, a^y, a^z).$$

In general, these two four-vectors are always orthogonal.

Definition 80

Define the **four-force**

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} = m\frac{d\mathbf{U}}{dt} = m\mathbf{A}.$$

We should be careful here: it is **not true** that $\vec{F} = m\vec{a}$ as three-vectors! One problem now is that if our velocity changes, γ will change, so taking derivatives will also need to involve γ from now on.

So how exactly is **f** related to our ordinary $\vec{F} = \frac{d\vec{p}}{dt}$? Let's write out the components in a generic frame: by definition, the spatial components

$$f^i = \frac{dp^i}{d\tau} = \frac{dt}{d\tau}\frac{dp^i}{dt} = \gamma \vec{F}^i.$$

So just like with the momentum, we just get an extra factor of γ in our equations! Meanwhile, note that

$$f^0 = \frac{dp^0}{d\tau} = \frac{dE}{d\tau} = \frac{d\sqrt{p^2 + m^2}}{d\tau}$$

(all of this is omitting the $c = 1$ factors). Calculating out the derivatives yields

$$f^0 = \frac{\vec{p} \cdot d\vec{p}}{E d\tau}.$$

Now we know how to find $\frac{d\vec{p}}{d\tau}$ from our discussion above, and $\frac{\vec{p}}{E} = \vec{u}$, the velocity of our particle (since $\vec{p} = \gamma m \vec{u}$ and $E = \gamma m$). Thus,

$$f^0 = \vec{u} \cdot \gamma \vec{F} = \boxed{\gamma \vec{u} \cdot \vec{F}}.$$

We can think of this as a kind of "power!" Putting everything together, we get the ugly expression

$$\mathbf{f} \underset{S}{\rightarrow} \gamma(u)\left(\vec{u} \cdot \vec{F}, \vec{F}\right).$$

Of course, we can use the normal Lorentz transformations from now if we want to change frame. One nice thing, though, is that if we calculate this in the momentarily comoving rest frame, $\gamma = 1$ and $\vec{u} = 0$, so

$$\mathbf{f} \xrightarrow[\text{MCRF}]{} (0, \vec{F}_{\text{MCRF}}).$$

Since we found that $\mathbf{A}$ is $(0, \vec{a})$ in this rest frame as well, we have found that

$$\vec{F} = m\vec{a}$$

only in the MCRF! In particular, if we change frames, the Newtonian force $\vec{F}$ looks different:

$$f^{\mu'} = \Lambda^{\mu'}{}_{\nu} f^{\nu},$$

so

$$\mathbf{f} \underset{S'}{\rightarrow} (f^{0'}, \gamma(u)\gamma(v)(F^x - v\vec{u} \cdot \vec{F}), \gamma(u)F^y, \gamma(u)F^z).$$

This is almost what we want: this doesn't directly give us the new force, but we know that in general

$$f^{x'} = \gamma(u')\vec{F}^{x'}.$$

Equating these two expressions,

$$(\vec{F})^{x'} = \frac{\gamma(v)\gamma(u)}{\gamma(u')}(F^x - v\vec{u} \cdot \vec{F}),$$

and now if we look back at last lecture's note to see how to write $\gamma(u')$ in terms of the new velocities, this yields

$$(\vec{F})^{x'} = \frac{F^x - v\vec{u} \cdot \vec{F}}{1 - u^x v}.$$

This is how the x-component of the three-force converts: we can also find that

$$\vec{F}^{y'} = \frac{F^y}{\gamma(v)(1 - u^x v)}, \vec{F}^{z'} = \frac{F^z}{\gamma(v)(1 - u^x v)}.$$

If we think back to electromagnetism and look at the Lorentz force

$$\vec{F} = q(\vec{u} \times \vec{B} + \vec{E}),$$

we can try calculating the "power"

$$\vec{F} \cdot \vec{u} = q\vec{u} \cdot \vec{E}.$$

The magnetic field doesn't do any work: it only makes the particles move in circles. However, note that when we change frames in this new context, $\vec{u} \cdot \vec{F}$ will change! So a magnetic field exists when we have moving charged particles, but we can always change frame so that any given charge isn't moving. What's going on here with the forces?

Basically, just like space and time are related, and energy and momentum are related, magnetic and electric fields are related too. However, we cannot put $\vec{E}$ and $\vec{B}$ in a four-vector because there are six components: we'll see in due time how to use a tensor to deal with them!

16 October 1, 2019 (Recitation)

There's a midterm coming up, so we should mention in email if we have any confusion!

Problem 81

Consider a photon γ that breaks up into a positron e_+ and an electron e_-. This is often known as **pair production.** Is this process allowed?

Solution. We should usually consider conservation of four-momentum, as well as whether there are any useful invariants in our problem. First of all, four-momentum tells us that

$$q^\mu = p_-^\mu + p_+^\mu :$$

squaring both sides will give us some Lorentz scalars, which gives us some invariants:

$$q^\mu q_\mu = (p_-^\mu + p_+^\mu)(p_{-\mu} + p_{+\mu})$$

Since a photon has zero mass, the left hand side is zero, which means that

$$0 = p_-^\mu p_{-\mu} + p_+^\mu p_{-\mu} + p_-^\mu p_{+\mu} + p_+^\mu p_{+\mu}.$$

The first and last terms are just the square of the four-momenta of the electron and positron, respectively. Empirically, we've found that both masses are equal, so this tells us that

$$2m_e^2 = p_+^\mu p_{-\mu} + p_-^\mu p_{+\mu} = \left(p_+^\mu p_-^\nu + p_-^\mu p_+^\nu\right)\eta_{\mu\nu},$$

and now rewriting our dummy indices (and using that the metric is symmetric) means that these two terms are actually the same. Thus,

$$m_e^2 = \eta_{\mu\nu} p_+^\mu p_-^\nu = p_+^\mu p_{-\mu}.$$

Remark 82. *By the way, we can always rearrange things like*

$$p_+^\mu p_-^\nu = p_-^\nu p_+^\mu,$$

as long as these components are all scalars! *However, if p is (for example) a gradient whose components are* *operators, then the order does matter.*

So is this last expression possible? Notice that it's a Lorentz scalar, so we can just go into whatever frame we want to! Let's pick the rest frame S of the electron: then

$$p_- \underset{S}{\rightarrow} (m_e, 0, 0, 0).$$

On the other hand, the momentum of the positron is

$$p_+ \underset{S}{\rightarrow} (\gamma m_e, p_+^x, p_+^y, p_+^z),$$

where we don't actually need anything other than the first component. Contraction then yields (don't forget the negative sign because of the upper versus lower index)

$$p_+^\mu p_{-\mu} = -\gamma m_e^2 + 0 + 0 + 0.$$

So

$$m_e^2 = -\gamma m_e^2,$$

which is not possible! Thus, the pair production process is not allowed. □

Pair production **can** be made legal, though, if we have a strong electromagnetic field or gravitational field which modifies our four-momenta. Basically, we can't just work in a vacuum of free space with no other interactions!

Problem 83 (Compton scattering)
Say a ray of light (in other words, a photon) comes to shine on a electron, and the photon and electron recoil. Compute the shift in wavelength of the light wave.

Say that the photon comes in horizontally, and the electron and photon recoil at angles θ and ϕ from the horizontal. Work on the lab frame (in other words, the frame where the electron is at rest before the collision). Before the collision, the energy of the photon is $E_1 = \frac{h}{\lambda_1} = hf_1$, and the energy of the electron is just its rest mass m_e (remember that we're using $c = 1$ here). Meanwhile, the energy of the photon after the collision is $E_2 = \frac{h}{\lambda_2} = hf_2$: we need the energy of the electron afterward.

Well, let's use conservation of four-momentum: first of all, let 1 denote before-collision and 2 denote after-collision. The four-momentum of the photon before collision is

$$\mathbf{p}_{1\gamma} = (E_1, p_{1\gamma}^x, 0, 0) = \boxed{(E_1, E_1, 0, 0)}$$

because the square of the four-momentum of a photon is zero, so $0 = E^2 - (p_{1\gamma}^x)^2$. The four-momentum of the electron is just

$$\mathbf{p}_{1e} = \boxed{(m, 0, 0, 0)}.$$

The four-momentum of the photon after the collision is still zero, so we can write it as

$$\mathbf{p}_{2\gamma} = (E_2, p_{2\gamma}^x, p_{2\gamma}^y, 0)$$

where $E_2^2 = [(p_{2\gamma})^x]^2 + [(p_{2\gamma})^y]^2$. Since $p_x = p\cos\phi$ and $p_y = p\sin\phi$ for some p, we can actually write this as

$$\mathbf{p}_{2\gamma} = \boxed{(E_2, E_2\cos\phi, E_2\sin\phi, 0)}$$

so that the dot product of $\mathbf{p}_{2\gamma}$ with itself is still zero. Similarly, we can write

$$\mathbf{p}_{2e} = \boxed{(E, p\cos\theta, p\sin\theta, 0)},$$

for some unknown energy and momentum E, p, and now we can start working with invariants and conservation laws! First of all, looking at conservation of momentum component by component yields

- $m_e + E_1 = E + E_2$
- $E_1 = p\cos\theta + E_2\cos\phi$
- $p\sin\theta = E_2\sin\phi$.

Doing a lot of algebra, the easiest way to simplify this is to square both sides: squaring the second of these equations yields

$$(E_1 - E_2\cos\phi)^2 = p^2\cos^2\theta,$$

and squaring the third yields

$$E_2\sin^2\phi = p^2\sin^2\theta,$$

so adding them gives us

$$(E_1 - E_2\cos\phi)^2 + E_2^2\sin^2\phi = p^2.$$

Using the invariant

$$p_{2e}^{\mu}p_{2e}^{\nu}\eta_{\mu\nu} = -m_e^2,$$

we find that

$$-E^2 + p^2 = -m_e^2 \implies p^2 = E^2 - m_e^2$$

Plugging this in, we now know E in terms of the other variables, and we also know that $E = (E_1 - E_2 - m_e)$. After a lot of algebra, we'll find that

$$\frac{1}{E_1} - \frac{1}{E_2} = \frac{1-\cos\phi}{m_e} > 0 \implies \lambda_1 - \lambda_2 = \frac{h}{m_e}(1-\cos\phi).$$

Next time, we'll get around to how to work with natural units!

17 October 2, 2019

We'll continue thinking about accelerating reference frames today! From this point onward in the course, we'll use natural units: this means that time will be treated as a length and we take $c = 1$. (For example, we use a "light-year" instead of a "year.")

We studied the four-velocity $\mathbf{U} \underset{S}{\rightarrow} (\gamma(\vec{u}), \gamma(\vec{u})\vec{u})$ last time, which generalized to the four-acceleration $\mathbf{A} = \frac{d\mathbf{U}}{d\tau}$. Remember that last time, we found that

$$A^{\mu}A^{\nu}\eta_{\mu\nu} = A^{\mu}U_{\mu} = 0.$$

Since these are Lorentz scalars, this is valid in any reference frame! The problem we have is that as soon as we're undergoing an acceleration, one comoving frame will not work: today, we're going to work more with the idea of a momentary comoving rest frame (MCRF). We'll also work with consequences of a uniform acceleration, and we'll explore the famous twin paradox.

Remember that any object moving at constant velocity in a spacetime diagram will have a comoving frame: letting its trajectory in the S-frame spacetime diagram be the t'-axis in the S' frame gives us an object that just sits at $x' = 0$ forever. But a momentary comoving rest frame corresponds to a curvy line in our spacetime diagram: how are we supposed to work with this? Well, at every point along this trajectory, we just associate a different t' axis! Specifically, associate a **different frame $S(\tau)$ for every point of proper time in the moving object's rest frame.** Momentarily for each $S(\tau)$, we have an inertial frame, so we can use the Lorentz transformations at any given time.

In particular, given any momentarily comoving rest frame, we can calculate the four-velocity and four-acceleration

$$\mathbf{U} \xrightarrow[\text{MCRF}]{} (1, 0, 0, 0), \quad \mathbf{A} \xrightarrow[\text{MCRF}]{} (0, \vec{a}_0(\tau)),$$

and thus

$$A^\mu A_\mu = \eta_{\alpha\beta} A^\alpha B^\beta = |\vec{a}_0(\tau)|^2$$

is a constant across all frames for a given τ. Today, to make calculations easier, we'll assume that this acceleration a_0 does not depend on τ: it is uniform **in the rest frame**. We'll also assume that everything is being done along the x-axis: this means that we can write the acceleration

$$\mathbf{A} \xrightarrow[\text{MCRF}]{} (0, g, 0, 0),$$

where g is an acceleration that does not change with time. (Note that the g measured from the MCRF may not be the same as the g measured from a different reference frame, though!)

Example 84

Consider a rocket leaving earth at a constant acceleration g (from its perspective), and we can track the velocity and acceleration $\vec{u}(\tau), \vec{a}(\tau)$ measured from the earth. We'll assume for simplicity that the rocket is moving from the x-axis: **remember that τ is the time measured from inside the rocket**.

We'll say that the rocket's clock and the Earth's clock are synchronized at $t = \tau = 0$, and that the rocket starts at rest. We'll assume here that $x(\tau = 0) = \frac{1}{g}$: this is just to save chalk later.

If $\vec{u}(\tau = 0) = 0$, then the four-velocity $\mathbf{U} \xrightarrow[S]{} (1, 0)$ (we'll toss the y- and z- components). We need a way to calculate the velocity and acceleration of the rocket in some frame, and then we can Lorentz transform to get to the answer.

We know that $A_\mu U^\mu = 0$ and $U_\mu U^\mu = -1$ are satisfied in all reference frames at all times. We also know that if g is the rest acceleration, $A^\alpha A_\alpha = g^2$, and this is true in all frames. Thus,

$$-U^0 U^0 + U^1 U^1 = -1, \quad U^0 A^0 + U^1 A_1 = 0, \quad -A^0 A^0 + A^1 A^1 = g^2.$$

We can parametrize the solutions to the first equation here via

$$U^0 = \cosh(k\tau + \alpha), \quad U^1 = \sinh(k\tau + \alpha).$$

How do we find k and α? We'll use initial conditions: we know that the rocket started at rest, so using properties of the four-velocity,

$$\vec{u}^1 = \frac{U^1}{U^0} = \tanh(k\tau + \alpha).$$

If $\vec{u} = 0$ at $\tau = 0$, this implies that $\boxed{\alpha = 0}$. Similarly, if we want to find k, notice that if we take derivatives,

$$A^0 = \frac{dU^0}{d\tau} = k \sinh(k\tau),$$

and

$$A^1 = \frac{dU^1}{d\tau} = k \cosh(k\tau),$$

so plugging in the fact that

$$g^2 = -(A^0)^2 + (A^1)^2 = -k^2 \sinh^2 k\tau + k^2 \cosh^2 k\tau,$$

we find that $k^2 = g^2 \implies \boxed{k = g}$.

Thus, we've already found our answer:

Proposition 85

For a particle starting at rest and accelerating at a constant acceleration g,

$$\mathbf{U} \underset{S}{\rightarrow} (\cosh g\tau, \sinh g\tau), \quad \mathbf{A} \underset{S}{\rightarrow} (g \sinh g\tau, g \cosh g\tau).$$

What can we do with this? Remember that general, the component of the four-velocity in any frame is

$$\mathbf{U} \underset{S}{\rightarrow} (\gamma(\vec{u}), \gamma(\vec{u})u).$$

Thus from the earth's frame, $\gamma(\vec{u}) = \cosh(g\tau)$, meaning that $\boxed{u = \tanh(g\tau)}$. Let's verify this in another way: how far does the rocket go from the point of view of the Earth frame? In other words, what is $x(\tau)$? We know that the four-vector

$$\mathbf{x} \underset{S}{\rightarrow} (t(\tau), x(\tau)),$$

and because we defined $\mathbf{U} = \frac{d\mathbf{x}}{dt}$, we have

$$d\mathbf{x} = \mathbf{U}(\tau)d\tau.$$

This is actually two equations: $\mathbf{x}$ consists of a time-component and a space-component. Let's first look at the time-component of these vectors: we find that

$$\int dt = \int \cosh(g\tau)d\tau.$$

This actually tells us how to relate the time coordinates of the two frames: since we have nice initial conditions, this just directly reduces to

$$\boxed{t = \frac{1}{g}\sinh(g\tau)}.$$

It's good to check that $t = \tau$ in the limit where τ is very small (at the very beginning). Similarly, if we relate the space coordinates,

$$\int_0^x dx = \int_0^\tau \sinh(g\tau)d\tau \implies x(\tau) - x(0) = \frac{1}{g}(\cosh(g\tau) - \cosh(0)).$$

But we chose $x(0) = \frac{1}{g}$, so the boundary terms cancel out! This leaves us with the simple form

$$\boxed{x(\tau) = \frac{1}{g}\cosh(g\tau)},$$

and now we can calculate the position of the rocket at any given (proper) time as well.

Remember that we had a constant velocity time dilation of the form $dt = \gamma d\tau$ between two different inertial frames: does that work here, too? Taking differentials of the boxed t expression,

$$dt = \frac{1}{g}g\cosh(g\tau)d\tau = \cosh(g\tau)d\tau,$$

and $\cosh(g\tau)$ is exactly $\gamma(\vec{u})$! So we still have time dilation: the only real difference is that γ depends on the current time.

Remark 86. *As $\tau \to \infty$, $u(\tau) = \tanh(g\tau)$ will go to 1. In other words, from the point of view of the earth, the rocket can keep accelerating and it will never go faster than the speed of light!*

Why is this happening? One simple way to understand this is to calculate the work needed to put on the rocket to keep accelerating it at the same rate g. Remember that we defined the kinetic energy

$$K = (\gamma - 1)m = (\gamma(\tau) - 1)m.$$

Remember that as $\tau \to 0$ and the velocities are all small, the rocket behaves like a Newtonian object: $K \approx \frac{1}{2}mu^2$. But as τ gets very large, cosh diverges, so $\gamma - 1$ goes to infinity! So more and more energy is needed to keep accelerating it.

Remark 87. *Checking Newtonian limits for the velocity, for example, show very familiar results. We can also check that the position $\frac{1}{g}\cosh(g\tau)$ will look like $\frac{1}{g} + \frac{1}{2}g\tau^2$ by Taylor expansion when τ is very small!*

Example 88

Now say that we started with a different initial velocity, so the rocket does not start at rest. How can we find the position and velocity in that case?

Remember that we still have the general form

$$U^0 = \cosh(k\tau + \alpha), \quad U^1 = \sinh(k\tau + \alpha)$$

for our four-velocity, but if the starting velocity is some σ, then

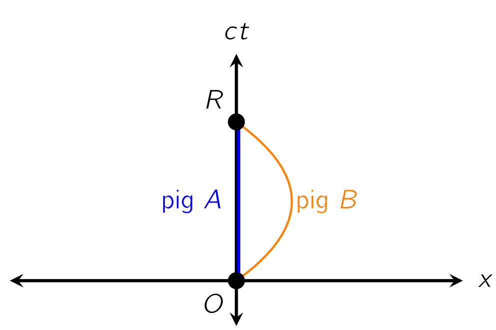
$$\vec{u}^1 = \tanh(\alpha) = \sigma \implies \alpha = \tanh^{-1}(\sigma).$$

Everything else stays exactly the same!

Problem 89 (Twin paradox)

Consider two identical pigs A and B. They are twin pigs, and they are put in an experiment. Pig A stays on the farm on Earth, and pig B is put in a rocket which leaves Earth for a while.

We'll put this in a spacetime diagram! Let's say that pig B actually experiences a constant acceleration as shown:



These two pigs are near each other at O and R, so they can compare themselves to each other at both times. We want to see which pig experiences more time between events O and R.

If we look at this naively, we may think that pig A will apply time dilation, but why can't pig B apply time dilation to pig A? The idea is that you get noninertial forces when you are in an accelerating reference frame! There's some loss of symmetry here. Let's try to show this mathematically: we'll show that pig A will be older when they meet at spot R.

Pig A on Earth measures some time t_R between events O and R, while pig B measures the proper time τ_R. Because pig B experiences a deceleration and has an initial velocity, we can write

$$u(\tau) = \tanh(-g\tau + \alpha)$$

for $\alpha = \tanh^{-1}(\sigma)$, where σ is the initial velocity of pig B. Plugging in $\tau = \tau_R$, since the velocity of the rocket is the negative of what pig B started with (by symmetry of the parabolic path),

$$-\sigma = \tanh(-g\tau_R + \alpha) \implies \tanh^{-1}(-\sigma) = -g\tau_R + \alpha.$$

Thus,

$$-g\tau_R + \alpha = -\tanh^{-1}(\sigma) = -\alpha \implies \boxed{\tau_R = \frac{2\alpha}{g}}.$$

This makes sense: for a fixed σ, a larger acceleration means that pig B gets back to its starting position faster.

But how long does this process take from the point of view of pig A? We just integrate, using the differential expression $dt = \gamma(\tau)d\tau$ we derived earlier:

$$t_R = \int_O^R dt = \int_O^R \cosh(-g\tau + \alpha)d\tau = -\frac{1}{g}\left(\sinh(-g\tau_R + \alpha) - \sinh(\alpha)\right).$$

Since $\tau_R = \frac{2\alpha}{g}$, this becomes

$$t_R = -\frac{1}{g}\left(\sinh(-\alpha) - \sinh(\alpha)\right) = \boxed{\frac{2}{g}\sinh(\alpha)}.$$

So which one is larger? Expanding $\sinh\alpha$ with a Taylor expansion,

$$t_R = \frac{2}{g}(\alpha + \frac{\alpha^3}{3} + \cdots),$$

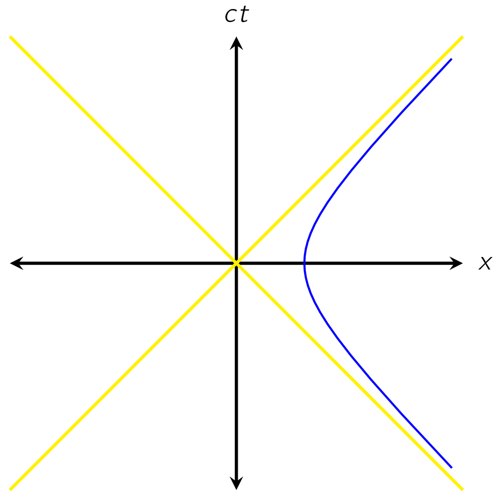
so indeed pig A, which stays on earth, will be older when they meet! This ratio can actually be very large if σ is large.

Remark 90. *If we've seen the movie Interstellar, this effect does play a role, but it is for a different reason which we might try to discuss later.*

Finally, how do we represent uniform acceleration in spacetime diagrams? We have the equations

$$\begin{cases} t(\tau) = \frac{1}{g}\sinh(g\tau) \\ x(\tau) = \frac{1}{g}\cosh(g\tau) \end{cases}$$

which are the parametric equations of the curve $x^2 - t^2 = \frac{1}{g^2}$. This is a hyperbola:



Divide this spacetime diagram into regions corresponding to the top, right, bottom, and left of the origin. If we are anywhere to the left of this origin, we will never be able to communicate with the rocket. But the top and bottom are more interesting: for example, the rocket can send information to anyone in the top part of the spacetime diagram, but the opposite is not true (and vice versa for the bottom part of the spacetime diagram)! This is similar to the idea of an **event horizon**: regions of spacetime where things can only enter but never leave. We'll see soon that there's actually a deep connection with gravity here!

18 October 3, 2019 (Recitation)

Let's start by talking a bit about natural units. When we set $c = 1$, we might get expressions like $E = m$: how do we figure out how to put c back in?

Well, we should think about the actual dimensions of our units: E has units of $\text{kg} \cdot \frac{\text{m}^2}{\text{s}^2}$, so we just need to put back the units of c^2. This gives us back $E = mc^2$.

Similarly, if we have an equation like $F = \frac{m}{d}$, we know that the units of force are $\text{kg} \cdot \frac{\text{m}}{\text{s}^2}$, while the units of $\frac{m}{d}$ are $\frac{\text{kg}}{\text{m}}$. Thus, we again just need a factor of $\frac{\text{m}^2}{\text{s}^2}$, so we just multiply in a factor of c^2. Central idea here: break into fundamental units and try to get the correct ratios!

There's a few quantities in nature, such as Newton's gravitational constant G_N, Planck's constant h, and the speed of light c, which we measure experimentally. So this natural unit idea is really a subset of the **fundamental (or Planck) units**: we set $\boxed{G_N = c = \hbar = \frac{1}{4\pi\varepsilon_0} = k_B = 1}$.

Can we really set all of these equal to 1 at once? It turns out that each constant contains some unit that doesn't exist in the others, so there's a unit system where all of these can be set to 1. It turns out that the way to do this is as follows: since c has units of length per time, $\hbar$ is energy times time, and so on, we can get expressions like the **Planck length**

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 10^{-35}\text{m},$$

the **Planck mass**,

$$m_P = \sqrt{\frac{\hbar c}{G_N}} \approx 10^{-8}\text{kg},$$

and the **Planck time**

$$t_P = \sqrt{\frac{\hbar G_N}{c^5}} \approx 10^{-44}\text{s}.$$

Then we can just define c as the ratio of the Planck length to the Planck time, and so on! What's great is that if an object of mass m_P moves a distance of ℓ_p in t_P, since this theory must involve $\hbar, G_N$, and c^3, you need a relativistic theory of quantum gravity!

So the takeaway is that setting $c = 1$ is really just using ratios of Planck units.

Fact 91 (Midterm stuff)

The following might be on the exam:

- Spacetime diagrams, light cones, past and future of a point
- Timelike and spacelike-separated events
- Lorentz transformations; addition of velocities
- Index notation, tensors
- Natural units
- Four-momentum (not four-acceleration)

Let's review some concepts that are requested! First of all, say that the four-velocity of an object in a rest frame is $\mathbf{U} = (c, 0, 0, 0)$. Looking at another rest frame, we have a four-momentum of $\gamma(c, \vec{v})$, so the time-like component of $\mathbf{U}$ seems to be larger than light. But didn't we say that nothing travels faster than light?

Well, remember that the four-velocity $\mathbf{v}_{\text{light}} = (1, 1, 0, 0)$ (so that the four-velocity dotted with itself is 0). What would a **superluminal** (faster than light) travel look like? It means that the slope in a spacetime diagram must be less than 1, so the component of the four-velocity in x-component must be larger than that in the t-component. Then this new velocity is $(a, b, 0, 0)$, where $b > a$, while the velocity of the ordinary object is $(a, b, 0, 0)$, where $a > b$.

But notice that

$$\mathbf{U}_{\text{ordinary}} \cdot \mathbf{U}_{\text{ordinary}} = -1 \implies -a^2 + b^2 = 1 \implies |a| > 1.$$

Really, all that's going on here is that we're moving faster than light through time, but not faster than light through space! (Light travels the same in space and time.) So moving slower than light means we move faster through time.

Next, let's do some problems involving collisions and decays. Remember that the **center of momentum frame** is the one where the sum of the (regular) three-momenta is equal to 0. This is **not necessarily the same as the center of mass**, which is just a weighted average of positions. The former here doesn't tell us anything about the origin!

Remark 92. *It's not always possible to write down a center of four-momentum (because there's an extra* 0*th component that we have to account for).*

Is there always a center of momentum frame, though? The answer is **no**: if we just have a single photon, the four-momentum always looks like something like $(E, E, 0, 0)$, so there's no center of three-momentum! (Remember that even though photons are massless, they can still have a finite nonzero four-momentum because γ is infinitely large.)

Problem 93

We have the decay of a pion into two photons: $\pi^0 \to \gamma_1 + \gamma_2$. Find the energies of γ_1 and γ_2 in the π^0's rest frame (which we also call the lab frame).

The pion's rest frame is a center of momentum frame:

$$\mathbf{p}_\pi = (m_\pi, 0, 0, 0).$$

By conservation of four-momentum, our final momenta will satisfy

$$\mathbf{p}_\pi = \mathbf{p}_{\gamma_1} + \mathbf{p}_{\gamma_2}.$$

In particular, we know that the sum of the spatial components is 0 after the collision as well. (Notice, for example, that this implies $\gamma_1 \vec{p}_1 = -\gamma_2 \vec{p}_2$ for the three-momenta.)

Remark 94. *A lot of discussion ensued about the differences between conservation and invariance here. The most important point is that conservation is something that relates a quantity before and after a process (for example, conservation of four-momentum), while invariances is some quantity, usually a Lorentz scalar, that is the same in all reference frames.*

So now let's dot the four-momenta with themselves:

$$(p_{\gamma_1})^\mu (p_{\gamma_1})_\mu = 0,$$

where we can write generically

$$\mathbf{P}_{\gamma_1} = (E_{\gamma_1}, P^x_{\gamma_1}, P^y_{\gamma_1}, P^z_{\gamma_1}).$$

We can write the last three coordinates in spherical coordinates as

$$(E_{\gamma_1}, P \sin\theta \cos\phi, P \sin\theta \sin\phi, P \cos\theta),$$

and then dotting this with itself gives 0, so

$$-E_{\gamma_1}^2 + P^2 = 0 \implies P = E_{\gamma_1}.$$

We can do the same thing with the second photon as well, and then remember that the spatial components add up to zero: thus, we can write

$$\mathbf{p}_{\gamma_2} = (E_{\gamma_2}, -P \sin\theta \cos\phi, -P \sin\theta \sin\phi, -P \cos\theta)$$

and since this also has norm zero, we find that

$$E_{\gamma_1}^2 = P^2 = E_{\gamma_2}^2.$$

So now we're almost done: looking at the zero component of the conservation equation,

$$m_\pi = E_{\gamma_1} + E_{\gamma_2} = 2E_{\gamma_1},$$

so the energy of each photon is just $\frac{1}{2} m_\pi$, and we're done!

To finish, we'll start talking a bit about four-acceleration. Most of what we've really been doing is **kinematics**, but physics is a predictive science: we want to be able to study **dynamics** instead, which means we want to be able to study forces. If we ever hope to predict how a system behaves under some kind of change, we need to be able to talk about acceleration.

We'll start this discussion by talking about uniform acceleration, and we'll give some intuition for that here. The general idea is that the object itself should feel like it has uniform acceleration. This is sort of important, because the acceleration doesn't quite look uniform from the outside point of view. One thing we can do is have a rocket moving, and an observer keeps dropping rocks outside the rocket every second from their point of view. Each rock corresponds to a momentary comoving rest frame: we can measure the velocity at each drop, and we say that the **proper acceleration** is constant if those velocities are changing uniformly.

Soon, we'll explore more about how to connect this to proper time and further kinematics!

19 October 9, 2019

Congratulations to all of us for surviving the midterm! We will know our grades by the end of the week.

In the next part of this class, we'll start looking into electromagnetism and how to make it consistent with relativity. We've been talking about how relativity unifies space and time, so that instead of dealing with t and $\vec{x}$ separately, we can write them together as a four-vector $\mathbf{x} \underset{S}{\rightarrow} (ct, \vec{x})$.

Similarly, we found that special relativity unifies energy and momentum, and we can write them together as a four-vector $\mathbf{p} \underset{S}{\rightarrow} (\frac{E}{c}, \vec{p})$. (Here, we add an extra γ factor to the momentum compared to what we expect in the Newtonian version.) From this, we could define the four-force, four-acceleration, and four-force: we find that $\mathbf{f} = m\mathbf{A}$ still holds in relativity.

Today, we'll start with a short review of tensors and index manipulation, building up to some useful tensor identities. Then we'll get into the basics of electromagnetism.

Recall that a **tensor** of rank $\begin{bmatrix} U \\ D \end{bmatrix}$ is a geometric object that, under coordinate transformations (like Lorentz transformations or other changes in frame) transform with U transformation matrices of the form $\Lambda^{\mu'}{}_{\nu}$ and D matrices of the form $\Lambda^{\alpha}{}_{\beta'}$.

Example 95

A $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ tensor's components can be described as $R^{\mu\nu\sigma}{}_{\rho}$ in some frame S. Then, if we want to find the components in some other frame,

$$R^{\alpha'\beta'\gamma'}{}_{\delta'} = \Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}\Lambda^{\gamma'}{}_{\sigma}\Lambda^{\rho}{}_{\delta'}R^{\mu\nu\sigma}{}_{\rho}.$$

(We get the transformation matrices by just matching up indices: free indices on one end need to be free on the other, and we need to make sure everything else cancels out by contraction.)

Fact 96

Not everything with indices is a tensor, though! For example, professor V has two indices and is not a tensor.

Let's look at some examples of tensors that we've already encountered!

- Scalars are $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ tensors: they are invariant under transformations.
- Four-vectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ tensors, which transform with one Lorentz matrix.
- Covariant vectors or one-forms are $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ tensors, which transform with the inverse transformation.
- The metric is a $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ tensor: it requires two inverse transformation matrices. (We can't just multiply transformations out as a matrix multiplication though!)

One thing to check for fun: we can take two $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ tensors and multiply them to get an object $A^{\mu}B^{\nu} = R^{\mu\nu}$, which is a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ tensor. (In general, we can multiply two tensors with independent (different) coordinates, and their ranks will add.)

We've also discussed the idea of lowering and raising indices: if we have a tensor of any rank, we can bring an index down or up using a metric $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$. For example,

$$T^{\alpha\beta}\eta_{\beta\gamma} = T^{\alpha}{}_{\gamma},$$

so in this case, a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ times a $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ tensor actually gives a $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ tensor. Similarly,

$$A^{\mu}{}_{\lambda\gamma}\eta^{\nu\lambda} = A^{\mu\nu}{}_{\gamma}.$$

We can always do multiple of these lowerings/raisings in succession. It's important to stress that lowering and raising does involve contraction, which is why we can't just add the ranks of the tensors to get the final rank! Also, we can contract over the same object: for example,

$$T^{\alpha\beta}\eta_{\alpha\beta} = T^{\alpha}{}_{\alpha}.$$

This is then a scalar, since we just have four terms in our sum, and there are no free indices. This scalar has a special name: since we're summing the terms with the same index, it's called the **trace**.

Remark 97. *Note that the tensors $T^{\mu}{}_{\nu}$, $T_{\mu\nu}$, $T^{\mu\nu}$, and so on are not the same! We can find the components of one of these from another by raising and lowering, but it's the same idea as how A^{μ} and A_{μ} do not have the same components (because the metric has a -1). But because these can be simply related to each other, we use the same letter for all of them.*

If we have an object $T^{\alpha\beta}$, specifying the object usually requires 16 different values, corresponding to each $0 \leq \alpha, \beta \leq 3$. But there might be some symmetries that constrain this! For example, $\eta_{\mu\nu}$ is symmetric, because

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \implies A^{\mu}B^{\nu}\eta_{\mu\nu} = B^{\mu}A^{\nu}\eta_{\mu\nu}$$

and then this tells us that $\eta_{\mu\nu} = \eta_{\nu\mu}$.

Definition 98

A tensor is **symmetric** in two coordinates α, β if we can swap the order of those two indices and still get the same number. A tensor is **antisymmetric** if swapping those two indices gives us an extra negative sign.

The reason this is important is that we need to specify less indices: for example, in a 2×2 matrix, we might force ourselves to satisfy $A^{01} = A^{10}$, so we only need three numbers to describe what's going on. It's good for us to prove that symmetric things stay symmetric under changes of frames! As a note, it is good to understand why in an anti-symmetric tensor, we have $C^{\mu\mu} = 0$ for all μ (in particular, this means we only need one number to describe it in our 2×2 matrix).

So the number of independent components changes if our tensors have special symmetry. One case that is relevant for us today is objects with two indices: it takes $16, 10, 6$ numbers to specify a tensor $T^{\mu\nu}$ with no symmetries, symmetry, and anti-symmetry, respectively.

So let's try to use these to write some tensorial identities: for example, what else can we do to relate our tensors beyond $f^{\mu} = mA^{\mu}$? The idea is that we want **all of our physical laws to be written in terms of tensors**, so that laws remain true. For example, transforming our coordinates by multiplying by a Λ matrix, we'd find that $f^{\nu'} = mA^{\nu'}$. So to enforce relativity, we must be able to transform our coordinates in a way that relates two tensors of the same type! As a small detail, remember that whenever we say two tensors are equal, we're just saying that all of the components are equal.

How can we turn statements about curl, divergence, and current density into statements with tensors? Well, if we did do a Lorentz transformation on Maxwell's equations, we'd indeed find that they do take the same form – it is just painful to calculate by brute force. One small tragic historical fact: we often say that the laws of physics are **covariant**, meaning that they don't change shape or form under a change of frame, which is an unfortunate conflict with the idea of a covariant vector.

So let's look at statements like the Lorentz force

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}).$$

We say here that a charged particle will feel a force that depends on the velocity of the charged particle. But this is a problem:

Example 99

If we think about a wire with a current going through it, and we think of a charge q moving with some velocity $\vec{u}$, then the Lorentz force tells us that there will a magnetic field acting on the charge (meaning $\vec{u} \times \vec{B} \neq 0$), but $\vec{E} = 0$ (because a wire is neutrally charged). In other words, the charge will indeed feel a force. But if we look at this situation in the rest frame of the charge, $\vec{u}' \times \vec{B}' = 0$ now, but that means that $\vec{E}'$ must be nonzero now, because the force being nonzero does depend on the reference frame!

So this means that $\vec{B}$ and $\vec{E}$ do depend on each other in some way, and we can't keep them completely separate. It's natural here to say that $\vec{E}$ and $\vec{B}$ should be put into a vector, just like $\vec{x}$ and t or $\vec{p}$ and E. But we have six components here (the x, y, z components of both $\vec{E}$ and $\vec{B}$), so if they belong to the same geometric object, that object must accommodate six numbers, and a scalar, four-vector, and one-form cannot do that.

So the next best thing is, perhaps, a $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ tensor. Remember that such an antisymmetric tensor only contains six independent quantities? It turns out that this is indeed what we want, and we can encode this in the antisymmetric **Faraday tensor** $F^{\mu\nu} = -F^{\nu\mu}$.

Let's try to figure that its components, and we'll start doing this by generalizing our Lorentz force. We want to find a four-dimensional version of this force, so let's replace each piece with objects that live in a four-dimensional space. The force is easy: we can associate that with the four-force f^μ. The charge is a scalar, but how do we account for the other terms?

Well, if $\vec{E}$ and $\vec{B}$ are part of a tensor, it makes sense to put that in. But we need the ranks to match up, so we need to contract one of our upper indices.

Well, we know that the four-velocity is also an important part of this equation, so maybe it makes sense to put that in. This motivates us to write

$$\boxed{f^\mu = qF^{\mu\nu}U_\nu},$$

and at least formally this has all of the pieces that we want! (Remember that U_ν is defined as $\eta_{\mu\nu}U^\mu$ here.)

Remark 100. *Notice that we contracted the second index ν of the Faraday tensor instead of the first. This is mostly a matter of convention, but we need to make sure this stays consistent to avoid random negative signs!*

To make this actually useful, we must try to find the components of the tensor.

Example 101

If we have a charge at rest under an electric and magnetic field, we should just have $\vec{F} = q\vec{E}$. Let's try to relate the tensorial identity above to this equation.

Well, the four-velocity in our rest frame is $\mathbf{U} \xrightarrow[\text{rest}]{} (c, 0, 0, 0)$ (we're putting the c back for today and next time just to make things simpler). Since we want the lower-index version, we have to contract to find that

$$U_\mu = (-c, 0, 0, 0).$$

Looking next at the four-force, note that

$$f^\mu = \frac{dp^\mu}{d\tau} = \frac{dp^\mu}{dt}$$

(because the particle isn't moving, $d\tau = dt$). Thus, if we plug these in, noting that the only component that is nonzero is the $\nu = 0$ component of the velocity,

$$f^\mu = \frac{dp^\mu}{dt} = q\left(F^{\mu 0}U_0\right) = -cqF^{\mu 0}.$$

We already know that $F^{00} = 0$, because F is an antisymmetric tensor. Taking $\mu = 1$,

$$\frac{dp^1}{dt} = \left(\vec{F}\right)^x = -cqF^{10},$$

where $\vec{F}^x = q\vec{E}_x$ is the Newtonian force. Thus,

$$q\vec{E}_x = -cqF^{10} \implies \boxed{F^{10} = -\frac{E^x}{c}}.$$

This also tells us that $F^{20} = -\frac{E^y}{c}$, $F^{30} = -\frac{E^z}{c}$, so we've now already determined half of the nonzero coordinates of our tensor! (We also get F^{01}, F^{02}, F^{03} by antisymmetry.)

To get the rest, let's look at another simple situation:

Example 102

Say we have a charge moving at constant velocity $\vec{u}$ with forces from both $\vec{E}$ and $\vec{B}$.

We now have

$$\vec{F} = q(\vec{E} + \vec{u} \times \vec{B}),$$

and the four-velocity now looks like

$$U^\mu = \gamma(u)(c, \vec{u}) \implies U_\mu = \gamma(u)(-c, \vec{u}).$$

In this case, we also get a factor due to time dilation: since $dt = \gamma d\tau$,

$$f^\mu = \frac{dp^\mu}{d\tau} = \gamma\frac{dp^\mu}{dt},$$

which means that

$$f^\mu = qF^{\mu 0}U_0 + F^{\mu i}U_i.$$

Taking $\mu = 1$,

$$\gamma\frac{dp^1}{dt} = q(F^{10}U_0 + F^{11}U_1 + F^{12}U_2 + F^{13}U_3) = q\gamma\left(-\frac{E^x}{c} \cdot -c + 0 + F^{12}u^y + F^{13}u^z\right),$$

so simplifying, we find that

$$\frac{dp^1}{dt} = \vec{F}^1 = q(E^x + u^y B^z - u^z B^y) = q(E^x + F^{12}u^y + F^{13}u^z).$$

This means that matching up the components, we must have

$$B^z = F^{12}, -B^y = F^{13},$$

and repeating for $\mu = 2$ gives us $F^{23} = B^x$

Putting this all together,

$$\boxed{F^{\mu\nu} = \begin{bmatrix} 0 & \frac{E^x}{c} & \frac{E^y}{c} & \frac{E^z}{c} \\ -\frac{E^x}{c} & 0 & B^z & -B^y \\ -\frac{E^y}{c} & -B^z & 0 & B^x \\ -\frac{E^z}{c} & B^y & -B^x & 0 \end{bmatrix}}.$$

and indeed, this contains all the degrees of freedom of $\vec{E}$ and $\vec{B}$.

If we're in the S frame, we'd call the first row the "electric field" and the rest the magnetic field. But if we're in an S' frame, because we have a tensor, we have to transform this via

$$F^{\mu'\nu'} = \Lambda^{\mu'}{}_{\alpha}\Lambda^{\nu'}{}_{\beta}F^{\alpha\beta}.$$

So the electric field that someone else measures is a combination of both our electric and magnetic field! Next time, we'll use this to make Maxwell's equations into tensorial identities.

20 October 10, 2019 (Recitation)

We'll be doing some practice problems with tensors today, primarily with index manipulation.

Start with a tensor $T_{\mu\nu}$: this means that it is a rank $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ tensor, so it takes in two vectors and returns a scalar.

Definition 103

A tensor $S_{\mu\nu}$ is **symmetric** if $S_{\mu\nu} = S_{\nu\mu}$ and **anti-symmetric** if $S_{\mu\nu} = -S_{\nu\mu}$.

Problem 104

Can we construct a symmetric tensor from $T_{\mu\nu}$?

One thing that we can consider is combining the tensor with its transpose (for example with multiplication). The transpose of $T_{\mu\nu}$ (when we think of it as a matrix) is $T^{\nu\mu}$, so

$$T_{\mu\nu}T^{\nu\mu}$$

is a scalar! So this isn't exactly the tensor that we want.

Instead, we can consider

$$T_{(\mu\nu)} = \frac{1}{2}\left(T_{\mu\nu} + T_{\nu\mu}\right)$$

and this is known as the **symmetrization** of T. We can symmetrize any two indices in a general tensor too: for example,

$$T_{\lambda(\mu\nu)} = \frac{1}{2}\left(T_{\lambda\mu\nu} + T_{\lambda\nu\mu}\right),$$

and we can also consider the totally symmetric tensor $T_{(\lambda\mu\nu)}$.

Another thing that we often like to do is to separate out the "trace" component of the tensor $T_{(\mu\nu)}$: for example,

$$T_{\mu\nu} = \text{symmetric traceless} + \text{trace} \cdot I.$$

(We do this because certain tensors have properties that act simply on, for example, the traceless part of a tensor.) Remember that

$$\text{Tr}[T] = \eta^{\alpha\beta} T_{\alpha\beta},$$

so we can just take the traceless part

$$\tilde{T}_{(\mu'\nu')} = T_{(\mu\nu)} - \frac{1}{4}\eta^{\alpha\beta} T_{\alpha\beta}\delta_{\mu\nu}.$$

This is because $T_{(\mu\nu)}$ has the same trace as $T_{\mu\nu}$ (because we take half of each $T_{\mu\nu}$ and $T_{\nu\mu}$), and then we subtract $\frac{1}{4}$ of the trace for each diagonal entry (and there are four such diagonal entries).

On the other hand, what if we want to take the component of $T_{\mu\nu}$ which is antisymmetric? Well, consider

$$T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu}):$$

it's antisymmetric because we get a negative sign by flipping μ and ν. (Analogously, this is called the **antisymmetrization**.) Notice that this is traceless (because $\eta^{\mu\nu}T_{[\mu\nu]} = \frac{1}{2}(\eta^{\mu\nu}T_{\mu\nu} - \eta^{\mu\nu}T_{\mu\nu})$, which vanishes), and thus **we can decompose**

$$\boxed{T_{\mu\nu} = T_{[\mu\nu]} + T_{(\mu\nu)} = T_{[\mu\nu]} + \tilde{T}_{(\mu\nu)} + \frac{1}{4}\eta^{\alpha\beta}T_{\alpha\beta}\delta_{\mu\nu}}$$

into its antisymmetric, symmetric traceless, and trace components. This is really helpful when we're manipulating tensors! Also, these components often correspond to different effects: for example, if this tensor is the Ricci curvature, then each component will correspond to a different type of behavior. The trace component tells us how light rays spread out, the antisymmetric part corresponds to distortion, and the symmetrization corresponds to shear.

So connecting this to what we covered in class, consider the field strength tensor of electromagnetism (or **Faraday tensor**) $F_{\mu\nu}$. This is anti-symmetric: in particular, if we contract it with a symmetric tensor, we get 0.

Remark 105. *Note that if $F_{\mu\nu}$ is antisymmetric, so is $F^{\mu\nu}$, because*

$$F^{\mu\nu}\eta_{\mu\alpha}\eta_{\nu\beta} = F_{\alpha\beta},$$

and swapping the μ and ν gives the same thing as swapping the α and β.

So let's put our tensor skills to the test and write Maxwell's equations in tensor form! Here they are again once we set $c = 1$:

- $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$
- $\vec{\nabla} \cdot \vec{B} = 0$
- $\vec{\nabla} \cdot \vec{E} = \rho$
- $\vec{\nabla} \times \vec{B} = \vec{j} + \frac{\partial \vec{E}}{\partial t}$

Remember that we can write down E and B in terms of potentials as well: for some vector potential $\vec{A}$ and scalar potential ϕ, we have

$$\vec{E} = -\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi,$$
$$\vec{B} = \vec{\nabla} \times \vec{A}.$$

We should also know about **gauge invariance**: if we make the transformation

$$\begin{cases} \phi' = \phi - \frac{\partial \varepsilon}{\partial t} \\ \vec{A'} = \vec{A} + \vec{\nabla}\varepsilon, \end{cases}$$

then the vector potential effectively doesn't change. So let's look at the field tensor

$$F = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}$$

If we're looking for a tensor at the end of the day, we want to be able to turn the three-vector potential into a four-vector, so we can deal with it in full four-dimensional spacetime. A somewhat naive thing to do is just to define a four-vector

$$\mathbf{A} = (\phi, A_x, A_y, A_z) :$$

it turns out this actually works out really nicely: we find that

$$\boxed{F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu}.$$

This means we've tied together the potential functions for $\vec{E}$ and $\vec{B}$! We can now see that $F_{\mu\nu}$ is indeed gauge invariant: doing the transformations we described above doesn't actually change F at all, so our physics stays constant.

Let's write that out. If our new potential four-vector satisfies

$$A'_\mu = A_\mu + \partial_\mu \varepsilon,$$

then

$$F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu = \partial_\mu (A_\nu + \partial_\nu \varepsilon) - \partial_\nu (A_\mu + \partial_\mu \varepsilon).$$

This simplifies to

$$F_{\mu\nu} - \partial_\mu \partial_\nu \varepsilon - \partial_\mu \partial_\nu \varepsilon = F_{\mu\nu}$$

because partial derivatives commute, and thus we've shown that F remains unchanged!

So let's look at the Maxwell equations. Consider the (somewhat arbitrary looking combination)

$$T_{\lambda\mu\nu} = \partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu}.$$

Plugging in the definition, this simplifies to

$$\partial_\lambda(\partial_\mu A_\nu - \partial_\nu A_\mu) + \partial_\mu(\partial_\nu A_\lambda - \partial_\lambda A_\nu) + \partial_\nu(\partial_\lambda A_\mu - \partial_\mu A_\lambda).$$

But then this simplifies by commutativity of partial derivatives: in fact, everything cancels out to 0. So $T_{\lambda\mu\nu} = 0$. (Notice that this is an **identity**: it does not actually depend on the properties of F or any other physics.) Plugging in some values of λ, μ, ν, this will now give us some interesting physics! if we take $\lambda = 0, \mu = 1, \nu = 2$,

$$0 = \partial_0 F_{12} \partial_1 F_{20} + \partial_2 F_{01} = \frac{\partial B_z}{\partial t} + \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y},$$

which can be rewritten as

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = -\frac{\partial B_z}{\partial t}.$$

This is just the z-component of one of Maxwell's equations, and we've found this by just writing out the four-potential and using tensor manipulations. We can verify that using some other components gives different components of all of the unsourced Maxwell's equations, which is a cool way to derive things!

Remark 106. *In higher dimensions, the cross product is not defined, so we can't use our normal Maxwell's equations in higher dimensions. But the point is that tensor form doesn't need dimensions at all!*

We'll finish by writing down the "sourced" Maxwell's equations: consider the vector

$$\mathbf{j} = (\rho, j^1, j^2, j^3).$$

Then writing down

$$\partial_\nu F^{\mu\nu} = j^\mu$$

actually gives us the sourced Maxwell equations, and these are actual **dynamics** (it's not just a mathematical fact)! But it's definitely easier to write down and remember than in the form that we're used to from 8.02.

21 October 16, 2019

Today, we'll finish making our Maxwell's equations explicitly tensorial in nature. As we mentioned last time, this makes it easier for us to change coordinates! Last time, we wrote down the classical Lorentz force on a charged particle $\vec{F} = q(\vec{E} + \vec{u} \times \vec{B})$ in terms of four-forces:

$$f^\mu = qF^{\mu\nu} U_\nu,$$

where the four-velocity U_ν is contracted with the second index. Here, F is the Faraday tensor

$$F^{\mu\nu} = \begin{bmatrix} 0 & E^x/c & E^y/c & E^z/c \\ & 0 & B^z & -B^y \\ & & 0 & B^x \\ & & & 0 \end{bmatrix},$$

and we can fill in the rest of the indices by noticing that F is antisymmetric. We'll continue working with this, and what's left to do is to consider a few questions:

- How do different observers see the electric and magnetic field $\vec{E}$ and $\vec{B}$?
- How do we deal with densities (e.g. charge density)?
- Maxwell's equations.

We've seen that in a frame S, we can introduce the Faraday tensor shown above, and this can encode the electric and magnetic field in our frame. But in some other inertial frame S', the components will change! Notably, because we have a rank $(2,0)$ tensor, we can relate this via

$$F^{\mu'\nu'} = \begin{bmatrix} 0 & E^{x'}/c & E^{y'}/c & E^{z'}/c \\ & 0 & B^{z'} & -B^{y'} \\ & & 0 & B^{x'} \\ & & & 0 \end{bmatrix} = \Lambda^{\mu'}{}_{\sigma}\Lambda^{\nu'}{}_{\rho}F^{\sigma\rho}.$$

Example 107

Consider the transformation where $\vec{v}$ is parallel to the x-axis: then

$$\Lambda^{\mu'}{}_{\nu} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then we can, for example, find the x-component of the electric field under a transformation:

$$F^{0'1'} = \frac{E^{x'}}{c} = \Lambda^{0'}{}_{\rho}\Lambda^{1'}{}_{\sigma}F^{\rho\sigma}.$$

This has 16 terms a priori, but because Λ is pretty simple, only a few of these terms contribute (where ρ, σ are either 0 or 1, and $\rho \neq \sigma$). This adds up to

$$\Lambda^{0'}{}_{0}\Lambda^{1'}{}_{1}F^{01} + \Lambda^{0'}{}_{1}\Lambda^{1'}{}_{0}F^{10} = \gamma^2\frac{E^x}{c} + (\beta^2\gamma^2)\cdot\left(-\frac{E^x}{c}\right) = \frac{E^x}{c}\gamma^2(1-\beta^2) = \frac{E^x}{c}.$$

So actually, if we boost along x, the electric field along the x-direction doesn't change! Let's try another component, then:

$$F^{0'2'} = \frac{E^{y'}}{c} = \Lambda^{0'}{}_{\rho}\Lambda^{2'}{}_{\sigma}F^{\rho\sigma}$$

only has nonzero terms when $\sigma = 2$ and $\rho \neq \sigma$, which ends up giving

$$\Lambda^{0'}{}_{0}\Lambda^{2'}{}_{2}F^{02} + \Lambda^{0'}{}_{1}\Lambda^{2'}{}_{2}F^{12} = \gamma\cdot\frac{E^y}{c} - \beta\gamma B^z = \gamma\left(\frac{E^y}{c} - \frac{v}{c}B^z\right)$$

So

$$\frac{E^{y'}}{c} = \gamma\left(\frac{E^y}{c} - \frac{v}{c}B^z\right) \implies \boxed{E^{y'} = \gamma(E^y - vB^z)}:$$

the $E^{y'}$ electric field in the S'-frame depends on both the electric and the magnetic field in the S-frame!

We won't do the rest of the calculations out, but we can just go ahead and bash: we'll find in general that for an arbitrary $\vec{v}$,

$$\begin{cases} \vec{E}'_{//} = \vec{E}_{//} \\ \vec{E}'_{\perp} = \gamma(\vec{v})\left(\vec{E}_{\perp} + \vec{v}\times\vec{B}\right) \\ \vec{B}'_{//} = \vec{B}_{//} \\ \vec{B}'_{\perp} = \gamma(\vec{v})\left(\vec{B}_{\perp} - \frac{\vec{v}\times\vec{E}}{c^2}\right) \end{cases}$$

So the key thing to remember is that **components along the direction of boost do not change, while the others do**.

Next, in preparation for Maxwell's equations, we need to figure out how to deal with densities. Often, we write (in differential form) Maxwell's equations as

- $\vec{\nabla} \cdot \vec{E} = \rho/\varepsilon_0$
- $\vec{\nabla} \times \vec{B} = \mu_0\varepsilon_0 \frac{\partial \vec{E}}{\partial t} + \mu_0 \vec{J}$
- $\vec{\nabla} \cdot \vec{B} = 0$
- $\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}$.

Obviously, turning this into a statement about densities isn't so easy: we want to get it to look like $d\rho = \frac{dq}{dV}$ or $d\vec{J} = \frac{dQ}{dAdt}\hat{n}$.

In order to work towards that, let's step back a bit: say we have a set of particles like dust, which are noninteracting and at rest. For any small box of volume V, if there are N particles, we can define the **number density** $m = \frac{N}{dV}$. This tells us how much stuff exists per cubic meter, and note that right now, there is no flux along any surface because none of the particles are moving.

But now let's introduce a new frame S' in which this box is moving in the positive x-direction. Then everything is moving with a velocity v towards the right, so all particles move with a speed of $\vec{v} = v\hat{e}_x$. Notice that by the Lorentz transformation,

$$dV' = dx'dy'dz' = \frac{dx}{\gamma}dydz = \frac{dV}{\gamma},$$

so

$$m' = \frac{\text{number of particles}}{\text{volume}} = \frac{N}{dV} = \gamma m_0.$$

The idea is that when we move the box, the volume is getting smaller, so the density must increase throughout the cube. There's more, too: consider a surface parallel to the $y'z'$-plane at rest in the S'-frame. Then particles in this frame are moving, so the flux is no longer zero. How many such particles will cross? In a time dt', particles in a box of base area dA' and vdt' will cross through the flux, so

$$\text{flux} = \frac{(vdt'dA') \cdot m'}{dt'dA'} = vm'.$$

So we've found a quantity that changes depending on the frame we're working in! In particular, if we pick some direction $\hat{n}$,

$$(\text{flux})^{\hat{n}} = m_0\gamma(\vec{v})^{\hat{n}} = m'(\vec{v})^{\hat{n}}.$$

So this motivates the following idea:

Definition 108

Define the four-vector **n** as

$$\mathbf{n} = n_0\mathbf{U} = (n_0\gamma c; n_0\gamma\vec{u}).$$

We can calculate the dot-product of **n** with itself: we find that it is

$$= m^\alpha m_\alpha = -n_0^2c^2.$$

This tells us that n_0, the number density in the rest frame, is a scalar: every frame will agree about its value, though everyone might find a different $m_0\gamma c$.

So now $d\rho$ and $d\vec{j}$ have a lot to do with this number density: say that all charges have charge q.

Definition 109

Define the **charge-current** four-vector via

$$\mathbf{J} = q\mathbf{n} = qn_0\mathbf{U} = \rho_0\mathbf{U}.$$

Again, we can calculate

$$J^\mu J_\mu = -\rho_0^2 c^2,$$

so the charge density ρ_0 is an invariant that everyone can calculate!

Note that $qn_0 = \rho_0$ is the charge density in the rest frame of the charge. If we want to know the components explicitly, notice that

$$\mathbf{J} \underset{S}{\rightarrow} (\rho c, \vec{J}),$$

because the first component is really coming from the definition of the number and charge density:

$$J^0 = qn_0 U^0 = qn_0\gamma c = n'qc = \rho c,$$

and the space components

$$J^i = n_0 q\gamma \cdot \vec{u}^i = \rho \cdot \vec{u}^i.$$

(After all, this is exactly how we define the current density: it's the density times the speed at which the particles are moving!)

So in summary, we introduced the idea of density and flux of particles, and we found a way to encode both in a single four-vector **n** and then into a four-vector that encoded both the charge and current. So now, if we know how to associate tensors to any and all objects in Maxwell's equation, we can understand what's going on. The first two equations that we wrote down relate the sources (densities) and our fields, and the left sides of Maxwell's equations are derivatives, so we write down something that looks of the right form:

$$\boxed{\partial_\nu F^{\mu\nu} = KJ^\mu}.$$

To show that this works, we need to show that this is actually the correct form of Maxwell's equations. First, plug in $\mu = 0$: this yields

$$\partial_\nu F^{0\nu} = KJ^0.$$

We know that $J^0 = \rho c$ (by the definition of the charge-current four-vector), and thus

$$Kc\rho = \partial_\nu F^{0\nu} = \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} = \partial_x \frac{E^x}{c} + \partial_y \frac{E^y}{c} + \partial_z \frac{E^z}{c} = \frac{1}{c}\vec{\nabla} \cdot \vec{E}.$$

Since $\frac{1}{c^2} = \mu_0\varepsilon_0$, this tells us that $\boxed{K = \mu_0}$, the magnetic constant.

Next, let's try plugging in $\mu = 1$: now

$$\partial_\nu F^{1\nu} = \mu_0 J^1 = \partial_0 F^{10} + \partial_2 F^{12} + \partial_3 F^{13} = \frac{\partial_t}{c}\left(-\frac{E^x}{c}\right) + \partial_y B^z + \partial_z(-B^y).$$

If we plug in the 1st component of the charge-current four-vector, this means that

$$-\frac{1}{c^2}\partial_t E^x + \partial_y B^z - \partial_z B^y = \mu_0 J^x,$$

which means that

$$(\vec{\nabla} \times \vec{B})_x = \mu_0(J^x + \varepsilon_0 \partial_t E^x).$$

This is indeed the x-component of the second Maxwell's equation! And now taking $\mu = 2, 3$ gives us the y, z component as well, and now we've derived the sourced Maxwell's equations in tensorial form.

But we still haven't derived the last two Maxwell's equations, and we'll do that by introducing a new tensor:

Definition 110

The **dual tensor** of the Faraday tensor, denoted $\tilde{F}^{\mu\nu}$ (this is a $(2, 0)$ tensor), is defined as

$$\tilde{F}^{\mu\nu} = F^{\mu\nu} \text{ except we swap } \frac{\vec{E}}{c} \to \vec{B}, \quad \vec{B} \to -\frac{\vec{E}}{c}.$$

This means we can write it as

$$\tilde{F}^{\mu\nu} = \begin{bmatrix} 0 & B^x & B^y & B^z \\ & 0 & -E^z/c & E^y/c \\ & & 0 & -E^x/c \\ & & & 0 \end{bmatrix}$$

(where we still have an antisymmetric tensor). If we want a more formal definition, this is actually just

$$\boxed{\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}}.$$

$\varepsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita symbol: it's 1 if we have an even permutation, -1 if we have an odd permutation, and 0 otherwise.

It turns out that the other two Maxwell's equations then follow by writing out a tensorial identity:

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}.$$

For example, if we take $\mu = 0$,

$$\partial_\nu \tilde{F}^{0\nu} = 0 = \partial_1 \tilde{F}^{01} + \partial_2 \tilde{F}^{02} + \partial_3 \tilde{F}^{03} = \vec{\nabla} \cdot \vec{B}.$$

Meanwhile, if we take $\mu = 1, 2, 3$, we'll find that this indeed gives us the final identity (Faraday's law).

So the three equations together give us basically all of electromagnetism:

$$\partial_\nu F^{\mu\nu} = \mu_0 J^\mu,$$

$$\partial_\nu \tilde{F}^{\mu\nu} = 0,$$

$$f^\mu = qF^{\mu\nu}U_\nu.$$

We also get something for free: $F^{\mu\nu}$ is antisymmetric, so if we contract it with any symmetric tensor $S^{\mu\nu}$, we get 0. Well, consider

$$\partial_\mu \partial_\nu F^{\mu\nu} :$$

derivatives commute, so this is a symmetric tensor. This means that

$$0 = \partial_\alpha \left(\partial_\nu F^{\alpha\nu}\right) = \partial_\alpha \left(\mu_0 J^\alpha\right),$$

so

$$\partial_t \rho + \partial_i J^i = \partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0$$

This is the **continuity equation**: charges and currents must be organized in a way so that they are conserved!

Soon, we'll try to play the same kind of game with gravity. We know that we can write the gravitational potential and equation of motionl via

$$\phi = -\frac{mG}{r}, \quad \nabla^2\phi = 4\pi G\rho(\vec{r}).$$

We have the same idea of "derivative of field equals source" here. Mass density is the source of gravity here, but if we try to apply the same procedure, we get a problem: while charge density and electric density are part of a four-vector, and ρ_0 is a scalar, the mass is not conserved in relativity. That's a bit of an issue, because we can make mass go away, and what happens to gravity? Next in 8.033, we'll understand what actually creates gravity – it's not just mass or mass density; it must involve momentum and energy too – and that will give us some deep nontrivial results.

22 October 17, 2019 (Recitation)

We're going to do two problems today: one relates to a loophole from the exam (parity inversion), and the other is an acceleration problem.

Problem 111

Consider the transformation

$$\begin{cases} t \to t \\ x \to -x \\ y \to -y \\ z \to -z \end{cases}$$

(parity inversion).

Is this Lorentz? It might seem like we can't use a γ-factor, but that's not what a Lorentz transformation is:

Definition 112 (Restatement)

A **Lorentz transformation** is a linear map between two coordinate systems that preserves the spacetime interval $ds^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$.

Notably, translations are included in this, even if they can't be written in the Lorentz matrix form we've been working with for boosts! We showed on the exam that

$$U^\mu V^\nu \eta_{\mu\nu} = U^{\mu'} V^{\mu'} \eta_{\mu'\nu'}.$$

If M is the parity inversion matrix, we know that

$$\eta_{\mu'\nu'} = {M^{-1}}^\mu{}_{\mu'} {M^{-1}}^\nu{}_{\nu'} \eta_{\mu\nu},$$

and then we can multiply the matrices to show that η's coordinates are the same in the new reference frame. (We do need to be careful about using a $(0, 2)$ tensor, though! Essentially, we just need to put the η in the middle between our transformation matrices.) The next step in the problem was to show that

$$U^\mu V^\mu \eta_{\mu\nu} = U^{\mu'} V^{\nu'} \eta_{\mu'\nu'} :$$

It was intended for us to just write out the right hand side with the M matrices, and then we could show that all we were left with was $\eta_{\mu\nu} U^\mu V^\nu$, instead of bashing everything out!

How do we show that this is a Lorentz transformation? It's indeed a linear map (we can write it as a matrix), and it does leave ds^2 invariant, because it leaves the inner product $\eta_{\mu\nu}dx^\mu dx^\nu$ invariant. So M is indeed a Lorentz transformation (though sometimes it'll be referred to as **inhomogeneous**).

Remark 113. *It's important for us to know that tensors are labeled with coordinates, but the tensor itself does not change between frames! So if we define a tensor in terms of linear maps, and it needs to transform in a particular way under Lorentz transformations.*

For example, we can **define** a (0, 2) tensor's coordinates via

$$T_{\mu'\nu'} = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}T_{\mu\nu}.$$

The whole point is that this ensures that the intrinsic "value" or "quantity" remains constant before frames.

Example 114

Consider the Levi-Civita symbol

$$\varepsilon_{ijk} = \begin{cases} 1 & ijk \text{ is an even permutation of } (213) \\ -1 & ijk \text{ is an odd permutation of } (1, 2, 3) \\ 0 & \text{otherwise.} \end{cases}$$

For illustration, let's try working in just two space dimensions and one time-dimension. Then we just define

$$\varepsilon_{\mu\nu} = \begin{cases} 1 & (\mu, \nu) = (0, 1) \\ -1 & (\mu, \nu) = (1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

If this were a tensor, we'd have

$$\varepsilon_{\mu'\nu'} = \Lambda^\mu{}_{\mu'}\Lambda^\nu{}_{\nu'}\varepsilon_{\mu\nu}.$$

It turns out that if we plug in boosts, nothing goes wrong: indeed the boost matrices cancel out, so the components of $\varepsilon_{\mu'\nu'}$ are the same as $\varepsilon_{\mu\nu}$. But look at parity inversion: if Λ is actually our parity inversion matrix M from above,

$$M = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \varepsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

and carrying out the computation, it turns out that our Levi-Civita symbol gets negated under parity inversion! As another example, ∂_μ is not a tensor either, so we need to be careful about how it transforms.

Let's now do a problem about four-acceleration:

Problem 115

A centrifuge of radius R in the lab frame starts out at rest in the lab frame. It spins up to $0.8c$: what's the radius of the centrifuge at that point?

Note that the velocity is orthogonal to the radius, so the radius is unchanged! That means that in the lab frame, it seems that we'd also measure the circumference to be $2\pi r$ as well (this is unchanged).

But it seems that the circumference is moving along the direction of motion, so it should be length contracted: what's going on here? Well, say that we wanted to measure the actual circumference of the relativistic centrifuge. We can do this by holding a bunch of sticks of length $d\ell$ above the centrifuge's circumference: how much of the rotating centrifuge's circumference would fit into each at-rest stick of length $d\ell$?

Well, for each infinitesimal stationary stick of length $d\ell$, we can fit $\gamma d\ell$ rotating sticks by length contraction. So the circumference is just the integral (along the rest sticks)

$$\int_{\ell=0}^{2\pi r} \gamma d\ell = \frac{10\pi r}{3}.$$

(This can be thought of as just multiplying the length of the stationary sticks by γ.) So we can no longer use the same formula for the circumference of a disk if it is rotating!

This is also weird, because we'd think naively that a moving object would get shorter when it moves relative to our frame. But things are different here because of the acceleration!

Problem 116

So now say we have a thread connecting two (non-diameter) points on our circumference (it's stretched maximally in a straight line in the lab frame). Once it's spinning at $0.8c$, does the thread break or not?

Since the circumference has increased, and the thread is stretched maximally in the stationary situation, the increase in circumference will make it stretch beyond capacity! It's true that the thread does contract as well, but it turns out it contracts less. This will all become a bit more intuitive as we work with this more in the coming weeks!

23 October 21, 2019

Let's start with a brief review of recent concepts. We've been finding that the right way to deal with the density and flux of a non-interacting particle is to consider it in a momentarily comoving rest frame (so in our system, **all particles are at rest**). This is often called **dust**.

We define d

$$n_{\text{MCRF}} = \frac{dN}{dV},$$

which is the number density of particles in a given region of space. This allows us to measure the flux along any given cross-section:

$$\vec{\text{flux}}^{\,i} = \frac{\text{number of particles that cross area } dA \text{ in } dt}{dAdt} = 0$$

(because the particles are not moving). This means that we can define a four-vector

$$\mathbf{n} \xrightarrow[\text{MCRF}]{} (n_{\text{MCRF}}, \vec{0}).$$

This means that in our MCRF,

$$\mathbf{n} = n_{\text{MCRF}} \mathbf{U},$$

and it turns out this is true in any inertial frame! Basically, we can define the particle density-flux four-vector in any frame explicitly via

$$\mathbf{n} = n_{\text{MCRF}} \mathbf{U} \xrightarrow[S]{} \boxed{n_{\text{MCRF}} \gamma(\vec{u})(1, \vec{u})}.$$

We can alternatively call this leading term $n = \gamma(\vec{u}) n_{\text{MCRF}}$ the number density in our frame S.

So now let's say our particles have an electric charge, and let's assume for sake of simplicity that all particles have a charge q. Then we can introduce

$$\mathbf{J} = q\mathbf{n} \xrightarrow[S]{} (\rho, \vec{J})$$

where ρ is the charge density and $\vec{J}$ is the current. This is a nice four-vector because it allows us to write the Maxwell equations and Lorentz force in tensorial form: we found last time that we can write the sourced Maxwell equations as

$$\partial_\beta F^{\alpha\beta} = \mu_0 \mathbf{J}$$

and the others as

$$\partial_\beta \tilde{F}^{\alpha\beta} = 0,$$

where $\tilde{F}$ is the dual tensor of the Faraday tensor and is defined via replacing $\vec{E}$ with $\vec{B}$ and $\vec{B}$ with $-\vec{E}$ (we're again using $c = 1$ in this class). Finally, the Lorentz force on a charged particle can be written in tensorial (or covariant) form

$$f^\mu = qF^{\mu\beta} U_\beta.$$

This is basically all of electromagnetism written on one blackboard! We have all the dynamics here: charges create a field, and the field tells the charges how to move. And we like this because they have the same form in all frames: the only thing that is different is that the individual components need to transform under the Lorentz matrix.

What we'll do next is to find similar equations for gravity! Wishful thinking tells us to find something of the form

$$\text{(Derivatives)(Field)} \propto \text{(Source of gravity)},$$

which is structurally the same as $\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$. Before we do that, though, we'll focus on the right hand side and try to figure out this **source**.

Fact 117

In Newtonian mechanics, it's clear that mass affects gravitational forces: if we define $\rho = \frac{dm}{dV}$ to be the mass density, the field ϕ satisfies

$$\nabla^2 \phi(\vec{r}) = 4\pi G \rho(\vec{r}).$$

This last equation looks a bit like what we want: we can solve this to find

$$\phi(\vec{r}) = -\frac{mG}{r},$$

where m is the integral of ρ over the volume V. But trying to generalize this to relativity doesn't quite work: mass isn't conserved in special relativity! We saw last time that $d_\alpha J^\alpha = 0$ gives us the continuity equation, so we have a conserved source there, and we definitely don't have a conserved mass here (because it can become energy and momentum via $E^2 = m^2 + p^2$).

We do know that p^μ is conserved: the four-vector of four-momentum (summed over all particles) remains constant. So maybe p^μ should be the source? Well, we need to make sure our equation fits the Galilean version in the limit, and in Newtonian mechanics, we have a mass density rather than a mass. So that motivates thinking of the **density of momentum or energy** here instead.

Let's start by looking at this for dust, so there exists a frame where all particles are identically at rest inside a box. Say all particles are identical. In the MCRF, we've seen that

$$n_{\text{MCRF}} = \frac{N}{V},$$

and the total energy of these particles is just

$$E = \sum_{p=1}^{N} E_{(p)} = Nm$$

because we're in the rest frame and therefore have no γ factor. The energy density is then

$$\rho = \frac{N}{V} m = nm,$$

just in the rest frame. So how do we translate this to any other frame S'? We know that the energy density will be equal to

$$\rho' = n'E' = (\gamma n_{\text{MCRF}})(\gamma m) = \gamma^2 n_{\text{MCRF}} m = \gamma^2 \rho_{\text{MCRF}}.$$

We know that ρ is not a four-vector: if it were, then changing frame should only give us one gamma factor. **So this means that ρ must be part of a rank $(2,0)$ tensor**, called the **stress-energy tensor** $T^{\mu\nu}$.

Definition 118

Formally, the component (μ, ν) of the stress-energy tensor T is the flux of the μth component of momentum $\mathbf{p}$ across a surface of constant x^μ.

So $T^{\mu\nu}$ tells us where p^μ is going (flux tells us that we're moving somewhere). An area of constant x^μ looks like a hyperplane! For example, T^{11} is the flux of p^x across the yzt-hyperplane. It tells us where the momentum is going, and the only nontrivial thing is that an area of constant time is just a volume. So let's write out a few of these components:

$$T^{00} = \text{flux of } E \text{ over a volume } V,$$

which is an energy density. Similarly,

$$T^{0i} = \text{flux of } E \text{ over an area of constant } x^i,$$

basically tells us how much energy leaves a certain cube. Flipping things around,

$$T^{i0} = \text{flux of } p^i \text{ over a volume } V$$

is again a density, and finally

$$T^{ij} = \text{flux of } p^i \text{ over an area of constant } x^j.$$

One important feature here is that

$$T^{\mu\nu} = T^{\nu\mu}.$$

This tells us that the change of energy across some surface is related to the change of momentum, which makes sense! We can prove this mathematically, but it takes some more advanced knowledge. By the way, this is a fact in Einstein relativity, which is what we're going to study in this class, not necessarily in more exotic theories.

Example 119

What does $T^{\mu\nu}$ look like for dust?

The beauty of relativity is that because this is a tensor, we can just calculate this in one frame: the **momentarily comoving rest frame**!

Since T^{00} is the energy density, this density is $n_{\text{rest}}m$, where m is the mass of all particles. Define this quantity to be ρ_{MCRF}.

Looking at all the other nine components (this is because $T^{\mu\nu}$ is symmetric, so there are ten independent components instead of sixteen), they are all zero! This is because the flux of energy is zero (nothing's moving) and the flux of $\vec{p}$ is zero because there's no three-dimensional momentum. So

$$T^{0i} = T^{i0} = T^{ij} = 0.$$

And now if we want to find the tensor in a frame S',

$$T^{\alpha'\beta'} = \Lambda^{\alpha'}{}_{\mu}\Lambda^{\beta'}{}_{\nu}T^{\mu\nu}.$$

We claim that in any frame,

$$T^{\mu\nu} = n^{\mu}p^{\nu}$$

for dust. This is clear in the rest frame, because p^{ν} is $(m, 0, 0, 0)$ and n^{μ} is $(n_{\text{MCRF}}, 0, 0, 0)$. If we do the Lorentz transformations explicitly, we'll see that this does indeed work in all frames! We can now write

$$n^{\mu}p^{\nu} = mU^{\nu}n_{\text{MCRF}}U^{\mu} = mn_{\text{MCRF}}U^{\mu}U^{\nu}$$

for **dust in all frames**. (Note that we can also write this expression geometrically as $T = \mathbf{np}$.)

So now, let's write the energy density out for more general systems (which aren't just dust).

Example 120

Can we find $T^{\mu\nu}$ for a **perfect fluid** that is in hydrostatic equilibrium? Say it has pressure p and energy density ρ – we do have to be careful about both of these quantities changing between frames, so we define them **in the momentarily comoving rest frame**.

In a fluid, we think about having a collection of particles which individually are moving (as gasses do, randomly), but overall, the box is not moving. (This is in contrast to what we've been saying before!)

We know that the particles are all moving, so there will be a nonzero pressure. Remember that the pressure is exactly what tells us "where momentum is going," because it relates to the force. It turns out

$$T^{\mu\nu} = (p_{\text{MCRF}} + \rho_{\text{MCRF}})U^{\mu}U^{\nu} + \eta^{\mu\nu}p_{\text{MCRF}}$$

is the general form for our tensor here. What does this look like in the rest frame? We know that

$$\mathbf{U} \xrightarrow[\text{MCRF}]{} (1, \vec{0}) \implies T^{00} = p_{\text{MCRF}} + \rho_{\text{MCRF}} - \eta^{00}p_{\text{MCRF}} = \rho_{\text{MCRF}}.$$

Similarly, we can find for (we can replace 1 with 2 or 3)

$$T^{11} = (p_{\text{MCRF}} + \rho_{\text{MCRF}})(U^1)^2 + \eta^{11}p_{\text{MCRF}} = p_{\text{MCRF}}.$$

This makes sense: T^{xx} is the flux of the x-component of the four-momentum over a region of x, which is just the pressure! It turns out that all non-diagonal components are zero, so

$$T^{\mu\nu}{}_{\text{MCRF}} = \text{diag}\left(\rho_{\text{MCRF}}, p_{\text{MCRF}}, p_{\text{MCRF}}, p_{\text{MCRF}}\right).$$

$p = 0$ is a special case here: **we can think of dust as a perfect fluid with no pressure.**

The last thing we'll do today is look at equations of state. Dust's equations look like (only in the rest frame)

$$\begin{cases} p = 0 \\ T^{\mu\nu} = \text{diag}(\rho, 0, 0, 0) \end{cases}$$

and it turns out this is a good model for galaxies! On the other hand, we can think of photons and neutrinos as a perfect fluid, with the equations of state

$$\begin{cases} p_{\text{MCRF}} = \frac{1}{3}\rho_{\text{MCRF}} \\ T^{\mu\nu} = \rho_{\text{MCRF}} \cdot \text{diag}(1, 1/3, 1/3, 1/3). \end{cases}$$

Next, a system that will be useful for us studying cosmology is dark energy: it has the interesting property that

$$\begin{cases} p_{\text{MCRF}} = -\rho_{\text{MCRF}} \\ T^{\mu\nu} = \rho_{\text{MCRF}} \cdot \text{diag}(1, -1, -1, -1). \end{cases}$$

This is weird because if we put dark energy in a box, it will contract: it has a negative pressure. We'll meet all of these equations again later when we study cosmology a bit more.

Finally, we'll look at one more example of $T^{\mu\nu}$: everything we've been doing so far is particles, but it turns out this concept extends to electromagnetism! This also has a stress-energy tensor (which means it can also create gravitational fields) in all frames

$$T^{\mu\nu} = \frac{1}{\mu_0}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}\eta^{\mu\nu}\right).$$

This basically implies that just because we have an $\vec{E}$ and a $\vec{B}$ field, we create gravity! We won't use this in the rest of class, but it's just a nice thing to know.

In general, if we think about a piece of the universe, we have a combination of all of these particles: photons, dust, electromagnetic waves, and so on. In general, given any collection of different species (for example, the four we've just described here), we can find the $T^{\mu\nu}$ tensor by summing everything together. Recall that we started this discussion by saying that we can't just use the mass m as the source of gravity, because mass is not conserved. The important idea is that

$$\partial_\mu T^{\mu\nu} = 0,$$

which implies that p^μ is conserved. This is nice, because four-momentum is what we want to play the role of an "extended mass!" This is very similar to how $\partial_\mu J^\mu = 0$ implies $\partial_t \rho + \vec{\nabla}\vec{J} = 0$, which is charge conservation. So that's the symmetry between the electromagnetism and gravity cases!

So $T^{\mu\nu}$ sounds like it has all the criteria that we want. It's related to mass and energy and momentum and density, and indeed $T^{\mu\nu}$ does give us what we want. So we now have the right hand side of the equation: what's left is to study the left-hand side of the equation, trying to figure out what the "field" of gravity is really about, and by extension how objects move!

24 October 22, 2019 (Recitation)

We'll continue the problem from last recitation, but we'll do things slowly and explicitly this time!

Problem 121

Say a merry-go-round with radius R is rotating in a sandbox such that the velocity of a point on the circumference is β.

First of all, R does not experience length contraction, because all motion is orthogonal to R. Thus, the radius of the merry-go-round is the same in both frames.

Say we're standing on the edge of a merry-go-round, and we drag a stick along the sand directly below us. The sand-circle that we draw in the sandbox also has radius R, and thus we have a sand-circle with circumference $2\pi R$ (the sand is not moving, so we don't have any relativistic effects). Specifically, if we integrate a differential length

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

is the spherical distance in Minkowski spacetime. This then gives us $2\pi R$, because

$$C = \int_{\text{circle}} \sqrt{ds^2} = \int_{\text{circle}} \sqrt{-dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2}$$

and since our circle has fixed θ, fixed radius, and $d\theta = dr = 0$, this can also be evaluated to

$$= \int_0^{2\pi} R d\theta = 2\pi R.$$

So how do we calculate the circumference of the actual merry-go-round? Suppose we put $\frac{2\pi R}{\ell}$ sticks of length ℓ (in the sand-frame) in the sand, along the sand circle that we just placed. Suppose we also put $\frac{2\pi R}{\ell}$ sticks of rest length ℓ on the merry-go-round before spinning it.

So in the reference frame of the sand, when we look at the sticks on the spinning merry-go-round, they are contracted to a length $\frac{\ell}{\gamma}$. So someone in the sand frame says that we can fit more of these sticks in a circle of circumference $2\pi R$: we can now fit $\frac{2\pi R\gamma}{\ell}$ rotating sticks around the disk, in the reference frame of the sand. (This means that yes, there will now be gaps between the sticks, but that gives rigidity issues which we won't quite get into.)

Since the number of sticks that we can fit around a disk doesn't change between our frames, this means we now have $\frac{2\pi R\gamma}{\ell}$ sticks around the rotating disk's frame, too (in which the sticks are not contracted). This then means the circumference is ineed $2\pi R\gamma$ in the rotating frame!

So this tells us that this system is not well described by flat spacetime: this is the first insight into spacetime being curved! Soon, we'll talk about the equivalence principle, which tells us how accelerating reference frames relate to gravity.

Problem 122

Now let's modify our situation a bit: place identical clocks around every point on the disk. Set off a firecracker at the center of the disk, which lets us synchronize the clock: when they receive the light ray from that event, let all clocks have $t = 0$. Say that we start spinning the disk after this synchronization.

If we look from the point of view of the sand, the clocks **do** remain synchronized, but all clocks are moving slower by a factor of $\gamma(\beta)$ than a clock in the sand. (This is because all clocks are symmetric from the perspective of the sand, and they are always moving at a velocity β with respect to the sand.)

So now let's say we have three observers Alice, Bob, and Charlie (in that order clockwise) on the merry-go-round, close to each other. Say the merry-go-round is rotating counterclockwise. From Bob's perspective in an **inertial reference frame**, Alice is moving with respect to Bob (because their velocities move in different directions), so they are no longer moving in the same direction at all times, meaning that their clocks are **not synchronized**. In fact, we can find that Alice sees her clock hit $t = 0$ **ahead of** Bob seeing his clock hit $t = 0$, and Charlie's clock will be another amount of time behind Bob.

Question 123. *Is it possible to choose a time coordinate $t' = 0$ on each clock, so that each observer (in their own reference frame) finds that they are synchronized with both the one ahead and the one behind them?*

The answer is no! Basically, this means we'll have to keep setting clocks further ahead, and if we go once around the circle, we can't get back to the initial time of the first observer. So there is no way to synchronize all clocks.

So this rotating disk is actually described by a new metric:

$$ds^2 = -dt^2 + r^2(d\phi^2 - \omega dt)^2 + r^2 d\theta^2 + dr^2.$$

Let's finish with a variant of the Twin Paradox:

Problem 124

Say that we identify points

$$(t, x, y, z) \sim (t, x + L, y, z)$$

so that if we travel from $x = 0$ to $x = L$, we end up back at $x = 0$ again. Say that twin A stays at $x = 0$, and twin B moves at constant velocity away and eventually ends up back at $x = 0$ again. Each one of them thinks the other experienced time dilation, so how do we get out of this?

Under this definition, we must have twin A sit at $x = 0$ for all time in their own frame, and twin B travels at some three-velocity $v\hat{x}$. The crucial observation here is that identification of points L apart is dependent on coordinates! When we do a Lorentz transformation between the two frames, we do need to change coordinates, and that takes

$$(t', x', y', z') \sim (t' - \gamma\beta L, x' + \gamma L, y', z').$$

So we do have a preferred reference frame here for the transformation, and it's the one where we don't have to change the time coordinate! So the situation is not indeed as symmetrical as we think it is.

25 October 23, 2019

Today, we'll finally have everything make sense! We'll complete, from one point of view, our reevaluation of gravity and gravitational forces. Recall our current goal is to find an update to the equations

$$\nabla^2\phi(\vec{r}) = 4\pi G\rho(\vec{r}),$$

which tells us how sources create gravitational fields, and

$$\vec{F} = -m_g \nabla\phi(\vec{r} = m_{\text{inertial}}\vec{a},$$

which tells us how gravitational fields generate forces and accelerations. Two notes: first of all, Newtonian gravity

$$\vec{F}(t) = \frac{mMG}{|\vec{r}_1(t) - \vec{r}_2(t)|}\hat{n}$$

looks rather suspicious if we try to use it as is in relativity: this tells us that the force at a certain time t requires us to measure both $\vec{r}_1$ and $\vec{r}_2$ at the same time, which is not something all observers agree on. Second of all, we know experimentally that $m_g = m_{\text{inertial}}$, at least within a factor of 10^{-13}, but this is not immediately obvious from Newtonian theory.

One final thing that is relevant here: last time, we introduced the source of gravity, which is the **stress-energy tensor** $T^{\mu\nu}$: this tells us where our energy and momentum currently are, as well as where the energy and momentum are going (flux). This has the property that

$$\partial_\mu T^{\mu\nu} = 0,$$

which implies that p^μ_{total} is conserved! The whole point is that this means all objects, even those without mass, can be affected (and affect) the stress-energy tensor.

The purpose of all of this is to find the field equations of the form

$$(\text{derivative})(\text{field}) = (\text{source}) = T^{\mu\nu}.$$

In addition, we want to describe the motion of an object along a gravitational field, which is a generalization of $\vec{F} = m\vec{a}$ for gravity.

Let's start with some immediate consequences of what we've already discussed.

Example 125

Consider this thought experiment: we are on Earth, and someone at the top of the Leaning Tower of Pisa drops a mass m from rest.

At the top, the object has $v(0) = 0, E = m$ (no γ factor). When this object hits the bottom, the object now has velocity $v = -\sqrt{2gh}\hat{z}$ (from mechanics) to the first linear order. Meanwhile, the energy of the object is now $\gamma m = m + mgh + O(v^4)$.

But now say we have a device at the bottom which emits a photon upward, and that photon has the energy of the object of mass m that was dropped earlier. At the bottom,

$$E_{\gamma,\text{bottom}} = m(1 + gh),$$

but the energy of the photon when it goes back up should have energy m (or else we would have perpetual motion), then the ratio

$$\frac{E_\gamma, \text{top}}{E_\gamma, \text{bottom}} = \frac{1}{1 + gh} \approx 1 - gh.$$

This isn't 1! Specifically, if we think of the gravitational field near the earth as $\phi(r) = -\frac{MG}{r}$, we can Taylor expand this ratio as

$$1 - c(\phi(h) - \phi(0)) < 1.$$

So the energy of the photon must have decreased when it went against gravity! The way to deal with this is to use the fact that the energy of a photon, $E = hf$, has changed between the top and the bottom, so the frequency is also different:

$$f(h) = f(0)(1 - gh) < f(0).$$

We've thus redshifted the photon! This is great, because we can now think of having an atomic clock on the ground which sends a photon upward every $\Delta\tau(0) = \frac{1}{f(0)}$ seconds. They're then received at the top with a frequency $f(h) \neq f(0)$, so

$$\Delta\tau(h) \neq \Delta\tau(0).$$

We can do the calculations to find that if we have objects a distance of r and s away from the Earth (which has mass

M) along a line,

$$\Delta\tau(s) = \Delta\tau(r)(1 - (\phi(r) - \phi(s))) \implies \Delta\tau(s) > \Delta\tau(r).$$

So **clocks must slow down near a mass**. We may ask why this is a big deal: we've already seen that time intervals are not universal, so why do we care about this? Last time, we found that different inertial frames would find different time intervals from events. But this is not quite the same: there is only one inertial frame.

Let's consider a spacetime diagram for this. The ground corresponds to the t-axis, but the curves corresponding to the first and second photon must just be translated copies of each other by a fixed vertical distance Δt in our diagram. So if we have flat spacetime, we'd believe that

$$\Delta\tau = -\eta_{00}\Delta t = \Delta t$$

for any x-coordinate. But this isn't true in the model we're using – we've just found that $\Delta t \neq \Delta\tau$! So we have to give something away, and it turns out the idea is that gravity is making our frame no longer inertial. That is, the metric is not η anymore!

So how do we still use inertial frames? Galileo found that in vacuum, all particles move in exactly the same way and hit the ground at some time. This is unique for gravity: for all other forces, they all have a subset of particles that do not interact with them. For example, massless particles don't interact with electromagnetism. But gravity doesn't have this, because everything has energy and momentum! So everything couples with gravity in the same way.

So what we should do is to consider a **free-falling frame**: if we jump off the Leaning Tower of Pisa with the mass m, then the mass will not move in our perspective. In addition, a second object that is also free-falling has the same acceleration, so it has some constant velocity. And this is exactly an inertial frame! Notice that this idea did not work in electromagnetism, because there are some objects that do not actually get affected.

So that's what made Einstein formulate the **principle of equivalence** in 1907:

Proposition 126

A **uniform** gravitational field (like the one near the Earth) is equivalent to frames that accelerate relative to inertial frames. In other words, anything we do in a gravitational field can be thought of as having an accelerating reference frame in the opposite direction.

Let's verify this by looking at our Leaning Tower of Pisa problem again. At the beginning of our scenario ($t = 0$), the FFF has no speed relative to the ground, and neither does the photon. This means that

$$f^{FFF}_{\text{bottom}} = f^{\text{ground}}_{\text{bottom}}.$$

We know that it takes $\Delta t = h$ to reach the top of the tower, so by the time the photon gets there, the velocity of the free-falling-frame is $-\Delta t g\hat{z}$. Meanwhile, we have a photon under an accelerating frame: then we experience a redshift of

$$f^{\text{FFF}}_{\text{top}} = f^{\text{ground}}_{\text{top}} \cdot \frac{1-\beta}{\sqrt{1-\beta^2}} \approx (1-\beta) f^{\text{ground}}_{\text{top}}.$$

And now because $\beta = 1 - gh$,this means that we can write (to first order)

$$f^{\text{FFF}}_{\text{top}} = (1+gh) f^{\text{ground}}_{\text{top}} = (1+gh)(1-gh) f^{\text{ground}}_{\text{bottom}} \approx f^{\text{FFF}}_{b,H}.$$

So the whole point is that we've managed to cancel our gravitational interactions by working in a good frame! But there is a catch: gravity is not necessarily universal, and in fact that's what happens in this particular universe. For example, if there is a particle which is not close to our initial object, the gravity is pointing in a different direction, so we will indeed notice a lack of inertial frame.

So to do physics around the different object, we need to define a new free-falling frame, one for each region of space. We can't just Lorentz transform everything at once, and the universe cannot be thought of as flat globally! Einstein realized this in 1912, when he realized that a free-family frame can be thought of as a set of coordinates.

Remark 127. *If we try to represent a sphere as a piece of paper, there will be some distortions. However, near each point of the earth, we can treat space as approximately flat! We can't use the same set of coordinates everywhere, so we can't use a unique set of flat coordinates. (Same with non-uniform gravity.) This is the same idea as having a gravity introduce a curvature on space and time itself.*

Here's another thought experiment to help with this thought. Suppose that we don't know that the earth is round: tell two observers on the equator to go north to the North Pole. If one of the observers believe that the earth is flat, they'd believe that the two observers got closer together because of some force that attracts objects together. The answer is gravity, and that's the real idea here! Because we're living in a spacetime which is not flat, we interpret our trajectories as being affected by a force called gravity.

So we should try to understand how objects move on curved surfaces, under the absence of forces. For example, what does it mean for us to move on a straight line on a sphere?

Well, we've seen how in Minkowski space, we can use the metric $\eta = (-1, 1, 1, 1)$ to take dot products

$$\mathbf{A} \cdot \mathbf{B} = \eta_{\mu\nu} A^\mu B^\nu.$$

It turns out that in general relativity, we can't quite do this anymore! **Space can be locally flat, but not globally flat**. To address this, we'll start using a different metric $g_{\mu\nu}$, where the metric now depends on our coordinate x. We can still say that

$$g_{\mu\nu}(x) = \mathbf{e}_\mu \cdot \mathbf{e}_\nu,$$

and we can still say that

$$\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu, \quad ds^2 = dx^\mu dx^\nu g_{\mu\nu}$$

so most of what we have been doing are still the same, just with a different metric.

Remark 128. *Just to emphasize this point again, from now on, gravity is not a force:* **it's just the curvature of spacetime**.

So how do we work with and find the value of $g_{\mu\nu}$? We need to get our field equations for gravity. It turns out that $g_{\mu\nu}$ is part of our "field" in our field equation, and finding it requires us to solve annoying, difficult nonlinear differential equations. Sp in most of the exercises in this class, we'll assume that we've already been given the metric $g_{\mu\nu}$: we'll then try to ask questions like "how a planet moves around a Sun."

By the way, it is not true that $g_{\mu\nu} \neq \eta_{\mu\nu}$ automatically means that there is gravity! For example, consider the change of coordinates

$$\begin{cases} \mu = t + x \\ \nu = x - t \\ y = y \\ z = z. \end{cases}$$

This means that

$$dx = \frac{d\nu + d\mu}{2}, \quad dt = \frac{d\mu - d\nu}{2},$$

and we will find that

$$ds^2 = -dt^2 + dx^2 + dt^2 + dz^2 = d\mu d\nu + dy^2 + dz^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

Comparing the expressions, we find that

$$g_{\mu\nu} = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and this is clearly not the Minkowski metric. But we haven't introduced gravity: we've just changed the coordinates! So the implication only goes in the direction

$$\text{If there is gravity, then } g_{\mu\nu} \neq \eta_{\mu\nu}.$$

So this leaves a problem: we were hoping that we could look at $g_{\mu\nu}$ and use it to see whether there is gravity. It's not quite this simple: we have to think a bit more carefully about how to see if there is a gravitational field (and thus curvature), and that's what we'll talk about over the next few days.

26 October 24, 2019 (Recitation)

We've finally gotten to general relativity! Today, we'll look at different types of geometries and how quantities change with time and space, so that we can have some intuition for what's going on.

One of the most beautiful insights in physics is the **equivalence principle**, which states that using local measurements, a gravitational field is completely equivalent to an accelerating reference frame. Einstein arrived at this by looking at all of the forces that were known at the time, and everything made a lot of sense except gravity. It seemed reasonable that the theory might just be modified a tiny bit, but it did take him 11 years to arrive at a final theory involving gravity!

The whole point here is that on a freely-falling frame, gravity has **no local effects**. Special relativity is basically a special case of general relativity: it's the case where there's a global inertial frame. In general, we don't have this: if we take any local observer on a spinning disk (like in last recitation!), we have a non-uniform acceleration as a function of position. We found that this meant we couldn't model this with a flat geometry: in this particular instance, we embedded this in a larger system that had a global inertial frame (the sandbox), but imagine that this spinning disk was all that existed. Then our geometry actually becomes non-Euclidean, and that explains why the $2\pi R\gamma$ was important!

Question 129. *What is the most basic object in geometry?*

Being able to measure a length is an important thing in figuring out properties of our underlying space, and that's actually what a geometry is: it's a way that lets us compute the length of a curve. We've already seen our Minkowski length

$$\sqrt{\pm ds^2} = \sqrt{-dt^2 + dx^2 + dy^2 + dz^2},$$

which measures the length of an infinitesimal piece of a curve. (Positive corresponds to a spacelike curve, and negative corresponds to a timelike curve). Let's try looking at some metrics that are not flat!

Fact 130

The reason Minkowski space is **flat** is because

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + dx^2 + dy^2 + dz^2$$

has the dx^2, dy^2, dz^2 terms of Pythagorean distance like in Euclidean space, and then we have a $-dt^2$ to be consistent with Maxwell's theory of electromagnetism. So this is really **Lorentzian flat space**, and we'll look at some ways of measuring curvature of a space later on – it'll turn out to be 0 for Minkowski.

Example 131

Consider the metric

$$ds^2 = -dt^2 + e^{2t}(dx^2 + dy^2 + dz^2).$$

We know that light rays just look like $x(t) = t$ in flat space, which gives $ds^2 = 0$: what do they do in this geometry? Well, we want

$$\ell = \int \sqrt{ds^2} = \int \sqrt{-dt^2 + e^{2t} dx^2} = \int_0^{t_1} dt \sqrt{-1 + e^{2t} x'(t)^2} = \int_0^{t_1} \sqrt{e^{2t} - 1} dt.$$

This is always nonnegative for $t \ge 0$, so the integral is positive! We can check that this gives a length of

$$\ell = \sqrt{e^{2t_1} - 1} - \tanh^{-1} \sqrt{e^{2t} - 1} \neq 0,$$

and this is because $x = t$ is only the path of light in flat space! What we've traced out is a spacelike curve instead. If we wanted to find a curve for light, we should instead start by setting $ds^2 = 0$ and solve

$$0 = -dt^2 + e^{2t} dx^2$$

and we can just solve this directly. The point is that when the geometry is different, we should figure out how light travels via $ds^2 = 0$.

Remark 132. *The length of a spacelike curve goes exponentially in t_1, and this is actually the metric for an inflationary universe! In other words, the spacial component of the universe is expanding extremely rapidly.*

For example, if we integrate over constant $t = t_0$ and look at $x \in [x_1, x_2]$,

$$\ell = \int \sqrt{ds^2} = \int_{x_1}^{x_2} \sqrt{e^{2t} dx^2} = e^{t_0}(x_2 - x_1).$$

So the same curve will have larger and larger length as time increases. This means the universe must be expanding very quickly!

Question 133 (Often asked)**.** *What is the universe expanding into?*

The answer here is that it's not expanding into anything: we just have stretched lengths. (For example, quantum fluctuations become the cosmic microwave background radiation!) This is what we suspect happened near the beginning of the universe.

Example 134

Let's try another metric of

$$ds^2 = -dt^2 + (t_1 - t)^6(dx^2 + dy^2 + dz^2).$$

Computing the length of the curve at some fixed time $t_0 < t_1$ and $x \in [x_1, x_2]$, we have

$$\ell = \int \sqrt{ds^2} = \int \sqrt{(t_1 - t_0)^6 dx^2} = \int_{x_1}^{x_2} (t_1 - t_0)^3 dx = (t_1 - t_0)^3 \Delta x.$$

If we take $t_0 \to t_1$, the curve length goes to 0. So that means that at time t_1, all lengths are 0: it's a universe that is shrinking as t_0 approaches t_1. So the spatial components are all becoming progressively smaller, and at $t = t_1$, the universe has zero size! This is known as a **big crunch**, which is the opposite of a big bang, and it causes a **gravitational singularity**. (That's what we find inside a black hole, too!)

Example 135

Finally, let's consider the metric

$$ds^2 = -(1 - \omega^2 r^2)dt^2 + 2\omega r^2 dt d\phi + dr^2 + r^2 d\phi^2 + dz^2.$$

Secretly, this is actually just a coordinate transformation of flat space: it's an exercise to find a coordinate transformation that takes us back to $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$. (These are called **Born coordinates**, and it's related to the rotating disk.) We might ask: how are we supposed to know if a metric is secretly just flat space? The right thing is to think about **tensor invariants**! (For example, if we replace the 6 in the exponent above with a 2, it actually just gives us flat space.)

Finally, let's do a problem on free fall:

Problem 136

Consider a lab frame on earth, and say that $g = 10$ m/s^2. In this lab frame, we draw four paths in a spacetime diagram: they all start at $(x, t) = (5, 1)$ and end up at $(5, 3)$, but the paths also pass through $(0, 2)$, $(5, 2)$, $(10, 2)$, and $(50, 2)$, respectively. Which has the greatest proper time?

The longest proper time is measured in a straight line, but it's measured in a straight line with respect to an inertial frame! So we need a curve that is straight with respect to a freely falling frame: the answer is thus the curve passing through $(10, 2)$, because that can correspond to the necessary acceleration that we want. Then we have an inertial frame, and the longest proper time comes from a straight line.

Here's a brain teaser to think about for later: start with two curves that start off perpendicular to the equator on Earth, and follow those curves until they intersect. So we started off with two parallel lines that eventually intersect, and that allows us to measure the curvature of the space: "how quickly" do the two lines cross? Specifically, let's say v^μ is the tangent vector: how quickly does that change, and what is $\partial_\nu v_\mu$?

Let's call that quantity $T_{\mu\nu}$. It seems that we can raise the indices via

$$T_{\mu\nu} g^{\mu\alpha} g^{\nu\beta} = T^{\alpha\beta} = \partial^\alpha V^\beta.$$

But we actually have to be careful here: what's happening here is that

$$\partial_\nu g^{\mu\alpha} V_\mu \neq g^{\mu\alpha} \partial_\nu V^\mu,$$

because the $g^{\mu\nu}$ is no longer a constant. The fundamental idea here is that ∂_μ is not a tensor! And we do actually need to fix this: by trying to get the commutativity to work, we can figure out how to fix this and turn it into a tensor.

27 October 28, 2019

This week, we'll try to study how objects move under the presence of gravity!

First of all, recall the (weak) **equivalence principle**, which tells us that in the presence of a gravitational field, we can find a small enough region of the universe such that the universe looks **locally** flat. This means we can introduce a free-falling frame at each local point in spacetime, and this lets us use Minkowskian coordinates in a small-enough space.

But we can't do this globally if we have gravity! Instead, general relativity deals with this problem by saying that energy, momentum, and mass (in other words, the $T^{\mu\nu}$ tensor) creates curvature in spacetime, and that's what we view as gravity. So gravity is not a force like electromagnetism: it's a manifestation of curved spacetime.

So we'll need to work with generic coordinates from here on, including coordinates that can be both position- and time-dependent! We won't stick to inertial frames anymore. In other words, our **metric** $g_{\mu\nu}$ now depends on our position $\mathbf{x}$, and therefore our basis vectors $\mathbf{e}_\mu(\mathbf{x})$ are also time- and position-dependent, because $g_{\mu\nu} = \mathbf{e}_\mu \cdot \mathbf{e}_\nu$.

Let's see what changes! Recall that we defined a contravariant four-vector to be an object that changes frame in a specific way (they change like the differentials dx^μ). In our earlier discussions, this specific way was just a Lorentz transform, and we can do a similar thing here:

$$dx^{\mu'} = \frac{\partial x^{\mu'}}{\partial x^\alpha} dx^\alpha = \Lambda^{\mu'}{}_\alpha dx^\alpha,$$

where the entries of the matrix Λ is now **dependent on time and space**: they are nonconstant!

Similarly, we can define one-forms to transform via

$$w_{\mu'} = \Lambda^\alpha{}_{\mu'} w_\alpha,$$

where this Λ is still the inverse matrix of the previous one:

$$\Lambda^{\alpha'}{}_\beta \Lambda^\beta{}_{\sigma'} = \delta^{\alpha'}{}_{\sigma'}$$

With this, all of the geometric ideas for changes of coordinates are still valid! The biggest difference is just that the transformation matrix doesn't just depend on the relative velocity between frames.

Dot products and raising/lowering are still obtained in he same way, as well:

$$\mathbf{A} \cdot \mathbf{B} = g_{\mu\nu} A^\mu B^\nu, \quad A_\mu = A^\nu g_{\mu\nu},$$

So what's different in this new formalism? If our objects are left in flat spacetime, they stay at constant velocity (meaning there is no acceleration), and our object will go straight. Meanwhile, in general relativity, objects can only go "as straight as possible:" this will become more defined soon. We'll use some principles to quantify that:

- Extremal aging: we take a path through spacetime that maximizes the proper time τ;
- Going straight means the velocity vector does not change (more geometrical).

Example 137

Neglect gravity for now, so we focus on the special relativity case. In general, say we have two timelike events A and B: there exists some frame S where $x_A = x_B$.

Consider two different ways we can get from A to B: one potential path (call this path I) stays at x_A the whole time, and the other (path II) has a small perturbation so that it moves around a bit (this should remind us of the twin paradox). Which path does the object actually take?

We can easily calculate the proper time of the first curve:

$$d\tau_I = dt \implies \tau_I = \int_A^B d\tau_I = t_B - t_A.$$

Meanwhile, notice that the second curve has

$$d\tau_{II} = \sqrt{dt^2 - dx^2},$$

so

$$\tau_{II} = \int_A^B \sqrt{dt^2 - dx^2} d\tau \leq \int_A^B d\tau = \tau_I$$

(notice that equality only occurs when $dx = 0$ everywhere.) So because the first type of path (the straight line) **maximizes the proper time,** it is preferred, and this is indeed the path that we see it follow in the absence of other forces.

With this idea in mind, let's add gravity into the mix. Say we have some curved surface: for each point, we can still introduce a free falling frame where things look flat locally. So one thing we can do with a curve on our curved surface is to break it up! We can state our results as the following:

Proposition 138

Throughout spacetime, objects without forces (except gravity) will move on a worldline corresponding to a local maximum of τ, known as a **geodesic**.

In general, the length along this curve can be described via

$$d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu.$$

How can we describe the geodesic $\mathbf{x}^\mu(\lambda)$ connecting two **time-like events**, where λ is a parameter for our curve? Well, let's say that we have some arbitrary nearby path

$$\mathbf{y}^\mu(\lambda) = \mathbf{x}^\mu(\lambda) + \delta x^\mu(\lambda)$$

so that we just have a **local perturbation** which is very small. (We also need to assume that the perturbation is 0 at the beginning and end, so that our perturbated path still goes from A to B.) We know that

$$\tau_{\text{geodesic}} = \int_A^B d\tau = \int_A^B \frac{d\tau}{d\lambda} d\lambda = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda.$$

If we now consider this other curve y and calculate the proper time along it, it will be some $\tau_{\text{geo}} + d\tau$. **But if we assume that the geodesic maximizes the proper time, the derivative should be zero!** In other words, we can say that locally, we have $d\tau = 0$.

How do we work with this? If we define the function

$$f = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda},$$

then we know that the function $\tau = \int_A^B \sqrt{-f}\,d\lambda$, the proper time, is maximized at our geodesic f. Then the perturbation along the curve y looks like

$$\boxed{\tau + d\tau} = \int_A^B \sqrt{-(f+\delta f)}d\lambda = \int_A^B \sqrt{-f\left(1+\frac{\delta f}{f}\right)}.$$

We can Taylor expand this out to

$$\approx \int_A^B \sqrt{-f}\left(1+\frac{1}{2}\frac{\delta f}{f}\right)d\lambda = \boxed{\int_A^B \sqrt{-f}\,d\lambda + \frac{1}{2}\int_A^B \sqrt{-f}\frac{\delta f}{f}d\lambda}.$$

The first piece here is τ, the proper time along the geodesic, so it cancels out. So if $d\tau = 0$ (because we're working at a local maximum), we have

$$0 = d\tau = \frac{1}{2}\int_A^B \frac{\delta f\,d\lambda}{\sqrt{-f}}.$$

Remark 139. *This should look similar to the Lagrangian equations from 8.223, if we've taken that class before. (We'll explain this in a bit more detail in a later class, too.)*

So far, we've been using a generic parameter λ to tell us where we are along the curve. But now we might as well make this easier for ourselves: **pick λ to be the proper time along the geodesic**, so $d\lambda = d\tau_{\text{geo}}$. So now we can get rid of the $\sqrt{-f}$ in the denominator:

$$-f = g_{\mu\nu}(x^\sigma + \delta x^\sigma)\cdot\frac{d}{d\tau}(x^\mu+\delta x^\mu)\frac{d}{d\tau}(x^\nu+\delta x^\nu).$$

Since we're working very close to the geodesic, we only need to keep the leading order terms:

$$\approx g_{\mu\nu}(x)\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau},$$

and this is just

$$= g_{\mu\nu}U^\mu U^\nu = -1,$$

because this is the expression for the norm of the four-velocity! **So if we only work at the linear order for perturbations**, $\sqrt{-f} = \sqrt{-(-1)} = 1$ goes away, and we're left with the equation

$$\boxed{0 = \frac{1}{2}\int_A^B \delta f\,d\lambda = \frac{1}{2}\delta\left(\int_A^B f\,d\lambda\right)}.$$

So that means we don't need to solve any nasty integrals: we perturb our path a tiny bit, and we want the variation of $\int_A^B f\,d\lambda$ to be equal to 0.

How do we do that? For a generic curve $x^\mu + \delta x^\mu$, we can expand out

$$g_{\mu\nu}(x^\sigma + \delta x^\sigma) \approx g_{\mu\nu}(x^\sigma) + \partial_\alpha g_{\mu\nu}\delta x^\alpha$$

(plus higher-order correction terms). Plugging this in, note that perturbing our function f by a little bit yields

$$I = \int_A^B f\,d\lambda \implies I = \delta I = \int_A^B g_{\mu\nu}(x^\sigma+\delta x^\sigma)\frac{d}{d\tau}(x^\mu+\delta x^\mu)\frac{d}{d\tau}(x^\nu+\delta x^\nu)\,d\tau.$$

Expand this function out and keep only linear terms in δx (zeroth order term on the right side cancels with the I on the left hand side). Then

$$0 = \delta I = \int_A^B \left[g_{\mu\nu}(x^\sigma) \frac{dx^\mu}{d\tau} \frac{d\delta x^\nu}{d\tau} + g_{\mu\nu}(x^\sigma) \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} + \partial_\alpha g_{\mu\nu} \delta x^\alpha \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \right] d\tau.$$

We want to say that this integrand needs to be 0 for any perturbation (like in Lagrangian mechanics), but we can't do this yet because some of the perturbations are inside the derivatives. So we have to do some more integration by parts, and we'll find that

$$\int_A^B g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = -\int_A^B \frac{d}{d\tau} \left[g_{\mu\nu} \frac{dx^\mu}{d\tau} \right] \delta x^\nu + (\text{correction})_A^B.$$

But the correction term has a δx term, and recall that we constrained our path to start at A and end at B, so that ends up being 0. Do the same thing for the second term, and the bottom line is that

$$\delta I = 0 \implies 0 = \int_A^B d\tau \delta x^\sigma \left(\partial_\sigma g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - \frac{d}{d\tau} \left(g_{\mu\sigma} \frac{dx^\mu}{d\tau} \right) - \frac{d}{d\tau} \left(g_{\sigma\nu} \frac{dx^\nu}{d\tau} \right) \right).$$

And now we can do the thing we wanted to do: the parenthetical term must evaluate to 0, because this expression has to be 0 for **any** perturbation. Grinding out the calculations yields

$$\frac{d^2 x^\beta}{d\tau^2} + \frac{1}{2} g^{\beta\sigma} \left(\partial_\sigma g_{\mu\nu} - \partial_\mu g_{\sigma\nu} - \partial_\nu g_{\sigma\nu} \right) \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0.$$

Definition 140

We define the **Christoffel symbols** via

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\sigma g_{\alpha\beta} - \partial_\alpha g_{\sigma\beta} - \partial_\beta g_{\sigma\alpha} \right).$$

(It's important to note that **this is not a tensor!**) This lets us write the **geodesic equation** we've just derived as

$$\boxed{\frac{d^2 x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} = 0}.$$

And notice that this is very simple in special relativity: in Lorentz spacetime, all Christoffel symbols are zero, because the derivatives of g are zero. So no force tells us that

$$\frac{d^2 x^\beta}{d\tau^2} = 0,$$

which is indeed the classical limit (no force means no acceleration). But these Γs will not be zero in general: in general, the coordinate accelerations are equal to some expression related to the Christoffel symbols. So this geodesic equation is kind of like $\vec{F} = m\vec{a}$, except that the force comes from curvature of spacetime rather than actual forces!

One other note: we can rewrite this equation as

$$\frac{d}{d\tau} \left(g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \right) = \frac{1}{2} \left(\partial_\beta g_{\sigma\rho} \right) \frac{dx^\rho}{d\tau} \frac{dx^\sigma}{d\tau}.$$

This is useful, because given a metric $g_{\mu\nu}$ (which depends on all four coordinates in general) that **does not depend on some x^σ**, $\partial_\sigma g = 0$, so the right hand side disappears and

$$\frac{d}{d\tau} \left(g_{\alpha\sigma} U^\alpha \right) = 0.$$

So the object inside the parentheses, which is the σ-component of the "covariant velocity," is constant!

Remark 141. *This last part has to do with Noether's theorem: every symmetry leads to a corresponding conserved quantity! If spacetime doesn't depend on one of the coordinates, there is a conserved quantity: the covariant four-velocity component.*

So we've finally arrived at our final equation in the absence of forces. Any forces that we add to this will appear on the right side of the geodesic equation, but we won't do that in this class.

Also, this equation applies for massive particles. What if we try to replace this with a photon? Then $d\tau = 0$, so we can't use the equation directly. But for photons, we have to go back to the derivation step where we replaced $d\lambda$ with $d\tau$, and use a different parametrization. Then we need to define

$$\frac{dx^\mu}{d\lambda} = p^\mu,$$

and then we can do through the derivation similarly.

On Wednesday, we'll arrive at the geodesic equation in a geometric way instead.

28 October 29, 2019 (Recitation)

We'll start today by talking about curved space: this helps us develop some intuition before jumping into curved spacetime. Remember that a **geodesic** can be thought of as the straightest path between two points, so we're always traveling in a straight line locally, even if this is not true globally!

Example 142

Consider two points diametrically opposite on the base of a hemisphere: if both trace a path in a "straight line" to the top of the hemisphere, they will have started off parallel to each other and still coincide eventually (this is called **positive curvature**, corresponding to **closed universes**). Similarly, we can have two paths in hyperbolic space that start off parallel but veer off eventually (**negative curvature**, corresponding to **open universes**).

Here are some fun aspects of curved space:

- The circumference of a circle is no longer $2\pi R$ (similar to the rotating disk). For example, if our space is a sphere, the equator is a circle centered at the north pole instead of the center of the sphere! And thus the radius is a quarter of the way along a great-circle, and the equator is a full-great circle, so in this specific example, $C = 4R$.
- The area of a circle is also not πR^2. Again, consider the equator as our circle: then the area of that circle is the area of the top half of the surface area of our sphere. Then if we embed this sphere as a sphere of radius r in our normal 3-dimensional space, the area of the circle is (half the total surface area)

  $$A = \frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 2\pi r^2.$$

  But because the radius of our sphere is $R = \frac{\pi}{2} r$ (one-quarter of the embedded circumference), we actually have

  $$A = \frac{8R^2}{\pi} < \pi R^2$$

  for this specific great-circle!

- The sum of the angles of a triangle is no longer 180 degrees. For example, if we pick two points on the equator of our sphere a quarter-circle away from each other and connect them to the north pole, we get a triangle with three right angles.

Note, though, that as our circles and traingles get smaller, space looks closer and closer to being locally flat, so we do approach the usual results of $C = 2\pi R$, $A = \pi R^2$, and the sum of the angles of a triangle being 180 degrees.

Remark 143. *Note that when we measure, for example, the distance between two points on a sphere, we have to make sure our path stays on our surface (because there's nothing inside the sphere unless we embed it in a higher-dimensional space)!*

So now we can move on. We've probably heard at some point the phrase "The shortest path between two points in curved spacetime is not a straight line." To understand this, let's think about how we find geodesics in space: if we have a straight line locally, and we perturb it by a little bit, that will give a larger distance ds^2 by the triangle inequality.

But this is not so true anymore in geodesics in spacetime: let's be a bit more careful now, because ds^2 can be negative! We know that we can describe the length along a spacelike curve via

$$L_{\text{spacelike}} = \int_{\text{curve}} \sqrt{ds^2} = \int_{\text{curve}} ds$$

or along a timelike curve via

$$L_{\text{timelike}} = \int_{\text{curve}} \sqrt{-ds^2} = \int_{\text{curve}} d\tau.$$

(Of course, we can have length 0 for a lightlike curve, too.) But what does it mean for a curve to be timelike? By definition, it means that the tangent vector (really, the four-velocity) has length less than 0:

$$U^\mu U_\mu < 0$$

(for a lightlike curve, this is equal to 0, meaning that it's normal to itself, and for a spacelike curve, it's greater to 0). So if a curve is spacelike somewhere and timelike in other spots, we have to break it up into the spacelike and timelike components and use the different formulas where appropriate.

So in curved space (no time), a geodesic between two points is the shortest curve. To understand what happens when we add time, consider the following metric:

$$ds^2 = -dt^2 + d\alpha^2 + \sin^2\alpha(d\theta^2 + \sin^2\theta d\phi^2).$$

So this actually corresponds to the metric on a three-sphere (that is, the lowest dimension we can embed it on is four dimensions). So in this universe, space just looks like a sphere at every moment in time, and thus our space looks like a cylinder when we add the time component.

Consider the equator of one of these three-spheres, and now perturb it a bit in the time direction. This has negative ds^2, so we have now decreased the total length ds^2. So we can decrease the length of such geodesics by slight deformation! So it's no longer true that in a Lorentzian geometry, the spacelike geodesic is the shortest distance between two points anymore. But it's also not true that it's the *longest* distance, either (we can always just wiggle it in the space direction). So what exactly makes these special? It turns out that spacelike geodesics are **saddle points** of the length in Lorentzian geometry.

We can make that a bit more precise, too. Say we have a family of curves from point a to b, and let's say that we can index them via $\gamma(\lambda)$, where λ varies continuously from 0 to 1. We're making the claim that $\boxed{\frac{dL}{d\lambda} = 0}$ if $\gamma(\lambda_0)$ is a spacelike geodesic.

Remark 144. *Also, we can prove that the* **signature** *of a geodesic (whether it's timelike or spacelike) never changes along the geodesic! So we don't have to worry about breaking up the integral in those cases.*

Because observers travel along timelike geodesics, we'll focus on those, but spacelike geodesics are important, too! We found during class that the right thing to do was to maximize $L_{\text{timelike}} = \int d\tau$ along our curve. One related note: **not all curves with $ds^2 = 0$ are geodesics**, because (for example) we might have the light ray curve or bounce back, and that's definitely not the "shortest path." We can rigorize this more:

Theorem 145 (Hawking)
A lightlike path between two points is a geodesic if and only if there exist no timelike curves between those points.

For example, we can show that there exist timelike paths near black holes between light rays that are bending!

As a reminder, the geodesic equation looks like (for timelike curves)

$$\frac{d^2x^\beta}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0.$$

The way in which we measure time along the geodesic shouldn't affect whether it's a geodesic or not. If we measure this with respect to some other parameter instead of proper time τ, this equation should still hold! Remember that

$$\Gamma^\beta_{\mu\nu} = \frac{1}{2}g^{\beta\alpha}\left(\partial_\nu g_{\mu\alpha} + \partial_\mu g_{\nu\alpha} - \partial_\alpha g_{\mu\nu}\right)$$

is the **Christoffel symbol** (it's not a tensor because the derivative operator ∂_ν is not a tensor). Let's write σ in terms of τ: how does the geodesic equation change? Well, the first term is

$$\frac{d}{d\tau}\left(\frac{dx^\beta}{d\sigma}\frac{d\sigma}{d\tau}\right) + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\frac{d\sigma}{d\tau}\frac{d\sigma}{d\tau} = 0$$

by the chain rule (noting that the Christoffel symbol does not care how we parameterize, because it's inherently about the metric). Now we can expand out the first term by the product rule: it becomes

$$\frac{d}{d\tau}\left(\frac{dx^\beta}{d\sigma}\right)\frac{d\sigma}{d\tau} + \frac{dx^\beta}{d\sigma}\frac{d^2\sigma}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\left(\frac{d\sigma}{d\tau}\right)^2 = 0,$$

and now we can just parse the first part with the chain rule again to get

$$\frac{d^2x\beta}{d\sigma^2}\left(\frac{d\sigma}{d\tau}\right)^2 + \frac{dx^\beta}{d\sigma}\frac{d^2\sigma}{d\tau^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{d\sigma}\frac{dx^\nu}{d\sigma}\left(\frac{d\sigma}{d\tau}\right)^2 = 0.$$

So dividing through by $\left(\frac{d\sigma}{d\tau}\right)^2$ and moving a term to the right side, our geodesic equation becomes

$$\frac{d^2x^\beta}{d\sigma^2} + \Gamma^\beta_{\mu\nu}\frac{dx^\mu}{dx^\sigma}\frac{dx^\nu}{dx^\sigma} = -\left(\frac{d\sigma}{d\tau}\right)^{-2}\frac{d^2\sigma}{d\tau^2}\frac{dx^\beta}{d\sigma} = \frac{d}{d\sigma}\ln\left(\frac{d\tau}{d\sigma}\right)\frac{dx^\beta}{d\sigma}.$$

This is the general geodesic equation for a general parameter σ! And indeed, if $\tau = \sigma$, the natural log term evaluates to 0, so the right hand side becomes zero. But that's not the only case where this happens: we can also have $\frac{d\tau}{d\sigma}$ be a constant, so $\tau = a\sigma + b$ works as well. In other words, we can use any affine function of τ and still have the simple form of the geodesic equation! And that's why we often call the simple form the **affinely parametrized geodesic equation**: the main idea here is that we have a whole family of parameters which are essentially equivalent.

29 October 30, 2019

Professor Vitale mistakenly thought we were more familiar with the principle of least action, but a few people raised concerns about the accessibility of the content on Monday. So we'll go through some of those ideas in a gentler way! And there will be extra office hours, too.

The two main things we want to keep from last time are that (1) in the absence of forces (where gravity doesn't count as a force), the path of a massive particle from point A to point B in spacetime **maximizes the proper time**

$$\tau_{\text{true}} = \int_A^B d\tau.$$

In other words, if we pick another path, then $\tau_{\text{true}} > \tau_{\text{other}}$, no matter what our perturbation looks like! And (2) doing some calculations, we find that the equation of motion that governs this motion (known as the **geodesic equation**) is

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0.$$

This is a differential equation, and the solution x^μ is the actual path that the particle will take from A to B! The Γ expressions are called the Christoffel symbols, and they depend on the metric and its derivatives only.

Notably, we can rewrite the geodesic equation as

$$\frac{d}{d\tau}\left(g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\right) = \frac{1}{2}\left(\partial_\beta g_{\sigma\lambda}\right)\frac{dx^\sigma}{d\tau}\frac{dx^\lambda}{d\tau}.$$

If g, the metric, has all components independent of some β, then the right hand side vanishes and we're left with $g_{\alpha\beta}\frac{dx^\alpha}{d\tau} = U_\beta$ being constant.

We'll go through a lot of the ideas in a less steep way today! First of all, let's define some relevant ideas from Lagrangian mechanics. We'll do this from a Newtonian point of view, so we have no gravity or relativity.

Say we have a mechanical problem where an object travels, and we know its starting and ending points. We can actually find the path that this particle takes without having to (for example) draw a free-body diagram by following these steps:

- For every point along the curve, calculate the value of K, the kinetic energy, and U, the potential energy.
- Define the **Lagrangian**

  $$L = K - U$$

  and the **action** of our path to be the integral of our Lagrangian

  $$S = \int_{t_1}^{t_2} L\,dt.$$

  The value of S here depends on the path $x(t)$ that we take from our starting to ending point.
- The **principle of least action** then tells us how to find the real path: we find the one such that S is minimized.

Many of us may wonder why this works. Is there any reason we want to minimize this action? One of the ways to think about this is that in quantum mechanics, a set of particles can be described with a wavefunction. If we ask for the probability that a particle goes from point A to point B, the probability it follows each path is proportional to $e^{iS/\hbar}$. So the total probability that we go from A to B is essentially related to the sum of $e^{iS/\hbar}$ over all paths. Quantum mechanics then tell us that particles want to go along extreme paths: if S is approximately constant near its maximum, ∂S is close to 0.

Remark 146. *Note that S is a scalar here, so this concept can be applied to a lot of ideas in physics.*

How do we actually find this path of least action? Say we have a function $f(x(t); \dot{x}(t), t)$, which tells us the value of the position and derivative of our current position x. So the mathematical problem we want is to solve the optimization problem

$$J = \int f(x, \dot{x}, t)$$

for the ideal $x(t)$. Here, we should think of $\dot{x}$ as an **independent variable** to x, so we are allowed to write expressions like derivatives with respect to $\dot{x}$:

Example 147

A function like $f = \frac{1}{2}m\dot{x}^2$ just has derivative $\frac{\partial f}{\partial \dot{x}} = m\dot{x}$.

With this, we can now formulate our problem explicitly: we know that our particle goes from (x_1, t_1) to (x_2, t_2), and our goal is to find the path x that gives us the least value of J.

Let $x_e(t)$ be the physical path that the particle follows. Intuitively, if this has the minimum value of J, the "derivative" at $x_e(t)$ should be 0. In other words, say that we perturb our path by a little bit, but we make sure that the starting and ending points are constant (these are the constraints we're trying to solve with):

$$x(t, \alpha) = x_e(t) + \alpha A(t), \alpha \in \mathbb{R}, A(t_1) = a(t_2) = 0.$$

If J were just a function of a single parameter, we could easily take the derivative with respect to that parameter and set it equal to 0. But in this case J is a **functional**, in terms of another function f. So to get around this, define a new function

$$J(\alpha) = \int f(x_e + \alpha A, \dot{x}_e + \dot{A}\alpha),$$

which just corresponds to the value of J at a slightly perturbed path. Notice that if $\alpha = 0$, $J(0) = J$ yields the actual path x_e that we are trying to solve for. So if this is a local extremum, the derivative with respect to α must be 0 here:

$$\boxed{\left.\frac{dJ}{d\alpha}\right|_{\alpha=0} = 0}.$$

So how do we work with this? We have to do some calculation with the chain rule, remembering that our variable f is in terms of both x and $\dot{x}$:

$$\frac{d}{d\alpha}J(\alpha) = \int_{t_1}^{t_2} dt \frac{df}{d\alpha} = \int_{t_1}^{t_2} dt \left(\frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial \dot{x}}\frac{d\dot{x}}{d\alpha}\right).$$

But we know that

$$\frac{dx}{d\alpha} = \frac{d}{d\alpha}(x_e(t) + \alpha A(t)) = A(t),$$

and similarly $\frac{d\dot{x}}{d\alpha} = \dot{A}$, so we can plug these in to find

$$\frac{dJ}{d\alpha} = \int_{t_1}^{t_2} dt \left(\frac{\partial f}{\partial x}A(t) + \frac{\partial f}{\partial \dot{x}}\dot{A}(t)\right).$$

We can do integration by parts on the second term:

$$\int \frac{\partial f}{\partial \dot{x}}\frac{dA}{dt}dt = -\int dt \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right)A(t)$$

(because the boundary term that we get from integration by parts goes to 0), and that turns this expression into

$$\int_{t_1}^{t_2} dt A(t) \left(\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) \right).$$

But here, A can be any function (remember that we defined it arbitrarily as some perturbation). So if we want this derivative $\frac{dJ}{d\alpha} = 0$ to hold for all functions A, we must satisfy the **Euler-Lagrange equations**

$$\frac{\partial f}{\partial x} - \frac{d}{dt} \left(\frac{\partial f}{\partial \dot{x}} \right) = 0.$$

Everything so far has been mathematical: it's telling us how to find the derivative with respect to any function $f(x, \dot{x}, t)$. But now let's go back to physics. If we're trying to maximize or minimize the function

$$S = \int_{t_1}^{t_2} L dt$$

for our Lagrangian $L = K - U$, our path must follow

$$\frac{\partial L}{\partial x} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) = 0.$$

Let's apply this to a situation where we have a potential field:

Example 148

The kinetic energy of a particle is always $K = \frac{1}{2} m \dot{x}^2$. Say that the potential energy is just some function of our position $U(x)$: what's the equation of motion that we get?

Then the Euler-Lagrange equation tells us that because $L(x, \dot{x}) = \frac{1}{2} m \dot{x}^2 - U(x)$, and remembering that x and $\dot{x}$ are independent of each other,

$$\frac{\partial L}{\partial x} = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) \implies -\frac{\partial U}{\partial x} = \frac{d}{dt}(m\dot{x}) = m\ddot{x},$$

which is exactly Newton's second law!

Remark 149. *In real life, most of our Lagrangians depend on more than one coordinate. But it turns out that if we have N coordinates instead of just one, we can write the Euler-Lagrange equation*

$$\frac{\partial L}{\partial q} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) = 0$$

for **each of the** *"generalized coordinates" q (they could be, for example, lengths or angles). Here,*

$$p_i = \frac{\partial L}{\partial \dot{q}_i}$$

is called the **generalized momentum** *for q_i. The bottom line is that we just solve N equations instead of one.*

The thing that's beautiful about this is that if our Lagrangian doesn't depend on one of the q_is, that means that

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0,$$

which means that we have the conserved quantity $\frac{\partial L}{\partial \dot{q}_i}$! This is one way to derive things like conservation of momentum, energy, and so on.

Example 150

Say we work in spherical coordinates, and we have a potential $U(r)$ that only depends on the radial distance.

The kinetic energy in spherical coordinates can be written as

$$K = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right),$$

so the Lagrangian only depends on $\dot{\phi}$, not ϕ. So by the Euler-Lagrange equation,

$$0 = \frac{\partial L}{\partial \phi} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{d}{dt}\left(mr^2\sin^2\theta\dot{\phi}\right),$$

meaning that the parenthetical term here is conserved!

So now let's reintroduce special relativity and gravity into the mix. In special relativity, a free particle will move along a path that maximizes the proper time along that path, $\tau = \int d\tau$, given that we know its starting and ending point. We motivated this last time by thinking about a frame on which A and B are colocated: it makes sense for the particle to stay in place. Let's show how to use this Euler-Lagrange approach to make the ideas of "inertial" and "moving along the straight path" related!

Remember that we parameterize our path using some generic variable λ, so if we use t here,

$$\tau = \int_A^B \frac{d\tau}{dt}dt = \int_A^B \sqrt{1 - \frac{dx^2}{dt^2}}dt$$

because $d\tau^2 = dt^2 - dx^2$. So now we're trying to do a similar kind of maximization problem, but we're just trying to do it with $f^{1/2} = \sqrt{1 - \frac{dx^2}{dt^2}}$ instead of with the Lagrangian, so we can apply the Euler-Lagrange equations! The thing is that sometimes we get some annoying square roots in the denominator, so (as we did last time) we can often maximize this integral by picking a convenient choice of λ. In particular, we can show that maximizing this complicated integral is the same as just maximizing $\int f d\lambda$ instead (using a change of variables). So because $f = 1 - \left(\frac{dx}{dt}\right)^2$,

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0 \implies \dot{x} = 0,$$

which means that we want $\dot{x}$ to be constant. So in other words, **we've shown using the Euler-Lagrange equations that a particle moves in a straight line in our spacetime diagram under special relativity**.

As an addendum to this, we can add an $-m$ to the quantity we're trying to minimize, and the new quantity we're doing optimization with is

$$-m\int d\tau = -m\int \frac{dt}{\gamma} = \int(-m\sqrt{1-v^2})dt.$$

We claim that this integrand is the Lagrangian in special relativity! To prove this, notice that the generalized momentum

$$\frac{\partial L}{\partial \dot{x}} = -m \cdot \frac{1}{2}\frac{-2\dot{x}}{\sqrt{1-\dot{x}^2}} = \frac{m\dot{x}}{\sqrt{1-v^2}} = \gamma m v,$$

which is indeed the momentum of a particle in relativity. So the principle of finding extrema to find the equation of motion is universal, but the exact thing to minimize depends on the scenario! For example, it wouldn't make sense to try to minimize the proper time, because that doesn't give us reasonable physical results.

So now let's try to generalize this to general relativity! Here, the proper time and coordinate time are related in a slightly different way:

$$d\tau^2 = -ds^2 = -g_{\mu\nu}dx^\mu dx^\nu.$$

Now if we ask ourselves for the proper time along some path, it will be the integral

$$\tau = \int_A^B d\tau = \int_A^B \frac{d\tau}{d\lambda} d\lambda = \int_A^B \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} d\lambda.$$

So again, we're trying to minimize this integral, and again we claim that it is sufficient mathematically to find the extremum for the integral

$$I = \int -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} d\lambda$$

instead, which saves us some work! If we call this our "Lagrangian," we can just apply the Euler-Lagrange equations to $L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}$. So we have

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x^\mu}}\right) = 0.$$

Let's find this term-by-term: $\frac{\partial L}{\partial x^\mu} = -\dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta}$, because the other terms don't depend on x. Meanwhile,

$$\frac{\partial L}{\partial \dot{x}^\mu} = -g_{\alpha\beta}\frac{\partial \dot{x}^\alpha}{\partial \dot{x}^\mu}\dot{x}^\beta - g_{\alpha\beta}\dot{x}^\alpha\frac{\partial \dot{x}^\beta}{\partial \dot{x}^\mu}.$$

But a term like $\frac{\partial \dot{x}^\beta}{\partial \dot{x}^\mu}$ just yields the Kronecker delta, because different components of $\dot{x}$ are independent! So this just reduces to

$$= -2g_{\mu\beta}\dot{x}^\beta.$$

If we put everything together, we find that

$$-\dot{x}^\alpha \dot{x}^\beta \partial_\mu g_{\alpha\beta} = \frac{d}{d\tau}\left(-2g_{\mu\beta}\right)\dot{x}^\beta.$$

This again tells us that if g again doesn't depend on one of the coordinates μ, we have the conserved quantity $g_{\mu\beta}\dot{x}^\beta$, which is the covariant velocity U_β. We can always expand the right hand side by the product rule to get the other familiar form of the geodesic equation (exercise for us). So hopefully now we can go back to last class and have a better idea of what's going on!

30 October 31, 2019 (Recitation)

If we come to office hours tonight, there will be candy.

Today, we're going to talk about the variational principle, so we'll do two problems that use the principles of least action. Note, though, that this material is not on Monday's exam! For that, we should know everything required to do problems up to tomorrow's problem set, and that's it.

Problem 151

Let's derive Maxwell's equations from the principle of least action!

When we first derived this in terms of the field strength $F^{\mu\nu}$, we had to assert two of the four laws (from empirical fact). We want to be able to write down an action which makes sense to us (for example, is Lorentz-invariant), which in turn gives us the sourced Maxwell equations.

This is often how we do physics – we write down a bunch of possible things that are consistent with what we already know, and then we pick the one that is further consistent when we do more tests. Remember that we also care about having **gauge-invariance**: the Maxwell equations are invariant under $A_\mu \to A_\mu + \partial_\mu \varepsilon$ for some ε, and since ε is not a physical object, the Lagrangian should also be gauge-invariant.

It turns out we want the Lagrangian to look like

$$L = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j^{\mu}A_{\mu},$$

which means that the action is just

$$S = \int L dt.$$

Here, we're using natural units, so $\mu_0 = c = 1$. First of all, there are no free indices, and each of the tensors transforms correctly under Lorentz transform, so L, our Lagrangian, is indeed a Lorentz scalar.

What about gauge-invariance? We know that $F^{\mu\nu}$ is gauge-invariant, but it looks like the Lagrangian changes by a $-j^{\mu}\partial_{\mu}\varepsilon$. But it turns out this works out: our new action is then

$$\int (L - j^{\mu}\partial_{\mu}\varepsilon)dt.$$

But ∂_{μ} is a total derivative, so that just gives the value of ε at the beginning and end of the path: since our perturbation requires ε to be 0 at the beginning and end, this indeed **preserves the action**.

So now let's work towards Maxwell's equations: our goal is to find an extremum of the action, with respect to one of the fundamental variables of the system. In the case of the Maxwell field, we're not trying to find a path: we're finding a field! So we're actually finding the maximum over all EM fields.

In class, we found that the Euler-Lagrange equations help us characterize the properties of this maximum. Let's use that, but first, what are we optimizing over? Really, A_{μ} is more fundamental than $F^{\mu\nu}$, because

$$F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$$

means that A is essentially the most fundamental thing we're dealing with here. So we want to apply Euler-Lagrange to our four-potential A_{μ}, by thinking of L as a function of A_0, A_1, A_2, A_3, and S as a functional of L (a function of a function). But here, **each A_{μ} depends on four different spacetime variables**, so the Euler-Lagrange equation looks more complicated:

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\partial L}{\partial\left(\frac{\partial A_{\nu}}{\partial x^{\mu}}\right)}\right) - \frac{\partial L}{\partial A_{\nu}} = 0.$$

Here, $\frac{\partial}{\partial x^{\mu}} = \partial_{\mu}$, so we can write this more compactly as

$$\partial_{\mu}\left(\frac{\partial L}{\partial\left(\partial_{\mu}A_{\nu}\right)}\right) - \frac{\partial L}{\partial A_{\nu}} = 0.$$

We can confirm that this has the right index structure: $\left(\frac{\partial L}{\partial(\partial_{\mu}A_{\nu})}\right)$ behaves like a tensor $Q^{\mu\nu}$, and $\frac{\partial L}{\partial A_{\nu}}$ behaves like a tensor V^{ν}, so our equation is really of the form

$$\partial_{\mu}Q^{\mu\nu} = V^{\nu},$$

which is indeed an equality of four-vectors! **Remember that we're in special relativity here**.

So now it's time to carry out all of our calculations. First of all, we want to lower the indices in $F^{\mu\nu}F_{\mu\nu}$ because we often are taking A_{ν} derivatives:

$$F^{\mu\nu}F_{\mu\nu} = F_{\rho\sigma}F_{\alpha\beta}\eta^{\rho\alpha}\eta^{\sigma\beta}.$$

(We're changing the dummy variables because we have ν in the upcoming expression, and we don't want to get confused.) So now let's compute this bit by bit: $Q^{\mu\nu}$ is taking derivatives with respect to $\partial_{\mu}A_{\nu}$, so the $j^{\mu}A_{\mu}$ term does not contribute here. Thus,

$$Q^{\mu\nu} = \frac{\partial L}{\partial(\partial_\mu A_\nu)} = -\frac{1}{4}\frac{\partial(F_{\rho\sigma}F_{\alpha\beta})}{\partial(\partial_\mu A_\nu)}\eta^{\rho\alpha}\eta^{\sigma\beta},$$

where we can pull out the η because it's not dependent on $\partial_\mu A_\nu$. Now we can do the product rule on the derivative:

$$= -\frac{1}{4}\eta^{\rho\alpha}\eta^{\sigma\beta}\left(F_{\alpha\beta}\frac{\partial(F_{\rho\sigma})}{\partial(\partial_\mu A_\nu)} + F_{\rho\sigma}\frac{\partial(F_{\alpha\beta})}{\partial(\partial_\mu A_\nu)}\right).$$

But now by definition, $F_{\rho\sigma} = \partial_\rho A_\sigma - \partial_\sigma A_\rho$, and similarly for $F_{\alpha\beta}$. So now

$$\frac{\partial(\partial_\rho A_\sigma)}{\partial(\partial_\mu A_\nu} = \begin{cases} 1 & \rho = \mu, \sigma = \nu \\ 0 & \text{otherwise} \end{cases} = \delta^\mu_\rho \delta^\nu_\sigma$$

because all sixteen expressions of the form $\partial_\rho A_\sigma$ are independent! So now plugging this back into our product rule expression,

$$Q^{\mu\nu} = -\frac{1}{4}\eta^{\rho\alpha}\eta^{\sigma\beta}\left(F_{\alpha\beta}(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho) + F_{\rho\sigma}(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\mu_\beta\delta^\nu_\alpha)\right).$$

And now it's easy to contract δ functions and ηs with F. First we distribute the ηs in:

$$= -\frac{1}{4}\left[(\eta^{\mu\alpha}\eta^{\nu\beta} - \eta^{\nu\alpha}\eta^{\mu\beta})F_{\alpha\beta} + (\eta^{\rho\mu}\eta^{\sigma\nu} - \eta^{\sigma\mu}\eta^{\rho\nu})F_{\rho\sigma}\right].$$

Now it's time to multiply everything out! The ηs actually just raise our indices one by one, which gives us

$$= -\frac{1}{4}[F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu}].$$

But now $F^{\mu\nu}$ is antisymmetric, this ends up giving

$$= -\frac{1}{4}\cdot 4F^{\mu\nu} = -F^{\mu\nu}.$$

So this means that the left-hand side of our equation is fairly simple: it's just

$$-\partial_\mu F^{\mu\nu}.$$

Finally, we just need to compute

$$V^\nu = \frac{\partial L}{\partial A_\nu}.$$

But now similar to how we discarded $j^\mu A_\mu$ in calculating $Q^{\mu\nu}$, we can discard $F^{\mu\nu}F_{\mu\nu}$ in calculating V^ν! And that means we're just left with

$$\frac{\partial(-j^\mu A_\mu)}{\partial A_\nu} = -j^\mu\delta^\nu_\mu = -j^\nu.$$

Substituting this back into our $\partial_\mu Q^{\mu\nu} = V^\nu$ yields the Maxwell equation

$$\boxed{\partial_\mu F^{\mu\nu} = j^\nu},$$

as desired!

Next time, we'll think about having an electromagnetic force on our particle as well. It turns out the action for a charged particle is

$$S = -m\int d\tau + q\int A_\mu \frac{dx^\mu}{d\tau}d\tau,$$

and we'll find the geodesic equation plus an extra term that gives the force from an electric field.

31 November 6, 2019

This lecture is being given by Professor Hughes.

We started this semester by developing special relativity: the fact that c, the speed of light in vacuum, is an invariant to all observers. This modifies our notions of space and time, and thus we need to adjust our notions of force, momentum, and so on.

Well, we learned about the Coulomb force early on in E&M

$$\vec{F}_c = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_2}\hat{r}$$

and the gravitational force early on in mechanics

$$\vec{F}_g = -\frac{Gm_1m_2}{r}\hat{r}:$$

up to some "cosmetic" differences, it seems like these are essentially the same equation. So it sounds like gravity should be easy to fix, because electromagnetism basically gave us special relativity.

But there's a key difference: q_1 and q_2 are Lorentz invariants, so we measure them to be the same in all reference frames. Meanwhile, it's not so obvious m_1 and m_2 are Lorentz invariant! If m_1 and m_2 designate the rest mass of an object, then they are indeed Lorentz invariant. But using rest mass in our equations of motion would contradict the gravitational redshift of light: even though a photon is massless, it still loses some energy as it climbs up against gravity.

So suppose that a light wave goes from the ground (at a height 0) to some higher location (at a height h). If we trace out the path of the first crest of our wave in spacetime, that should be congruent to the path of the second crest, if we can describe this whole situation in a single Lorentz frame.

But that would mean that the period at the bottom and top are the same, which is equivalent to saying that the frequencies at the bottom and top are the same. But we know that

$$\frac{\nu_T}{\nu_B} = 1 - \frac{gh}{c^2}$$

is supposed to be different between the top and bottom! So we can't use a global Lorentz frame here: we're going to need something more powerful.

So that's where Einstein comes in: fun fact, he was a great physicist, but only an okay mathematician. Getting the physics right didn't take him that long. His insight was that a freely-falling frame, all objects feel the same gravitational acceleration, so all of the objects freely floating along with an observer are following straight lines. But the mathematics was a bit more complicated than that!

By the way, a key point here is that **gravitational mass is identical to inertial mass**, meaning that

$$m\vec{a} = m\left(\frac{-GM_E\hat{r}}{r^2}\right).$$

We should be a bit careful: gravity is not actually uniform, so there's some variation, which we call gravity's **tide**, as we move along the region. Our freely falling frame only works locally: it's not true that all gravitational effects will cancel out.

So let's think about some principles, which were initially developed by Einstein, which will help us put gravity and relativity together!

Fact 152

In a freely falling frame, things look inertial in the absence of non-gravitational effects. So a Lorentz transform should almost perfectly work in a finite region, which we call a **local Lorentz frame**.

Fact 153

The size of that finite region is controlled by tides: very strong tides, giving big derivatives associated with the gravitational field, means our local Lorentz frame is small, and vice versa.

So how do we turn these into principles with mathematics? We'll talk about the **principle of equivalence**, and we can describe it in a few simple ways:

- (Weak equivalence principle) Over sufficiently small regions, the motion of freely falling bodies due to gravity cannot be distinguished from uniform acceleration.
- (Einstein's equivalence principle) Over sufficiently small regions, the laws of physics in a freely falling frame reduce to those of special relativity.

Einstein's equivalence principle gives us a tool for discussing the motion of a body in relativistic gravity! If we're in our freely falling frame, and there's no gravity, then the motion is unaccelerated (because everything else that is falling with us experiences the same acceleration as us), and this means that observers are maximally aging: out of all possible trajectories, the one we're trying to travel through accumulates **the most proper time**.

So given any random spacetime (we're often going to be given a spacetime), it's a reasonable question to ask where that spacetime "comes from.,"' and analougously how gravity arises from a source!

Remember that we know that

$$\Phi(g) = \frac{-GM}{r}$$

is the gravitational potential associated with a point mass, and we can write this in the continuous case as

$$\nabla^2 \Phi(g) = 4\pi G \rho_m$$

Here, the first derivative of Φ yields our field, and the second derivatives have to do with the tide (or matter density)! But notice that this equation is formulated in terms of a non-Lorentz fiew: we should start by rethinking the equation.

Well, our goal is to formulate our equations of motion in tensorial form, but we can't write down anything with just the mass density ρ_E (it's not a scalar)! Instead, we need to work with the stress-energy tensor $T^{\mu\nu}$. This is difficult, and it turns out that over 10 years, Einstein figured out the correct equation:

$$\boxed{G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}}.$$

This equation is deceptively simple: what's $G^{\mu\nu}$? Well, we start with our length in spacetime

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta,$$

and we take one derivative of the metric to get our Christoffel symbols

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\gamma} \left(\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\gamma\alpha} - \partial_\gamma g_{\alpha\beta} \right)$$

Take another derivative, and we get the **Riemann curvature tensor**

$$R^{\alpha}{}_{\mu\lambda\sigma} = \partial_\lambda \Gamma^{\alpha}{}_{\sigma\mu} - \partial_\sigma \Gamma^{\alpha}{}_{\lambda\mu} + \Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\nu}{}_{\sigma\mu} - \Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\nu}{}_{\lambda\mu}$$

which is very nonlinear. If we define $R_{\mu\nu}$ to be the result of contracting α with λ, and then define R to be the contraction R^{μ}_{μ},

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R.$$

So this is a second order nonlinear operator acting on the metric, and Einstein's equation tells us that we can set it equal to the source $T^{\mu\nu}$, which can sometimes depend on our metric $g^{\mu\nu}$ as well! So this equation is incredibly hard to solve, and we won't be working with it very much here.

Remark 154. *A month after Einstein published the field equations in 1915, Karl Schwarzschild showed that a spacetime with the line element*

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)(cdt)^2 + \left(1 - \frac{2GM}{rc^2}\right) dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

is actually an exact solution to the field equations! (He did this while he was serving as an active-duty officer on the Russian front, and he was also very ill. So then he died after doing this...)

So we'll play with that solution above: it's a very interesting spacetime. It has interesting behavior: if we consider $r = \frac{2GM}{c^2} \approx 3\text{km} \cdot \frac{M}{M_{\text{sun}}}$, this actually tells us about the spacetime of a black hole! (In particular, if we collapse the sun into a 3 kilometer ball, it would exhibit black hole-like behavior.) This is actually the solution for $T^{\mu\nu} = 0$, and we'll talk about it a little bit more later.

Right now, though, we'll talk about a slightly different solution, and it's only an approximate one:

$$ds^2 = -(1 + 2\Phi)(cdt)^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2)$$

where we make the assumption that

$$\Phi = -\frac{GM}{rc^2}$$

(basically our gravitational field scaled by c^2 to be unitless) is very small. So we basically ignore all second-order terms here: this is called the **weak gravity solution**, and let's try to think a little bit about what motion looks like! Remember that we have a principle of **maximal aging**: it turns out the Lagrangian for our system here can be written as

$$L = \frac{m}{2} g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau},$$

and we apply the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\alpha} = \frac{d}{d\tau} \frac{\partial L}{\partial \dot{x}^\alpha}$$

to our system. Let's plug that in here and see what happens! Well, $g_{\mu\nu}$ only has diagonal entries here, and they take on a pretty simple form: a dot above a variable corresponds to a derivative with respect to $d\tau$:

$$L = \frac{m}{2} \left[-(1 + 2\Phi) c^2 \dot{t}^2 + (1 - 2\Phi)(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \right].$$

We apply the Euler-Lagrange equation for $\alpha = 1$, which means we only care about derivatives with respect to x and $\dot{x}$. First, let's calculate the derivative of L with respect to x: remember that $\dot{x}$ and the other derivatives are constant with respect to this partial derivative, but Φ is a function in terms of x, so

$$\frac{\partial L}{\partial x} = \frac{m}{2} \left[-c^2 \dot{t}^2 \frac{\partial}{\partial x}(1 + 2\Phi) + (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \frac{\partial}{\partial x}(1 - 2\Phi) \right].$$

We can make another simplification here: as long as our motion is fairly nonrelativistic, $\dot{t}$ is going to be approximately 1, and $\dot{x}, \dot{y}, \dot{z}$ are much smaller than c. (And by the way, this means we've now introduced a particular frame that is special: this means we're making our motion nonrelativistic to whatever is creating our field Φ.) So this basically means that if we're okay doing nonrelativistic motion on this first pass, we can toss the second term here completely! That means

$$\frac{\partial L}{\partial x} \approx \frac{m}{2} \cdot -c^2 \frac{2\partial \Phi}{\partial x}.$$

Remembering that we know exactly what the functional form Φ is, it turns out we end up with

$$\frac{\partial L}{\partial x} = -\frac{GMmx}{r^3}.$$

The other part of the Euler-Lagrange equation is simpler, because Φ is constant with respect to $\dot{x}$:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{d}{d\tau}\left(m(1-2\Phi)\dot{x}\right) = m\ddot{x}(1-2\Phi) - 2m\dot{x}\frac{d\Phi}{d\tau}$$

(by the product rule). So if we use the chain rule on this last part, we find that

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} = m\ddot{x}(1-2\Phi) - \frac{2GMm}{r^2}\left(\frac{x}{r}\left(\frac{\dot{x}}{c}\right)^2 + \frac{y}{r}\left(\frac{\dot{x}\dot{y}}{c^2}\right) + \frac{z}{r}\left(\frac{\dot{x}\dot{z}}{c^2}\right)\right).$$

Remember that we're working in a nonrelativistic limit here, so the terms with a $\frac{1}{c^2}$ die. In particular, remember that

$$1 - 2\Phi = 1 + \frac{2GM}{rc^2} \approx 1$$

if we toss the c^2 terms! So if we put everything together, we find that

$$m\ddot{x} = \frac{d}{d\tau}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} = -\frac{GMmx}{r^3},$$

and indeed this simplifies to

$$m\ddot{x} = -\frac{GMm}{r^3}x,$$

which looks a lot like Newtonian gravity! Repeating this for the y and z-components gives us back

$$m\frac{d^2\vec{r}}{d\tau^2} = -\frac{GMm\vec{r}}{r^3}.$$

Remark 155. *When Einstein first derived the field equation, he left that $\frac{8\pi G}{c^4}$ as some constant κ. And he figured out what κ was by looking at this nonrelativistic limit!*

So this

$$1 - 2\Phi = 1 + \frac{2GM}{rc^2} \approx 1$$

approximation is pretty sketchy. If we keep it as is, that modifies the motion of our bodies a little bit: Newton's equations predict that bodies move in ellipses. This additional correction actually fixes Kepler's laws a little bit, and Einstein knew that Mercury's orbit wasn't quite a Keplerian ellipse. And indeed, when he did the approximation to the next order, that actually explained the motion: Mercury's equation of motion processes at a rate

$$\frac{d\phi_{\text{peri}}}{dt} = \frac{6\pi GM_{\text{sun}}}{a(1-e^2)Pc^2}$$

where P is the orbital period and e is the eccentricity. Plugging in the numbers here, we find that this gives a rate of 42.9 arcseconds per century. And the actual value is measured to be around 43 arcseconds per century!

32 November 7, 2019 (Recitation)

The average on this year's midterm was 72, and the spread is pretty wide. (The first midterm's average was maybe around 80.) So we shouldn't feel like we have to drop if we scored in the 50s – we can still get a very good grade if we do well on the final. And we should always feel like we can get help through tutoring or office hours or something else.

Today, we'll go over the midterm, so that we'll be good to go on the final.

Problem 156

Two people are arguing about the nature of light. One person says that we have an electromagnetic wave satisfying

$$\vec{E}(z,t) = A\sin(\omega t - kz)\hat{x}$$

$$\vec{B}(z,t) = A\sin(\omega t - kz)\hat{y}$$

(where we set $c = 1$).

We are supposed to boost along the z-direction with a relative three-velocity of $v = \beta c$ and see what the new electromagnetic waves look like!

First of all, we notice that the argument of the sine function $\omega t - kx$ can be written as

$$\phi(z,t) = k_\mu x^\mu = \omega t - kz,$$

where $k_0 = \omega, k_1 = k_2 = 0, k_3 = -k$, and $x_0 = t, x^1 = x, x^2 = y, x^3 = z$ (as usual). Thus, we can compute (in the new primed reference frame)

$$\Phi(z', t') = \omega' t' - k' z'$$

by computing ω' and k'. We like to transform four-vectors, so one way for us to do this is to first turn the 1-form into a four-vector: thus, we replace ω with $-\omega$ (because $\eta^{\mu\nu}$ gives us a negative sign in the zeroth component), and we get

$$\begin{bmatrix} -\omega' \\ k_x' \\ k_y' \\ k_z' \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} -\omega \\ 0 \\ 0 \\ -k \end{bmatrix}$$

and we can do out the matrix multiplication to get the correct answer:

$$\omega' = \gamma(\omega - \beta k), \quad k' = \gamma(k - \beta\omega).$$

The main problem here was not knowing what four-vector to transform! To finish the Lorentz transformation from here, we also need to transform $\vec{E}$ and $\vec{B}$ themselves. Since $\vec{v} = \beta\hat{z}$, we can use the transformation of fields: the parallel components are 0, so $E_z' = B_{z'} = 0$. From here, we need to find the perpendicular components:

$$\vec{E}_\perp' = \gamma(\vec{E} + \vec{v} \times \vec{B}_\perp).$$

If we denote $A_s = A\sin(k_\mu x^\mu)$, A_s is a Lorentz scalar, so we don't need to transform it:

$$\vec{E}_{\perp'} = \gamma A_s(\hat{x} + \beta\hat{z} \times \hat{y}) = \gamma A_s(1 - \beta)\hat{x}$$

and we can write $\gamma(1-\beta) = \sqrt{\frac{1-\beta}{1+\beta}}$. Very similarly,

$$\vec{B}_{\perp'} = \gamma(\vec{B} - \vec{v} \times \vec{E}_\perp) = \gamma A_s(\hat{y} - \beta\hat{z} \times \hat{x}) = \gamma A_s(1-\beta)\hat{y}.$$

So this tells us that

$$(E'_x, E'_y, E'_z) = \left(A_s\sqrt{\frac{1-\beta}{1+\beta}}, 0, 0\right), \quad (B'_x, B'_y, B'_z) = \left(0, A_s\sqrt{\frac{1-\beta}{1+\beta}}, 0\right),$$

and we had to make sure to write out everything explicitly in terms of the given variables!

So now if we want to find the energy flux, we can look at the Poynting vector

$$\langle\vec{S}\rangle = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

(these are all three-vectors), and find the time-averaged value. Well, because $\langle\sin^2(x)\rangle = \frac{1}{2}$,

$$\langle\vec{S}\rangle = \frac{1}{\mu_0}\langle A^2\sin^2(\omega t - kx)\rangle\hat{x} \times \hat{y} = \frac{A^2}{2\mu_0}.$$

And then the energy flux in the primed reference frame looks similar, just scaled:

$$\langle\vec{S}'\rangle = \sqrt{\frac{1-\beta}{1+\beta}}^2\langle\vec{S}\rangle = \frac{1-\beta}{1+\beta}\frac{A^2}{2\mu_0}.$$

(the argument of the sine is more complicated, but the time-average remains the same).

Problem 157

But another person says that light is a particle, and we describe this wave in terms of a four-momentum and number density

$$p^\mu = (E, 0, 0, E), n^\mu = (N/V, 0, 0, N/V).$$

Again, we boost this along the z-direction: we can just say that

$$p^{\mu'} = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} p^\mu = \begin{bmatrix} \gamma(1-\beta)E \\ 0 \\ 0 \\ \gamma(1-\beta)E \end{bmatrix}.$$

The number density transforms the same way, just replacing E with $\frac{N}{V}$. How do we describe p^μ and n^μ here? These both describe density for a lightlike-particle! Specifically, we know that

$$T^{0i} = \text{momentum density / energy flux} = p^0 n^i = \frac{EN}{V}\delta_{iz}.$$

And we can see now that this is consistent with the first description: T^{03} changes by a factor of $\gamma^2(1-\beta)^2 = \frac{1-\beta}{1+\beta}$, so the energy flux there also changes by the same constant.

Problem 158

(Geometry of uniformly accelerating observers.) The trajectory of such an observer can be written as

$$(t, x, y, z) = (W\sinh gT, W\cosh gT, y, z).$$

First, we're asked to find a combination of t and x to solve for W and T. First of all,

$$t = W\sinh(gT), x = W\tanh(gT) \implies x^2 - t^2 = W^2(\cosh^2(gT) - \sinh^2(gT)) = W^2.$$

Thus, $W = \pm\sqrt{x^2 - t^2}$. We can also divide these to find that

$$\frac{t}{x} = \tanh(gT) \implies T = \frac{1}{g}\tanh^{-1}(t/x).$$

We can find when these are real and differentiable: as long as $x^2 - t^2 > 0$, meaning $|x| > |t|$, W is real and differentiable, and as long as $|t/x| < 1$, meaning $|x| > |t|$ again, T is also real and differentiable.

So how do we **write the metric** here in terms of W and T? We know that

$$ds^2 = -dt^2 + dx^+dy^2 + dz^2$$

and many of us wrote this as

$$-(W\sinh(gT))^2 + (W\cosh(gT))^2 + dy^2 + 0 + 0,$$

but this is **not valid**! Remember that we're taking dt, not t, just like we can't say

$$\int t^2 dt = \int t^3.$$

One way we can actually interpret this is that

$$\frac{ds^2}{d\lambda^2} = -\frac{dt^2}{d\lambda^2} + \frac{dx^2}{d\lambda^2},$$

and now we won't blindly plug in $x = W\cosh(gT)$! The correct way for us to approach this problem is to use the multivariable chain rule:

$$dt = \frac{\partial t}{\partial W}dW + \frac{\partial t}{\partial T}dT,$$

and similar for dx. So we can just take derivatives and expand out:

$$ds^2 = -\left(\sinh(gT)^2 dW^2 - W^2\cosh^2(gT)dT^2 - 2W\cosh(gT)\sinh(gT)dWdT\right)$$

$$+\left(\cosh(gT)^2 dW^2 + W^2\sinh^2(gT)dT^2 + 2W\cosh(gT)\sinh(gT)dWdT\right)^2.$$

So this means

$$ds^2 = dW^2 - W^2 dT^2,$$

or that

$$g_{\mu\nu} = \begin{bmatrix} -W^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

This is one of the most important things for us to understand going forward!

So the rest of this problem is conceptual: if we draw the rocket's path on a spacetime diagram, it follows a hyperbola. If we're dropped off the rocket, we are tangent to the rocket at first, and we keep going in a straight line.

If light rays are fired off from us, the rocket will see those at a **lower** frequency from when we fire them: we can see this through a Doppler shift, because the rocket is moving away from us, or we can think of this through gravitatational time delay. But conceptually, it takes an infinite amount of time for the rocket to get all of our finite number of crests (before we cross $x = t$)! So there must be a redshift. Similarly, the rocket sees us aging much slower, because it takes infinitely long for them to see us age a finite amount of time. Finally, the rocket never actually sees us cross $x = t$, because the rocket never crosses that (it's an asymptote)!

33 November 12, 2019 (Recitation)

We'll spend some time thinking about a few problems today. First of all, recall the geodesic equation

$$\frac{d^2x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = 0$$

(we use λ instead of τ to show that we're not committing to a timelike path). The first term here is the coordinate acceleration (second derivative of x^α), and the other term is something else. We can think of this as the contribution from gravity or spacetime curvature! Let's play around with some coordinates that cancel out the second term, so that it looks like we indeed have coordinate acceleration. It turns out we can always do this locally, but if we try to extend this everywhere, we'll find that we can't do this unless we're actually in Minkowski space.

Here, we let $\Gamma^\alpha_{\beta\gamma}$ be the Christoffel symbols associated with the metric $g_{\sigma\lambda}(x)$ (remember the metric is position-dependent) for some coordinate system x^α:

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\gamma\beta,\mu}\right).$$

Remark 159. *Recall the shorthand notation*

$$g_{\mu\beta,\gamma} = \partial_\gamma g_{\mu\beta} = \frac{\partial g_{\mu\beta}}{\partial x^\gamma}.$$

We define a set of new coordinates with primed indices

$$\boxed{y^{\alpha'} = \delta^{\alpha'}_\beta x^\beta + \frac{1}{2}\delta^{\alpha'}_\sigma \Gamma^\sigma_{\beta\gamma}x^\beta x^\gamma}.$$

This is okay as long as the metric is differentiable (which we assume is true).

Problem 160

Find x^β in terms of $y^{\alpha'}$ around $x = 0$, up to order y^2.

We're trying to invert the relation, so we can solve for x in terms of y. To first order, we know that $y^{\alpha'}$ is just $\delta^\alpha_\beta x^\beta + O(x^2)$ (the second term disappears because it's second order), so $x^\beta = \delta^\beta_{\alpha'}y^{\alpha'}$ to first order.

To include second order terms, we have to do more work: use a Taylor expansion

$$x^\alpha = \delta^\alpha_{\beta'}y^{\beta'} + Q^\alpha_{\beta'\gamma'}y^{\beta'}y^{\gamma'},$$

where $Q^\alpha_{\beta'\gamma'}$ is some constant for each α, β, γ. (We should make sure we understand why this accounts for all possibly second-order terms!) And now we can plug in our expression for x^α into our definition of $y^{\alpha'}$: this yields

$$y^{\alpha'} = \delta^{\alpha'}_\beta\left(\delta^\beta_{\beta'}y^{\beta'} + Q^\beta_{\beta'\gamma'}y^{\beta'}y^{\gamma'}\right) + \frac{1}{2}\delta^{\alpha'}_\sigma\Gamma^\sigma_{\beta\gamma}\left(\delta^\beta_{\beta'}y^{\beta'} + Q^\beta_{\beta'\gamma'}y^{\beta'}y^{\gamma'}\right)\left(\delta^\gamma_{\beta'}y^{\beta'} + Q^\gamma_{\beta'\gamma'}y^{\beta'}y^{\gamma'}\right).$$

This may look complicated, but the first-order terms already match up, and we only need to keep the second-order terms in y! And we should be careful: we repeat dummy indices here, so we really need to write this as

$$y^{\alpha'} = \delta^{\alpha'}_{\rho}\left(\delta^{\rho}_{\beta'}y^{\beta'} + Q^{\beta}_{\beta'\gamma'}y^{\beta'}y^{\gamma'}\right) + \frac{1}{2}\delta^{\alpha'}_{\sigma}\Gamma^{\sigma}_{\beta\gamma}\left(\delta^{\beta}_{\beta'}y^{\beta'} + O(y^2)\right)\left(\delta^{\gamma}_{\gamma'}y^{\gamma'} + O(y^2)\right).$$

Now extracting the y^2 terms only yields

$$0 = \delta^{\alpha'}_{\rho}Q^{\rho}_{\beta'\gamma'}y^{\beta'}y^{\gamma'} + \frac{1}{2}\delta^{\alpha'}_{\sigma}\Gamma^{\sigma}_{\beta\gamma}\delta^{\beta}_{\beta'}y^{\beta'}\delta^{\gamma}_{\gamma'}y^{\gamma'}.$$

Relabeling a few indices makes this equal to (just drop all of the primes

$$= \delta^{\alpha'}_{\rho}\left(Q^{\rho}_{\beta\gamma} + \frac{1}{2}\Gamma^{\rho}_{\beta'\gamma'}\delta^{\beta'}_{\beta}\delta^{\gamma'}_{\gamma}\right)y^{\beta}y^{\gamma}.$$

Since this needs to vanish for any β, γ, that tells us that

$$Q^{\rho}_{\beta\gamma} + \frac{1}{2}\Gamma^{\rho}_{\beta'\gamma'}\delta^{\beta'}_{\beta}\delta^{\gamma'}_{\gamma} = 0 \implies Q^{\rho}_{\beta\gamma} = -\frac{1}{2}\Gamma^{\rho}_{\beta'\gamma'}\delta^{\beta'}_{\beta}\delta^{\gamma'}_{\gamma}.$$

We can then plug this into our Taylor expansion for x^{α}, and we're done:

$$x^{\alpha} = \delta^{\alpha}_{\beta'}y^{\beta'} - \frac{1}{2}\Gamma^{\alpha}_{\beta\gamma}\delta^{\beta'}_{\beta}\delta^{\gamma'}_{\gamma}y^{\beta'}y^{\gamma'}$$

Problem 161

Consider the curves $y^{\alpha'}(\lambda) = v^{\alpha'}\lambda$ (which is some constant). Show that any curve of this form satisfies the geodesic equation at the event $y = 0$: in other words, these look like straight lines!

The idea is to plug things in to the geodesic equation directly: we want to show that

$$0 = \frac{d}{d\lambda}\frac{d}{d\lambda}x^{\alpha}(y(\lambda)) + \Gamma^{\alpha}_{\beta\gamma}\frac{dx^{\beta}(y(\lambda))}{d\lambda}\frac{dx^{\gamma}(y(\lambda))}{d\lambda}$$

around $y = x = 0$. First we need to write everything in terms of y, instead of x: by the chain rule,

$$\frac{d}{d\lambda}x^{\alpha}(y(\lambda)) = \frac{dy^{\alpha'}}{d\lambda}\frac{\partial x^{\alpha}}{\partial y^{\alpha'}}.$$

(Remember that x is a function of $y^{\alpha'}$ for all α', so this is a multivariable chain rule!) And this simplifies (by looking at our expression for x in terms of y) to

$$= v^{\alpha'}\left(\delta^{\alpha}_{\alpha'} - \frac{1}{2}\Gamma^{\alpha}_{\beta'\gamma'}\frac{\partial}{\partial y^{\alpha'}}(y^{\beta'}y^{\gamma'})\right)$$

(the first term because $\frac{dy^{\beta'}}{dy^{\alpha'}}$ is a Kronecker delta), where we're allowed to treat the Christoffel symbol as constant near $y = 0$. Now using the product rule on the last part here yields

$$= v^{\alpha'}\left(\delta^{\alpha}_{\alpha'} - \frac{1}{2}\Gamma^{\alpha}_{\beta'\gamma'}\left(\delta^{\beta'}_{\alpha'}y^{\gamma'} + \delta^{\gamma'}_{\alpha'}y^{\beta'}\right)\right) = v^{\alpha'}\left(\delta^{\alpha}_{\alpha'} - \frac{1}{2}\left(\Gamma^{\alpha}_{\alpha'\gamma'}y^{\gamma'} + \Gamma^{\alpha}_{\beta'\alpha'}y^{\beta'}\right)\right).$$

Relabeling dummy indices yields that this is just

$$\boxed{\frac{\partial x^{\alpha}}{\partial y^{\alpha'}} = \delta^{\alpha}_{\alpha'} - \Gamma^{\alpha}_{\beta'\alpha'}y^{\beta'}}.$$

But we need to take a second derivative of this: by the chain rule again,

$$\frac{d}{d\lambda}\frac{dx^\alpha}{d\lambda} = \frac{dy^\sigma}{d\lambda}\frac{\partial}{\partial y^\sigma}\left(\frac{dx^\alpha}{d\lambda}\right).$$

The first term here is v^σ, and the second term is a derivative of something we just found with respect to y^σ: looking at the boxed thing, the Kronecker delta is constant with respect to y^σ, and the second term gives another Kronecker delta! So

$$\frac{d}{d\lambda}\frac{dx^\alpha}{d\lambda} = v^\sigma \frac{\partial}{\partial y^\sigma}\left(v^{\alpha'} \cdot -\Gamma^\alpha_{\beta'\alpha'} y^{\beta'}\right) = -v^\sigma v^{\alpha'} \Gamma^\alpha_{\beta'\alpha'}\delta^{\beta'}_\sigma = -v^\sigma v^{\alpha'}\Gamma^\alpha_{\sigma\alpha'},$$

and now we can plug everything into the geodesic equation. It's nice here because we're **evaluating everything at** $y = 0$, so the term $\frac{dx^\beta}{d\lambda} = v^{\alpha'}\delta^\beta_{\alpha'}$ (the second term goes away)! And we just have (at $y = 0$)

$$0 = -v^\sigma v^{\alpha'}\Gamma^\alpha_{\sigma\alpha'} + \Gamma^\alpha_{\beta\gamma}\frac{dx^\beta}{d\lambda}\frac{dx^\gamma}{d\lambda} = -v^\sigma v^{\alpha'}\Gamma^\alpha_{\sigma\alpha'} + \Gamma^\alpha_{\beta\gamma}v^\beta v^\gamma,$$

and indeed this is zero.

The important physics content here is that the geodesic equation tells us what the coordinate acceleration looks like. These new coordinates y cancel out the coordinate acceleration at the origin, so that straight lines are actually straight (in fact, the metric looks like flat space)!

34 November 13, 2019

Today, we'll continue what Professor Hughes talked about last time, which is the geodesic equation for a gravitational field around a mass M.

The central concept we've been discussing so far is that an object left in a gravitational field (with no external forces) will follow geodesics that maximize the proper time, which corresponds to trajectories of free particles or photons that satisfy the Euler-Lagrange equations under the Lagrangian

$$L = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}.$$

Basically, if we have a path in spacetime, we can parameterize this with some coordinate λ (often we use the proper time for particles with mass), which gives us a path $x^\mu(\lambda)$. (As a sidenote, if we have massless particles, we often define λ such that $\frac{dx^\mu}{d\lambda} = p^\mu$.) Then if we solve the Euler-Lagrange equations

$$\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \dot{x}^\mu}\right) = 0,$$

for all possible values of μ, and where we define $\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$, we find for this Lagrangian that

$$\boxed{\frac{d}{d\lambda}(g_{\alpha\mu}\dot{x}^\mu) = \frac{1}{2}\dot{x}^\mu\dot{x}^\nu\partial_\alpha g_{\mu\nu}}.$$

And we can write this in a slightly different way, with **Christoffel symbols**, to

$$\boxed{\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0},$$

which isolates the second derivative and makes it more clear that we have an equation of motion. Here,

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2}g^{\mu\sigma}\left(\partial_\alpha g_{\sigma\beta} + \partial_\beta g_{\sigma\alpha} - \partial_\sigma g_{\alpha\beta}\right).$$

As we've said a few times in this class, consider the first boxed statement. If we have $\partial_\alpha g_{\mu\nu} = 0$ for all α, which means that g doesn't depend on that coordinate, then the whole left hand side is zero, meaning that

$$\dot{x}_\alpha = g_{\alpha\mu}\dot{x}^\mu,$$

known as the covariant four-velocity, is constant! So if we get $g_{\mu\nu}$ for some configuration, we're done and we can solve all of the geodesic equations (they're just coupled differential equations).

In general, finding $g_{\mu\nu}$ is really hard, and we'll never do it here in this class. We've found that Γ depends on ∂g (first derivatives of g), and we can build a tensor $R^\alpha{}_{\mu\beta\nu}$, the Riemann tensor, which is related to the second derivative of g. From this, we can find the Ricci tensor, and then we can build a scalar quantity by contracting the two indices to get the Ricci scalar, both of which depend on the second derivative of g. So if we find a metric for a distribution of masses in our universe, we need to solve the Einstein equations, which look like

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^\alpha{}_\alpha \sim T^{\mu\nu}.$$

So being given $T^{\mu\nu}$, which tells us the sources of gravity, doesn't make it easy to find $g_{\mu\nu}$ at all! One key point is that the second derivative of g is zero in flat space, so if we're given a metric and want to tell if it's just flat space, just take the second derivative.

As mentioned last time, Schwarzschild found an explicit solution here: we can use the Schwarzschild metric, which describes a universe **outside** a spherical object of mass M: we're putting that object at the center of our coordinate system, and then we find that

$$ds^2 = g_{\mu\nu}x^\mu x^\nu = -\left(1 - \frac{2MG}{rc^2}\right)c^2dt^2 + \left(1 - \frac{2MG}{rc^2}\right)^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2.$$

This is very close to flatspace spherical coordinates: let's make our life easier by defining $r_s = \frac{2MG}{c^2}$. Notice that if the source $M = 0$, this spacetime interval becomes

$$ds^2 = -c^2dt^2 + dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2,$$

which is the Minkowski metric in spherical coordinates! We don't actually want to make M zero, but one thing we can do is show that $\frac{r_s}{r}$ is small for a lot of objects: $r_s \approx 3000$ meters for the Sun, 9×10^{-3} meters for the earth, and 3×10^{-28} meters for a Big Mac. These correspond to values for $\frac{r_s}{R} = 4 \times 10^{-6}, 1.5 \times 10^{-8}, 6 \times 10^{-27}$, respectively, which are much less than 1.

So because this is so small, we can Taylor expand the Schwarzschild metric to first order with good accuracy if we're (for example) looking at the earth's gravitational field, and we look at the weak field limit:

$$ds^2 \approx -\left(1 - \frac{r_s}{r}\right)(c^2dt^2) + \left(1 + \frac{r_s}{r}\right)dr^2 + r^2(d\Omega^2), \quad d\Omega^2 = d\theta^2 + \theta^2\sin^2\phi.$$

Since this is radial, $g_{\mu\nu}$ does not depend on θ and ϕ, so really our geodesic equation is most interesting for $\mu = r$ ($\mu = t$ is also interesting, but not in this context). Then out of the Christoffel symbols, the only ones that come out to be nonzero are

$$\Gamma^r_{tt} = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right), \quad \Gamma^r_{rr} = -\frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)^{-1}.$$

So if we plug this in to the geodesic equation for $\mu = r$, we want to solve

$$\boxed{\frac{d^2x^r}{d\tau^2} + \Gamma^r_{tt}\left(\frac{cdt}{d\tau}\right)^2 + \Gamma^r_{rr}\left(\frac{dr}{d\tau}\right)^2 = 0}.$$

We can make a physicist's approximation now: remember we're taking the weak field limit, so because

$$v = \frac{dr}{dt} \ll c \implies \frac{dr}{d\tau} \ll c\frac{dt}{d\tau}.$$

In other words, we can neglect the third term in our geodesic equation completely! And also, because $v \ll c, \gamma(v) \approx 1$ to first order, and thus $dt = d\tau$ at this approximation:

$$\frac{d^2r}{dt^2} = \frac{r_s}{2r^2}\left(1 - \frac{r_s}{r}\right)c^2 = 0.$$

And since $\frac{r_s}{r}$ is very small in all the situations we're considering, we can toss $1 - \frac{r_s}{r}$ as well, which just means we have

$$\frac{d^2r}{dt^2} \approx -\frac{r_s}{2r^2}c^2 = -\frac{2MG/c^2}{2r^2}c^2 = \frac{-MG}{r^2}.$$

And this is exactly the Newtonian gravity that we want! The physical interpretation here is that the earth creates a nontrivial metric (curvature in spacetime), and any object near the earth is just following a path. There are other equations for θ and ϕ too if we want to work those out.

Well, things get much more nontrivial when we are near black holes and do not have the Newtonian limit: we'll talk about those some other time.

We'll now move on to the concept of **tides**. Let's start without the general relativity for now: first of all, how do tides work on Earth?

Example 162

Say we have the earth and the moon, which have some (Newtonian) gravitational pull on each other.

Notice that because gravitational force is not uniform, some parts of the earth behave differently under the moon's gravitational field: in particular, the earth distorts slightly into an ellipse, where the major axis is along the line towards the moon. And that's why we have high tide near the moon! The idea is that the force of gravity depends on our position, so tides arise out of non-uniformity.

Example 163

To consider another situation, say we have two nearby objects A and B that are dropped towards the earth (ignore gravitational pulls between A and B).

Since gravity pulls us towards the center of the earth, these two objects will actually get a little closer to each other. So we can actually track the nonuniformity of gravity based on how quickly these two particles get closer! Specifically, say that $\vec{x}_A(t)$ is the position of particle A, and $\vec{x}_B(t) = \vec{x}_A(t) + \vec{\zeta}(t)$ is the position of particle B. Then the acceleration $\ddot{\vec{\zeta}}(t)$ is the important quantity here for determining curvature! And this separation vector $\vec{\zeta}$ and its derivative $|\partial_t \vec{\zeta}|$ can be thought of as being very small, because our two vectors x_A and x_B are nearly parallel and very close to each other.

Well,

$$\ddot{\zeta}^i = \ddot{x}_B^i - \ddot{x}_A^i = -\partial_i \phi(B) + \partial_i \phi(A),$$

where ϕ is the gravitational potential, and then we should remember that x_B is close to x_A, so we can calculate to first order

$$\partial_i \phi(x_B^j) = \partial_i(\phi(x_A^j) + \zeta^k \partial_k \phi) = \partial_i \phi(x_A^j) + \zeta^k \partial_{ki}^2 \phi$$

where we split up the derivatives, making the approximation that $\partial_i(\zeta^k \partial_k \phi) = \zeta^k \partial_k^2 \phi$ by neglecting ∂_ζ terms. This yields an equation for the relative position ζ between the two objects:

$$\frac{d^2\zeta^i}{dt^2} = -\left(\partial_i\phi(A) + \zeta^k\partial_k^2\phi\right) + \partial_i\phi(A) = -\zeta^k\partial_{ki}^2\phi.$$

So if we know that the second derivative of our field is nonzero, the second derivative of ζ is nonzero, which means that we do have a non-uniform gravity! (We often call the object that encodes these second partial derivatives the **tidal tensor**.) The whole point here is to motivate us considering the second derivative of the metric.

How does general relativity explain this? We have to use time parameterized by τ instead of t, and we need to use paths in spacetime instead of just paths in space. And now we just have a four-position for A and B! We won't go through the calculations here, but we end up with

$$\frac{d^2\zeta^\mu}{d\tau^2} = -R^\mu{}_{\alpha\nu\beta}U^\alpha\zeta^\nu U^\beta$$

where U^α, U^β are the four-velocity for our particles. The main point here is that around the earth, since we don't have a very strong gravitational field, we can consider a frame where things look very simple: the free-falling frame! If we look at the free-falling frame of A, the four-velocity is just $(1, 0, 0, 0)$, and we can also show that ζ only has space components, not time components! (Intuitively, ζ is orthogonal to U.) And since things are moving slowly, $dt = d\tau$, and our equation simplifies greatly to

$$\frac{d^2\zeta^i}{dt^2} = -R^i{}_{tjt}c^2\zeta^j.$$

And this Riemann tensor depends on the second derivative of the gravitational field, just like the tidal tensor!

Now this approach actually shows us why the Earth has tides. Again take an object of mass M (like the Earth) with center O, and consider two objects C, D at the same height dropped towards the center. Since C moves along line CO and D moves along DO, we can say that the angle COD is a constant $\delta\phi$ throughout the motion. Then

$$\zeta^\phi = r\delta\phi,$$

where r is the distance of the objects to the center of the earth, and we find that this tells us

$$\delta\ddot{\phi} = -\frac{MG}{r^3}\delta\phi,$$

since indeed the points are getting closer and closer faster and faster as C and D approach the earth. Simultaneously, though, consider objects A and B that are along the same ray towards the center of the earth, but start off separated by a **radial** distance δr. Then it turns out the equation we get is

$$\delta\ddot{r} = \frac{2MG}{r^3}\delta r,$$

which means that objects A and B actually get farther away! So this indeed tells us that the Earth gets stretched along the radial direction and squished along the angular direction, as we expect.

35 November 14, 2019 (Recitation)

Today, we're going to continue the problem with what turn out to be called **locally inertial coordinates** from last recitation! Just to review: if we're given a metric $g_{\sigma\rho}$ with respect to some coordinate system x, we define a new coordinate system at $x = 0$ via

$$y^\alpha = \delta^\alpha_{\beta'}x^{\beta'} + \frac{1}{2}\delta^{\alpha'}_\sigma\Gamma^\sigma_{\beta\gamma}x^\beta x^\gamma,$$

where

$$\Gamma^\sigma_{\beta\gamma} = \frac{1}{2} g^{\sigma\mu}\left(\partial_\beta g_{\sigma\mu} + \partial_\gamma g_{\beta\mu} - \partial_\mu g_{\beta\gamma}\right).$$

Last time, we inverted this and found that

$$\boxed{x^\alpha = \delta^\alpha_{\beta'} y^{\beta'} - \frac{1}{2}\Gamma^\alpha_{\beta\gamma}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'} y^{\beta'} y^{\gamma'} + O(y^3)}$$

(to second order), so this is valid near $x = y = 0$. We also showed that any curve of the form $y^{\alpha'}(\lambda)$ satisfies the geodesic equation at $y = 0$, so our flat spacetime intuition is correct: intuitively straight lines in the y-coordinates are indeed geodesics.

We'll continue to do some index manipulation today so we can be more comfortable with the ideas!

Problem 164

Calculate the metric $g_{\alpha'\beta'}$ (in the y-coordinates) near $y = 0$ by expanding $g_{\lambda\sigma}$ (in the x-coordinates) in a Taylor series and then transforming to the y-coordinates, ignoring y^2 and higher-order contributions. Then show that $\partial_{\gamma'} g_{\alpha'\beta'} = 0$ at $y = 0$, so the metric is constant at $y = 0$. (Throughout this problem, it's helpful to remember that g is symmetric.)

Here, we can think of g as a function of x. We want to start by expanding $g_{\lambda\sigma}$ and we want to expand near $y = 0$, which in this case is the same as expanding around $x = 0$:

$$g_{\alpha\beta}(x) = g_{\alpha\beta}(0) + (\partial_\rho g_{\alpha\beta})_{x=0} \cdot x^\rho + O(x^2).$$

by the multivariable chain rule.

Remark 165. *The $O(x^2)$ term consists of two derivatives, so it would take the form*

$$\frac{1}{2}\partial_\rho\partial_\sigma g_{\alpha\beta} x^\sigma x^\rho$$

where we evaluate the derivative at 0.

Now that we've expanded g as a Taylor series, we transform to the y-coordinates: **remember that the metric transforms via**

$$g_{\alpha'\beta'}(y) = \frac{\partial x^\sigma}{\partial y^{\alpha'}}\frac{\partial x^\lambda}{\partial y^{\beta'}} g_{\sigma\lambda}(x).$$

(We can confirm that this makes sense by multiplying both sides by $\partial x^{\alpha'}\partial x^{\beta'}$, and now both sides give the length ds^2.) And now we know how to write x in terms of y (to second order), so we can plug into the boxed expression above:

$$\frac{\partial x^\sigma}{\partial y^{\alpha'}} = \delta^\sigma_{\alpha'} - \frac{1}{2}\Gamma^\sigma_{\beta\gamma}\delta^\beta_{\beta'}\delta^\gamma_{\gamma'}(\delta^{\beta'}_{\alpha'} y^{\gamma'} + \delta^{\gamma'}_{\alpha'} y^{\beta'})$$

where the last parenthetical term comes again from the product rule on $y^{\beta'}y^{\gamma'}$, just like last class. This then simplifies (by clearing out some of the delta functions) by relabeling indices to

$$\frac{\partial x^\sigma}{\partial y^{\alpha'}} = \delta^\sigma_{\alpha'} - \Gamma^\sigma_{\alpha'\gamma'} y^{\gamma'}.$$

So we can plug this in:

$$g_{\alpha'\beta'} = \left(\delta^\sigma_{\alpha'} - \Gamma^\sigma_{\alpha'\gamma'} y^{\gamma'}\right)\left(\delta^\lambda_{\beta'} - \Gamma^\lambda_{\beta'\rho'} y^{\rho'}\right) g_{\sigma\lambda}(x),$$

and then plug **that** into our Taylor series expansion:

$$= \left(\delta^\sigma_{\alpha'} - \Gamma^\sigma_{\alpha'\gamma'} y^{\gamma'}\right)\left(\delta^\lambda_{\beta'} - \Gamma^\lambda_{\beta'\rho'}(y^{\rho'}\right)\left(g_{\sigma\lambda}(0) + (\partial_\rho g_{\sigma\lambda})_{x=0} \cdot x^\rho + O(x^2)\right).$$

The delta functions can commute through the derivative, and we can expand this: using the fact that $x^\rho = y^\rho$ **at first order**,

$$= \delta^\sigma_{\alpha'}\delta^\lambda_{\beta'} g_{\sigma\lambda}(0) + (\partial_\rho g_{\alpha'\beta'})y^\rho - g_{\sigma\lambda}\left(\Gamma^\sigma_{\alpha'\gamma'} y^{\gamma'}\delta^\lambda_{\beta'} + \Gamma^\lambda_{\beta'\rho'} y^{\rho'}\delta^\sigma_{\alpha'}\right) + O(y^2).$$

This simplifies after contraction (and relabeling the $y^{\rho'}$ to $y^{\gamma'}$) to

$$= g_{\alpha'\beta'}(0) + y^\rho \partial_\rho g_{\alpha'\beta'} - y^{\gamma'}\left(g_{\sigma\beta'}\Gamma^\sigma_{\alpha'\gamma'} + g_{\alpha'\lambda}\Gamma^\lambda_{\beta'\gamma'}\right).$$

Now we need to compute the last parenthetical term! Plugging in the definition of the Christoffel symbol,

$$g_{\sigma\beta'}\Gamma^\sigma_{\alpha'\gamma'} = \frac{1}{2} g_{\sigma\beta'} g^{\sigma\mu'}\left(\partial_{\alpha'} g_{\gamma'\mu'} + \partial_{\gamma'} g_{\alpha'\mu'} - \partial_{\mu'} g_{\alpha'\gamma'}\right)$$

$$g_{\alpha'\lambda}\Gamma^\lambda_{\beta'\gamma'} = \frac{1}{2} g_{\alpha'\lambda} g^{\lambda\mu'}\left(\partial_{\beta'} g_{\gamma'\mu'} + \partial_{\gamma'} g_{\beta'\mu'} - \partial_{\mu'} g_{\beta'\gamma'}\right)$$

and now note that $g_{\sigma\beta'} g^{\sigma\mu'} = \delta^{\mu'}_{\beta'}$**, because** g **is a raising/lowering operator** (this is important!). So this greatly simplifies our terms: their sum is

$$\frac{1}{2}\delta^{\mu'}_{\beta'}\left(\partial_{\alpha'} g_{\gamma'\mu'} + \partial_{\gamma'} g_{\alpha'\mu'} - \partial_{\mu'} g_{\alpha'\gamma'}\right) + \frac{1}{2}\delta^{\mu'}_{\alpha'}\left(\partial_{\beta'} g_{\gamma'\mu'} + \partial_{\gamma'} g_{\beta'\mu'} - \partial_{\mu'} g_{\beta'\gamma'}\right)$$

and then we can contract the delta functions; many things end up canceling out, and this actually just leaves us with $\partial_{\gamma'} g_{\alpha'\beta'}$! Now we can plug this back into our expression to find that

$$g_{\alpha'\beta'} = g_{\alpha'\beta'}(0) + y^\rho \partial_\rho g_{\alpha'\beta'} - y^{\gamma'}\partial_{\gamma'} g_{\alpha'\beta'} + O(y^2).$$

And these last two terms are identical, so

$$\boxed{g_{\alpha'\beta'}(y) = g_{\alpha'\beta'}(0) + O(y^2)}.$$

This indeed shows that the first derivative is always zero, which completes our problem! And $\Gamma^\beta_{\alpha\gamma}$, the Christoffel symbol, **only** contains first derivatives of the metric, which means that $y = 0$, all Christoffel symbols are zero. So our geodesic equation reduces to

$$\frac{d^2 y^{\beta'}}{d\tau^2} = 0,$$

which is exactly what dictates flat spacetime: geodesics are literally straight lines!

The point here is that we can only ignore the $O(y^2)$ terms exactly at $y = 0$, so these inertial coordinates are only **local**. But this does show the existence of local flat spacetime!

36 November 18, 2019

Today will be the first of two classes on black holes! We'll first start with a bit of review: note that all massive objects have a four-velocity **U** which satisfies $U^\mu U_\mu = g_{\mu\nu} U^\mu U^\nu = -1$, because we can always go to a frame where the four-velocity is $(1, 0, 0, 0)$.

We also know that all massive objects must move along timelike curves, so $ds^2 < 0$ for any two points. In particular, if the object is freely falling (so only dependent on gravity), its trajectory is further constrained: it must also satisfy the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0 \iff \frac{d}{d\lambda}\left(g_{\mu\alpha}\frac{dx^\alpha}{d\lambda}\right) = \frac{1}{2}\partial_\mu g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

On the other hand, note that massless particles (like photons) always have $ds^2 = 0$, but if they are freely falling as well, they **also** satisfy the geodesic equation!

Today, we'll explore a little more about the Schwarzschild metric, especially when we have a very compact mass: define $r_s = \frac{2MG}{c^2}$ (except we'll work with natural units), and then define $g_{\mu\nu}$ for the Schwarzschild metric via

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2.$$

We'll first discuss what it means to use (t, r, θ, ϕ) here instead of (t, x, y, z) (especially in terms of creating a black hole), and then we'll look at some simple trajectories under this metric.

First of all, our choice of coordinates is a bit confusing, so let's spend a bit of time explaining what they mean. Say we use some set of coordinates x^μ, and we have our metric $g_{\mu\nu}$: is there a way to do a change of coordinates from x^μ to $x^{\alpha'}$? Well,

$$g_{\alpha'\beta'} = \Lambda^\mu{}_{\alpha'}\Lambda^\nu{}_{\beta'}g_{\mu\nu},$$

where we define the transformation matrix via $\Lambda^\mu{}_{\alpha'} = \frac{\partial x^\mu}{\partial x^{\alpha'}}$.

Question 166. *But is there any way for us to relate the variables t, r, θ, ϕ to more physical quantities?*

The key is to use **invariants.** One note is that the dt^2 term is the only one with a negative coefficient, which is why it is the "time" component here! If we have an observer at rest, which means $dr, d\theta, d\phi$ (our space coordinates) are all zero, then

$$d\tau^2 = -ds^2 = \left(1 - \frac{r_s}{r}\right)dt^2.$$

Notice here that the coefficient $1 - \frac{r_s}{r}$ is position-dependent, but as $r \to \infty$, this term goes to 1! So far away from the source of gravity, $dt = d\tau$, which means that t is the time that we'd measure if we're a very far away observer.

So what about the space components? If we consider two events at a fixed radius $r = R_0$ and a fixed time $t = T$, then we can write the separation between them along the sphere via

$$ds^2 = R_0^2 d\theta^2 + R_0^2\sin^2\theta dp^2$$

This allows us to compute the surface area of the sphere by integrating, and thus it makes sense to define the **radius**

$$r = \sqrt{\frac{A}{4\pi}}$$

to be related to the surface area. Notably, r is **not just the "distance to the source"** in this curved spacetime! But it is true that as we get farther away from the center, r does get larger and larger.

Example 167

Let's show why this is necessary (in particular, why r is not just an ordinary distance like in polar coordinates). Suppose we have two events in the Schwarzchild metric which occur at (t, r_1, θ, ϕ) and (t, r_2, θ, ϕ), so the only separation between these events is along r.

Solution. Note that we need to use an invariant notion of distance here: since these are spacelike separated, we use the metric $g_{\mu\nu}$ to deduce that (because $dt = d\theta = d\phi = 0$)

$$ds^2 = \left(1 - \frac{r_s}{r}\right)^{-1}dr^2,$$

so the distance is just an integral here:

$$\Delta s = \int_{r_1}^{r_2} \frac{dr}{(1 - r_s/r)^{1/2}} \neq r_2 - r_1.$$

So we should be careful about how we're interpreting our polar coordinates here! $\square$

Taking a closer look, the Schwarzschild metric breaks at $r = 0$ and $r = r_s$. These are known as **singularities** of spacetime or of the metric, and although it may not look evident, these have slightly different properties. The singularity at $r = 0$ is a **true singularity**, which means we can't actually make it go away by changing coordinates. (In particular, something is bad about invariants: the **curvature**, which is related to the tidal pull, goes to ∞, which is bad.) However, $r = r_s$ is actually just a **coordinate singularity**: we claim that write our metric in alternate coordinates so that the points at $r = r_s$ look fine. (In particular, we could get in a rocket and cross $r = r_s$, and nothing would really happen to us.)

The region $r = r_s$ is known as the **event horizon**. Usually, r_s is very small for objects like the sun, earth, or a Big Mac: in fact, it is inside those objects, so the Schwarzschild metric does not apply in those regions. But we can speculate that if nature produces objects that are very compact, meaning we can pack enough mass within a radius r_s, there is a region where the metric is valid! And that's something we'll talk about soon.

We'll now consider some **simple orbits** under the Schwarzschild metric, and we will no longer assume r_s is very small! (Solving the general case is very challenging, so we won't do it in this class.)

Example 168

Can we just place an observer at rest (so that, for example, we can synchronize a clock here)? We're allowed to have external forces, so we can't use the geodesic equation here.

To stay still, our trajectory must only travel in the t-component: $dr = d\theta = d\phi = 0$, so

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2.$$

Since we have a massive object, we know that $ds^2 < 0$ for its trajectory, and therefore we must have $1 - \frac{r_s}{r} > 0$ (and therefore $\boxed{r > r_s}$). In particular, what this seems to say is that as long as we are far enough away from the source of the Schwarzschild metric, we can indeed stay still as long as we use things like rockets. But it's interesting that **if we're inside the event horizon, there is no way to stay still!** Instead, it turns out we'll always fall deeper and deeper into our black hole. So that's why we might have heard that the event horizon is the "point of no return."

Example 169

What happens to time measurement under the Schwarzschild metric?

We know that the proper time between two events satisfies

$$-d\tau^2 = g_{00} dt^2 = -\left(1 - \frac{r_s}{r}\right) dt^2,$$

because the objects are colocated. But notice here that as r gets closer and closer to r_s,

$$\lim_{r \to r_s} d\tau = 0.$$

So from the perspective of an outside observer, time passes very slowly at the event horizon: in fact, it freezes at $r = r_s$ itself!

And because the frequency of a light wave depends on how time passes, light beams behave interestingly. An invariant way of talking about light is that $ds^2 = 0$ for all trajectories. In the Minkowski (flat) spacetime, this just tells us that

$$ds^2 = 0 = -(cdt)^2 + dx^2 \implies \left|\frac{dx}{dt}\right| = c,$$

so light travels at c in all frames in flat spacetime. **The key is that this isn't true anymore in curved spacetime!** If we plug in $ds^2 = 0$ into the Schwarzschild metric and focus only on radially moving photons (to toss the $d\phi$ and $d\theta$ terms), we find that

$$ds^2 = 0 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 \implies \left|\frac{dr}{dt}\right| = 1 - \frac{r_s}{r}.$$

In other words, as light approaches the event horizon, it actually gets slower and slower from the point of view of an outside observer. (And this is indeed what we call the speed of light, because r and t are the coordinates that we're using here. So the key is that something weird is happening to our coordinates because of gravity.)

So in a spacetime diagram, our rays of light are no longer lines at 45 degree angles. Far away from the Schwarzschild source (when r is very large), the lines do look like basically straight lines at 45 degree angles. But as we get closer and closer to r_s, the cones close up: they approach vertical lines. In other words, our motion becomes much more constrained! (And we'll ignore the part where $r < r_s$ for now.)

Remark 170. *And at $r = r_s$, we again see that light cannot escape the event horizon, because it is constrained to move vertically in our spacetime diagram. Note that we're assuming our black holes don't have angular momentum or electric charge: we do actually know how to deal with those cases, but the metric is just much uglier.*

Example 171

One final thing with photons: let's confirm the gravitational redshift formula under the Schwarzschild metric. Say we have two points along the same radial direction at distances $r = r_A$ and r_B. A photon leaves A and eventually arrives at B: what is the change in energy (and therefore frequency)?

The first thing we need to confirm is that a photon has a four-momentum $\mathbf{p}$, and the 0-component of this momentum has something to do with the energy. So which frame should we be using here? We need the energies E_A and E_B, and this depends on an observer at A and B, respectively. Probably the frame that we want to calculate the components of $\mathbf{p}$ are the rest frames: they'll look like $(E, \vec{p})$. In this frame, the four-velocity of the observer is $(1, \vec{0})$, so

$$\mathbf{U}_{\text{observer}} \cdot \mathbf{p} = g_{\mu\nu} U^{\mu}_{\text{obs}} p^{\nu},$$

and we know that we can always find a frame that there is no gravity by the weak equivalence principle! So in this freely falling frame, the metric is just the Minkowski metric, and thus this will just evaluate to (because most components of $\mathbf{U}$ are zero)

$$= \eta_{00} \cdot 1 \cdot E = -E.$$

Proposition 172

In other words, $\mathbf{U}_{\text{obs}} \cdot \mathbf{p}$ is a **frame-independent** way to evaluate the energy E of a particle.

So let's apply this equation twice: once at A and once at B. The energy at point A is

$$-\mathbf{U}_{\text{obs}}(A) \cdot \mathbf{p}(A)$$

and similar for B. And because we're using observers at rest during the measurements, $U^1 = U^2 = U^3 = 0$, so

$$E_A = -U^0_{\text{obs}}(A) p_0(A).$$

Here, though, we have gravity, **so the t-component of a nonmoving observer is not 1 anymore**: we use the fact that $U^\mu U_\mu = -1$, so we actually have $g_{00}(U^0)^2 = -1$ in this case. And the ratio of the energies now satisfies

$$\frac{E_B}{E_A} = \sqrt{\frac{g_{00}(A)}{g_{00}(B)}} \cdot \frac{p_t(B)}{p_t(A)}.$$

We know how to calculate the first term by directly plugging in the g_{00} term of the metric, and it turns out **the second ratio is just 1!** This is because we can use the geodesic equation here (no external forces), and our metric here does not depend on t. (It only depends on r, θ.) Thus, we have the conserved quantity $g_{0\alpha} \frac{dx^\alpha}{d\lambda}$ along the entire geodesic, and since the metric is diagonal, this is just

$$= g_{tt} \frac{dt}{d\lambda} = g_{tt} p^t = p_t$$

because we defined λ in such a way for photons such that the derivatives of the four-vector $\mathbf{x}$ give the four-momentum $\mathbf{p}$! And so putting everything together, we get the following:

$$\boxed{\frac{E_B}{E_A} = \sqrt{\frac{1 - r_s/r_A}{1 - r_s/r_B}}}.$$

In particular, if we take $r_B \to \infty$, the energy of a photon infinitely far away from a black hole

$$E(\infty) = E_A \sqrt{1 - \frac{r_s}{r_a}} < E_A,$$

and indeed photons will lose energy: in fact, if the source $r_A \to r_s$, the energy of the photon goes to 0.

37 November 19, 2019 (Recitation)

There will be a substitute teacher for the week of December 2.

We get to talk about black holes now! Remember that the metric we're using now is

$$ds^2 = -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

where we often shorten $d\theta^2 + \sin^2\theta d\phi^2 = d\Omega^2$, and where

$$f(r) = 1 - \frac{r_s}{r}, \quad r_s = 2GM.$$

(By the way, note that Schwarzschild means "black shield," which is pretty fun.) This is particularly interesting when we have objects smaller than their Schwarzschild radius: this means the object needs to be dense enough, and it is called a **black hole**.

Question 173. *People say "Not even light can escape" from a black hole, but what does it mean to escape?*

Remark 174. *People also say "the singularity is at the center of the black hole." But we* **should not say this**.

These two ideas are actually linked to each other, and we'll talk about that today.

Problem 175

We'll start by looking at what's going on at $r = r_s$, the **event horizon**.

We showed that there's no way to hold an observer at constant $r \le r_s$ at rest, because that means $dr = d\theta = d\phi = 0$, so

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 \ge 0,$$

which doesn't make sense for a massive observer! Specifically, at $r = r_s$, we have $ds^2 = 0$, which is the path of a light ray. In order for an observer to travel at the speed of light, they have to be accelerated by an infinite amount. $r = r_s$ here is a three-dimensional surface in spacetime, because we haven't specified t, θ, ϕ, and it's "ruled" by light rays.

So what is the boundary of light-like paths for the Schwarzschild geometry? Near $r = \infty$, the Schwarzschild metric goes to the Minkowski metric, so the light cones just look like ordinary light cones. At $r = r_s$, though, light travels along the surface $r = r_s$, and this means that the rest of the light cone must be entirely contained inside $r \le r_s$ or entirely within $r \ge r_s$.

Well, let's look at the metric at a small perturbation $r_s + \delta r$:

$$ds^2 = -\left(1 - \frac{r_s}{r_s + \delta r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r_s + \delta r}} \approx -\frac{\delta r}{r_s} dt^2 + \frac{r_s}{\delta r} dr^2$$

(we're ignoring the angular parts for now). If $\delta r > 0$ is small, the coefficient of dt^2 is very small, while the coefficient of dx^2 is very large. So this corresponds to $ds^2 > 0$, which is not allowed! This means the (future) light cone must point towards the middle of a black hole.

Well, the light cone tells us the areas where causal effect is possible, because information can only travel at the speed of light. This means that once light is at $r = r_s$, it can't affect an observer at $r > r_s$! This is a lot like the uniformly accelerating observer, who can't signal the accelerating observer past a certain finite time.

In particular, this means that if we're an observer approaching the event horizon, our light cones have to "tip over:" by the time we get to $r = r_s$, the light cones will never be able to reach an observer at any fixed $r > r_s$ at any future time.

Problem 176

What's happening beyond $r = r_s$, then?

Notice that $1 - \frac{r_s}{r}$ is now negative, so space and time coordinates have flipped! The timelike coordinate is the dr^2 one, and the spacelike coordinate is the dt^2 one, and we can deal with this by just calling $t = r'$ and $r = t'$.

Our first question: does r need to keep decreasing, or is it just important that we've crossed past $r = r_s$? The time coordinate dt' must keep moving forward: we can't keep moving back and forth, because that is the direction of time. So $r = t'$ must keep moving towards 0 once we're inside the event horizon! (Specifically, we move towards 0 instead of towards ∞, because remember that 0 is in our future light cone.)

Remark 177. *The whole point is that we can think of the inside and outside as being two different metrics.*

So what is the "center of the black hole," exactly? $r = 0$ corresponds to $t' = 0$, so **that's actually just a moment in time**, not the center of anything. Every single observer in a black hole has the $t' = 0$ event in their future, so they will eventually cross it no matter what (and go into the singularity)!

So now we can draw the light cones in our spacetime diagram for $r < r_s$: we know that we must progressively move toward $r = 0$, so the light cones must continue to tip until the singularity is the only thing we can see in the future.

Remark 178. *We should search up Penrose diagrams for more information about this pictorally.*

38 November 20, 2019

We're going to finish the main discussion of non-rotating black holes today. One main point from last week is the shape of the metric

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2,$$

where

$$r_s = \frac{2MG}{c^2}$$

is the Schwarzschild radius. We saw last time that for any massive particle moving in spacetime, $U^\mu U_\mu = 1$, and this fact alone tells us a lot about trajectories in spacetime under this metric near and around $r = r_s$.

Last time, we derived the gravitational redshift equation

$$\frac{E(r_B)}{E(r_A)} = \sqrt{\frac{1 - r_s/r_A}{1 - r_s/r_B}},$$

which tells us that a photon emitted near the event horizon will have much lower energy.

Something pretty extreme happens to time, too: we found that if we have a watch calculating proper time, then

$$\frac{d\tau(r_A)}{\sqrt{1 - r_s/r_A}} = dt.$$

This means that as the watch moves closer to the event horizon, a far-away observer will see events take larger and larger amounts of time.

Today, we'll review the Newtonian two-body problem, and we'll go from that to more simple orbits around a black hole!

Problem 179 (Two-body problem)

Say we have two objects of mass m and $M \gg m$: how can we describe the motion of the orbit of m around M?

The total energy of this system is the sum of the potential and kinetic energies of m:

$$E = \frac{1}{2}mv^2 - \frac{MmG}{r}.$$

There are lots of simplifications we can make here: specifically, the orbit will stay within a plane, so we can characterize its motion with just the radius r and angle ϕ (no θ necessary).

Because we're working in these polar coordinates,

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

by the Pythagorean theorem (radial versus tangential velocity). We can plug this into our expression for E, and we can also use the fact that $L = mr^2\dot{\phi}$ is a constant for our motion, to get a differential equation

$$E = \frac{1}{2}m\dot{r}^2 - \frac{Mgm}{r} + \frac{1}{2}\frac{L^2}{mr^2}.$$

We can divide through by m, and then define $\hat{E} = \frac{E}{m}$ and $\hat{L} = \frac{L}{m}$ (energy and momentum per unit mass) to get

$$\frac{1}{2}\dot{r}^2 = \hat{E} - V_{\text{eff}}(r)$$

where the effective potential

$$V_{\text{eff}}(r) = \frac{1}{2}\frac{\hat{L}^2}{r^2} - \frac{MG}{r}.$$

But we can take a derivative of this (with respect to t), and because the total energy is conserved, $\hat{E}$ goes away: we get $\ddot{r} = -\frac{dV}{dr}$, and then we can solve the equation of motion that arises.

But there's some interesting things we can discover without solving for the explicit force: points where $\dot{r} = 0$ are **turning points**, because it usually means the object is switching from moving away to moving closer. So this gives us a qualitative understanding of the orbit! Graphing V_{eff} as a function of r, it goes to ∞ as $r \to 0$ and 0 as $r \to \infty$, so there will be a single local minimum: this tells us that in Newtonian mechanics, if a particle has energy greater than 0, it is not in a bound orbit (because it can bounce back to $r = \infty$), and if a particle has energy less than 0, it is in a bound orbit.

Remark 180. *Because the potential goes to ∞ whenever $\hat{L} > 0$ and $r \to 0$, then any object will not be bound by the orbit: we will have a hyperbolic path.*

In particular, if we want to find the radius of a circular orbit, we can just find the local minimum where the derivative is zero:

$$\frac{dV_{\text{eff}}}{dr} = 0 \implies -\frac{\hat{L}^2}{r^3} + \frac{MG}{r^2} = 0 \implies \boxed{r = \frac{\hat{L}^2}{MG}}.$$

This is a stable circular orbit (unless $\hat{L} = 0$)!

Many of these concepts will work when we're trying to describe orbits in the Schwarzschild metric as well. However, there are going to be some correction terms!

Notice that the metric doesn't depend on t or ϕ, so there will be two conserved quantities: the covariant velocity U_t and U_ϕ. Thus, by extension, so are mU_t and mU_ϕ, which we will write as

$$p_t = g_{tt}p^t = -\left(1 - \frac{r_s}{r}\right)p_t = -\left(1 - \frac{r_s}{r}\right)m\frac{dt}{d\tau} = em$$

(here we define $e = -\left(1 - \frac{r_s}{r}\right)\frac{dt}{d\tau}$, e for energy) and

$$p_\phi = g_{\phi\phi}p^\phi = r^2\sin^2\theta\frac{d\phi}{d\tau}m = r^2 m\frac{d\phi}{d\tau} = \ell m.$$

($\ell = r^2\frac{d\phi}{d\tau}$ for angular momentum) where **we are placing the orbit at $\theta = \frac{\pi}{2}$, in an equatorial orbit**, because again in general relativity we can constrain an orbit to a plane.

But as before in the Newtonian case, the big thing we care about is r, the radial distance. One thing we could do is to plug in the geodesic equation for r, but let's instead just try to use a qualitative description. We know that

$$U^\mu U_\mu = -1 \implies g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{rr}\left(\frac{dr}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{d\tau}\right)^2$$

(here we assume $d\theta = 0$ by construction). One thing we can do is to get rid of the non r-derivatives by plugging in the definition of our conserved quantities e and ℓ: we'll find that

$$\left(\frac{dr}{d\tau}\right)^2 = e^2 - \left(\frac{\ell^2}{r^2} + 1\right)\left(1 - \frac{r_s}{r}\right).$$

If we now define the variables $\varepsilon = \frac{e^2-1}{2}$ (the reason for this is that there's no rest energy equivalent in Newtonian gravity) and $V_{\text{eff}}(r) = \frac{-MG}{r} + \frac{\ell^2}{2r^2} - \frac{MG\ell^2}{r^3}$ (the first two terms are Newtonian, and **this last term is the GR correction**), the GR equation becomes

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \varepsilon - V_{\text{eff}}(r),$$

which now looks formally identical to the case above!

Fact 181

If we plot V_{eff} versus r again, the behavior as $r \to \infty$ looks basically the same. **The big difference here** is that for small r, the potential goes to $-\infty$ instead of $+\infty$! So there is a finite-sized barrier, and there are two critical points: one is a stable minimum, and the other is an unstable maximum.

So let's find the critical points:

$$\frac{dV}{dr} = 0 \implies \frac{MG}{r^2} - \frac{\ell^2}{r^3} + \frac{3MG\ell^2}{r^4}.$$

We need to solve a quadratic equation for this:

$$r_\pm = \frac{\ell^2}{r_s}\left(1 \pm \sqrt{1 - \frac{12M^2G^2}{\ell^2}}\right).$$

If the term under the square root is positive, then there is indeed a maximum and a minimum. However, if

$$1 - \frac{12M^2G^2}{\ell^2} \implies \ell < \sqrt{12}MG,$$

then there are no extrema at all! In other words, if our particle has a small angular momentum, and we throw it at a black hole, it will fall from any radius.

In general, the idea is that there are more types of orbits that are allowed in GR than in Newtonian gravity, but the main idea is still the same: we compare the energy ε to this effective potential. Let's look at the case where $\ell > \sqrt{12}MG$:

- If the energy is larger than the maximum value of the potential V_{eff}, there is no point where ε and V are the same. Thus, the derivative $\frac{dr}{d\tau}$ is never zero: the particle will not change direction, so a particle coming in from far away **will fall**, even if it has a nonzero angular momentum.
- If the energy is larger than 0 but less than the maximum value of V_{eff}, r will "bounce back" at some point: basically, the black hole will deflect the orbit, but the object will not fall in.
- If the energy is larger than the local minimum value of V_{eff}, but smaller than 0, we'll be in a bound orbit: it won't quite be an ellipse, but it'll be slowly **processing** (so the angular position will slightly change)!
- If the energy is exactly equal to the local minimum, we will have a circular orbit, where r is constant.

If we want to calculate the procession $\Delta\Phi$ depending on the angular momentum, or something like that, we do have to solve the geodesic equation. Instead, we'll focus on a few more simple orbits that we can use with the current mathematical framework.

Problem 182

What is the radius of a circular orbit? Is it true that for any choice of angular momentum ℓ, we can find a radius r?

Well, remember that we need to have $\ell < \sqrt{12}MG$ in order to have this local minimum and maximum. **So the**

minimum radius for a circular orbit corresponds to this value of ℓ: we find that

$$r_{\min} = \frac{\ell^2}{r_s} = \frac{12M^2G^2}{2MG} = \boxed{6MG} = 3r_s.$$

So unlike in Newtonian mechanics, there is a minimum value for stable circular orbits!

Problem 183

Let's now look at radial orbits. From an observer outside, it takes an infinite amount of time for a person to fall into the black hole, but the person experiences this in a finite amount of time. How can we justify this mathematically?

Say we're moving from a radius of r_B to a radius of r_A. Here, ϕ is not changing, so our angular momentum $\ell = 0$:

$$V_{\text{eff}} = -\frac{MG}{r}.$$

Remember that $\varepsilon = \frac{e^2-1}{2}$ in our equation of motion: let's say we drop our object from ∞ with zero velocity, so we have less terms to worry about. Then

$$e(\infty) = -\left(1 - \frac{r_s}{\infty}\right)\frac{dt}{d(\tau = t)} = -1 \implies \varepsilon = 0,$$

and this is a constant throughout the motion (so it makes our equation even easier)!

Thus, the equation of motion is just

$$\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 = \frac{MG}{r} \implies \sqrt{r}dr = -\sqrt{2MG}d\tau$$

(we take the negative square root because we have inward motion). This simplifies to

$$\tau_B - \tau_A = \frac{2}{3\sqrt{r_s}}\left(r_B^{3/2} - r_A^{3/2}\right).$$

Interestingly, nothing bad happens even if r_B or r_A is equal or less than r_s! So from a falling person's point of view, any arrival and starting point takes a finite amount of time.

But an observer at ∞'s point of view, we can derive that

$$t_B - t_A \propto \log\left(\frac{1}{\sqrt{r_A/r_s} - 1}\right).$$

As $r_A \to r_s$, this goes to ∞! So it takes an infinite amount of time to get to r_s, and then time freezes.

Finally, the last thing we'll do is to mention that photons can orbit around a black hole as well and form a **photon sphere**. Again, we'll still take $\theta = \frac{\pi}{2}$: we can exploit the fact that (in a circular orbit, $dr = 0$)

$$0 = ds^2 = -\left(1 - \frac{r_s}{r}\right)dt^2 + R^2 d\phi^2.$$

We can solve this to find that

$$\frac{d\phi}{dt} = \frac{\pm\sqrt{1 - r_s/R}}{R},$$

and the $\pm$ just tells us whether we're rotating clockwise or counterclockwise. But the question we're interested in is to find the actual value of R: to do that, we need to use the r-component of the geodesic equation:

$$\frac{d}{d\lambda}\left(\frac{dx_r}{d\lambda}\right) = \frac{1}{2}\partial_r(g_{\alpha\beta})\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}.$$

The left hand side is zero, and the $\frac{dx^\alpha}{d\lambda}$ only has contributions from t and ϕ: this yields

$$\left(\frac{d\phi}{dt}\right)^2 = -\frac{\partial_r g_{tt}}{\partial_r g_{\phi\phi}}.$$

Combine this with the other equation we just found, and we find that this yields $\boxed{R = 3MG}$. But it turns out this is an unstable orbit: any small perturbation will make our photon fall in or out of the black hole. But general relativity says that photons can get closer to a black hole than massive object! And that's saying that we'd be able to see the back of our head, theoretically, which is not something that exists in Newtonian mechanics.

39 November 21, 2019 (Recitation)

We'll continue talking about black holes today. Recall that the definition of a black hole is a region of spacetime for which light cannot escape to any infinite future time outside the black hole. Also, the singularity at the center of the black hole is a moment in time.

Remark 184. *Note that there is no mass in this black hole right now, because our stress-energy tensor is zero! The Schwarzschild solution yields a particular kind of black hole which always exists, so it's kind of a toy model.*

Today, we'll take a look at the two singularities at $r = r_s$ and $r = 0$, and we'll see why they're actually different! The singularity at $r = r_s$ is a fake one, and we're going to show this by making an invertible change of coordinates that make the metric "good."

Let's consider lightlike curves with θ, ϕ constant (so the $d\Omega^2$ term goes away). Then

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{r_s}{r}} = 0 \implies \frac{dt}{dr} = \pm\frac{1}{1 - \frac{r_s}{r}}.$$

We can solve this differential equation to find that

$$t(r) = \pm r_s \ln\left(\frac{r}{r_s} - 1\right).$$

The problem is that as $r \to r_s$, the slope goes to ∞. We'll do this by introducing the **tortoise coordinate**

$$r^* = r + r_s \ln\left(\frac{r}{r_s} - 1\right).$$

Taking the differentials and using the chain rule, we find that

$$dr^* = dr + r_s d\left(\ln\left(\frac{r}{r_s} - 1\right)\right) = dr\left(1 + \frac{r_s}{r_s\left(\frac{r}{r_s} - 1\right)}\right) = dr\left(\frac{1}{1 - \frac{r_s}{r}}\right).$$

We can then square both sides of this: plugging this into the metric,

$$ds^2 = \left(1 - \frac{r_s}{r}\right)(-dt^2 + dr^{*2}) + r^2 d\Omega^2.$$

This is particularly nice, because light rays in this new coordinate system have slope 1 again! So now we're back to the 45 degree light cones.

Remark 185. *Notice that in the Minkowski metric, where*

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2,$$

it makes sense to define $u = t - r, v = t + r$ (which trace out the light rays): $du = dt - dr, dv = dt + dr \implies dt = \frac{du+dv}{2}, dr = \frac{dv-du}{2}$, and plugging this in yields

$$ds^2 = -dudv + \frac{(u-v)^2}{4} d\Omega^2.$$

These are known as **null** *or* **lightlike** *coordinates: the 45-degree angle lines are now the u- and v-axis.*

Let's apply this to our new coordinates: define

$$u = t - r^*, \quad v = t + r^*.$$

Along constant u, t and r^* increase at the same time, so these are called **outgoing**. Similarly, constant v is called **infalling**. But the important thing is that we'll plug these back into our Schwarzschild metric: substituting in $t = v - r^*$ yields

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)(dv - dr^*)^2 + \frac{dr^2}{1 - \frac{r_s}{r}} + r^2\Omega^2 = \left(1 - \frac{r_s}{r}\right)dv^2 - dr^{*2}\left(1 - \frac{r_s}{r}\right) + 2\left(1 - \frac{r_s}{r}\right)dvdr + dr^{*2}\left(1 - \frac{r_s}{r}\right)^{-1} + r^2 d\Omega^2.$$

Now using the relation between dr^2 and dr^{*2},

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)dv^2 + 2dvdr + r^2 d\Omega^2.$$

It already looks like we have progress here: at $r_s = r$, nothing is broken! One concern we might have is that the coefficient in front of dv^2 equal to 0, so it seems like we may not have an invertible metric. Luckily, we have off-diagonal terms $dvdr$, and the determinant of $g_{\mu\nu}$ turns out to be $-r^4 \sin^2\theta \neq 0$ at $r = r_s$!

Remark 186. **Kroskal coordinates** *fix this even more by making the g_{vv} term not equal to 0 at $r = r_s$, but that's just a lot more manipulation.*

So if we look at the initial coordinate system, our metric does not depend on t, so replacing $t \to -t$, nothing changes. So if we draw a spacetime diagram with our r^* and t coordinates, the picture should look reflection-symmetric across the t-axis, and all of our light rays rule (lie along the boundary of) the $r = r_s$ event horizon. There's a few important properties of this diagram:

- Both light rays correspond to $r = r_s$, and these light rays break up our diagram into a top, bottom, left, and right region. (The top and bottom correspond to $r < r_s$.)
- Since there's a singularity at future time at the top of our diagram, by symmetry, there must be a **big bang** or **white hole** at the bottom of our diagram.
- The left and right parts of our diagram are "independent universes:" they are causally disconnected.
- Every point in our $r^* - t$ diagram is actually a sphere (in the directions ϕ, θ). In particular, if we slice a cross-section of this diagram at $t = 0$, the sphere has larger radius for large values of r^*, and it's small for smaller values of r^* (at $r^* = 0$, it has a radius of $r = r_s$). This is actually known as a **wormhole** or an **Einstein-Rosen bridge**!

So what's actually happening at the singularity here? Our timelike geodesics actually move forward in r here instead of t (because r is the timelike coordinate inside $r < r_s$), and we can pick two geodesics along constant t: $t = t_1, t = t_2$.

As the geodesics approach the singularity $r = 0$, we can compute the proper length of a constant-r curve between them: well, $dr = 0$, and we'll assume $\Omega^2 = 0$, so

$$ds^2 = \left(\frac{r_s}{r} - 1\right)dt^2 \implies s = \int \sqrt{\frac{r_s}{r} - 1}dt = \sqrt{\frac{r_s}{r} - 1}(t_1 - t_2).$$

Notice that as $r \to 0$, this goes to ∞! So proper distance between these two geodesics goes to infinity: the light rays accelerate away from each other at an infinite rate, which can be phrased as **proper acceleration diverges at the singularity**.

Well, proper acceleration is closely related to the gravitational field by the equivalence principle, so indeed the curvature of spacetime blows up as $r \to 0$. Notice that this means that matter gets ripped apart, too: electrons and protons will move away from each other.

But the strong nuclear force gets stronger if two particles are farther away from each other: therefore, a proton moving towards the singularity will have conflicting forces between gravity and the strong force! We'll talk about this more next time.

40 November 25, 2019

This lecture is being given by Professor Hughes. We'll spend some time talking about **gravitational radiation** today.

Proposition 187

In any theory that has a maximum speed for information, there must be radiation.

For instance, say we have two objects separated by some distance D. If there is a maximum speed c for which information can occur, there is always a time delay of $\frac{D}{c}$. One way to put this is that if we've done something at the origin and wait some time Δt after that action, there is a sphere of radius $c\Delta t$ inside which the universe "knows" that this action was taken.

Example 188

Say we start with the source-free Maxwell equations, and we take the curl of the curl equations.

A vector identity tells us that

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{F}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{F}) - \nabla^2 \vec{F},$$

and plugging this in yields

$$-\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\nabla^2 \vec{E} = -\mu_0 \varepsilon_0 \frac{\partial^2 \vec{E}}{\partial t},$$

(the last equality comes from swapping the $\vec{\nabla}$ and $\frac{\partial}{\partial t}$ from the first expression), and a similar equations for the $\vec{B}$ terms. One solution for this is the propagating **electromagnetic wave**

$$\vec{E} = \vec{E}(\vec{r} - ct\hat{k}), \quad \vec{B} = \vec{B}(\vec{r} - ct\hat{k}),$$

where $c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$. Further applying Maxwell's equations tells us that $\vec{E}$ and $\vec{B}$ are orthogonal to each other and the direction of propagation (which is $\vec{E} \times \vec{B}$).

Let's put this into a slightly more useful form. We've been trying to write things in a spacetime-covariant way, and we found that we could do this using the **four-potential**: if we define $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, we can write our source-free Maxwell's equations via

$$\partial^\mu \partial_\nu A^\nu - \Box A^\mu = \mu_0 J^\mu.$$

But by gauge invariance (pset 6), we can choose a ψ such that $A^\mu \to A^\mu + \partial^\mu \psi$ implies that $\partial_\nu A^\nu = 0$. This gives us the **sourced wave equation**

$$\Box A^\mu = -\mu_0 J^\mu, \quad \Box = \partial_\mu \partial^\mu.$$

what's nice here is that we can **connect the properties of the outgoing radiation to the dynamics of the source**. If we have no J^μ somewhere, we just get our ordinary 8.02 equations. Unfortunaetly, trying to solve this inside a region where there is a nonzero J^μ is very hard.

Instead, we'll use a **multipolar expansion**, which is a mathematical distribution of how the charges are distributed! This is essentially a leading-order approximation to describe our system. Specifically, we can calculate the **monopole**

$$\int_R \rho(\vec{x}) d^3x = Q,$$

which gives us the charge in our total distribution, and we can also find a **dipole**

$$\int \rho(\vec{x})\vec{x} d^3x = \vec{p}.$$

(Quadripole moments and so on also make sense.) But the leading order of radiation here will take the form

$$\vec{A} = \frac{\mu_0}{4\pi} \frac{1}{r} \frac{d\vec{p}}{dt}\bigg|_{t=t_R}$$

where $t_R = t - \frac{r}{c}$. The $\frac{1}{r}$ factor is important here: if the $\vec{E}$ and $\vec{B}$ fields both fall off in that way, then the flux is constant even as $r \to \infty$! ($\vec{E} \times \vec{B}$ falls off as $\frac{1}{r^2}$, and the surface area is proportional to r^2. So that's a sign that we do have radiation: energy is being conserved.)

With this, let's move to general relativity! Recall that we have the field equation

$$G_{\mu\nu}(g_{\alpha\beta}) = \frac{8\pi G}{c^4} T_{\mu\nu},$$

and we want to repeat the same thing as what we did for electromagnetic radiation. Well, it's very hard, but we'll make our life easier by looking at

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$$

to be a small perturbation from the flat spacetime. This is known as **weak gravity**, and Einstein's equation turns out to reduce to

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}.$$

This looks pretty similar to the sourced Maxwell's equations: the only difference is really that we have two indices instead of one!

Remark 189. *There's some small technical details that are being ignored here, but it turns out to not make a difference.*

Well, we have some mass density $\rho(t, x^i)$ which correspond to our "source." We can similarly define our monopole density

$$\int \rho(t, x^i) d^3x = M,$$

which just gives us the total mass of our object. If we only include this, it turns out that our perturbation $|h_{\mu\mu}|$ has magnitude $\frac{2GM}{r}$, which recovers the Newtonian limit! But now let's introduce a dipole moment

$$\int \rho(t, x^i)\vec{x} d^3x.$$

This dipole moment isn't exactly what we want, though! We get a nonzero dipole moment in electromagnetism because we put positive charge on one side and negative charge on the other. There's only one sign for mass (we don't have negative mass), so this dipole moment turns out to not give us anything other than the center of mass, which has no important dynamical properties.

Well, next is the **quadripole moment**

$$I_{ij} = \int \rho(t, \vec{x}) \left[x_i x_j - \frac{1}{3}\delta_{ij} r^2 \right] d^3x$$

where $r^2 = x^2 + y^2 + z^2$, and it turns out that this actually will give us our first sign of radiation:

$$h_{ij} = \frac{2G}{c^4} \frac{1}{r} \frac{d^2 I_{ij}}{dt^2}.$$

Note the $\frac{1}{r}$ making another appearance, and note that we have two time-derivatives instead of one (like in $\frac{d\vec{p}}{dt}$) because we have two indices! Here, G is small and c is huge, so we need a lot of time-varying quadripole moment to get any noticeable h_{ij} at all. (And the $h_{00} = h_{0i} = 0$ terms disappear.)

Example 190

Consider a binary star system in the xy-plane, where the two stars are in circular orbit and are a distance R away from each other.

This is equivalent to the **reduced mass** picture, where we have a mass $M = m_1 + m_2$ at rest and another mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ orbiting at radius R away from it. So we'll use that instead to avoid needing to look at two different moving parts!

We can say that the coordinates we're working with are

$$M = (0, 0, 0), \quad \mu = R(\cos \Omega t, \sin \Omega t, 0).$$

We'll assume that to first order, we can approximate their motion using Newton's gravity (and a Keplerian orbit), so $\Omega = \sqrt{\frac{GM}{R^3}}$. If we want to figure out how the gravitational radiation looks, we just need to compute I_{ij} and take two derivatives:

$$I_{ij} = \mu R^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2 \Omega t - \frac{1}{3} & \cos \Omega t \sin \Omega t & 0 \\ 0 & \sin \Omega t \cos \Omega t & \sin^2 \Omega t - \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Taking some derivatives (first using a trick that all terms here can be expressed with $\cos 2\Omega t$ and $\sin 2\Omega t$):

$$\ddot{I}_{ij} = -2\mu\Omega^2 R^2 \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos(2\Omega t) & \sin(2\Omega t) & 0 \\ 0 & \sin(2\Omega t) & -\cos(2\Omega t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Throwing this into our equation, we immediately find that if we're directly above the system on the z-axis, the perturbations from η look like

$$h_{ij} = -\frac{4G\mu/c^2}{r} \cdot \frac{\Omega^2 R^2}{c} \ddot{I}_{ij}.$$

The way we've written this, the first term gives us a distance ratio, and the second term gives a β^2 ratio. Both of these terms are going to be really small!

So how do we find out whether there are gravitational waves in our system?

Example 191

Let's say we have two objects at $A(0,0,0)$ and $B(L,0,0)$, under the metric $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, both starting at rest.

We have to be careful: we can't use the typical Lagrangian procedure, because our coordinates actually wiggle along with the perturbation of h (because radiation is part of spacetime!). Instead, let's do a direct measurement: let's see how long it takes for light to travel from A to B: we know that for light under this metric,

$$0 = ds^2 = (\eta_{\alpha\beta} + h_{\alpha\beta})dx^\alpha dx^\beta = -dt^2 + (1 + h_{xx})dx^2.$$

Thus, our speed of light

$$\frac{dt}{dx} = \pm\sqrt{1 + h_{xx}} \approx \pm\left(1 + \frac{h_{xx}}{2}\right).$$

If we integrate this from A to B, the time it takes to travel this distance is

$$\int_0^L \left(\frac{dt}{dx}\right) dx = L + \frac{1}{2}\int_0^L h_{xx} dx.$$

Suppose that h_{xx} is slowly varying, so in the time it takes to travel a distance L, h doesn't change much. Then the time it takes for this to occur is just

$$T_{A\to B} = L\left(1 + \frac{h_{xx}}{2}\right).$$

So the entire effect of the gravitational wave is in the slight difference $\Delta t = \frac{L}{2}h_{xx}$ measured!

Is this something that we could calculate with an experiment? Astrophysics tell us that out of things we can reasonably observe, m_{12} is about 1 to 100 solar masses, ΩR is about 0.1 to 0.5 times the speed of light, and r is about 20 to 2000 megaparsecs (where 1 parsec is 3.26 light years). Throwing all of these together, we're looking for $h \sim 10^{-19}$ to 10^{-24}, which is really, really small.

There's a few things we can do to mitigate this: first of all, we can try to make our separation L really big. But the big idea is that in 1972, Rainer Weiss proposed using laser interferometry for measurements: if we use a beam splitter, we can measure the phase shift between waves traveling in the x- and y-direction. It turns out that we can measure a shift of 10^{-12} of the laser wavelength if we use a laser of 100 watts!

The last aspect that we can bring into this is that our waves carry some amount of energy: in electromagnetism, the source loses energy E governed by the equation

$$\frac{dE}{dt} = \frac{1}{4\pi\varepsilon_0}\frac{2}{3}\frac{1}{c^3}\frac{d^2\vec{p}}{dt}\cdot\frac{d^2\vec{p}}{dt},$$

and in gravity, we get something that looks similar:

$$\frac{dE}{dt} = \frac{1}{5}\frac{G}{c^5}\frac{d^3 I_{ij}}{dt^3}\cdot\frac{d^3 I_{ij}}{dt^3}.$$

This basically tells us how much energy the source is losing because of gravitational radiation! Where is this energy coming from? If we look at this at a Newtonian level, we have some kinetic energy for each object, and then there's a potential energy term between the two of them. In our reduced mass picture, this just looks like $E = -\frac{G\mu M}{2R}$, so as our gravitational waves take energy away from our system, the magnitude R must get smaller (because masses don't change at this order).

Well, as R shrinks, Ω goes up, so the amplitude of our waves actually gets larger! This means that the "in-spiral" will give us a characteristic form for gravitational waves that we can measure, and that's actually what was detected just a few years ago.

41 November 26, 2019 (Recitation)

Last time, we studied the causal structure of the Schwarzschild metric: we found a future singularity as well as a past singularity at $r = 0$, because there is a $t \to -t$ symmetry of the metric.

This isn't physical, because we know that black holes haven't existed since $t = -\infty$ (because our universe has the beginning). So to what extent is the Schwarzschild metric actually physical at all? Today, we'll get a bit more physics out of what's going on.

Remember that the Einstein field equation takes the form

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}$$

where $G_{\mu\nu}$ is the Einstein tensor, depending on second derivatives of the metric, and Λ is the cosmological constant. Then the Schwarzschild metric is a **vacuum solution**, where $T_{\mu\nu} \approx 0$. In other words, it's good if we're in an area where there isn't much matter! If $T_{\mu\nu}$ is large near a star and small everywhere else, that means it's only reasonable to use the approximation farther away from the star.

The picture for the $r = 0$ area looks a bit confusing in Eddington-Finklestein coordinates (where light travels at 45 degree angles), so let's instead use Schwarzschild coordinates. Then the region $r = r_s$ looks like a cylinder, and the singularity $r = 0$ is just a line.

Example 192

If we have a collapsing star, once the entire mass disappears below $r = r_s$, it's okay to use Schwarzschild outside that collapsing mass but not inside.

Recall that Eddington-Finklestein coordinates follow the metric

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2,$$

where $v = t + r^*$ and $r^* = r + r_s \ln\left|\frac{r}{r_s} - 1\right|$ corresponds to infalling light rays. If we instead want outgoing light rays, we should instead use $t - r^* = v - 2r^*$.

We want to add some matter to this spacetime, because now we're simulating a **collapsing black hole**: if we throw an infalling light ray (of constant v) into our black hole, we see more mass for larger v, because more mass has fallen in. So we actually change M to be a function of v, and this is known as the **Vaidya metric**.

Proposition 193

Earlier light rays see a smaller M than lower ones, so $\frac{dM}{dv} > 0$.

This actually turns out to solve the Einstein equation if

$$T_{\mu\nu} = \frac{1}{4\pi G r^2} \frac{dM(v)}{dv} K_\mu K_\nu,$$

where

$$K_\mu dx^\mu = -dv \implies \mathbf{K} = (-1, 0, 0, 0).$$

Constant v means dv is constant, and we should have $ds^2 = 0$ for such lightlike paths. So $ds^2 = K_\mu K^\mu$ should be zero if K is a lightlike vector: indeed, this is just $g^{vv} K_v K_v$ (because K has no components in the other components).

So that seems problematic, because it looks like the vv-component is nonzero in the Vaidya metric. Turns out, though, because we have a non-diagonal metric, the matrix corresponding to the upper-index g does have a 0 entry in the top left corner! So indeed $\mathbf{K}$ is a light-like vector, as we expect.

Remark 194. *$K_\mu K_\nu$ here looks similar to the stress-energy tensor for a fluid when $P = 0$ ($T_{\mu\nu} = \rho U_\mu U_\nu$). In other words, a collapsing black hole is the solution when the stress-energy tensor*

$$T_{\mu\nu}^{\text{Vaidya}} = \frac{1}{4\pi G r^2} \frac{dM}{dv} K_\mu K_\nu$$

tells us something about a pressureless fluid moving at the speed of light. The thing is that we won't see this fluid until it hits us!

So as a sanity check, we indeed have a positive energy density, which corresponds to the black hole increasing its size.

Example 195

Consider the mass profile

$$M(v) = \begin{cases} 1 & v > 4 \\ \frac{v}{4} & 0 < v < 4 \\ 0 & v < 0 \end{cases}$$

What does this look like?

When $M = 0$, we just have the metric

$$ds^2 = -dv^2 + 2dvdr + r^2 d\Omega^2,$$

and this means $r_s = 0 \implies r^* = r$. Plugging in $v = t + r^* = t + r$ just yields $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$, so $M = 0$ looks like Minkowski space! So prior to $v = 0$, this is Minkowski space. Similarly, we can check calculations such that after $v = 4$ ($M = 1$), we have Schwarzschild with $r_s = 2G$ (the black hole just sits there, and the collapse has already happened).

But between $0 < v < 4$, matter is collapsing into the black hole. We can actually calculate the event horizon analytically in this region, and one weird feature is that even in some parts of the Minkowski space region, it's possible we're already inside the event horizon!

The way to further analyze this is to look at the outgoing light rays, where $t - r^* = v - 2r^*$ is constant. So

$$\frac{dv}{dr} - 2\frac{dv^*}{dr} = 0,$$

and we already know $\frac{dv^*}{dr}$ (it's nicely in terms of the Schwarzschild metric), so we can check that

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2GM(v)}{r}\right)$$

for outgoing light rays. Then we can see which light rays are able to reach $r = \infty$ and which can't: then the last light ray that makes it out to $r = \infty$ is the boundary of the event horizon!

Well, note that the event horizon starts at $r = 0$, and the area of that event horizon increases with time (because we have growing mass). This turns out to be a deep fact about general relativity (the Hawking area law), and this gives some interesting results related to black hole radiation (because we need to make sure entropy increases in the universe)! It turns out that black holes have entropy: **quantum gravity** doesn't actually look like what we've described so far in this class.

42 November 27, 2019

Today's lecture is being given by **Maximiliano Isi** from the LIGO laboratory.

First, a quick review of what's going on: gravitational waves are the solutions to the linearized Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu},$$

These waves have two main effects: first of all, by the equivalence principle, gravity "couples to all mass-energy," so we cannot detect a constant g-field. However, we can measure **tidal forces**, which are variations in a field (the gradient of g), and that's what is often called the **tidal field**. One effect of this is that there is some differential stretching: a gravitational wave propagates at the speed of light, and it will squeeze space **transversally**. We can describe this squeezing with a linear combination of polarizations: pick some arbitrary xy-plane, and decompose it into the "plus" and "cross" components:

$$h_{ij} = h_+ e_{ij}^+ + h_\times e_{ij}^\times, \quad e_{ij}^+ = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, e_{ij}^\times = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

assuming that the wave propagates in the z-direction so we only look at stretching in the xy-plane.

Definition 196

The **strain amplitude** of a wave is defined as $h = \frac{\Delta L}{L}$, where ΔL is the amount of stretch that we see in an object with length L. (Note that this means the stretch ΔL is proportional to the length L.)

The idea is that spacetime is unfortunately very hard to bend: gravitational waves are very weak! One way we generate gravitational waves is with something that is not spherically symmetric, very dense, and with high acceleration: a **binary system** is good for this. The frequency of such a wave is twice the frequency of the system, and the amplitude is proportional to $\frac{m_1 m_2}{Rr}$, where R is our distance from the system. So we want something that's very massive and compact, so the two objects are very close to each other and moving very fast.

Well, the most compact objects in the universe are black holes: these are indeed the sources that were found first. They radiated a huge amount of energy, but the strain amplitude h was only on the order of 10^{-21}, which is extremely small!

Black holes aren't the only sources of gravitational waves, though. Other systems have characteristic forms for their signals:

- Systems of two black holes, two neutron stars, or a neutron star and black hole are called **compact binary coalescences**: they have a quickly increasing frequency and amplitude, which gives a characteristic chirp. They're well-understood from calculations, so the search has been simplified.
- There's some transient signals from supernovae, dark matter, and unknown sources: there are no accurate waveforms here, but they can still be detected if we look at coherent power across detectors.
- Persistent signals, known as **continuous waves**, may come from neutron stars that aren't spherically symmetric, dark matter, and so on.
- Stochastic background could come from random process or cosmological processes.

LIGO and Virgo have detected 10 black hole pairs to date: the reason we know the mass of the objects involved is that the strain is directly related to the dynamics of the object. One point is that even though we only have two objects, there is no analytic solution to the two-body problem in general relativity. So it's usually required to put everything into a computer and simulate the whole process.

Example 197

Two spinning black holes create a waveform with three regimes:

- Inspiral: two objects relatively far away and moving slowly has a frequency around twice the orbital frequency.
- Intermediate: the two objects get faster and faster, slowly approaching each other.
- Merger ringdown: the objects merge into one, and the final product vibrates a bit until it settles down

We can use a Newton approximation here: using the linearized form of the field equations, we find that the power

$$\dot{E} \propto -\frac{(m_1 m_2)^2 (m_1 + m_2)}{r^5}$$

and the approximations for the coordinates of h are

$$h_+ = A(\tau)\frac{1}{2}(1 + \cos^2 \iota)\cos \Phi(\tau), \quad h_\times = A(\tau)\cos\iota \sin\Phi(\tau)$$

where τ is the time to coalescence (the end of the inspiral).

- $A(\tau)$ increases as τ gets closer to 0, and it's inversely proportional to R. The overall magnitude depends on $\mathcal{R}$, which is known as the **chirp radius** $\mathcal{R} = \frac{G\mathcal{M}}{c^2}$, where $\mathcal{M}$ is defined as $\mu^{3/5}M^{2/5}$ (it's a useful quantity that happens to appear in the equations).
- $\Phi(\tau)$, the phase, also depends on the chirp mass:

$$\Phi(\tau) = 2\tau \left(\frac{5}{256\tau}\right)^{3/8} \left(\frac{\mathcal{R}}{c}\right)^{-5/8}.$$

- ι is the **inclination**, which tells us which angle we're looking at the binary star, and this gives us a relative ratio between h_+ and $h_\times$. The key here is that we're defining these relative to some frame (aka our detector)! Here, we suppress the celestial coordinates α and δ here.

Remark 198. *These binary systems actually output gravitational strain as a sum of different angular modes: the emission pattern is non-isotropic, and each mode has an intrisic direction! Usually, the quadrupole is the dominant term for a binary source.*

So this approximation is good when we're far away from the merger, but it's bad when $\tau \to 0$: instead, we need to make a lot of corrections, which vary in accuracy and computing cost. Hybridization between different methods happens pretty often here!

In general, there are at least 15 parameters for a quasi-circular orbit that an experiment may want to capture. But it turns out that many of these can be studied with a single frequency-versus-time plot!

Remark 199. *Everything here assumes that our source is localized, though: if we have faraway contributions, then all quantities with a factor of m actually gain a factor of $(1 + z)$. For example, the frequency f_s gains a factor of $(1 + z)^{-1}$, and the chirp mass $\mathcal{M}$ gains a factor of $(1 + z)$.*

So what would an ideal gravitational wave detector look like? We can't measure stretches with a ruler, because rulers will also be stretched by the gravitational wave. Instead, the point is to use light as a clock instead of a ruler and to do **interferometry**! Each LIGO instrument uses a 4 kilometer interfereometer, and the strain

$$d(t) = \frac{\Delta L}{L}$$

is measured as a function of time. Not all stretches are detectable by the two arms: some signals are invisible because they're at the wrong angle! So there's a characteristic pattern of how sensitive each detector is to certain directions, irrespective of the actual amplitude, and this is represented mathematically as a detector tensor D_{ab}.

So if there are a series of detectors, each in a different location, it's easy to mathematically write down the output of that detector. But there's some instrumental noise that comes up, so what we actually see looks like

$$d = F_+ h_+ + F_\times h_\times + n.$$

Real life detectors had a peak strain of $h \approx 10^{-21}$ for the first detected gravitational waves. To observe that, the laser was enclosed in a 10^{-9} Torr vacuum, there was a stable infrared laser, the mirrors were polished atom by atom, and there was lots of seismic isolation. Even with this, the order of magnitude of the noise was still on the order of 10^{-22} to 10^{-23}.

43 December 2, 2019

We'll spend the rest of 8.033 talking about an introduction to cosmology: the structure and evolution of the universe. First, though, we'll review a few points (because some of us have asked questions, or because we'll need them later on). We'll review the ideas of **proper time and length**, as well as the description and **speed of light**.

Remark 200. *We should note that coordinates are arbitrary, so we generally want to use invariants instead.*

For example, instead of trying to describe light as traveling at some speed c, which depends on our coordinates, or length as a difference of coordinates, we should use $ds^2 = 0$ or the proper length, respectively.

1. Earlier in the class, we found that for any pair of **timelike** events, we can describe their separation via an invariant $ds^2 < 0$, where

   $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu.$$

   We found that massive particles always travel along timelike paths: one way to track the **proper time** along this path is to break up the path into infinitesimally small chunks. Along each chunk, we can go into the local free-falling rest frame of the object, and because space is locally flat,

   $$g_{\mu\nu} \implies \eta_{\mu\nu}, d\vec{x} = 0,$$

   so

   $$ds^2 = \eta_{00} dt \cdot dt = -dt^2.$$

   We **define** this to be $d\tau^2$, and now if we sum over all chunks,

   $$s = \int_{\text{path}} ds$$

   is an invariant, so it has the same value in all frames. This proper time is then extremized along a geodesic that a particle takes through spacetime, but the key idea is that we've written it in terms of an invariant ds.
2. The next topic we'll review a bit about light and the way in which it is characterized (independent of coordinates).

In Minkowski space, we saw that the coordinate speed of light

$$\left|\frac{d\vec{x}}{dt}\right| = c$$

is constant, and this is a vector statement. But this isn't exactly the characterizing statement – we know that under the Schwarzschild metric, it's dependent on G, M, r – so we should instead characterize lightlike paths as $\boxed{ds^2 = 0}$. Indeed, in flat spacetime, $ds^2 = -dt^2 + dx^2 = 0$ gives $\left|\frac{dx}{dt}\right| = 1$, so the definition is constant with our initial motivation.

Basically, the right way to approach this situation is to look at a toy example with flat spacetime: for example, characterize our coordinates via $t = u + w, x - u - w$. In this new system,

$$ds^2 = -dt^2 + dx^2 = -4dudw.$$

If we're asking what the speed of light in these new coordinates looks like, it doesn't really make a lot of sense to make a statement like $\frac{dx}{dt} = 1$ anymore, because our variables are now in terms of some other arbitrary variables! So the right way to describe this propagation is to think about a worldline where $ds^2 = 0$: if $du = 0$, then u is constant, and if $dw = 0$, then w is a constant – plotting this on an $x - t$ plane, we see that these are indeed the same standard 45 degree light cones that we know (we've just done a rotation). $ds^2 = 0$ is therefore a better description of light paths, because we don't need to rely as heavily specific values of our coordinates!

3. Finally, let's discuss proper length. The context for this is the same as with the previous cases: we can't just take differences in coordinates and expect this to be a consistent distance in all frames! The proper length is equal to ds^2 if the time separation is zero (as we showed earlier on in class), but typically the notion of distance should always be ds^2 rather than a distance in coordinates.

Example 201

For example, consider a 2D sphere (in 3D space), and draw a meridian (from the north pole to the south pole).

Consider two points along the surface of the sphere along a meridian, and say that we want to calculate their distance. **Whenever we calculate a distance**, we want to find ds^2, and the metric along a unit sphere looks like

$$ds^2 = d\theta^2 + \sin^2 d\phi^2.$$

This is not just a simple Pythagorean distance from $d\theta$ and $d\phi$: what matters here is the line element ds^2 for calculating lengths!

If we add in a time variable, our line element now starts looking like ($\mathbf{x}$ here is a four-vector)

$$ds^2 = A(\mathbf{x})d(\text{time})^2 + B(\mathbf{x})d(\text{time})d(\text{space}) + C(\mathbf{x})d(\text{space})^2,$$

and the B (off-diagonal) term is often zero for simplicity in this case. If we're given two events that are spacelike-separated, and we want to find an invariant distance, one thing we can consider is the case where $dt = 0$: then the metric just simplifies to

$$ds^2 = C\mathbf{x})d(\text{space})^2,$$

which gives us the **proper length** $d\sigma^2$. So in a specific set of coordinates where the time separation is zero, we do look only at the space coordinates, but just like before we need the metric term $C(\mathbf{x})$ to actually compute the distance.

So if we go back to the Schwarzschild metric again, we can see why the r-coordinate is not just an invariant distance (it's instead a coordinate distance):

$$ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

This looks kind of like polar coordinates, but r isn't just a distance! For example, consider two events that occur at the same time, so $dt = 0$: then we can simplify the invariant distance

$$ds^2 = \left(1 - \frac{2MG}{r}\right)^{-1} dr^2 + r^2 d\Omega^2.$$

Specifically, let's say that $P_1 = (T, r_1, \Theta, \Phi)$ and $P_2 = (T, r_2, \Theta, \Phi)$ are only radially separated. Then the invariant distance is an integral:

$$ds = \left(1 - \frac{2MG}{r}\right)^{-1/2} dr,$$

and then we integrate this from $r = r_1$ to $r = r_2$ to find the total distance. We'll find that it's **larger than** $r_2 - r_1$, which is the coordinate distance. One way to intuitively understand how gravity affects distances and coordinates is to imagine a mattress, where objects of mass create a "dip" downward in the surface! And this allows us to qualitatively relate the results we've found to a geometric picture: ds^2 is then the distance along the surface of our mattress, but our coordinates only have a top-down view: they don't see the dips in spacetime at all.

One final point about proper distances is a point that helps us start thinking about how the universe works: let's consider points in a 3D sphere (in 4D space). We live spatially in 3 dimensions, and let's assume that instead of having a rectangular geometry, we have this other curved distance instead. **In particular, what if this radius keeps increasing over time?**

Well, the best way to describe the line element ds^2 along this sphere is to **embed it** in a general space. For example, in the 2-D sphere case, we can describe points on the sphere via

$$x^2 + y^2 + z^2 = 1, \quad ds^2 = dx^2 + dy^2 + dz^2,$$

and then we parameterize points on the sphere with our two coordinates θ and ϕ (with the usual spherical coordinates), and then define $ds^2 = dx^2 + dy^2 + dz^2$ to be the distance by plugging in x, y, z in terms of θ, ϕ.

So a 3-D sphere can be described via

$$x^2 + y^2 + z^2 + w^2 = 1, \quad ds^2 = dx^2 + dy^2 + dz^2 + dw^2$$

as an **embedding in 4-dimensional space**. If we now introduce appropriate polar coordinates

$$x = \sin\chi \sin\theta \cos\phi, y = \sin\chi \sin\theta \sin\phi, z = \sin\chi \cos\theta, w = \cos\chi,$$

and then plug in all the differentials, we find that the line element

$$ds^2 = d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2),$$

and this gives us a way to measure distances on a surface of radius 1. But if our universe is expanding, how does this metric look if we have a changing radius instead? Well, all of our coordinates x, y, z, w are scaled by a, the metric just gains a factor of a^2 everywhere. So in full generality, we have a pre-factor of $a(t)^2$ in front of this space element, and then we also gain a $(-dt^2)$ term from the space distance (analogous to the Minkowski case)! With this, we have a way of describing an expanding three-dimensional sphere.

But the point is that coordinates are bad: if we consider two events that occur at

$$P_0 = (T_0, \chi_0, \theta_0, \phi_0), P_1 = (T_0, \chi_0 + d\chi, \theta_0 + d\theta, \phi_0 + d\phi),$$

we get a separation exactly as the metric predicts for us:

$$ds^2 = a^2(T_0)(d\chi^2 + \sin^2\chi d\Omega^2).$$

But if we calculate the distance of these points a little bit later, the distance has changed, even though we're looking at the same position! We'll continue to explore these ideas later on in class.

44 December 3, 2019 (Recitation)

This recitation is being taught by **Halston Lim**.

In previous lectures, we've been talking about **compact objects**: this means that a body is localized in space, so very far away, we don't see its effects. For example, if we have a source of mass M at our origin, but $\frac{GM}{rc^2}, \frac{v}{c} \ll 1$, then everything looks basically Newtonian, and we often say the metric is **asymptotically flat** (we just have $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$). If we write down the Einstein equation

$$G^{\mu\nu} = \frac{8\pi G}{c^4}T^{\mu\nu},$$

this indeed only holds when $T^{\mu\nu} = 0$ and we have no source.

Cosmology is different from this, because the universe is not one object: matter fills the whole universe, so we need to modify our description a little bit! A lot of space is filled with **galaxies**, and many galaxies are arranged in clusters in a way that is hard to describe with such a simple model.

But what we do have is the cosmic microwave background: looking there, the fluctuations of matter density are within a factor of 10^{-5} of the average, as long as we considered things above 3 degrees Kelvin. One way to think about this as a fluid: we can treat everything as a continuous medium if the time scale is large enough, and similarly in cosmology here, we're going to ignore the small fluctuations and look at some bulk assumptions about symmetry to get something that's workable.

So in a **spatially uniform** situation, the metric looks like

$$ds^2 = -c^2dt^2 + a(t)^2\left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \bar{r}^2d\theta^2 + \bar{r}^2\sin^2\theta d\phi^2\right].$$

This is called the **Robertson–Walker** metric: notice that the last two terms here look a bit like the 2-sphere, and the $a^2(t)$ is known as the **scale factor**, with units of length. If $k = 0$, this reduces to the flat space metric except with an extra $a(t)^2$ term, and the distance

$$(\bar{r}_1, \theta_1, \phi_1) \iff (\bar{r}_2, \theta_1, \phi_1), L = a(t)(\bar{r}_2 - \bar{r}_1).$$

One way we can think of this is that we have an inflating balloon, and our two points are on the surface of that balloon. As the balloon blows up, even if our coordinates (the actual points on the balloon marked) are the same, the distance still changes!

Remark 202. *There's a lot of different forms of the Robertson–Walker metric: in this one, we pick $\bar{r}$ to be unitless and often "normalized by the curvature," in such a way that $a\bar{r}$ will give a distance. This metric actually has the most amount of rotational and translation symmetry (assuming an isotropic and homogeneous universe) possible in four* *dimensions.*

As we mentioned earlier, k, the **spatial curvature parameter**, yields a **spatially flat** (not necessarily flat) spacetime when $k = 0$. There's two other cases:

- If $k = 1$, we can solve for an equivalent notion of distance: again if θ, ϕ are fixed, we have

$$d\xi = \frac{d\overline{r}}{\sqrt{1 - \overline{r}^2}} \implies \overline{r} = \sin \chi,$$

  so $\overline{r}$ is bounded. This means there is a maximum distance possible on the space (for example, we can imagine this as the diametrically opposite distance on our balloon), and this is known as a **closed universe**.
- In contrast, $k = -1$ is an **open universe**, which yields

$$\frac{d\overline{r}}{\sqrt{1 + \overline{r}^2}} = d\chi \implies \overline{r} = \sinh \chi.$$

Within error of measurement, we think that $k = 0$, but we're still not sure why this is or whether it's actually flat!

So we have a metric ds^2, which tells us the value of $g_{\alpha\beta}$: if we plug this into Einstein's equations, we get the **Friedmann equations**: the idea is that if we have $\dot{a}(t)$ and $a(t)$, we can make some observations about k and the matter content of the universe. (And we should expect that in this very simplified case, we just have some constant matter density ρ_M: playing with this ρ changes our scale factor a.)

If we look along a geodesic, consider an observer at rest, so $u^t = c, u^{\overline{r}} = u^\theta = u^\phi = 0$. In a cosmological sense, what this means is that the $\overline{r}, \theta, \phi$ coordinates are constant, but we need to **comove** (as the balloon expands), so the proper distance between two points "at rest" may change.

For now, let's deal with $k = 0$, and let's think about how light travels here. Suppose a person emits light at some time t_e, and it is received by some person at some time t_r (this is the Robertson-Walker time, which is some **"global" time independent of observers**). Say that the light travels radially, so

$$\mathbf{p}_\gamma = (p^t, p^{\overline{r}}, 0, 0).$$

We want to do a redshift-type calculation: how does the energy of light compare when it is emitted versus when it is received?

Well, we can use the comoving notion:

$$E_e = -\mathbf{p}_{\gamma,e} \cdot \mathbf{u}_e = p^t_{\text{emitted}} \cdot c.$$

If we propagate this until time t_r, we know that $\mathbf{p} \cdot \mathbf{p} = p^\alpha p^\beta g_{\alpha\beta} = 0$, so we can verify that

$$p^{\overline{r}} = \frac{p^t}{a(t)}.$$

Also, we use the fact that light travels along a geodesic, which gives us another equation. We'll do this in lecture tomorrow, so we won't go through the details, but we can look at the key result: we find that

$$\frac{p^t(t_r)}{p^t(t_e)} = \frac{a(t_e)}{a(t_r)},$$

so the energy ratio

$$\frac{E_R}{E_e} = \frac{a(t_e)}{a(t_r)}$$

only depends on the scale factor! The idea is that if we know the frequency of light when it was emitted, and we know the frequency when we receive it, that tells us the ratio of scale factors at these different times, which directly tells us how quickly the universe is expanding.

Astrophysically, the idea here is that atoms and molecules are emitting light at specific spectral lines: if we plot the flux (photons per area per unit time) as a function of wavelength, it's peaked at a few specific points: we can say that those are the hydrogen, carbon dioxide, etc. line in our lab picture. If we then actually measure the actual peaks in some dusty part of the Milky Way, we'll see something that is shifted a bit: now we can just pick out the peaks, match them, and then the ratios should be all the same:

$$\frac{\lambda_1'}{\lambda_1} = \frac{\lambda_2'}{\lambda_2} = \frac{a(t_r)}{a(t_e)}.$$

We define this to be $1 + z$, giving us the **redshift**: this then allows us to figure out how far away things really are! (In our local universe, $z = 0$, and we have to go to megaparsecs or larger to find large z or large effects. This is consistent with the metric that we have.)

45 November 4, 2019

Today, we'll start to study cosmology more seriously. We'll start with a general description of the universe, so that we can describe and justify a metric from our observations.

Last time, we discussed how distances should be measured using the line element $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$, rather than coordinate distances. $ds^2 < 0$ (timelike separation) corresponds to a **proper time** $d\tau^2 = -ds^2$, and $ds^2 > 0$ (spacelike separation) corresponds to a **proper distance** $ds^2 = d\sigma^2$.

This led us to the description of an expanding 3D sphere:

$$ds^2 = -dt^2 + a^2(t)\left[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)\right],$$

where the parenthetical term is, as always, $d\Omega^2$, and $a^2(t)$ refers to $a(t)$ squared. Notably, this means the distance ds^2 can change even if the coordinates χ, θ, ϕ stay constant (because of the **scale factor** a).

Recall that $\frac{MG}{c^2R}$, the factor for black holes, only gives us a significant factor if it is comparable to 1, and this only occurs with large masses M. But the universe seems mostly empty: do we really need relativity to give a general description of the universe? Here's what we know from observations:

- The universe is generally **homogeneous**: any point is equivalent to any other point, so physics should be translation-invariant (on average). This may not seem true around us – obviously things look different near the Earth and outside the Milky Way – but this is a general **average** over very large boxes, on the scale of 100 megaparsecs.
- The universe is also **isotropic**: no matter which angle we choose to look out into the night sky, we see (on average) the same behavior.

With this, the best way to describe how much stuff is in the universe is to determine the constant (average) **matter density** ρ_m. And observationally, we've found that

$$\rho_m \approx 10^{-26}\frac{\text{kg}}{\text{m}^3},$$

which is indeed extremely sparse! So if we calculate how much stuff matter there is in a sphere around us of radius R, we just have

$$M = \frac{4}{3}\pi R^3 \rho_m.$$

And this means that as long as R is sufficiently large, $\frac{MG}{c^2R} = O(R^2)$ will eventually be big enough that we can't ignore it: doing a back-of-the-envelope calculation, we find that

$$R_{\text{relativity}} \sim 6 \text{ Gpc}$$

(here 1 parsec is about 3.26 light years). This is a huge distance, but it's not big enough to be outside the range of our telescope. So we do need to take into account relativity in what we observe in the sky!

Remark 203. *For a sense of scale, our distance to the sun is on the order of* 5×10^{-6} *parsecs, to the next nearest star is on the order of* 1 *parsec, and the size of the Milky Way is about* 15000 *parsecs. The distance to Andromeda is* 800 *kiloparsecs, but the size of the visible universe is* 14 *gigaparsecs, which is far bigger than any of the other numbers we've found here.*

And there's one more complication, too:

- The universe is **expanding**: this was discovered in 1929 by Hubble. He found this by measuring the velocity of galaxies around us, and all galaxies seemed to be receding away from us. In fact, the farther away a galaxy was from us, the faster it seemed to be receding (this was basically linear). **Hubble's law** then just tells us that $v = Hd$ for some constant H. And because we're not special, this means that everything is receding from everything else: the only explanation for this is that the universe must be expanding. However, this also means that there must have been a beginning where all of the expansion started: that's what we often call the Big Bang.

This means that something like the Schwarzschild metric cannot describe our universe, because the metric must depend on time! We can start by writing generically that the metric can look like

$$ds^2 = A(\vec{x}, t)d(\text{time})^2 + B(\vec{x}, t)d(\text{time})d(\text{space}) + C(\vec{x}, t)d(\text{space})^2.$$

However, we can make some simplifications: since our universe is homogeneous, A, B, and C cannot depend on $\vec{x}$ at all. Also, since all directions are the same, $B(t) = 0$ as well, or else some direction would be preferred over another in the cross term. Finally, the term $A(t)d(\text{time})^2$ can be made simpler if we introduce a scaled time parameter where $A(t) = -1$! So now **we can pick coordinates** in such a way that (picking for historical reasons $C(t) = a^2(t)$)

$$ds^2 = -dt^2 + a^2(t)d(\text{space})^2.$$

These are known as **co-moving coordinates**. If we plug in the metric into the geodesic equation, we find that

$$\ddot{x}^i = 0 \implies \dot{x} = \text{constant}$$

(for a free-falling object). So objects in our spacetime only under the influence of gravity (for example, galaxies), have no acceleration! And if we pick the velocity $\vec{v}(t = 0) = 0$, the velocity will be zero forever and the coordinate position is constant: thus, objects will be co-moving with the coordinates. (In a balloon analogy, we could imagine that grid lines expand along with the balloon, even if the distances are changing.)

What about the time coordinate t here? If we put a clock on a galaxy in our universe, and we look at two time ticks, they will happen at the same coordinate position. So t is **time measured by a free-falling object in our galaxy**, and it just looks like $ds^2 = -dt^2 = -d\tau^2$.

Remark 204. *The* **3D sphere** *is not the only shape that is both homogeneous and isotropic: while that gives a metric of* $d(\text{space})^2 = d\chi^2 + \sin^2\chi d\Omega^2$*, we can also just have an* **(infinite) 3D plane** *where* $d(\text{space})^2 = d\chi^2 + \chi^2 d\Omega^2$ *(this is the normal description of a flat space in polar coordinates), or a* **3D hyperboloid***, where* $d(\text{space})^2 = d\chi^2 + \sinh^2\chi d\Omega^2$*. These have positive, zero, and negative curvature, respectively.*

And it turns out that we can do a trick to unify all three of these:

Definition 205

Introduce the variable

$$\overline{r} = \begin{cases} \sin\chi & \text{for a sphere,} \\ \sinh\chi & \text{for a hyperboloid,} \\ \chi & \text{for a plane.} \end{cases}$$

Then we can just write

$$d(\text{space})^2 = \frac{d\overline{r}^2}{1 - k\overline{r}^2} + \overline{r}^2 d\Omega^2,$$

where

$$k = \begin{cases} 1 & \text{for a sphere,} \\ 0 & \text{for a plane,} \\ -1 & \text{for a hyperboloid.} \end{cases}$$

So now writing our metric for all three possible universes can be written identically, and plugging it in has given us a metric that describes the (homogeneous, isotropic, non-static) universe, known as the **Friedmann–Robertson–Walker** metric:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{d\overline{r}^2}{1 - k\overline{r}^2} + \overline{r}^2 d\Omega^2\right).$$

Today, there is overwhelming evidence that $k = 0$, within error bars. But it's important to note that $k = 0$ does not mean we have a Minkowski space: it's just that a slice **at constant** t looks flat, but we still have nonzero curvature! Note also that $\overline{r}$ has been chosen to be dimensionless: a is the quantity which carries dimension.

So let's go back to the idea that galaxies are receding from each other: just to emphasize the point again, this does not mean that $\overline{r}$ is changing for those galaxies! It just means that $a(t)^2$ is increasing as a function of t. Say two galaxies are only separated by this $\overline{r}$ coordinate:

$$G_1 = (t, \overline{r}_1, \theta, \phi), G_2 = (t, \overline{r}_2, \theta, \phi).$$

The invariant distance here has $dt = d\phi = d\theta = 0$:

$$d\sigma^2 = ds^2 = a^2(t)\frac{d\overline{r}^2}{1 - k\overline{r}^2}.$$

This is true infinitesimally, so we can take the square root of both sides to find the proper distance between the two galaxies at some time t:

$$\sigma(t) = \int_{\overline{r}_1}^{\overline{r}_2} \frac{a(t)d\overline{r}}{\sqrt{1 - k\overline{r}^2}}.$$

If the universe is flat, and $k = 0$, this just evaluates to $a(t)(\overline{r}_2 - \overline{r}_1)$. And indeed, this shows us that the distance will scale linearly with $a(t)$, the scale factor of the size of the universe!

Fact 206

If we want to find $a(t)$ itself, we need to solve Einstein's equations, which we won't do explicitly in the class – we'll mention next time how we expect a to scale with t, though.

For the rest of today, we'll do something less ambitious. Hubble's law $v = Hd$ is a bit deceiving here: d is actually the proper distance, and v is a kind of **proper velocity**: given the proper distance $\sigma(t)$, we can just take

$$\frac{d\sigma}{dt} = \frac{da}{dt} \int_{\bar{r}_1}^{\bar{r}_2} \frac{d\bar{r}}{\sqrt{1 - k\bar{r}^2}}$$

(the integral does not depend on t). But notice that this can be written in terms of σ as well:

$$\frac{\dot{\sigma}}{\sigma} = \frac{\dot{a}}{a},$$

so the rate of change of the proper distance is proportional to the distance: we can defining the **Hubble parameter**

$$H(t) = \frac{\dot{a}(t)}{a(t)}.$$

Note here that $\frac{d\sigma}{dt}$ is invariant, because in these coordinates, coordinate and proper time are the same thing, and σ is related to ds^2, which is invariant. The Hubble parameter is a function of time, and the current value of H is known as H_0 (this is the measurement of $\dot{a}$ today, divided by the measurement of a today).

This is one of the most important measurements (because we can say something about the evolution of a), but it's hard to measure distances of stars and galaxies! This is because anything that we see is a bunch of photons, which do not carry information about how far they came from.

- One thing we can do is look at different measurements when the Earth is at different parts of the orbit around the Sun and use trigonometry, and this is called using the **parallax**. Unfortunately, this only works for very nearby objects (otherwise we can't even tell the difference).
- So an alternative method is to look at **variable stars**, whose luminosity changes a lot as a function of time. Such stars have a very precise relationship between their period of variability and brightness: if we can find the period, and we know the true brightness, we can compare it to our observed brightness to find the value of r. Ideally, there is a set of such stars close enough for us to use the parallax method to double-check. But even this method only gives us distances on the order of tens of megaparsecs: what's the next step?
- Use supernovae! A certain type, known as type 1A supernovae, are those where a **white dwarf** absorbs material from some other source. At a certain point (around 1.4 solar masses), this white dwarf explodes, and all such explosions produce the same true brightness (because they come from the same mass). So we can again find the true brightness (by knowing the real distance of one of these sources), again by comparing this method to the cepheid method: this gives us distances up to 1 gigaparsec.

So the main idea here is that we want to measure v and d for a galaxy: we've shown that measuring d is hard. The uncertainty of H got smaller and smaller for a while, but now we have different methods that yield very different values of the Hubble constant (statistically 5σ apart)! So we still don't know the value of H_0 today.

46 December 5, 2019 (Recitation)

This recitation is being covered by a graduate TA.

We've been discussing the FRW metric

$$ds^2 = -c^2 dt^2 + a^2(t) \left[\frac{d\bar{r}^2}{1 - k\bar{r}^2} + \overline{r^2} d\Omega^2 \right]:$$

there are different conventions that we may see for this metric. In lecture, we've taken a to have dimensions of length, so $k \in \{-1, 0, 1\}$. However, it's also possible to take a dimensionless, so that $\bar{r}$ has dimensions of length, and

$k \sim (\text{length})^{-2}$. Today, we'll use the second convention for simplicity of calculations.

This metric is used for large scale structure of the universe, and the key idea is that we don't know the exact value of the scale factor a. If we plug things into the Einstein equations, though, we can derive $G^{\mu\nu}$ from the metric to get a $T^{\mu\nu}$ which tells us how the matter is distributed: let's assume here that it's a perfect fluid. This will lead us to the **Friedmann equation**

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3c^2} - \frac{kc^2}{a^2}.$$

So if we know ρ, the energy density, as a function of time, we can solve for a. If we define the critical density $\rho_{\text{crit}} = \frac{3H^2c^2}{8\pi G}$, and define Ω to be the ratio $\frac{\rho}{\rho_{\text{crit}}}$, we can rewrite our equation as

$$\Omega - 1 = \frac{kc^2}{a^2H^2}.$$

This allows us to determine k, the overall curvature of the universe: more specifically, if $\rho < \rho_{\text{crit}}$, then $\Omega - 1$ is negative, so k is negative (open or hyperbolic universe), and similarly, if $\rho > \rho_{\text{crit}}$, we have a closed or spherical universe. Equality gives us a flat universe, and this is what we've observed experimentally so far.

Let's look at a few different instances:

Example 207 (Matter dominated universe)

In this universe, most of the contribution to ρ comes from regular matter.

Because we have an expanding universe, the matter will get diluted, which means that

$$\rho(t) = \rho_0 \left(\frac{a_0}{a(t)}\right)^3$$

(this is because the universe expanding by some scale factor changes the volume by the cube of that factor). Plugging this into the Friedmann equation,

$$\frac{\dot{a}}{a} = \frac{8\pi G}{3c^2}\rho_0\frac{a_0^3}{a^3},$$

and now we just have a differential equation that we can solve directly:

$$a(t) \propto t^{2/3}.$$

Example 208 (Radiation dominated universe)

In this universe, most of the contribution to ρ comes from radiation or photons.

As the universe expands, then, we get a redshift for the photons: the frequency

$$f \sim \frac{1}{a},$$

so we have both the density effect and an energy reduction. It turns out that this will yield

$$\rho = \rho_0 \left(\frac{a_0}{a(t)}\right)^4 \implies a(t) \propto t^{1/2},$$

which is a slower expansion.

Example 209 (Vacuum energy)

In this universe, ρ is a constant independent of a, so the energy does not scale with the scale factor.

Plugging this into our equations gives us an exponential function

$$a(t) \propto \exp\left[\sqrt{\frac{8\pi G\rho}{3c^2}}t\right].$$

Intuitively, if the energy does not dilute during the expansion process, it should expand much faster than the previous two examples.

Our measurements actually yield (from matter and radiatin)

$$\Omega_{MR} = \frac{\rho_{mk}}{\rho_{\text{crit}}} = 0.279 \pm 0.014,$$

and (from vacuum energy)

$$\Omega_\Lambda = 0.721 \pm 0.015.$$

So the total energy is something like

$$\Omega_{\text{total}} = 1.0052 \pm 0.0084.$$

This is pretty close to 1, so our universe is possibly flat within measurement error! And this raises other possible questions: what is this vacuum energy, actually? Quantum field theory tried calculating this value, but it gave a total energy value of 10^{120} times larger than the observed value, so we still have no real idea.

47 December 9, 2019

In the last two lectures of this class, we'll look a bit more at what we can say about the evolution of our universe. We won't look at questions like "what is the meaning of the universe" or "why is their life," but rather something less ambitious.

We've been working with the line element

$$ds^2 = -dt^2 + a^2(t)\left(\frac{d\overline{r}^2}{1-k\overline{r}^2} + \overline{r}^2 d\Omega^2\right)$$

which has the nice properties of isotropy and homogeneity that the universe seems to have. This is the Friedmann-Robertson-Walker metric: here, $k \in \{-1, 0, 1\}$ tells us about the curvature, and $\overline{r}$ is dimensionless. Again, note that just because we have **co-moving coordinates** does not mean the proper distance between objects at constant coordinate distance stays constant, because of that scale factor a:

$$\sigma(t) = a(t)\int_{\overline{r}_1}^{\overline{r}_2} \frac{d\overline{r}}{\sqrt{1-k\overline{r}^2}}.$$

Taking a derivative gives us **Hubble's law**,

$$\frac{d\sigma}{dt} = \left.\frac{\dot{a}}{a}\right|_t \sigma(t) = H(t)\sigma(t).$$

(A dot here means a derivative with respect to time.) Here, the value of H today is one of the most important quantities in cosmology! Today, we'll focus a bit on the cosmological redshift, as well as the $T^{\mu\nu}$ stress-energy tensor for our universe.

Cosmological redshift was the main way we could study the evolution of the universe before gravitational waves were discovered recently! Consider a galaxy where a photon is emitted at some time t_e, and say that this is at some coordinate $\overline{r}_e$. Meanwhile, we're at a spot $\overline{r}_o$: say the photon reaches us at some time t_o. (We're assuming the two objects are radially separated, so $d\phi = d\theta = 0$.) Then the metric element

$$ds^2 = -dt^2 + a^2(t)\frac{d\bar{r}^2}{1-k\bar{r}^2} = 0,$$

because light moves along lightlike paths. One trick that we've used is that an observer with four-velocity U^μ observing a particle with momentum p^ν has energy

$$E = -p^\mu E_\mu.$$

But we're dealing with galaxies here, which are free-falling, so their coordinates to not change:

$$U_o^\mu = U_e^\mu = (1, \vec{0}).$$

(We did something similar with Schwarzschild.) So the energy of a photon at the initial point is

$$E_e = -p^t U_e^t g_{tt} = p_e(r_e)^t,$$

the t-component of the four-momentum at the initial point, and the energy at the final point is (similarly) $E_o = p_o^t(r_o)$. (We're using a t superscript instead of a 0 to not be confused with o.) So to calculate the ratio of energies, we just have

$$\frac{E_e}{E_o} = \frac{P_e^t}{p_o^t}$$

(all of these are **component** times), and all we need to do is understand the four-momentum of a particle along the path!

Well, the photon is traveling along a geodesic, so we can just use the geodesic equation (here using $\sigma = t$)

$$\boxed{\frac{d}{d\lambda}(p_t)} = \frac{1}{2}\left(\partial_t g_{\alpha\beta}\right)\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}.$$

Remember that in this path, we're assuming θ and ϕ do not change, so the right hand side simplifies a lot: each of α and β can only be t or $\bar{r}$. And the only component of the metric that varies with time is g_{rr}, so now the right hand side is just

$$\frac{1}{2}\partial_t g_{\bar{r}\bar{r}}\left(\frac{d\bar{r}}{d\lambda}\right)^2 = \boxed{\frac{a(t)\dot{a}(t)}{1-k\bar{r}^2}(p^{\bar{r}})^2}$$

(the last term here by definition of the four-momentum for photons). So now we know how the time-component of the four-momentum p evolves, but it also depends on $p^{\bar{r}}$. But we also know that $ds^2 = 0$, so we also have the equation

$$dt^2 = \frac{a^2}{1-k\bar{r}^2}d\bar{r}^2.$$

We can divide through by $d\lambda^2$ to get an equation relating p_t^2 and $p_{\bar{r}}^2$:

$$\boxed{(p^t)^2 = \frac{a^2}{1-k\bar{r}^2}(p^{\bar{r}})^2}$$

(note that while this equation has upper indices, there's a p_t on the top equation). Plugging the second equation into the first, we find that

$$\frac{dp_t}{d\lambda} = \frac{\dot{a}}{a}(p^t)^2.$$

Luckily, our metric is diagonal, and g_{tt} is very simple, so $p_t = -p^t$. And now we have an equation for the time-component of our four-momentum:

$$\frac{\dot{a}}{a}(p^t)^2 = -\frac{dp^t}{d\lambda}.$$

Now we can use the chain rule to rewrite

$$\dot{a} = \frac{da}{d\lambda}\frac{d\lambda}{dt} = \frac{da/d\lambda}{dt/d\lambda} = \frac{da/d\lambda}{p^t},$$

because $t = x^0$! So we can plug this back in to find that

$$\frac{dp^t}{d\lambda} = -p^t \frac{da}{d\lambda}\frac{1}{a} = -p^t \frac{d}{d\lambda}(\log a).$$

Doing a few more steps to solve this differential equation yields

$$p^t(t) \propto \frac{1}{a(t)}.$$

So as the scale factor increases, the p^t decreases! And this is enough to give us the ratio of energies at emission and observation

$$\frac{E_e}{E_o} = \frac{a(t_o)}{a(t_e)}.$$

Since $t_o > t_e$ (we observe after emission), and the universe is expanding, this fraction is larger than 1, which means that we **lose energy from emission to observation.**

Fact 210

There are some emissions that we never get to see, but for different reasons than the cosmological redshift. We might discuss this later.

This means that for a photon,

$$\frac{E(t_e)}{E(t_{\text{now}})} = \frac{a(t_{\text{now}})}{a(t_e)},$$

which we defined to be $1 + z(t_e)$. z is always positive here, and it is larger for smaller values of t_e.

What do we know about properties of this photon? Since $E = hf$, the frequency of the photon has been reduced by a factor of $1 + z(t_e)$:

$$f(\text{now}) = \frac{f(z)}{1+z},$$

which is smaller than the emitted frequency. This explains the name of **redshift**! By extension, this means that the wavelength λ has been stretched (increased) by a factor of $1 + z$, and this can also be rewritten as

$$z = \frac{|\Delta\lambda|}{\lambda(z)}.$$

It's easy for us to measure frequencies, so this gives us the redshift $1+z$ of a galaxy! This is because those photons are coming from atomic transitions of hydrogen and helium, which have discrete (and universal) emission patterns. So comparing the frequencies that we expect and the frequencies that we actually measure gives us the value of z.

So looking at Andromeda or our favorite quasar on Wikipedia will often gives us the redshift factor z, rather than the actual distance.

Remark 211. *Here are some numbers. $z = 0$ means that we're here on Earth (or in our galaxy), and the edge of the observable universe has $z \approx 1000$. The farthest quasar (very big black holes) has $z \approx 8$, which is much larger of a factor than local disturbances or energy loss due to dust! What we do have to worry about, though, is that photons rarely make it to us because there is so much stuff in between.*

How is this related to the distance between our objects or other dynamical quantities? Returning to the Hubble parameter $H(t) = \frac{\dot{a}(t)}{a(t)}$, note that we can rearrange to

$$\frac{da}{a} = H(t)dt \implies a(t) = a(t^*) \exp\left[\int_{t^*}^{t} H(t)dt\right]$$

for some arbitrary time t^*. This can then be rewritten (by plugging in the above expression) as

$$1 + z(t_e) = \frac{a(\text{now})}{a(t_e)} = \exp\left[\int_{t^*}^{t} Hdt - \int_{t^*}^{t_e} Hdt\right] = \exp\left[\int_{t_e}^{t_{\text{now}}} Hdt\right].$$

Let's look at the case where the difference in time is very small (so we're reasonably close to our galaxy). Then we can write $t_e = t_{\text{now}} - \delta t$, so we can use the Taylor expansion of e^t to find that

$$\boxed{1 + z(t_e)} \approx 1 - \int_{t_o}^{t_e} H(t)dt \approx \boxed{1 + H(t_{\text{now}})|\delta t|}.$$

So it now remains to find this time difference δt: how can we measure it? Again, we rely on the fact that $ds^2 = 0$: this means that

$$dt = a(t)\frac{d\overline{r}}{1 - k\overline{r}^2} = d\sigma,$$

and therefore $\delta t = \sigma(t_{\text{now}})$ to first order, and therefore

$$z(t_e) = H(t_{\text{now}})\sigma(t_{\text{now}}).$$

That's why the redshift is important: if we can measure the frequency shift, we can find the distance to a nearby galaxy if we know the current value of H!

Remark 212. *"Where does the energy go?" Well, energy is only conserved in Minkowski spacetime! In the presence of gravity, ds^2 is nonconstant, so we shouldn't expect things to be so nice. The actual answer is that the energy has gone "into the spacetime."*

What's left for us to do, then, is to understand how the universe evolves over time. We discover something about a, the scale factor, by solving the Einstein equations: to do that, we need the value of $T^{\mu\nu}$, which appears on the right hand side of the equation.

Luckily, because the stress-energy tensor is very symmetric (because the universe is very symmetric), it turns out that we can describe the universe very well by thinking of it as a **perfect fluid** (random motion with the same energy and velocity). There's no preferred direction, pressure is the same in all directions, and so on, and this means that we just have the expression from earlier in the class

$$T^{\mu\nu} = (p + \rho)u^\mu u^\nu + pg^{\mu\nu},$$

as seen from the point of view of an observer with four-velocity u^μ. But we're the observer here, and we're comoving with the coordinates! Thus, $u^\mu = (1, \vec{0})$ as before, and $T^{\mu\nu}$ becomes simpler: it's described by a pressure p and an energy density ρ.

So this is the form we'll be using for the rest of the class. We'll consider simple fluids with the equation of state

$$p = w\rho$$

for some constant of proportionality w. For the three types of species we'll be considering, w will have the following values:

- Dust (that is, pressureless objects) with $w = 0$: this describes galaxies well.

- Radiation: $w = \frac{1}{3}$, which describes light traveling around the universe.
- **Dark energy**: $w = -1$ (so the equivalent of tension in a rubber band), which describes something related to the **cosmological constant**.

There are many more exotic things out there, but for our purposes, we'll consider these three. Looking back to our original point, if we have a simple relationship between pressure and density, we can calculate $T^{\mu\nu}$ in terms of just ρ, and we can plug that into our Einstein's equation to find a!

So the total stress-energy tensor for our purposes is

$$T^{\mu\nu} = T^{\mu\nu}_{\text{Dust}} + T^{\mu\nu}_{\text{Light}} + T^{\mu\nu}_{\text{DE}},$$

which all depend on the $\rho(t)$ for that particular species. Einstein's equation relates the second derivative of the metric to this stress-energy tensor, and it turns out we only need the tt-component for these equations: we find that

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2}$$

(here ρ means the sum of the ρs). This is called the first Friedmann equation: if we know how much stuff there is in the universe, we can find out the behavior of a! And the k term depends on curvature: not only does the stuff in the universe affect its evolution, but so does the way the spacetime is curved.

48 December 10, 2019 (Recitation)

(Today's material is not examinable.)

Let's start with a recap of this class. We started with a universal speed limit c for light, derived from the Michelson-Morley experiment. With this one fact, we found that a lot of quantities – length, time, and so on – depend on our frame of reference! However, even if we go from one frame to another, physics is still immutable, and causality remains intact. So we've found a way (using index notations, tensors, and so on) to go back and forth between reference frames, so that the core physics stays the same.

From there, we considered accelerating observers, and we found that we no longer see flat spacetime. This led to the principle of equivalence (associating curved space with gravity), and this gives us a beautiful structure for things like the beginning and end of spacetime!

Today, we'll talk a bit about where we go from here: this is mostly by popular demand. We'll talk about quantum gravity and discuss some current research in gravitational physics!

Example 213

Consider a thought experiment. The stress-energy tensor determines our spacetime by looking at objects large enough to affect our curvature, while quantum mechanics deals with objects at small scales. Both describe our universe with remarkable accuracy (they hold up to experiments). Now let's say we are falling into a black hole, so spacetime is contracting in such a way so that there is a lot of spacetime curvature on smaller length scales. So eventually we hit the scale where quantum mechanics is important: how can we put these two theories together?

We can quantize things well with electromagnetism (quantum electrodynamics) and the strong nuclear force, but there was a problem when we tried to quantize gravity (the theory says that we can never determine the fundamental constant in a finite number of experiments). So the normal method of quantizing theories does not work for quantize gravity! (This is the **nonrenormalizability** problem.)

Why exactly are we running into a problem here? In physics, we're often given a problem with initial data, and we need to figure out what happens at some future time (it's a predictive science). But quantum gravity asks us to predict the **existence of time** itself, so we need to do physics differently to do that!

Professor Engelhardt's favorite way to do this is to consider the problem indirectly: what must a consistent theory of gravity do? It has to look like quantum mechanics and general relativity in the respective realms, so we'll talk about three general programs that take this approach.

Fact 214 (Thermodynamics of a black hole)
After a black hole reaches equilibrium and nothing's falling into it anymore, we can completely describe a black hole by its mass and angular momentum once we wait long enough.

In other words, general relativity says that there's only one consistent behavior for how the black hole behaves given those two quantities! In particular, the **entropy** S of a black hole, which is a measure of our ignorance of the system at the microscopic level (higher entropy means more uncertainty about the individual molecules), is zero. (We know exactly what the black hole looks like, as long as we know its mass and its angular momentum.)

But now take an object with positive entropy (like a hot cup of tea), and throw that into a black hole. After a while, the black hole's entropy goes to 0 again – it's a slightly different black hole because its mass might be different. But that means we've decreased the entropy of the universe, which is not allowed by the second law of thermodynamics!

The idea is that the **microscopic behavior of the black hole is not governed by general relativity**, but rather by quantum gravity. This is Bekenstein's equation:

$$S_{\text{BH}} = \frac{\text{Area[event horizon]}}{4\ell_p^2}.$$

This increases over time by a theorem by Hawking, but for 50 years people couldn't really tell what the entropy was measuring. Well, professor Engelhardt started thinking about this as a grad student in 2015! First of all, how do we calculate the area of this event horizon? We know that some light rays reach $t = \infty$, and other light rays reach $r = 0$ (the singularity): the event horizon is then the last light ray that reaches $t = \infty$. But this takes an infinite amount of time to figure out. Also, if the universe has a Big Crunch, then nothing makes it to $t = \infty$: does the event horizon even make sense there?

Basically, it seems that S is only well-defined at a very far away future time. So maybe there's a better thing that we can do in a short amount of time: **gravitational lensing!** Say that we have a sphere, and we can fire light rays from that sphere. In some regions (like far away from a black hole), the light rays will make the light cone cross-sectional area increase, and in other regions (like inside), the light rays will all curve inward. Well, pick the sphere where the cross-sectional area is constant: this is called the **apparent horizon**.

It turns out the apparent horizon also satisfies a monotonicity theorem like Hawking's, but it's still not quite clear whether this is an entropy. Well, it turns out that Bekenstein's equation doesn't hold for certain models in string theory, and recently, professor Engelhardt found a way to describe what this entropy actually means: it's the number of **possible wormholes inside the black hole, consistent with certain observations**. The next question is then how to relate this to cosmology as well!

Fact 215 (Cosmic censorship)
Here are some important things to know:

- Generic solutions to the Einstein equation have singularities.
- Our universe looks like a generic solution to the Einstein equation.
- We have not actually observed such a singularity in our universe.

Roger Penrose suggested the following resolution: perhaps "nature abhors an uncloaked singularity," so it always has to be behind an event horizon. Except that last year, there was a counterexample: it involves an electromagnetic field with an ordinary scalar field with a numerical evolution. It was shown that this is generic, so any small perturbation also gives a singularity not behind an event horizon. So our universe needs to be special if it doesn't satisfy this constraint.

Well, one way to get around this problem is that Penrose's statement holds in quantum gravity, because there are some quantum dynamics in string theory that enforce this cosmic censor. And we can probably read information about this topic on Wikipedia or in some journals at this point!

Fact 216 (Black hole information paradox)
We know that black holes satisfy some entropy relation: note here that ℓ_p is really, really small, so any black hole has a huge amount of entropy (because of the different wormholes that could be hidden behind the event horizon). It turns out the temperature of the black hole satisfies

$$T_{\text{BH}} = \frac{\hbar}{8\pi GM} \sim \frac{1}{M}.$$

For example, the temperature of the black hole at the center of the Milky Way is about 10^{-14} Kelvin. (This is very different from the temperature of the surroundings.)

But if we put a black hole that has mass less than the moon, so it is hotter than 3 Kelvin (the ambient temperature), it should radiate heat. But this means the black hole should lose mass, so the temperature should get hotter! So the more it radiates, the hotter it gets.

Well, what does quantum mechanics tell us? Suppose we have two **entangled** particles near a black hole, which means they are correlated in some way. This means that we can store some kind of information in that entanglement: we can store quantum information here, such as how the two particles are spinning relative to each other. Also, it's a fact that quantum mechanics conserves quantum information (for all time).

But now let's add general relativity into the mix: say one of these two particles is behind the event horizon, but the other isn't. Say that particle goes into the singularity, and then the black hole evaporates: now information seems to have been lost, and entanglement is broken! How does quantum gravity deal with this?

Well, the black hole can evaporate, while the information is not destroyed. So the biggest open question right now in quantum gravity is **how the quantum information gets out**! And right now, it looks like the answer has something to do with the wormholes.

49 December 11, 2019

We'll start by discussing the feedback after pset 8. Overall, it sounds like people like the class, and the recitations are very useful (particularly professor Engelhardt's). One thing that we should realize: the professors do their best to make everyone happy, but sometimes this is impossible. For example, 3 or 4 people said the class is too fast, and another 3 or 4 said it is too slow.

One thing that was useful, though, is that a few people (3 to 5) mentioned that some of the pset problems had a ratio of math to physics that was too high. So discussions have been started about how to phase out those problems!

In the last day of class, we'll focus mostly on qualitative understanding of the beginning of the universe, as well as some open problems in physics right now. Recently, we've been discussing the scale factor $a(t)$, which governs the scale of the universe, and we've been modeling the universe generally as a perfect fluid with a simple equation of state $p_s(t) = w_s \rho_s(t)$ for a given species s (radiation, matter, dark energy). Here, $w = 0$ (and therefore $p = 0$) corresponds to dust, $w = \frac{1}{3}$ corresponds to radiation, and $w = -1$ corresponds to dark energy. In each of these cases, we get the differential **Friedmann equation** (from the t-component of the Einstein equation)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho(t) - \frac{k}{a^2},$$

where k dictates the curvature of spacetime and $\rho = \rho_m + \rho_\gamma + \rho_\Lambda$ is the sum of the ρs from matter, radiation, and dark energy. Here, the left hand side is often written as just $H(t)^2$, where H is the **Hubble parameter**.

Another equation which is relevant and important, because it tells us how these densities ρ evolve, comes from the conservation of $T^{\mu\nu}$ (the **covariant derivative** is 0). Taking the time-component of this conservation equation gives us the **first law of cosmological thermodynamics**:

$$\frac{d}{dt}\left(\rho a^3\right) = -p\frac{da^3}{dt}.$$

(This may remind us of the equation $\frac{dE}{dt} = -pdV$ from thermodynamics.) What's great is that we now have a relation between the energy density and the scale factor! And for any given species, we can write p in terms of ρ:

$$\frac{d}{dt}\left(\rho a^3\right) = -w\rho\frac{da^3}{dt}.$$

We can solve this explicitly: first of all, by the product rule,

$$a^3 + \frac{d\rho}{dt}3\rho a^2\frac{da}{dt} = -3w\rho a^2\frac{da}{dt}.$$

Rearranging and simplifying yields

$$\frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{a}}{a} \implies \rho(t) = \rho^*\left(\frac{a(t)}{a^*}\right)^{-3(w+1)},$$

where ρ^*, a^* denote the values of ρ, a at some arbitrary time (initial conditions). If we further assume that the individual species don't interact very much, we can solve this equation for one species at a time and get solutions for each relevant w!

- For dust, $w = 0$, so $\rho_m(t) \propto a(t)^{-3}$. This tells us that the energy density of matter decreases with the cube of the scale factor, and that makes sense: If we have the same amount of matter that is being stretched out over twice as much volume, the density should be halved, which is exactly what the equation tells us.
- For radiation, $w = \frac{1}{3}$, so $\rho_\gamma(t) \propto a(t)^{-4}$ (it goes down faster with a than matter does). This extra factor comes from redshift: not only is the density of photons decreasing, just like for matter, but the energy of each photon goes like $\frac{1}{a}$.
- For dark energy, $w = -1$, so $\rho_\Lambda(t)$ is constant for all time.

Before we analyze these a little more, though, here's another note: we can combine the cosmological thermodynamic law and the Friedmann equation to get the **second Friedmann equation**

$$\left(\frac{\ddot{a}}{a}\right) = -\frac{4}{3}\pi G(\rho + 3p).$$

This is primarily useful in justifying why we need this whole concept of dark energy! If $\rho + 3p$ were positive, then $\ddot{a}$ would be negative, which means that the universe must be decelerating. But if we look out into the universe today, measurements tell us that the universe is actually accelerating with $\ddot{a} > 0$. So we need some way of making $\rho + 3p < 0$, which only occurs for $w < -\frac{1}{3}$: it can't be radiation (positive w) or dust ($w = 0$).

Remark 217. *Measuring the actual acceleration is pretty difficult, so we should take a class on this if we're curious! The short answer is that we can look at the photons from the cosmic microwave background.*

So if we return to our equation $H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\gamma + \rho_\Lambda) - \frac{K}{a^2}$, we can define a few auxiliary variables to make this more compact: our goal is to say something about k, the curvature of spacetime.

Definition 218

Define the quantities

$$\rho_H = \frac{3H^2}{8\pi G}, \rho_k = -\frac{3k}{8\pi G a^2}.$$

Then the Friedmann equation can be written as

$$\rho_H = \rho_m + \rho_\gamma + \rho_\Lambda + \rho_k \implies 1 = \frac{\rho_m}{\rho_H} + \frac{\rho_\gamma}{\rho_H} + \frac{\rho_\Lambda}{\rho_H} + \frac{\rho_k}{\rho_H} = \Omega_m + \Omega_\gamma + \Omega_\Lambda + \Omega_k.$$

ρ_H is known as the **critical energy density:** after all, if the conbtributions from matter, radiation, and dark energy add up 1, then $\Omega_k = \frac{\rho_k}{\rho_H} = 0 \implies k = 0$. Here, we define $\Omega_A = \frac{\rho_A}{\rho_H}$ to be the "fraction" of contribution from non-curvature.

Well, what is the value of these values? Experiments today say that about 72 percent of the universe is dark energy, about 23 percent is **dark matter**, and the last 5 percent or so is the things we already know of, like baryos, firmeoons, and so on.

Remark 219. *Dark matter is called that because we can't see it. But we still don't know what it (or dark energy) is! (It also has $w = 0$.) But there is other evidence for the existence of dark matter, such as the gravitational pull of certain galaxies extending to a larger radius than the matter that we can see with our telescope.*

So with that, let's go back in time and think about how the universe must have looked near its beginning! To do that, we'll try to find energy densities of various species: we have a simple form for ρ_m, ρ_γ, and ρ_Λ. What if we take our solutions for ρ and look at different regimes for $a(t)$? For a specific species s, we can plug in our explicit functional form for ρ and just solve the differential equation from there:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho_s^*\left(\frac{a(t)}{a_s^*}\right)\right)^{-3(w_s+1)}.$$

If $w = 0$, this tells us that $a(t) \propto t^{2/3}$, and therefore $\ddot{a} \propto -t^{-4/3}$ (so this is deceleration). So in a regime where dust is the dominant contribution, we get that particular scale factor! Similarly, $w = \frac{1}{3}$ yields $\alpha(t) \propto \sqrt{t}$, and $\ddot{a} \propto -t^{3/2}$. But $w = -1$ is completely different: we now have $a(t) \propto e^{Ht}$! So dark energy must be the dominant factor in the evolution of the universe right now.

As a gets smaller and smaller, the energy density of ρ_m and ρ_γ get larger: at some point, because p_Λ stays constant, the energy densities will increase above that one of the constant four-momentum from dark energy. So we can think of the history of the universe as divided into three parts: those where radiation, matter, and dark matter are the most important sources.

Remark 220. *However, we cannot measure anything before 400000 years after the Big Bang, because everything was too dense and ionized: photons would bounce back and forth without significantly moving. But radiation stopped being the primary factor around 90000 years after the universe started, so we have no observational records of it. Instead, we have the cosmic microwave background, which is the first time the universe was transparent to us.*

Observations do actually predict that the universe is flat, so $k = 0$ today. But that's a problem: it's very hard to make the universe be flat! This is known as the **flatness problem**: if we start from the Friedmann equation and assume that there is no dark energy:

$$H^2 = \frac{8\pi G}{3}(\rho_m + \rho_\gamma) - \frac{k}{a^2},$$

then multiplying through by a^2 yields

$$a^2 \left(H^2 - \frac{8\pi G}{3}(\rho_m + \rho_\gamma) \right) = -k.$$

Rearranging a bit yields

$$a^2 \rho_c(t) - a^2(\rho_m + \rho_\gamma) = -\frac{3k}{8\pi G}.$$

where ρ_c is the critical energy density. We can then write this as

$$a^2(\rho_m + \rho_\gamma) \left[1 - \frac{\rho_c}{\rho_m + \rho_\gamma} \right] = \frac{3k}{8\pi G}.$$

But the right hand side is a constant, so if one term on the left side increases, the other should decrease. As the universe expands, $a^2(\rho_m + \rho_\gamma)$ gets smaller and smaller (because $\rho_m \propto a^{-3}$ and $\rho_\gamma \propto a^{-4}$). But that means that as time goes on, the second term on the left side (which is basically $1 - \Omega^{-1}$) must increase: this means that Ω_k must get more and more different from 0. So we must be at some very special initial conditions, which is a bit suspicious!

Well, the idea is to look very early on in the universe, when the behavior was dominated by dark energy (which may or may not be the same dark energy that we see now). If we're in a period of the universe where dark energy is the dominant factor, $a(t)$ grows exponentially. And now during this period of **inflation**, the first term $a^2\rho$ grows exponentially, the second term must decrease exponentially: this means Ω^{-1} must become very close to 1 no matter what. And that explains one of the puzzles about how our universe could be flat, no matter what the initial conditions were!

To conclude, here's one more situation where we must have an issue unless inflation actually occurred near the beginning of the universe. Let's use coordinates so that light cones travel at 45 degrees in our spacetime diagram: if we're at some time t right now, we can only access information that originated from our past light-cone, not the whole universe.

But it doesn't make sense that the cosmic microwave background we see today is so uniform from 400000 years: if we see two photons that look identical to us right now, they must have reached thermodynamic equilibrium by communicating with each other. But how is that possible if they are separated by a huge distance (and only had 400000 years after the Big Bang)? This is the **horizon problem**, and this is also fixed by inflation: if the scale factor a blew up exponentially at the beginning, it changes the behavior of spacetime so that everything grows from a single point. So it is indeed possible for two faraway photons from the CMB to "communicate."