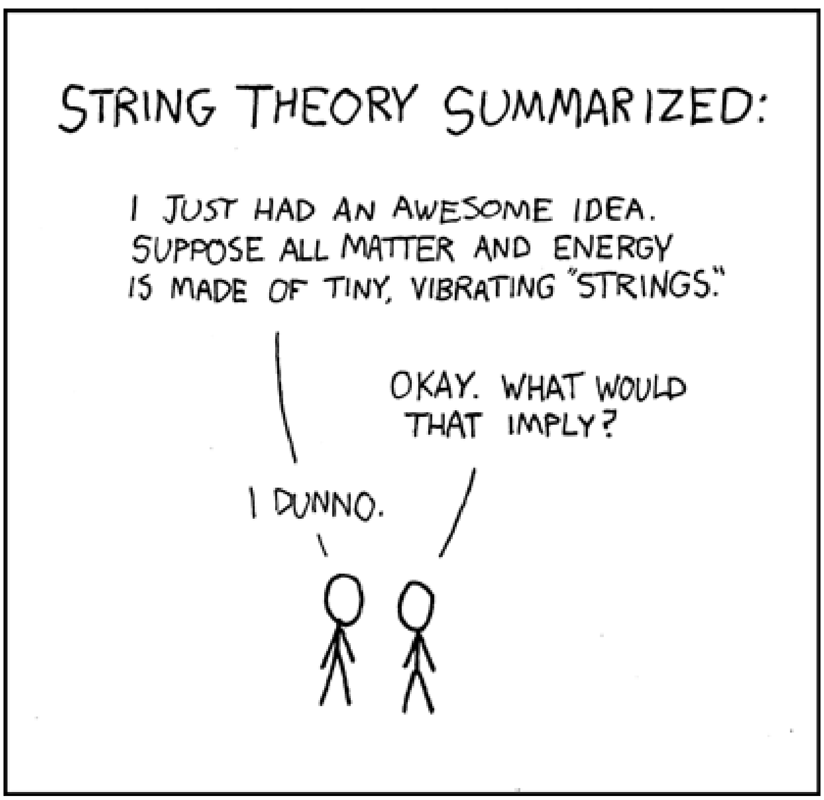
MIT Physics 8.251: String Theory



8.251: String Theory for Undergraduates

Lecturer: Professor Hong Liu

Notes by: Andrew Lin

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Introduction

It's good for all of us to have our videos on, so that we can all recognize each other in person and so that the pace can be adjusted based on face feedback. In terms of logistics, we should be able to find three documents on Canvas (organization, course outline, and pset 1).

Sam Alipour-fard will be our graduate TA, doing recitations and problem set grading.

- There are no official recitations for this class, but Sam has offered to hold a weekly Friday 11am recitation.
- Office hours are Thursday 5-6 (Professor Liu) and Friday 12-1 (Sam).

If any of those times don't work for us, we should email both of the course staff for a separate meeting time.

Professor Zwiebach's "A First Course in String Theory" is the main textbook for this course, which we'll loosely follow (and have assigned readings and problems taken from). We won't have any tests or exams for this class – grading is based on problem sets (75%) and a final project (25%) where we read Chapter 23 and do calculations and problems on our own. The idea is that we'll learn some material on our own so that we can understand the material deeply! (Problem sets will be due at midnight on Fridays, with 50% credit for late submissions within 2 weeks.)

Fact 1

The "outline" document gives us a roadmap of what we'll be doing in this class.

In particular, we'll be trying to answer the following four questions:

- Why do quantum strings demand a particular spacetime dimension?
- Why is string theory a theory of quantum gravity?
- Why does string theory have the potential to unify all fundamental interactions?
- How do we derive the AdS/CFT duality from string theory?

The final topic here will require a bit of general relativity, but for those of us who have not studied it, we shouldn't be too concerned.

Finally, lectures will focus on main messages and concepts rather than mathematical calculations (unless those calculations are very important, we'll be told to read them ourselves). Questions are particularly important in these times, because we need to make sure not to get lost.

February 17, 2021

We'll start with the basic ideas before diving into the main questions. Before strings, we had **field theory**: it was believed that the building blocks of the universe are "fundamental particles" like electrons, photons, gravitons, and quarks (where "fundamental" means that they have no internal structure and can be described as zero-dimensional points). In principle, these different particles are completely independent – photons have nothing to do with gravitons.

But in string theory, the fundamental objects are instead (one-dimensional) **strings** (which can be open or closed. We can still have different particles, like photons and gravitons, but they arise from **different oscillation modes of strings**. So different particles are different patterns for the same fundamental string, and this gives us a **unified approach for all particles and interactions**. (If photons are just a certain oscillatory pattern, and photons propagate through electromagnetic interactions, then E&M is then also part of string theory.) One of the biggest projects in physics is to unify quantum mechanics and gravity, and that's one thing string theory aims to achieve.

Fact 2

Our understanding of string theory is still incomplete, similar to the story of the blind man and the elephant. We see different aspects from different angles, but the global picture is still lacking today because the field is very young.

There are a variety of other physical contexts in which strings arise, including the following:

0. Strings on a violin (not a very serious example),
1. **Flux tubes** that connect quarks in a proton,
2. **Vortex strings** in superfluids,
3. **Cosmic strings** in cosmology.

But the strings that arise in these physical situations are all different from the **fundamental string** that will be used to obtain quantum gravity: the four examples above are all string-like objects with **internal structure** and internal physics, so they are **composite strings**. The point is that the fundamental string is a one-dimensional object with no internal structure – it can only vibrate.

Fact 3

String theory has a reputation for being very mathematical and difficult to understand, but the motto we should keep in mind is that **nothing is difficult if you understand it in the right way.**

If we find certain concepts unintuitive, we should find another angle or perspective to use instead. In particular, we can ask ourselves two questions:

- Do we understand the basic concept correctly? (Usually, confusions stem from misunderstandings in these fundamental ideas.)
- If we're confused about formalism, what are the questions that the formalism is trying to address?

After all, theories in physics, including string theory, were initially invented to address concrete physical questions, even if there is lots of mathematical formalism. So understanding those questions also goes a long way towards resolving our difficulties (for example, why we're using certain mathematical tricks).

With that, we're ready to jump into the first chapter of the class, and we'll start by reviewing **special relativity** (partially to set up notation for relativistic systems).

1: Basics

1.1: Special relativity

Proposition 4 (Important elements of SR)

- A **reference frame** is a system of coordinates that we use to describe or label physical processes (events in spacetime). We often use notation like $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$.
- An **inertial frame** is one where a freely moving body moves with constant velocity. Two inertial frames always move uniformly relative to each other.
- The same event A may have different coordinates $x^\mu, x^{\mu'}$ in two frames S, S' (for both time and space).
- (**Principle of relativity**) Physical laws, as well as the speed of light, are the same in all inertial frames (so we can't tell inertial frames apart by doing experiments).

Suppose we have two events A and B, labeled with coordinates x^μ, y^μ. Then the **invariant interval**

$$s_{AB} = -(x^0 - y^0)^2 + \sum_{i=1}^{3}(x^i - y^i)^2$$

is the same in all inertial frames. (So if we describe this system in another frame S' with coordinates $x^{\mu'}, y^{\mu'}$, then $s'_{AB} = s_{AB}$.)

Since s_{AB} is agreed upon by all inertial frames, it can be a meaningful physical quantity, and in particular we say that the events A, B are **spacelike separated** if $s_{AB} > 0$, **timelike separated** if $s_{AB} < 0$, and **null-like separated** if $s_{AB} = 0$.)

If we consider two infinitesimally separated events $A : x^\mu$ and $B : x^\mu + dx^\mu$, then we have

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{3}(dx^i)^2,$$

and we write this in the shorthand notation

$$= \eta_{\mu\nu} dx^\mu dx^\nu,$$

where repeated indices in a product are summed (Einstein convention) from 0 to 3, and

$$\eta_{\mu\nu} = \begin{bmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}.$$

When $ds^2 > 0$, we write it as $d\ell^2$ for the (infinitesimal) **proper distance** $d\ell$, and when $ds^2 < 0$, we write it as $-c^2 d\tau^2$ for the **proper time** $d\tau > 0$. The point of all of these infinitesimal elements is that we can now assign a proper distance or proper time for any arbitrary curve in spacetime. If we have a particle which moves in spacetime from point A to point B along some curve C, it might take some arbitrary path (depending on the forces acting on that particle), but regardless of that path, we can find a proper time for that path. To do this, we divide this curve into many small intervals and applying the $ds^2 = -(cdt)^2 + d\vec{x}^2$ definition to each one to find its proper length. We then have

$$ds^2 = -c^2dt^2\left(1 - \frac{d\vec{x}^2}{dt^2}\right) = -c^2dt^2\left(1 - \frac{\vec{v}^2}{c^2}\right) = -c^2d\tau^2$$

(this quantity is indeed negative because $v < c$) and thus we've found the proper-time $d\tau$, and we get our final answer by integrating $\tau_{AB} = \int_C d\tau$ from A to B. Similarly, if we have two events that are spacelike separated along a curve C', we can find the path's proper length by integrating $\ell = \int_{C'} d\ell$, where $d\ell = \sqrt{ds^2}$.

We'll next turn to **Lorentz transformations**, which are ways of getting between coordinates $x^\mu, x^{\mu'}$ in different frames. We need to make sure that these transformations leave ds^2 invariant: we can write down the new coordinates

$$x'^\mu = L^\mu{}_\nu x^\nu,$$

where x^ν, x'^μ can be thought of as column vectors and L as a matrix. If we substitute in our new coordinates into the definition of $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, it turns out that the relation that L must satisfy is

$$L^T\eta L = \eta.$$

This is a matrix identity, and we can fully classify Lorentz transformations by studying it. For example, if a frame S' moves with velocity v in the x^1 direction relative to the frame S, then we can show the corresponding Lorentz transformation is

$$L = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

where we use the standard notation

$$\beta = \frac{v}{c}, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

The reason for switching between frames is that certain frames might be more convenient for doing mathematical calculations than others depending on the particular physics question that we're trying to solve (even if the physical description is always the same).

We've been using the x^μ notation a fair bit, and this is an example of a **four-vector**. A general four-vector a^μ has four indices $\mu = 0, 1, 2, 3$, and it also transforms in the same way as we saw above under Lorentz transformations:

$$a'^\mu = L^\mu{}_\nu a^\nu,$$

We can also introduce the inner product between two four-vectors

$$a \cdot b = \eta_{\mu\nu}a^\mu b^\nu = a_\mu b^\mu,$$

where η can be used to lower the indices on the four-vectors: we define

$$a_\mu = \eta_{\mu\nu}a^\nu.$$

(The inner product is important because this quantity is the same in all frames.) Another important four-vector is the momentum four-vector

$$p^\mu = m\frac{dx^\mu}{d\tau} = (p^0, p^1, p^2, p^3),$$

which is also a four-vector because τ is an invariant quantity. We define the **energy** via $p^0 = \frac{E}{c}$, and for a freely moving particle of mass m, we have

$$p^2 = p \cdot p = \eta_{\mu\nu} p^\mu p^\nu = -\frac{E^2}{c^2} + \vec{p}^2 = m^2 c^2.$$

(This is often called the **mass-shell condition**, and we'll see it again in the future – it's often used to identify the mass of a particle.)

In this class, we'll be using a special set of coordinates called **light-cone coordinates**, which will play a key role:

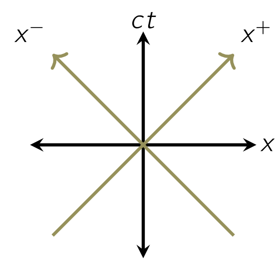
Definition 5

Pick a spatial direction – we'll use x^1 – and define

$$x^+ = \frac{1}{\sqrt{2}}(x^0 + x^1), \quad x^- = \frac{1}{\sqrt{2}}(x^0 - x^1).$$

We then use $x^\mu = (x^+, x^-, x^2, x^3)$ as our set of **light-cone coordinates**.

Basically, the x^+ and x^- axes are the "diagonal lines" in the following picture:



If we have a line of constant x^+, that means that a light ray is traveling in the negative x^1-direction, and similarly, if we have a line of constant x^-, that means a light ray travels in the positive x^1-direction. (In these two cases, we can use x^- or x^+ to parameterize "time," respectively.) So heuristically, this is the "time that we experience as we traveling along a light-ray," and conventionally we treat x^+ as "light-cone time" and x^- as a "spatial coordinate," even though both are really null coordinates. That means that we have a frame of observers in which all observers move to the right in the x^1 direction.

In these new coordinates, we have

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 = -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2,$$

meaning that we can write the metric as

$$\eta = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The off-diagonal structure may seem annoying, but using it cleverly will actually make certain problems much easier to simplify! And the inner product in this new set of coordinates is

$$a \cdot b = \eta_{\mu\nu} a^\mu b^\nu = a_\nu b^\nu = -a^+ b^- - a^- b^+ + a^2 b^2 + a^3 b^3,$$

with raising and lowering now given by $a_+ = -a^-$, $a_- = -a^+$. Additionally, we find that if we define $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1)$,

$$p^2 = -2p^+ p^- + \vec{p}_\perp^2,$$

where $\vec{p}_\perp^2 = (p^2)^2 + (p^3)^2$.

February 22, 2021

We'll start with a quick recap of last lecture: last time, we talked through the key elements of special relativity, and the fundamental structure in SR is the Minkowski metric

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu.$$

We also talked about various 4-vectors, like the position four-vector (and its corresponding one-form)

$$x^\mu = (ct, x^1, x^2, x^3), \quad x_\mu = \eta_{\mu\nu} x^\nu = (-ct, x^1, x^2, x^3).$$

We also mentioned that a change of coordinates we'll often use in this class is to use light-cone coordinates

$$x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$$

instead of x^0, x^1, and we'll be using x^+ as "light-cone time" and x^- as "light-cone spatial" (even though both are really null coordinates). Remembering that x^+-axis direction corresponds to observers moving in the $+x$ direction at the speed of light, we can imagine that light-cone coordinates essentially correspond to those observers' point of view.

In these new coordinates, our metric becomes

$$ds^2 = -2dx^+ dx^- + (dx^2)^2 + (dx^3)^2 \eta_{\mu\nu} dx^\mu dx^\nu,$$

where we now have $x^\mu = (x^+, x^-, x^2, x^3)$ and

$$\eta_{\mu\nu} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We'll see many benefits of these coordinates over time: for example, the off-diagonal structure in fact turns out to be very helpful, and the Lorentz boosts in the x^1-direction **do not mix** $x^\pm$: if S' moves at velocity v in the x^1 direction relative to S, and we take $\beta = \frac{v}{c} \in [-1, 1]$, it makes sense to write $\beta = \tanh \lambda$ (where $\lambda \in (-\infty, \infty)$ is called the **rapidity**). The reason this is nice is that we actually have

$$x'^+ = e^\lambda x^+, \quad x'^- = e^{-\lambda} x^-,$$

and thus we're just multiplying by a scalar factor in each component: we've diagonalized the boosts along the x^1-direction. The dot-product in these new coordinates is now

$$a \cdot b = -a^+ b^- - a^- b^+ + a^2 b^2 + a^3 b^3,$$

and lowering our indices gives us $a_- = -a^+$ and $a_+ = -b^-$.

At the end of last lecture, we talked about the momentum four-vector

$$p^\mu = (p^+, p^-, p^2, p^3) = (p^+, p^-, \vec{p}_\perp),$$

where $p^\pm = \frac{1}{\sqrt{2}}(p^0 \pm p^1)$, $p^2 = -2p^+ p^- + \vec{p}_\perp^2$, and $p \cdot x = -p^+ x^- - p^- x^+ + p^2 x^2 + p^3 x^3$. But we can understand more physically what p^+ and p^- actually mean now: recall that in quantum mechanics, a free particle has wavefunction

$$e^{i/\hbar(-Et+\vec{p}\cdot\vec{x})} = e^{-i/\hbar p\cdot x}.$$

If we interpret x^+ as "time," then the conjugate variable to the time (that is, what term is being multiplied against x^+ in the expression $p \cdot x$) is p^-. So p^- will be interpreted as the **light-cone energy**, and p^+ corresponds to the momentum in the x^- direction, so we call it the **light-cone momentum**. (Then the other components $\vec{p}_\perp$ are often known as the **tranverse momentum**.)

And also recall that we have the condition $p^2 = -m^2c^2$, which means that

$$-2p_+ + p_- + \vec{p}_\perp^2 = -m^2c^2 \implies \boxed{p^- = \frac{m^2c^2}{2p^+} + \frac{\vec{p}_\perp^2}{2p^+}}.$$

A particularly interesting feature here is in the second term: it looks like we have a non-relativistic energy term with momentum $\vec{p}_\perp$ and mass p^+. This is a feature of the light-cone frame: there will be an energy that looks like a non-relativistic kinetic energy term.

Remark 6. *Since p^μ is a four-vector, we can also extract components like $p_+ = -p^-$ and $p_- = -p^+$ by using the usual $p_\mu = \eta_{\mu\nu}p^\nu$. (This is pretty annoying to say and think about, but we'll get used to it.)*

1.2: Compact extra dimensions

From our current understanding, we live in a $(3+1)$-dimensional world, but this does not exclude the possibility that **at short (microscopic) distances, the world can actually have more dimensions**. Let's consider a toy model:

Example 7

Suppose we have a $(2+1)$-dimensional world, where we have spatial manifold given by (the surface of) a cylinder. In other words, we have an x^1 coordinate which takes values in $(-\infty, +\infty)$, but we have an x^2 coordinate which only ranges from $[0, 2\pi R]$, with 0 and $2\pi R$ **identified** with each other.

If we want to be able to see the x^2-direction (in experiments and in general), we need to have a probe (ruler) whose size is smaller than the length $L = 2\pi R$. (This is vaguely like how we can't see bacteria with the naked eye.) And to understand how a probe like this might work, we can use the uncertainty principle: because the domain of x^2 is a compact space, we know that the uncertainty in the coordinate is on the order $\Delta x^2 \sim 2\pi R = L$, and thus uncertainty tells us that

$$E \sim cp \sim \frac{\hbar}{L}c.$$

So we need to have an energy at least on the order of $\frac{\hbar}{L}c$, but the highest energy we can achieve on earth is about 14 TeV. So the minimum length scale we can probe right now is about

$$L_{\text{min}} = \frac{\hbar c}{E_{\text{max}}} \sim 10^{-18} \text{ cm}.$$

So if $L \ll L_{\text{min}}$, we cannot use our current tools to observe this x^2 dimension. And as we'll see, extra dimensions are indeed necessary for string theory.

In the above situation, we can describe the spacetime surface as

$$\text{cylinder} = R^{1,1} \times S_1$$

where S_1 is a circle. And the idea is that our world can be described as

$$\text{world} = R^{1,3} \times M,$$

where M could (logically) be some compact manifold with size smaller than L_{min}, like S_1 or $S_1 \times S_1$ (which is also sometimes called the **torus**), or S_n (an n-dimensional sphere), or something more exotic. And string theory generally predicts (**and selects**) certain candidates for this compact manifold, but there are many possibilities in general.

Fact 8

A common procedure that we use to build a compact space from an uncompact one (like $\mathbb{R}^n$) is to use **identification**.

Example 9

To explain what's going on here, consider how we get from the real line $\mathbb{R}$ to the circle S_1.

Even though x can take on values in $(-\infty, \infty)$, what we can do is to make the identification

$$x \sim x + 2\pi n R$$

for all integers $n \in \mathbb{Z}$. So any two points that are a multiple of $2\pi R$ are treated as the same point (we can also think of this as having an "equivalence class" of such points), and what happens then is that **any point in the real line is identified with a point in** $[0, 2\pi R]$, and in additional 0 and $2\pi R$ are "glued together."

Example 10

Next, consider the identification

$$x \sim x + 2\pi R_1, \quad y \sim y + 2\pi R_2$$

for all $x, y \in \mathbb{R}$.

This time, any point can be brought inside the **fundamental domain** $x \in [0, 2\pi R_1]$, $y \in [0, 2\pi R_2]$, and additionally, the top and bottom edges are identified and so are the left and right edges (and we can imagine folding this up to get to a torus). In general, the **fundamental domain** is a subset of the original space which contains one point from each identified class.

1.3: Electromagnetism in various dimensions

First, we'll start by recalling that the dynamical variables in E&M are the electric field $\vec{E}$ and the magnetic field $\vec{B}$, and inputs into the system include the charge density ρ and current density $\vec{J}$. The dynamical equations are then **Maxwell's equations** (which we won't write down yet because we'll have a relativistic formulation soon).

Often, we solve Maxwell's equations by introducing the scalar and vector potentials $\phi, \vec{A}$ so that

$$\vec{E} = -\nabla\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A}.$$

The variables ϕ and $\vec{A}$ are actually redundant here, in the sense we can have different sets of ϕ and $\vec{A}$ which give us the same $\vec{E}$ and $\vec{B}$. In fact, we can set

$$\vec{A}' = \vec{A} + \nabla\lambda, \quad \phi' = \phi - \frac{1}{c}\frac{\partial\lambda}{\partial t}$$

for any function $\lambda(t, \vec{x})$ – this is called a **gauge transformation**, and because Maxwell's equations are invariant under these transformations, we say that they have **gauge symmetry**. And these gauge transformations are an example of **local transformations**, in that the transformation parameters can be different at different places in time (because λ is an **arbitrary** function of t and $\vec{x}$).

Remark 11. *It turns out that all fundamental interactions of nature are all defined using theories with local symmetries, and E&M is just one of them – there's a fundamental concept going on here!*

Remark 12. *When we do E&M classically, the definition of $(\phi, \vec{A})$ are mostly a mathematical device to simplify Maxwell's equations. But in quantum mechanics, these potentials are actually fundamental variables (even more so than $\vec{E}$ and $\vec{B}$).*

And now we're ready for the relativistic formulation: the point of writing ϕ and $\vec{A}$ together is to put them into a four-vector

$$A^\mu = (\phi, \vec{A})$$

(meaning that $A^0 = \phi, A^i = \vec{A^i}$), and since we're still using the Minkowski metric, we have $A_\mu = (-\phi, \vec{A})$. Similarly, we also define the four-vector

$$J^\mu = (c\rho, \vec{J}),$$

and we can use this to define the **field strength**

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

where we're defining the shorthand $\partial_\mu A_\nu = \frac{\partial A_\nu}{\partial x^\mu}$. We can notice (by inspection) that $F_{\mu\nu} = -F_{\nu\mu}$, and we often call this the (antisymmetric) **Lorentz tensor**. We can in fact write down the components of this tensor as a matrix, and what we'll find that

$$F_{\mu\nu} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix}.$$

Then the gauge transformations can also be written in a simpler way as

$$A'_\mu = A_\mu + \partial_\mu \lambda :$$

we can now check (by the definition) that

$$\partial_\mu F_{\nu\lambda} + \partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} = 0,$$

which we can compute to give us the source-free part of Maxwell's equations, and we also have that

$$\partial_\nu F^{\mu\nu} = \frac{1}{c} J^\mu,$$

which gives the rest of Maxwell's equations. In the vacuum case, this last equation becomes $\boxed{\partial_\nu F^{\mu\nu} = 0}$, and in fact Maxwell's equations are still nontrivial even in this case – this gives us the familiar (but important) **electromagnetic waves**. And what we'll find is that

$$A_\mu = e_\mu e^{ik\cdot x},$$

where the four-vector k satisfies $k^2 = 0$ (because the wave must travel at the speed of light), and where e_μ has only two independent components. (We can solve this by fixing the **Lorentz gauge** $\partial_\mu A^\mu = 0$, and then we can turn the equation into a wave equation. But then we need to get rid of the gauge redundancies – this might be on our pset.)

Fact 13

Quickly turning to quantum mechanics, we know that there's a wave-particle duality, so the electromagnetic wave should correspond to some particle – this is what we call the **photon**. And the photon is a massless, spin-1 particle with two independent polarizations (normally a spin-1 particle would have three, but here we have gauge symmetry and the masslessness of the particle), and this is related to the two independent components e_μ.

And now we're ready to generalize E&M to arbitrary dimensions, instead of just $3+1$. In d-dimensional Minkowski spacetime, we still have vectors of the form $x^\mu = (ct, \vec{x})$, but now $\vec{x}$ has $(d-1)$ spatial components and our metric becomes

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + (dx^1)^2 + \cdots + (dx^{d-1})^2.$$

Next, we generalize by defining $A_\mu = (-\phi, \vec{A})$ and $J^\mu = (c\rho, \vec{J})$, where $\vec{A}$ and $\vec{J}$ are $(d-1)$-component vectors, and then the definition of $F_{\mu\nu}$, the gauge transformation expression, and the other tensor equations we wrote down are all directly generalized (except our index runs from 0 to $(d-1)$).

It would have been much more difficult if we tried to generalize electromagnetism using $\vec{E}$ and $\vec{B}$ directly, but we can still define the spatial vector $\vec{E}$ via $E_i = F_{i0}$. On the other hand, the magnetic field $\vec{B}$ is only a vector in $d = 4$ – in other dimensions d, the magnetic field generalizes to F_{ij} (basically $F_{\mu\nu}$ but only with spatial coordinates $i, j \in \{1, 2, \cdots, (d-1)\}$). This quantity F_{ij} is then associated with both the i and j directions, so (for example) if we work in $2+1$ dimensions, the only relevant quantity is F_{12}: there's only one component of the magnetic field. (And indeed, the only direction of the magnetic field is perpendicular to the plane!)

In general in d-dimensional space, our E&M waves have $(d-2)$ independent components, and that's something we can show ourselves: the same derivation for E&M waves in $(3+1)$ dimensions still generally goes through. So in particular, photons will have $(d-2)$ polarizations in d dimensions.

Finally, we'll close this lecture by talking a bit about another force:

1.4: Gravity in various dimensions

Classically, if we have two particles of mass M and m, then the force exerted by one particle on the other is

$$\vec{F} = -\frac{GMm}{r^2}\hat{r},$$

where $\hat{r}$ is a unit vector in the direction along the two particles and G is Newton's constant. More generally, the force experience by an object of mass m can be written as

$$\vec{F} = m\vec{g},$$

where $\vec{g}$ is the (Newtonian, non-relativistic) gravitational field obtained via $\vec{g} = -\nabla V_g$, and we have the equation

$$\nabla^2 V_g = 4\pi G \rho_m$$

for some mass density ρ_m. So if we're given ρ_m, we can work out V_g, and everything is theoretically solved.

And we can generalize all of this quickly to $\mathbb{R}^{1,d-1}$ with the corresponding (∇, ∇^2) in $\mathbb{R}^{d-1}$ Euclidean space – including more spatial directions makes this work in any dimension. And the Newton constant G is a fundamental parameter characterizing the strength of gravity, and in nature we observe that $G = 6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}$ (which has dimensions $\frac{L^3}{MT^2}$). The number doesn't mean very much on its own (especially because it depends on the units that we use), so if we want to characterize strength of gravity, we should use a dimensionless number. But we'll continue the discussion on Wednesday!

February 24, 2021

Last lecture, we reviewed the basic formalism of electromagnetism: recall that the basic dynamical variable is the four-vector A_μ, which lets us construct the antisymmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We can then define the electric field in terms of the tensor components $F_{i0} = E_i$, and in three spatial dimensions, we also have a magnetic field $F_{ij} = \varepsilon_{ijk} B_k$. And we found that all of this generalizes to any number of dimensions: Maxwell's equations like $\partial_\nu F^{\nu\mu} = \frac{1}{c} J^\mu$ still hold, and the main noticeable difference is that there is no longer a magnetic field vector (of the same dimension as the electric field vector).

Of course, what we've described is just a mathematical formulation of electromagnetism – other generalizations could also make sense. But the version we've described will actually come up in string theory!

From there, we started talking about Newtonian gravity, mentioning that the gravitational field $g = -\nabla V_g$ can be deduced from the mass density via $\nabla^2 V_g = 4\pi G \rho_m$, and this generalizes to any number of dimensions. In our world, G is some particular number with dimensions $\frac{L^3}{MT^2}$, so it depends on what units we're using for length, mass, and time.

To make the meaning of G more transparent, though, we can proceed in two different ways. First of all, since G depends on our units, we can use a particular set of units:

Definition 14

The constants $G, c, \hbar$ are fundamental constants in nature. When using **natural units**, we pick a particular set of three fundamental units ℓ_p, t_p, m_p (called the Planck length, time, and mass) so that $G = c = \hbar = 1$.

In particular, based on the units for G that we described above,

$$G = 1 \frac{\ell_p^3}{m_p t_p^2}, \quad c = 1 \frac{\ell_p}{t_p}, \quad \hbar = 1 \frac{m_p \ell_p^2}{t_p}.$$

We can invert the above equations to express these units in terms of the ordinary units that we use:

$$\ell_p = \sqrt{\frac{\hbar G}{c^3}} \approx 1.6 \times 10^{-33} \text{ cm},$$

$$t_p = \frac{\ell_p}{c} \approx 5.4 \times 10^{-44} \text{ s},$$

$$m_p = \sqrt{\frac{\hbar c}{G}} \approx 2.2 \times 10^{-5} \text{ g}.$$

Compared to our ordinary units, we notice that the Planck time and Planck length are very short. But we'll claim that the Planck mass is actually very large, and we'll understand that soon.

Alternatively, we can work with dimensionless numbers and quantities.

Example 15

Suppose we have two masses of mass m that are at a distance r apart from each other. Then the potential energy is $V = \frac{Gm^2}{r}$, and we can compare this to the static energy of each particle to get a dimensionless quantity.

We seem to run into an issue, because V changes with r, but because of quantum mechanics (and the uncertainty principle) there is some "minimum distance" that the two masses can be brought together, coming from the **Compton**

wavelength $\ell_c = \frac{\hbar}{mc}$. So we can calculate the "maximal gravitational energy"

$$\alpha_G = \frac{\text{max gravitational energy}}{\text{static energy}} = \frac{Gm^2/\ell_c}{mc^2} = \frac{m^2}{m_p^2} = \frac{\ell_p^2}{\ell_c^2}.$$

Here, we should be using elementary particles to characterize the fundamental strength of particles.

Example 16

The mass of an electron is $m_e \approx 10^{-27}$ g, so $\alpha_G \sim 10^{-44}$ (which is very small). So gravity is very weak, and in particular, the mass m_p is huge compared to the mass of elementary particles. (Even the heaviest elementary particle, the top quark, has energy 170 GeV, which is about $10^{-17} m_p$.) The fact that the Planck length ℓ_p is much smaller than the Compton wavelengths of ℓ_c of known elementary particles or the conventional units is a reflection of the weakness of gravity.

If we go to general d-dimensional space, $\nabla^2 V_g = 4\pi G \rho_M$ is still valid, and the units of G are then $[a/L]/[\rho_M]$ (where a is the acceleration), which means the units of G are $\frac{L^{d-1}}{MT^2}$ and thus

$$G = \frac{\ell_p^{d-1}}{m_p t_p^2}.$$

And because $\hbar$ and c don't change when we switch between dimensions (we still have $c = \frac{\ell_p}{t_p}$ and $\hbar = \frac{m_p \ell_p^2}{t_p}$ from above), we get the relation

$$\ell_p^{d-2} = \frac{\hbar G}{c^3}.$$

In practice, it is often convenient to set $c = \hbar = 1$, and thus $m_p = \frac{1}{\ell_p} = \frac{1}{t_p}$. In these units, we then have $G = \ell_p^{d-2}$.

For more motivation about the effects of compact extra dimensions, let's now consider a situation involving gravity:

Example 17

Suppose that the world is 5-dimensional with coordinates t, x, y, z, w, but $w \in [0, 2\pi R]$ lives on a circle, with R sufficiently small that we can't detect it. (Thus, our spacetime is the space $R^{1,3} \times S_1$.)

We then have a 5-dimensional gravity theory with Newton constant G_5, but if R is tiny, we effectively observe a 4-dimensional world and see 4-dimensional gravity with a Newton constant G_4. Our goal is to **find the relation** between G_4 and G_5, and we'll do this exercise to get more intuition for what higher dimensions do for us. Fundamentally, the Newtonian gravity equation should look like

$$\nabla_5^2 V_g^{(5)} = 4\pi G_5 \rho_m^{(5)}.$$

for some 5-dimensional gravitational potential, Newton constant, and mass density, and $V_g^{(5)}$ and $\rho_m^{(5)}$ should be periodic functions of w.

Fact 18

Whenever the gradient operators ∇ and ∇^2 appear, they are only being taken in the **spatial directions**.

However, we observe a 4-dimensional analog

$$\nabla_4^2 V_g^{(4)} = 4\pi G_4 \rho_m^{(4)},$$

and we basically need to understand how the quantities in these two equations relate to each other. We know that $\nabla_5^2 = \nabla_4^2 + \partial_w^2$, and because we can't resolve the difference in w-values, we must not be able to resolve the difference between $V_g^{(5)}(x, y, z, w_1)$ and $V_g^{(5)}(x, y, z, w_2)$, and $V_g^{(4)}$ should be the **average value** of $V_g^{(5)}$ over the w-direction:

$$V_g^{(4)}(x, y, z) = \frac{1}{2\pi R} \int_0^{2\pi R} V_g^{(5)}(x, y, z, w) dw.$$

Similarly, because we can't resolve the mass density at different w-coordinates, we must have

$$\rho_m^{(4)}(x, y, z) = \int_0^{2\pi R} \rho_m^{(5)}(x, y, z, w) dw$$

(notice that we're taking a sum instead of an average here). So plugging these relations back into our 4-dimensional gravity equation yields

$$\frac{1}{2\pi R} \nabla_4^2 \int_0^{2\pi R} V_g^{(5)} dw = 4\pi G_4 \int_0^{2\pi R} \rho_m^{(5)} dw.$$

We can bring all terms on the left-hand side into the integral (because ∇_4^2 doesn't act on w) to get

$$= \frac{1}{2\pi R} \int_0^{2\pi R} \nabla_4^2 V_g^{(5)} dw,$$

and because $V_g^{(5)}$ is periodic in w, we can actually replace ∇_4^2 with ∇_5^2 (because we have a total derivative inside the integral). So applying the 5-dimensional gravity equation gives us

$$= \frac{1}{2\pi R} \int_0^{2\pi R} 4\pi G_5 \rho_m^{(5)} dw,$$

and comparing this now with the right-hand side gives us

$$\boxed{G_4 = \frac{G_5}{2\pi R}}.$$

This is actually an example of a more general result:

Proposition 19

Suppose we have a d-dimensional spacetime of the form $\mathbb{R}^{1,3} \times M$ for a compact manifold M. Then if V_M denotes the volume of the manifold, we have

$$G_4 = \frac{G_d}{V_M}.$$

This result turns out to also be valid in general relativity, and we now have a fundamental relation between the intrinsic gravity and the effective Newtonian gravity that we observe.

1.5: General relativity

Basically, Newtonian gravity is a special limit of general relativity in the case where we have low velocity and weak gravity. The key dynamical variable for general relativity is then the symmetric **spacetime metric** $g_{\mu\nu}$, a generalization of our Minkowski metric: our distances are now calculated via

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu,$$

and the **Einstein equations**, along with the matter distribution, gives us $g_{\mu\nu}(x)$. And this works in any dimension, as long as our index runs from 0 to $d - 1$. In fact, in the absence of gravity, we have special relativity

$$g_{\mu\nu} = \eta_{\mu\nu}$$

in any dimension. But **weak** gravity yields a $g_{\mu\nu}$ which can be expanded around the Minkowski metric

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x),$$

In the low velocity limit, only one component of this $h_{\mu\nu}$ survives, given exactly by the Newtonian potential V_g that we've been previously discussing. Here are a few other key elements of GR as well:

- Newtonian gravity is instantaneous: the equation $\nabla^2 V_g = 4\pi G \rho_m$ doesn't have any time-component at all, so it is "action at a distance" (which is incompatible with relativity). In contrast, propagation of interactions in GR, including gravity, is **limited by the speed of light**.
- If we're in vacuum (with no mass density), gravity is actually still highly nontrivial: for example, there can be **gravitational waves**, which were first detected in 2015 and have given rise to many new experimental discoveries in the past few years. Gravitational waves can be described as $h_{\mu\nu} = e_{\mu\nu}e^{ik\cdot x} + c.c.$ for some $k^2 = 0$, where the polarization $e_{\mu\nu}$ again only has two independent components in $d = 4$. These waves become particles in quantum mechanics, and we call the corresponding particles **gravitons** – they are massless and of spin 2 (because of the 2 indices) in $d = 4$. And in general d, there are $\frac{1}{2}d(d-3)$ independent components (polarizations) in $e_{\mu\nu}$ (but the spin of the graviton is still 2). We will in fact derive the graviton spectrum from string theory, and we'll see this number arise later on in the class!

Finally, we have one more piece of preparation before we get to strings:

1.6: Action principle

Example 20
Consider a **non-relativistic** particle of mass m in a potential $V(x)$.

Then we can write down the Newton's second law expression $m\ddot{x} = -V'(x)$, but there's also an alternative formulation, the **principle of stationary action**, which we will talk about now.

Definition 21
Let $L(t) = T - V$ be the **Lagrangian** of a system (where T is the kinetic energy and V is the potential energy), and define the **action**

$$S = \int_{t_i}^{t_f} L(t)dt.$$

This S is known as a **functional**, because it takes in a function $x(t)$ and evaluates to some number.

Proposition 22 (Principle of stationary action)
The action S is stationary for the physical path.

In other words, if we have a trajectory $x(t)$ from (t_i, x_i) to (t_f, x_f), and we infinitesimally perturb the path to $x(t) + \delta x(t)$ (without changing the initial and final points, meaning $\delta x(t_i) = \delta x(t_f) = 0$). Then if we expand in the perturbation, we have $S[x(t) + \delta x(t)] = S[x(t)] + O(\delta x)$, and what we're claiming for the **physical path** is that

$$S[x(t) + \delta x(t)] = S[x(t)] + O((\delta x)^2).$$

We can do this variation explicitly for the Newton's law case above: we have

$$S[x(t)+\delta x(t)] = \int_{t_i}^{t_f} \left[\frac{1}{2}m(\dot{x}(t)+\delta\dot{x}(t))^2 - V(x+\delta x)\right] dt,$$

and expanding this in δx gives us

$$= S[x(t)] + \int_{t_i}^{t_f} [m\dot{x}(t)\delta\dot{x}(t) - V'(x)\delta x(t)]\, dt + O((\delta x)^2).$$

So if the path is stationary, the linear piece we've written out should be zero: defining $\delta S = \int_{t_i}^{t_f} [m\dot{x}(t)\delta\dot{x}(t) - V'(x)\delta x(t)]\, dt$, we can do integration by parts to remove the derivative on the first δx term – we find that

$$\delta S = \int_{t_i}^{t_f} [\partial_t(m\dot{x}(t)\delta x(t)) - m\ddot{x}(t)\delta x - V'(x)\delta x]\, dt.$$

But the total derivative term goes to 0, because $\delta x(t)$ vanishes at the initial or final position. So what we're left with is

$$-\int_{t_i}^{t_f} dt\, [m\ddot{x} + V'(x)]\, \delta x(t) = 0,$$

and because $\delta x(t)$ is arbitrary, this can only be zero for all perturbations if $\boxed{m\ddot{x} + V'(x) = 0}$, and we've derived Newton's law.

But this action principle is very useful even when we can't write down the kinetic and potential energy explicitly in the Newtonian framework! And as strings enter the picture, we'll be able to derive the equation of motion using this powerful method.

Remark 23. *There is a close connection between the action principle and quantum mechanics as well, but we won't go into this.*

With this, chapter 1 of our class (review and preparation) is complete – we've now set up all of the notation, concepts, and basic techniques that we'll need to use later on. And in chapter 2, we'll discuss **classical strings**, but we're not quite there yet: we need a bit of warmup first.

2: Classical strings

2.1: The relativistic free particle

We'll first see the action principle and a few other important techniques being used here. First of all, this problem is trivial in the non-relativistic case: the action of the particle (which has no potential energy) is

$$S = \int L(t)dt = \int \frac{1}{2}m\dot{x}^2 dt,$$

and the result of the action principle is that $\frac{d\vec{v}}{dt} = 0$.

But in the relativistic case, the action looks a little bit different. We're going to use a strategy that theoretical physics use often to find new theories:

- First, list all of the requirements that our theory must satisfy.
- Next, guess the theory by **writing down an action** (if we can't figure it out from the constraints).
- Study properties that result from the action we've written down, and make sure that everything is self-consistent.

We'll start applying this strategy to our relativistic free particle next time!

March 1, 2021

Last time, we discussed the **action principle**, whose basic idea is to introduce a functional

$$S[x(t)] = \int L(\dot{x}, x, t) dt$$

for a function L known as the **Lagrangian**, which (for example) is $\frac{1}{2}m\dot{x}^2 - V(x)$ in the nonrelativistic case. We can then derive the equation of motion by setting $\delta S = 0$. We can also define the canonical momentum $p = \frac{\partial L}{\partial \dot{x}}$, and then we can define the **Hamiltonian**

$$H = \dot{x}p - L(\dot{x}, x),$$

where we are considering $\dot{x}$ as a function of p, so that H should be understood as a function of x, p. We know that the Hamiltonian plays a large role in quantum mechanics. In classical mechanics it gives the energy of a conservative system.

At the end of last lecture, we started the application of this formalism to the relativistic free particle, which is a warmup for our dive into string theory. We'll use this classical relativistic particle to then lead into a discussion about classical strings, and we'll do the same thing for quantum particles and strings as well.

Fact 24

The history of how string theory was created is rather contrived, but it started here at MIT.

In the late 1960s, people were puzzled by strong interactions why they led to having huge numbers of particles from particle collisions, and the only tool they had at the time was perturbation theory (which didn't really apply). People then suggested that strong interactions didn't need a full theory to be fully studied – we could guess the answers for problems like scattering amplitudes directly, and many ideas came out of this approach. At first, there weren't any answers that satisfied various requirements that people wanted such amplitudes to have, until a young professor named Veneziano at MIT saw the beta function in a math textbook! From there, various further guesses emerged, and then slowly a physical interpretation of these amplitudes in terms of scatterings of strings appeared, thus giving rise to the string theory. (In this course, we will find the spectrum of a string, but will not be able to tell you how to calculate scattering amplitudes as that will involve too much technicalities.)

So now we'll return to the relativistic free particle and write down the action promised at the end of last lecture. Recall that the strategy we suggested is basically to list all of the requirements that need to be satisfied (both theoretical and experimental), to guess or deduce some answer, and then work out the physical properties of the theory. After making sure everything is self-consistent, one can then make predictions for experiments.

Let's list out some of the requirements:

1. In special relativity, physics does not depend on the (Lorentz) frame that we're choosing, which means that our action S must be invariant under Lorentz transformations.

2. Let $\vec{x}(t)$ be a spatial vector (remember that we'll use notation like x^μ for a spacetime vector). Then $S[\vec{x}(t)]$ should only contain first order derivatives of $\vec{x}$, so things like $\frac{d\vec{x}}{dt}$ are okay but not $\frac{d^2\vec{x}}{dt^2}$. This is because the equation of motion should be at most a second-order differential equation, based on our **empirical experiences** – we can always determine the trajectory of a particle with two initial conditions. (We can see https://en.wikipedia.org/wiki/Ostrogradsky_instability for more details!)

3. The action S should have the correct non-relativistic limit of $\frac{1}{2}m\dot{\vec{x}}^2$ (leading to Newtonian mechanics) for $|\vec{v}| \ll c$.

With this information, we'll deduce the answer – it turns out Lorentz invariance is already a pretty constraining requirement. We know that the Lagrangian is a function of $\dot{\vec{x}}$ and $\vec{x}$, and because we're dealing with a free particle (meaning that the transformation $\vec{x} \mapsto \vec{x} + \vec{a}$ with $\vec{a}$ constant should leave L invariant), there can't actually be any $\vec{x}$-dependence in L. The first two points above help us out a lot here – if we have a trajectory $\vec{x}(t)$ along a curve C, the distance along that curve is controlled by the invariant distance $ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$, and this is the only Lorentz invariant object we can construct along the trajectory. Therefore the answer must depend on **proper time** (because our trajectory $ds^2 = -c^2dt^2 + d\vec{x}^2 = -c^2d\ell^2$ must be timelike), and thus our guess is

$$S = k\int_C d\ell.$$

In other words, we get the action by integrating the infinitesimal proper time along the curve! This is in fact not the only action satisfying requirements 1 and 2 above, but the simplest one. We'll comment more on this later.

To determine the constant k, we can use the third point. We know that

$$-c^2dt^2 + d\vec{x}^2 = -c^2d\ell^2 \implies d\ell = dt\sqrt{1 - \frac{\vec{v}^2}{c^2}},$$

so that the action can also be written as $S = k\int dt\sqrt{1 - \frac{\vec{v}^2}{c^2}}$. In the nonrelativistic limit where $|\vec{v}| \ll c$, this square root expands to $1 - \frac{\vec{v}^2}{2c^2} + O(\frac{\vec{v}^4}{c^4})$:

$$S = k\int dt\left(1 - \frac{\vec{v}^2}{2c^2} + \cdots\right),$$

and in order for this to match $S = \int \frac{1}{2}m\vec{v}^2dt$, we need $k = -mc^2$. So we've arrived at our result:

Proposition 25

The action of a relativistic free particle can be written as

$$S = -mc^2\int d\ell = -mc^2\int dt\sqrt{1 - \frac{\vec{v}^2}{c^2}}.$$

We can notice that the Lagrangian expands out to $-mc^2 + \frac{1}{2}m\vec{v}^2 + O\left(mv^2\frac{v^2}{c^2}\right)$, and here the first term $-mc^2$ can be thought of as a constant potential (with our particle's rest energy mc^2). So our consistency checks work out, and let's now shift to studying the properties of this system. First of all, the conjugate momentum is

$$\frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}},$$

which is indeed the usual expression we expect, and then the Hamiltonian $H = \vec{p}\cdot\vec{v} - L$ (expressed in terms of $\vec{v}$ instead of $\vec{x}$) is

$$H = \frac{mc^2}{\sqrt{1 - \frac{\vec{v}^2}{c^2}}},$$

which again is what we expect of the relativistic energy.

But note that the way we've been thinking about this so far is not Lorentz covariant; we have been using variables $\vec{x}(t)$ for intuitive reasons. To be compatible with relativistic spirit, $(ct, \vec{x})$ should be treated on equal footing. We can introduce a variable τ to parameterize the path of the particle and describe the path C with a Lorentz-vector $x^\mu = (ct, \vec{x})$ as

$$x^\mu(\tau) : \mathbb{R} \to \mathbb{R}^{1,d-1}$$

which is a mapping from a one-dimensional parameter space τ to d-dimensional Minkowski spacetime $\mathbb{R}^{1,d-1}$ (which is a notion that will be very relevant when we think about strings later on). We will often refer to the trajectory C as a **world line**. This trajectory has some initial condition $x_i^\mu = x^\mu(\tau_i), x_f^\mu = x^\mu(\tau_f)$, and now we can try to write the action in terms of $x^\mu(\tau)$ instead of $\vec{x}(t)$. We can do this by going to the definition of the proper time $c^2 d\ell^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$, which we can rewrite as

$$c^2 d\ell^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2,$$

so that if we plug in $d\ell = \frac{1}{c}\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}} d\tau$ into the action, we find that

$$S = -mc \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}}.$$

We now have an important physical requirement – **the physics should not depend on how we parameterize the path** C, so it should be invariant if we replace τ with some arbitrary monotonic function $\tau' = f(\tau)$. Our mapping must be such that

$$x'^\mu(\tau') = x^\mu(\tau)$$

(the above equation means that τ and τ' are mapped to the same spacetime point), and in particular this means by the chain rule that

$$\frac{dx'^\mu}{d\tau'} = \frac{dx^\mu}{d\tau} \frac{d\tau}{d\tau'}.$$

So now we have

$$\boxed{S[x'^\mu(\tau')]} = -mc \int d\tau' \sqrt{-\eta_{\mu\nu} \frac{dx'^\mu}{d\tau'} \frac{dx'^\nu}{d\tau'}},$$

and the chain rule simplifies this to the original expression:

$$= -mc \int \frac{d\tau'}{d\tau} d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} \left(\frac{d\tau}{d\tau'}\right)^2} = -mc \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}},$$

which is exactly the original expression for $\boxed{S[x^\mu(\tau)]}$, so this whole situation is indeed reparameterization-invariant (and in fact, our original setup $\vec{x}(t)$ is using the parametrization variable $\tau = t$).

We can now turn to the equations of motion that come from $S = -mc^2 \int d\ell$: since $c^2 d\ell^2 = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} d\tau^2$, if we apply a variation $x^\mu(\tau) \mapsto x^\mu(\tau) + \delta x^\mu(\tau)$, then variational calculus gives us

$$2c^2 d\ell \, \delta d\ell = -2\eta_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} d\tau^2.$$

To compute $\delta d\ell$, note that

$$\delta d\ell = -\frac{1}{c^2} \eta_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\tau} \frac{d\tau}{d\ell} d\tau.$$

But the last two derivative terms on the right combine to $\frac{dx^\nu}{d\ell}$ by the chain rule, so we can plug this in to find that

$$\delta S = m \int d\tau \eta_{\mu\nu} \frac{d\delta x^\mu}{d\tau} \frac{dx^\nu}{d\ell} = \int d\tau \eta_{\mu\nu} \frac{d\delta x^\mu}{d\tau} p_\mu,$$

where we introduce the standard definition of momentum

$$p_\mu = m\eta_{\mu\nu} \frac{dx^\nu}{d\ell} = m\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{1}{d\ell/d\tau} = m\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{1}{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}}.$$

We can now do integration by parts on the expression for δS – remembering that δx^μ doesn't change at the endpoints, this expression becomes

$$\delta S = -\int d\tau \delta x^{\mu} \frac{dp_{\mu}}{d\tau}.$$

For this to be zero, we need $\boxed{\frac{dp_{\mu}}{d\tau} = 0}$, and that gives us the equation of motion – p_{μ} must be constant.

Remark 26. *Notice that p_{μ} is also reparameterization invariant, and that's what we expect because it's something we can measure. And in particular we can check that* $\boxed{p_{\mu}p^{\mu} = -m^2}$ *(which is the familiar mass-shell relation).*

We can also observe that the following action

$$S = \int d\tau f(\dot{x}^2),$$

where $\dot{x}^2 = -\eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}$ and f is an arbitrary function, satisfies the requirements 1 and 2 we listed earlier. But such an action is in general not reparameterization invariant except for $f(\dot{x}^2) = \sqrt{\dot{x}^2}$: the power of $d\tau$ that shows up needs to balance out in the calculation that we did above!

Formulating the action in this way is still annoying, though, because the square roots cause problems especially in the quantum theory. So we'll rewrite the action in another way now by introducing a new dynamical degree of freedom. Specifically, we'll write

$$\boxed{S = \frac{1}{2}\int d\tau \left[-\frac{1}{e(\tau)}\dot{x}^2 - e(\tau)m^2c^2\right]}$$

(with $\dot{x}^2$ as above), where $e(\tau)$ is a new dynamical degree of freedom. We claim that (at least classically) this formulation is equivalent to the one above, and this is because $e(\tau)$ has no derivative term, meaning that finding the equation of motion $e(\tau)$ is algebraic (no second-order differential equation). We can then plug in the solution of $e(\tau)$ back in, and we'll see that we get the previous action S with a square-root. We'll see that happen next time!

March 3, 2021

We'll start with a quick summary of what happened last lecture: we wrote down our action for the relativistic free particle in a variety of ways as

$$S = -mc^2\int_C d\ell = -mc^2\int_C dt\sqrt{1-\frac{\vec{v}^2}{c^2}} = -mc\int_C d\tau\sqrt{-\eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}}. \tag{1}$$

We'll introduce some notation as shorthand: set

$$\frac{dx^{\mu}}{d\tau} = \dot{x}^{\mu}, \quad \eta_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = \dot{x}^{\mu}\dot{x}_{\mu} = \dot{x}^2,$$

so that now we can also rewrite the action as

$$= -mc\int_C d\tau\sqrt{-\dot{x}^2}. \tag{2}$$

Notice that all of the equations we've written above in Eq. (1) and Eq. (2) are **manifestly** Lorentz invariant except for $-mc^2\int_C dt\sqrt{1-\frac{\vec{v}^2}{c^2}}$, which is written in terms of standard velocity instead of relativistic velocity. And furthermore, we have two ways of writing down the action in a **reparameterization-invariant way** (namely those that used an arbitrary parameter τ). But if we plug in particular parameters like the proper length (setting $d\tau = d\ell$) or the time (setting $\tau = t$), we recover the first two forms of the action that we wrote down.

The point of writing down an action is often to get an equation of motion: in this case, we found that $\frac{dp^\mu}{d\tau} = 0$, where

$$p^\mu = m\frac{dx^\mu}{d\ell} = m\frac{dx^\mu}{d\tau}\frac{d\tau}{d\ell}.$$

And now we can read off the relation between τ and ℓ from above: since $c d\ell = \sqrt{-\dot{x}^2} d\tau$ (by comparing the forms in Eq. (1) and Eq. (2)), we find that $\frac{d\tau}{d\ell} = \frac{c}{\sqrt{-\dot{x}^2}}$, and therefore our four-momentum can be written as $p^\mu = m\dot{x}^\mu \frac{c}{\sqrt{-\dot{x}^2}}$. And now we can check that our components are not independent – we have the relation

$$p^\mu p_\mu = -m^2 c^2.$$

At the end of the lecture, we mentioned that square roots can often be inconvenient (especially in quantum theory), and thus we wrote the action in yet another way:

$$S' = \frac{1}{2}\int d\tau \left(\frac{1}{e(\tau)}\dot{x}^2 - e(\tau) m^2 c^2\right).$$

Lemma 27

The action $S' = \frac{1}{2}\int d\tau \left(\frac{1}{e(\tau)}\dot{x}^2 - e(\tau)m^2c^2\right)$ is equivalent to the other forms of the action above.

Proof. The equation of motion that we get from our new dynamical variable $e(\tau)$ is derived as follows: because e and x are independent variables, we can treat x as constant while finding the equation of motion for e, and this yields

$$\delta S = \frac{1}{2}\int d\tau \left[-\frac{1}{e^2}\delta e\ \dot{x}^2 - \delta e\ m^2c^2\right] \implies \frac{1}{e^2}\dot{x}^2 + m^2c^2 = 0,$$

which means that $e = \frac{1}{mc}\sqrt{-\dot{x}^2}$ is the physical solution. And plugging this back in yields the original action, because we have

$$S' = \frac{1}{2}\int d\tau \left(\frac{1}{\sqrt{-\dot{x}^2}}\dot{x}^2 - \sqrt{-\dot{x}^2}m^2c^2\right) = -mc\int d\tau \sqrt{-\dot{x}^2} = S.$$

□

Now that we have our action written in a new way, we'll examine some properties of it. There are two types of **symmetry** that we can see in our action:

- **Global symmetries** are symmetries whose transformation parameters don't depend on τ (which we can think of as our one-dimensional "fundamental space"). For example, translations $x^\mu \to x^\mu + a^\mu$ are symmetries, because the action here only depends on the derivative. Lorentz symmetries $x^\mu \to L^\mu{}_\nu x^\nu$ are also global symmetries because the transformations are constant matrices (not depending on τ).
- **Local symmetries** are symmetries where different parts of the trajectory (parameterized by τ) may transform differently. The reparameterization symmetry $\tau \to \tau' = f(\tau)$ is one such example (as we derived last lecture), and our transformation is then $x^\mu(\tau) \to x'^\mu(\tau') = x^\mu(\tau)$. If we look at the new action, it's worth thinking about how $e(\tau)$ transforms under these kinds of reparameterization: we need

  $$\int d\tau e(\tau) m^2 c^2 = \int d\tau' e'(\tau') m^2 c^2,$$

  which implies that we must have

  $$e'(\tau')\frac{d\tau'}{d\tau} = e(\tau).$$

In other words, e must transform in a particular way so that the second term in our new action behaves properly under reparameterization. If we enforce this rule, we can similarly check that the first term in our new action is invariant – this is left as an exercise for us.

Remark 28. *Heuristically, $e(\tau)$ can be thought of as a local metric along the path C, since we have the relation $e'(\tau')d\tau' = e(\tau)d\tau$. Basically, the length along the path does not change under reparameterization.*

But note that τ is a dummy variable here, so instead of changing τ, we can consider a transformation of the form $e(\tau) \to e'(\tau)$ and $x^\mu(\tau) \to x'^\mu(\tau)$ in a way that keeps the action invariant. So this is another way to think about the transformation, and we'll see that on our problem set.

Recall that gauge symmetries in electromagnetism implied some redundancy in our variables, and that's similar to what's happening here: the existence of local symmetries implies redundancy of variables. And just like we fix a particular gauge in electromagnetism to make problems easier to solve, we can also remove the redundancy in our action here. But the reason we still write our action in this way is that it is a more general form, and we can simplify the action in a few different ways and get different insights.

To see that, let's now revisit our equations of motion using the new action. This time, instead of solving for $e(\tau)$ and substituting things back into the original action, let's treat this as a new problem by first **fixing a gauge** (meaning that we pick a convenient $e'(\tau') = e(\tau)\frac{d\tau}{d\tau'}$). We want to make it as simple as possible – we can't make it zero, but we can choose a gauge so that $e'(\tau')$ is some constant. Checking the dimensions of e, we see that it should be $\frac{1}{[M]}$, and since the only mass parameter we have in our system is m, we'll pick the gauge so that

$$e'(\tau') = \frac{1}{m}.$$

In other words, we pick τ' so that $\frac{d\tau'}{d\tau} = me(\tau)$, and this is always possible because we can integrate that equation over τ to get τ'. Then the equation of motion from e becomes

$$m^2\dot{x}^2 + m^2c^2 = 0 \implies \dot{x}^2 = -c^2.$$

Remembering that $\dot{x} = \frac{dx^\mu}{d\tau}$, we find that our choice of τ' has actually turned it into the **proper time**, because

$$-c^2 = \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = \frac{-c^2 d\ell^2}{d\tau^2}.$$

So now a lot of other things simplify nicely, and we get equations like $p^2 = -m^2c^2$ (which look familiar to us). So the equation of motion from e is basically the mass-shell equation again.

And if we substitute $e = \frac{1}{m}$, the action now looks like

$$S = \frac{1}{2}\int d\tau \left(m\dot{x}^2 - mc^2\right).$$

And this is now looking like a "kinetic minus potential" term, but remember that $\dot{x}^2 = \eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$ is not the ordinary three-velocity! Instead, we can use the action principle on x^μ, which immediately gets us the equation of motion (with constraint)

$$\boxed{\frac{d^2x^\mu}{d\tau^2} = 0, \quad \eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -c^2}.$$

Fact 29
In electromagnetism, fixing a gauge often results in the question of "are there any redundancies left in the system." For example, as we'll see in our problem set, the Lorentz gauge does not actually fix all of the degrees of freedom.

We'll ask the same question here – does this choice of $e = \frac{1}{m}$ fix all of the gauge freedom? And the answer is **no** – we can still do a constant translation $\tau \mapsto \tau' = \tau + b$, and this does not change the value of $e(\tau)$. This freedom will actually be crucial for us (and we'll see this later on)!

To solve the boxed equations of motion above, note that the first equation tells us that

$$x^\mu(\tau) = v^\mu \tau + x_0^\mu,$$

where $v^\mu = \frac{p^\mu}{m}$ and x_0^μ are constant vectors. And now plugging this into the constraint tells us that

$$v^\mu v_\mu = -c^2,$$

which is the same as the usual $p^2 = -m^2c^2$ condition. This condition can be rewritten as (because $p^\mu = (p^0, \vec{p})$))

$$p^0 = \sqrt{m^2c^2 + \vec{p}^2},$$

and now we can think about what independent parameters exist in our system: it might seem like (naively) we have p_i (where $i \in [1, 2, \cdots, d-1]$ and x_0^μ (where $\mu \in [0, 1, \cdots, d-1]$). But there's a mismatch between position and momentum parameters, and now we have to remember the freedom in shifting the parameter τ – we can set $\boxed{x_0^0 = 0}$, so that

$$\boxed{x^0(\tau) = \frac{p^0}{m}\tau, \quad x^i(\tau) = \frac{p^i}{m}\tau + x_0^i}.$$

We now indeed have x_0^i, p_i as our $2(d-1)$ independent parameters. Thus fixing all the gauge freedom is important for getting the correct number of free parameters. This becomes particularly important in quantum theory.

And now remembering that $x^0 = ct$, with $x_0^0 = 0$ we're implying that τ is proportional to the physical time t.

Classically, it's okay that $p^0 = \sqrt{m^2c^2 + \vec{p}^2}$, but this is inconvenient in quantum mechanics because p's become operators, and square roots are awkward in that context. So to avoid this problem, **we go to the light-cone frame**: the relativistic free particle's solutions can be alternatively written as $x^\mu = (x^+, x^-, x^a)$ (where $a \in [2, \cdots, d-1]$) and $p^\mu = (p^+, p^-, p^a)$. So taking our solution from above and putting it in the light-cone frame gives us

$$x^+(\tau) = \frac{p^+}{m}\tau + x_0^+, \quad x^-(\tau) = \frac{p^-}{m}\tau + x_0^-, \quad x^a(\tau) = \frac{p^a}{m}\tau + x_0^a.$$

Then the mass-shell condition becomes

$$p^- = \frac{m^2c^2}{2p^+} + \frac{p_\perp^2}{2p^+},$$

and we've removed the square roots at the price of having the reciprocal of $\frac{1}{p^+}$ (which is a slightly better operator in quantum mechanics). If we fix the remaining gauge freedom so that $x_0^+ = 0$, then our free parameters are $(p^+, \vec{p}_\perp, x_0^-, \vec{x}_{\perp 0})$, and this makes sense because p^+ and x_0^- are conjugate in light-cone coordinates.

We emphasize that choosing $x_0^+ = 0$ is a different gauge fixing from choosing $x_0^0 = 0$. Now τ is proportional to the lightcone time x^+ rather than t.

With this, we're ready to see what happens to a relativistic particle under forces – specifically, we'll try to understand a particle with electric charge under an electromagnetic field, as the electromagnetism is the only force we have learned so far to be consistent with special relativity. Suppose that we have a charged particle in a background field described by $A_\mu(x^\nu)$: then we have

$$S[x^\mu(\tau)] = S_0[x^\mu(\tau)] + S_{\text{EM}}[x^\mu(\tau)]$$

where S_0 is the action for a free particle. S_{EM} should contain A_μ in a Lorentz-covariant way, meaning that indices should be properly contracted using $\eta_{\mu\nu}$. Furthermore, it should not contain higher derivatives than $\frac{dx^\mu}{d\tau}$, and it should also be reparameterization invariant. These requirements essentially uniquely fix

$$S_{\text{EM}} = b \int_C d\tau A_\mu(x(\tau)) \frac{dx^\mu}{d\tau}.$$

(For reparameterization invariance we can't have more than one copy of $\frac{dx^\mu}{d\tau}$ because there's only one $d\tau$ in this integral.) And dimensional analysis tells us that b should have dimensions of charge over velocity, so we will write $b = \frac{q}{c}$ (which can be understood as the **definition of the charge** q). This means our full action is

$$S = -mc \int_C d\ell + \frac{q}{c} \int A_\mu \frac{dx^\mu}{d\tau} d\tau.$$

We can now derive the equation of motion, and we will find that

$$\boxed{\frac{dp^\mu}{d\tau} = \frac{q}{c} F_{\mu\nu} \frac{dx^\nu}{d\tau}}$$

(which we can check reproduces the Lorentz force equations if we write out the contraction).

With that, we've finished discussing the particle, and we'll finally start talking about strings next time! But if it feels like there's a lot of formalism with the particle, we should be aware that there will be more of that for the string.

March 9, 2021

Today, we'll finally get to strings (as promised):

2.2: Relativistic strings

Recall that we describe the trajectory of a particle by a worldline, which we can parameterize with some parameter τ (so the position along the trajectory is some four-vector $x^\mu(\tau)$). If we think of a particle as a zero-dimensional object traveling along a one-dimensional line, then it makes sense that a one-dimensional string will travel along a **two-dimensional surface**, called a **worldsheet**, in spacetime. (And in particular, if we have a closed string that moves through spacetime, we can think of the worldsheet as a cylinder in spacetime.)

We will thus parameterize our worldsheet with two parameters and denote the spacetime position as $X^\mu(\xi^0, \xi^1)$. In other words, to determine the physical coordinates x^μ (in the **target spacetime**), we only need to feed in these two parameters ξ^0, ξ^1 (called the **worldsheet parameters** or **worldsheet coordinates**), and we'll use the capital X^μ to denote this mapping from worldsheet to spacetime.

Fact 30

One way of phrasing this situation is that $X^\mu(\xi^0, \xi^1)$ describes the **embedding of a 2-dimensional surface in a d-dimensional Minkowski spacetime** (much like a particle trajectory was an embedding of a 1-dimensional line).

Even though the difference is mathematical at first, there will be lots of physical differences in the theories between strings and particles that we'll see soon! We now wish to write down the action for the string $S_{\text{string}}[X^\mu(\xi^\alpha)]$

(where $\alpha = 0, 1$), with the requirements again that S_{string} must be Lorentz invariant and reparameterization invariant. Furthermore, just like before, we'll say that the action should only contain at most first-order time-derivatives of X^μ.

These requirements uniquely determined the action S_{particle} for us to be proportional to proper time along the trajectory x^μ, which is basically the proper "length" of that worldline. So the natural generalization is that the action for the string is proportional to the **proper area of the worldsheet**, and we need to do some work to prepare for that.

Example 31

For a similar question, let's try to calculate the area of a 2D surface in Euclidean (rather than Minkowski) space.

Remark 32. *Since everything is Euclidean space, we don't need to worry about lower and upper indices here.*

Similar to the setup above, let's say that we can parameterize our surface as $\vec{X}(\xi^1, \xi^2)$. An example of this could be the sphere, in which

$$\vec{X}(\theta, \phi) = (X_1(\theta, \phi), X_2(\theta, \phi), X_3(\theta, \phi)) = (R \sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta).$$

(Again, this is a function from the parameter space of points (ξ^1, ξ^2) to the target space of points (x^1, x^2, x^3).) If we wish to calculate the area of this surface, we can look at the infinitesimal area elements: letting $d\vec{V}_1$ correspond to the variation in $d\xi^1$ and $d\vec{V}_2$ correspond to $d\xi^2$, we have

$$d\vec{V}_1 = \frac{\partial \vec{X}}{\partial \xi^1} d\xi^1, \quad d\vec{V}_2 = \frac{\partial \vec{X}}{\partial \xi^2} d\xi^2,$$

where the area element is $dA = |d\vec{V}_1||d\vec{V}_2|\sin\theta$ and we know that $\cos\theta = \frac{d\vec{V}_1 \cdot d\vec{V}_2}{|d\vec{V}_1||d\vec{V}_2|}$. So what we find is that

$$dA = |d\vec{V}_1||d\vec{V}_2|\sqrt{1 - \frac{(d\vec{V}_1 \cdot d\vec{v}_2)^2}{|d\vec{V}_1|^2|d\vec{V}_2|^2}} = \sqrt{|d\vec{V}_1|^2|d\vec{V}_2|^2 - |d\vec{V}_1 \cdot d\vec{v}_2|^2},$$

and we can now rewrite this as

$$= d\xi^1 d\xi^2 \sqrt{\left|\frac{\partial \vec{X}}{\partial \xi^1}\right|^2 \left|\frac{\partial \vec{X}}{\partial \xi^2}\right|^2 - \left(\frac{\partial \vec{X}}{\partial \xi^1} \cdot \frac{\partial \vec{X}}{\partial \xi^2}\right)^2},$$

which gives us the final answer for the total area

$$\boxed{A = \int d\xi^1 d\xi^2 \sqrt{\left|\frac{\partial \vec{X}}{\partial \xi^1}\right|^2 \left|\frac{\partial \vec{X}}{\partial \xi^2}\right|^2 - \left(\frac{\partial \vec{X}}{\partial \xi^1} \cdot \frac{\partial \vec{X}}{\partial \xi^2}\right)^2}}.$$

We will now show that the above expression for the area can be written more transparently using the concept of **induced metric**, which arises when we embed a space inside another space. For example, if we have a sphere in 3D Euclidean space, the shortest distance between two points is the usual Euclidean distance $ds^2 = d\vec{x}^2$, but if we **restrict ourselves to the sphere**, we have a different shortest distance given by the **induced metric**, which is defined by taking the Euclidean metric and restricting it to the sphere. In particular, what this means is that because the ith component satisfies

$$dX^i = \frac{\partial X^i}{\partial \xi^\alpha} dx^\alpha,$$

we can find the induced metric via

$$ds^2_{\text{ind}} = dX^i dX^i = \frac{\partial X^i}{\partial \xi^\alpha} d\xi^\alpha \frac{\partial X^i}{\partial \xi^\beta} d\xi^\beta = g_{\alpha\beta} d\xi^\alpha d\xi^\beta,$$

where we have a 2×2 matrix

$$g_{\alpha\beta} = \frac{\partial X^i}{\partial \xi^\alpha} \cdot \frac{\partial X^i}{\partial \xi^\beta} = \frac{\partial \vec{X}}{\partial \xi^\alpha} \cdot \frac{\partial \vec{X}}{\partial \xi^\beta} \implies g_{\alpha\beta} = \begin{bmatrix} \frac{\partial \vec{X}}{\partial \xi^1} \cdot \frac{\partial \vec{X}}{\partial \xi^1} & \frac{\partial \vec{X}}{\partial \xi^1} \cdot \frac{\partial \vec{X}}{\partial \xi^2} \\ \frac{\partial \vec{X}}{\partial \xi^2} \cdot \frac{\partial \vec{X}}{\partial \xi^1} & \frac{\partial \vec{X}}{\partial \xi^2} \cdot \frac{\partial \vec{X}}{\partial \xi^2} \end{bmatrix}.$$

In particular, if we now go back to the expression for the area A, we find that there is a geometrically transparent way of understanding how the area and the induced metric relate to each other:

$$\boxed{A = \int d\xi^1 d\xi^2 \sqrt{\det g_{\alpha\beta}} = \int d^2\xi \sqrt{g}}$$

This result turns out to extend to other numbers of dimensions as well – we can find volumes of embedded spaces by looking at the determinant of the induced metric in general! And it turns out this number is reparameterization invariant (which is good, because the area should not depend on the way we parameterize the surface). To check that, suppose that we write $\xi^\alpha = \xi^\alpha(\tilde{\xi}^\beta)$ in terms of some other parameters $\tilde{\xi}^\beta$, so that

$$\boxed{\vec{X}(\xi^\alpha) = \vec{X}'(\tilde{\xi}^\alpha)}$$

(the new embedding function should map to the same point as the original embedding). Then the measure transforms via

$$d\xi^1 d\xi^2 = \left|\det\left(\frac{\partial \xi^i}{\partial \tilde{\xi}^j}\right)\right| d\tilde{\xi}^1 d\tilde{\xi}^2,$$

where the matrix M given by $M_{ij} = \frac{\partial \xi^i}{\partial \tilde{\xi}^j}$ is the Jacobian matrix. And meanwhile, our induced metric transforms as

$$g_{\alpha\beta} = \frac{\partial \vec{X}}{\partial \xi^\alpha} \cdot \frac{\partial \vec{X}}{\partial \xi^\beta} = \left(\frac{\partial \vec{X}'}{\partial \tilde{\xi}^\gamma} \frac{\partial \tilde{\xi}^\gamma}{\partial \xi^\alpha}\right) \cdot \left(\frac{\partial \vec{X}'}{\partial \tilde{\xi}^\delta} \frac{\partial \tilde{\xi}^\delta}{\partial \xi^\beta}\right)$$

where we're using the chain rule and the boxed equation above. Everything here is numbers, so we can rearrange terms to write this as

$$= g'_{\gamma\delta}(\tilde{\xi}) \frac{\partial \tilde{\xi}^\gamma}{\partial \xi^\alpha} \frac{\partial \tilde{\xi}^\delta}{\partial \xi^\beta}.$$

So if we take determinants of both sides, we find that

$$\det g = \det g' \left|\det \frac{\partial \tilde{\xi}}{\partial \xi}\right|^2 \implies \sqrt{g} = \sqrt{g'} \left|\det \frac{\partial \tilde{\xi}}{\partial \xi}\right| = \sqrt{g}(\det M)^{-1}.$$

So substituting everything back into the area formula indeed gives us back the same A, as desired, and we've shown reparameterization invariance.

Example 33

Now that we've finished our preparation (in Euclidean space), we'll look at the action for a relativistic string in Minkowski space.

We basically need to generalize what we did into a Lorentzian signature, and the method for that is essentially to be careful with locations of indices. If we have our parameters $\xi^\alpha = (\xi^0, \xi^1)$, we can use the parameter $\xi^1 = \sigma$ to point "along the string" and the parameter $\xi^0 = \tau$ to essentially represent the string's evolution through spacetime. (We'll use these coordinates interchangeably.)

Fact 34

By convention, we take $\sigma \in [0, \pi]$ when we have an **open string**, so that $\sigma = 0$ and $\sigma = \pi$ are endpoints. (We can then define an orientation for the string by taking σ from 0 to π.) On the other hand, closed strings have $\sigma \in [0, 2\pi]$ with periodic boundary conditions. (The other variable τ always takes on values in $(-\infty, \infty)$.)

So now we can consider the surface Σ described by $X^\mu(\tau, \sigma)$ that's embedded in our spacetime: we do the same thing as above, finding the induced metric on Σ and calculating the determinant to find the proper area. Then our induced metric is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \implies ds^2|_\Sigma = \eta_{\mu\nu} dX^\mu dX^\nu = \eta_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta} d\xi^\alpha d\xi^\beta$$

(basically the same as above but with an extra $\eta_{\mu\nu}$), and we define this to be

$$= \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta.$$

One tricky point to note, though, is that if we take an infinitesimal area element in our worldsheet, one direction should be timelike and the other should be spacelike. So the induced metric $\gamma_{\alpha\beta}$ on the worldsheet Σ **will also be Lorentzian**: it has one spacelike and one timelike eigenvector, and $\gamma = \det \gamma_{\alpha\beta} < 0$. So then the proper area on our worldsheet will look like

$$\boxed{A = \int d^2\xi \sqrt{-\gamma}}$$

(since the determinant will be smaller than 0).

We can check that this is reparameterization-invariant: we basically follow the same argument as before, also using the fact that

$$\gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X_\mu}{\partial \xi^\beta}$$

(one lower and one upper index to get rid of the η) is manifestly Lorentz-invariant, so its determinant is also Lorentz invariant. And this expression only contains first-order derivatives (as we want), so this proper area does indeed satisfy all of our desired properties! We'll thus define the action to be some constant times this proper area, and we take the following convention:

Proposition 35 (Nambu–Goto action)

The action of a relativistic string is defined as

$$S_{\text{NG}} = -\frac{T_0}{c} \int_\Sigma d^2\xi \sqrt{-\gamma}.$$

Remember that the particle action looked like $S_{\text{particle}} = -mc^2 \int d\ell$, and the negative sign was needed so that the answer agreed with the non-relativistic story. And **heuristically**, because we can think of a string (worldsheet) as a collection of particles (worldlines), **this is why we have a negative sign** in the action above! (And this can be justified by taking the non-relativistic limit of the string action.) Also, we know that the dimension of the action is $\frac{[M][L]^2}{[T]}$, so if we plug in the dimensions of c and of the proper area, we find that we must have $[T_0] = \frac{[E]}{[L]}$. So this T_0 can be physically interpreted as the **energy per unit length** of the string.

This is not the only possible string action, but it is the simplest one and one of the first that was written down!

Fact 36

From now on, we will take $c = 1$, and in these units we have $[T] = [L]$ and $[T_0] = [E] = \frac{[M]}{[L]}$.

Just like with the previous action that we wrote down, square roots are often annoying to deal with, so we will introduce a new dynamical variable to get rid of it here:

Proposition 37 (Polyakov action)

The string action can also be written in terms of the (symmetric) variable $h_{\alpha\beta}(\xi^\gamma)$ via

$$S_p = -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\,\gamma_{\alpha\beta}h^{\alpha\beta} = -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\,h^{\alpha\beta}\eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}.$$

(The action is now polynimial in X^μ. The square root for h will not matter because we will see that fixing a gauge makes that expression very simple!)

Fact 38

By convention, we denote $h^{\alpha\beta}$ to be the inverse matrix of $h_{\alpha\beta}$. In particular, the contraction $h^{\alpha\gamma}h_{\gamma\beta} = h_{\beta\gamma}h^{\gamma\alpha} = \delta^\alpha_\beta$ gives the identity.

Before showing that this is the same as the Nambu-Goto action, we should comment on what the $h_{\alpha\beta}$ means: it is an **intrinsic metric on the worldsheet**. This interpretation should be obvious for those who have studied general relativity before. For those who have not studied GR before, here is a heuristic way to understand it. We know that X^μ, X^ν coordinates represent our physical spacetime, and those are being contracted with $\eta_{\mu\nu}$ (which represents the metric in that physical spacetime). So if $h^{\alpha\beta}$ is being contracted with the ξ^α, ξ^β coordinates, it makes sense to interpret $h_{\alpha\beta}$ as the metric on the worldsheet. Basically, by introducing $h_{\alpha\beta}$ we're making the parameter space curved!

Proof of equivalence of actions. The equation of motion we get from $h_{\alpha\beta}$ when we keep X^μ fixed can be obtained by adding a small variation $\delta h_{\alpha\beta}$ and applying the action principle. It's equivalent to vary $h^{\alpha\beta}$, and that'll be more convenient for us here.

When $h^{\alpha\beta}$ varies, the determinant varies as well, and this is a bit tricky: it turns out that

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\,h_{\alpha\beta}\delta h^{\alpha\beta}.$$

So the variation of S_p looks like (by the product rule)

$$\delta S_p = -\frac{T_0}{2}\int d^2\xi\left[\sqrt{-h}\,\gamma_{\alpha\beta}\delta h^{\alpha\beta} - \frac{1}{2}\sqrt{-h}\,h_{\alpha\beta}\delta h^{\alpha\beta}h^{\lambda\delta}\gamma_{\lambda\delta}\right].$$

In order for this to always be zero, we find that our equation of motion is

$$\gamma_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}\left(h^{\lambda\delta}\gamma_{\lambda\delta}\right) = 0.$$

This equation can now be solved because the $\left(h^{\lambda\delta}\gamma_{\lambda\delta}\right)$ term is fully contracted (and is thus just some function). And it turns out that we can have an arbitrary function there:

$$h_{\alpha\beta} = f(\xi)\gamma_{\alpha\beta}$$

is a solution for any function $f(\xi)$ (the contributions will cancel in the initial expression), as long as we don't change the signature of the metric. So it's basically okay to say that $f(\xi)$ can be any positive function here.

This form of $h_{\alpha\beta}$ tells us that (remembering that upper indices mean inverse matrices)

$$\sqrt{-h} = f(\xi)\sqrt{-\gamma}, \quad h^{\alpha\beta} = f^{-1}(\xi)\gamma^{\alpha\beta},$$

and thus plugging things back into the action yields

$$S_p = -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\,\gamma_{\alpha\beta}h^{\alpha\beta} = -\frac{T_0}{2}d^2\xi\sqrt{-\gamma}\,ff^{-1}\gamma_{\alpha\beta}\gamma^{\alpha\beta},$$

and now $\gamma_{\alpha\beta}\gamma^{\alpha\beta} = 2$ because we have a 2×2 matrix. So everything simplifies back into

$$= -T_0\int d^2\xi\sqrt{-\gamma},$$

which is indeed the Nambu-Goto action, as desired. $\square$

March 10, 2021

Last time, we introduced the concept of a **worldsheet**, which is a two-dimensional surface Σ embedded inside spacetime via the parameterization $X^\mu(\xi^\alpha)$. The embedding gives us an **induced metric** on the worldsheet

$$ds^2|_\Sigma = \eta_{\mu\nu}dX^\mu dX^\nu,$$

and to do computations on the worldsheet we should convert this to a differential expression involving the worldsheet coordinates:

$$= \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}d\xi^\alpha d\xi^\beta = \gamma_{\alpha\beta}d\xi^\alpha d\xi^\beta,$$

where $\gamma_{\alpha\beta} = \eta_{\mu\nu}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta} = \frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X_\mu}{\partial\xi^\beta}$ is a 2×2 matrix that tells us how to measure distances and proper times along the worldsheet. With this, we can define the string action

$$S_{\text{string}} = -T_0\int d^2\xi\sqrt{-\gamma},$$

which (as proved last lecture) we can equivalently write as

$$= -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\,\gamma_{\alpha\beta}h^{\alpha\beta},$$

where $h_{\alpha\beta}$ is a new (symmetric) dynamical variable that we introduced, meant to be interpreted as an intrinsic metric on the space ξ^α. (Here, remember that $h^{\alpha\beta}$ in the expression above is the inverse matrix of $h_{\alpha\beta}$.) Varying this alternate form of the action gave us the equation of motion for $h_{\alpha\beta}$, $\gamma_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}\gamma_{\lambda\rho}h^{\lambda\rho} = 0$, which can be solved by $h_{\alpha\beta} = f(\xi)\gamma_{\alpha\beta}$ for an arbitrary positive function f. Plugging this back in results in the original action.

Example 39

As we've done with other actions in this class already, we'll try to study the symmetries of the string action and use them to extract some structure.

Plugging the explicit form of $\gamma_{\alpha\beta}$ we have

$$S = -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\,h^{\alpha\beta}\frac{\partial X^\mu}{\partial\xi^\alpha}\frac{\partial X^\nu}{\partial\xi^\beta}\eta_{\mu\nu}\ .$$

Let us first examine the global symmetries of the above action. We can first see that (just like in the particle case) we have translation symmetry: if we take $X^\mu \mapsto X^\mu + a^\mu$, the derivatives that appear in S do not change. Also, all of our indices are contracted, so (again, like the particle case) this action is Lorentz-invariant because the ξ-derivatives do not act on constant Lorentz matrices. So the global symmetries are the same as the particle case.

We now consider the local symmetries: we again have the reparameterization symmetry discussed last time, which comes from a coordinate transformation of the form

$$\sigma, \tau \mapsto \sigma'(\tau, \sigma), \tau'(\tau, \sigma)$$

(which can alternatively written as $\xi^\alpha \mapsto \xi'^\alpha(\xi^\beta)$), so that the corresponding embedding transform $X^\mu \mapsto X'^\mu$ satisfies $X'^\mu(\xi') = X^\mu(\xi)$. So now if we want to think about how h transforms under this reparameterization, one thing we can do is look at the action, understanding how to make h transforms given how X and ξ transform and knowing that S needs to stay constant.

But another thing we can do is see how γ transforms, using the fact that h is proportional to γ, and that's what we'll do here. Remembering that γ tells us about the induced metric,

$$ds^2|_\Sigma = \gamma_{\alpha\beta} d\xi^\alpha d\xi^\beta = \gamma'_{\alpha\beta}(\xi^1) d\xi'^\alpha d\xi'^\beta.$$

Applying the chain rule yields

$$\gamma'_{\alpha\beta}(\xi') = \gamma_{\delta\lambda}(\xi) \frac{\partial \xi^\delta}{\partial \xi'^\alpha} \frac{\partial \xi^\lambda}{\partial \xi'^\beta}.$$

So now that we know how $\gamma'_{\alpha\beta}$ transforms, we can find that

$$h'_{\alpha\beta}(\xi') = h_{\delta\lambda}(\xi) \frac{\partial \xi^\delta}{\partial \xi'^\alpha} \frac{\partial \xi^\lambda}{\partial \xi'^\beta}.$$

It will be left to you as an exercise to check explicitly that the Polyakov action is indeed reparameterization invariant under the above transformation (it's an instructive exercise to get used to this kind of tensor analysis).

And remember that this was the only local symmetry that existed for the particle, but it turns out there's another one for the string: we mentioned that the function $f(\xi)$ can be taken to be arbitrary, because it always canceled out in the action. So this gives us the **Weyl scaling**: if we consider a transformation

$$X^\mu \mapsto X^\mu, \quad h_{\alpha\beta} \mapsto e^{2\sigma(\xi)} h_{\alpha\beta},$$

where the $e^{2\sigma(\xi)}$ term is some arbitrary positive function, that implies that the terms in the action transform as

$$h^{\alpha\beta} = e^{-2\sigma(\xi)} h^{\alpha\beta} \implies \sqrt{-h} \mapsto e^{2\sigma(\xi)} \sqrt{-h}.$$

And because $h^{\alpha\beta}$ and $\sqrt{-h}$ each show up once in the action, their changes cancel out! So we do have a new local symmetry here.

Fact 40

Because this local symmetry only affects h and not X, it is not present in the Nambu-Goto action. And in order to actually extract the genuine physical degrees of freedom, we need to fix all of these symmetries to avoid overcounting. But we'll talk about that soon.

2.3: Equations of motion and boundary conditions

We're now ready to follow a similar strategy as in the particle case: we'll derive the equations of motion, then fix the gauge so that those equations of motion are easy to solve, and finally solve the equations in that particular gauge.

Example 41

First of all, let's consider **closed strings**, where the σ coordinate is parameterized by $\sigma \in [0, 2\pi]$ and our embedding must satisfy $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$ for all τ, σ (and our $h_{\alpha\beta}$ must be similarly periodic).

We can first get an equation by varying $h^{\alpha\beta}$, which have already done: remember that the resulting equation $\gamma_{\alpha\beta} - \frac{1}{2}h_{\alpha\beta}\gamma_{\lambda\rho}h^{\lambda\rho} = 0$ is actually three different equations, and we can write out those equations explicitly as

$$\boxed{\partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}h_{\alpha\beta}h^{\lambda\rho}\partial_\lambda X^\nu \partial_\rho X_\nu = 0}$$

(where α, β range over $\{0, 1\}$). We solved this equation and then plugged in h back into the action, but remember that an alternative strategy is to keep this equation as is and fix the gauge for h later.

So now let's turn to the equation of motion for X^μ: we find that varying X^μ gives us

$$\delta S = -\frac{T_0}{2} \cdot 2 \int d^2\xi \sqrt{-h}\ h^{\alpha\beta}\partial_\alpha \delta X^\mu \partial_\beta X_\mu.$$

Now an integration by parts splits this into two terms:

$$= -T_0 \int d^2\xi \left[\partial_\alpha(\sqrt{-h}\ h^{\alpha\beta}\partial_\beta X_\mu \delta X^\mu) - \partial_\alpha(\sqrt{-h}\ h^{\alpha\beta}\partial_\beta X_\mu)\delta X^\mu\right].$$

All of the α indices here are summed from 0 to 1, so ∂_α either means ∂_σ or ∂_τ. But we can always assume the ∂_τ total derivative term is zero (this is like fixing the initial and final conditions for the particle case), and the ∂_σ derivative term goes away by periodicity! So the blue boundary term above is zero, and thus the condition for stationary action is

$$\boxed{\partial_\alpha\left(\sqrt{-h}\ h^{\alpha\beta}\partial_\beta X_\mu\right) = 0}.$$

Example 42

Next, let's look at the open string case, where we have boundaries in the σ direction.

The first boxed equation from the closed-string case is still the same, since we haven't done any integration by parts, but the variation of X^μ now gives us a total derivative term in the ∂_σ direction which we cannot ignore! So what we find this time is that we get an extra term

$$\delta S = \int d\tau \int_0^\pi d\sigma\ \partial_\sigma(\sqrt{-h}\ h^{1\beta}\partial_\beta X_\mu \delta X^\mu) - \partial_\alpha\left(\sqrt{-h}\ h^{\alpha\beta}\partial_\beta X_\mu\right),$$

since we're taking σ to be the ξ^1 coordinate. Now notice that one of these two terms is a total derivative evaluated at the endpoints, while the other depends on the entire X^μ trajectory. So for the open string, we find that we have a third equation that also needs to be satisfied **in addition** to the first two if we want δS to always be zero:

$$\boxed{\sqrt{-h}\ h^{1\beta}\partial_\beta X_\mu \delta X^\mu\Big|_{\sigma=\pi} - \sqrt{-h}\ h^{1\beta}\partial_\beta X_\mu \delta X^\mu\Big|_{\sigma=0} = 0}.$$

And in fact, because the two ends of the open string should be able to move freely on their own (we don't want to impose correlations), we will require both of these two terms to separately be zero. In other words, we have

$$\sqrt{-h}\ h^{1\beta}\partial_\beta X_\mu \delta X^\mu\Big|_{\sigma=0,\pi} = \Pi^1_\mu \delta X^\mu\big|_{\sigma=0,\pi} = 0.$$

Remember that μ is summed here over all of the spacetime indices, so there are d terms there. But the motion of the endpoints **in each direction of spacetime** should also be independent, and thus we now say that the condition is

$$\boxed{\Pi^1_\mu \delta X^\mu\Big|_{\sigma=0,\pi} = 0 \text{ for each index } \mu}.$$

In other words, for each given μ (and setting $\sigma = 0$ or π), this means that either $\boxed{\delta X^\mu|_{\sigma=\sigma_*} = 0}$ (the **Dirichlet** boundary condition) or $\boxed{\Pi^1_\mu|_{\sigma=\sigma_*} = 0}$ (the **Neumann** boundary condition). In other words, there are four possible boundary conditions for each spacetime coordinate μ, and thus there are 4^d total boundary conditions overall.

Let's understand a little more about what this means: suppose that $\delta X^\mu|_{\sigma=0} = 0$. Then because X^μ is a function of τ and σ, we're saying that X^μ needs to be a constant function of τ (...secretly it's possible for $f(\tau)$ to be a **fixed function**, but we'll talk about that later). So that means that the motion of the **string end point** in the X^μ direction does not change. On the other hand, if $\Pi^1_\mu = 0$ for a given μ, we're saying that some combination of derivatives of the X_μs must be zero (but the value of X_μ itself is not fixed, so it can be anything in principle). It'll turn out that the Neumann distribution tells us that the **momentum flow out of the string is zero**.

We can now think more specifically about the $\mu = 0$ coordinate: for example, let's think about what the Dirichlet boundary condition

$$X^0(\tau)\Big|_{\sigma=0} = \text{const.}$$

But this doesn't actually make sense: X^0 is supposed to be the time-coordinate (since we took $c = 1$), and we know that time cannot stop. So we need the Neumann bounadry conditions for $\mu = 0$ (for both $\sigma_* = 0, \pi$):

$$\Pi^1_0\Big|_{\sigma_*} = 0.$$

That still leaves 4^{d-1} possible boundary conditions for the other $(d-1)$ dimensions, but now let's suppose that boundary conditions also respect Lorentz symmetry. Then time and space are put on equal footing (we can choose an appropriate Lorentz transformation), which implies that we must have Neumann boundary conditions everywhere:

$$\boxed{\Pi^1_\mu\Big|_{\sigma=\sigma_*} = 0}.$$

This was an acceptable answer for about 30 years or so, but we can now ask whether Dirichlet boundary conditions can make sense at all (for some spatial directions, for example). And it turns out the answer is yes. Let us consider a simple example:

Example 43

Suppose we're working in $d = 3+1$ dimensional space, so that $x^\mu = (x^0, x^1, x^2, x^3)$. and we impose the Neumann boundary condition in the $0, 1, 2$ directions but the Dirichlet condition in the 3 direction (meaning $X^3\Big|_{\sigma_*} = 0$).

This basically says that the string's two endpoints can take arbitrary positions in the x^1 and x^2 directions, but those endpoints must be fixed in the x^1x^2-plane: both endpoints have $x^3 = 0$. This generally violates Lorentz symmetry, because it violates boosts and rotations in the x^3 direction, unless there is a physical object that actually holds the string endpoints there (in other words, there must be some physical object at $x^3 = 0$). And in those situations, it's not a big deal anymore that the symmetry is violated, but we still need a way to interpret such physical objects. The answer turns out to be yes, but we won't get into the details for now. We'll instead work with the following postulates:

Proposition 44

1. Strings exist naturally as closed strings.
2. Open strings can only have endpoints on special objects, which we will call **D-branes**. Furthermore, closed strings can break open at the location of a D-brane.

We'll explain more about how these D-branes give us physical explanations for all 4^{d-1} boundary conditions next time – they just correspond to different configurations of the D-branes!

March 15, 2021

Last lecture, we considered the Polyakov action for the string

$$S_p = -\frac{T_0}{2} \int d^2\xi \sqrt{-h}\; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\mu,$$

where we have the two dynamical variables h and X with equation of motion from h given by

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} h_{\alpha\beta} h^{\lambda\delta} \partial_\lambda X^\nu \partial_\delta X_\nu = 0.$$

(If we've studied general relativity, $T_{\alpha\beta}$ is the **stress tensor** on the worldsheet.) We also have the equation of motion from X given by

$$\partial_\alpha (\sqrt{-h}\; h^{\alpha\beta} \partial_\beta X_\mu) = 0.$$

Last time, we discussed boundary equations for the open and closed string: for a closed string, we have periodicity constraints

$$h_{\alpha\beta}(\tau, \sigma + 2\pi) = h_{\alpha\beta}(\tau, \sigma), \quad X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma).$$

But there are four possible boundary conditions for each μ in the open string case, since each endpoint can either have Neumann boundary conditions $\Pi^1_\mu|_{\sigma=\sigma_*} = 0$ or Dirichlet boundary conditions $\delta X^\mu|_{\sigma_*} = 0$. We know that the time dimension $\mu = 0$ can only have Neumann boundary conditions (because time always flows), so there are 4^{d-1} possible boundary conditions left.

One special case is that we have Neumann boundary conditions in all directions for both ends. It is the only one that respects Lorentz symmetry. But we can actually make sense of the Dirichlet boundary conditions by introducing the concept of a D-brane:

Definition 45
Let p be an integer. A **D*p*-brane** is an object with p spatial dimensions on which an open string can end.

Example 46
Suppose $d = 4$ (so that $x^\mu = (x^0, x^1, x^3, x^3)$), and consider the Neumann boundary condition $\Pi^1_{0,1,2}|_{\sigma=0,\pi} = 0$ but use the Dirichlet boundary condition $X^3|_{\sigma=0,\pi} = 0$.

Geometrically, this means that our string moves in 3-dimensional space over time, and there are no constraints on the location of the endpoint in the x- or y-direction, but the two endpoints of our string must end in the xy-plane with

$z = 0$. (It's okay for the body of the string to be at any z-coordinate, though.) Then this string ends on a **D2-brane** which sits at $x^3 = 0$, and the breaking of Lorentz symmetry is consistent with physics.

Example 47

Suppose that we have Neumann boundary conditions in the time direction (so that $\Pi^1_0|_{\sigma_*} = 0$), but Dirichlet boundary conditions in the other directions $X^i|_{\sigma^*} = 0$ for all $i \in \{1, 2, \cdots, d-1\}$.

This means that both endpoints of the string end at the origin, which makes it look like an oriented loop. And thus there is a **D0-brane** sitting at $x^i = 0$ in this case.

Example 48

Finally, suppose that we have Neumann boundary conditions in all directions, and there are no restrictions on where the endpoints can lie in space. In other words, $\Pi^1_\mu|_{\sigma_*} = 0$.

Then we have a space-filling **D$(\mathbf{d-1})$-brane**, since both endpoints of the string can be anywhere in our space.

Example 49

More generally, suppose that we have Neumann boundary conditions $\Pi^1_{0,1,\cdots,p}|_{\sigma_*} = 0$, but we have Dirichlet boundary conditions $X^{p+1,\cdots,d-1}|_{\sigma=0} = \vec{a} = (a^{p+1}, \cdots, a^{d-1})$ and $X^{p+1,\cdots,d-1}|_{\sigma=0} = \vec{b}$.

Now, we have a string stretched between two planes, and there must be a D-brane at each of the two planes that force our string endpoints on the appropriate coordinates. But if we are just told about the D-brane configuration, **there are other strings that can be drawn** (for example, it's also possible to have strings that have both endpoints living on one of the two D-branes).

Example 50

Finally, take $d = 3$, and consider the configuration where $\Pi^1_{0,1}|_{\sigma=0} = 0$, and $X^2|_{\sigma=0} = a$. But for the other string endpoint, we have $\Pi^1_0|_{\sigma=\pi} = 0$ and $X^{1,2}|_{\sigma=\pi} = (b_1, b_2)$.

This means that the $\sigma = 0$ endpoint can freely move only in the 0 and 1 directions, and the $\sigma = \pi$ endpoint cannot freely move at all. So if we were to draw the (spatial component) of the D-branes, we'd have one endpoint constrained to the point (b_1, b_2), and the other endpoint constrained to a line. Thus, this setup means that we have a D1-brane and a D0-brane, and an open string is stretched between the two D-branes.

Remark 51. *Using this kind of argument, we can convince ourselves that any collection of boundary conditions corresponds to some collection of D-branes. And note that any Dp-brane should be actually understood as a $(p+1)$-dimensional object (in p space dimensions and 1 time dimension). As we will discuss later it is a dynamical object and thus automatically fill all of the time dimension.*

These D-branes may mostly seem like they are for mathematical bookkeeping right now – they help us interpret the boundary conditions, but they don't seem to be very dynamical at the moment. Similar statement can be made about the spacetime, which is non-dynamical at the moment. But after we quantize the string, and we'll find that both the spacetime and D-branes will become dynamical!

Remark 52. *Before we solve the equations of motion, notice that we've been using the Polyakov action, and we could have also tried using the Nambu-Goto action $S_{NG} = \int d^2\xi\sqrt{-\gamma}$. The derivation becomes a little bit more complicated,* *and the analysis requires more mathematical ingenuity, but it may be more intuitive to deal with the string embedding directly without the extra dynamical variable h. So if we want to consider that perspective, we can read Chapter 6 in Professor Zwiebach's textbook.*

To simplify our lives for the equations of motion, we'll first fix a gauge to make the expressions less complicated. Recall that in the **Nambu-Goto** action, the local symmetries are all reparameterizations: $\tau \to \tau'(\sigma, \tau)$ and $\sigma \to \sigma'(\sigma, \tau)$ are two independent reparameterization degrees of freedom, and we define our dynamical variables as $X^\mu(\sigma, \tau)$ (where there are d indices μ). So in principle, reparameterization gives us $(d-2)$ independent degrees of freedom there. A common choice (though the choice depends on the particular problem we're solving) is to choose $\tau = \lambda X^0 = \lambda t$ for some constant λ, usually 1 – this is called the **static gauge**. But that only fixes one of the degree of freedom, so there is still a choice for σ. One example that will come up in our problem set is to consider a string with Neumann boundary conditions except for

$$X^1|_{\sigma=0} = 0, \quad X^1|_{\sigma=\pi} = a,$$

meaning that our string is stretching between two surfaces. For that kind of problem, it then makes sense to pick $\sigma = \lambda' x^1$.

But we're solving with the **Polyakov** action, not the Nambu-Goto action, and the situation isn't exactly the same there. We still have the local symmetries $\tau \to \tau'(\sigma, \tau)$ and $\sigma \to \sigma'(\sigma, \tau)$, but we also have the Weyl symmetry

$$h_{\alpha\beta} \to e^{2f(\sigma,\tau)} h_{\alpha\beta}$$

for an arbitrary function f. So there are three independent functions of τ and σ here. Let us now count our dynamical degrees of freedom. We know that $h_{\alpha\beta}(\sigma, \tau)$ has three components (since $\alpha, \beta \in \{0, 1\}$ and $h_{\alpha\beta}$ is symmetric), and X^μ has d degrees of freedom, so the $(d+3)$ degrees of freedom, minus the 3 degrees from the symmetries above, naively gives us d degrees of freedom in our system. But there are additional constraints here, **because we have extra equations of motion** in the Polyakov action! Indeed, the extra $h_{\alpha\beta}$ equations of motion turn out to have 2 independent equations (we'll see why soon), and thus the counting works out as we want.

Fact 53

We won't go through the mathematical details, but we should be convinced (other than some fine topological details) that two of the reparameterizations, plus one Weyl symmetry, can simplify

$$h_{\alpha\beta} = \eta_{\alpha\beta} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(Not all of the local symmetries are fixed yet, but we'll talk about the residual symmetries later.) And now that $h_{\alpha\beta}$ is a constant matrix (so that $\sqrt{-h} = 1$), we can look at our equations of motion again. The equation from X^μ gives us

$$\eta^{\alpha\beta} \partial_\alpha \partial_\beta X_\mu = 0 \implies \boxed{\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0}.$$

In other words, we've arrived at a dynamical wave equation (which we already know how to solve). Similarly, the equation of motion from h gives us

$$T_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2} \eta_{\alpha\beta} \eta^{\lambda\delta} \partial_\lambda X^\nu \partial_\delta X_\nu = 0,$$

and if we write out all of the indices, we find that

$$T_{11} = T_{00} = \boxed{\frac{1}{2}\left(\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu\right) = 0}, \quad T_{01} = \boxed{\dot{X}^\mu X'_\mu = 0},$$

where we're using the notation as if τ, σ are "time" and "space" variables:

$$\dot{X}^\mu = \partial_\tau X^\mu, \quad X'^\mu = \partial_\sigma X^\mu.$$

And because the last two equations here only contain first-order time derivatives, they're **constraint equations**. In the current case, the two equations are often referred to as the **Virasoro constraints** (while the wave equation above is a **dynamical equation**).

If we return to the Neumann boundary conditions for the open string now, we can simplify the equation $0 = \Pi^1_\mu|_{\sigma_*} = \sqrt{-h}\ h^{1\beta}\partial_\beta X_\mu$ as well: we find that

$$\boxed{\partial_\sigma X_\mu|_{\sigma_*} = 0}.$$

Furthermore, we can return to the Polyakov action itself and find that we now have

$$S_p = -\frac{T_0}{2}\int d^2\xi \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

with contraction done by the two-dimensional Minkowski metric! And indeed doing variations of this simplified action gives us the dynamical equation, but notice that it misses the constraint equations that come from $h_{\alpha\beta}$.

We'll now find the general classical solutions for the system. This is not only for the purpose of understanding classical motions of a string, but in fact we'll see that this plays an important role also for obtaining the quantum dynamics of a string. First of all, we solved the wave equation in 8.03: since the X^μ solutions are all decoupled, let's just call the coordinate X, and then the solutions look like

$$X(\tau, \sigma) = X_R(\sigma^-) + X_L(\sigma^+),$$

where $\sigma^\pm = \tau \pm \sigma$ and x_R, x_L are arbitrary one-variable functions (x_R describes the right-moving wave, and x_L describes the left-moving wave). Then we need to impose boundary conditions in one of a few ways, depending on the setup.

For a closed string, we must have

$$X(\tau, \sigma + 2\pi) = X(\tau, \sigma),$$

but this doesn't necessarily mean X_L and X_R need to be separately periodic with period 2π. Instead, we need

$$X_R(\sigma^- - 2\pi) + X_L(\sigma^+ + 2\pi) = X_R(\sigma^-) + X_L(\sigma^+),$$

which implies that

$$X_R(\sigma^- - 2\pi) - X_R(\sigma^-) = X_L(\sigma^+) - X_L(\sigma^+ + 2\pi),$$

and the left-hand side is a function of σ^-, while the right-hand side is a function of σ^+. Since those two are independent variables, this means that **both sides must be constant**, and thus

$$X'_R(\sigma^- - 2\pi) - X'_R(\sigma^-) = 0 \implies X'_R \text{ periodic.}$$

In symbols, this means that $\boxed{X_R(\sigma^-) = \frac{v}{2}\sigma^- + f_R(\sigma^-)}$ for a periodic function f_R and a constant v. Similarly, we can find that $\boxed{X_L(\sigma^+) = \frac{v}{2}\sigma^+ + f_L(\sigma^-)}$. And here, we can decompose into Fourier modes as

$$f_R = \tilde{f}_0 + \sum_{n\neq 0} \tilde{f}_n e^{-in\sigma^-}, \quad f_L = f_0 + \sum_{n\neq 0} f_n e^{-in\sigma^+},$$

and finally add everything back up to get (combining the $f_0, \tilde{f}_0$, and $\frac{v}{2}\sigma^-$ and $\frac{v}{2}\sigma^+$ all together in the front, using that $\sigma^+ + \sigma^- = 2\tau$)

$$\boxed{X^\mu(\sigma, \tau) = X_0^\mu + v^\mu\tau + \sum_{n\neq 0}\left(f_n e^{-in\sigma^+} + \tilde{f}_n e^{-in\sigma^-}\right)}.$$

And using the fact that X^μ must be real, we also find that $\boxed{f_{-n}^\mu = (f_n^\mu)^*}$ and $\boxed{\tilde{f}_{-n}^\mu = (\tilde{f}_{-n}^\mu)^*}$.

We can interpret this solution as follows: we find that

$$\frac{1}{2\pi}\int_0^{2\pi} d\sigma X^\mu(\sigma, \tau) = X_0^\mu + v^\mu(\tau)$$

tells us about the **center-of-mass motion** for the string, and the rest of the terms are the oscillatory terms (in other words, they are the different oscillatory modes of the string). Remember that $X_0^\mu, v^\mu, f_n, \tilde{f}_n$ are all integration constants, and they are independent with respect to the dynamical wave equation, but the constraint equations will relate the coefficients to each other. We'll look at this, as well as how to solve the open-string case, next time!

March 17, 2021

Last lecture, we fixed the gauge for the string action by setting $h_{\alpha\beta} = \eta_{\alpha\beta}$, so that the intrinsic metric on the worldsheet is given by

$$ds_\Sigma^2 = h_{\alpha\beta}d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta}d\xi^\alpha d\xi^\beta = -d\tau^2 + d\sigma^2;$$

that is, we have two-dimensional Minkowski spacetime. Our action will simplify significantly to

$$S_p = -\frac{T_0}{2}\int d^2\xi\ \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

and the equations of motion for $h^{\alpha\beta}$ become

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^\mu \dot{X}_\mu + X'^\mu + X'_\mu) = 0, \quad T_{01} = \dot{X}^\mu X'_\mu = 0$$

(where X'^μ is the σ-derivative of X^μ, and $\dot{X}^\mu$ is the τ-derivative of X^μ). The equation of motion for X^μ then tells us that

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0,$$

and notice that this means the wave is traveling at the speed of light $c = 1$.

Last time, we solved this wave equation and found that the most general solution is

$$X^\mu(\tau, \sigma) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+),$$

where $\sigma^\pm = \tau \pm \sigma$. But there are additional constraints we have as well: for example, for the closed string, from that $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$, we found last time that we must have the form

$$X^\mu(\tau, \sigma) = X_0^\mu + v^\mu\tau + \sum_{n\neq 0}\left(f_n^\mu e^{-in\sigma^+} + \tilde{f}_n^\mu e^{-in\sigma^-}\right),$$

where $(f_n^\mu)^* = f_{-n}^\mu$ and $\left(\tilde{f}_n^\mu\right)^* = \tilde{f}_{-n}^\mu$). The way to interpret this expression is that the first two terms are the center-of-mass motion, and the rest are vibrational modes of the string. So x_0^μ should be treated as a center-of-mass initial location, v^μ should be treated as the (covariant Lorentz) center-of-mass velocity, and $f_n^\mu, \tilde{f}_n^\mu$ are the left-moving and right-moving vibrational amplitudes for the nth harmonics of the string.

But there are still constraints that we need to impose, namely the **Virasoro constraints** from $h^{\alpha\beta}$'s equations of motion. These are nonlinear (quadratic) in X, and they couple all of the directions of the string together, so they can be pretty nasty to solve – they cannot be solved in closed form as stated.

Example 54

For now, let's first turn to the open string and understand how to impose those boundary conditions.

Recall that in the gauge where $h_{\alpha\beta} = \eta_{\alpha\beta}$, the Neumann boundary condition becomes $\partial_\sigma X|_{\sigma_*} = 0$, and for each μ we can have a mix of Neumann and Dirichlet boundary conditions. We'll first solve the case here where we have Neumann boundary conditions at both ends, which means that $\partial_\sigma X|_{\sigma=0,\pi} = 0$.

If we plug in $\sigma = 0$, we find that the condition required is

$$X_L'(\tau) - X_R'(\tau) = 0 \implies X_L = X_R + c$$

(remembering that X_L, X_R are functions of just one variable). If we then plug this into the equation for $\sigma = \pi$, letting $X_R = F$, we find that

$$-F'(\tau - \pi) + F'(\tau + \pi) = 0,$$

and thus F' must be a periodic function with period 2π. So again like last lecture, we find that $F(x) = \frac{v}{2}x + f(x)$, where x is a dummy index, v is some arbitrary constant, and f is some periodic function which we can now expand out by Fourier transform as $f_0 + \sum_{n\neq 0} \frac{f_n}{2} e^{-inx}$. So now the general form for our solution is

$$X_R(\sigma^-) = f_0 + \frac{v}{2}\sigma^- + \sum_{n\neq 0} \frac{f_n}{2} e^{-in\sigma^-},$$

and then X_L as a function of σ^+ only differs by a constant:

$$X_L(\sigma^+) = \tilde{f}_0 + \frac{v}{2}\sigma^+ + \sum_{n\neq 0} \frac{f_n}{2} e^{-in\sigma^+}.$$

Putting this all together, we get our wave equation solution

$$\boxed{X^\mu(\tau, \sigma) = X_0^\mu + v^\mu \tau + \sum_{n\geq 0} f_n e^{-in\tau} \cos(n\sigma)},$$

where $(f_n^\mu)^* = f_{-n}^\mu$ and we've used the fact that

$$\frac{f_n}{2}(e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}) = f_n e^{-in\tau} \cos(n\sigma).$$

This time, notice that we only have **one set of vibration modes**, compared to the two that showed up in the closed string.

Similarly, we can find the answers for other boundary conditions as well: for example, if we have two Dirichlet boundary conditions $X(\tau, 0) = a, X(\tau, \pi) = b$, the most general solution is given by

$$X(\tau, \sigma) = a + \frac{b-a}{\pi}\sigma + i\sum_{n\neq 0} f_n e^{-in\tau} \sin(n\sigma).$$

Notice that there is **no linear τ term** this time, because there is no linear motion like there was with Neumann boundary conditions or with the closed string. And this is because we're being told that our string is fixed on the two endpoints $x = a$ and $x = b$, so we can't expect any (center-of-mass) linear motion in the x-direction!

And again, we need to impose the Virasoro constraints in the open string case, and we still cannot solve those equations in closed form because of nonlinearities and coupling.

To solve the Virasoro constraints, we will need help from fixing the **residual gauge freedoms** that we mentioned last lecture.

Example 55

Looking at the worldsheet metric $dS_\Sigma^2 = -d\tau^2 + d\sigma^2$, our question is whether there is further freedom to redefine σ, τ, or perform the Weyl scaling and do so in a way that leaves the worldsheet metric invariant. (That would mean that we haven't fully fixed the gauge yet.)

The answer turns out to be yes, and the key is to rewrite

$$dS_\Sigma^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-.$$

If we now consider the transformation

$$\sigma^+ \to \tilde{\sigma}^+ = f(\sigma^+), \quad \sigma^- \to \tilde{\sigma}^- = g(\sigma^-)$$

for some arbitrary functions f, g, then the metric becomes

$$dS_\Sigma^2 = -d\tilde{\sigma}^+ d\tilde{\sigma}^- = f'(\sigma^+) g'(\sigma^-) d\sigma^+ d\sigma^-,$$

and now it looks like we have an extra prefactor, but this is an overall expression that can be removed by a further Weyl scaling, which gets us back to the original metric. And the reason that such freedoms exist is that even though we removed 3 degrees of freedom when turning $h_{\alpha\beta}$ to $\eta_{\alpha\beta}$, these new parameterizations don't remove further degrees of freedom – since they only depend on **specific** combinations of σ and τ, and they only depend on one rather than two variables, they are in some sense of measure zero. Thus, we can redefine σ and τ to

$$\tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) = \frac{1}{2}(f(\sigma + \tau) - g(\tau - \sigma)),$$

$$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2}(f(\sigma + \tau) + g(\tau - \sigma)),$$

for functions f and g of a single variable. But now this redefinition needs to respect the closed string's periodicity or the open string's endpoints!

Example 56

For example, let's look at the conditions for the closed string.

For a closed string, we need $\tilde{\tau}(\tau, \sigma + 2\pi) = \tilde{\tau}(\tau, \sigma)$ and $\tilde{\sigma}(\tau, \sigma + 2\pi) = \tilde{\sigma}(\tau, \sigma) + 2\pi$. Adding or subtracting these together tells us that

$$\tilde{\sigma}^\pm(\tau, \sigma + 2\pi) = \tilde{\sigma}^\pm(\tau, \sigma) \pm 2\pi,$$

and remember that $\tilde{\sigma}^\pm$ are defined as certain functions of $\sigma + \tau$ and $\tau - \sigma$: this leads us to

$$f(\sigma^+ + 2\pi) = f(\sigma^+) + 2\pi, \quad g(\sigma^- - 2\pi) = g(\sigma^-) - 2\pi.$$

So f and g need to be periodic functions up to linear shifts, and thus

$$f(\sigma^+) = \sigma^+ + \tilde{f}(\sigma^+)$$

for some periodic function $\tilde{f}$ (and we know that the coefficient in front of σ^+ is 1 because changing σ by 2π adds 2π to $\tilde{\sigma}$), and similarly

$$g(\sigma^-) = \sigma^- + \tilde{g}(\sigma^-).$$

Therefore, we can redefine our reparameterization in terms of periodic functions as

$$\boxed{\tilde{\tau} = \tau + \frac{1}{2}(\tilde{f}(\tau+\sigma) + \tilde{g}(\tau-\sigma))}$$

for periodic functions $\tilde{f}, \tilde{g}$.

Fact 57

Similarly, we can do a calculation for the open string, where the constraints are $\tilde{\sigma}(\tau, 0) = 0$ and $\tilde{\sigma}(\tau, \pi) = \pi$. We can then show that we must have $f = g$ for the parameterization to be valid, and thus we actually have $\tilde{\tau} = \tau + \frac{1}{2}(\tilde{f}(\tau+\sigma) + \tilde{f}(\tau-\sigma))$ for a single periodic function $\tilde{f}$.

For both the open and closed strings, we can then use this freedom to set either τ or σ to be one of the X^μs. Here are two examples:

- The **static gauge** turns the general solution for the closed string

  $$X^0 = X_0^0 + v^0\tau + \sum_{n\neq 0} f_n^0 e^{-in\sigma^+} + \tilde{f}_n^0 e^{in\sigma^-}$$

  much simpler as follows: write this above expression as

  $$= v_0\left(\tau + \frac{x_0^0}{v^0} + \frac{1}{v^0}\left[\sum_{n\neq 0} f_n^0 e^{-in\sigma^+} + \tilde{f}_n^0 e^{in\sigma^-}\right]\right),$$

  which is precisely τ plus a periodic function, so it's valid to use this whole expression as our $\tilde{\tau}$. Thus, we now have $\boxed{X^0 = v^0\tilde{\tau}}$, which is a much simpler expression. Note that we do not have freedom to set $X^0 = \tilde{\tau}$, i.e. to set $v^0 = 1$. This makes sense as v^0 is a component of the spacetime four-velocity, which is a physical quantity, and thus cannot be gauged away.

- The **light-cone gauge** instead sets $\boxed{X^+ = v^+\tilde{\tau}}$ (where remember that $X^\pm = X^0 \pm X^1$). The idea is again to get all of the constant and oscillating modes into the τ, and this can also be a useful gauge to use.

In the static gauge, we have

$$t = X^0 = v^0\tau$$

(we're now using $\tilde{\tau}$ as our τ-variable, so we can drop the tildes), and thus $\partial_\tau X^\mu = \left(v^0, v^0\frac{\partial\vec{X}}{\partial t}\right)$ (the derivative is now on t instead of τ) and $\partial_\sigma X^\mu = \left(0, \partial_\sigma \vec{X}\right)$. We also have

$$\sigma^\pm = \tau \pm \sigma = \frac{t}{v_0} \pm \sigma = \frac{1}{v_0}\left(t \pm v^0\sigma\right) = \frac{1}{v^0}(t \pm \hat{\sigma}),$$

where $\hat{\sigma} = v^0\sigma$ ranges from $[0, \pi v^0]$ for open strings and $[0, 2\pi v^0]$ for closed strings – we'll denote this as $\sigma \in [0, \sigma_1]$ in both cases. And now we're ready to return to the Virasoro constraints: plugging in the explicit expressions for $\partial_\tau X^\mu$

and $\partial_\sigma X^\mu$ yields

$$\boxed{-1 + (\partial_t \vec{X})^2 + (\partial_{\hat{\sigma}} \vec{X})^2 = 0},$$

$$\boxed{\partial_t \vec{X} \cdot \partial_{\hat{\sigma}} \vec{X} = 0}.$$

Remark 58. *We still have coupling among the different spatial directions, but this static gauge is convenient for visualization because we've expressed the string evolution in terms of physical time.*

Example 59

Consider a rotating open string in $2+1$ dimensions moving with angular velocity ω, where the string is of length 2ℓ, and the origin is the center of rotation (so it is distance ℓ away from the endpoints $\sigma = 0$ and $\sigma = \pi$).

We want to check whether this describes a consistent string motion – clearly, we have Neumann boundary conditions in both directions because the string endpoints can take on various x- and y-coordinates. So our most general solution is

$$\vec{X}(t, \hat{\sigma}) = \vec{X}_0 + \frac{\vec{v}}{v_0} t + \sum_{n>0} \vec{f}_n e^{-int/v_0} \cos \frac{n\hat{\sigma}}{v_0},$$

and we can determine the parameters by setting $\sigma = 0$ and compare it with what we know: since

$$\vec{X}(t, 0) = (\ell \cos \omega t, \sin \omega t),$$

We conclude that $\vec{X}_0 = 0$ and $\vec{v} = 0$ (there is no center-of-mass motion in a rotation), we only have rotational components, and only one of the n (call it m) can be nonzero: we must have $\frac{m}{v_0} = \omega \implies \boxed{v_0 = \frac{m}{\omega}}$. Plugging in the value of v_0, our worldsheet coordinate is

$$\vec{X}(t, \hat{\sigma}) = (\ell \cos \omega t, \ell \sin \omega t) \cos \omega \hat{\sigma},$$

and we can now check whether it satisfies the constraints. We find that

$$\partial_t \vec{X} = (-\omega \ell \sin \omega t, \omega \ell \cos \omega t) \cos \omega \hat{\sigma},$$

and similarly

$$\partial_{\hat{\sigma}} \vec{X} = (-\omega \ell \cos \omega t, -\omega \ell \sin \omega t) \sin \omega \hat{\sigma}.$$

The first Virasoro constraint now requires us to check that the sum of the squares of the two expressions above is 1, and that forces $\boxed{\omega^2 \ell^2 = 1}$. (And condition 2 is automatically satisfied by this functional form.) In other words, this is only a consistent rotating string motion if the endpoints are traveling at the speed of light (because $\omega \ell$ is the linear velocity)! And this is in fact a general phenomenon: we can show that **open string endpoints always need to move at the speed of light**.

Later on in the class, we will see that the energy of a string is given by

$$E = \pi T_0 v^0,$$

where we can recall that v^0 is the 0-component of the four-velocity and T_0 is the string tension per unit length. And therefore, we have $E = \frac{\pi T_0 m}{\omega}$ for the rotating string described above, for some positive integer m.

March 24, 2021

In the last few lectures, we've been working with the Polyakov action $S_p = -\frac{T_0}{2}\int d^2\xi\sqrt{-h}\ h^{\alpha\beta}\partial_\alpha X^\mu \partial_\beta X_\mu$, fixing the gauge so that we have a flat metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ so that our equations of motion for become the constraint equations

$$\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu = 0, \quad \dot{X}^\mu X'_\mu = 0$$

as well as the wave equation

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0.$$

We described previously that the general solution for this wave equation can be written as a Fourier decomposition: for a closed string, we have

$$X^\mu(\tau,\sigma) = x_0^\mu + v^\mu \tau + \sum_{n\neq 0}\left(f_n e^{-in\sigma^+} + \tilde{f}_n e^{-in\sigma^-}\right),$$

and for an open string, we have

$$X^\mu(\tau,\sigma) = x_0^\mu + v^\mu\tau + \sum_{n\neq 0} f_n e^{-in\tau}\cos(n\sigma).$$

This most general solution then allows us to encode all of the classical motion of the string, and it will also help us quantize the string (as we will soon see). But we need to impose our two constraints as well, and we can make this easier by fixing the gauge with our additional freedoms. Recall that we can either set $X^0 = v^0\tau$ (the **static gauge**), which simplifies our equations but still doesn't allow us to solve things in a closed form, or $X^+ = v^+\tau$ (the **light-cone gauge**, where $(X^+, X^-, \vec{X}_\perp)$ are our coordinates). Recall that in this latter case, we define $X^\pm = X^0 \pm X^1$, and the other coordinates in $\vec{X}_\perp$ will often be denoted X^I, where I ranges from 2 to $d-1$.

We'll dive into using this light-cone gauge now: setting $X^+ = v^+\tau$ for some constant physical parameter v^+, recall that we have the dot product

$$\dot{X}^\mu \dot{X}_\mu = -2\dot{X}^+\dot{X}^- + \dot{X}^I \dot{X}_I$$

(noting that X^I and X_I are the same because we're now using spatial indices in Minkowski space), and similar dot products if we replace $\dot{X}$ with X'. But we know that

$$\dot{X}^+ = \partial_\tau X^+ = v^+, \quad X'^+ = \partial_\sigma X^+ = 0,$$

so now we can look at the first constraint $\dot{X}^\mu\dot{X}_\mu + X'^\mu X'_\mu = 0$ and write it as

$$\left(-2v^+\partial_\tau X^- + (\partial_\tau \vec{X}_\perp)^2\right) + \left(\partial_\sigma \vec{X}_\perp)^2\right) = 0.$$

Simplifying slightly, we find that

$$\boxed{\partial_\tau X^- = \frac{1}{2v^+}\left(\partial_\tau \vec{X}_\perp)^2 + \partial_\sigma \vec{X}_\perp)^2\right)}.$$

Similarly, the second constraint $\dot{X}^\mu X'_\mu = 0$ now simplifies to

$$-\partial_\tau X^+ \partial_\sigma X^- - \partial_\sigma X^+ \partial_\tau X^- + \partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp = 0,$$

which simplifies after some rearranging to

$$\boxed{\partial_\sigma X^- = \frac{1}{v^+}\partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp}.$$

And what these two boxed equations tell us is that we can solve X^- completely in terms of the spatial coordinates X^I

(up to a constant), since we know both derivatives of X^-! So we've now solved the Virasoro constraints completely, and we've found that (other than X_0^- and v^+) the X^Is are the only independent degrees of freedom here, and again this will be important for quantization.

Remark 60. *Notice that the off-diagonal structure of the light-cone coordinates helped us out a lot here: our solution then became linear in X^-, so that we could solve for it.*

If we take the two boxed equations and add and subtract them together, we find a nice relation because we have a perfect square:

$$(\partial_\tau \pm \partial_\sigma)X^- = \frac{1}{2v^+}\left(\partial_\tau \vec{X}_\perp \pm \partial_\sigma \vec{X}_\perp\right)^2.$$

If we then introduce the operators

$$\partial_- = \partial_{\sigma^-} = \frac{1}{2}(\partial_\tau - \partial_\sigma), \quad \partial_+ = \partial_{\sigma^+} = \frac{1}{2}(\partial_\tau + \partial_\sigma),$$

we get the more transparent form

$$\boxed{\partial_+ X^- = \frac{1}{v^+}(\partial_+\vec{X}_\perp)^2, \quad \partial_- X^- = \frac{1}{v^+}(\partial_-\vec{X}_\perp)^2},$$

and what this says is that the left-moving and right-moving parts for the X^- solution are decoupled and evolve separately! We can then use these two equations and go back to the Fourier decomposition, expressing v^-, f_n^-, and $\tilde{f}_n^-$ in terms of quantities for the X^Is.

Remark 61. *Notice that if we integrate the $\partial_\sigma X^- = \frac{1}{v^+}\partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp$ equation around a closed string from 0 to 2π, periodicity makes the left-hand side go away. We thus find the constraint equation*

$$0 = \int_0^{2\pi} d\sigma \partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp \tag{3}$$

for closed strings, rather than a relation for X^- in terms of the $\vec{X}_\perp$s.

If we plug in our equations and do solve for those unknown coefficients, we get the following results:

Proposition 62

For an open string, we have

$$v^- = \frac{1}{2v^+}\left(\vec{v}_\perp^2 + \sum_{n\neq 0} n^2 f_n^I f_{-n}^I\right), \tag{4}$$

and for a closed string, we have

$$v^- = \frac{1}{2v^+}\left(\vec{v}_\perp^2 + 2\sum_{n\neq 0} n^2 (f_n^I f_{-n}^I + \tilde{f}_n^I \tilde{f}_{-n}^I)\right). \tag{5}$$

It is also straightforward to find the explicit expressions of f_n^-, $\tilde{f}_n^-$ in terms of parameters of X^I, but we will not give them here.

Notice in particular that we must have

$$\sum_{n\neq 0} n^2 f_n^I f_{-n}^I = \sum_{n\neq 0} n^2 \tilde{f}_n^I \tilde{f}_{-n}^I$$

for the closed string because of the equation in Remark 76.

To understand the physical interpretation of the identities above, we can go back to the Polyakov action again. We know that a closed string must satisfy $X^\mu(\tau, \sigma) = X^\mu(\tau, \sigma + 2\pi)$, and let's assume that we have an open string with Neumann boundary conditions, so that $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$. Expanding the X^μs in terms of the complete set of Fourier modes, the closed string now has

$$X^\mu(\tau, \sigma) = \sum_{n=-\infty}^{\infty} X_n^\mu(\tau) e^{in\sigma},$$

so that we now care about the dynamical variables $X_n^\mu(\tau)$, and similarly for the open string we need the full set of modes where the boundary condition is satisfied:

$$X^\mu(\tau, \sigma) = \sum_{n=0}^{\infty} X_n^\mu(\tau) \cos(n\sigma).$$

(We also have the conditions $(X_n^\mu)^* = X_{-n}^\mu$ for the closed string, and $(X_n^\mu)^* = X_n^\mu$ for the open string.) Then plugging this back into the Polyakov action gives us

$$\boxed{S = \frac{T_0}{2}\pi \left(\int d\tau \dot{X}_0^\mu \dot{X}_{0\mu} + \frac{1}{2} \sum_{n=1}^{\infty} \int d\tau \left(\dot{X}_n^\mu \dot{X}_{n\mu} - n^2 X_n^\mu X_{n\mu} \right) \right)} \tag{6}$$

(where we used the fact that different Fourier modes are orthogonal).

Fact 63

From this equation, we then find that, the action for X_0^μ, which describes our center-of-mass motion, looks like the action for a nonrelativistic particle of mass $m = T_0\pi$. And the oscillatory modes where $n \neq 0$ become **harmonic oscillators** with frequencies $\omega_n = n$ and mass $\frac{1}{2}T_0\pi$.

And because we know how to quantize a free particle and a harmonic oscillator, equation (11) provides a convenient way to think about quantization of a string later! But keep in mind that τ is the worldsheet time τ, rather than the proper time in spacetime. We can also look again at the solutions for a free particle

$$X_0^\mu(\tau) = X_0^\mu + v^\mu \tau$$

and the harmonic oscillator

$$X_n^\mu(\tau) = f_n e^{-in\tau} + f_n^* e^{in\tau},$$

and we indeed see that these components are exactly what come up in our solutions for X^μ. If we now turn our attention to the **canonical momentum** for each mode X_n^μ: thinking about the action as $\int d\tau L$ for some Lagrangian L, we want to calculate $p_\mu = \frac{\partial L}{\partial \dot{X}_n^\mu}$. We find that for $n = 0$, we have

$$p_\mu = \frac{\partial L}{\partial \dot{X}_0^\mu} = \pi T_0 \dot{X}_{0\mu} = v_\mu \pi T_0, \tag{7}$$

and otherwise we have

$$p_{n\mu} = \frac{\partial L}{\partial \dot{X}_n^\mu} = \frac{\pi T_0}{2} \dot{X}_{n\mu}$$

(we can then plug in the expression we have above if we want). If we interpret p_μ again as the center-of-mass momentum, we can think of this vector as the (Lorentz covariant) **spacetime d-momentum for the string**. This may seem like a natural thing to do, but we may also be concerned because τ is the worldsheet time, not the physical time – that's why this is a heuristic argument, but we'll actually justify why it is valid later on!

Remark 64. *The rewriting of our Polyakov action into different Fourier modes is not quite like the usual action, though, because our $\dot{X}_0^\mu \dot{X}_{0\mu}$ in the boxed expression above is explicitly equal to*

$$= \frac{\pi T_0}{2}\left(-\dot{X}_0^0 \dot{X}_0^0 + \dot{X}_0^i \dot{X}_0^i\right).$$

The $\dot{X}_0^i \dot{X}_0^i$ term has a positive sign (which is correct for the kinetic energy), but we have the wrong sign specifically for the $\dot{X}_0^0 \dot{X}_0^0$ term, which appears to imply that the worldsheet energy can be unbounded from below! This can cause some trouble when we get to quantum theory, but we'll resolve it later.

If we now go from the worldsheet Lagrangian to the worldsheet Hamiltonian, we want

$$H = H_0 + \sum_{n=1}^{\infty} H_n,$$

where

$$H_0 = \frac{p^\mu p_\mu}{2\pi T_0} = \frac{1}{2\pi T_0}(-(p^0)^2 + (p^i)^2)$$

is the Hamiltonian for the center-of-mass component, and

$$H_n = \frac{p_n^\mu p_{n\mu}}{\pi T_0} + \frac{1}{2} n^2 \frac{\pi T_0}{2} X_n^\mu X_{n\mu}$$

is the Hamiltonian for an individual harmonic oscillator component. Using (12) we can now express (9) in terms of the light-cone momentum p^+ and light-cone energy p^-, we find that

$$p^- = \frac{1}{2p^+}\left[(\vec{p}_\perp)^2 + 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp\right],$$

where $H_n^\perp$ is the Hamiltonian associated with the $X^\perp$ direction (summing over only Is instead of the μs in the definition of H_n). This can then be rewritten in the form

$$-2p^- p^+ - (p^I)^2 = 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp,$$

and now the left-hand side is just $-p^\mu p_\mu$, which we can interpret as the squared mass M^2 associated to the center-of-mass motion of a string. So we have a formula for the mass of a string now:

Proposition 65 (Mass-shell condition for open strings)

For the open string, we have the equation

$$M^2 = -p^\mu p_\mu = 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp,$$

so that the squared mass associated to a string is $2\pi T_0$ times the total energy of its transverse-direction oscillatory modes on the worldsheet.

We can get similar results for closed strings, except that we have both left-moving and right-moving modes and some factors of 2 that come up. From (12) and (10) we end up with the following:

Proposition 66 (Mass-shell condition for closed strings)
For the closed string, we have the equation

$$M^2 = 4\pi T_0 \sum_{n=1}^{\infty} H_n^{\perp},$$

where $H_n^{\perp}$ contains the energy from both left-moving and right-moving modes.

3: Quantum strings

Before we talk about the quantum string, we need to set up some theory first:

3.1: Global symmetries and conservation laws

Recall that we interpreted the canonical momentum p_μ associated with the $n = 0$ mode with the spacetime center-of-mass. We'll be able to justify things in a different way at the end of this secion.

Recall that in classical mechanics a conservation law takes the form $\frac{dQ}{dt} = 0$ for some conserved quantity Q along the trajectory of our particle (e.g. Q can be energy or momentum). In electromagnetism, conservation laws are more sophisticated because it is a classical field theory. For example, the charge conservation takes the form of the continuity equation

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0\ .$$

Integrating the above equation inside a volume V we

$$\partial_t Q = -\int_V d^{d-1}x\, \vec{\nabla} \cdot \vec{j} = -\int_\Sigma \vec{j} \cdot d\vec{S}, \quad Q \equiv \int_V d^{d-1}x\, \rho(x)$$

where Σ is the surface enclosing V. We can write the continuity equation in a more compact form $\partial_\mu j^\mu = 0$, where $j^\mu = (\rho, \vec{j})$.

We will now show that there are deep connections between **global symmetries** and conservation laws as realized by Emmy Noether in 1918.

Recall that **global symmetries** are transformations of our dynamical variable which keep the action invariant (and thus do not change the dynamics), and the transformation parameters must be independent of coordinates.

Example 67
For a single particle with $L = \frac{1}{2}mv^2 - V(x,t)$, if V is independent of t, then the Lagrangian is invariant under time translation, and thus we get energy conservation in the system. And if $V(x)$ is independent of x, then the Lagrangian is invariant under spatial translation, and we have momentum conservation.

Example 68
If we go to higher dimensions so that we now have a potential $V(\vec{x})$, and V is only dependent on the magnitude of $\vec{x}$, then the system has rotational symmetry, and we have angular momentum conservation.

We'll discuss this in more detail next lecture and understand why we indeed always get conservation laws from global symmetries, and we'll then apply this to our string theory action to understand the implications!

March 17, 2021

Last lecture, we fixed the gauge for the string action by setting $h_{\alpha\beta} = \eta_{\alpha\beta}$, so that the intrinsic metric on the worldsheet is given by

$$ds_\Sigma^2 = h_{\alpha\beta} d\xi^\alpha d\xi^\beta = \eta_{\alpha\beta} d\xi^\alpha d\xi^\beta = -d\tau^2 + d\sigma^2;$$

that is, we have two-dimensional Minkowski spacetime. Our action will simplify significantly to

$$S_p = -\frac{T_0}{2} \int d^2\xi \, \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

and the equations of motion for $h^{\alpha\beta}$ become

$$T_{00} = T_{11} = \frac{1}{2}(\dot{X}^\mu \dot{X}_\mu + X'^\mu + X'_\mu) = 0, \quad T_{01} = \dot{X}^\mu X'_\mu = 0$$

(where X'^μ is the σ-derivative of X^μ, and $\dot{X}^\mu$ is the τ-derivative of X^μ). The equation of motion for X^μ then tells us that

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0,$$

and notice that this means the wave is traveling at the speed of light $c = 1$.

Last time, we solved this wave equation and found that the most general solution is

$$X^\mu(\tau, \sigma) = X_R^\mu(\sigma^-) + X_L^\mu(\sigma^+),$$

where $\sigma^\pm = \tau \pm \sigma$. But there are additional constraints we have as well: for example, for the closed string, from that $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$, we found last time that we must have the form

$$X^\mu(\tau, \sigma) = X_0^\mu + v^\mu \tau + \sum_{n \neq 0} \left(f_n^\mu e^{-in\sigma^+} + \tilde{f}_n^\mu e^{-in\sigma^-} \right),$$

where $(f_n^\mu)^* = f_{-n}^\mu$ and $\left(\tilde{f}_n^\mu\right)^* = \tilde{f}_{-n}^\mu$). The way to interpret this expression is that the first two terms are the center-of-mass motion, and the rest are vibrational modes of the string. So x_0^μ should be treated as a center-of-mass initial location, v^μ should be treated as the (covariant Lorentz) center-of-mass velocity, and $f_n^\mu, \tilde{f}_n^\mu$ are the left-moving and right-moving vibrational amplitudes for the nth harmonics of the string.

But there are still constraints that we need to impose, namely the **Virasoro constraints** from $h^{\alpha\beta}$'s equations of motion. These are nonlinear (quadratic) in X, and they couple all of the directions of the string together, so they can be pretty nasty to solve – they cannot be solved in closed form as stated.

Example 69

For now, let's first turn to the open string and understand how to impose those boundary conditions.

Recall that in the gauge where $h_{\alpha\beta} = \eta_{\alpha\beta}$, the Neumann boundary condition becomes $\partial_\sigma X|_{\sigma_*} = 0$, and for each μ we can have a mix of Neumann and Dirichlet boundary conditions. We'll first solve the case here where we have Neumann boundary conditions at both ends, which means that $\partial_\sigma X|_{\sigma=0,\pi} = 0$.

If we plug in $\sigma = 0$, we find that the condition required is

$$X'_L(\tau) - X'_R(\tau) = 0 \implies X_L = X_R + c$$

(remembering that X_L, X_R are functions of just one variable). If we then plug this into the equation for $\sigma = \pi$, letting

$X_R = F$, we find that

$$-F'(\tau - \pi) + F'(\tau + \pi) = 0,$$

and thus F' must be a periodic function with period 2π. So again like last lecture, we find that $F(x) = \frac{v}{2}x + f(x)$, where x is a dummy index, v is some arbitrary constant, and f is some periodic function which we can now expand out by Fourier transform as $f_0 + \sum_{n \neq 0} \frac{f_n}{2} e^{-inx}$. So now the general form for our solution is

$$X_R(\sigma^-) = f_0 + \frac{v}{2}\sigma^- + \sum_{n \neq 0} \frac{f_n}{2} e^{-in\sigma^-},$$

and then X_L as a function of σ^+ only differs by a constant:

$$X_L(\sigma^+) = \tilde{f}_0 + \frac{v}{2}\sigma^+ + \sum_{n \neq 0} \frac{f_n}{2} e^{-in\sigma^+}.$$

Putting this all together, we get our wave equation solution

$$\boxed{X^\mu(\tau, \sigma) = X_0^\mu + v^\mu \tau + \sum_{n \geq 0} f_n e^{-in\tau} \cos(n\sigma)},$$

where $(f_n^\mu)^* = f_{-n}^\mu$ and we've used the fact that

$$\frac{f_n}{2}(e^{-in(\tau-\sigma)} + e^{-in(\tau+\sigma)}) = f_n e^{-in\tau} \cos(n\sigma).$$

This time, notice that we only have **one set of vibration modes**, compared to the two that showed up in the closed string.

Similarly, we can find the answers for other boundary conditions as well: for example, if we have two Dirichlet boundary conditions $X(\tau, 0) = a, X(\tau, \pi) = b$, the most general solution is given by

$$X(\tau, \sigma) = a + \frac{b-a}{\pi}\sigma + i \sum_{n \neq 0} f_n e^{-in\tau} \sin(n\sigma).$$

Notice that there is **no linear τ term** this time, because there is no linear motion like there was with Neumann boundary conditions or with the closed string. And this is because we're being told that our string is fixed on the two endpoints $x = a$ and $x = b$, so we can't expect any (center-of-mass) linear motion in the x-direction!

And again, we need to impose the Virasoro constraints in the open string case, and we still cannot solve those equations in closed form because of nonlinearities and coupling.

To solve the Virasoro constraints, we will need help from fixing the **residual gauge freedoms** that we mentioned last lecture.

Example 70

Looking at the worldsheet metric $dS_\Sigma^2 = -d\tau^2 + d\sigma^2$, our question is whether there is further freedom to redefine σ, τ, or perform the Weyl scaling and do so in a way that leaves the worldsheet metric invariant. (That would mean that we haven't fully fixed the gauge yet.)

The answer turns out to be yes, and the key is to rewrite

$$dS_\Sigma^2 = -d\tau^2 + d\sigma^2 = -d\sigma^+ d\sigma^-.$$

If we now consider the transformation

$$\sigma^+ \to \tilde{\sigma}^+ = f(\sigma^+), \quad \sigma^- \to \tilde{\sigma}^- = g(\sigma^-)$$

for some arbitrary functions f, g, then the metric becomes

$$dS_\Sigma^2 = -d\tilde{\sigma}^+ d\tilde{\sigma}^- = f'(\sigma^+)g'(\sigma^-)d\sigma^+ d\sigma^-,$$

and now it looks like we have an extra prefactor, but this is an overall expression that can be removed by a further Weyl scaling, which gets us back to the original metric. And the reason that such freedoms exist is that even though we removed 3 degrees of freedom when turning $h_{\alpha\beta}$ to $\eta_{\alpha\beta}$, these new parameterizations don't remove further degrees of freedom – since they only depend on **specific** combinations of σ and τ, and they only depend on one rather than two variables, they are in some sense of measure zero. Thus, we can redefine σ and τ to

$$\tilde{\sigma} = \frac{1}{2}(\tilde{\sigma}^+ - \tilde{\sigma}^-) = \frac{1}{2}(f(\sigma + \tau) - g(\tau - \sigma)),$$

$$\tilde{\tau} = \frac{1}{2}(\tilde{\sigma}^+ + \tilde{\sigma}^-) = \frac{1}{2}(f(\sigma + \tau) + g(\tau - \sigma)),$$

for functions f and g of a single variable. But now this redefinition needs to respect the closed string's periodicity or the open string's endpoints!

Example 71

For example, let's look at the conditions for the closed string.

For a closed string, we need $\tilde{\tau}(\tau, \sigma + 2\pi) = \tilde{\tau}(\tau, \sigma)$ and $\tilde{\sigma}(\tau, \sigma + 2\pi) = \tilde{\sigma}(\tau, \sigma) + 2\pi$. Adding or subtracting these together tells us that

$$\tilde{\sigma}^\pm(\tau, \sigma + 2\pi) = \tilde{\sigma}^\pm(\tau, \sigma) \pm 2\pi,$$

and remember that $\tilde{\sigma}^\pm$ are defined as certain functions of $\sigma + \tau$ and $\tau - \sigma$: this leads us to

$$f(\sigma^+ + 2\pi) = f(\sigma^+) + 2\pi, \quad g(\sigma^- - 2\pi) = g(\sigma^-) - 2\pi.$$

So f and g need to be periodic functions up to linear shifts, and thus

$$f(\sigma^+) = \sigma^+ + \tilde{f}(\sigma^+)$$

for some periodic function $\tilde{f}$ (and we know that the coefficient in front of σ^+ is 1 because changing σ by 2π adds 2π to $\tilde{\sigma}$), and similarly

$$g(\sigma^-) = \sigma^- + \tilde{g}(\sigma^-).$$

Therefore, we can redefine our reparameterization in terms of periodic functions as

$$\boxed{\tilde{\tau} = \tau + \frac{1}{2}(\tilde{f}(\tau + \sigma) + \tilde{g}(\tau - \sigma))}$$

for periodic functions $\tilde{f}, \tilde{g}$.

Fact 72
Similarly, we can do a calculation for the open string, where the constraints are $\tilde{\sigma}(\tau, 0) = 0$ and $\tilde{\sigma}(\tau, \pi) = \pi$. We can then show that we must have $f = g$ for the parameterization to be valid, and thus we actually have $\tilde{\tau} = \tau + \frac{1}{2}(\tilde{f}(\tau + \sigma) + \tilde{f}(\tau - \sigma))$ for a single periodic function $\tilde{f}$.

For both the open and closed strings, we can then use this freedom to set either τ or σ to be one of the X^μs. Here are two examples:

- The **static gauge** turns the general solution for the closed string

$$X^0 = X_0^0 + v^0\tau + \sum_{n\neq 0} f_n^0 e^{-in\sigma^+} + \tilde{f}_n^0 e^{in\sigma^-}$$

  much simpler as follows: write this above expression as

$$= v_0 \left(\tau + \frac{x_0^0}{v^0} + \frac{1}{v^0}\left[\sum_{n\neq 0} f_n^0 e^{-in\sigma^+} + \tilde{f}_n^0 e^{in\sigma^-} \right] \right),$$

  which is precisely τ plus a periodic function, so it's valid to use this whole expression as our $\tilde{\tau}$. Thus, we now have $\boxed{X^0 = v^0\tilde{\tau}}$, which is a much simpler expression. Note that we do not have freedom to set $X^0 = \tilde{\tau}$, i.e. to set $v^0 = 1$. This makes sense as v^0 is a component of the spacetime four-velocity, which is a physical quantity, and thus cannot be gauged away.

- The **light-cone gauge** instead sets $\boxed{X^+ = v^+\tilde{\tau}}$ (where remember that $X^\pm = X^0 \pm X^1$). The idea is again to get all of the constant and oscillating modes into the τ, and this can also be a useful gauge to use.

In the static gauge, we have

$$t = X^0 = v^0\tau$$

(we're now using $\tilde{\tau}$ as our τ-variable, so we can drop the tildes), and thus $\partial_\tau X^\mu = \left(v^0, v^0 \frac{\partial \vec{X}}{\partial t}\right)$ (the derivative is now on t instead of τ) and $\partial_\sigma X^\mu = \left(0, \partial_\sigma \vec{X}\right)$. We also have

$$\sigma^\pm = \tau \pm \sigma = \frac{t}{v_0} \pm \sigma = \frac{1}{v_0}\left(t \pm v^0\sigma\right) = \frac{1}{v^0}(t \pm \hat{\sigma}),$$

where $\hat{\sigma} = v^0\sigma$ ranges from $[0, \pi v^0]$ for open strings and $[0, 2\pi v^0]$ for closed strings – we'll denote this as $\sigma \in [0, \sigma_1]$ in both cases. And now we're ready to return to the Virasoro constraints: plugging in the explicit expressions for $\partial_\tau X^\mu$ and $\partial_\sigma X^\mu$ yields

$$\boxed{-1 + (\partial_t \vec{X})^2 + (\partial_{\hat{\sigma}} \vec{X})^2 = 0},$$

$$\boxed{\partial_t \vec{X} \cdot \partial_{\hat{\sigma}} \vec{X} = 0}.$$

Remark 73. *We still have coupling among the different spatial directions, but this static gauge is convenient for visualization because we've expressed the string evolution in terms of physical time.*

Example 74
Consider a rotating open string in $2+1$ dimensions moving with angular velocity ω, where the string is of length 2ℓ, and the origin is the center of rotation (so it is distance ℓ away from the endpoints $\sigma = 0$ and $\sigma = \pi$).

We want to check whether this describes a consistent string motion – clearly, we have Neumann boundary conditions in both directions because the string endpoints can take on various x- and y-coordinates. So our most general solution is

$$\vec{X}(t, \hat{\sigma}) = \vec{X}_0 + \frac{\vec{v}}{v_0} t + \sum_{n>0} \vec{f}_n e^{-int/v_0} \cos \frac{n\hat{\sigma}}{v_0},$$

and we can determine the parameters by setting $\sigma = 0$ and compare it with what we know: since

$$\vec{X}(t, 0) = (\ell \cos \omega t, \sin \omega t),$$

We conclude that $\vec{X}_0 = 0$ and $\vec{v} = 0$ (there is no center-of-mass motion in a rotation), we only have rotational components, and only one of the n (call it m) can be nonzero: we must have $\frac{m}{v_0} = \omega \implies \boxed{v_0 = \frac{m}{\omega}}$. Plugging in the value of v_0, our worldsheet coordinate is

$$\vec{X}(t, \hat{\sigma}) = (\ell \cos \omega t, \ell \sin \omega t) \cos \omega \hat{\sigma},$$

and we can now check whether it satisfies the constraints. We find that

$$\partial_t \vec{X} = (-\omega\ell \sin \omega t, \omega\ell \cos \omega t) \cos \omega \hat{\sigma},$$

and similarly

$$\partial_{\hat{\sigma}} \vec{X} = (-\omega\ell \cos \omega t, -\omega\ell \sin \omega t) \sin \omega \hat{\sigma}.$$

The first Virasoro constraint now requires us to check that the sum of the squares of the two expressions above is 1, and that forces $\boxed{\omega^2\ell^2 = 1}$. (And condition 2 is automatically satisfied by this functional form.) In other words, this is only a consistent rotating string motion if the endpoints are traveling at the speed of light (because $\omega\ell$ is the linear velocity)! And this is in fact a general phenomenon: we can show that **open string endpoints always need to move at the speed of light**.

Later on in the class, we will see that the energy of a string is given by

$$E = \pi T_0 v^0,$$

where we can recall that v^0 is the 0-component of the four-velocity and T_0 is the string tension per unit length. And therefore, we have $E = \frac{\pi T_0 m}{\omega}$ for the rotating string described above, for some positive integer m.

March 24, 2021

In the last few lectures, we've been working with the Polyakov action $S_p = -\frac{T_0}{2} \int d^2\xi \sqrt{-h}\; h^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X_\mu$, fixing the gauge so that we have a flat metric $h_{\alpha\beta} = \eta_{\alpha\beta}$ so that our equations of motion for become the constraint equations

$$\dot{X}^\mu \dot{X}_\mu + X'^\mu X'_\mu = 0, \quad \dot{X}^\mu X'_\mu = 0$$

as well as the wave equation

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0.$$

We described previously that the general solution for this wave equation can be written as a Fourier decomposition: for a closed string, we have

$$X^\mu(\tau, \sigma) = x_0^\mu + v^\mu \tau + \sum_{n \neq 0} \left(f_n e^{-in\sigma^+} + \tilde{f}_n e^{-in\sigma^-} \right),$$

and for an open string, we have

$$X^\mu(\tau,\sigma) = x_0^\mu + v^\mu\tau + \sum_{n\neq 0} f_n e^{-in\tau}\cos(n\sigma).$$

This most general solution then allows us to encode all of the classical motion of the string, and it will also help us quantize the string (as we will soon see). But we need to impose our two constraints as well, and we can make this easier by fixing the gauge with our additional freedoms. Recall that we can either set $X^0 = v^0\tau$ (the **static gauge**), which simplifies our equations but still doesn't allow us to solve things in a closed form, or $X^+ = v^+\tau$ (the **light-cone gauge**, where $(X^+, X^-, \vec{X}_\perp)$ are our coordinates). Recall that in this latter case, we define $X^\pm = X^0 \pm X^1$, and the other coordinates in $\vec{X}_\perp$ will often be denoted X^I, where I ranges from 2 to $d-1$.

We'll dive into using this light-cone gauge now: setting $X^+ = v^+\tau$ for some constant physical parameter v^+, recall that we have the dot product

$$\dot{X}^\mu \dot{X}_\mu = -2\dot{X}^+\dot{X}^- + \dot{X}^I\dot{X}_I$$

(noting that X^I and X_I are the same because we're now using spatial indices in Minkowski space), and similar dot products if we replace $\dot{X}$ with X'. But we know that

$$\dot{X}^+ = \partial_\tau X^+ = v^+, \quad X'^+ = \partial_\sigma X^+ = 0,$$

so now we can look at the first constraint $\dot{X}^\mu\dot{X}_\mu + X'^\mu X'_\mu = 0$ and write it as

$$\left(-2v^+\partial_\tau X^- + (\partial_\tau \vec{X}_\perp)^2\right) + \left(\partial_\sigma \vec{X}_\perp)^2\right) = 0.$$

Simplifying slightly, we find that

$$\boxed{\partial_\tau X^- = \frac{1}{2v^+}\left(\partial_\tau \vec{X}_\perp)^2 + \partial_\sigma \vec{X}_\perp)^2\right)}.$$

Similarly, the second constraint $\dot{X}^\mu X'_\mu = 0$ now simplifies to

$$-\partial_\tau X^+ \partial_\sigma X^- - \partial_\sigma X^+ \partial_\tau X^- + \partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp = 0,$$

which simplifies after some rearranging to

$$\boxed{\partial_\sigma X^- = \frac{1}{v^+}\partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp}.$$

And what these two boxed equations tell us is that we can solve X^- completely in terms of the spatial coordinates X^I (up to a constant), since we know both derivatives of X^-! So we've now solved the Virasoro constraints completely, and we've found that (other than X_0^- and v^+) the X^Is are the only independent degrees of freedom here, and again this will be important for quantization.

Remark 75. *Notice that the off-diagonal structure of the light-cone coordinates helped us out a lot here: our solution then became linear in X^-, so that we could solve for it.*

If we take the two boxed equations and add and subtract them together, we find a nice relation because we have a perfect square:

$$(\partial_\tau \pm \partial_\sigma)X^- = \frac{1}{2v^+}\left(\partial_\tau \vec{X}_\perp \pm \partial_\sigma \vec{X}_\perp\right)^2.$$

If we then introduce the operators

$$\partial_- = \partial_{\sigma^-} = \frac{1}{2}(\partial_\tau - \partial_\sigma), \quad \partial_+ = \partial_{\sigma^+} = \frac{1}{2}(\partial_\tau + \partial_\sigma),$$

we get the more transparent form

$$\boxed{\partial_+ X^- = \frac{1}{v^+}(\partial_+ \vec{X}_\perp)^2, \quad \partial_- X^- = \frac{1}{v^+}(\partial_- \vec{X}_\perp)^2},$$

and what this says is that the left-moving and right-moving parts for the X^- solution are decoupled and evolve separately! We can then use these two equations and go back to the Fourier decomposition, expressing v^-, f_n^-, and $\tilde{f}_n^-$ in terms of quantities for the X^Is.

Remark 76. *Notice that if we integrate the $\partial_\sigma X^- = \frac{1}{v^+}\partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp$ equation around a closed string from 0 to 2π, periodicity makes the left-hand side go away. We thus find the constraint equation*

$$0 = \int_0^{2\pi} d\sigma \partial_\tau \vec{X}_\perp \cdot \partial_\sigma \vec{X}_\perp \tag{8}$$

for closed strings, rather than a relation for X^- in terms of the $\vec{X}_\perp$s.

If we plug in our equations and do solve for those unknown coefficients, we get the following results:

Proposition 77

For an open string, we have

$$v^- = \frac{1}{2v^+}\left(\vec{v}_\perp^2 + \sum_{n\neq 0} n^2 f_n^I f_{-n}^I\right), \tag{9}$$

and for a closed string, we have

$$v^- = \frac{1}{2v^+}\left(\vec{v}_\perp^2 + 2\sum_{n\neq 0} n^2 (f_n^I f_{-n}^I + \tilde{f}_n^I \tilde{f}_{-n}^I)\right). \tag{10}$$

It is also straightforward to find the explicit expressions of f_n^-, $\tilde{f}_n^-$ in terms of parameters of X^I, but we will not give them here.

Notice in particular that we must have

$$\sum_{n\neq 0} n^2 f_n^I f_{-n}^I = \sum_{n\neq 0} n^2 \tilde{f}_n^I \tilde{f}_{-n}^I$$

for the closed string because of the equation in Remark 76.

To understand the physical interpretation of the identities above, we can go back to the Polyakov action again. We know that a closed string must satisfy $X^\mu(\tau,\sigma) = X^\mu(\tau,\sigma+2\pi)$, and let's assume that we have an open string with Neumann boundary conditions, so that $\partial_\sigma X^\mu|_{\sigma=0,\pi} = 0$. Expanding the X^μs in terms of the complete set of Fourier modes, the closed string now has

$$X^\mu(\tau,\sigma) = \sum_{n=-\infty}^{\infty} X_n^\mu(\tau) e^{in\sigma},$$

so that we now care about the dynamical variables $X_n^\mu(\tau)$, and similarly for the open string we need the full set of modes where the boundary condition is satisfied:

$$X^\mu(\tau,\sigma) = \sum_{n=0}^{\infty} X_n^\mu(\tau)\cos(n\sigma).$$

(We also have the conditions $(X_n^\mu)^* = X_{-n}^\mu$ for the closed string, and $(X_n^\mu)^* = X_n^\mu$ for the open string.) Then plugging this back into the Polyakov action gives us

$$\boxed{S = \frac{T_0}{2}\pi\left(\int d\tau \dot{X}_0^\mu \dot{X}_{0\mu} + \frac{1}{2}\sum_{n=1}^{\infty}\int d\tau \left(\dot{X}_n^\mu \dot{X}_{n\mu} - n^2 X_n^\mu X_{n\mu}\right)\right)} \tag{11}$$

(where we used the fact that different Fourier modes are orthogonal).

Fact 78

From this equation, we then find that, the action for X_0^μ, which describes our center-of-mass motion, looks like the action for a nonrelativistic particle of mass $m = T_0\pi$. And the oscillatory modes where $n \neq 0$ become **harmonic oscillators** with frequencies $\omega_n = n$ and mass $\frac{1}{2}T_0\pi$.

And because we know how to quantize a free particle and a harmonic oscillator, equation (11) provides a convenient way to think about quantization of a string later! But keep in mind that τ is the worldsheet time τ, rather than the proper time in spacetime. We can also look again at the solutions for a free particle

$$X_0^\mu(\tau) = X_0^\mu + v^\mu \tau$$

and the harmonic oscillator

$$X_n^\mu(\tau) = f_n e^{-in\tau} + f_n^* e^{in\tau},$$

and we indeed see that these components are exactly what come up in our solutions for X^μ. If we now turn our attention to the **canonical momentum** for each mode X_n^μ: thinking about the action as $\int d\tau L$ for some Lagrangian L, we want to calculate $p_\mu = \frac{\partial L}{\partial X_n^\mu}$. We find that for $n = 0$, we have

$$p_\mu = \frac{\partial L}{\partial \dot{X}_0^\mu} = \pi T_0 \dot{X}_{0\mu} = v_\mu \pi T_0, \tag{12}$$

and otherwise we have

$$p_{n\mu} = \frac{\partial L}{\partial \dot{X}_n^\mu} = \frac{\pi T_0}{2}\dot{X}_{n\mu}$$

(we can then plug in the expression we have above if we want). If we interpret p_μ again as the center-of-mass momentum, we can think of this vector as the (Lorentz covariant) **spacetime d-momentum for the string**. This may seem like a natural thing to do, but we may also be concerned because τ is the worldsheet time, not the physical time – that's why this is a heuristic argument, but we'll actually justify why it is valid later on!

Remark 79. *The rewriting of our Polyakov action into different Fourier modes is not quite like the usual action, though, because our $\dot{X}_0^\mu \dot{X}_{0\mu}$ in the boxed expression above is explicitly equal to*

$$= \frac{\pi T_0}{2}\left(-\dot{X}_0^0 \dot{X}_0^0 + \dot{X}_0^i \dot{X}_0^i\right).$$

The $\dot{X}_0^i \dot{X}_0^i$ term has a positive sign (which is correct for the kinetic energy), but we have the wrong sign specifically for the $\dot{X}_0^0 \dot{X}_0^0$ term, which appears to imply that the worldsheet energy can be unbounded from below! This can cause some trouble when we get to quantum theory, but we'll resolve it later.

If we now go from the worldsheet Lagrangian to the worldsheet Hamiltonian, we want

$$H = H_0 + \sum_{n=1}^{\infty} H_n,$$

where

$$H_0 = \frac{p^\mu p_\mu}{2\pi T_0} = \frac{1}{2\pi T_0}(-(p^0)^2 + (p^i)^2)$$

is the Hamiltonian for the center-of-mass component, and

$$H_n = \frac{p_n^\mu p_{n\mu}}{\pi T_0} + \frac{1}{2}n^2 \frac{\pi T_0}{2} X_n^\mu X_{n\mu}$$

is the Hamiltonian for an individual harmonic oscillator component. Using (12) we can now express (9) in terms of the light-cone momentum p^+ and light-cone energy p^-, we find that

$$p^- = \frac{1}{2p^+}\left[(\vec{p}_\perp)^2 + 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp\right],$$

where $H_n^\perp$ is the Hamiltonian associated with the $X^\perp$ direction (summing over only Is instead of the μs in the definition of H_n). This can then be rewritten in the form

$$-2p^- p^+ - (p^I)^2 = 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp,$$

and now the left-hand side is just $-p^\mu p_\mu$, which we can interpret as the squared mass M^2 associated to the center-of-mass motion of a string. So we have a formula for the mass of a string now:

Proposition 80 (Mass-shell condition for open strings)

For the open string, we have the equation

$$M^2 = -p^\mu p_\mu = 2\pi T_0 \sum_{n=1}^{\infty} H_n^\perp,$$

so that the squared mass associated to a string is $2\pi T_0$ times the total energy of its transverse-direction oscillatory modes on the worldsheet.

We can get similar results for closed strings, except that we have both left-moving and right-moving modes and some factors of 2 that come up. From (12) and (10) we end up with the following:

Proposition 81 (Mass-shell condition for closed strings)

For the closed string, we have the equation

$$M^2 = 4\pi T_0 \sum_{n=1}^{\infty} H_n^\perp,$$

where $H_n^\perp$ contains the energy from both left-moving and right-moving modes.

3: Quantum strings

Before we talk about the quantum string, we need to set up some theory first:

3.1: Global symmetries and conservation laws

Recall that we interpreted the canonical momentum p_μ associated with the $n = 0$ mode with the spacetime center-of-mass. We'll be able to justify things in a different way at the end of this secion.

Recall that in classical mechanics a conservation law takes the form $\frac{dQ}{dt} = 0$ for some conserved quantity Q along the trajectory of our particle (e.g. Q can be energy or momentum). In electromagnetism, conservation laws are more sophisticated because it is a classical field theory. For example, the charge conservation takes the form of the continuity equation

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0 \ .$$

Integrating the above equation inside a volume V we

$$\partial_t Q = -\int_V d^{d-1}x\, \vec{\nabla} \cdot \vec{j} = -\int_\Sigma \vec{j} \cdot d\vec{S}, \quad Q \equiv \int_V d^{d-1}x\, \rho(x)$$

where Σ is the surface enclosing V. We can write the continuity equation in a more compact form $\partial_\mu j^\mu = 0$, where $j^\mu = (\rho, \vec{j})$.

We will now show that there are deep connections between **global symmetries** and conservation laws as realized by Emmy Noether in 1918.

Recall that **global symmetries** are transformations of our dynamical variable which keep the action invariant (and thus do not change the dynamics), and the transformation parameters must be independent of coordinates.

Example 82

For a single particle with $L = \frac{1}{2}mv^2 - V(x, t)$, if V is independent of t, then the Lagrangian is invariant under time translation, and thus we get energy conservation in the system. And if $V(x)$ is independent of x, then the Lagrangian is invariant under spatial translation, and we have momentum conservation.

Example 83

If we go to higher dimensions so that we now have a potential $V(\vec{x})$, and V is only dependent on the magnitude of $\vec{x}$, then the system has rotational symmetry, and we have angular momentum conservation.

We'll discuss this in more detail next lecture and understand why we indeed always get conservation laws from global symmetries, and we'll then apply this to our string theory action to understand the implications!

March 29, 2021

Last lecture, we started talking about symmetries and conservation laws in preparation for talking about quantum strings: we mentioned some examples from classical mechanics, such as time translation and energy conservation or spatial translation and momentum conservation. And we can formulate this more generally:

Theorem 84 (Noether)

In any Lagrangian field theory, any continuous global symmetry corresponds to a conservation law of the form $\partial_\alpha j^\alpha = 0$ (for some current j^α).

For some setup, we can write our general field theory action in the form

$$S = \int d\xi^0 d\xi^1 \cdots d\xi^k \mathcal{L}(\phi^a, \partial_\alpha \phi^a),$$

where ξ^α are our coordinates and the $\phi^a(\xi^\alpha)$ are dynamical fields which are functions of the coordinates. (Notice that we use a different label a for the fields, because we can have any number of them). The term $\mathcal{L}(\phi^a, \partial_\alpha \phi^a)$ is known as the **Lagrangian density**, and we should notice that it's a function of our fields ϕ^a and their first derivatives.

Example 85

The string action is a $(1+1)$-dimensional field theory with $\xi^\alpha = (\tau, \sigma)$ and $\phi^a = \{X^\mu, h_{\alpha\beta}\}$ where the fields are the set of dynamical variables we used in the Polyakov action.

Example 86

The particle action is a $(0+1)$-dimensional field theory with $\xi^\alpha = \tau$ (only one coordinate) and fields given by $\phi^a = \{X^\mu, e\}$ (remembering that e was our new dynamical variable for the relativistic free particle).

Example 87

Electromagnetism is a $(d+1)$-dimensional field theory with $\xi^\alpha = (X^0, \vec{X})$ and $\phi^a = \{A_\alpha(X^0, \vec{x})\}$.

In all of these cases, we vary the action to find equations of motion. But we can also find a general formulation for the equation of motion coming out of $\delta S = 0$: for any field theory action S, we have

$$\delta S = \int d^{k+1}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \partial_\alpha \delta \phi^a \right)$$

by the chain rule on $\mathcal{L}$. Applying integration by parts on the second term turns this into

$$= \int d^{k+1}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) \delta \phi^a \right)$$

plus some boundary terms, but we'll always assume that those boundary terms can be made to vanish. And if this needs to be true for any ϕ^a variation, we get the **Euler-Lagrange equations**

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) = 0}$$

for each field a, where we're summing over α. (And we can use this to derive the equations of motion for our classical actions as well.)

Proof sketch for Theorem 104. We can now talk about what we mean when we have a **symmetry**: basically, if we have transformations $\phi^a \to \phi'^a$ which leave the action S invariant, we call those symmetries. And we can also be more specific with the vocabulary of Theorem 104:

Definition 88

A **continuous symmetry** can be described by transformations $\phi'^a(\lambda, \xi^\alpha)$ that depend on a continuous parameter λ, so that $\phi'^a(0) = \phi^a$ and $\phi'^a(\lambda)$ is a symmetry for all λ. Symmetries with λ that depend on ξ^α are **local symmetries**, and symmetries with λ that don't are **global symmetries**.

Infinitesimal transformations are those with λ being equal to an infinitesimal ε, we can expand out the transformations in small ε and find

$$\phi'^a(\varepsilon, \xi^\alpha) = \phi^a + \delta\phi^a = \phi^a(\xi^\alpha) + \varepsilon h^a(\phi) + O(\varepsilon^2),$$

where h^a is some function. We will consider global symmetries with ϵ being a constant. In the simplest situation, the Lagrangian density itself is invariant under the symmetric transformations (so does the action)

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = 0$$

from which we have

$$\frac{\partial \mathcal{L}}{\partial \phi^a}\varepsilon h^a + \frac{\partial L}{\partial(\partial_\alpha \phi^a)}\varepsilon \partial_\alpha h^a = 0\ .$$

Now we can cancel out the εs and use our Euler-Lagrange equation to find that

$$\partial_\alpha \left(\frac{\partial L}{\partial(\partial_\alpha \phi^a)} \right) h^a + \frac{\partial L}{\partial(\partial_\alpha \phi^a)} \partial_\alpha h^a = 0, \tag{13}$$

and now what we have is actually a total derivative: we can write it as

$$\partial_\alpha \left(\frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a \right) = 0,$$

and we've found our desired current

$$\boxed{j^\alpha = \frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a}$$

(where we're summing over a). □

Fact 89

We assumed above that $\mathcal{L}$, the Lagrangian density, is invariant. More generally, for the action to be invariant, the Lagrangian density only needs to be invariant up to a total derivative

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = \varepsilon \partial_\alpha K^\alpha.$$

We then end up with $\partial_\alpha K^\alpha$s on the right-hand side of (15) instead of 0, so the current is instead

$$\boxed{j^\alpha = \frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a - K^\alpha}\ .$$

Remark 90. *In general, there can be more than one symmetry for an action, so that our transformations may look like $\phi^a \to \phi'(a) = \phi(a) + \varepsilon_i h_i^a$, where the i's label the different symmetries. We then get a conserved current j_i^α for each symmetry, given by $\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} h_i^a - K_i^\alpha$,*

Remark 91. *The above discussion only applies when there is a continuous parameter λ. A discrete symmetry like $\phi^a \mapsto -\phi^a$ in general does not lead to a conservation law.*

The above discussion can be applied to a local symmetry but in general does not result in a nontrivial j^α.

The discussion is rather abstract, but it is extremely general, applicable to any Lagrangian field theories.

Example 92

Recall that global symmetries of the Polyakov action include translations $X^\mu \mapsto X^\mu + a^\mu$ and Lorentz transformations $X^\mu \mapsto L^\mu{}_\nu X^\nu$.

We'll set $h_{\alpha\beta} = \eta_{\alpha\beta}$, because h does not appear in any of the global symmetries. So then our action simplifies in form to

$$S_P = -\frac{T_0}{2} \int d^2\xi \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

and now we can work on the conservation laws: there are d different symmetries of the form $X^\mu \mapsto X^\mu + a^\mu$ (one for each dimension), so we'll look specifically at the translations along one of the dimensions:

$$X^\mu \mapsto X^\mu + \delta^\mu_\nu a^\nu.$$

Now, a^ν is ε_i, and δ^μ_ν is our h^a_i. So the current that comes out of this looks like

$$\Pi^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\nu)} \delta^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} = \boxed{-T_0 \partial^\alpha X_\mu},$$

and more explicitly we have two components

$$\Pi^0_\mu = T_0 \partial_\tau X_\mu, \quad \Pi^1_\mu = -T_0 \partial_\sigma X_\mu \ .$$

We can check that the currents are indeed conserved

$$\partial_\alpha \Pi^\alpha_\mu = 0 \implies \partial^2_\tau X_\mu - \partial^2_\sigma X_\mu = 0.$$

provided the equations of motion for X^μ are satisfied.

Recall that when we have a conserved current j^α, we can integrate j^0 over all spatial directions to get a conserved charge $Q = \int d^k\xi j^0$. In this case, we have

$$p_\mu = \int d\sigma \Pi^0_\mu = \boxed{T_0 \int d\sigma \partial_\tau X_\mu},$$

which is the conserved charge on the worldsheet corresponding to translation symmetry in the target spacetime. And remembering that translation symmetry gave us conservation of momentum, we will "define" p_μ as the **spacetime center of mass momentum** for the string, and Π^1_μ is then the spacetime **momentum current** on the worldsheet.

Consider now a **closed string**. We have

$$\partial_\tau p_\mu + \int_0^{2\pi} d\sigma \partial_\sigma \Pi^1_\mu = 0 \quad \to \quad \partial_\tau p_\mu = 0$$

where the second term of the first equation integrates to zero due to Π^1_μ being periodic. Recall the general solution for a closed string

$$X^\mu = x^\mu_0 + v^\mu \tau + \sum_{n \neq 0} (\text{oscillatory modes}),$$

and thus

$$p^\mu = T_0 \int_0^{2\pi} d\sigma \partial_\tau X^\mu = 2\pi T_0 v^\mu,$$

where the contributions from oscillatory modes disappear upon integration. The above expression is manifest τ-independent and confirms our earlier identification of spacetime momentum as the canonical momentum on the worldsheet for the center of mass motion.

For an **open string**, if we have the Neumann boundary conditions for both ends, meaning that $\Pi^1_\mu|_{\sigma_*} = 0$, we have

$$\partial_\tau p_\mu + \int_0^\pi d\sigma \partial_\sigma \Pi^1_\mu = 0 \quad \to \quad \partial_\tau p_\mu + \Pi^1_\mu\big|_{\sigma=\pi} - \Pi^1_\mu\big|_{\sigma=0} = 0 \quad \to \quad \partial_\tau p_\mu = 0 \ .$$

In this case, we have momentum conservation as there is **no momentum flux** out of the string endpoints. Furthermore, evaluating p_μ on the general solution to equations of motion we find

$$p_\mu = \pi T_0 v^\mu$$

which is a constant and confirms the expression obtained earlier from canonical momentum on the worldsheet for ceneter of mass motion. When one or both end points have Dirichlet condition, we have $\Pi^1_\mu|_{\sigma_*} \neq 0$, and thus

$$\partial_\tau p_\mu + \Pi^1_\mu\big|_{\sigma=\pi} - \Pi^1_\mu\big|_{\sigma=0} = 0 \quad \to \quad \partial_\tau p_\mu \neq 0 \ .$$

So if any of the string endpoints are in the Dirichlet boundary conditions, we no longer have momentum conservation – it will leak out of the string endpoints.

Example 93

Consider a D-brane in the x^1x^2 plane, so that we have NN boundary conditions in $X^{0,1,2}$ but DD boundary conditions in X^3.

The string can then move freely in the x^1, x^2 directions but not in the x^3 direction. That means that translation symmetry is broken in the x^3 direction because of the D-brane, and thus we do not have momentum conservation in the x^3 direction (momentum flows out from the endpoints into the D-brane). And this makes sense if we remember that in the DD and ND most general solutions for the wave equation, we do not have the $v^\mu \tau$ linear term.

Fact 94

And remember that we must have NN boundary conditions in the X^0 direction, so p_0 (which we can interpret as the energy) is still conserved.

We'll now turn our attention to the other kind of global symmetry for the Polyakov action, which is the **Lorentz transformations**. We then need an infinitesimal version of the transformation $X'^\mu = L^\mu{}_\nu X^\nu$, and we'll look at a few examples to understand how to do this.

- Infinitesimal rotations in the 2-dimensional plane take the form

$$\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^1 \\ x^2 \end{bmatrix}.$$

  for **small** θ. Taylor expanding to first order tells us that for $\theta \ll 1$, we have

$$\begin{bmatrix} \tilde{x}^1 \\ \tilde{x}^2 \end{bmatrix} = \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} + \begin{bmatrix} \theta x^2 \\ -\theta x^1 \end{bmatrix},$$

  meaning that $\delta x^1 = -\theta x^2$ and $\delta x^2 = -\theta x^1$.

- An infinitesimal boost comes from a small velocity parameter β in the equation

$$\begin{bmatrix} \tilde{x}^0 \\ \tilde{x}^1 \end{bmatrix} = \begin{bmatrix} \cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \end{bmatrix},$$

  which (again by Taylor expansion) gives us $\delta x^0 = \beta x^1$ and $\delta x^1 = \beta x^0 = -\beta x_0$.

And we can unify the two examples above, even though there's an extra negative sign in one of them, by writing things with lower indices instead of upper indices: all of the transformations of rotations and boosts take an antisymmetric form

$$\boxed{\delta x^\mu = \theta^{\mu\nu} x_\nu},$$

where $\theta^{\mu\nu}$ is **antisymmetric**. (So then the rotation would correspond to $\theta^{12} = -\theta^{21} = \theta$, and the boost would correspond to $\theta^{01} = -\theta^{10} = \beta$.)

The above expression applies to any infinitesimal rotations and boosts, with θ^{ij} being an infinitesimal rotation angle in $x_i - x_j$ plane and θ^{0i} being an infinitesimal boost velocity along x_i-direction. Altogether we have $\frac{1}{2}d(d-1)$ independent parameters, so there are $\frac{1}{2}d(d-1)$ different conservation laws that will come out of Lorentz transformations. (In $d = 4$, these correspond to the three rotations and three boosts.)

For any pair (μ, ν), we can look at the specific infinitesimal transformation

$$X^\lambda = X^\lambda + \eta \varepsilon^{\lambda\rho} X_\rho,$$

where $\varepsilon^{\mu\nu} = 1 = -\varepsilon^{\nu\mu}$, i.e. $\epsilon^{\lambda\rho}$ is nonzero only if λ, ρ are equal to μ, ν. If we then plug everything back into our formula from Noether's theorem, we get the conserved current

$$M^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\lambda)} \varepsilon^{\lambda\rho} X_\rho = \boxed{\Pi^\alpha_\mu X_\nu - \Pi^\alpha_\nu X_\mu}$$

(where summing λ, ρ while fixing μ, ν). Note the above expression is manifestly antisymmetric in μ, ν. We'll elaborate more what these conserved currents mean next time!

March 31, 2021

Last time, we studied symmetries and conserved quantities. Suppose we have an action of the form

$$S = \int d^{k+1}\xi \, \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$$

for some Lagrangian density $\mathcal{L}$, and this action is invariant under an infinitesimal transformation

$$\phi^a \to \phi^a + \varepsilon h^a(\phi)$$

for a small constant parameter ε (corresponding to global symmetries). Accordingly $\mathcal{L}$ can at most change by a total derivative: letting ∂_α denote derivatives with respect to ξ^α, we have

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = \varepsilon \partial_\alpha K^\alpha,$$

and then we get the **conserved current**

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} h^a - K^\alpha$$

(meaning that $\partial_\alpha j^\alpha = 0$). And using the these conserved quantities can make calculations much easier than solving equations of motion directly! For example, we saw last time that translations $X^\mu \to X^\mu + a^\mu$ keep the string action invariant, which gives us the conserved current $\Pi^\alpha_\mu = -T_0 \partial_\alpha X_\mu$. The corresponding **conserved charge**

$$p_\mu = \int d\sigma \, \Pi^0_\mu$$

is interpreted as the **spacetime momentum** for our string. (Analogously, we also have the **momentum current** on the worldsheet Π^1_μ, which tells us how spacetime momentum flows along the string. Remember that when we have Neumann boundary condition at a string end point, this momentum current is fixed to be 0 at that point.)

Invariance of the string action under Lorentz symmetries leads to the current

$$M^{\alpha}_{\mu\nu} = \Pi^{\alpha}_{\mu} X_{\nu} - \Pi^{\alpha}_{\nu} X_{\mu},$$

which is manifestly antisymmetric and also satisfies $\partial_{\alpha} M^{\alpha}_{\mu\nu} = 0$ from the equations of motion (last part left as an exercise).

To understand why this form is particularly suggestive, let's consider the Lorentz transformations corresponding to spatial rotations, meaning that $(\mu, \nu) = (i, j)$ for some spatial coordinates i, j. (M^{α}_{ij} is the current corresponding to rotations in $x^i - x^j$ plane.) Then we have

$$M^{\alpha}_{ij} = \Pi^{\alpha}_{i} X_j - \Pi^{\alpha}_{j} X_i.$$

This looks a lot like our standard $p_i x_j - p_j x_i$ angular momentum definition, and the conserved quantity associated with spatial rotations should indeed be interpreted as the angular momentum:

Definition 95

The **angular momentum** of a string for rotations in $x^i - x^j$ plane is defined as

$$m_{ij} = \int d\sigma M^{0}_{ij}.$$

(In three spatial dimensions we often associate rotations in ij-plane with its orthogonal direction, but this no longer works in general d-dimensional space!)

We can then also look at the corresponding quantities M^{α}_{0i}, corresponding to Lorentz boosts in the x^i direction: we have the similar form

$$M^{\alpha}_{0j} = \Pi^{\alpha}_{0} X_j - \Pi^{\alpha}_{j} X_0,$$

and then this current leads to a **boost charge**

$$m_{0i} = \int d\sigma\, M^{0}_{0i}.$$

We will later see that all of these charges will play important roles in quantum mechanics!

Remark 96. *Notice that for a given classical string configuration, we now have a reliable way to compute its spacetime momentum and angular momentum (our previous methods involved looking at the Hamiltonian along the worldsheet, but we now have a proper definition).*

Remark 97. *If we move to quantum mechanics, conserved charges become conserved operators, and then the role of these operators becomes even more significant: the commutators of the charges generate the algebra of the symmetry. So these Lorentz charges will give us the Lorentz algebra in quantum mechanics!*

3.2: Quantization of the relativistic particle

We're now ready to move on to quantization, but we'll do the relativistic particle as a warmup for the string (just like we did when we wrote down the action).

3.2.1: Canonical quantization

We'll start by reminding ourselves about the **canonical quantization**, which is the method of going from a classical theory to a quantum theory.

Fact 98

From here on, we'll have $\hbar = 1$ to make our notation simpler.

Example 99

Consider a theory where we just have one degree of freedom (x, p).

One way to do this quantization is through the **Schrodinger picture**: if we go from classical theory to quantum theory, we promote x and p to operators, and we turn the Hamiltonian $H(x, p)$ into $\hat{H}(\hat{x}, \hat{p})$, so that our system's state is described by wavefunction solutions to the Schrodinger equation $i\partial_t \psi(t, x) = \hat{H}\psi(t, x)$.

An equivalent formulation is the **Heisenberg picture**: the dynamical variables of the system are now the operators instead of the wavefunctions. We then have the evolution of an operator $\hat{O}$ given by the Heisenberg equation of motion

$$\frac{d}{dt}\hat{O}(t, x) = \frac{\partial}{\partial t}\hat{O}(t, x) - i[\hat{O}(t, x), \hat{H}],$$

and states ψ do not evolve. (In some sense, this means that the states provide a frame of measurement, much like the operators did in the Schrodinger picture.) There is a natural connection between classical equations of motion and quantum Heisenberg equations. Recall that classical equation of motions can be obtained from the Hamiltonian by

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x},$$

and it turns out there is a precise correspondence from these classical Hamiltonian equations to the quantum operator equations if we replace the classical variables with the corresponding quantum operators (setting x to $\hat{x}$ and p to $\hat{p}$), up to some operator ordering ambiguity because the x and p don't commute anymore in the quantum world. So what that means is that if we have the classical solutions for some Hamiltonian, we also basically have the solutions to the operator equations, and thus we've solved the corresponding quantum mechanics problem. (And the integration constants in the classical equations of motion correspond to the "integration constants" that come up as quantum operators.) That's a good sign, because we can then take the solutions to the classical string and convert them to the quantum string!

But we'll start by reviewing some simple examples in the Heisenberg picture, because many of us are less familiar with this way of treating quantum mechanics.

Example 100

Consider the free non-relativistic particle, in which we have $H = \frac{p^2}{2m}$.

The classical equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = 0;$$

solving, we find that $p = p_0$ and $x = \frac{p_0}{m}t + x_0$, where x_0, p_0 are our two integration constants. If we want to turn this into a quantum setup, we turn x, p to $\hat{x}, \hat{p}$ and impose $[\hat{x}, \hat{p}] = i$ (at any time t). We have no operator ambiguity problems here – $\hat{H} = \frac{\hat{p}^2}{2m}$ – so the Heisenberg equations of motion are

$$\frac{d}{dt}\hat{x}(t) = -i\left[\hat{x}(t), \hat{H}\right] = \frac{\hat{p}}{m},$$

(because x has no explicit dependence on t, $\frac{\partial x}{\partial t} = 0$) and similarly

$$\frac{d}{dt}\hat{p}(t) = -i[\hat{p}(t), H] = 0.$$

So the quantum operator equations are solved by $\hat{p}(t) = \hat{p}_0$ for some constant operator $\boxed{\hat{p}_0}$, and then $\boxed{\hat{x}(t) = \frac{\hat{p}_0}{m}t + \hat{x}_0}$. So as we said, everything just becomes an operator, and the commutation relation also implies that we must have

$$[\hat{x}(t), \hat{p}(t)] = i \implies \boxed{[\hat{x}_0, \hat{p}_0] = i}.$$

Since $\hat{H} = \frac{\hat{p}^2}{2m} = \frac{\hat{p}_0^2}{2m}$ it's convenient to choose a basis for our Hilbert space so that we have an eigenbasis of $\hat{p}_0$ (so that we're looking at "momentum eigenstates"). Indexing our states by p so that

$$\hat{p}_0 |p\rangle = p |p\rangle,$$

we know that $|p\rangle$ is a plane wave, and we can normalize it with the usual normalization $\langle p|p'\rangle = 2\pi\delta(p - p')$.

We have now completely solved the quantum theory and we can now compute anything: we know the full time-dependence of $\hat{x}$ and $\hat{p}$, and notice that the quantum story throughout this problem has been very parallel to the classical story.

Example 101

Next, consider the harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2$, where we'll set $m = 1$ for convenience.

The classical equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\omega^2 x.$$

We'll solve these equations in a slightly different way from how we usually do: because these are two first-order linear differential equations, we can diagonalize and decouple by introducing

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x + \frac{i}{\sqrt{\omega}}p\right), \quad a^* = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x - \frac{i}{\sqrt{\omega}}p\right).$$

We can plug these back in and find that

$$\dot{a} = -i\omega a, \quad \dot{a}^* = -i\omega a^*,$$

and then we can solve these equation to find $a(t) = a_0e^{-i\omega t}$ and $a^*(t) = a_0^*e^{i\omega t}$, so that

$$x(t) = \frac{1}{\sqrt{2\omega}}(a(t) + a^*(t)) = \frac{1}{\sqrt{2\omega}}\left(a_0e^{-i\omega t} + a_0^*e^{i\omega t}\right).$$

The integration constants are a_0, a_0^*. We can also calculate the value of the Hamiltonian (i.e the energy of the system) for the classical motion. Plugging the above solution into the expression for H we have $H = \omega a^*(t)a(t) = \omega a_0^* a_0$, where the last expression shows that energy is conserved (as expected) as the value of H is time-independent.

Now let's turn to quantum mechanics, with

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{1}{2}\omega^2\hat{x}^2$$

and we still have the canonical commutation relation $[\hat{x}(t), \hat{p}(t)] = i$. The Heisenberg equations of motion then become

$$\frac{d}{dt}\hat{x}(t) = -i\left[\hat{x}(t), \hat{H}\right] = \hat{p}, \quad \frac{d}{dt}\hat{p}(t) = -\omega^2\hat{x}$$

which have exactly the same form as the classical equations. And again, we can introduce the quantum operators $\hat{a}(t) = \hat{a}_0 e^{-i\omega t}$, so that $\hat{a}_0$ is now an operator. We can now just take our solution form from above and make everything into operators:

$$\hat{x}(t) = \frac{1}{\sqrt{2\omega}} \left(\hat{a}_0 e^{-i\omega t} + \hat{a}_0^\dagger e^{i\omega t} \right). \tag{14}$$

We can obtain the expression for $\hat{p}(t)$ by taking time derivative on the above equation. Now canonical commutation $[\hat{x}(t), \hat{p}(t)] = i$ implies that

$$[\hat{a}_0, \hat{a}_0^\dagger] = 1 \ .$$

Plugging (16) and the corresponding equation for $hatp(t)$ into $\hat{H}$ and being careful about **quantum operator ordering**, we have

$$\hat{H} = \omega \left(a_0^\dagger a_0 + \frac{1}{2} \right),$$

where the $\frac{1}{2}$ came from non-commutativity of a_0 and $a_0^\dagger$. (this part we have to calculate explicitly, and we'll see many versions of the $\frac{1}{2}$ when we quantize strings later on).

A convenient basis for the Hilbert space (just like we're used to) is to define a ground state $|0\rangle$ so that $a_0 |0\rangle = 0$, and then inductively define

$$|n\rangle = \frac{1}{\sqrt{n!}} (a_0^\dagger)^n |0\rangle \ .$$

Again, the point of this discussion is that there are a lot of parallels with the classical case, and once we solve the classical equations of motion, we can often recognize the correct basis to use for indexing our Hilbert space.

Now that we've seen two examples of canonical quantization using the Heisenberg picture, we now elaborate about the **role of conserved charges** in the quantum theory. If we start with a **classical** action $S = \int dt \ L(q_i, \dot{q}_i)$, and suppose that the classical system is invariant under some symmetry $\delta q_i = \varepsilon f_i(q_j)$, we can find a conserved charge (there's nothing to integrate here because we have no spatial directions in the expression for S)

$$Q = \frac{\partial L}{\partial \dot{q}_i} f_i(q_j) = p_i f_i(q_j),$$

where we sum over i, and in the second equality write it in terms of the canonical momentum p_i.

Now we move to the quantum theory in which $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$. Then $\hat{Q}$ becomes a conserved **operator** instead, of the form

$$\hat{Q} = \hat{p}_i f_i(\hat{q}_j) \text{ (up to some ordering)}.$$

Again, because p and q don't commute, there might be some operator ambiguity here, but we'll find later on that there is always a physical principle we can use to fix the form of this ordering (for example, if we require $\hat{Q}$ to be Hermitian). Usually, we basically just need to choose a location of the different ps and qs. And now we can calculate

$$\left[i\varepsilon\hat{Q}, \hat{q}_i \right] = i\varepsilon[\hat{p}_j f_j(\hat{q}_k), \hat{q}_i] = \varepsilon f_i(\hat{q}_k) = \delta\hat{q}_i,$$

so this conserved charge $\hat{Q}$ **generates the symmetry transformations** when we're working in the quantum theory! More generally, if we have multiple symmetries $\hat{Q}_i$, they will form a closed algebra of the form

$$[\hat{Q}_i, \hat{Q}_j] = i f_{ijk} \hat{Q}_k,$$

because $\hat{Q}_i\hat{Q}_j - \hat{Q}_j\hat{Q}_i$ is the difference of two conserved charges (since the composition of two symmetries is a symmetry itself). So the key is that we form a **closed Lie algebra**, and that will be the algebra of the corresponding symmetries (all of which will commute with the Hamiltonian).

Example 102
Consider a Lagrangian of the form $L = \frac{1}{2}m\dot{x}^2 - V(x)$, with corresponding Hamiltonian $\frac{p^2}{2m} + V(x)$. Suppose that we have a continuous translation symmetry $x \mapsto x + a$, which means that V must be a constant.

In this case, we find that the conserved charge $\hat{Q}$ is just $\hat{p}$, and we have $[\hat{x}, \hat{p}] = i$. If we then look at the commutator of $\hat{Q}$ with $\hat{q}_i$ as above, then we find that

$$\left[i\varepsilon\hat{Q}, \hat{x}\right] = [i\varepsilon\hat{p}, \hat{x}] = \varepsilon = \delta\hat{x}.$$

So $\hat{Q} = \hat{p}$ indeed generates the translations, which is what we expect in quantum mechanics.

Example 103
Next, consider a three-dimensional Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + V(r),$$

which has rotational symmetry.

We know (from our study of quantum mechanics) that the the angular momentum operators $\hat{L}_i$ should be the conserved operators (where $\hat{L}_i$ is the rotation around the x^i-axis), where $\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k$. We can check that $[\hat{L}_i, \hat{H}] = 0$, and now we can evaluate the commutators as above:

$$\left[i\varepsilon_i\hat{L}_i, \hat{x}_j\right] = \delta x_j,$$

and furthermore the $\hat{L}_i$s form the Lie algebra of angular momentum, so that $\left[\hat{L}_i, \hat{L}_j\right] = i\varepsilon_{ijk}\hat{L}_k$. So again in this case, we find that the conserved charges corresponding to symmetries become conserved operators at the quantum level, which then generate the corresponding symmetry transformations.

Next time, we'll do this process for the case of the relativistic point particle, which will allow us to quantize the point particle (since we already know solutions to the equations of motion for the particle). But the realization of the algebra will turn out to be less simple, because of gauge symmetries, and that gauge fixing will occur in the quantum string study as well!

March 29, 2021

Last lecture, we started talking about symmetries and conservation laws in preparation for talking about quantum strings: we mentioned some examples from classical mechanics, such as time translation and energy conservation or spatial translation and momentum conservation. And we can formulate this more generally:

Theorem 104 (Noether)
In any Lagrangian field theory, any continuous global symmetry corresponds to a conservation law of the form $\partial_\alpha j^\alpha = 0$ (for some current j^α).

For some setup, we can write our general field theory action in the form

$$S = \int d\xi^0 d\xi^1 \cdots d\xi^k \mathcal{L}(\phi^a, \partial_\alpha \phi^a),$$

where ξ^α are our coordinates and the $\phi^a(\xi^\alpha)$ are dynamical fields which are functions of the coordinates. (Notice that we use a different label a for the fields, because we can have any number of them). The term $\mathcal{L}(\phi^a, \partial_\alpha \phi^a)$ is known as the **Lagrangian density**, and we should notice that it's a function of our fields ϕ^a and their first derivatives.

Example 105

The string action is a $(1+1)$-dimensional field theory with $\xi^\alpha = (\tau, \sigma)$ and $\phi^a = \{X^\mu, h_{\alpha\beta}\}$ where the fields are the set of dynamical variables we used in the Polyakov action.

Example 106

The particle action is a $(0+1)$-dimensional field theory with $\xi^\alpha = \tau$ (only one coordinate) and fields given by $\phi^a = \{X^\mu, e\}$ (remembering that e was our new dynamical variable for the relativistic free particle).

Example 107

Electromagnetism is a $(d+1)$-dimensional field theory with $\xi^\alpha = (X^0, \vec{X})$ and $\phi^a = \{A_\alpha(X^0, \vec{x})\}$.

In all of these cases, we vary the action to find equations of motion. But we can also find a general formulation for the equation of motion coming out of $\delta S = 0$: for any field theory action S, we have

$$\delta S = \int d^{k+1}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a + \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \partial_\alpha \delta \phi^a \right)$$

by the chain rule on $\mathcal{L}$. Applying integration by parts on the second term turns this into

$$= \int d^{k+1}\xi \left(\frac{\partial \mathcal{L}}{\partial \phi^a} \delta \phi^a - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) \delta \phi^a \right)$$

plus some boundary terms, but we'll always assume that those boundary terms can be made to vanish. And if this needs to be true for any ϕ^a variation, we get the **Euler-Lagrange equations**

$$\boxed{\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\alpha \left(\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} \right) = 0}$$

for each field a, where we're summing over α. (And we can use this to derive the equations of motion for our classical actions as well.)

Proof sketch for Theorem 104. We can now talk about what we mean when we have a **symmetry**: basically, if we have transformations $\phi^a \to \phi'^a$ which leave the action S invariant, we call those symmetries. And we can also be more specific with the vocabulary of Theorem 104:

Definition 108

A **continuous symmetry** can be described by transformations $\phi'^a(\lambda, \xi^\alpha)$ that depend on a continuous parameter λ, so that $\phi'^a(0) = \phi^a$ and $\phi'^a(\lambda)$ is a symmetry for all λ. Symmetries with λ that depend on ξ^α are **local symmetries**, and symmetries with λ that don't are **global symmetries**.

Infinitesimal transformations are those with λ being equal to an infinitesimal ε, we can expand out the transformations in small ε and find

$$\phi'^a(\varepsilon, \xi^\alpha) = \phi^a + \delta\phi^a = \phi^a(\xi^\alpha) + \varepsilon h^a(\phi) + O(\varepsilon^2),$$

where h^a is some function. We will consider global symmetries with ϵ being a constant. In the simplest situation, the Lagrangian density itself is invariant under the symmetric transformations (so does the action)

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = 0$$

from which we have

$$\frac{\partial \mathcal{L}}{\partial \phi^a}\varepsilon h^a + \frac{\partial L}{\partial(\partial_\alpha \phi^a)}\varepsilon \partial_\alpha h^a = 0 \ .$$

Now we can cancel out the εs and use our Euler-Lagrange equation to find that

$$\partial_\alpha \left(\frac{\partial L}{\partial(\partial_\alpha \phi^a)} \right) h^a + \frac{\partial L}{\partial(\partial_\alpha \phi^a)}\partial_\alpha h^a = 0, \tag{15}$$

and now what we have is actually a total derivative: we can write it as

$$\partial_\alpha \left(\frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a \right) = 0,$$

and we've found our desired current

$$\boxed{j^\alpha = \frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a}$$

(where we're summing over a). □

Fact 109

We assumed above that $\mathcal{L}$, the Lagrangian density, is invariant. More generally, for the action to be invariant, the Lagrangian density only needs to be invariant up to a total derivative

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = \varepsilon \partial_\alpha K^\alpha.$$

We then end up with $\partial_\alpha K^\alpha$s on the right-hand side of (15) instead of 0, so the current is instead

$$\boxed{j^\alpha = \frac{\partial L}{\partial(\partial_\alpha \phi^a)} h^a - K^\alpha} \ .$$

Remark 110. *In general, there can be more than one symmetry for an action, so that our transformations may look like $\phi^a \to \phi'(a) = \phi(a) + \varepsilon_i h_i^a$, where the i's label the different symmetries. We then get a conserved current j_i^α for each symmetry, given by $\frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} h_i^a - K_i^\alpha$,*

Remark 111. *The above discussion only applies when there is a continuous parameter λ. A discrete symmetry like $\phi^a \mapsto -\phi^a$ in general does not lead to a conservation law.*

The above discussion can be applied to a local symmetry but in general does not result in a nontrivial j^α.

The discussion is rather abstract, but it is extremely general, applicable to any Lagrangian field theories.

Example 112

Recall that global symmetries of the Polyakov action include translations $X^\mu \mapsto X^\mu + a^\mu$ and Lorentz transformations $X^\mu \mapsto L^\mu{}_\nu X^\nu$.

We'll set $h_{\alpha\beta} = \eta_{\alpha\beta}$, because h does not appear in any of the global symmetries. So then our action simplifies in form to

$$S_P = -\frac{T_0}{2}\int d^2\xi \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

and now we can work on the conservation laws: there are d different symmetries of the form $X^\mu \mapsto X^\mu + a^\mu$ (one for each dimension), so we'll look specifically at the translations along one of the dimensions:

$$X^\mu \mapsto X^\mu + \delta^\mu_\nu a^\nu.$$

Now, a^ν is ε_i, and δ^μ_ν is our h^a_i. So the current that comes out of this looks like

$$\Pi^\alpha_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\nu)}\delta^\nu_\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\mu)} = \boxed{-T_0 \partial^\alpha X_\mu},$$

and more explicitly we have two components

$$\Pi^0_\mu = T_0 \partial_\tau X_\mu, \quad \Pi^1_\mu = -T_0 \partial_\sigma X_\mu \ .$$

We can check that the currents are indeed conserved

$$\partial_\alpha \Pi^\alpha_\mu = 0 \implies \partial^2_\tau X_\mu - \partial^2_\sigma X_\mu = 0.$$

provided the equations of motion for X^μ are satisfied.

Recall that when we have a conserved current j^α, we can integrate j^0 over all spatial directions to get a conserved charge $Q = \int d^k \xi \, j^0$. In this case, we have

$$p_\mu = \int d\sigma \Pi^0_\mu = \boxed{T_0 \int d\sigma \partial_\tau X_\mu},$$

which is the conserved charge on the worldsheet corresponding to translation symmetry in the target spacetime. And remembering that translation symmetry gave us conservation of momentum, we will "define" p_μ as the **spacetime center of mass momentum** for the string, and Π^1_μ is then the spacetime **momentum current** on the worldsheet.

Consider now a **closed string**. We have

$$\partial_\tau p_\mu + \int_0^{2\pi} d\sigma \partial_\sigma \Pi^1_\mu = 0 \quad \to \quad \partial_\tau p_\mu = 0$$

where the second term of the first equation integrates to zero due to Π^1_μ being periodic. Recall the general solution for a closed string

$$X^\mu = x^\mu_0 + v^\mu \tau + \sum_{n \neq 0} (\text{oscillatory modes}),$$

and thus

$$p^\mu = T_0 \int_0^{2\pi} d\sigma \partial_\tau X^\mu = 2\pi T_0 v^\mu,$$

where the contributions from oscillatory modes disappear upon integration. The above expression is manifest τ-independent and confirms our earlier identification of spacetime momentum as the canonical momentum on the worldsheet for the center of mass motion.

For an **open string**, if we have the Neumann boundary conditions for both ends, meaning that $\Pi^1_\mu|_{\sigma_*} = 0$, we have

$$\partial_\tau p_\mu + \int_0^{\pi} d\sigma \partial_\sigma \Pi^1_\mu = 0 \quad \to \quad \partial_\tau p_\mu + \Pi^1_\mu\big|_{\sigma=\pi} - \Pi^1_\mu\big|_{\sigma=0} = 0 \quad \to \quad \partial_\tau p_\mu = 0 \ .$$

In this case, we have momentum conservation as there is **no momentum flux** out of the string endpoints. Furthermore, evaluating p_μ on the general solution to equations of motion we find

$$p_\mu = \pi T_0 v^\mu$$

which is a constant and confirms the expression obtained earlier from canonical momentum on the worldsheet for ceneter of mass motion. When one or both end points have Dirichlet condition, we have $\Pi^1_\mu|_{\sigma_*} \neq 0$, and thus

$$\partial_\tau p_\mu + \Pi^1_\mu\big|_{\sigma=\pi} - \Pi^1_\mu\big|_{\sigma=0} = 0 \quad \rightarrow \quad \partial_\tau p_\mu \neq 0\ .$$

So if any of the string endpoints are in the Dirichlet boundary conditions, we no longer have momentum conservation – it will leak out of the string endpoints.

Example 113

Consider a D-brane in the x^1x^2 plane, so that we have NN boundary conditions in $X^{0,1,2}$ but DD boundary conditions in X^3.

The string can then move freely in the x^1, x^2 directions but not in the x^3 direction. That means that translation symmetry is broken in the x^3 direction because of the D-brane, and thus we do not have momentum conservation in the x^3 direction (momentum flows out from the endpoints into the D-brane). And this makes sense if we remember that in the DD and ND most general solutions for the wave equation, we do not have the $v^\mu\tau$ linear term.

Fact 114

And remember that we must have NN boundary conditions in the X^0 direction, so p_0 (which we can interpret as the energy) is still conserved.

We'll now turn our attention to the other kind of global symmetry for the Polyakov action, which is the **Lorentz transformations**. We then need an infinitesimal version of the transformation $X'^\mu = L^\mu{}_\nu X^\nu$, and we'll look at a few examples to understand how to do this.

- Infinitesimal rotations in the 2-dimensional plane take the form

$$\begin{bmatrix}\tilde{x}^1 \\ \tilde{x}^2\end{bmatrix} = \begin{bmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x^1 \\ x^2\end{bmatrix}.$$

  for **small** θ. Taylor expanding to first order tells us that for $\theta \ll 1$, we have

$$\begin{bmatrix}\tilde{x}^1 \\ \tilde{x}^2\end{bmatrix} = \begin{bmatrix}x^1 \\ x^2\end{bmatrix} + \begin{bmatrix}\theta x^2 \\ -\theta x^1\end{bmatrix},$$

  meaning that $\delta x^1 = -\theta x^2$ and $\delta x^2 = -\theta x^1$.

- An infinitesimal boost comes from a small velocity parameter β in the equation

$$\begin{bmatrix}\tilde{x}^0 \\ \tilde{x}^1\end{bmatrix} = \begin{bmatrix}\cosh\beta & \sinh\beta \\ \sinh\beta & \cosh\beta\end{bmatrix}\begin{bmatrix}x^0 \\ x^1\end{bmatrix},$$

  which (again by Taylor expansion) gives us $\delta x^0 = \beta x^1$ and $\delta x^1 = \beta x^0 = -\beta x_0$.

And we can unify the two examples above, even though there's an extra negative sign in one of them, by writing things with lower indices instead of upper indices: all of the transformations of rotations and boosts take an antisymmetric form

$$\boxed{\delta x^\mu = \theta^{\mu\nu} x_\nu},$$

where $\theta^{\mu\nu}$ is **antisymmetric**. (So then the rotation would correspond to $\theta^{12} = -\theta^{21} = \theta$, and the boost would correspond to $\theta^{01} = -\theta^{10} = \beta$.)

The above expression applies to any infinitesimal rotations and boosts, with θ^{ij} being an infinitesimal rotation angle in x_i–x_j plane and θ^{0i} being an infinitesimal boost velocity along x_i-direction. Altogether we have $\frac{1}{2}d(d-1)$ independent parameters, so there are $\frac{1}{2}d(d-1)$ different conservation laws that will come out of Lorentz transformations. (In $d = 4$, these correspond to the three rotations and three boosts.)

For any pair (μ, ν), we can look at the specific infinitesimal transformation

$$X^\lambda = X^\lambda + \eta \varepsilon^{\lambda\rho} X_\rho,$$

where $\varepsilon^{\mu\nu} = 1 = -\varepsilon^{\nu\mu}$, i.e. $\epsilon^{\lambda\rho}$ is nonzero only if λ, ρ are equal to μ, ν. If we then plug everything back into our formula from Noether's theorem, we get the conserved current

$$M^\alpha_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha X^\lambda)} \varepsilon^{\lambda\rho} X_\rho = \boxed{\Pi^\alpha_\mu X_\nu - \Pi^\alpha_\nu X_\mu}$$

(where summing λ, ρ while fixing μ, ν). Note the above expression is manifestly antisymmetric in μ, ν. We'll elaborate more what these conserved currents mean next time!

March 31, 2021

Last time, we studied symmetries and conserved quantities. Suppose we have an action of the form

$$S = \int d^{k+1}\xi \, \mathcal{L}(\phi^a, \partial_\alpha \phi^a)$$

for some Lagrangian density $\mathcal{L}$, and this action is invariant under an infinitesimal transformation

$$\phi^a \to \phi^a + \varepsilon h^a(\phi)$$

for a small constant parameter ε (corresponding to global symmetries). Accordingly $\mathcal{L}$ can at most change by a total derivative: letting ∂_α denote derivatives with respect to ξ^α, we have

$$\mathcal{L}(\phi^a + \varepsilon h^a) - \mathcal{L}(\phi^a) = \varepsilon \partial_\alpha K^\alpha,$$

and then we get the **conserved current**

$$j^\alpha = \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \phi^a)} h^a - K^\alpha$$

(meaning that $\partial_\alpha j^\alpha = 0$). And using the these conserved quantities can make calculations much easier than solving equations of motion directly! For example, we saw last time that translations $X^\mu \to X^\mu + a^\mu$ keep the string action invariant, which gives us the conserved current $\Pi^\alpha_\mu = -T_0 \partial_\alpha X_\mu$. The corresponding **conserved charge**

$$p_\mu = \int d\sigma \, \Pi^0_\mu$$

is interpreted as the **spacetime momentum** for our string. (Analogously, we also have the **momentum current** on the worldsheet Π^1_μ, which tells us how spacetime momentum flows along the string. Remember that when we have Neumann boundary condition at a string end point, this momentum current is fixed to be 0 at that point.)

Invariance of the string action under Lorentz symmetries leads to the current

$$M^\alpha_{\mu\nu} = \Pi^\alpha_\mu X_\nu - \Pi^\alpha_\nu X_\mu,$$

which is manifestly antisymmetric and also satisfies $\partial_\alpha M^\alpha_{\mu\nu} = 0$ from the equations of motion (last part left as an exercise).

To understand why this form is particularly suggestive, let's consider the Lorentz transformations corresponding to spatial rotations, meaning that $(\mu, \nu) = (i, j)$ for some spatial coordinates i, j. (M_{ij}^α is the current corresponding to rotations in $x^i - x^j$ plane.) Then we have

$$M_{ij}^\alpha = \Pi_i^\alpha X_j - \Pi_j^\alpha X_i.$$

This looks a lot like our standard $p_i x_j - p_j x_i$ angular momentum definition, and the conserved quantity associated with spatial rotations should indeed be interpreted as the angular momentum:

Definition 115

The **angular momentum** of a string for rotations in $x^i - x^j$ plane is defined as

$$m_{ij} = \int d\sigma M_{ij}^0.$$

(In three spatial dimensions we often associate rotations in ij-plane with its orthogonal direction, but this no longer works in general d-dimensional space!)

We can then also look at the corresponding quantities M_{0i}^α, corresponding to Lorentz boosts in the x^i direction: we have the similar form

$$M_{0j}^\alpha = \Pi_0^\alpha X_j - \Pi_j^\alpha X_0,$$

and then this current leads to a **boost charge**

$$m_{0i} = \int d\sigma\, M_{0i}^0.$$

We will later see that all of these charges will play important roles in quantum mechanics!

Remark 116. *Notice that for a given classical string configuration, we now have a reliable way to compute its spacetime momentum and angular momentum (our previous methods involved looking at the Hamiltonian along the worldsheet, but we now have a proper definition).*

Remark 117. *If we move to quantum mechanics, conserved charges become conserved operators, and then the role of these operators becomes even more significant: the commutators of the charges generate the algebra of the symmetry. So these Lorentz charges will give us the Lorentz algebra in quantum mechanics!*

3.2: Quantization of the relativistic particle

We're now ready to move on to quantization, but we'll do the relativistic particle as a warmup for the string (just like we did when we wrote down the action).

3.2.1: Canonical quantization

We'll start by reminding ourselves about the **canonical quantization**, which is the method of going from a classical theory to a quantum theory.

Fact 118

From here on, we'll have $\hbar = 1$ to make our notation simpler.

Example 119

Consider a theory where we just have one degree of freedom (x, p).

One way to do this quantization is through the **Schrodinger picture**: if we go from classical theory to quantum theory, we promote x and p to operators, and we turn the Hamiltonian $H(x, p)$ into $\hat{H}(\hat{x}, \hat{p})$, so that our system's state is described by wavefunction solutions to the Schrodinger equation $i\partial_t \psi(t, x) = \hat{H}\psi(t, x)$.

An equivalent formulation is the **Heisenberg picture**: the dynamical variables of the system are now the operators instead of the wavefunctions. We then have the evolution of an operator $\hat{O}$ given by the Heisenberg equation of motion

$$\frac{d}{dt}\hat{O}(t, x) = \frac{\partial}{\partial t}\hat{O}(t, x) - i[\hat{O}(t, x), \hat{H}],$$

and states ψ do not evolve. (In some sense, this means that the states provide a frame of measurement, much like the operators did in the Schrodinger picture.) There is a natural connection between classical equations of motion and quantum Heisenberg equations. Recall that classical equation of motions can be obtained from the Hamiltonian by

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x},$$

and it turns out there is a precise correspondence from these classical Hamiltonian equations to the quantum operator equations if we replace the classical variables with the corresponding quantum operators (setting x to $\hat{x}$ and p to $\hat{p}$), up to some operator ordering ambiguity because the x and p don't commute anymore in the quantum world. So what that means is that if we have the classical solutions for some Hamiltonian, we also basically have the solutions to the operator equations, and thus we've solved the corresponding quantum mechanics problem. (And the integration constants in the classical equations of motion correspond to the "integration constants" that come up as quantum operators.) That's a good sign, because we can then take the solutions to the classical string and convert them to the quantum string!

But we'll start by reviewing some simple examples in the Heisenberg picture, because many of us are less familiar with this way of treating quantum mechanics.

Example 120

Consider the free non-relativistic particle, in which we have $H = \frac{p^2}{2m}$.

The classical equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = 0;$$

solving, we find that $p = p_0$ and $x = \frac{p_0}{m}t + x_0$, where x_0, p_0 are our two integration constants. If we want to turn this into a quantum setup, we turn x, p to $\hat{x}, \hat{p}$ and impose $[\hat{x}, \hat{p}] = i$ (at any time t). We have no operator ambiguity problems here – $\hat{H} = \frac{\hat{p}^2}{2m}$ – so the Heisenberg equations of motion are

$$\frac{d}{dt}\hat{x}(t) = -i\left[\hat{x}(t), \hat{H}\right] = \frac{\hat{p}}{m},$$

(because x has no explicit dependence on t, $\frac{\partial x}{\partial t} = 0$) and similarly

$$\frac{d}{dt}\hat{p}(t) = -i[\hat{p}(t), H] = 0.$$

So the quantum operator equations are solved by $\hat{p}(t) = \hat{p}_0$ for some constant operator $\boxed{\hat{p}_0}$, and then $\boxed{\hat{x}(t) = \frac{\hat{p}_0}{m}t + \hat{x}_0}$.

So as we said, everything just becomes an operator, and the commutation relation also implies that we must have

$$[\hat{x}(t), \hat{p}(t)] = i \implies \boxed{[\hat{x}_0, \hat{p}_0] = i}.$$

Since $\hat{H} = \frac{\hat{p}^2}{2m} = \frac{\hat{p}_0^2}{2m}$ it's convenient to choose a basis for our Hilbert space so that we have an eigenbasis of $\hat{p}_0$ (so that we're looking at "momentum eigenstates"). Indexing our states by p so that

$$\hat{p}_0 |p\rangle = p |p\rangle ,$$

we know that $|p\rangle$ is a plane wave, and we can normalize it with the usual normalization $\langle p|p'\rangle = 2\pi\delta(p - p')$.

We have now completely solved the quantum theory and we can now compute anything: we know the full time-dependence of $\hat{x}$ and $\hat{p}$, and notice that the quantum story throughout this problem has been very parallel to the classical story.

Example 121

Next, consider the harmonic oscillator $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$, where we'll set $m = 1$ for convenience.

The classical equations of motion are

$$\dot{x} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial x} = -\omega^2 x.$$

We'll solve these equations in a slightly different way from how we usually do: because these are two first-order linear differential equations, we can diagonalize and decouple by introducing

$$a = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x + \frac{i}{\sqrt{\omega}}p\right), \quad a^* = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x - \frac{i}{\sqrt{\omega}}p\right).$$

We can plug these back in and find that

$$\dot{a} = -i\omega a, \quad \dot{a}^* = -i\omega a^*,$$

and then we can solve these equation to find $a(t) = a_0 e^{-i\omega t}$ and $a^*(t) = a_0^* e^{i\omega t}$, so that

$$x(t) = \frac{1}{\sqrt{2\omega}}\left(a(t) + a^*(t)\right) = \frac{1}{\sqrt{2\omega}}\left(a_0 e^{-i\omega t} + a_0^* e^{i\omega t}\right).$$

The integration constants are a_0, a_0^*. We can also calculate the value of the Hamiltonian (i.e the energy of the system) for the classical motion. Plugging the above solution into the expression for H we have $H = \omega a^*(t)a(t) = \omega a_0^* a_0$, where the last expression shows that energy is conserved (as expected) as the value of H is time-independent.

Now let's turn to quantum mechanics, with

$$\hat{H} = \frac{\hat{p}^2}{2} + \frac{1}{2}\omega^2\hat{x}^2$$

and we still have the canonical commutation relation $[\hat{x}(t), \hat{p}(t)] = i$. The Heisenberg equations of motion then become

$$\frac{d}{dt}\hat{x}(t) = -i\left[\hat{x}(t), \hat{H}\right] = \hat{p}, \quad \frac{d}{dt}\hat{p}(t) = -\omega^2\hat{x}$$

which have exactly the same form as the classical equations. And again, we can introduce the quantum operators $\hat{a}(t) = \hat{a}_0 e^{-i\omega t}$, so that $\hat{a}_0$ is now an operator. We can now just take our solution form from above and make everything into operators:

$$\hat{x}(t) = \frac{1}{\sqrt{2\omega}}\left(\hat{a}_0 e^{-i\omega t} + \hat{a}_0^\dagger e^{i\omega t}\right). \tag{16}$$

We can obtain the expression for $\hat{p}(t)$ by taking time derivative on the above equation. Now canonical commutation $[\hat{x}(t), \hat{p}(t)] = i$ implies that

$$[\hat{a}_0, \hat{a}_0^\dagger] = 1 \ .$$

Plugging (16) and the corresponding equation for $hatp(t)$ into $\hat{H}$ and being careful about **quantum operator ordering**, we have

$$\hat{H} = \omega\left(a_0^\dagger a_0 + \frac{1}{2}\right),$$

where the $\frac{1}{2}$ came from non-commutativity of a_0 and $a_0^\dagger$. (this part we have to calculate explicitly, and we'll see many versions of the $\frac{1}{2}$ when we quantize strings later on).

A convenient basis for the Hilbert space (just like we're used to) is to define a ground state $|0\rangle$ so that $a_0 |0\rangle = 0$, and then inductively define

$$|n\rangle = \frac{1}{\sqrt{n!}}(a_0^\dagger)^n |0\rangle \ .$$

Again, the point of this discussion is that there are a lot of parallels with the classical case, and once we solve the classical equations of motion, we can often recognize the correct basis to use for indexing our Hilbert space.

Now that we've seen two examples of canonical quantization using the Heisenberg picture, we now elaborate about the **role of conserved charges** in the quantum theory. If we start with a **classical** action $S = \int dt\ L(q_i, \dot{q}_i)$, and suppose that the classical system is invariant under some symmetry $\delta q_i = \varepsilon f_i(q_j)$, we can find a conserved charge (there's nothing to integrate here because we have no spatial directions in the expression for S)

$$Q = \frac{\partial L}{\partial \dot{q}_i} f_i(q_j) = p_i f_i(q_j),$$

where we sum over i, and in the second equality write it in terms of the canonical momentum p_i.

Now we move to the quantum theory in which $[\hat{q}_i, \hat{p}_j] = i\delta_{ij}$. Then $\hat{Q}$ becomes a conserved **operator** instead, of the form

$$\hat{Q} = \hat{p}_i f_i(\hat{q}_j) \text{ (up to some ordering)}.$$

Again, because p and q don't commute, there might be some operator ambiguity here, but we'll find later on that there is always a physical principle we can use to fix the form of this ordering (for example, if we require $\hat{Q}$ to be Hermitian). Usually, we basically just need to choose a location of the different ps and qs. And now we can calculate

$$\left[i\varepsilon\hat{Q}, \hat{q}_i\right] = i\varepsilon[\hat{p}_j f_j(\hat{q}_k), \hat{q}_i] = \varepsilon f_i(\hat{q}_k) = \delta\hat{q}_i,$$

so this conserved charge $\hat{Q}$ **generates the symmetry transformations** when we're working in the quantum theory! More generally, if we have multiple symmetries $\hat{Q}_i$, they will form a closed algebra of the form

$$[\hat{Q}_i, \hat{Q}_j] = i f_{ijk}\hat{Q}_k,$$

because $\hat{Q}_i\hat{Q}_j - \hat{Q}_j\hat{Q}_i$ is the difference of two conserved charges (since the composition of two symmetries is a symmetry itself). So the key is that we form a **closed Lie algebra**, and that will be the algebra of the corresponding symmetries (all of which will commute with the Hamiltonian).

Example 122

Consider a Lagrangian of the form $L = \frac{1}{2}m\dot{x}^2 - V(x)$, with corresponding Hamiltonian $\frac{p^2}{2m} + V(x)$. Suppose that we have a continuous translation symmetry $x \mapsto x + a$, which means that V must be a constant.

In this case, we find that the conserved charge $\hat{Q}$ is just $\hat{p}$, and we have $[\hat{x}, \hat{p}] = i$. If we then look at the commutator of $\hat{Q}$ with $\hat{q}_i$ as above, then we find that

$$\left[i\varepsilon\hat{Q}, \hat{x}\right] = [i\varepsilon\hat{p}, \hat{x}] = \varepsilon = \delta\hat{x}.$$

So $\hat{Q} = \hat{p}$ indeed generates the translations, which is what we expect in quantum mechanics.

Example 123

Next, consider a three-dimensional Hamiltonian

$$H = \frac{\vec{p}^2}{2m} + V(r),$$

which has rotational symmetry.

We know (from our study of quantum mechanics) that the the angular momentum operators $\hat{L}_i$ should be the conserved operators (where $\hat{L}_i$ is the rotation around the x^i-axis), where $\hat{L}_i = \varepsilon_{ijk}\hat{x}_j\hat{p}_k$. We can check that $[\hat{L}_i, \hat{H}] = 0$, and now we can evaluate the commutators as above:

$$\left[i\varepsilon_i\hat{L}_i, \hat{x}_j\right] = \delta x_j,$$

and furthermore the $\hat{L}_i$s form the Lie algebra of angular momentum, so that $\left[\hat{L}_i, \hat{L}_j\right] = i\varepsilon_{ijk}\hat{L}_k$. So again in this case, we find that the conserved charges corresponding to symmetries become conserved operators at the quantum level, which then generate the corresponding symmetry transformations.

Next time, we'll do this process for the case of the relativistic point particle, which will allow us to quantize the point particle (since we already know solutions to the equations of motion for the particle). But the realization of the algebra will turn out to be less simple, because of gauge symmetries, and that gauge fixing will occur in the quantum string study as well!

April 5, 2021

Last lecture, we reviewed the concept of **canonical quantization** in the Heisenberg picture, looking at particular systems that are simple but demonstrate the ideas behind quantization of particles and strings. In such a quantization, we turn classical conserved charges Q_i into quantum conserved operators $\hat{Q}_i$ (which commute with the Hamiltonian), and what's important is that these $\hat{Q}_i$s generate the symmetry transformations which gave us the charges in the first place: for example, we get transformations of our dynamical variables via

$$\left[i\varepsilon_i\hat{Q}_i, \hat{q}_j\right] = \delta\hat{q}_j.$$

Furthermore, the commutator between two such charges is itself a linear combination of charges:

$$[\hat{Q}_i, \hat{Q}_j] = if_{ijk}\hat{Q}_k.$$

Today, we'll move into the **quantization of the relativistic particle**. Let's first do a review: we start with the action

$$S = \frac{1}{2}\int d\tau\left(e^{-1}\dot{x}_\mu\dot{x}^\mu - em^2\right)$$

(remembering that $c = 1$) and choose the gauge so that $e = \frac{1}{m}$. We then get

$$S = \frac{1}{2}\int d\tau\left(m\dot{x}_\mu\dot{x}^\mu - m\right),$$

and the resulting equations of motion from the action are $\frac{dp^\mu}{d\tau} = 0, \quad p^\mu = m\dot{x}^\mu$. We can then write the classical solution in two different gauges: in the **static gauge**, we set $x^0 = \frac{p^0}{m}\tau$ and $x^i = \frac{p^i}{m}\tau + x_0^i$ (where $(p^0)^2 = \vec{p}^2 + m^2$, so that our independent dynamical variables are p^i and x_0^i). But we can also write our solutions in the **light-cone** gauge, in which $x^+ = \frac{p^+}{m}\tau$, so that

$$x^- = x_0^- + \frac{p^-}{m}\tau, \quad x^I = x_0^I + \frac{p^I}{m}\tau.$$

This time, we get the condition $p^- = \frac{1}{2p^+}\left(\vec{p}_\perp^2 + m^2\right)$, so that our independent variables are again x_0^-, x_0^I, p^+, p^I. (In both cases, we get the same number of degrees of freedom.)

So now if we move to the quantum theory, recall that we need to turn our independent parameters into independent operators, imposing the canonical commutation relations. Our classical solutions should then become solutions to operator equations – remember that in the nonrelativistic particle and the quantum harmonic oscillator, once we did those steps, we were done. But in this case, we have some extra dependent variables, like the relation for p^0 or p^- in the two different gauges above, so we need to **turn those into operator equations as well**.

Fact 124

We'll use the light-cone gauge from now on, because solving the relation $(p^0)^2 = \vec{p}^2 + m^2$ in the static gauge requires taking a square root, which is very nontrivial.

So we'll turn our attention to the equation

$$\hat{p}^- = \frac{1}{2\hat{p}^+}(\hat{p}_\perp^2 + m^2)$$

(here, $\hat{p}^-$ is a **composite operator** as it is expressed in terms of other operatoes). Then we know we have the canonical commutation relations for the independent variables

$$[x_0^I, p^J] = i\delta^{IJ} = i\eta^{IJ}, \quad [x_0^-, p^+] = i\eta^{-+} = -i$$

(remembering that x^- and p^+ are conjugate variables because of the off-diagonal nature of the light-cone frame), and otherwise things commute $[x_0^-, x_0^I] = 0, [p^+, p^I] = 0$ (we'll drop the operator hats from now on unless there is ambiguity).

Fact 125

Notice that **we don't write down a commutator with** p^- here because it's a composite operator – this will soon make a difference for us!

We can now write down operator equations like

$$\hat{x}^-(\tau) = \hat{x}_0^- + \frac{\hat{p}^-}{m}\tau, \cdots,$$

which will characterize the motion of every dynamical variable. A convenient basis for us to use for this Hilbert space is the momentum basis $|p^+, \vec{p}_\perp\rangle$ (so eigenstates of both p^+ and p^I) with the standard normalization

$$\langle p^+, p^I | p'^+, p'^J\rangle = (2\pi)^{d-1}\delta(p^+ - p'^+)\prod_I \delta(p^I - p'^I),$$

where we should remember that p^I indicates the collection of all transverse indices I simultaneously. Such states are also eigenstates of $\hat{p}^-$ as

$$\hat{p}^- \left|p^+, p^I\right\rangle = \frac{1}{2p^+}\left(\vec{p}_\perp^2 + m^2\right)\left|p^+, p^I\right\rangle$$

with eigenvalue $p^- = \frac{1}{2p^+}\left(\vec{p}_\perp^2 + m^2\right)$, so we have the mass-shell condition $2p^+p^- - \vec{p}_\perp^2 = m^2$.

We're now ready to look at the conserved charges of this system: recall that we have translational symmetries $\delta x^\mu = \varepsilon^\mu$, which give us p^μ conservation, and Lorentz transformations $\delta x^\mu = \varepsilon^{\mu\nu}x_\nu$, which give us $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ conservation. So as we've been saying, these charges now become quantum operators $\hat{p}^\mu$ and $\hat{M}^{\mu\nu}$, and we want to figure out the properties for these operators.

Example 126

For the momentum operators, we may first expect that p^μ will generate the translations, so that

$$[i\varepsilon_\nu p^\nu, x^\mu] \stackrel{?}{=} \varepsilon^\mu = \delta x^\mu.$$

To check this, first note that we can explicitly write out

$$i\varepsilon_\nu p^\nu = i\varepsilon_- p^- + i\varepsilon_+ p^+ + i\varepsilon_I p^I.$$

We'll compute each commutator in turn:

1. First of all, because we have the special relation $x^+ = \frac{p^+}{m}\tau$, we have

$$\left[i\varepsilon_\nu p^\nu, x^+\right] = 0$$

   because the ps commute with each other. But this already violates our expectation of how the commutators should act, and something interesting is already happening here.

2. On the other hand, we can then find that

$$\boxed{[i\varepsilon_\nu p^\nu, x^I]} = \left[i\varepsilon_\nu p^\nu, x_0^I + \frac{p^I}{m}\tau\right] = \left[i\varepsilon_\nu p^\nu, x_0^I\right],$$

   which we can further write out as

$$= [i\varepsilon_- p^- + i\varepsilon_+ p^+ + i\varepsilon_J p^J, x_0^I] = i\varepsilon_-[p^-, x_0^I] + i\varepsilon_+[p^+, x_0^I] + i\varepsilon_J[p^J, x_0^I].$$

   Now the second term is zero, and the third term only gives us an ε_I, but the first term is a bit more complicated because it is a composite operator: we can thus write out the commutator as

$$\varepsilon_I + i\varepsilon_-\left[\frac{1}{2p^+}(\vec{p}_\perp^2 + m^2), x_0^I\right].$$

   This further simplifies to (pulling the p^+ out of the commutator)

$$= \varepsilon_I + i\varepsilon_-\frac{1}{2p^+}[p_J p_J, x_0^I]$$

   and because this commutator is $-2i\delta_{IJ}p_J$, we end up with

$$= \boxed{\varepsilon^I + \varepsilon_-\frac{p^-}{p^+}} \tag{17}$$

   So we find an additional term in the commutator from what we might expect when generating the translations!

3. Finally,

$$\boxed{\left[i\varepsilon_\nu p^\nu, x^-\right]} = \left[i\varepsilon_+ p^+ + i\varepsilon_- p^- + i\varepsilon_j p^J, x_0^- + \frac{p^-}{m}\tau\right]$$

can be simplified because the p^J term commutes with $x_0^- + \frac{p^-}{m}\tau$, and the p^+ doesn't commute with x_0^- so that we get

$$= -\varepsilon_+ + i\varepsilon_- \left[p^-, x_0^-\right].$$

Again because p^- is a composite operator, we have to be careful with this last expression: we then get

$$= -\varepsilon_+ + i\varepsilon_- \left[\frac{1}{2p^+}(p^I p^I + m^2), x_0^-\right],$$

and for that we need to calculate

$$\left[\frac{1}{p^+}, x_0^-\right] = \frac{1}{p^+}x_0^- - x_0^-\frac{1}{p^+} = \frac{1}{p^+}x_0^- p^+\frac{1}{p^+} - \frac{1}{p^+}p^+ x_0^-\frac{1}{p^+} = \frac{1}{p^+}\left[x_0^-, p^+\right]\frac{1}{p^+} = -\frac{i}{(p^+)^2}.$$

So putting this back into our commutator gives us

$$= -\varepsilon_+ + i\varepsilon_- \left(\vec{p}_\perp^2 + m^2\right)(-i)\frac{1}{2(p^+)^2} = -\varepsilon_+ + \varepsilon_- \frac{p_-}{p^+} = \boxed{\varepsilon^- + \varepsilon_- \frac{p^-}{p^+}}, \tag{18}$$

and yet again we get an additional term.

It appears that the p^μ do not generate the expected translations – there are extra or missing terms in each of the commutators that we found. To understand this, note that there is a fundamental conflict with how translations work with the light-cone gauge: since x^+ is just $\frac{p^+}{m}\tau$, with no other terms, if we want to translate $x^+ \to x^+ + \varepsilon^+$, we will violate the light-cone gauge!

So we'll want to do a translation in a way that doesn't change x^+: we can imagine doing a new gauge fixing so that we go from $\frac{p^+}{m}\tau + \varepsilon^+$ to $\frac{p^+}{m}\tau'$. And then we can maintain the gauge with a reparameterization, and the infinitesimal form will be

$$x^\mu(\tau) \to x'^\mu(\tau) = x^\mu(\tau) - f(\tau)\partial_\tau x^\mu(\tau)$$

for some infinitesimal function $f(\tau)$ so that $\delta x^+ = \varepsilon^+ - f(\tau)d_\tau x^+ = 0$ (so that we can stay inside the light-cone gauge). Solving, this requires $\varepsilon^+ = f(\tau)\frac{p^+}{m} \implies f = \frac{\varepsilon^+ m}{p^+}$. So such a reparameterization for the worldline needs to be done, and this will act on the other coordinates as well:

$$\delta x^- = \varepsilon^- - f(\tau)\partial_\tau x^- = \varepsilon^- - \frac{\varepsilon^+ m}{p^+}\frac{p^-}{m} = \varepsilon^- - \varepsilon^+\frac{p^-}{p^+} = \varepsilon^- + \varepsilon_-\frac{p^-}{m}.$$

This looks **exactly the commutator that we found** above! And similarly, the updated reparameterization and translation will give us

$$\delta x^I = \varepsilon^I - f(\tau)\partial_\tau x^I = \varepsilon^I - \frac{\varepsilon^+ m}{p^+}\frac{p^I}{m} = \varepsilon^I + \varepsilon_-\frac{p^I}{m},$$

and we've gotten what we expect – we now know what the quantum version of translation and reparameterization are if we want to stay in the light-cone gauge.

Remark 127. *The extra terms in* (17) *and* (18)*, which we see in the above can be interpreted as coming from reparameterizations of worldline so as to remain in the light-cone gauge, arose from commutators with p^-, thus p^- plays the role of* **generating reparameterization** *here, and what's interesting is that* **quantum mechanics already knows about these extra terms** *that we get out of the commutators (without needing the classical justification with the reparameterization and staying in the light-cone gauge)!*

Example 128
Next, let's turn to our Lorentz symmetries, and now we may expect that the $M^{\mu\nu}$s should generate our symmetries via

$$\left[-\frac{i}{2}\varepsilon_{\mu\nu}M^{\mu\nu}, x^\rho\right] = \delta x^\rho = \varepsilon^{\rho\nu}x_\nu.$$

But again, the fact that we're working in the light-cone gauge gives us some constraints: it explicitly breaks Lorentz symmetry, just like translations did. So Lorentz transformations will be more complicated – we need to do some extra reparameterizations. Furthermore, the commutators of these conserved charges should obey the **Lorentz algebra**:

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\eta^{\mu\rho}M^{\nu\sigma} - i\eta^{\nu\rho}M^{\mu\sigma} + i\eta^{\mu\sigma}M^{\rho\nu} - i\eta^{\nu\sigma}M^{\rho\mu}.$$

(Remember that the angular momentum operators M^{IJ} are a subset of Lorentz operators, and the algebra of angular momentum operators is a subalgebra of the above Lorentz algebra.) Since we're working in light-cone coordinates, we can separate out the $M^{\mu\nu}$ operators into a few different groups: we have M^{IJ} (rotations in the transverse directions), as well as

$$M^{\pm I} = \frac{1}{\sqrt{2}}(M^{0I} + M^{1I}),$$

and finally $M^{+-} = M^{10}$ (the boost in the x^1-direction). We'll now write these in terms of our independent operators: we know that

$$\boxed{M^{IJ} = x^I p^J - x^J p^I}$$

(no ambiguity because $I \neq J$ for any nonzero M^{IJ}),

$$\boxed{M^{+I} = x^+ p^I - x^I p^+},$$

(again no ordering ambiguity because everything commutes), but then we come to

$$M^{-I} \stackrel{?}{=} x^- p^I - x^I p^-.$$

This one is tricky because p^- does not commute with x^I, and in this kind of situation we can resolve the ambiguity as follows: we require M^{-I} to be a Hermitian operator (since a Lorentz charge should be a physical observable), so we want the symmetrized version

$$\boxed{M^{-I} = x^- p^I - \frac{1}{2}(x^I p^- + p^- x^I)}.$$

A similar process gives us

$$\boxed{M^{+-} = x^+ p^- - x^- p^+ = x^+ p^- - \frac{1}{2}(x^- p^+ + p^+ x^-)}.$$

And because p^- generates reparameterizations for the worldline of the relativistic particle, we expect M^{+-} and M^{-I} (which both contain p^-) to generate not just standard Lorentz transformations but also some reparameterizations. (And we'll check this fact for ourselves by checking commutators with various coordinates explicitly!)

We stress that despite that the light-cone gauge does not have manifest Lorentz covariance, the Lorentz algebra should always be satisfied at the quantum level. Checking it is indeed so is an important consistency requirement.

Fact 129
We can check in the particle case that this algebra works in any dimension for the particle. But in the string theory case, the algebra is **only satisfied in the case** $d = 26$, and that may be something that we will check ourselves!

April 7, 2021

Last lecture, we discussed symmetries in the light-cone gauge, in which we fix $x^+ = \frac{p^+}{m}\tau$. The important feature of this gauge is that **manifest translation and Lorentz invariances are lost** (since we fix the coordinate x^+), but those symmetries are still present – they are just realized in a more subtle way. Translations and Lorentz transformations now also involve certain reparameterizations of the particle's worldline needed to remain in the light cone gauge. The important point is that **the symmetry algebra is indeed still preserved** (otherwise we wouldn't have a mathematically self-consistent theory).

3.3: Quantum open strings

We're now ready to do the quantization of strings, and again we will do everything in the light-cone gauge (because we can avoid square roots and write down the constraints explicitly). Conceptually, this will be very similar to the point particle, but the calculations will be more complicated.

Example 130

We'll consider open strings with NN boundary conditions in all directions, which is equivalent to thinking about open strings on a space-filling D-brane.

Fact 131

We calculated that the angular momentum of a rotating string is given by $J = \frac{1}{2\pi T_0 n}E^2$, so the maximal J for a given E is $\frac{E^2}{2\pi T_0}$. For a moment, we will restore the constant c and write this as $\boxed{J = \frac{E^2}{2\pi T_0 c}}$ – this will turn out to be a general fact in string theory.

If we go to quantum theory, we know that angular momentum is quantized in units of $\hbar$, so we'll define the notation

$$\frac{J}{\hbar} = \frac{1}{2\pi T_0 \hbar c}E^2 = \alpha' E^2$$

by setting $\alpha' = \frac{1}{2\pi T_0 \hbar c}$ (where this term has units $\frac{1}{[E]^2}$).

Remark 132. *A few decades ago, in the 1950s and 1960s, physicists tried to organize a large number of particles discovered when probing strong interactions and found that the squared mass was linearly related to their angular momentum (this is called the* **Regge trajectory***). That's exactly what we're noticing here, where the energy E can be identified as the mass M of a string (as the string does not have any center of mass motion). This observation was one of the motivations for string theory.*

We'll now again set $\hbar = c = 1$, so that M has the same units as $\frac{1}{L}$ and L has the same units as T. We then find that $\alpha' = \frac{1}{2\pi T_0} \iff T_0 = \frac{1}{2\pi\alpha'}$ for α' having units of L^2, and thus we sometimes write $\alpha' = \ell_s^2$. The point is that by convention, we will stop using T_0, and we'll do everything in terms of α'.

Our process for quantizing strings in the light-cone gauge will be as follows (very similar to the string):

1. Identify the **independent** classical degrees of freedom, and impose the canonical commutation relations while promoting classical solutions to quantum operator solutions.
2. Identify the dependent degrees of freedom (such as $X^-(\sigma, \tau)$), which will become composite operators.

3. Check that this resulting theory is self-consistent by verifying that the Lorentz algebra behaves properly.
4. Work out the spectrum, meaning that we find the string theory excitations.

For the first step, recall that we have

$$X^+ = v^+\tau = \frac{p^+}{\pi T_0}\tau = 2\alpha' p^+ \tau$$

in the light-cone gauge, and we also have the Virasoro constraints

$$\partial_\tau X^- = \frac{1}{4\alpha' p^+}\left[(\dot{X}^I)^2 + (X'^I)^2\right], \quad \partial_\sigma X^- = \frac{1}{2\alpha' p^+}\dot{X}^I X'^I,$$

from which we can solve X^- up to a constant term x_0^-. So we find that our set of independent dynamical variables $\{X^I(\sigma,\tau), x_0^-, p^+\}$, and thus when we impose the canonical commutation relations, we must have $[x_0^-, p^+] = -i = i\eta^{i+}$, and then we need to also impose canonical commutation relations for $X^I(\sigma,\tau)$. The idea here is to write our solutions in terms of Fourier modes

$$X^I(\sigma,\tau) = X_0^I(\tau) + \sum_{n=1}^{\infty} X_n^I(\tau)\cos n\sigma,$$

and then the particle action basically just gives us things in the tranverse direction:

$$S_p = \frac{1}{2}\frac{1}{2\alpha'}\int d\tau (\dot{X}_0^I)^2 + \frac{1}{2}\frac{1}{4\alpha'}\sum_{n=1}^{\infty}\int d\tau(\dot{X}_n^I\dot{X}_n^I - n^2 X_n^I X_n^I).$$

We can now quantize our string by thinking about the action as the components for a nonrelativstic free particle, plus an infinite number of harmonic oscillators with frequencies of $\omega_n = n$. The canonical momenta conjugate to X_0^I and X_n^I can be readily found from the above action by taking derivative with respect to $\dot{X}_0^I$ and $\dot{X}_n^I$. We find

$$p^I = p_0^I = \frac{1}{2\alpha'}\dot{X}_0^I, \qquad p_n^I = \frac{1}{4\alpha'}\dot{X}_n^I$$

where as we discussed before the center-of-mass canonical momentum p_0^I also coincides with the spacetime momentum of the string p^I.

At quantum level, we impose the canonical commutation relations,

$$\boxed{[X_n^I, p_m^J] = i\delta_{mn}\delta^{IJ}}, \quad , m, n = 0, 1, \cdots \ .$$

The solutions to operator equations of $X_0^I(\tau)$ and $X_n^I(\tau)$ can also be directly written down from our previous discussion of a free non-relativistic particle and harmonic oscillator

$$X_0^I(\tau) = x_0^I + 2\alpha' p^I \tau,$$

$$X_n^I(\tau) = \frac{1}{\sqrt{2m_n\omega_n}} i\left(a_n^I e^{-i\omega_n\tau} - (a_n^I)^+ e^{i\omega_n\tau}\right), \quad m_n = \frac{1}{4\alpha'}, \quad \omega_n = n, \quad n = 1, 2, \cdots$$

with

$$[x_0^I, p^J] = i\delta_{IJ}, \qquad [a_n^I, (a_m^J)^\dagger] = \delta_{nm}\delta_{IJ} \ .$$

Remark 133. *To connect this with our previous notation, we used to write the coefficients of the Fourier modes as f_n. And equating the coefficients, we find that*

$$f_n^I = i\sqrt{2\alpha'}\frac{a_n^I}{\sqrt{n}}.$$

The convention in string theory is to use the notations

$$\alpha_n^I = \sqrt{n} a_n^I, \qquad \alpha_{-n}^I = \sqrt{n}(a_n^I)^+$$

and thus the commutation relations among α_n^I are given by

$$[\alpha_n^I, \alpha_m^J] = n\delta_{n+m,0}\delta^{IJ}.$$

In terms of our new notations the full operator equation for X^I can be written as

$$\boxed{X^I(\tau, \sigma) = x_0^I + 2\alpha' p^I \tau + i\sqrt{2\alpha'} \sum_{n\neq 0} \frac{\alpha_n^I}{n} e^{-in\tau} \cos n\sigma},$$

and now we're done – we have quantized the string!

So now we can look at the second step: previously by solving the Virasoro constraints we have

$$v^- = \frac{1}{2v^+}(\vec{v}_\perp^2 + \sum_{n\neq 0}\sum_I n^2 f_n^I f_{-n}^I) \tag{19}$$

and similar for f_n^-. Again, we want to promote these classical equations into operator equations, we need to be careful with operator ordering issues. We first write things in new notation, not worrying about ordering, and we get that

$$p^- \stackrel{?}{=} \frac{1}{2\alpha' p^+} L_0^\perp, \quad L_0^\perp \stackrel{?}{=} \alpha (p^I)^2 + \sum_{n=1}^\infty \alpha_n^I \alpha_{-n}^I. \tag{20}$$

Similarly, we find that

$$\alpha_k^- = \frac{L_k^\perp}{\sqrt{2\alpha'} p^+}, \quad L_k^\perp = \frac{1}{2} \sum_{n=-\infty}^\infty \alpha_n^I \alpha_{k-n}^I.$$

There is no ordering issue for $L_k^\perp$ with $k \neq 0$ as α_n^I and α_{k-n}^I commute. But we have some ordering issues between α_n^I and α_{-n}^I in $L_0^\perp$ – that is why we put question marks in (20). Since the commutators between α_n^I and α_{-n}^I are constants, different choices of orderings are related by constant shifts.

The way we usually deal with this is to define $L_0^\perp$ to be a particular ordering

$$L_0^\perp = \alpha' (p^I)^2 + \sum_{n=1}^\infty \alpha_{-n}^I \alpha_n^I, \tag{21}$$

and then we have

$$p^- = \frac{1}{2\alpha' p^+}(L_0^\perp + a) \tag{22}$$

for some constant a. Lorentz symmetry can be used to determine what a should be, as we will see later. Here we present another way. Recall that equation (19) can also be written as

$$2p^+p^- - (p^I)^2 = 2\pi T_0 \sum_{n=1}^\infty \sum_I H_n^I = \frac{1}{\alpha'} \sum_{n=1}^\infty \sum_{I=2}^{d-1} H_n^I$$

where H_n^I is the Hamiltonian for our harmonic oscillator X_n^I. At quantum level we know that $H_n^I = \omega_n \left((a_n^I)^\dagger a_n^I + \frac{1}{2}\right)$ (no sum over I), with zero-point energy $n/2$ ($\omega_n = n$). Since the terms in (21) are precisely of the form $\omega_n \left((a_n^I)^\dagger a_n^I\right)$, we thus conclude that the constant a in (22) should be given by the sum of the zero-point energies for all the oscillators, i.e.

$$a = \left(\sum_{n=1}^\infty \frac{n}{2}\right) \cdot (d-2).$$

This is a divergent expression, but we can nevertheless find a finite value

$$a = \frac{d-2}{2}\left(\sum_{n=1}^{\infty} n\right) = \frac{d-2}{2} \cdot -\frac{1}{12} = -\frac{1}{24}(d-2) \ . \tag{23}$$

That the sum of the positive integers is $-\frac{1}{12}$ can be derived in the following way:

Definition 134

The **Riemann zeta function** is defined as $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

This function is convergent for all $s > 1$ (and diverges for example at $s = 1$), but we can **analytically continue** $\zeta(s)$ to all complex-valued s to get $\zeta(-1) = -\frac{1}{12}$.

So now we have a full expression for X^-, now that we know p^- and α_k^-. Recall in the relativistic particle case, p^- generates the worldline reparameterization (which keeps us in the light-cone gauge). Here, $X^-(\tau, \sigma)$ (which basically consists of p^- and α_k^-) will generate the worldsheet reparameterization (needed to remain in the lightcone guage). (For example, we can check that the $L_n^\perp$s, which are proportional to the α_k^-s, give us such reparameterizations when we calculate their commutators with $X^I(\tau, \sigma)$.)

We can verify the following properties of the $L_n^\perp$ operator:

- We have $(L_n^\perp)^\dagger = L_{-n}^\perp$.
- The commutator $[L_m^\perp, \alpha_n^J] = -n\alpha_{m+n}^J$.
- The commutator $[L_m^\perp, x_0^I] = -i\sqrt{2\alpha'}\alpha_m^I$.
- We have
$$[L_m^\perp, L_n^\perp] = (m-n)L_{m+n}^\perp + \frac{d-2}{12}(m^3 - m)\delta_{m+n,0}$$
(this is the **Virasoro algebra**, and this basically gives us reparameterization on a circle).

We're now ready for step 3, which is to discuss the Lorentz algebra. We need to do this consistency check at the quantum level, and that will imply that it's okay to fix the light-cone gauge. Classically, recall that we have our conserved charges

$$M^{\mu\nu} = \frac{1}{2\pi\alpha'}\int d\sigma(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu),$$

and then if we promote this to a quantum version, we need to check whether there are operator ordering issues, how they act on our $X^\mu(\sigma, \tau)$s, and whether they satisfy the Lorentz algebra. Because we have solved for each X^μ explicitly, we can plug those in and get an explicit form of $M^{\mu\nu}$, and what we find is that

$$M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-)$$

(we need to do this symmetrization process because of a nonzero commutator), and similarly we can write down M^{+I}, M^{-I}, M^{IJ}. For example,

$$M^{-I} = x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I) - i\sum_{n=1}^{\infty} \frac{1}{n}(\alpha_{-n}^- \alpha_n^I - \alpha_{-n}^I \alpha_n^-)$$

Now checking commutators, we find that $[M^{\mu\nu}, X^\lambda]$ indeed gives us a Lorentz transformation plus worldsheet reparameterization, and then we can check the algebra of Lorentz symmetry – the most complicated case is the commutator

$$[M^{-I}, M^{-J}] \stackrel{?}{=} 0.$$

But if we plug in the expressions we have for $M^{\mu\nu}$, since we'll get commutators involving $L_m^\perp$. What we find instead is that

$$[M^{-I}, M^{-J}] = -\frac{1}{\alpha'(p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\alpha_{-m}^I\alpha_m^J - \alpha_{-m}^J\alpha_m^I),$$

where

$$\Delta_m = m\left(1 - \frac{1}{24}(d-2)\right) + \frac{1}{m}\left(\frac{1}{24}(d-2) + a\right).$$

So for the commutator to vanish, we need this expression to be zero for all m, and this happens exactly if $a = -\frac{1}{24}(d-2)$ and $d = 26$. In other words, $d = 26, a = -1$ is a requirement for string theory to be self-consistent, and then we have

$$p^- = \frac{1}{2\alpha' p^+}\left(L_0^\perp - 1\right) \iff M^2 = 2p^+p^- - (p^I)^2 = \frac{1}{\alpha'}\left(\sum_{n=1}^{\infty}\alpha_{-n}^I\alpha_n^I - 1\right).$$

So this is the quantum version of the mass-shell condition – again, the oscillator excitations increase the spacetime mass of the string, and this is how we determine the spectrum.

April 12, 2021

Last lecture, we quantized the open string by expanding the solution in terms of Fourier modes as

$$X^\mu(\tau,\sigma) = X_0^\mu(\tau) + \sum_{n=1}^{\infty}X_n^\mu(\tau)\cos(n\sigma).$$

The $X_0^\mu(\tau)$ term behaves like a center-of-mass motion for the free nonrelativistic particle, and each of the nonzero n terms correspond to a harmonic oscillator with frequency $\omega_n = n$. We know how to quantize a harmonic oscilator and a free particle, and we did so in the light-cone gauge where $X^+(\sigma,\tau) = X^+(\tau) = 2\alpha' p^+\tau$. The spatial coordinates X^I are then independent variables with solution given by

$$X^I(\sigma,\tau) = x_0^I + 2\alpha' p^I\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^I}{n}e^{-in\tau}\cos(n\sigma),$$

where the x_0^I, p^I, α_n^I are constant operators (the equivalents of integration constants). Similarly, we find that

$$X^-(\sigma,\tau) = x_0^- + 2\alpha' p^-\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha_n^-}{n}e^{-in\tau}\cos(n\sigma).$$

p^- and α_n^- are composite operators: they can be expressed in terms of the following **elementary operators**

$$\{p^+, x_0^-, x_0^I, p^I, \alpha_n^I\}$$

which have the commutation relations after quantization:

$$[x_0^-, p^+] = -i, \quad [x_0^I, p^J] = i\delta^{IJ}, \quad [\alpha_n^I, \alpha_n^J] = n\delta_{n+m,0}\delta^{IJ}.$$

(This is because $\alpha_n^I = \sqrt{n}a_n^I$ and $\alpha_{-n}^I = \sqrt{n}(a_n^I)^\dagger$ for all positive n, so we basically have scaled versions of the raising and lowering operators.) If we then solve the Virasoro constraints, we find that

$$p^- = \frac{1}{2\alpha' p^+}(L_0^\perp + a),$$

where a is the sum of the zero-point energies and is equal to $(d-2)\frac{1}{2}\sum_{n=1}^{\infty} n = -\frac{d-2}{24}$ (using a mathematical trick), and

$$L_0^{\perp} = \alpha'(p^I)^2 + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I.$$

We also find:

$$\alpha_k^- = \frac{L_k^{\perp}}{\sqrt{2\alpha'}p^+}, \quad L_k^{\perp} = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_n^I \alpha_{k-n}, \quad \alpha_0^I = \sqrt{2\alpha'}p^I.$$

The set of operators $L_n^{\perp}$ (over all $n \in \mathbb{Z}$) are often called the **Virasoro operators**, and they play an important role in calculating the algebra of the Lorentz conserved charges.

Fact 135

We can show that these $L_n^{\perp}$s generate **residual worldsheet parameterizations** (when acting on the $X^I(\tau,\sigma)$s) in the gauge $h_{\alpha\beta} = \eta_{\alpha\beta}$, and these parametrizations allow us to further fix the light-cone gauge.

Remark 136. *Since $L_n^{\perp}$ are contained in p^- and α_n^-, they are contained in the generators for translations and Lorentz transformations. Thus as in the relativistic particle case, translation and Lorentz symmetries are realized in the light-cone gauge by combining translations and Lorentz transformations with appropriate worldsheet reparameterizations.*

Remark 137. *Geometrically, $L_n^{\perp}$s generate reparmeterizations on a circle – this is because the reparameterizations*

$$\tau \to \tau + \frac{1}{2}(f(\tau+\sigma) + f(\tau-\sigma)), \quad \sigma \to \sigma + \frac{1}{2}(f(\tau+\sigma) - f(\tau-\sigma)),$$

are specified by function f, which is a periodic function with period 2π (i.e. an arbitrary function on a circle).

As previously mentioned, it turns out that these operators obey the Virasoro algebra

$$[L_m^{\perp}, L_n^{\perp}] = (m-n)L_{m+n}^{\perp} + \frac{d-2}{12}(m^3 - m)\delta_{m+n,0}.$$

The first part of the right-hand side gives us the Lie algebra of reparameterizations of a circle, and the extra term is the quantum correction for this particular story! And with that in mind, we can study the Lorentz algebra of $M^{\mu\nu}$s, meaning that the commutators of $M^{\mu\nu}$ and $M^{\rho\sigma}$ should be some linear combination of Ms. In lecture, we showed that this line of reasoning led us to the requirement $d = 26$ for string theory (so that we have $[M^{-I}, M^{-J}] = 0$, which is a requirement of the Lorentz algebra).

3.4: Open string spectrum

If we now look at the Hilbert space for this system, we have

$$\mathcal{H} = \mathcal{H}_{\text{free particle}} \otimes \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots,$$

where the $\mathcal{H}_i$s are the Hilbert spaces for the individual harmonic oscillators with $n = i$. We know how to construct these individual Hilbert spaces on the right side: we can index our states so that the ground states look like

$$\left|p^+, p^I\right\rangle \otimes |0\rangle_1 \otimes |0\rangle_2 \otimes \cdots,$$

for some p^+, p^I (remembering that p^I encodes all spatial dimensions). For notational simplicity, we'll denote these states as just $\left|p^+, p^I\right\rangle$. A **generic basis state** in the Hilbert space can be obtained by applying creation operators:

$$|\lambda\rangle \propto \prod_{n=1}^{\infty}\prod_{I=2}^{d-1}\left(a_n^{I+}\right)^{\lambda_{nI}}\left|p^+, p^I\right\rangle,$$

where the λ_{nI}s are the oscillator numbers for a_n^I and are always nonnegative integers. Recalling that a state $(a^\dagger)^k|0\rangle$ of a single harmonic oscillator has energy $(k+\frac{1}{2})\omega$, that tells us that $(a_1^+)^{k_1}(a_2^\dagger)^{k_2}\cdots|0\rangle$ has energy $E = k_1\omega_1 + k_2\omega_2 + \cdots + \frac{1}{2}(\omega_1+\omega_2+\cdots)$.

But now if we try to find the (light-cone) energy eigenvalue of $\hat{p}^-$ on our state, we find that

$$\hat{p}^-|\lambda\rangle = p^-|\lambda\rangle, \quad p^- = \frac{1}{2\alpha' p^+}\left((\alpha'(p^I)^2 + \sum_{n=1}^{\infty}\sum_{I=2}^{d-1} n\lambda_{nI} - \frac{d-2}{24}\right).$$

We'll call the double sum $N_\perp$, and thus rearranging gives us the **quantum mass-shell condition** below:

Proposition 138

We have the mass for a given string excitation given by

$$M^2 = 2p^+p^- - (p^I)^2 = \frac{1}{\alpha'}\left(N_\perp - \frac{d-2}{24}\right),$$

The key point here is that states $|\lambda\rangle$ have well-defined spacetime momenta and masses. They should thus describe spacetime particles. In other words, states for excitations of an open string correspond to spacetime particles!

Remark 139. *In the light-cone gauge, the set of $|\lambda\rangle$s (over all allowed $p{\cdot}p^I, \lambda_{nI}$) form a basis of the Hilbert space, and they inherit a* **well-defined inner product** *based on Hamiltonians from harmonic oscillators and the free particle. This is an important advantage of the light cone gauge.*

In contrast, if we consider a covariant quantization with manifest Lorentz covariance, we would have to impose $[x^\mu, p^\nu] = i\eta^{\mu\nu}$ and $[a_n^\mu, (a_m^\nu)^\dagger] = \eta^{\mu\nu}\delta_{m,n}$. Then commutator in the time direction, $[a_n^0, (a_k^0)^\dagger] = -\delta_{nk}$, has the extra negative sign due to $\eta^{00} = -1$, which can lead to negative norm states. We then need to find ways to get rid of these negative norm states.

In the light-cone gauge, all states automatically have positive norms as we only need to deal with X^I directions where such negative sign does not arise.

Before we go further, we need to make a digression and tell you how to classify particles in a Minkowski spacetime. There is some group theory involved here – the representations of the **Poincare group** (the combination of the Lorentz transformations and the translations) mathematically help us talk about this topic, but we'll just describe the results directly.

Recall that in the relativistic particle case, our basis states are spanned by $\left|p^+, p^I\right\rangle$. Other than momentum, there is no other quantum number. Such a particle is called a scalar.

But there are other kinds of particles as well, which have additional "spins" quantum numbers. Quantum states for these particles can be written as $|p^\mu, \sigma\rangle$, where p^μ denote momentum eigenvalues and σ denote collectively other spin quantum numbers.

It turns out that there are important distinctions between massive or massless particles.

Let us consider massive particles first, for which we can go to the rest frame where $p^\mu = (m, \vec{0})$, and then we can classify them (specifically, the σs) based on how they **transform under spatial rotations**.

Example 140
A spin 1 particle is described by basis vectors $|p^\mu, i\rangle$, where $i \in [1, 2, \cdots, d-1]$, and then this object transforms as a vector under spatial rotations. Similarly, a spin 2 particle is described as $|p^\mu, i, j\rangle$, where i, j range over spatial directions and is symmetric and traceless in i, j. That means there are $(d-1)$ polarizations for the spin 1 particle, but there are $\frac{1}{2}d(d-1)-1$ polarizations for the spin 2 particle.

On the other hand, for massless particles, there is no rest frame, and the best we can do is to go into the frame $(p^0, p^1, p^I) = (p^+, p^-, \vec{0})$. Then we only have $(d-2)$ spatial directions left, and we can now only classify our particles by transformations in those X^I directions: for example, a spin 1 particle is labeled as $\left|p^+, p^I, I\right\rangle$, where $I \in [2, 3, \cdots, d-1]$ and this object transforms as a vecotor under rotation among the different Is, and thus there are only $(d-2)$ polarizations in the massless case. (So for example, we find two polarizations for $d = 4$, which makes with what we know about photons.) Meanwhile, spin 2 particles are then labeled $\left|p^+, p^I, I, J\right\rangle$ for some symmetric traceless operator in I, J, and then we have only $\frac{1}{2}(d-1)(d-2)-1 = \frac{1}{2}d(d-3)$ polarizations. (For example, gravitons have 2 polarizations in $d = 4$.)

So now we're ready to go back to the states we described above as

$$|\lambda\rangle \propto \prod_{n=1}^{\infty}\prod_{I=2}^{d-1}(a_n^{I+})^{\lambda_{nI}}\left|p^+, p^I\right\rangle,$$

with the mass-shell condition

$$M^2 = \frac{1}{\alpha'}\left(N_\perp - \frac{d-2}{24}\right), \quad N_\perp = \sum_{n=1}^{\infty}\sum_{I=2}^{d-1} n\lambda_{nI}.$$

We now want to identify what kind of spacetime particles they correspond to. We will start with states with lowest M^2.

- First of all, the lowest possibility is where $N_\perp = 0$, so that $\lambda_{nI} = 0$ for all n, I. So we must have $\left|p^+, p^I\right\rangle$ be a scalar particle with no other quantum numbers, and the mass-shell condition evaluates to

  $$M^2 = -\frac{1}{\alpha'}\frac{d-2}{24} = -\frac{1}{\alpha'}$$

  with a negative mass square. A particle with a negative mass square is called a **tachyon**. Naively, a tachyon travels faster than the speed of light as from the energy relation $E^2 = \vec{p}^2 + M^2$, we have $E^2 < p^2$ with $M^2 < 0$. But this naive expectation is false as it turns out such a propagating particle does not exist. Instead the negative mass square implies existence of an instability in the system. More explicitly, for small enough momenta $p^2 < |M^2|$, we find that $E^2 < 0$, so that the energy E is pure imaginary $i\omega$, and thus the time-evolution e^{-iEt} becomes $e^{\omega t}$ or $e^{-\omega t}$. So these types of excitations do not correspond to a propagating particle – they are just some objects that grow exponentially in time, implying an instability. (For example, the electroweak phase transition can be understood as being due to such a tachyon.)

- The next excited state is $N_\perp = 1$, and thus $\lambda_{1I} = 1$ for one of the Is (and all other λ_{nI}s are zero). This then corresponds to the state $a_{-1}^I\left|p^+, p^\perp\right\rangle$ (using $p^\perp$ instead of p^I to avoid confusion), and this state can be labeled as $\left|p^+, \vec{p}^\perp, I\right\rangle$ for some $I \in [2, \cdots, d-1]$. This object is a vector under rotations of I (since X^I transform as a vector and thus so are a_{-n}^I). The mass square of this particle is:

  $$M^2 = \frac{1}{\alpha'}\left(1 - \frac{d-2}{24}\right).$$

In $d = 26$, this means $M^2 = 0$, so we have a **massless vector**, but in other dimensions we have $M^2 \neq 0$, and thus we end up with a **massive vector**. But the $(d-2)$ states we've written down for $N_\perp = 1$ are the only states, while massive particles are supposed to have $(d-1)$ polarizations (independent basis states)! So that gives us a contradiction for $d \neq 26$, which implies the theory is inconsistent. This is an alternative way to the Lorentz algebra in seeing the inconsistency in the theory for other dimensions.

- If we now go to $N_\perp = 2$, we can have the states

$$a_{-1}^I a_{-1}^J \left|p^+, \vec{p}^\perp\right\rangle, \quad a_{-2}^I \left|p^+, \vec{p}^\perp\right\rangle.$$

  at this energy level. We then have $M^2 = \frac{1}{\alpha'}(2-1) = \frac{1}{\alpha'}$, and counting the number of gives us agreement with a spin-2 particle (we can check this ourselves).

Remark 141. *The inconsistency that we described at the $N_\perp = 1$ level only shows up there with the "massive" versus "massless" particles (not in the other $N_\perp$ levels), so it is very subtle!*

More generally, we can generalize this to any Dp-brane for $p \geq 1$, where we have NN boundary conditions for the $0, 1, \cdots, p$ directions, and DD boundary conditions in the other directions. Then our p^μs are p^+, p^-, p^a, where a only ranges from 2 up to p (since there is no momentum in the DD directions). So now our particles essentially live in $(p+1)$-dimensional Minkowski spacetime instead of d-dimensional spacetime, and everything else follows through the same way.

April 14, 2021

Last lecture, we described how to classify particles in Minkowski spacetime: they are specified by mass and spin quantum numbers. We then label our particle states as $|p^\mu, \sigma\rangle$, where σ collectively denotes spin polarizations of the particle, and momentum p^μ satisfies the mass shell condition $p^2 = -m^2$. The number of independent polarizations depends the spin and whether we have massive or massless particles (spin-1 corresponds to a vector with $(d-1)$ components for massive particles and $(d-2)$ components for massless particles, spin-2 corresponds to a symmetric traceless tensor with $\frac{1}{2}(d-2)(d+1)$ components for massive particles and $\frac{1}{2}d(d-3)$ components for massless particles, and so on).

We then discussed how to apply this to interpret states in the Hilbert space of a quantum open string – a nice choice of basis states can be labeled with creation operators

$$\left|p^+, \vec{p}_\perp, \{\lambda_{nI}\}\right\rangle = \prod_{n=1}^{\infty} \prod_{I=2}^{d-1} \left(a_n^{I\dagger}\right)^{\lambda_{nI}} \left|p^+, \vec{p}_\perp\right\rangle.$$

(We can also label with α_{-n}^I operators instead of $a_n^{I\dagger}$ operators, which just changes the normalization.) The squared mass of the string state is then given by

$$M^2 = \frac{1}{\alpha'}\left(N_\perp - \frac{d-2}{24}\right),$$

where $N_\perp$ sums over all energies except the zero-point energies for the oscillators:

$$N_\perp = \sum_{n=1}^{\infty} \sum_{I=2}^{d-1} n\lambda_{nI}.$$

Last time, we discussed interpretations of those of smallest mass $N_\perp = 0, 1, 2, \cdots$. The $N_\perp = 0$ case gives us states $|p^+, \vec{p}_\perp\rangle$ (corresponding to instability and **tachyons**), and $N_\perp = 1$ gives us $|p^+, \vec{p_\perp}, I\rangle$ and a mass of 0 in $d = 26$ (for other values of d we have an inconsistent theory). So $N_\perp = 1$ describes a photon (a massless vector). Finally, $N_\perp = 2$ has a positive mass given by $M^2 = \frac{1}{\alpha'}$. Interestingly the inconsistency with $d \neq 26$ only appears at level $N_\perp = 1$.

Fact 142

At a general level $N_\perp = m = \sum_n \sum_I n\lambda_{nI}$, we have largest angular momentum when only $n = 1$ oscillators are excited, i.e. $\alpha_{-1}^{I_1} \cdots \alpha_{-1}^{I_m} |p^+, \vec{p}_\perp\rangle$ (as in this case we get the most possible indices). Such a state has angular momentum $J = m$. We then find that $M^2 = \frac{1}{\alpha'}(N_\perp - 1) = \frac{1}{\alpha'}(m-1) = \frac{1}{\alpha'}(J-1)$, which is close to the classical expression $M^2 = \frac{J}{\alpha'}$ we found before (the -1 can be viewed as a quantum correction).

To summarize, a single string can give rise to an infinite tower of particles (of increasing masses), and each oscillating pattern for the string corresponds to a particular species of particle (of a particular mass and spin). In particular, the larger the oscillation number, the string is more excited and thus the corresponding spacetime particle is more massive.

Remark 143. *An open string with NN boundary conditions in all directions live on a space-filling D-brane, with $p = d - 1 = 25$. An open string living on a Dp-brane with $p < d - 1$ can only have momenta in the directions along the brane, as they cannot move away from it. String excitations then correspond to particles in $(p+1)$-dimensional spacetime spanned by the worldvolume of the Dp-brane. Note that a string can still oscillate in all $d = 26$ spacetime directions.*

And now we need to make a conceptual leap: since open strings live on a D-brane, then it is natural to think of **open strings themselves** as excitations of the D-brane (so particles corresponding to different string vibrations are excitations as well). This means that D-branes are dynamical objects (rather than just the background on which our strings move), and we're going to understand their dynamics through our open strings. We will elaborate more on this a bit later.

Fact 144

In particular, this allows us to interpret the presence of a tachyon with $M^2 = -\frac{1}{\alpha'}$ as indicating that a D-brane is unstable and can decay (to closed strings). Furthermore, because D-brane excitations also contain photons, quantum electrodynamics are included in this theory. We will see a bit later that the Yang-Mills fields also arise when we consider the dynamics of multiple D-branes.

So all of the non-gravitational interactions in nature arise from the open string dynamics on D-branes, so this is a sign that the different interactions are being unified by string theory!

Remark 145. *We don't have any fermionic (e.g. spin 1/2) particles yet, but when we discuss superstrings later on, we'll see fermionic particles as well, and the tachyons will also be removed from the picture.*

3.5: Quantum closed strings

The story for quantum closed string is very much parallel to that of open strings with the main difference being that closed strings have two sets of oscillatory modes instead of one set. There are also various numerical factor differences due to that σ goes to 2π instead of π.

We first briefly remind you of the classical story. We again have the Polyakov action

$$S_P = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X^\mu \partial_\alpha X_\mu,$$

together with the Virasoro constraints. Now $\sigma \in [0, 2\pi)$ with periodic boundary conditions on X^μ. We again expand X^μ in Fourier modes

$$X^\mu(\sigma, \tau) = \sum_{n=-\infty}^{\infty} X_n^\mu(\tau) e^{-in\sigma},$$

where the coefficients are complex with $(X_n^\mu)^* = X_{-n}^\mu$, except for $n = 0$ which is real. Substituting the expansion into the action gives us

$$S_P = \frac{1}{2}\frac{1}{\alpha'}\int d\tau \left(\dot{X}_0^\mu\right)^2 + \frac{1}{\alpha'}\sum_{n=1}^{\infty}\int d\tau \left(\dot{X}_n^\mu \dot{X}_{n\mu}^* - n^2 X_n^\mu X_{n\mu}^*\right),$$

where this time we have **complex harmonic oscillators** (so if we write things in terms of real and imaginary parts, we get two sets of harmonic oscillators). The most general solution is then

$$X_0^\mu = x_0^\mu + \alpha' p^\mu \tau$$

where $v^\mu = \alpha' p^\mu$ gives us the usual center-of-mass motion, and for all $n \geq 1$ we have

$$X_n^\mu = i\sqrt{\frac{\alpha'}{2}}\left(\frac{\alpha_n^\mu}{n}e^{-in\tau} - \frac{\tilde{\alpha}_n^{\mu *}}{n}e^{in\tau}\right),$$

where $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ and $\tilde{\alpha}_n^{\mu *} = \tilde{\alpha}_{-n}^\mu$. Since X_n^μ is complex we have two sets of integration constants $\alpha_n^\mu, \tilde{\alpha}_n^\mu$.

In the light-cone gauge, we have $X^+ = \alpha' p^+ \tau$, and the Virasoro constraints are given by

$$\partial_\sigma X^- = \frac{1}{\alpha' p^+}\dot{X}^I X^{I'}, \quad \partial_\tau X^- = \frac{1}{2\alpha' p^+}((\dot{X}^I)^2 + (X^{I'})^2).$$

We use these equations to express $p^-, \tilde{\alpha}_n^-, \alpha_n^-$ in terms of those of X^I. More explicitly,

$$p^- = \frac{1}{2p^+}\left(\vec{p}_\perp^2 + \frac{2}{\alpha'}\sum_{n=1}^{\infty}\sum_I H_n^I\right)$$

where H_n^I is the Hamiltonian for the complex harmonic oscillator X_n^I. The equation for $\partial_\sigma X^-$ also leads to a constraint for X^I's: integrate the equation from 0 to 2π, and the left-hand side is zero by periodicity so that we get

$$\boxed{\sum_{n=1}^{\infty}\alpha_{-n}^I \alpha_n^I = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I}. \tag{24}$$

We now proceed to quantize the closed string: we have our set of independent operators

$$\{x_0^-, p^+, x_0^I, p^I, \alpha_n^I, \tilde{\alpha}_n^I\}$$

(the last of these being the only new operators from the open string case), and the non-vanishing commutation relations are the same as the open string case with the additional set

$$[\tilde{\alpha}_n^I, \tilde{\alpha}_m^J] = n\delta_{n+m,0}\delta^{IJ}.$$

The full operator equation is then

$$X^I(\tau, \sigma) = x_0^I + \alpha' p^I \tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\left[\frac{\alpha_n^I}{n}e^{-in(\tau+\sigma)} + \frac{\tilde{\alpha}_n^I}{n}e^{in(\tau-\sigma)}\right]$$

where again the main difference is that we have one more set of modes, and some numerical factors. We can further separate the above equation into left- and right-moving parts

$$X^I(\tau, \sigma) = X_L^I(\tau + \sigma) + X_R^I(\tau - \sigma)\ .$$

Fact 146

In the open string case, we also had left- and right-moving parts, but the boundary conditions on the open string meant they weren't independent of each other! And that's why we ended up with "standing wave"-like solutions with the $\cos n\sigma$ factors.

Promoting the equation for p^- to an operator equation again gives us ordering issues. We do the same "sum over zero-point energies" trick and find that

$$p^- = \frac{1}{\alpha' p^+}\left(L_0^\perp + a + \tilde{L}_0^\perp + \tilde{a}\right), \quad a = \tilde{a} = -\frac{d-2}{24}$$

where

$$L_0^\perp = \frac{\alpha'}{4}(p^I)^2 + \sum_{n=1}^{\infty} \alpha_{-n}^I \alpha_n^I \equiv \frac{\alpha'}{4}(p^I)^2 + N_\perp$$

and the analogous expression for $\tilde{L}_0^\perp$, just with tildes everywhere. Our mass-shell condition then becomes

$$M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp + a + \tilde{a}).$$

The crucial difference from the open string case is (24): if we treat it as an operator equation, then it implies that not all $\alpha_m^I, \tilde{\alpha}_n^I$ are independent, which would make everything break down: we've been treating $\alpha, \tilde{\alpha}$s as independent from each other by imposing commutation relations on every pair. This extra dependence would invalidate the commutation relations.

This constraint comes from the periodic nature of X^I as a function σ and implies that no point on the σ circle is special. In fact, we can derive that equation from the symmetry $\sigma \mapsto \sigma + c$. So the constraint is topological in nature and does not reflect redundancy of degrees of freedom. So quantum mechanically, we'll impose this condition on the state rather than as an operator equation:

$$(N_\perp - \tilde{N}_\perp)\left|\psi\right\rangle = 0,$$

where $\left|\psi\right\rangle$ is any of the physical states. (And if we study gauge theory more systematically, we will understand that this is a natural thing to do.) In particular, the eigenvalues of $N_\perp$ and $\tilde{N}_\perp$ must be the same, which is often referred to as the **level-matching condition**.

So now the closed string has a convenient basis given by

$$\left(\prod_{n=1}^{\infty}\prod_{I=2}^{d-1}(\alpha_{-n}^I)^{\lambda_{nI}}\right)\left(\prod_{n=1}^{\infty}\prod_{I=2}^{d-1}(\tilde{\alpha}_{-n}^I)^{\tilde{\lambda}_{nI}}\right)\left|p^+, \vec{p}_\perp\right\rangle$$

and the mass-shell condition is again given by

$$M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp + a + \tilde{a}), \quad N_\perp = \tilde{N}_\perp,$$

with eigenvalues of the $N_\perp, \tilde{N}_\perp$ operators given by

$$N_\perp = \sum_n \sum_I n\lambda_{nI}, \quad \tilde{N}_\perp = \sum_n \sum_I n\tilde{\lambda}_{nI}.$$

Remembering that for $d = 26$ (the only allowed spacetime dimension), we have $a = \tilde{a} = -1$, we can now think about the spectrum for the closed string.

- The lowest mode is where $N_\perp = \tilde{N}_\perp = 0$, so that our states are $|p^+, \vec{p}_\perp\rangle$ (corresponding to scalars). Then we have $M^2 = -\frac{4}{\alpha'}$, and again this is a tachyon signaling instability
- Our next lowest mode is where $N_\perp = \tilde{N}_\perp = 1$ (remember the eigenvalues have to be the same). So our states here look like $\alpha^I_{-1}\tilde{\alpha}^J_{-1}|p^+, \vec{p}_\perp\rangle$, where we can label them alternatively as $|p^+, \vec{p}_\perp; I, J\rangle$. Our mass is then $M^2 = \frac{2}{\alpha'}(1 + 1 - 1 - 1)$ again (so we have a massless particle for $d = 26$ and a massive particle for $d \neq 26$), and we have $(d-2)^2$ polarization because I and J are independent. We can break up these polarizations into **(a)** the $\frac{1}{2}d(d-3)$ symmetric traceless polarizations, giving us the standard spin-2 particle (**graviton**), **(b)** the $\frac{1}{2}(d-2)(d-3)$ antisymmetric polarizations (**anti-symmetric tensor**), and **(c)** the single trace polarization, which gives us a scalar particle (**dilaton**). Adding these together gives us the total $(d-2)^2$ polarizations, and notably each group here is closed under rotations among I, J. So this energy level corresponds to three different particles!

  And the point here is that we again get inconsistency if $d \neq 26$: we cannot split $(d-2)^2$ states into different massive particles with the number of polarizations required.
- Finally, higher levels for $N_\perp, \tilde{N}_\perp$ give us an infinite tower of particles, and all of those will be massive.

We've only studied the energy spectrum so far, but we can also study the interactions between strings – the idea is that because our theory contains gravitons, exchanging gravitons should give us gravitational forces, and that's what we will find. So **quantum gravity** is entering the picture here!

Fact 147

Recall that we interpreted **open strings as excitations of D-branes**. We will now interpret **closed strings as excitations of spacetime**. This is natural as closed string excitations include a graviton and the presence of graviton means the spacetime becomes dynamical.

Classically, closed strings propagate in Minkowski spacetime, and the spacetime is rigid (there are no fluctuations), merely providing a background for closed strings to propagate. But in the quantum theory, closed strings become dynamical excitation of the spacetime, and the spacetime becomes dynamical. Similarly, classically, D-branes only provide boundary conditions for open strings. In quantum theory, open strings become dynamical excitations of the D-branes, and D-branes are dynamical objects.

April 21, 2021

Today, we'll start with a summary of the key points that we've discussed in the past few lectures:

1. We think of open strings as excitations of D-branes, and we think of closed strings as excitations of spacetime. (So the quantization of the string leads D-branes and spacetime to become dynamical.)
2. In particular, the open string spectrum includes photon, and the closed string spectrum includes a graviton. So if string theory is consistent, we will have a consistent quantum theory of gravity.
3. In both the open and closed string cases, the lowest-mass particle is a tachyon with negative squared mass M^2. In particular, this implies that there is instability for both D-branes and for $d = 26$ Minkowski spacetime. (Being "unstable" means that the object can decay, not that the theory is inconsistent.) Decay of D-branes is a

well-understood problem (we get radiation of closed strings), but it is still an open question to understand how 26-dimensional Minkowski spacetime decays.

4. Some particles are massless, meaning that $M^2 = 0$, while other particles are massive and have $M^2 = \frac{n}{\alpha'}$ for some positive integer n.

5. The closed-string spectrum has two other types of massless particles in addition to the graviton, called the anti-symmetric tensor and the scalar dilaton. Meanwhile, the open-string spectrum has only the massless photon if we have a space-filling brane, but for $p < 25$ we have $(d - p - 1)$ massless scalars.

Example 148

To elaborate more on this last point, recall that the states $\alpha_{-1}^I \tilde{\alpha}_{-1}^J |p^+, \vec{p}_\perp\rangle = |p^+, \vec{p}_\perp; I, J\rangle$ give us $(d-2)^2$ basis states. We can separate these states into three groups by how they transform under rotations of I and J. Each of these three groups transforms into themselves, and corresponds to a particular kind of particle. States within each group correspond to different polarizations of the particle that the group represents. One group consists of basis states which are traceless and symmetric in I, J (like $|p^+, \vec{p}_\perp; I, J\rangle + |p^+, \vec{p}_\perp; J, I\rangle$ for $I \neq J$). A second group consists of states which are antisymmetric in I, J (like $|p^+, \vec{p}_\perp; I, J\rangle - |p^+, \vec{p}_\perp; J, I\rangle$). The last group consists of one basis state, obtained by the taking the trace in I, J, i.e. the basis element $\sum_I |p^+, \vec{p}_\perp; I, I\rangle$.

So far, everything in our discussion has been self-contained, but we'll now need to start including points outside of the scope of this class. But they're still important for us to understand the context of what's going on, so we'll just accept them for now:

Fact 149

The framework of quantum field theory is that particles can be associated with fields, and when we quantize fields, we get the dynamics of the particles.

All fundamental interactions can be described in this way – for example, photons correspond to the abelian gauge field A_μ, and gravitons correspond to metric perturbations $h_{\mu\nu}$.

Fact 150

If we consider low energies, meaning that $E^2 \ll \frac{1}{\alpha'} \sim M^2$, then the massive string states cannot be excited (just like how we only discover particles by going to high enough energies).

This will mean that the low energy physics will be dominated by massless particles, which is why such massless particles play a key role in string theory.

Fact 151

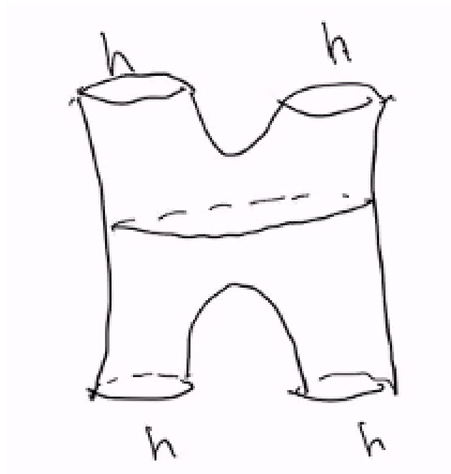
From broad principles of quantum mechanics and quantum field theory (Lorentz symmetry, unitarity, and so on), it can be shown that any massless **vector** is described by Maxwell's theory (in other words, quantum electrodynamics must show up), and any massless **spin-2 particle** is described by Einstein gravity.

We can verify explicitly that the massless vector appearing in the open string theory and the massless spin-2 particle in the closed string theory do indeed behave like photons and gravitons, and we can explain this at a general physical level here with an example

Example 152

If we consider two particles h and scatter them, we get some scattering amplitude A_4 for $h + h \to h + h$.

Any scattering processes in string theory can be reduced to compositions of two fundamental processes that are time-verse of each other: a string splits into two strings or two strings combine into one string. The amplitudes of these two fundamental processes are described by a dimensionless constant g_s, called **string coupling constant**. We will normally assume that g_s is small, so that the interactions of strings are weak. The simplest process for scattering $h + h \to h + h$ is (where two strings join into a single string and then splits again)



Since there is a joining and then a splitting, the amplitude of this type of process is proportional to g_s^2. More complicated processes can occur with more joinings and splittings of string, which are higher order g_s.

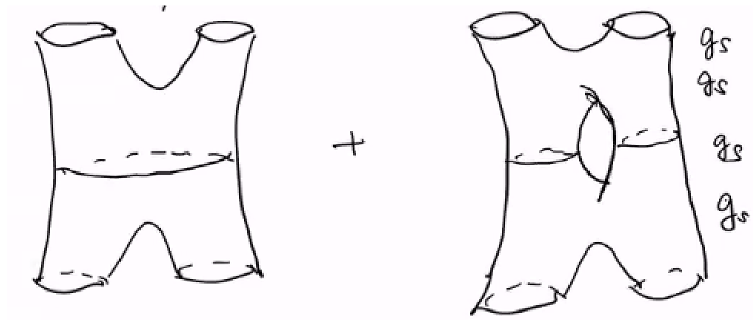
At low energies $E^2 \ll \frac{1}{\alpha'}$ such a scattering process should be dominated by contributions from exchanges of massless particles (as massive particles are too heavy), i.e. from exchanging the graviton, the anti-symmetric tensor and the dilaton. Since the dynamics of graviton at low energies should be described by the Einstein gravity, we indeed find that such string scattering amplitudes can be reproduced by the Einstein gravity coupled to the anti-symmetric tensor and dilaton! This shows that the Einstein gravity indeed arises as low energy limit of string theory. In particular, in Einstein gravity at lowest order A_4 is proportional to $G_N m^2$ (where m is the mass of each particle), we find that $G_N \propto g_s^2$.

Fact 153

There are no dimensional parameters in string theory, and it turns out that the string coupling constant satisfies $g_s \propto e^{\phi_0}$, where $\phi_0 = \langle \phi \rangle$ is the expectation value of the scalar field corresponding to the dilaton.

Fact 154

If we use Einstein gravity to study how two particles interact gravitationally, we get an amplitude proportional to G_N if we exchange one graviton, but we get divergent results when two more gravitons are exchanged (the Einstein theory is non-normalizable). But if we use string theory, we always get finite results.



This is another indication of why string theory gives us a consistent theory of gravity!

3.6: More on D-branes

Suppose we now look at a situation where we have a Dp-brane with $p < 25$. We'll then label our coordinates as

$$(X^+, X^-, X^i, X^a), \quad i \in [2, 3, \cdots, p], \quad a \in [p+1, \cdots, d-1],$$

where X^a are the coordinates perpendicular to the brane. We'll use $\alpha = (+, -, i)$ to denote the $(p+1)$ directions, including time, along the D-brane. In other words, we have NN boundary conditions along the X^α directions and DD boundary conditions along the X^a directions.

We can expand our solutions in terms of Fourier modes as

$$X^a(\tau, \sigma) = \sum_{n=1}^{\infty} X_n^a(\tau)\sin(n\sigma),$$

and we can solve this with a similar "quantization of harmonic oscillators" procedure to find that

$$X_n^a = i\sqrt{2\alpha'}\left(\frac{\alpha_n^a}{n}e^{-in\tau} - \frac{\alpha_{-n}^a}{n}e^{in\tau}\right),$$

where the key point is that there is no X_0^a and no center-of-mass motion p^a. The commutation relations for oscillation modes are the same as before, with $[\alpha_m^I, \alpha_n^J] = m\delta_{m+n,0}\delta^{IJ}$. In other words, we still have vibrational modes in all directions, just not center-of-mass motion in X^a.) Our mass-shell condition is also similar, in that

$$M^2 = 2p^+p^- - (p^i)^2 = \frac{N_\perp}{\alpha'} + a,$$

but only looking at the momenta in $(p+1)$-dimensional Minkowski space. So our basis states

$$\prod_{n=1}^{\infty}\prod_I (\alpha_{-n}^I)^{\lambda_{nI}} \left|p^+, \vec{p}_\perp\right\rangle$$

should now be interpreted as **particles living in** $(p+1)$**-dimensional Minkowski space**, along the D-brane. So the spectrum of the string now contains the same tachyon with $N_\perp = 0$, but at $N_\perp = 1$, it also contains states of the form $\alpha_{-1}^i \left|p^+, p_j\right\rangle$, which transform as a **vector under rotations** in $(p+1)$-dimensional Minkowski spacetime, and $\alpha_{-1}^a \left|p^+, p^j\right\rangle$, which transform as **scalars** (rotations along i don't transform a).

So as we mentioned before, we do indeed get 1 vector particle and $(d-p-1)$ massless scalar particles (the number of perpendicular directions to the brane). The associated vector field for the particle then looks like $A_\alpha(x^\beta)$ (where x^β are the coordinates along the brane), and the scalar fields look like $\phi^a(x^\beta)$.

Fact 155

The way we can interpret the scalar fields is that they describe **locations of the brane** in the perpendicular directions, specifically the motion of the brane when it is excited.

Deriving this result requires us to go beyond the techniques we've learned so far, but we can still try to heuristically motivate the answer here: we will argue that the motions of the D-brane in perpendicular directions have to be described by **massless** fields, and the ones we just described are the only massless excitations (the number of such fields match up, and they're the only ones we have).

To see that the motions of the D-brane in perpendicular directions have to be described by **massless** fields, let is consider a simple example where there is only one perpendicular direction x^{d-1}. Consider the D-brane at different values of x^{d-1}, e.g.$x^{d-1} = 0$ and $x^{d-1} = a$. Because of translation symmetry of Minkowski spacetime, these D-brane configurations must have the same energy (analogous to that a desk at different locations have the same mass), i.e. there is no potential barrier for the perpendicular motion of the D-brane, meaning that we only have kinetic energies. That's the same thing as saying that the corresponding field theory description must be a massless scalar field. (If it were described by a massive field, whose excitations would then have dispersion relation $E^2 = \vec{p}^2 + m^2$, and it would mean that we have some potential energy term even witn $p = 0$.)

For the D-brane sitting at $x^{d-1} = a$ we have $\phi^{d-1}(x^\alpha) = a$. The motion of the D-brane in x^{d-1} directions is captured by the time dependence of ϕ^{d-1} and the spatial dependence of ϕ^{d-1} along D-brane directions allow D-brane to have shape deformations, i.e. the location of x^{d-1} can be different at different positions along the brane (i.e. the brane does not have to be flat any more).

Fact 156

New physics arises when we have multiple D-branes sitting at the same location (on top of each other).

As an illustration let us first consider two D-branes sitting at the same location. Labeling our two branes by 1 and 2, there are then four different types of strings between the two D-branes (1-1, 1-2, 2-1, 2-2). The quantization procedure is exactly the same for all four type sof strings, because the two branes sit on top of each other, and what this means is that our string excitations can now be written in terms of $|\psi; m, n\rangle$ for $m, n \in \{1, 2\}$: for example, massless modes look like $\alpha^i_{-1} |p, mn\rangle$ or $\alpha^a_{-1} |p, mn\rangle$. (Basically, we have $2 \times 2 = 4$ copies of the same spectrum compared to before.)

More generally, if we have N D-branes that all coincide, we get N^2 copies of the string spectrum, since m, n can range in $[1, 2, \cdots, N]$ now. We will then arrange these in a matrix: instead of a vector field A_α, we now get $(A_\alpha)^m{}_n$, where m, n indicate which brane our left and right end of the string are on. And this generalization with multiple D-branes is what leads us to the **non-abelian gauge field**, which gives us the weak and strong interactions!

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April 26, 2021

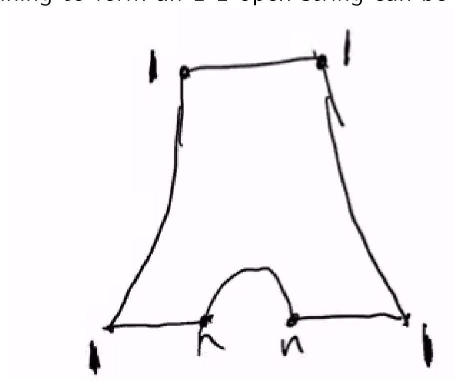
Recall from last time that a single Dp-brane has various **massless modes**: if x^a are the directions perpendicular to the D-brane, and x^α are the directions along (parallel to) the brane, then the massless modes split into a massless vector field $A_\alpha(x^\beta)$, as well as one scalar field $\phi^a(x^\alpha)$ for each of the transverse directions. Those transverse scalar fields then describe the motion (and therefore the shape) of our D-brane.

Example 157

For illustration, we can imagine that the only spatial directions are x and y, and we have a D-brane along the x-direction. Then we have one scalar field $\phi(t, x)$, which describes the y-component of the brane at a given x-coordinate and a given time t. (Any t-dependence corresponds to the movement of the brane.)

We can then think about situations where we have multiple D-branes sitting on top of each other, so that our states are labeled by $|\lambda, m, n\rangle$ instead of $|\lambda\rangle$ with $m, n = 1, 2, \cdots N$ (meaning that the string starts on the mth D-brane and ends on the nth D-brane). This gives us N^2 different kinds of strings. That is, for each string state, we now have N^2 copies of it, which we can view as forming an $N \times N$ matrix. For example, the massless fields become $(A_\alpha)^m{}_n$ and $(\phi^a)^m{}_n$.

Recall that two strings can join together to form a single string, or a string can split apart into two strings. For example, the process of two strings joining to form an 1-1 open string can be visualized as below:



Note that the end points of the two initial strings which are joined together can be on any D-brane, and we should sum over all possibilities, i.e. we should be able to write

$$|\lambda, 1, 1\rangle = \sum_n |\lambda_1, 1, n\rangle \, |\lambda_2, n, 1\rangle \ .$$

The above equation looks like a matrix multiplication: therefore, open strings interacting on multiple D-branes interact via matrix products. Based on this it can be shown that at low energies the dynamics of massless fields $(A_\alpha)^m{}_n$ and $(\phi^a)^m{}_n$ are described by Yang-Mills theory (namely **non-abelian gauge theory**).

Remark 158. *Detailed descriptions of how strings interact are outside the scope of this course. Instead, we've given a heuristic picture to give you some basic intuitions.*

3.7: Superstring theory

So far, the string theory that we've been considering has had the following properties (for both open and closed strings):

- We require the spacetime dimension to be $D = 26$.
- The system is unstable, because there is a tachyon in the spectrum.
- All excitations are **bosonic** (spins have been integers, and if we want fermions, we need half-integer spins).

The main point is that our excitations have been built from the raising operators α^I_{-n} or $\tilde{\alpha}^I_{-n}$, which transform as a **vector under rotations** and are thus spin 1. And we've learned in quantum mechanics that spin satisfies $1 \otimes 1 = 2 \oplus 1 \oplus 0$, and so on – the point is that combining states with integer spins gives us other states with integer spins. (Furthermore, fermions are related to **anticommutation relations** rather than commutation relations.) So we're going to need to build up a slightly different theory if we want to see fermions come up.

We mentioned that wanting certain properties for string action built from $X^\mu(\tau,\sigma)$ essentially gave us the unique Nambu-Goto string action, so we can't just modify our action directly. So instead, we'll **add new degrees of freedom** to our system. Since $X^\mu(\tau,\sigma)$ already tells us the spacetime motion of our string, **any additional degrees of freedom must be non-geometric**, which means that they can be regarded as internal degrees of freedom of our strings.

We know that $X^\mu(\tau,\sigma)$ behaves as a $1+1$-dimensional scalar field from the point of view of the **worldsheet**, and a natural thing to add to this theory is a **fermionic field** $\psi(\tau,\sigma)$. This is a good thing to do for the following reasons:

- In the mass-shell equation
$$M^2 = \frac{1}{\alpha'}(N_\perp + a), \quad a = -1,$$
the zero-point energy a is negative. This is the mathematical reason why we get a tachyon for $N_\perp = 0$. Getting rid of this tachyon can be done by making a non-negative. Adding fermions can help with changing the sign of a, because fermions have opposite zero-point energies to bosons and thus the energies may cancel out with each other.
- In addition, if we add fermions on our **worldsheet** (internal degrees of freedom on the string), it may make sense that we'll get fermions in **spacetime** (corresponding to string excitations).

These expectations are heuristic and somewhat naive, but adding fermions does indeed work, and we'll now see how to develop that theory. We'll need to understand how many and what kinds of fermions are added, and we also need to see how they affect the action of the string. Just like when we first wrote up the bosonic action, we want the action to be Lorentz invariant, reparameterization invariant, and also contain at most one derivative. But there's also an additional requirement: we ask that the action should be such that **it can exactly cancel out the zero-point energy of** X^μ, to get rid of the tachyon problem. This will make the combined action of X^μ along with the fermions supersymmetric.

Definition 159
Supersymmetry refers to a symmetry that transforms bosons to fermions and vice versa.

We're now going from the old **bosonic string** to the new **superstring**. We can again find an essentially unique action under these constraints, and then we perform a similar procedure as before: fix the gauge, quantize the theory (checking for consistency in things like the Lorentz algebra), and find the spectrum of the result. But coming up with a reparameterization invariant action is rather intricate, so instead, we're going to start with a gauge-fixed action (which can be found basically through guesswork), and this is indeed how string theory was done at the beginning – supersymmetry was basically discovered by accident!

3.7.1: Fermionic harmonic oscillators

We start with the **standard** harmonic oscillator action
$$S = \int dt \left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2\right),$$
where we have $[x,p] = i$, we can define x and p linear combinations of the raising and lowering operators $a, a^\dagger$, $[a, a^\dagger] = 1$, and we have the Hamiltonian $H = \omega(a^\dagger a + \frac{1}{2})$.

To write down a theory for **classical fermionic** harmonic oscillators, we first need to introduce a new mathematical object: **anti-commuting variables** that satisfy the relations

$$b_1 b_2 = -b_2 b_1 \implies \quad b_1^2 = -b_1^2 = 0$$

their complex conjugates are defined as

$$(b_1 b_2)^\dagger = b_2^\dagger b_1^\dagger \ .$$

These are usually known as the **Grassmann variables**. We can then write the action for a fermionic harmonic oscillator in terms of a complex anti-commuting variable ψ as

$$S = \int L dt, \quad L = i\psi^\dagger \partial_t \psi - \frac{\omega}{2}(\psi^\dagger \psi - \psi \psi^\dagger).$$

(We can check that putting an i in the first term on the right-hand side ensures that the Lagrangian is actually real.)

Remark 160. *Notice that there's only one time-derivative here – in the bosonic case, having just one time-derivative would give us nothing useful in the action, because that would be a total derivative. But the simplest kinetic term here is a single derivative of the form $\psi^\dagger \dot{\psi}$, and similarly the $\psi^\dagger \psi - \psi \psi^\dagger$ term is the simplest potential term.*

The corresponding equation of motion and solution are

$$\partial_t \psi = -i\omega\psi \implies \psi(t) = e^{-i\omega t}\psi(0).$$

Now let us consider the quantum theory. To quantize a theory of classical anticommuting variables, we need to impose **anticommutation relation** on a variable and its canonical momentum, i.e.

$$\{\psi, p_\psi\} = \psi p_\psi + p_\psi \psi = i.$$

For the above action we have

$$p_\psi = i\psi^\dagger, \quad \to \quad \{\psi, \psi^\dagger\} = 1 \ .$$

(Taking $\hbar \to 0$ recovers the anticommutativity of classical variables, just like it recovered the classical commutativity in the ordinary harmonic oscillator.)

The Hamiltonian can be obtained from the Lagrangian using the standard Legendre transform and is given by

$$H = \frac{\omega}{2}(\psi^\dagger \psi - \psi \psi^\dagger) = \omega \left(\psi^\dagger \psi - \frac{1}{2} \right),$$

which indeed has the opposite zero-point energy as that of an ordinary h.o.

As a result, a theory of an ordinary harmonic oscillator plus a fermionic harmonic oscillator with the same frequency ω gives us Hamiltonian

$$H = \omega(a^\dagger a + \psi^\dagger \psi),$$

with no zero-point energy! This is the simplest supersymmetric system. In your pset, you will have chance to verify that the theory is supersymmetric yourself. In a supersymmetric theory, each bosonic degree of freedom has a fermionic "partner."

3.7.2: Worldsheet fermions

We now turn to the string action and understand how to introduce fermions there. First of all, recall the **partially gauge-fixed** Polyakov action

$$S_p = -\frac{1}{4\pi\alpha'} \int d^2\sigma \partial^\alpha X^\mu \partial_\alpha X_\mu,$$

which has equations of motion

$$\partial_\tau^2 X^\mu - \partial_\sigma^2 X^\mu = 0$$

with general solutions having the form

$$X^\mu(\tau, \sigma) = X_R^\mu(\tau - \sigma) + X_L^\mu(\tau + \sigma) \ .$$

We will now add fermionic variables ψ's which are functions of τ and σ – here, τ, σ are ordinary numbers and commute in the usual way, but ψs are now anticommuting. In order to have Lorentz symmetry, they should transform as a Lorentz vector, so we write them as $\psi^\mu(\tau, \sigma)$. In order to have supersymmetry we also need to make sure that each X_L^μ and X_R^μ mode has a **fermionic partner**. So we should have real fermionic fields ψ_L^μ, ψ_R^μ that are respectively left and right moving. Also recall that our action in the fermionic harmonic oscillator only used one time-derivative. These considerations lead to the following guess for the gauge-fixed fermonic action

$$S_\psi = \frac{i}{4\pi} \int d\tau d\sigma \left[\psi_L^\mu (\partial_\tau - \partial_\sigma)\psi_{L\mu} + \psi_R^\mu(\partial_\tau + \partial_\sigma)\psi_{R\mu}\right] .$$

Varying this action and for the moment ignoring boundary terms coming from integration by parts, we find the equations of motion can be written as

$$(\partial_\tau - \partial_\sigma)\psi_{L\mu} = 0, \qquad (\partial_\tau + \partial_\sigma)\psi_{R\mu} = 0.$$

In other words, ψ_L^μ is a function of $\tau + \sigma$, and ψ_R^μ is a function of $\tau - \sigma$, so they are indeed left-moving and right-moving in the same way as our original string modes $X_{L,R}^\mu$.

Next time, we'll see how boundary conditions are imposed on these fermionic "partners" to ensure that we indeed get the above equations of motion.

April 28, 2021

Last time, we started discussing how to add fermions to our worldsheet to build a new string theory. We first discussed a **fermionic harmonic oscillator**, which is a theory of anticommuting variables

$$L = i\psi^\dagger \partial_t \psi - \frac{\omega}{2}(\psi^\dagger\psi - \psi\psi^\dagger),$$

with anticommuting relations (the analogs of the standard creation and annihilation operators) satisfying

$$\{\psi, \psi^\dagger\} = 1.$$

The Hamiltonian is given by $H = \omega\left(\psi^\dagger\psi - \frac{1}{2}\right)$, where the zero-point energy is the negative of that for the ordinary harmonic oscillator.

The Hilbert space of the theory can be found by starting with a vacuum state $|0\rangle$, saying that ψ annihilates that vacuum, and then defining $\psi^\dagger|0\rangle = |1\rangle$. But **that's all we can do** – notice that $(\psi^\dagger)^2|0\rangle = 0$ because of anticommutativity, and $\psi|1\rangle = |0\rangle$. So we have a two-state system – indeed, as the **Pauli exclusion principle** tells us, we can only have zero or one fermions in the state, and that corresponds to the basis states $|0\rangle$ and $|1\rangle$ respectively.

If we combine our standard harmonic oscillator with our fermionic harmonic oscillator, we find that

$$H = \omega(a^\dagger a + \psi^\dagger\psi).$$

The Hilbert space for the combined system then looks like $|m\rangle_b \otimes |n\rangle_f$, where $m \in \mathbb{Z}_{\geq 0}$ and $n \in \{0, 1\}$. This is a theory with no zero-point energy, which can be attributed to the existence of supersymmetry.

We construct superstring theory by adding fermions to the string worldsheet. For the full string action to be supersymmetric (so as to have no zero-point energy) we need to add partners for the right-moving and left-moving modes X_R^μ and X_L^μ, which we denote by ψ_R^μ and ψ_L^μ. Last lecture we discussed that the action which fulfills various requirements is

$$S_f = \frac{i}{4\pi}\int d^2\xi \left[\psi_L^\mu(\partial_\tau - \partial_\sigma)\psi_{L\mu} + \psi_R^\mu(\partial_\tau + \partial_\sigma)\psi_{R\mu}\right],$$

which gives the equations of motion $(\partial_\tau - \partial_\sigma)\psi_{L\mu} = 0, (\partial_\tau + \partial_\sigma)\psi_{R\mu} = 0$ (so that the fermionic modes are also left- and right-moving).

If we combine S_f and S_X (the bosonic and fermionic actions), the full action is invariant under the following **supersymmetric transformations** (up to boundary terms from integration by parts)

$$\delta X = i(\varepsilon_L\psi_L + \varepsilon_R\psi_R), \quad \delta\psi_L = -2\varepsilon_L\partial_v X, \quad \delta\psi_R = -2\varepsilon_R\partial_u X,$$

where $\varepsilon_L, \varepsilon_R$ are anticommuting constant parameters, $v = \tau + \sigma, u = \tau - \sigma$, and for simplicity we have restricted to one direction (i.e. suppressed μ indices). Since there are two independent fermionic tranformation parameters $\epsilon_{L,R}$, we say the theory has **two supersymmetries**.

3.7.3: Boundary conditions

We'll now return to the boundary conditions that should be satisfied by fermionic variables.

Taking the complex conjugate of S_f, we have

$$S_F^* = -\frac{i}{4\pi}\left[(\partial_\tau - \partial_\sigma)\psi_{L\mu}\psi_L^\mu + (\partial_\tau + \partial_\sigma)\psi_{R\mu}\psi_R^\mu\right]$$

(where the derivatives only act on one ψ term each, and we use the fact that ψ is real). Now integrating by parts we find

$$S_F^* = \frac{i}{4\pi}\int d^2\xi \left[\psi_{L\mu}(\partial_\tau - \partial_\sigma)\psi_L^\mu + \psi_{R\mu}(\partial_\tau + \partial_\sigma)\psi_R^\mu\right] + \frac{i}{4\pi}\left(\psi_{L\mu}\psi_L^\mu - \psi_{R\mu}\psi_R^\mu\right)_0^{\sigma_1},$$

where $\sigma_1 = \pi$ for open strings and 2π for closed strings. The first term above is simply S_f, and the second term comes from the total derivative term in σ direction (as always we assume the boundary terms in time direction vanish). For the action to be real we need the boundary terms to vanish, i.e.

$$\left(\psi_{L\mu}\psi_L^\mu - \psi_{R\mu}\psi_R^\mu\right)_0^{\sigma_1} = 0. \tag{25}$$

In varying the action S_f to obtain equations of motion we also need the boundary terms in σ direction to vanish, which gives

$$\left(\psi_R^\mu\delta\psi_{R\mu} - \psi_L^\mu\delta\psi_{L\mu}\right)_0^{\sigma_1} = 0 \tag{26}$$

where here $\delta\psi$'s are arbitrary variations.

We now proceed to find boundary conditions $\psi_{L,R}^\mu$ have to satisfy to ensure (25) and (26). We will consider (26) and we will see (25) will automatically be satisfied for the boundary conditions which ensure (26).

Example 161

For the open strings, we have two endpoints at $\sigma = 0, \pi$, and we want conditions at the left and right end to vanish separately (because we want the two ends of the string to behave independently).

In other words, we want (for both $\sigma_* = 0$ and $\sigma_* = \pi$) the constraint

$$\left(\psi_{R\mu}\delta\psi_R^\mu - \psi_{L\mu}\delta\psi_L^\mu\right)_{\sigma_*} = 0.$$

The key point now is that ψ_L and ψ_R are both a function of only **one variable**. Thus, if we set $\delta\psi_R^\mu|_{\sigma_*}$ or $\delta\psi_L^\mu|_{\sigma_*} = 0$, then the corresponding ψ_R or ψ_L must be a constant, which is not very interesting. Otherwise, this equation will be satisfied for arbitrary $\delta\psi_R^\mu, \delta\psi_L^\mu$, only if we have

$$\psi_R^\mu(\tau, \sigma_*) = \pm\psi_L^\mu(\tau, \sigma_*).$$

In other words, the **left- and right-moving waves must be correlated** in a particular way. It may seem initially like there are four possibilities for how this plays out, since we have a $\pm$ that needs to be specified at each of the two string endpoints, but we have an extra degree of freedom in multiplying either ψ_R or ψ_L by -1 (since both the action and $\psi\delta\psi$ will remain invariant). So in fact we can break up the situation into just the two cases

$$\psi_R^\mu(\tau, \sigma = 0) = \psi_L^\mu(\tau, \sigma = 0), \tag{27}$$

$$\psi_R^\mu(\tau, \sigma = \pi) = \pm\psi_L^\mu(\tau, \sigma = \pi) \tag{28}$$

(the $+$ corresponds to the **R (Ramond) boundary conditions**, while the $-$ corresponds to the **NS (Neveu-Schwarz) boundary conditions**). But now remember that we can write ψ_R^μ as $\psi_R^\mu(\tau - \sigma)$ and ψ_L^μ as $\psi_L^\mu(\tau + \sigma)$. So equation (27) implies that

$$\psi_L^\mu(\tau) = \psi_R^\mu(\tau) \equiv \psi^\mu(\tau)$$

i.e. ψ_R^μ and ψ_L^μ are given by the same function ψ^μ. Equation (28) now gives

$$\psi^\mu(\tau - \pi) = \pm\psi^\mu(\tau + \pi),$$

and this means that the R boundary condition requires 2π-periodicity, while the NS boundary condition requires 2π-antiperiodicity.

Example 162

For the closed strings, we now have $\sigma_1 = 2\pi$, but we must remember that $\sigma = 0$ and $\sigma = 2\pi$ are the same point on the circle.

In this case, we don't need the $\sigma = 0$ and $\sigma = 2\pi$ parts of our boundary condition to vanish independently anymore: our equation is

$$\psi_R^\mu\delta\psi_{R\mu}\big|_{\sigma=2\pi} - \psi_R^\mu\delta\psi_{R\mu}\big|_{\sigma=0} - \psi_L^\mu\delta\psi_{L\mu}\big|_{\sigma=2\pi} + \psi_L^\mu\delta\psi_{L\mu}\big|_{\sigma=0} = 0.$$

and this time the condition is that our **left and right moving modes vanish independently**. This means the L part of the equation must vanish, and so must the R part. So we have that

$$\psi_R^\mu(\sigma = 2\pi) = \pm\psi_R^\mu(\sigma = 0), \quad \psi_L^\mu(\sigma = 2\pi) = \pm\psi_L^\mu(\sigma = 0).$$

This time, we do indeed have four different cases – both the left- and right-moving modes can either be in the R or NS boundary conditions, independently, and again we have either periodic or anti-periodic functions, but this time they are defined on the circle. (Here, the point to notice is that **fermions are internal degrees of freedom**, so it's okay for fermions to gain a -1 factor when we increase σ by 2π – there are no geometric constraints to impose the periodic boundary conditions. But this is not okay for our X^μ string solution.)

In all cases, we can now further fix the light-cone gauge. What happens this time is that in addition to the

Virasoro constraints, we also get a fermionic partner for the Virasoro constraints. The analogy for the light-cone gauge condition is that we set $\psi^+ = 0$, and we can also write ψ^- in terms of the ψ^Is (which helps us write down Lorentz generators and so on). We can then quantize the system and check the Lorentz algebra – in other words, calculate the commutators of the $M^{\mu\nu}$s, which are now involving both fermions and bosons – and this time what we find is that $\boxed{d = 10}$ is the requirement for superstring theory.

Our goal from here is to find the spectrum of the string, but to do that we first need to write the solutions to the equations of motion and see how the quantization works out. Much like in the bosonic case, we reduce the fermionic action to a bunch of fermionic harmonic oscillators and quantize those as we've already done. We'll focus on the open string here.

Example 163

First of all, suppose we have the NS boundary conditions, so that

$$\psi_R^I = \psi_L^I = \psi^I$$

takes on a functional form which is antiperiodic every 2π.

We can then expand this solution in Fourier modes as (using u as the variable for this functional form)

$$\psi^I(u) = \sum_{n \in \mathbb{Z}+\frac{1}{2}} \left(\psi_n^I e^{-in\sigma} + (\psi_n^I)^\dagger e^{-inu}\right),$$

and if we plug in this ψ^I back into the action, our fermionic action becomes (separating out the σ- and τ-dependence)

$$S_f = \int d\tau \sum_{n \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\}} \left[i(\psi_n^I)^\dagger \partial_\tau \psi_n^I - \frac{n}{2}\left((\psi_n^I)^\dagger \psi_n^I - \psi_n^I (\psi_n^I)^\dagger\right)\right].$$

So just like in the bosonic case, we have an action which is a combination of (fermionic) harmonic oscillators. The quantization relation $\{\psi_n^I, (\psi_n^J)^\dagger\} = \delta_{nm}\delta^{IJ}$ must then be imposed, and we have the Hamiltonian

$$\sum_{n \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\}} \sum_I n\left((\psi_n^I)^\dagger \psi_n^I - \frac{1}{2}\right).$$

Our creation operators in this theory are $\{(a_n^I)^\dagger, (\psi_m^I)^\dagger\}$, where $n = 1, 2, \cdots$ and $m = \frac{1}{2}, \frac{3}{2}, \cdots$, so our total Hamiltonian for the oscillators then looks like

$$H_{\text{NS}} = \sum_I \left[\sum_{n=1}^{\infty} n(a_n^I)^\dagger a_n^I + \sum_{m \in \{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \cdots\}} m(\psi_m^I)^\dagger \psi_m^I\right] + a_b + a_f,$$

where our zero-point energy now has some contribution from bosonic and fermionic ground-state energies

$$a_b = \sum_I \sum_{n=1}^{\infty} \frac{n}{2}, \quad a_f = \sum_I \sum_{m \in \{\frac{1}{2}, \frac{3}{2}, \cdots\}} -\frac{m}{2},$$

and we can calculate these using our usual tricks. We have

$$a_b = \frac{d-2}{2} \sum_{n=1}^{\infty} n = -\frac{d-2}{24},$$

and now similarly we need to do a "sum over all odd integers," which is the sum over integers minus the sum over all even integers:

$$a_f = -\frac{d-2}{2}\sum_{m\in\{\frac{1}{2},\frac{3}{2},\cdots\}} m = -\frac{d-2}{2}\cdot\frac{1}{2}\left(\sum_{m=1}^{\infty} m - 2\sum_{m=1}^{\infty} m\right) = -\frac{d-2}{48}.$$

So the total zero-point energy is

$$a = a_b + a_f = -(d-2)\left(\frac{1}{24}+\frac{1}{48}\right) = -\frac{d-2}{16},$$

and the reason that this is a nonzero sum (i.e. bosons and fermions do not cancel out with each other) is that the **boundary conditions are different**: an NS fermion has antiperiodic boundary conditions. Indeed, we'll find that the supersymmetric transformations give us boundary terms that don't vanish, and thus **supersymmetry is broken by boundary conditions** in the NS case. You will check this yourself in your pset 10.

Example 164

Next, let's consider the R boundary conditions: everything will be very similar, but our Fourier modes are periodic instead of antiperiodic.

This time, we have a sum over integers instead of half-integers:

$$\psi^I = \sum_{n\in\mathbb{Z}} \psi_n^I e^{-inu} = \psi_0^I + \sum_{n=1}^{\infty}\left(\psi_n^I e^{inu} + (\psi_n^I)^\dagger e^{-inu}\right),$$

where u is replaced by either $\tau+\sigma$ or $\tau-\sigma$ depending on the direction of the moving mode. Our oscillators must then have integer frequencies $\omega = n$, and then we have the anticommutation conditions $\{\psi_0^I, \psi_0^J\} = \delta^{IJ}, \{\psi_n^I, (\psi_n^J)^\dagger\} = \delta^{IJ}\delta_{mn}$ (there's no dagger in the first of these relations because ψ_0 is already real and is thus its own conjugate). Our fermionic Hamiltonian is then (the zero modes don't add to the energy because ψ_0^I is just a constant)

$$H_f = \sum_{n=1}^{\infty}\sum_I n\left((\psi_n^I)^\dagger\psi_n^I - \frac{1}{2}\right),$$

and thus the creation operators in the full theory are $\{(a_n^I)^\dagger, (\psi_n^I)^\dagger, \psi_0^I\}$ for integers $n \geq 1$. We can combine this with the bosonic Hamiltonian to get

$$H^{(R)} = \sum_I\sum_{n=1}^{\infty}(n(a_n^I)^\dagger a_n I + n(\psi_n^I)^\dagger\psi_n^I) + a_f + a_b,$$

where now $a_f = -\frac{1}{2}\sum_I\sum_{n=1}^{\infty} n = -a_b$, so the zero-point energies cancel out. In this case, since we have identical periodic boundary conditions for $X_{L,R}^I$ and $\psi_{L,R}^I$, we do have supersymmetry between them.

Fact 165

The discussion also looks very similar in the closed string case – the main difference is that we have two copies of the ψs (left-moving and right-moving), and thus we need to consider NS-NS, NS-R, R-NS, and R-R boundary conditions.

For example, $\{(a_n^I)^\dagger, (\tilde{a}_n^I)^\dagger, (\psi_m^I)^\dagger, (\tilde{\psi}_m^I)^\dagger\}$ are the creation operators in the NS-NS case, where n is a positive integer and m is a positive half-integer, and $\{(a_n^I)^\dagger, (\tilde{a}_n^I)^\dagger, (\psi_m^I)^\dagger, (\tilde{\psi}_n^I)^\dagger, \tilde{\psi}_0^I\}$ are our creation operators in the NS-R case (basically, R boundary conditions give us integer indices, and NS boundary conditions give us half-integer indices).

Now that we've arrived at our Hamiltonian, just like in the bosonic case, we're going to try to arrive at physical spectrum for superstring theory as well. One question is whether different boundary conditions for fermions correspond to different theories, but in this case they actually correspond to **different sectors** of the same theory. Specifically, if we find the Hilbert space for each set of boundary conditions, and then add them together, that gives us the theory we're looking for. This is a difficult thing to justify physically, and the reason for doing it is that there are extra self-consistency checks we need to do. We'll talk more about this next time!

May 3, 2021

In the last few lectures, we've been discussing superstrings, which have action given by $S = S_X + S_f$, where

$$S_X = -\frac{1}{4\pi\alpha'}\int d^2\xi \partial_\alpha X^\mu \partial^\alpha X_\mu$$

is the bosonic action, and

$$S_f = \frac{i}{4\pi}\int d^2\xi \left[\psi_L^\mu(\partial_\tau - \partial_\sigma)\psi_{L\mu} + \psi_R^\mu(\partial_\tau + \partial_\sigma)\psi_{R\mu}\right]$$

is the fermionic action. The important point is that S is invariant, up to boundary terms in integration by parts, under certain supersymmetries which transform boson and fermion solutions in terms of each other.

We've solved the fermionic action explicitly as well: the equation of motion tells us that ψ_L^μ and ψ_R^μ are functions of $\tau + \sigma$ and $\tau - \sigma$, respectively, and the boundary conditions are

$$\psi_L^\mu(\tau, \sigma = 0) = \psi_R^\mu(\tau, \sigma = 0), \quad \pm\psi_L^\mu(\tau, \sigma = \pi) = \psi_R^\mu(\tau, \sigma = \pi)$$

for an open string, meaning that in either case we must have ψ_L^μ and ψ_R^μ having the same functional form ψ^μ, and we also need that function ψ to be either periodic or antiperiodic of period 2π (corresponding to R or NS boundary conditions, respectively). On the other hand, for the closed string, ψ_L^μ and ψ_R^μ are independent functions of a single variable which can be periodic or antiperiodic, and we can have either NS-NS, NS-R, R-NS, or R-R boundary conditions in this case.

It turns out that there is a fermionic version of the light-cone gauge as well: in both the open and closed string cases, we can set $\psi^+ = 0$, and we can then solve for ψ^- in terms of the ψ^I's much like we've done before. So our next step is to understand how we quantize the ψ^I variables, and this depends on the boundary conditions.

We'll use u as our dummy variable. For the R boundary conditions, we know that

$$\psi^I(u) = \psi_0^I + \sum_{n=1}^{\infty}\left(\psi_n^I e^{-inu} + (\psi_n^I)^\dagger e^{inu}\right),$$

meaning that (just like in the boson case) each ψ_n^I corresponds to a fermionic harmonic oscillator with frequency $\omega = n$, and thus the appropriate anticommutation relations are

$$\{\psi_n^I, (\psi_m^J)^\dagger\} = \delta_{nm}\delta^{IJ}$$

for all positive n, m. But the zero mode is special, because for $n \geq 1$ we have $p_\psi = i\psi^\dagger$, but for $n = 0$ the effective action comes from $L_0 = i\psi_0^I \partial_\tau \psi_0^I$ (so there are no oscillatory parts, just the kinetic term and ψ_0^I are real). So here, $p_{\psi_0} = i\psi_0^I$, and thus our anticommutation relation is

$$\{\psi_0^I, \psi_0^J\} = \delta^{IJ}.$$

We can also find the zero-point energy for the fermionic action – for R boundary conditions, we can find that a_f is the negative of the bosonic zero-point energy, which is $a_f = -a_b = \frac{d-2}{24}$. But for the NS boundary conditions, where we have antiperiodicity $\psi^I(u+2\pi) = -\psi^I(u)$, we find that expansion in Fourier modes gives

$$\psi^I(u) = \sum_{n=\frac{1}{2},\frac{3}{2},\cdots} (\psi^I_n e^{-inu} + (\psi^I_n)^\dagger e^{inu}),$$

where notably the modes are all indexed by half-integers and everything is harmonic oscillators. We again have the anticommutation relation $\{\psi^I_n, (\psi^J_m)^\dagger\} = \delta^{IJ}\delta_{nm}$, but our zero-point energy now becomes $a_f = -\frac{d-2}{48}$ instead of $\frac{d-2}{24}$ (since we're summing $-\frac{1}{2}n$ over half-integers instead of over integers).

In the open string case, the mass-shell condition now tells us that

$$M^2 = \frac{1}{\alpha'}(N_\perp + a),$$

where $N_\perp = \sum_I \left(\sum_{n=1}^\infty n(a^I_n)^\dagger a^I_n + \sum_m m(\psi^I_n)^\dagger \psi^I_n\right)$ now includes excitations from both bosonic and fermionic oscillators, and $a = a_b + a_f$ is $-\frac{d-2}{16}$ for the NS boundary conditions and 0 for the R boundary conditions. And for the closed string, we find

$$M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp + a + \tilde{a}),$$

where we get two copies corresponding to the left-moving and right-moving modes, and we also have the level-matching condition $N_\perp + a = \tilde{N}_\perp + \tilde{a}$. And it's also good to remember that a single fermionic harmonic oscillator with $\{\psi, \psi^\dagger\}$ has a two-dimensional Hilbert space, with $\psi|0\rangle = 0, \psi^\dagger|0\rangle = |1\rangle, \psi|1\rangle = |0\rangle$

3.7.4: Physical spectrum

We can now find the spectrum of the superstring. As previously mentioned, we should treat the different boundary conditions (NS and R for open, NS-NS, NS-R, R-NS, and R-R for closed) as **different sectors of the same theory**. Explicitly, we can work out the different Hilbert spaces $\mathcal{H}_X \otimes \mathcal{H}_{NS}, \mathcal{H}_X \otimes \mathcal{H}_R$, and so on, and it turns out that the physical spectrum corresponds to a direct sum of **subspaces** of those Hilbert spaces. The reason this is necessary is that fermions require more subtle quantum consistency conditions, and this "patching together" is how we resolve those conditions. We'll talk about this more later on!

Example 166

We'll first find the spectrum for the open string, in both the NS and R sectors.

For the NS sector, we have the mass-shell condition

$$M^2 = \frac{1}{\alpha'}\left(N_\perp - \frac{d-2}{16}\right),$$

where $N_\perp$ is the sum of the boson and fermion oscillator numbers. Our lowest energy states are again the unexcited $|p^+, \vec{p}_\perp\rangle$ states with $N_\perp = 0$, and what we find is that $M^2 = -\frac{1}{\alpha'}\frac{d-2}{16}$. We should recall that the Lorentz algebra requires $d = 10$ for superstrings, but we can also find out that fact from the spectrum directly, so we will not employ it here.

The next level is then $N_\perp = \frac{1}{2}$, which occurs when we have a state of the form

$$(\psi^I_{1/2})^\dagger |p^+, \vec{p}_\perp\rangle = |p^+, \vec{p}_I, I\rangle, \quad I \in \{2, \cdots, d-1\}.$$

These states then have energy $M^2 = \frac{1}{\alpha'}\left(\frac{1}{2} - \frac{d-2}{16}\right)$, and further notice that even though these are generated by fermionic modes, these polarizations transform as a **spacetime vector** (and thus this is actually a spin 1 particle). Since we have $(d-2)$ degrees of freedom for our spin 1 particle, that means this must be a massless vector, and thus we must have

$$\frac{1}{2} - \frac{d-2}{16} = 0 \implies d = 10.$$

Thus, we again have a tachyon at the $N_\perp = 0$ level – in other words, adding worldsheet fermions give us spacetime bosons, so things don't cancel out to remove tachyons as we initially may have expected. But as we will see momentarily spacetime fermions can also be generated – in the R-sector. (And all higher $N_\perp \geq 1$ correspond to massive modes with positive M^2.)

The R sector corresponds to periodic boundary conditions, and the total zero-point energy is $0 = a_f + a_b$ (with supersymmetry not broken). We thus have $M^2 = \frac{1}{\alpha'} N_\perp$, where

$$N_\perp = \sum_I \sum_{n=1}^{\infty} (n(a_n^I)^\dagger a_n^I + n(\psi_n^I)^\dagger \psi_n^I).$$

This system still has modes ψ_0^I of zero energy (with the anticommutation condition $\{\psi_0^I, \psi_0^J\} = \delta^{IJ}$, and this time the massless modes correspond to $N_\perp = 0$ (meaning that the corresponding states are only generated by ψ_0^I). So we need to understand how the ψ_0^Is act on our states $|p^+, \vec{p}_\perp\rangle$. Recall that $(\psi_0^I)^2 = \frac{1}{2}$ (by plugging $I = J$ into the anticommutation relation above), and notice that this algebra has commutation relations that are very different from the ones we're used to (it goes by the name "Clifford algebra," and the fermions go by the name of **Majorana fermions**).

Fact 167

If we write $|p^+, \vec{p}_\perp\rangle$ as $|0\rangle$, and we look at ψ_0^I acting on $|0\rangle$ for $I = 2, \cdots, 9$, we will generate **sixteen** different states, in two groups of eight, where each group corresponds to a spacetime spin 1/2 **particle**. (We'll denote this as $8 + \overline{8}$.)

We will now show this explicitly. We introduce

$$\xi_1 = \frac{1}{\sqrt{2}}(\psi_0^2 + i\psi_0^3), \quad \xi_1^\dagger = \frac{1}{\sqrt{2}}(\psi_0^2 - i\psi_0^3)$$

and we similarly define ξ_2, ξ_3, ξ_4 to be those definitions but with $\{4,5\}, \{6,7\}, \{8,9\}$ instead of $\{2,3\}$. From anticommutation relation of ψ_0^I we find that

$$\{\xi_a, \xi_b\} = \{\xi_a^\dagger, \xi_b^\dagger\} = 0, \quad \{\xi_a, \xi_b^\dagger\} = \delta_{ab}, \quad a, b = 1, 2, 3, 4\ .$$

Now $\xi_a, \xi_a^\dagger$ behave as the standard fermionic annihilation and creation operators. Acting $\xi_a^\dagger$'s on $|0\rangle$ generates $2^4 = 16$ different states. It is convenient to introduce a new notation, denoting $|0\rangle$ as $|----\rangle$. When $\xi_a^\dagger$ acts on $|----\rangle$ it turns the a-th $-$ into $+$. For example.

$$\xi_1^\dagger \xi_2^\dagger |----\rangle = |++--\rangle\ .$$

The 16 states can thus be written as $|\pm\pm\pm\pm\rangle$. Restoring momentum we have $|p^+, \vec{p}_\perp; \pm\pm\pm\pm\rangle$.

Fact 168

We now denote the set of 8 states with even number of $-$'s as $|R_\alpha\rangle$ with $\alpha = 1, 2 \cdots 8$ and those with odd number of $-$'s as $|R_{\bar{\alpha}}\rangle$ with $\bar{\alpha} = 1, 2 \cdots 8$. We will now show that these states have spin 1/2 and Lorentz transformations only mix states within the same group.

The full action including fermions is Lorentz invariant, and in the light-cone gauge we can again separate the corresponding charges as $M^{\pm}, M^{\pm I}, M^{IJ}$. Let us focus on M^{IJ}, which are angular momentum operators associated with rotations in $I-J$ plane. Applying the Noether procedure we discussed before, we find that M^{IJ} is given by

$$M^{IJ} = -\frac{i}{2}(\psi_0^I\psi_0^J - \psi_0^J\psi_0^I) + \cdots$$

where we have only shown the contribution of ψ_0^I's with $\cdots$ denoting the contributions from other fermionic modes and bosonic modes. It can now be checked that $|R_\alpha\rangle$ and $|R_{\overline{\alpha}}\rangle$ are eigenstates of M^{23}, M^{45}, M^{67}, M^{89} with eigenvalues $\pm 1/2$. Let us use M^{23} as an illustration, which can be written in terms of $\xi_1, \xi_1^\dagger$ as

Example 169

$M^{23} = -\frac{i}{2}(\psi_0^2\psi_0^3 - \psi_0^3\psi_0^2) = \frac{1}{2}(\xi_1^\dagger\xi - \xi_1\xi^\dagger) = \xi_1^\dagger\xi_1 - \frac{1}{2}$.

Since $\xi_1^\dagger|-\rangle = |+\rangle$ and $\xi_1|-\rangle = 0$, so that $M_{23}|-\rangle = -\frac{1}{2}|-\rangle$ and $M_{23}|+\rangle = \frac{1}{2}|+\rangle$. In other words, any state of the form $|-,\pm,\pm,\pm\rangle$ will have eigenvalue $-\frac{1}{2}$ under M^{23}, and states of the form $|+,\pm,\pm\pm\rangle$ have eigenvalue $\frac{1}{2}$. Similarly with M^{45}, M^{67}, M^{89}. Basically, by defining ξ_1 through ξ_4 by picking out four pairs of indices in a particular way, we have picked a basis that makes $M^{23}, M^{45}, M^{67}, M^{89}$ all diagonal. And the other M^{IJ} operators other than those will not necessarily be diagonal – instead, they transform some of these states into others.

But they **will not transform states with odd and even numbers of $-$s into each other**, and thus we have two fermions with 8 polarizations each. This is because M^{IJ} is always quadratic in the ξs, and it always changes the number of $-$s by an even number.

Finally, remembering that angular momentum generates rotations, $e^{i\theta M_{23}}$ should generate rotations by angle θ in the 23-plane. Indeed,

$$e^{i\theta M_{23}}|\pm\rangle = e^{\pm i\frac{\theta}{2}}|\pm\rangle = -|\pm\rangle$$

when $\theta = 2\pi$, and this is similar to the situation we've seen in with the spin of an atom.

In summary, in the R boundary conditions, we have $8+\overline{8}$ massless spacetime fermions that are generated through zero modes. We'll talk more about the string spectrum next time!

May 5, 2021

Last lecture, we talked about the spectrum of the superstring, focusing on the massless excitations (which are the most important at low energies). The spectrum of the open string $\mathcal{H}_X \otimes \mathcal{H}_\psi$ can be split into the NS sector and the R sector. In the NS sector, we have the mass shell condition

$$M^2 = \frac{1}{\alpha'}\left(N_\perp - \frac{1}{2}\right)$$

in $d=10$ dimensions. At $N_\perp = 0$ we have a scalar particle described by $|p^+, \vec{p}_\perp\rangle$ (tachyon with negative $M^2 = -\frac{1}{2\alpha'}$). Above that, we have $N_\perp = \frac{1}{2}$, which is a massless vector described by $\psi_{-1/2}^I|p^+, \vec{p}_\perp\rangle$ with 8 polarizations. Above this, we have massive modes, which we won't consider too much.

On the other hand, for R boundary conditions, we have $M^2 = \frac{1}{\alpha'}N_\perp$, and the relevant operators we can apply to our ground state are $\alpha_{-n}^I, \psi_{-n}^I$, and ψ_0^I. So at the $N_\perp = 0 \implies M^2 = 0$ level, we get the different allowed states by having $\{\psi_0^I\}$ act on $|p^+, \vec{p}_\perp\rangle$: these are the Majorana fermions, and we can separate them into four pairs to get the standard creation and annihilation operators $\xi_a, \xi_a^\dagger$ for $a = 1,2,3,4$. We can label these states as $|\pm\pm\pm\pm\rangle$, and $\xi_a^\dagger$ flips a-th $-$ to $+$. These 16 states can be split into 8 with an even number of raising operators (that is, an even number of $-$s), and another 8 with an odd number of $-$s. So overall, states with $M^2 = 0$ can be labeled as

$$\left|p^+, \vec{p}_\perp, R_\alpha\right\rangle, \quad \left|p^+, \vec{p}_\perp, R_{\overline{\alpha}}\right\rangle,$$

where $\alpha, \overline{\alpha} \in \{1, 2, 3, 4, 5, 6, 7, 8\}$ just labels the different states that we have.

But we notice that even though we have gone from the bosonic string to superstring, we still haven't removed the tachyon that comes up in the NS sector. And in fact, we get an inconsistent theory unless we are careful and take only a subspace of allowed states from the NS sector and a subspace of allowed states from the R sector. We do so using a **projection**.

For this purpose we introduce a **worldsheet fermionic number operator**, denoted as $(-1)^F$, which has eigenvalues ± 1. This operator commutes with bosonic operators a_n^I, but it anticommutes with a fermonic operator

$$(-1)^F \psi_n^I = -\psi_n^I (-1)^F .$$

Also by convention we define the lowest state to have eigenvalue -1, i.e.

$$(-1)^F \left|p^+ \vec{p}_\perp\right\rangle = -\left|p^+, \vec{p}_\perp\right\rangle.$$

Because of those facts, we find that **any** basis state, whether in the NS or the R sector, are eigenstates of $(-1)^F$ (because constructing other basis states by applying a_n or ψ_n just gives us an extra factor of 1 or -1). So under this $(-1)^F$ operator, NS splits into an NS+ and an NS- subspace (corresponding to having an eigenvalue of 1 or -1 under $(-1)^F$, respectively), and similarly R splits into R+ and R-.

Example 170

By construction, the tachyon state $|p^+, \vec{p}_\perp\rangle$ has eigenvalue -1, while the photon states $\psi_{-1/2}^I |p^+, \vec{p}_\perp\rangle$ has eigenvalue 1 due to anticommutativity. Also, notice that all states $|p^+, \vec{p}_\perp, R_\alpha\rangle$ have eigenvalue -1, and all states $|p^+, \vec{p}_\perp, R_{\overline{\alpha}}\rangle$ have eigenvalue 1.

It turns out the way to get a consistent theory without a tachyon is to restrict to the 1 eigenvalue: in other words, we take the Hilbert space for string excitations to be

$$\mathcal{H} = \mathcal{H}_{\text{NS}\,+} \oplus \mathcal{H}_{\text{R}\,+}.$$

Now the tachyon has been projected out and the lowest mass square is zero: we have the **photon** $\psi_{-1/2}^I |p^+, \vec{p}_\perp\rangle$ and the **fermion** $|p^+, \vec{p}_\perp, R_{\overline{\alpha}}\rangle$, each with 8 polarizations. Thus at massless level we find an equal number of bosonic and fermionic polarizations. This is not an accident: it turns out that the number of fermionic and bosonic states will always be the same at any level of M^2 (a consequence of restricting to $NS+ \oplus R+$), and **we have supersymmetry** for the **spacetime** theory! In particular, this is very different from the previous **worldsheet** supersymmetry that we found, and this supersymmetry gives us "super-QED."

Our next step is to look at the closed string, for which the story is very similar but slightly more intricate. The lowest state is still $|p^+, \vec{p}_\perp\rangle$, but now we have four sectors instead of two (NS-NS, NS-R, R-NS, and R-R). The mass-shell condition is now

$$M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp + a + \tilde{a}),$$

where $\tilde{a} = a$ is 0 for R boundary conditions and $-\frac{1}{2}$ for NS boundary conditions, and we also require $N_\perp + a = \tilde{N}_\perp + \tilde{a}$.

Much like in the open string, we can split NS into NS+ and NS- (using fermionic number operators $(-1)^F$ and $(-1)^{\tilde{F}}$ for the left- and right-moving modes).

Example 171

For example, the NS-NS sector has mass=shell and level matching conditions

$$M^2 = \frac{2}{\alpha'}\left(\tilde{N}_\perp + N_\perp - \frac{1}{2} - \frac{1}{2}\right), \quad \tilde{N}_\perp = N_\perp \ .$$

According to the fermionic numbers it splits into four subsectors, $(NS+, NS+), (NS+, NS-), (NS-, NS+), (NS-, NS-)$.

Let's now try to understand where some of our different states live. At the $N_\perp = \tilde{N}_\perp = 0$ level, we have the state $\left|p^+, p_\perp^+\right\rangle$, which has $M^2 = -\frac{2}{\alpha'}$ and eigenvalues $-1, -1$ for $(-1)^F$ and $(-1)^{\tilde{F}}$. At the next level $N_\perp = \tilde{N}_\perp = \frac{1}{2}$, we get states of the form $\psi^I_{-1/2}\tilde{\psi}^J_{-1/2}\left|p^+, \vec{p}_\perp\right\rangle$, which has $M^2 = 0$ and eigenvalues $1, 1$ for $(-1)^F$ and $(-1)^{\tilde{F}}$. Since I and J can range from 2 to 9, we get 64 states, and the decomposition is the same as before: we get the graviton, anti-symmetric tensor, and dilaton.

Example 172

Next, let's think about the R-R sector, where $M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp)$ and $\tilde{N}_\perp = N_\perp$.

At the $N_\perp = \tilde{N}_\perp = 0$ level, we have $M^2 = 0$, and our states are indexed by those zero-energy modes from last lecture. So there are four possibilities: we can have $\left|R_\alpha, \tilde{R}_\alpha\right\rangle$, giving us eigenvalues $-1, -1$, or $\left|R_{\overline{\alpha}}, \tilde{R}_\alpha\right\rangle$, giving eigenvalues $1, -1$, or $\left|R_\alpha, \tilde{R}_{\overline{\alpha}}\right\rangle$, giving eigenvalues $-1, 1$, or $\left|R_{\overline{\alpha}}, \tilde{R}_{\overline{\alpha}}\right\rangle$, giving eigenvalues $1, 1$. In particular, notice that we're taking the tensor product of two fermions, which gives us a boson, so these are giving us **spacetime bosonic particles** (this is related to how the sum of two spin $\frac{1}{2}$ particles is either spin 1 or 0).

Example 173

Finally, we can think about the NS-R sector, for which we have $M^2 = \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp - \frac{1}{2})$, and we have level-matching condition $N_\perp - \frac{1}{2} = \tilde{N}_\perp$.

In this case, the smallest level that we're allowed to have is $\tilde{N}_\perp = 0, N_\perp = \frac{1}{2}$, which gives us $M^2 = 0$ and states of the form $\psi^I_{-1/2}$ (left-moving NS) acting on $\left|p^+, \vec{p}_\perp, \tilde{R}_\alpha\right\rangle$ or on $\left|p^+, \vec{p}_\perp, \tilde{R}_{\overline{\alpha}}\right\rangle$, which both yield eigenvalues of 1 for $(-1)^F$ (since we applied one raising operator) but have eigenvalues -1 and 1 respectively for $(-1)^{\tilde{F}}$. And what we have here are **spacetime fermions**, since we have a boson and fermion tensored together – it turns out that we have spin $\frac{3}{2}$ and spin $\frac{1}{2}$ particles. Spin-$\frac{3}{2}$ particles are called gravitinos. (And the R-NS sector looks similar, but we just swap the tildes.)

Just like with the open string, we'll need to do a nontrivial projection to get a consistent theory for the closed string. There are two that do not have tachyons, called respectively type IIA and type IIB. In type **IIB**, we restrict to eigenvalues $(-1)^F = (-1)^{\tilde{F}} = 1$, and in type **IIA**, we restrict to eigenvalues $(-1)^F = 1$ and $(-1)^{\tilde{F}} = 1$ for NS, -1 for R. More explicitly, we have the subspaces

$$(\text{NS+ NS+}), (\text{R+ NS+}), (\text{NS+ R+}), (\text{R+ R+})$$

extracted in type IIB, and

$$(\text{NS+ NS+}), (\text{NS+ R-}), (\text{R+ NS+}), (\text{R+ R-})$$

extracted in type IIA. Notice neither of them has NS- (so we avoid the tachyon). This gives us **two independent superstring theories**, and they yield different spectrums (we can go back to all of the work we did above and see which particles actually make it into the theory). In the IIB case, we get 128 boson states and 128 fermion states, so we again get spacetime supersymmetry. The states explicitly look as follows (where I, J range from 2 to 9): we have the bosons from (NS+, NS+)

$$\psi^I_{-1/2}\tilde{\psi}^J_{-1/2}\left|p^+, \vec{p}_\perp\right\rangle,$$

the fermions from (NS+, R+)

$$\psi^I_{-1/2}\left|p^+, \vec{p}_\perp, \tilde{R}_{\overline{\alpha}}\right\rangle,$$

the fermions from (R+, NS+)

$$\tilde{\psi}^I_{-1/2}\left|p^+, \vec{p}_\perp, R_{\overline{\alpha}}\right\rangle,$$

and the bosons from (R+, R+)

$$\left|p^+, \vec{p}_\perp, R_{\overline{\alpha}}, \tilde{R}_{\overline{\alpha}}\right\rangle.$$

And from here, we can list out the particles in the theory. We'll just do the bosons here: from (NS+, NS+), we get $h_{\mu\nu}$ (graviton), $B_{\mu\nu}$ (anti-symmetric tensor), and ϕ (dilaton), and from (R+, R+) we get χ (scalar), $C^{(2)}_{\mu\nu}$ (another anti-symmetric tensor with two indices), and $C^{(4)}_{\mu\nu\sigma\rho}$ (an anti-symmetric tensor with four indices). We can also list out the fermions in the IIB theory, and we can list out the bosons and fermions in the IIA theory as well, but we'll omit this here. Overall, both theories have a bunch of massless particles, and we have spacetime supersymmetry in both cases.

It makes sense to ask now whether there are other supersymmetric string theories that also don't include tachyons, and it turns out that there are three others as well.

- In type IIB strings, left-moving and right-moving modes look the same – there's a symmetry between moving to the left or right on the worldsheet, and thus we get a symmetry under exchange of left-moving and right-moving degrees of freedom. We can take advantage of this symmetry to construct an **unoriented** version of type IIB string theory, which is referred to as type **I** superstring theory.
- To get the other two string theories, recall that in bosonic string theory, we have right-moving and left-moving modes (X^μ_R, X^μ_L, where $d = 26$), and in superstring theory, we have X_R, ψ_R, X_L, ψ_L modes, where $d = 10$. It turns out that we can combine **half of each** to get **heterotic string theory**: if we use the modes

$$X^\mu_R, X^a_R, \quad X^\mu_L, \psi^\mu_L,$$

  where we take $d = 10$ (i.e. $\mu \in (0, 1, \cdots 9)$) and have $a \in \{10, 11, \cdots, 25\}$. Essentially we have a right-moving bosonic string and a left-moving superstring. This kind of approach gives us the last two superstring theories.

So we now have five different theories but only one universe, and the different theories all have different spectra. But it turns out that **all five superstring theories are equivalent**, and we might see more of the duality in future lectures!

3.8: Interaction of strings

We'll start this section with an alternative formulation of nonrelativistic quantum mechanics. We've mentioned before that we have the Schrodinger and the Heisenberg formulations, but there's a third called the **path integral formulation**, which we'll give an overview of right now.

We start with the fundamental observable

$$\langle x, t | x', t'\rangle$$

which is the amplitude for the system to be in an eigenvector of $\hat{x}$ with eigenvalue x at time t, if at time t', the system is in an eigenvector of $\hat{x}$ with eigenvalue x'. All of the physics of nonrelativistic quantum mechanics is encoded in this amplitude. Here, we're using the Heisenberg notation. In Schrodinger picture notation it has the form

$$\left\langle x \middle| e^{-iH(t-t')} \middle| x' \right\rangle.$$

An alternative way to compute this object, though, is to think about the physical meaning of going from (x, t) to (x', t'). Classically, we have trajectories from one point to the other using Newton's equations, and quantum mechanically, we use Schrodinger's equation. But the path integral formulation is to write this probability amplitude as

$$\sum_{\text{path } (x',t') \to (x,t)} e^{\frac{i}{\hbar} S},$$

where $S = \int dt \left(\frac{1}{2} m\dot{x}^2 - V(x)\right)$ is the standard nonrelativistic action. In other words, we sum over all possible paths between the initial and final points with a weight given in terms of the action, and this can be interpreted as "adding quantum fluctuations to the classical path."

In quantum mechanics, this is just an alternative way to do things – the Schrodinger equation is often easier for doing computations than doing this functional integral. But in some quantum field theory problems or string theory, the path integral becomes crucial.

Before going to string theory, Let us first turn to the **relativistic** particle case. We still sum over all possible paths to get the amplitude of getting from our initial to our final point:

$$\sum_{\text{path}} e^{\frac{i}{\hbar} S_{\text{particle}}},$$

where S_{particle} is the relativistic particle action that we've derived earlier on in the semester. But all of this is for a free particle – once we introduce interactions into the picture, we need to allow particles to split apart and join together along their path, and we still sum over all possible paths: not splitting gives us the **tree-level** diagram, splitting once and joining back gives us the **1-loop**, and so on. (The actual calculations are very technical, but the main point is still to "sum over all paths.") The resulting theory can be viewed as an alternative formulation (so-called first-quantized formulation) of quantum field theory. Note that different choices of how a particle can split into multiple particles ("interaction vertices") give us different theories.

We now turn to string theory. The difference is that instead of having a particle traveling in spacetime and summing over trajectories of that particle, we must sum over worldsheets with specified initial and final conditions.

Example 174

Suppose we start with a closed string $t = t_0$ in the graviton state and with some given momentum p. We may want to know about the probability that we're still in the graviton state at time $t = t_1$ with some other momentum p'.

And to find this amplitude, we sum over all possible possibilities of evolution: the string may stay closed and not split or join (tree-level), or it may split into two strings, creating a hole in the middle of our worldsheet (1-loop). Next time, we'll understand how all of this relates back to the self-consistency check needed for our superstring theory!

May 10, 2021

Last time, we briefly introduced the main elements of the path integral formulation. Basically, in nonrelativistic quantum mechanics, particles cannot be created or destroyed. A fundamental equaiton is

$$\langle x, t | x', t' \rangle = \sum_{\text{paths}} e^{\frac{i}{\hbar} S},$$

where we sum over all paths from (t', x') to (t, x), and S is the standard action $\int_{t'}^{t} dt \left(\frac{1}{2} m\dot{x}^2 - V(x)\right)$. (If we have multiple particles, we just increase the dimension of x, so interactions can all be encoded into this action S.) In relativistic systems, the propagation of a free particle can be described as

$$\sum_{\text{paths}} e^{\frac{i}{\hbar} S_{\text{particle}}},$$

where we now sum over the action for a free relativistic particle which we discussed earlier in the class. To describe interactions among particles, we need to include **interaction vertices** as well (where particles can split into two or particles, or two or more particles can join together) – different sets of allowed interaction vertices (for example, choosing how many particles a particle may split into) will give rise to different theories, since we have a different set of trajectories to "sum over." Quantum relativistic systems are described by quantum field theory. This "sum-over-trajectories" formulation is often referred to as the "first-quantized" approach quantum field theory. This is in contrast to the second quantized formulation where we quantize a classical field theory.

Fact 175

The simplest observable of a system is **vacuum energy**. In relativistic systems, virtual particles can be created from the vacuum and then annihilate. To find the vacuum energy, we need to sum over all such processes, which corresponds to sum over all **vacuum diagrams**, i.e. closed trajectories without any external legs.

The simplest vacuum diagram is a single closed loop, which can be interpreted by viewing its evolution from "bottom to top" as time-evolution: basically, we nucleate two particles out of the vacuum at some time, they propagate, and eventually they annihilate at some later time. And more generally, diagrams may have additional "legs" coming out of those vacuum diagrams, corresponding to additional splittings or nucleations.

So connecting this all back to string theory, the ideas will be similar: instead of summing over all trajectories for a particle, we sum over all possible worldsheets (which are trajectories for strings). More explicitly, for the quantum dynamics of a string, we sum over all two-dimensional surfaces with appropriate initial and final conditions, weighted by a factor of $e^{\frac{i}{\hbar} S_{\text{string}}}$ for the usual string action S_{string}. (So we'll want to sum over all possible $\gamma_{\alpha\beta}$s and X^μs that are compatible with those conditions.)

We've mentioned previously that closed strings are spacetime excitations, so strings can be nucleated out of vacuum in the same way as we described for particles earlier. So string diagrams for **vacuum energy** should correspond to closed two-dimensional surfaces without external legs.

Example 176

The simplest vacuum diagram for a string can be represented as a **sphere**: again, letting time evolve from bottom to top, the different "time slices" tell us that a small closed string is nucleated out of vacuum, that string expands and then contracts, and then it disappears back into the vacuum at some later time.

Example 177

If we next imagine a vacuum diagram represented as a torus (standing up), then we can interpret the physical dynamics as creating a string, splitting that string into two (creating a hole), bringing those strings again, and then having everything annihilate.

Example 178

String propagation can be similarly represented: if we want a closed string at the initial and final times, then evolution can be represented as a cylinder, or a cylinder with a hole, and so on. And scattering processes also correspond to splitting and joining of strings in a similar manner.

Basically, the methods for actually "summing over these diagrams" is very intricate, but the main concept remains the same as for particles.

Remark 179. *Each diagram (sphere, torus, and so on) that we've mentioned should be thought of as "summing over all possible continuous deformations of that corresponding surface." So each diagram basically tells us about how the strings are qualitatively created, joined, split, and destroyed, but those diagrams can be arbitrarily stretched (and different stretchings, corresponding to different $\gamma_{\alpha\beta}$s and X^μs, will have different actions S).*

We haven't talked about the details, but topologically, stretching a two-dimensional surface (for the string) is very different from stretching a one-dimensional line (for the particle). So string theory will give us many important new features. For example, the splitting of a single particle into two particles will give us a Y shape, and the splitting vertex is a sharp spacetime event. When we sum over splittings and joinings in quantum field theory, we find divergences, coming from collisions of interaction vertices. Since each vertex corresponds to a sharp spacetime event, so are collisions of vertices. But when a string splits into two strings, there is no sharp spacetime event corresponding to the splitting. This can be seen by viewing the corresponding worldsheet in different Lorentz frames: in different frames the string splits at **different spacetime points** (which we drew in lecture). Since different frames must be physically equivalent, there is no way to associate a sharp spacetime event with the splitting. Thus in string theory collisions of string vertices are must **softer** than those in a theory of particles. As a result, "string theory is finite," and we won't run into the same divergences in calculations!

Fact 180

Another important aspect is that in a particle theory, we need to specify our interaction vertices, and thus there are an infinite number of quantum field theories (because we can specify an arbitrary set of vertices and they are all inequivalent). But in string theory, **interactions are essentially unique**, because all two-dimensional surfaces can be built from joining and splitting by stretching the surface.

For example, if a closed string splits into three strings, we can stretch it so that the initial string splits into two strings, and then one of those products splits further into two strings. So "interactions are unique" in string theory.

Fact 181

In a theory of particles, for a given process, there can be many different channels contributing to the process, each corresponding to a distinct diagram. All the channels should be summed over. For example, $2 \to 2$ scatterings, there are so-called s,t,u channels (which we drew in lecture). In string theory all different channels are included in the same diagram (again we illustrated in lecture using diagrams).

Fact 182

In string theory, diagrams with different string loops have different topologies, i.e. they cannot be deformed to one another with continuous stretchings. Vacuum diagrams of sphere and torus topologies correspond respectively to 0 and 1 loop.

Fact 183

A theorem of topology is that orientable two-dimensional closed surfaces are classified by an integer h, known as the **genus** of the surface (which is essentially the number of handles).

So vacuum diagrams can be broken up into those of genus 0, 1, 2, and so on, corresponding to $0, 1, 2, \cdots$ loops. Diagrams for other processes can be obtained by attaching "external legs" to the vacuum diagrams, and we can still make these kinds of classifications: **string loops correspond to different topologies**, and each loop only corresponds to a single diagram.

Remark 184. *As was mentioned in a previous lecture, we can also have nonorientable string theories, leading to nonorientable surfaces – we can take a look in Professor Zweibach's textbook for more details there.*

We have also previously mentioned the **string coupling constant** g_s which gives the amplitudes for a string splitting into two strings (or vice versa). We normally assume g_s is small, so that physical quantities can be expanded in it. Notice that adding a loop (i.e. increasing the genus by 1) adds 1 splitting and 1 joining to our string dynamics, so we get an additional factor of g_s^2 for each additional loop in our diagram. So a genus of h gives us a factor of g_s^{2h} compared to the tree-level string diagram. By counting the number of splittings and joinings, we can also readily find that for a process of n ($n \geq 3$) external strings, the tree-level diagram has a factor g_s^{n-2}. Therefore, the amplitude for a process containing n external strings can be expanded in g_s as

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+2h} A_n^{(h)},$$

where $A_n^{(h)}$ is the contributions coming from surfaces of genus h and with n external legs. And this formula turns out to also be valid for $n = 0, 1, 2$ as well. This can derived, for example, for $n = 2$ as follows. Consider the $n = 2$, $h = 1$ diagram, which is a cylinder with a handle. The same diagram can also be thought of as sewing together a joining and a splitting process (see lecture video for a diagram), which gives $g_s \cdot g_s = g_s^2$. (So the cylinder with no hole is then $\frac{g_s^2}{g_s^2} = g_s^0$.) We thus see the formula indeed applies to $n = 2$. Similarly we can demonstrate its correctness for $n = 0, 1$ as well. Note that the vacuum sphere diagram is proportional to g_s^{-2}.

For each two-dimensional surface with n legs and genus h, we can associate a topological invariant known as the **Euler number**

$$\chi_{n,h} = 2 - 2h - n\ .$$

Thus the amplitude can also be written as $A_n = \sum_{h=0}^{\infty} g_s^{-\chi_{n,h}} A_n^{(h)}$.

Parallel discussion can be applied to open strings: associating an open string coupling constant g_o with the joining and splitting processes for open strings, we will also similarly find

$$A_n^{(\text{open})} = \sum_{h=0}^{\infty} g_o^{n-2+2h} A_n^{(h),\text{open}}$$

where h is now the number of holes in an open string diagram.

Now we describe a very simple, but profound observation. Consider a vacuum one-loop diagram for open string, as given by the LHS of the following diagram:

On the right hand side we have relabelled τ and σ by exchanging them. The resulting diagram can then be interpreted as the tree-level propagation of a closed string! Since physics is independent of parameterizations of the worldsheet, we thus conclude a 1-loop open string vacuum process can be interpreted alternatively as the propagation of a closed string. This, in particular, implies that any open string theory must actually include closed strings.

In fact all open string vacuum processes (i.e. for any loop) can be reinterpreted as closed string processes, which is known as the **open-closed duality**. Now recall that open strings give rise to gauge theories (which do not include gravity), and closed strings give rise to gravity, we learn that there is an equivalence in which **gravity is equivalent to a theory without gravity**. This simple geometric picture can be considered as the root of the **holographic duality**, the equivalence between gauge theories and gravity, which we'll explore more in the AdS-CFT correspondence for our final project.

And finally, getting back to our consistency check for superstring theory, we must be able to rewrite our 1-loop open string result in terms of the tree-level propagation in the closed string. This is straightforward for bosonic string theory, but it turns out this is not straightforward when we have worldsheet fermions – it's only satisfied when we impose the $(-1)^F = 1$ condition.

Now on the other hand, if we look at the one-loop of a closed string (which we can describe as a torus), the surface can be thought of as a parallelogram with opposite sides identified, and we can choose many different sets of bases for $\{\tau, \sigma\}$ (and this is known as **modular invariance**). These different ways of building the torus are equivalent, and that equivalence is automatic for bosons but not when including fermions. So that's where the GSO projection, giving us the IIA and IIB theories, comes in.

May 12, 2021

Last time, we discussed interactions of strings: the important point is that for any process of closed strings with n external strings, we can write the amplitude as a sum

$$A_n = \sum_{h=0}^{\infty} g_s^{n-2+h} A_n^{(h)},$$

where the $h = 0$ term corresponds to the tree-level diagram with genus 0, the $h = 1$ term corresponds to the one-loop diagram with genus 1, and so on. This gives us the general structure of string theory: it naturally lends itself as a perturbative expansion in terms of the string coupling onstant g_s. When we have $n = 0$, we get the vacuum process (where strings are created and destroyed into the vacuum), leading us to the vacuum energy. And when we have $n = 2$, we have the propagation of a single string, and more general scattering processes will correspond to different values of n.

Similar statements can be made about open strings as well, whose worldsheets correspond to surfaces with boundaries (from end points of a string). We will have disks with holes instead of spheres with handles, and there is a different coupling constant g_o.

By reinterpreting one-loop open string vacuum diagram as a closed string tree-level diagram we concluded that a theory containing open strings must also contain closed strings. In such a theory, there is a simple relation between g_s and g_o.

Proposition 185
For consistency, we must have $g_s \propto g_o^2$.

We can see this through an example: the simplest vacuum process for the open string is represented with a disk (corresponding to an open string being created and then disappearing), and the corresponding amplitude should be proportional to g_o^{-2} because $n = h = 0$. But we can also view this as a closed-string process, because a disk is a half-sphere. So this disk also represents a closed string nucleating out of vacuum, and that process (having $n = 1, h = 0$) is proportional to g_s^{-1}.

In the case of superstring reinterpreting the one-loop open string vacuum diagram in terms of a closed string process imposes a nontrivial consistent constraint on the theory. It can be shown that imposing the projection $(-1)^F = 1$ satisfies the constraint.

And we started discussing last time, we get the type IIA and type IIB superstrings due to modular invariance:

Example 186
A torus can be defined as the fundamental domain when we identify points in the complex plane which are related by

$$z \sim z + mu + nv \quad \forall m, n \in \mathbb{Z}$$

where $u = 1$, $v = \alpha$ and α is a complex number. (This basically means we take the parallelogram spanned by u and v and identify opposite sides.)

We may view the u and v as representing σ and τ directions respectively. There is in fact a countably infinite set of choices of u, v that give rise to the same torus (for example, we may use $u = 1 + \alpha, v = \alpha$ or $u = \alpha, v = 1$), and these different choices can be thought of as different choices for the directions of σ and τ. We need to make sure the vacuum energies that are obtained from these different choices of u, v are the same if our theory has any chance of being consistent. When we have worldsheet fermions, it turns out that IIA and IIB projections are the only ways to make sure that does occur and there is no tachyon. (If we allow tachyon, there are other possible projections.) (Also it turns out that this consistency on the torus will also guarantee consistency on all other surfaces.)

3.9: T-duality

We mentioned last lecture that stretching two dimensional surfaces is very different from stretching two-dimensional lines, giving us certain features in string theory that aren't there in field theory. It turns out that while spacetime viewed by a particle is described using points, the string perspective of spacetime is very different. We'll describe one instance of that now, and we'll only be talking about it for **closed strings**.

In general, **dualities** are powerful because they are different descriptions of the same physics – for example, the wave-particle duality in quantum mechanics is a useful tool for bringing different concepts together. And the point of **T-duality** is that **different spacetimes can be the same** – that is, strings on some manifold M_1 are the same as strings on some other manifold M_2.

Example 187

The IIA superstring theory on $M^9 \times S^1$ (the product of a nine-dimensional Minkowski space and a circle) is the same as the IIB superstring theory on $M^9 \times \tilde{S}^1$ (where $\tilde{S}^1$ is some other circle).

Basically, the spectra for IIA and IIB look completely different in ten dimensional Minkowski spacetime, but we'll see they become equivalent if we put both of them in spaces containing circles! This equivalence suggests that these theories should be viewed as different regimes of the same theory.

We'll describe another duality like this, which is simpler:

Example 188

The bosonic string on $M^{25} \times S^1$ is equivalent to the bosonic string on $M^{25} \times \tilde{S}^1$, where $S^1, \tilde{S}^1$ are circles of radius R and $\frac{\alpha'}{R}$.

This result is particularly striking when we take $R \to \infty$, because S^1 becomes an infinite line, but $\tilde{S}^1$ becomes very small. So the claim is that (taking $R \to \infty$) the bosonic string theory on M^{26} is equivalent to the theory on M^{25} times a point.

Remark 189. *Having an equivalence between theories means that all physical observables are the same (such as the spectrum and all the scattering amplitudes), but not necessarily the actions are the same.*

We'll work with $M^{25} \times S^1$ by labeling our string embedding functions as $X^\mu = (X^+, X^-, X^I)$, where $X^I = (X^i, X)$ (where i ranges from 2 to 24 and $X = X^{25}$ is a coordinate living on a circle, meaning that $X^{25} \sim X^{25} + 2\pi n R$ for arbitrary integer n). Visually, we can think of this spacetime spatial manifold as a cylinder, where the circle is the X^{25} direction and the other directions are X^1 through X^{24}. And what's important is strings can now wind around the cylinder, and those are topologically different from strings that don't wind around the cylinder! One way to represent this mathematically is to represent the string in a hyperplane, where we identify points whose x^{25} are related by integer multiples of $2\pi R$. Then closed strings can form closed loops, or they can travel 2π units in the x^{25} direction (corresponding to one wrap-around), and so on (see pictures in lecture video).

For the **contractible** string which does not wrap around the cylinder, we know that $X(\sigma + 2\pi, \tau) = X(\sigma, \tau)$ (remember X now denotes the 25th coordinate). But for the string which winds around once, we have $X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi R$, and more generally we can have $X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + 2\pi m R$ for any integer m. (But the 0 through 24 directions must all be periodic no matter what.) This **winding number** m therefore labels different topological sectors for our string when we're living on this space $M^{25} \times S^1$.

The $m = 0$ sector is the same as what we're used to, but we now need to add in the contributions from other winding strings. (Notice that particles cannot wind, so they do not have these additional sectors like we have with strings!) Because oscillatory modes are always periodic, the extra $2\pi m R$ factor will not do anything to them, and for $m \neq 0$ the only noticeable change in the other sectors is that the zero mode

$$X(\tau, \sigma) = x_0(\tau, \sigma) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n}{n} e^{-in(\tau - \sigma)} + \frac{\tilde{\alpha}_n}{n} e^{-in(\tau + \sigma)} \right)$$

will look different: instead of $x_0(\tau, \sigma) = x_0 + \alpha' p \tau$, we now have

$$x_0(\tau, \sigma) = x_0 + \alpha' p \tau + \alpha' w \sigma,$$

where we must have $w = \frac{mR}{\alpha'}$ (for some integer m). This is still linear in τ, σ, so it still satisfies the wave equation that it's supposed to satisfy. Note that x_0 lives on the circle, i.e. we should identify x_0 with $x_0 + 2\pi n R$. In the quantum theory, everything becomes operators, and thus x_0, p, w will all need to be operators (we can check that there's nothing conjugate with w so it commutes with everything, and we have the standard commutation relation $[x_0, p] = i$). We can then solve the Virasoro conditions in the light-cone gauge, repeating our previous procedures, so we can now label our states as

$$\prod_{I,n} (\alpha^I_{-n})^{\lambda_{nI}} (\tilde{\alpha}^I_{-n})^{\tilde{\lambda}_{nI}} \left| p^+, p^i, p, w \right\rangle,$$

where the eigenvalue corresponding to w is just the $\frac{mR}{\alpha'}$ that we found above, and p is the momentum conjugate to x_0. Eigenstates $|p\rangle$ are then plane waves of the form

$$|p\rangle \sim e^{ipx_0} \implies p = \frac{n}{R}$$

by required periodicity of x_0. By working out the Virasoro constraints for the current case explicitly we find that the mass-shell condition and level matching condition are modified as follows: This can then be written as

$$M_{25}^2 = 2p^+p^- - (p^i)^2 = p^2 + w^2 + \frac{2}{\alpha'}(N_\perp + \tilde{N}_\perp + a + \tilde{a}),$$

and

$$N_\perp - \tilde{N}_\perp = \alpha' p w.$$

Note that M_{25}^2 is the mass square viewed as a particle in 25 dimensional Minkowski spacetime. Note that anything related to the oscillatory modes are the same as before because we haven't changed any oscillatory modes. We notice that p and w appear symmetrically in M_{25}^2 and the level-matching condition.

Recalling that $p = \frac{n}{R}$ and $w = \frac{mR}{\alpha'}$, where m, n are arbitrary integers, substituting in yields

$$\boxed{M_{25}^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'} + \frac{2}{\alpha'}\left(N_\perp + \tilde{N}_\perp - 2\right), \quad mn = N_\perp - \tilde{N}_\perp}.$$

Remark 190. *If we now have a state with $n = 0$ but a nonzero winding number m, then $N_\perp = \tilde{N}_\perp = 0$ and we get a tachyon mass $M^2 = \frac{m^2R^2}{\alpha'^2} - \frac{4}{\alpha'}$. (So there can be multiple different scalar particles for various values of m!) Notice that*

$$\frac{|m|R}{\alpha'} = \frac{1}{2\pi\alpha'}|m| \cdot 2\pi R = T_0|m| \cdot 2\pi R,$$

which is $T_0 L$, the string tension times the length of the winding string. That means this first term for M_{25}^2 is given by $(T_0L)^2$. Also note that when m is such that $\frac{m^2R^2}{\alpha'^2} > \frac{4}{\alpha'}$, we will no longer have tachyons.

Remark 191. *When we have $N_\perp = \tilde{N}_\perp = 0$, level-matching tells us that we must have $mn = 0$, meaning that we cannot have both $n \neq 0$ and $m \neq 0$ at the same time. In other words, a string cannot oscillate around the x^{25} direction and also move uniformly in that direction, and this is a point in support of the fact that* **string motion must be transverse**.

And now we can get to T-duality: the boxed equations above remain the same if we let S^1 have a radius $\tilde{R} = \frac{\alpha'}{R}$. Indeed,

$$M_{25}^2 = \frac{n^2}{R^2} + \frac{m^2R^2}{\alpha'^2} \to \frac{n^2R^2}{\alpha'^2} + \frac{m^2}{R^2},$$

and the only difference is that m, n's roles have switched! So indeed, the string spectrum on $M^{25} \times S^1$ with radius R is the same as the string spectrum on $M^{25} \times \tilde{S}^1$ with radius $\tilde{R} = \frac{\alpha'}{R}$, where the momentum number becomes the winding number and vice versa. It can be shown that interactions among string excitations also look the same, so these two descriptions are really the same way to describe a string.

Finally, taking $R \to \infty$ gives us M^{26} on the left-hand side (because a circle becomes an infinite line), with momentum $p = \frac{n}{R}$ becoming smaller (lighter), while winding modes become infinitely heavy. Thus the winding modes decouple and all that's left is momentum modes. And $\tilde{R}$ goes to 0 on the right-hand side, so our circle has shrunk to a point, our momentum modes will decouple because $p \to \infty$, and our winding modes become light and form a continuum. Our conventional point of view is more used to the former concept, but the point is that both viewpoints are equally good!

May 17, 2021

Last lecture, we discussed T-duality, which showed us that string theory looks the same physically on two different spacetimes. We discussed an explicit example of this for the bosonic string, seeing the equivalence on the spacetimes $M^{25} \times S^1$ and $M^{25} \times \tilde{S}^1$, where the two circles have radii R and $\frac{\alpha'}{R}$. (Essentially, the equivalence comes from momentum and winding in the two spacetimes being mapped to each other, since S^1 is a fundamental domain of the real line under the equivalence $x \sim x + 2\pi R$.)

It turns out that this kind of phenomenon is very general – for example, it turns out that a type IIA superstring theory on $M^9 \times S^1$ is equivalent to a type IIB superstring theory on $M^9 \times \tilde{S}^1$, where the radii are again related by $R \mapsto \frac{\alpha'}{R}$ and we again map momentum and winding to each other. And this is interesting, because in ten-dimensional Minkowski spacetime, type IIA and type IIB look very different! And if we take $R \to \infty$, this tells us that IIA on M^{10} is equivalent to IIB on M^9 times a point, and IIA on M^9 times a point is equivalent to IIB on M^{10}. In other words, the difference is not in the two theories but in the two spacetime manifolds.

Example 192

Closely related to T-duality is the concept of **S-duality**, in which we take the string coupling constant g_s (usually assumed to be small so that scattering amplitudes can be expanded in powers of g_s) to infinity.

It's difficult to work with this rigorously, but it turns out that the limit as $g_s \to \infty$ and the limit as $g_s \to 0$ look the same for the IIB theory: in other words, we get an equivalence when we take $\tilde{g}_s = \frac{1}{g_s}$. And similarly, if we take $g_s \to \infty$ in a type IIA superstring theory (which is ten-dimensional), we get an eleven-dimensional theory called **M-theory**. M-theory is not a string theory: instead, it turns out that M-theory on $M^{10} \times S^1$ (with circle of radius R) corresponds to the IIA superstring theory in M^{10} with coupling constant $g_s = \frac{R}{\ell_s}$ (where $\alpha = \ell_s^2$).

To summarize, we can draw connections between these different kinds of theories just by looking at different spacetime manifolds, and it turns out that all of these different kinds of dualities also get us relations between type I and the heterotic string theories. So we've arrived at the promised equivalence between all five superstring theories and M-theory – they're all connected, and this suggests that there may only be a single quantum gravity theory. But we don't really know much about the general structure yet – each of the six theories mentioned above are really only a corner of the whole picture.

3.10: Strings on an orbifold

Einstein's classical theory of gravity explains a lot of physical phenomena in the world very well, but we know that general relativity should break down because there are solutions with certain singularities (for example with black holes). So there must be something beyond our current understanding of general relativity and quantum field theory.

String theory can help us understand these singularities to some degree, but there are many different kinds of singularities and they can be complicated to deal with. So today, we'll just deal with the simplest case of such a singularity and see that string theory will make sense.

Definition 193
An **orbifold** is a space obtained by identifications with fixed points.

Example 194
If we start with the real line $\mathbb{R}$ and quotient out by $\mathbb{Z}_2$ by identifying $x \sim -x$, the resulting space (a half-infinite line $\mathbb{R}_{\geq 0}$) is an orbifold because we have the fixed point $x = 0$.

Example 195
Similarly, if we define a space $\mathbb{R}^2/\mathbb{Z}_2$ by identifying by reflection $((x, y) \sim (-x, -y))$, we again have a fixed point $(0, 0)$. This can be thought of as the upper half-plane, but again identifying the left and right halves of the boundary together, so this gives us a cone and a conical singularity (infinite curvature) at the fixed point.

But it turns out that physical quantities will still be finite for strings even on orbifolds – from the perspective of the strings, these fixed points will basically look like normal points. We'll see an example of this now:

Example 196
Consider a bosonic string on $M^{25} \times \mathbb{R}/\mathbb{Z}_2$ – we'll focus on the closed string case here. The identification $x^{25} \sim -x^{25}$ on the coordinates means that we now have a boundary at $x^{25} = 0$.

We'll consider string theory on this space – since there is a fixed hyperplane, we would normally get singularities in quantum field theory. But let's work in the light-cone gauge and use coordinates (X^+, X^-, X^i, X) (where i ranges from 2 to 24): we want to do our usual Fourier mode expansion and find the spectrum, and we'll do so by introducing a **parity operator** U on the worldsheet. This U is the unitary operator satisfying

$$UX^{\pm}U^{\dagger} = X^{\pm}, \quad UX^iU^{\dagger} = X^i, UXU^{\dagger} = -X.$$

(Such a U does exist, because X is a symmetry of the string action. But note that this is a discrete symmetry, so it doesn't have all of the properties like the continuous symmetries in Noether's theorem.) Looking at our string solution in the 25th dimension

$$X = x_0(\tau) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \left(\frac{\alpha_n}{n} e^{-in(\tau - \sigma)} + \frac{\tilde{\alpha}_n}{n} e^{-in(\tau+\sigma)} \right),$$

where $x_0(\tau) = x_0 + p\tau$ is our usual center-of-mass motion, we find that we must have $Ux_0U^{\dagger} = -x_0$, $Upu^{\dagger} = -p, U\alpha_nU^{\dagger} = -\alpha_n, U\tilde{\alpha}_nU^{\dagger} = -\tilde{\alpha}_n$ if we want the entire solution for X to always gain this negative sign. And if we want to define a theory on the orbifold that's consistent with everything we've said so far, our **physical states must be invariant** under U as well.

So now suppose we are looking at states with $\tilde{N}_{\perp} = N_{\perp} = 0$, so that our states are labeled as $|p^+, p^i, p\rangle$. Because our states are eigenstates of p, we must have

$$U|p^+, p^i, p\rangle = |p^+, p^i, -p\rangle$$

(we can check this by thinking about how states interact with the p operator). So we can get a state that's invariant under U via

$$\left|p^+, p^i, p\right\rangle + \left|p^+, p^i, -p\right\rangle,$$

and this is a physical state of the orbifold theory (though it does not have a definite momentum). In other words, if $x^{25} = 0$ is the hyperplane at the edge of our universe, then any wave coming into that edge is also reflected back into the spacetime.

We can next look at states with $N_\perp = \tilde{N}_\perp = 1$: such states can look like $\alpha^i_{-1}\tilde{\alpha}^j_{-1}(\left|p^+, p^i, p\right\rangle + \left|p^+, p^i, -p\right\rangle)$ when the i, j indices in the raising operators are not in the 25th dimension, or they can look like $\alpha^i_{-1}\tilde{\alpha}_{-1}(\left|p^+, p^i, p\right\rangle - \left|p^+, p^i, -p\right\rangle)$ because $\tilde{\alpha}_{-1}$ is now also odd under the action of U, and so on.

Remark 197. *Notice that momentum in the* 25*th direction is no longer conserved (and thus is not a good quantum number), because we don't have translation symmetry anymore!*

It may seem like we're now done, but it turns out that doing this process **does not give us a consistent theory**. In particular, the modular invariance that we described last time with the torus no longer holds because we've thrown away some of our states (because we originally wanted to calculate the vacuum energy, which sums up contributions across the spectrum). The issue comes in that we assumed the periodicity $X(\sigma + 2\pi, \tau) = X(\sigma, \tau)$, so that our strings start and end at the same point, but now that we've identified x^{25} with $-x^{25}$, we can now create strings which start at some x^{25} coordinate and end up at the negative of that coordinate – that will still be a closed string!

Therefore, our theory will now have two sectors: the one in which we have $X(\sigma + 2\pi, \tau) = -X(\sigma, \tau)$ is called the **twisted sector**. If we add the Hilbert spaces of the regular and twisted sectors together, and we project into the space that is invariant under U, then we indeed get a consistent theory. And in the twisted sector, we have solutions of the form

$$X(\tau, \sigma) = \sum_{n=0}^{\infty} \left(x_n(\tau) e^{i\left(n+\frac{1}{2}\right)\sigma} + \tilde{x}_n(\tau) e^{-i\left(n+\frac{1}{2}\right)\sigma} \right)$$

(we have half-integers here so that we get antiperiodicity with period 2π). In particular, there will be no zero modes – indeed, if one endpoint of our twisted string has positive x^{25} and the other has negative x^{25}, we can't have any translational center-of-mass motion in the x^{25} direction without breaking $X(\sigma + 2\pi) = -X(\sigma)$. Writing in a more familiar form, we have

$$X(\sigma, \tau) = i\sqrt{\frac{\alpha'}{2}} \sum_{\text{half-integers } m} \frac{1}{m} \left(\alpha_m e^{-im(\tau-\sigma)} + \tilde{\alpha}_n e^{-im(\tau+\sigma)} \right),$$

with everything else the same as before. So indeed, strings do not end up with the singularity issues that general relativity or quantum field theory have: instead, we get some extra degrees of freedom (which are localized near the singularity), but our theory remains well-defined. One way to think about this is that particles do not see the necessary additional degrees of freedom, but strings are able to incorporate them naturally!

May 19, 2021

Last time, we discussed singularities coming from an orbifold, such as $\mathbb{R}$ or $\mathbb{R}^2$ under identification of diametrically opposite points $x \sim -x$. We mentioned that general relativity and quantum field theory become singular at such points, but string theory is able to make sense of the physics on such spacetimes – we find a **regular sector** and a **twisted sector** for closed strings, where the idea is that a string can start at a and end at $-a$ and still be "closed."

3.11: Holographic (AdS/CFT) duality

In this last lecture, we'll discuss an important equivalence and motivate some of the main points. The central idea is that we have an equivalence between a (quantum) non-gravitational theory and a (quantum) gravity theory, where the gravity theory lives in **anti-de Sitter spacetime** (AdS) with a negative cosmological constant (so not quite the same as our world) and the non-gravitational theory is a **conformal field theory** (CFT).

We'll start with a toy example of this non-gravitational theory:

Example 198

Consider the multivariable integral

$$Z = \int \prod_{i,j} dM_{ij} e^{-\frac{1}{2}\mathrm{Tr}(M^2) - g\mathrm{Tr}(M^4)},$$

where M_{ij} are the entries of an $M \times M$ Hermitian matrix.

Essentially, we integrate over all matrix elements, and we get a Gaussian-like factor with an additional quartic term in the exponent. This can be written more explicitly as

$$= \int \prod_{i,j} dM_{ij} \exp\left(-\frac{1}{2}M_{ij}M_{ji} - gM_{ij}M_{jk}M_{k\ell}M_{\ell i}\right).$$

The surprising result is that this integral, which is just a function of g and N, turns out to be secretly equivalent to a one-dimensional string theory. (Though this is a toy example, we will see that our string theories all tend to involve matrix degrees of freedom.) To see this equivalence, notice that we don't know how to do integrals beyond Gaussian integrals, so we'll do this by expansion in a power series:

$$Z = \prod_{i,j} dM_{ij} e^{-\frac{1}{2}\mathrm{Tr}(M^2)} \sum_{n=0}^{\infty} \frac{(-g)^n}{n!} (\mathrm{Tr}(M^4))^n.$$

In other words, if we can sum over all of these terms, then we can compute the desired integral.

Example 199

To see a simpler version, we can do the one-dimensional real analog of it:

$$\int dx e^{-\frac{1}{2}x^2} \sum_{n=0}^{\infty} a_n x^n.$$

We can compute each term here, because we know how to integrate powers of x against a Gaussian.

But in this case, we're going to use a **diagram technique** to compute the integral. All odd terms are zero, the zeroth order term is

$$S_0 = \int dx e^{-\frac{1}{2}x^2} = \sqrt{2\pi},$$

and then the second order term is

$$S_2 = \frac{1}{S_0} \int dx e^{-\frac{1}{2}x^2} x^2 = 1.$$

More generally, we have

$$S_n = \frac{1}{S_0} \int dx e^{-\frac{1}{2}x^2} x^n = \frac{\binom{n}{2}\binom{n-2}{2}\cdots\binom{2}{2}}{(n/2)!}.$$

We can understand this integral by thinking of x^n as "$n/2$ groups of x^2s:" essentially, by **Wick's theorem** or other methods, we can imagine that we have n vertices, and we need to contract them into 2 pairs. Then there are $\binom{n}{2}$

ways to pick the first pair, $\binom{n-2}{2}$ ways to pick the next, and so on, but we overcount our total pairings by a factor of $(n/2)!$ because the order of pairing doesn't matter. This gives us the expression above.

We can now return to our matrix integral, which looks like

$$\int dM_{ij} e^{-\frac{1}{2}\mathrm{Tr}(M^2)} \left(1 - g\mathrm{Tr}(M^4) + \frac{g^2}{2}\mathrm{Tr}(M^4)\mathrm{Tr}(M^4) + \cdots\right).$$

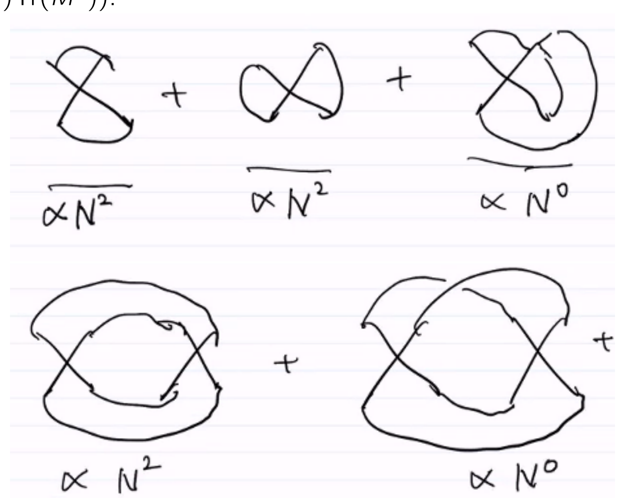
Computing the trace of M^4 can then be thought of as doing the integral over all possible ways to contract our four indices, and then higher order terms correspond to needing to contract multiple times. But different contractions will give us different answers here, so we need to more explicitly sum over all of the diagrams, and computing that series is very difficult.

Fact 200

We can't just keep only the first few terms when g is not small, and this is actually the issue that we run into with strongly interacting theories in general.

So what we'll do instead is to use another approximation which will work better – we'll expand in $\frac{1}{N}$, where N is very large. This turns out to capture the key physics for many situations, and we're now going to heuristically sketch out where the equivalence to a string theory comes in.

The way we'll take this limit is by keeping $\lambda = gN$ fixed and sending $N \to \infty$ (it turns out that we get lots of mathematical structure in this regime). If we do this, we can look at each contraction diagram and find those which are lowest order in $\frac{1}{N}$, and here are some examples (the top row consists of diagrams for $\mathrm{Tr}(M^4)$, and the bottom row consists of diagrams for $\mathrm{Tr}(M^4)\mathrm{Tr}(M^4)$):



Basically, we can draw the diagrams proportional to N^2 without crossing lines (outside of the X's), meaning that they are **planar diagrams**, but we cannot do so for the diagrams proportional to N^0. More generally, it turns out that all planar diagrams (regardless of order) give contributions proportional to N^2, and they are the leading contributions. Then there are some diagrams which are not planar but can be drawn on a torus, and those give contributions proportional to N^0.

Fact 201

More generally, if we can draw the diagram on a two-dimensional surface of genus h, meaning that there are h holes (and h is minimal), then the contribution to the integral looks like N^{2-2h}. (Notice that $\chi = 2 - 2h$ is the Euler characteristic of the surface.)

Remark 202. *We can get more complicated surfaces by identifying sides of polygons, much like a torus can be gotten from identifying opposite sides of a square.*

The key point of proving this result is by thinking about the N-dependence of this diagram and calculating that it is N^{F+V-E}, where F, V, E are the number of faces, vertices, and edges, respectively. And it is a geometrical fact (Gauss-Bonnet) that $F+V-E$ is exactly the Euler characteristic of the surface.

But returning now to the integral we wanted to compute, we find that we can write the integral in the form

$$\log Z = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) = N^2 f_0(\lambda) + f_1(\lambda) + \frac{1}{N^2} f_2(\lambda) + \cdots .$$

This quantity turns out to be a "vacuum energy," and we've computed it by doing a topological expansion in terms of diagrams of topologies. But that should sound familiar: we calculate the vacuum energy in spacetime for a string theory by summing over two-dimensional surfaces, organized in terms of topologies! So there is a parallel we can draw here: the vacuum energy in string theory looks like

$$\mathcal{A}_0 = \sum_{n=0}^{\infty} g_s^{2h-2} A_0^{(h)},$$

and the vacuum energy in this "matrix theory" looks like

$$\log Z = \sum_{n=0}^{\infty} N^{2-2h} f_h,$$

where $A_0^{(h)}$ and f_h both come from summing over diagrams of genus h. In particular, the topology of the string worldsheet maps to the topology of our diagrams, and we map g_s to $\frac{1}{N}$.

This may seem like a mathematical coincidence, but there is some deep physics behind this: both setups are ways to describe the same system. And because this matrix theory does not have gravity, we've found a "baby" version of the correspondence we wanted to explore.

To understand more about why these are the same theory mathematically, remember that $A_0^{(h)}$ is obtained by summing $\sum e^{\frac{i}{\hbar} S_{\text{string}}}$ over all two-dimensional surfaces of genus h. We can now **discretize our surfaces** by using triangulation (tiling using polygons) and corresponding each polygon to a point, so that this sum is thought of as a sum over triangulations of genus-h surfaces. And in the corresponding version in the matrix theory side, we obtain f_h by summing integral contributions over diagrams of genus h. But as we draw our diagrams, we are in fact tiling our genus-h surface with polygons (the aforementioned "faces")! So this diagram can be considered as a triangulation of that surface, and thus in both cases we're summing over triangulations and get equivalent theories.

Therefore, hidden behind each theory of matrices is a quantum gravitational string theory, including the non-abelian Yang-Mills theory that we've previously discussed, and relating N to a string coupling constant g_s means that there are certain regimes in which we can do perturbation theory. For more complicated theories, the key for finding explicit examples turns out to be to look at the D-branes.

Example 203

Consider a IIB superstring theory, and suppose we have N D3-branes in the spacetime, so that open string endpoints live in $(3+1)$-dimensional Minkowski spacetime.

Each open string can be written as an $N \times N$ matrix (because they can start and end on any of the branes). The massless excitations that we get here are known as **non-abelian gauge bosons** (which are matrix generalizations of photons). But let's now switch to an alternative description of the D-branes: because D-branes have tension, they have mass, and thus they carry some charge and energy, meaning that their presence bends spacetime. So we can equivalently describe the D-branes by using the curved spacetime that they generate (instead of thinking of Minkowski spacetime).

Fact 204

We can then associate a radial direction perpendicular to the branes – if r is very large, we get ten-dimensional Minkowski space, but if r is small and we're close to the brane, we get the space $\text{AdS}_5 \times S_5$.

In this new description, we only have closed strings in a particular spacetime geometry, and those closed strings can interact with the open strings on the D-brane by equivalently interacting with the spacetime geometry! Unfortunately, both descriptions here are very complicated, so the way we can extract useful information out of it is by looking at the **low-energy limit** $E^2\alpha' \ll 1$. In that limit, the full string-theory description gives us the $N = 4$ super-Yang-Mills theory, and the curved spacetime description gives us a decoupling of the Minkowski spacetime from the $\text{AdS}_5 \times S_5$ spacetime: thus, we get string theory just in $\text{AdS}_5 \times S_5$, and that's the equivalence that we're after.