Inversion

National Team Training 2017-18

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1 Introduction

This section is dedicated solely to the various properties that you may encounter while doing inversion. The various paradigms in which inversion is applied is covered later.

1.1 The inversive plane

Basically, Euclidean plane. But add in P_∞ ("point at infinity"), such that every line also passes through P_∞, and if a circle passes through P_∞ it has to be a line. This is mainly to prevent awkwardness when we think about what happens when we invert the center.

Notice that this is different from the *projective plane*, which requires a point at infinity for "each direction".

1.2 Inversion

When we say we invert a diagram, we actually mean we invert a diagram [with respect to some circle ω, centered at O and with radius r]. What this means is that we think about this function f, which maps the inversive plane to itself according to the following:

- $f(O) = P_\infty$
- $f(P_\infty) = 0$
- For any point $P \neq O$, let P' be on ray $\overrightarrow{OP}$ such that $OP \cdot OP' = r^2$. Then $f(P) = P'$.

Then we apply f to the whole diagram and see what we get. Simple, right?

1.3 Properties

For any point X, et X' denote X after inversion about a circle centered at O with radius r.

1. (Basics)
   (a) For any point A, $A'' = A$
   (b) For any points A, B which are not O, $\angle OAB = \angle OB'A'$
   (c) If l is a line passing through O, then its image is precisely itself.
   (d) If l is a line not passing through O, then its image is a circle passing through O (denoted as l'). Conversely, the image of a circle passing through O is a line.
   (e) Let l be a line, and OX be the diameter of l'. Then $OX \perp l$.
   (f) If w is a circle not passing through O, then its image is a circle (not passing through O as well).

2. (Conformity)

   Define the angle between two clines (short for circles/lines) to be the acute or right angle formed by the tangents to each cline one of the intersection points.

   (a) Let w_1, w_2 be tangent clines. Then w_1', w_2' are tangent as well.

   (b) Let k be a line passing through O, and l be another line. Then the angle between k and l is equal to the angle between k and l.

   (c) Show that the above is true if l is a cline instead.

   (d) Let w_1, w_2 be clines. Show that the angle between w_1 and w_2 is preserved under inversion.

   (e) If the angle between clines w_1, w_2 is a right angle then we say w_1 and w_2 are *orthogonal.* In particular, orthogonal pairs of circles are preserved under inversion.

3. (Metrics)

   (a) $A'B' = \frac{r^2}{OA \cdot OB} AB$ and $AB = \frac{r^2}{OA' \cdot OB'} A'B'$.

   (b) The cross-ratio of any four points (not necessarily collinear) is preserved under inversion. That is, for any distinct points A, B, C, D:

   $$\frac{AB}{BC} \bigg/ \frac{AD}{DC} = \frac{A'B'}{B'C'} \bigg/ \frac{A'D'}{D'C'}$$

   This generalizes the fact the cross-ratio is preserved by projecting from a circle to itself through any point.

   (c) Harmonic quadrilaterals stay has harmonic quadrilaterals.

2 How to apply it

There are two main schools of thought on how inversion is usually applied:

2.1 Classic

Inversion takes a complicated diagram and makes it simple.

For this method, we pick some circle to invert about, and complete redraw the diagram by inverting each geometric feature of the problem (points, lines, circles etc.) until we essentially have new but simpler problem. In this case, we do not care too much about the exact radius of the circle ("invert about O", no radius specified), since the result is the same up to scaling.

This method is feasible because by choosing the appropriate point (to be the center of the circle), we can make the following happen:

1. Make circles disappear. Remember, circles passing through O become lines!
2. Turn tangent circles into parallel lines (by picking O to be the point of tangency).
3. Simplify awkward length/angle conditions.
4. Induce symmetry in an otherwise unsymmetric diagram.

These rules also work conversely: if there are too many lines in the original diagram that do not pass through the point we are inverting about, then we will create a lot of unnecessary circles. For instance, the following problem is definitely not feasible for inverting:

Example. In triangle ABC, AD, BE, CF are three concurrent Cevians. Let $X = EF \cap BC, Y = DF \cap AC, Z = DE \cap AB$. Prove that X, Y, Z are collinear

These types of problems are usually more suited for other methods (projective/synthetic).

Here's an example of inversion in practice:

Example. Circles k_1, k_2, k_3, k_4 are such that k_2, k_4 are each tangent to k_1, k_3. Show that the tangency points are either collinear or concyclic.

Sketch. Invert about the tangency points of k_1, k_2. k_1', k_2' are now parallel lines. The condition we wish to prove becomes showing that the 3 remaining tangency points are collinear. This is obvious by considering the (negative) homothety mapping k_3' to k_4' through their tangency points, which must map k_1' and k_2'. □

Problems

1. (Ptolemy) Show that for any four points A, B, C, D,

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD$$

2. (ISL 2003) Let Γ_1, Γ_2, Γ_3, Γ_4 be distinct circles such that Γ_1, Γ_3 are externally tangent at P, and Γ_2, Γ_4 are externally tangent at the same point P. Suppose that Γ_1 and Γ_2; Γ_2 and Γ_3; Γ_3 and Γ_4; Γ_4 and Γ_1 meet at A, B, C, D, respectively, and that all these points are different from P. Prove that

$$\frac{AB \cdot BC}{AD \cdot DC} = \frac{PB^2}{PD^2}.$$

3. (IMO 1996) Let P be a point inside a triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

   Let D, E be the incenters of triangles APB, APC, respectively. Show that the lines AP, BD, CE meet at a point.

4. Let ω be the circumcircle of ABC, l be the tangent line to the circle ω at point A. The circles ω_1 and ω_2 touch lines l, BC and the circle ω externally. Denote by D, E the points where ω_1, ω_2 touch BC. Prove that the circumcircles of triangles ABC and ADE are tangent.

5. (Sharygin 2014 CR) We say points A, B, C, D are *triharmonic* if

$$AB \cdot CD = AC \cdot BD = AD \cdot BC$$

   Let W, X, Y, Z be triharmonic, and let W_1 be a point distinct from W_1 such that W_1, X, Y, Z are triharmonic, and similarly define X_1, Y_1, Z_1. Prove that W_1, X_1, Y_1, Z_1 are triharmonic.

6. (Shoemaker's Knife) Let A, B, C be three collinear points (in that order) and construct semicircles $\Gamma_{AC}, \Gamma AB, w_0$ on the same side of AC with diameters AC, AB, BC respectively. For each positive integer k, let w_k be the circle tangent to Γ_{AC}, Γ_{AB} and w_{k-1}. Show that for any natural n, the distance from the center of w_n to AC is n times its diameter.

7. Circle w is tangent to segment PQ (at C) and circle (PQ) internally. Let AB be a chord in (PQ) perpendicular to PQ but tangent to w (such that P, w lie on the same side of AB). Show that AC bisects $\angle PAB$.

8. (Steiner's Porism) Given two nonintersecting circles Ω_1, Ω_2 where one contains the other in its interior, let $\omega_1, \omega_2, ..., \omega_n$ be circles such that ω_k is tangent to $\omega_{k+1}, \omega_{k-1}, \Omega_1, \Omega_2$, where the indices are taken modulo n. Show that if given (Ω_1, Ω_2, n), there exists one possible set of $\omega_1, \omega_2, ..., \omega_n$, then such a set can be constructed regardless of the position of ω_1.

2.2 Symmetry

Inversion is like reflection for circles; objects in the same diagram invert to one another.

Instead of drawing a completely new diagram, the big idea here is that with the correct choice of point and power, we can overlay the results of inversion onto the original diagram. That is, we can say that features in the diagram have additional symmetry due to inversion.

The advantage of having symmetric things are of course, you get to make additional symmetric properties for free.

This is a fundamentally different way to think about inversion. I remember when I first learnt inversion, I only knew how to do it the classic way. So when we saw this problem, we didn't really know what to do with it.

Example. $\triangle ABC$ is inscribed in circle Γ, and circle ω is tangent to sides AB, AC and internally tangent to Γ at D. Let the A-excircle touch BC at E. Show $\angle BAD = \angle CAE$.

Can you see why? Inverting about A, we get the exact same diagram! If we think about this the classic way, the diagram did not become simpler (since it is exactly the same diagram) and so inversion doesn't give any advantage. However, the way we want to think about this is: if we can get the same diagram, why not just have a single diagram, and make things symmetric under inversion?

Example. (NTST 2018 / ISL 2017 G3) Let O and H be the circumcenter and orthocenter of $\triangle ABC$ respectively. The altitudes from B, C (in $\triangle ABC$) intersect AO at P, Q respectively. Show that the circumcenter of $\triangle PQH$ lies on the median from A (in $\triangle ABC$).

Sketch. Let B', C' be the feet of the altitudes from C, B respectively. Invert about A with power $AB \cdot AB'$. Then $B, C \leftrightarrow B', C'$ and H' is the foot of the altitude from A. Subsequently, P' must lie on AO while satisfying $\angle AP'C = 90°$, so it is the foot of the perpendicular dropped from C to AO. Analogously, Q' is the foot of the perpendicular dropped from B to AO.

It suffices to show that the circumcenter of $\triangle P'Q'H'$ still lies on the median. From here, it is easy to show that the circumcenter is precisely the midpoint of BC. $\square$

Extras

1. In a "$\triangle ABC$" setting, it is often useful to consider instead an *inverflection*: that is, an inversion about A with power $AB \cdot AC$ composed with a reflection about the angle bisector of C. What happens is:

   (a) $B \longleftrightarrow C$

   (b) line $BC \longleftrightarrow (ABC)$

   (c) $I \longleftrightarrow I_A$, where I, I_A are the incenter and A-excenter respectively.

   (d) $O \longleftrightarrow A'$, where A' is the reflection of A about BC.

   (e) For points X, Y, $\triangle AXY \sim \triangle AX'Y'$.

   (f) The list goes on, try to come up with your own!

   For example, let's use this on the previous example:

   Example. $\triangle ABC$ is inscribed in circle Γ, and circle ω is tangent to sides AB, AC and internally tangent to Γ at D. Let the A-excircle touch BC at E. Show $\angle BAD = \angle CAE$.

   Sketch. Consider an inverflection about A (as defined above). Since $BC \leftrightarrow (ABC)$, and AB, AC are constant, thus $\omega \leftrightarrow A$-excircle, hence $D \leftrightarrow E$. $\square$

   Of course, this is not the only power that gives interesting results. Go experiment!

2. If $ABCDEF$ is a complete quadrilateral ($ABCD$ is a convex quadrilateral) with Miquel point M, then there is an inverflection that swaps $A \leftrightarrow C, B \leftrightarrow D, E \leftrightarrow F$.

   More generally, any configuration good for spiral similarity will be good for inverflection as well.

3. Sometimes it's good to invert about the orthocenter! But to do that, you need to use a negative power (i.e. $\overrightarrow{OA}$ and $\overrightarrow{OA'}$ go in opposite directions). Then ($\triangle ABC$, D, E, F feet of altitudes) you can have $A \leftrightarrow D, B \leftrightarrow E, C \leftrightarrow F$ among other things.

Problems

1. Circles ω_1, ω_2 are externally tangent to each other at P. Let their common external tangents intersect at A, and let ω_1, ω_2 touch an external tangent at B, C respectively. Show that AP is tangent to the circumcircle of BPC.
2. (Kürschák 2012) Let the excenters of triangle ABC opposite vertices A,B be I_a, I_b respectively. Let PQ be a chord of the circumcircle of ABC which is parallel to AB, and let CP intersect line AB at R. Show that $\angle I_aQI_b + \angle I_aRI_b = 180°$.
3. Let H be the orthocenter of triangle ABC. Let P, Q, R be the second points of intersection of circles with diameters AH, BH, CH and the circumcircle of ABC, respectively. Points X, Y, Z are the reflections of H with respect to BC, CA, AB, respectively. Show that the lines PX, QY and RZ are concurrent.
4. (CWMI 2016) Let $\odot O_1$ and $\odot O_2$ intersect at P and Q, their common external tangent touches $\odot O_1$ and $\odot O_2$ at A and B respectively. A circle Γ passing through A and B intersects $\odot O_1$, $\odot O_2$ at D, C. Prove that $\dfrac{CP}{CQ} = \dfrac{DP}{DQ}$.
5. (IMO 2010) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, M respectively. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.
6. Let I, I_A be the incenter and A-excenter of $\triangle ABC$ respectively. Let M be the midpoint of minor arc BC (on (ABC)) and let G be the point of tangency between the A-excircle. If (AI_A) meets (ABC) at $P(\neq A)$, prove that M, G, P are collinear.
7. Let H be the orthocenter of scalene $\triangle ABC$. Let P, Q, R be the second points of intersection of circles with diameters AH, BH, CH and the circumcircle of ABC, respectively. Points X, Y, Z are the reflections of H with respect to BC, CA, AB, respectively. Show that the lines PX, QY and RZ are concurrent.
8. (Serbia MO 2013) Let M, N and P be midpoints of sides BC, AC and AB, respectively, and let O be circumcenter of acute-angled triangle ABC. Circumcircles of triangles BOC and MNP intersect at two different points X and Y inside of triangle ABC. Prove that
$$\angle BAX = \angle CAY.$$
9. (ELMO Shortlist 2014) In triangle ABC with incenter I and circumcenter O, let A', B', C' be the points of tangency of its circumcircle with its A, B, C-mixtilinear circles, respectively. Let ω_A be the circle through A' that is tangent to AI at I, and define ω_B, ω_C similarly. Prove that $\omega_A, \omega_B, \omega_C$ have a common point X other than I, and that $\angle AXO = \angle OXA'$.

More Problems

For the people who do 5 geom problems a second.

1. (Feuerbach) Show that the nine-point circle (circumcircle of the midpoints) of a triangle is tangent to the incircle and the three excircles of the triangle.

2. (ELMO 2016/2) Given $\triangle ABC$ and a point D such that DB and DC are tangent to the circumcircle of ABC, let B' be the reflection of B over AC and C' be the reflection of C over AB. If O is the circumcenter of $DB'C'$, prove that AO is perpendicular to BC.

3. (USAMO 2009/5) Trapezoid $ABCD$, with $AB//CD$, is inscribed in circle ω and point G lies inside $\triangle BCD$. Rays AG and BG meet ω again at points P and Q, respectively. Let the line through G parallel to AB intersects BD and BC at points R and S, respectively. Prove that quadrilateral $PQRS$ is cyclic if and only if BG bisects $\angle CBD$.

4. (RMM 2011/3) A triangle ABC is inscribed in a circle ω. A variable line ℓ chosen parallel to BC meets segments AB, AC at points D, E respectively, and meets ω at points K, L (where D lies between K and E). Circle γ_1 is tangent to the segments KD and BD and also tangent to ω, while circle γ_2 is tangent to the segments LE and CE and also tangent to ω. Determine the locus, as ℓ varies, of the meeting point of the common inner tangents to γ_1 and γ_2.

5. (China TST 2014) Given circle O with radius R, the inscribed triangle ABC is an acute scalene triangle, where AB is the largest side. AH_A, BH_B, CH_C are heights on BC, CA, AB. Let D be the symmetric point of H_A with respect to H_BH_C, E be the symmetric point of H_B with respect to H_AH_C. P is the intersection of AD, BE, H is the orthocentre of $\triangle ABC$. Prove: $OP \cdot OH$ is fixed, and find this value in terms of R.

6. (ISL 2011 G4) Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.

7. (IMO 2014/3) Convex quadrilateral $ABCD$ has $\angle ABC = \angle CDA = 90°$. Point H is the foot of the perpendicular from A to BD. Points S and T lie on sides AB and AD, respectively, such that H lies inside triangle SCT and

   $$\angle CHS - \angle CSB = 90°, \quad \angle THC - \angle DTC = 90°.$$

   Prove that line BD is tangent to the circumcircle of triangle TSH.

8. (IMO 2015/3) Let ABC be an acute triangle with $AB > AC$. Let Γ be its cirumcircle, H its orthocenter, and F the foot of the altitude from A. Let M be the midpoint of BC. Let Q be the point on Γ such that $\angle HQA = 90°$ and let K be the point on Γ such that $\angle HKQ = 90°$. Assume that the points A, B, C, K and Q are all different and lie on Γ in this order.

   Prove that the circumcircles of triangles KQH and FKM are tangent to each other.

9. (ISL 2012 G8) Let ABC be a triangle with circumcircle ω and ℓ a line without common points with ω. Denote by P the foot of the perpendicular from the center of ω to ℓ. The side-lines BC, CA, AB intersect ℓ at the points X, Y, Z different from P. Prove that the circumcircles of the triangles AXP, BYP and CZP have a common point different from P or are mutually tangent at P.

10. (ISL 2010 G7) Three circular arcs γ_1, γ_2, and γ_3 connect the points A and C. These arcs lie in the same half-plane defined by line AC in such a way that arc γ_2 lies between the arcs γ_1 and γ_3. Point B lies on the segment AC. Let h_1, h_2, and h_3 be three rays starting at B, lying in the same half-plane, h_2 being between h_1 and h_3. For $i, j = 1, 2, 3$, denote by V_{ij} the point of intersection of h_i and γ_j (see the Figure below). Denote by $\widehat{V_{ij}V_{kj}}\widehat{V_{kl}V_{il}}$ the curved quadrilateral, whose sides are the segments $V_{ij}V_{il}$, $V_{kj}V_{kl}$ and arcs $V_{ij}V_{kj}$ and $V_{il}V_{kl}$. We say that this quadrilateral is *circumscribed* if there exists a circle touching these two segments and two arcs. Prove that if the curved quadrilaterals $\widehat{V_{11}V_{21}}\widehat{V_{22}V_{12}}, \widehat{V_{12}V_{22}}\widehat{V_{23}V_{13}}, \widehat{V_{21}V_{31}}\widehat{V_{32}V_{22}}$ are circumscribed, then the curved quadrilateral $\widehat{V_{22}V_{32}}\widehat{V_{33}V_{23}}$ is circumscribed, too.